NONLINEAR UNITY-FEEDBACK SYSTEMS AND Q-PARAMETRIZATION
(Improved Version)

by

C. A. Desoer and C. A. Lin

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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C. A. Desoer and C. A. Lin
Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

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Abstract†

This paper concerns nonlinear systems, defines a new concept of stability and extends to nonlinear unity-feedback systems the technique of Q-parametrization introduced by Zames and developed by Desoer, Chen and Gustafson. We specify 1) a global parametrization of all controllers that $\mathcal{S}$-stabilize a given $\mathcal{S}$-stable plant; 2) a parametrization of a class of controllers that stabilize an unstable plant; 3) necessary and sufficient conditions for a nonlinear controller to simultaneously stabilize two nonlinear plants.

† This is an improved version (i.e., better theorem statements and streamlined proofs) of the paper with the same title to be published in International Journal of Control in the summer of 1984. Research sponsored by the National Science Foundation grant ECS-8119763.

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I. INTRODUCTION

The purpose of this paper is to obtain the broadest generalization within the context of nonlinear systems of a number of recent results pertaining to linear feedback systems. For the unity feedback configuration and for a given linear stable plant, Zames (1981) proposed a parametrization of the stabilizing linear controllers in terms of a stable proper transfer function Q. This idea was further developed as a design procedure by Desoer and Chen (1981) and was used for computer aided design by Gustafson and Desoer (1983). In this paper we use also a Q-parametrization but in a nonlinear context. We first generalize the concept of finite-gain stability (incremental stability) to that of $\mathcal{S}$-stability (incremental $\mathcal{S}$-stability, resp.). In Theorem 1, we establish for the nonlinear case, a global parametrization of all I/O maps and of all compensators that result in an $\mathcal{S}$-stable configuration. This theorem generalizes to the nonlinear case, the original linear results of Zames, and in view of the more general stability concept, it also generalizes Desoer and Liu (1981).

$$\|Hx-Hx'\|_T \leq \tilde{\phi}(\|x-x'\|_T)$$

It can be shown that if the nonlinear causal maps H_1 and H_2 are $\mathcal{S}$-stable, (incr. $\mathcal{S}$-stable), then H_1+H_2 and $H_1 \circ H_2$ are $\mathcal{S}$-stable, (incr. $\mathcal{S}$-stable, resp.). (For simplicity, we drop in the following the symbol "$\circ$" denoting the composition of the maps.)

A feedback system is said to be <u>well-posed</u> iff the relation from the exogenous inputs into each subsystem† variable (i.e., subsystem input and subsystem output) is a well-defined nonlinear causal map between the corresponding extended spaces. More precisely, the system ${}^1S(P,C)$ of Fig. 1, where $P:\mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$, $C:\mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ are causal maps, is said to be <u>well-posed</u> iff $H:(u_1,u_2) \mapsto (e_1,e_2,y_1,y_2)$ is well-defined and causal. Note that ${}^1S(P,C)$ is well-posed implies that†† $(I+PC)^{-1}$ and $(I+CP)^{-1}$ are well-defined and causal. We say that a well-posed nonlinear feedback system is <u>$\mathcal{S}$-stable</u> (incr. $\mathcal{S}$-stable) iff the map from the exogenous inputs to any subsystem variable is $\mathcal{S}$-stable (incre. $\mathcal{S}$-stable, resp.). For the system ${}^1S(P,C)$, since $e_1 = u_1 - y_2$, $e_2 = u_2 + y_1$, we see that $H_{yu}:(u_1,u_2) \mapsto (y_1,y_2)$ is $\mathcal{S}$-stable iff $H_{eu}:(u_1,u_2) \mapsto (e_1,e_2)$ is $\mathcal{S}$-stable iff ${}^1S(P,C)$ is $\mathcal{S}$-stable. The same equivalence holds for <u>incr. $\mathcal{S}$-stability</u>. These concepts of $\mathcal{S}$-stability and incr. $\mathcal{S}$-stability are generalizations of finite-gain stability and incremental stability (Desoer and Vidyasagar 1975); they are in spirit closer to Safonov's work (Safonov 1980). The map $H:\mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_o}$ is said to be <u>an achievable I/O map</u> of the nonlinear feedback system ${}^1S(P,C)$ iff by some appropriate choice of $C:\mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$, (i) $H_{y_2u_1} = H$; (ii) ${}^1S(P,C)$ is $\mathcal{S}$-stable.

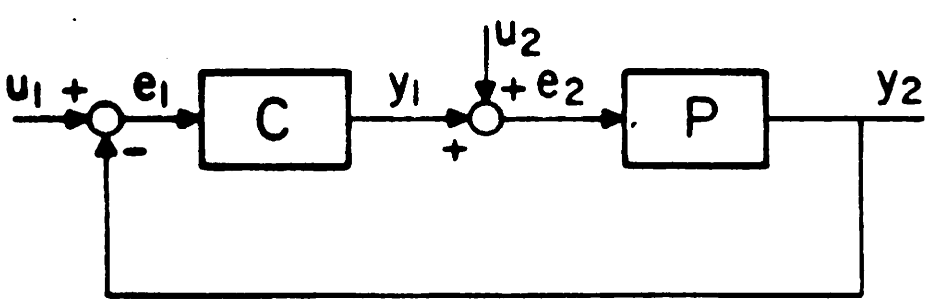

Fig. 1. Shows the system ${}^1S(P,C)$.

†By subsystem we mean any block of the block diagram of the feedback system.

††The meaning of $(I+PC)^{-1}$ deserves clarification: the map C is composed with P then the identity is added, and the resulting map is inverted. Although this formula has the same form as the linear case, it has a completely different interpretation.

In Section IV we consider the case where the plant is unstable. For the linear case, Zames established his "decomposition principle," i.e., stabilize the given linear plant P with a stable linear compensator F, and then proceed with the Q-parametrization as above. Anantharam and Desoer (1982) established a nonlinear version of this result. In Theorem 2 we establish a similar result in the more general concept of $\mathcal{S}$-stability and we weaken the requirement on the stabilizing feedback F: it need not be itself stable but need only lead to a stable feedback configuration of P and F. Note that Theorem 2 generalizes our previous work, first it uses the more general stability concept and, second, the method of proof is greatly improved (Desoer and Lin 1983a).

The problem of simultaneous stability has been formulated and solved in the linear case by Saeks and Murray (1982). Vidyasagar and Viswanadham (1982) also have interesting results along this line. In Section V we consider the nonlinear case: we are given two (possibly unstable) nonlinear plants $\bar{P}_1$ and $\bar{P}_2$ and we derive necessary and sufficient conditions for the existence of a fixed compensator that stabilizes both plants. Theorem 4 is a generalization for nonlinear plants and within the $\mathcal{S}$-stability concept of the linear results of Vidyasagar et al., and of our previous work (Desoer and Lin 1983b).

II. DEFINITIONS AND NOTATIONS

Let $(\mathcal{L},\|\cdot\|)$ be a normed space of "time functions": $\mathcal{T} \to \mathcal{V}$ where $\mathcal{T}$ is the time set (typically $\mathbb{R}_+$ or $\mathbb{N}$), $\mathcal{V}$ is a normed space (typically $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}^n$, $\cdots$) and $\|\cdot\|$ is the chosen norm in $\mathcal{L}$. Let $\mathcal{L}_e$ be the corresponding extended space (see e.g. Willems 1971, Desoer and Vidyasagar 1975, Vidyasagar 1978).

A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class K iff ϕ is continuous and increasing. ϕ is said to belong to class K_o iff $\phi \in K$ and $\phi(0) = 0$. If ϕ_1 and $\phi_2 \in K_o$, then $\phi_1 + \phi_2$ and $\alpha \mapsto \phi_1(\phi_2(\alpha)) \in K_o$. A nonlinear causal map $H : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ is said to be $\mathcal{S}$-stable iff $\exists\ \phi \in K$ s.t. $\forall x \in \mathcal{L}_e^{n_i}$, $\forall T \in \mathcal{T}$,

$$\|Hx\|_T \leq \phi(\|x\|_T)$$

H is said to be incrementally $\mathcal{S}$-stable (incr. $\mathcal{S}$-stable) iff (i) H is $\mathcal{S}$-stable, (ii) $\exists\ \tilde{\phi} \in K_o$ s.t. $\forall x, x' \in \mathcal{L}_e^{n_i}$, $\forall T \in \mathcal{T}$, We assume throughout this paper that all the nonlinear maps under consideration are causal and that all the nonlinear feedback systems under consideration are well-posed. We use "s.t." to abbreviate "such that," and "u.t.c." to abbreviate "under these conditions."

III. GLOBAL PARAMETRIZATION OF NONLINEAR $\mathcal{S}$-STABLE I/O MAPS

Consider the well-posed nonlinear unity feedback system ${}^1S(P,C)$ shown in Fig. 1, where $P:\mathcal{L}_e^{n_i}\to\mathcal{L}_e^{n_o}$, $C:\mathcal{L}_e^{n_o}\to\mathcal{L}_e^{n_i}$ are nonlinear causal maps, and (u_1,u_2), (y_1,y_2) and (e_1,e_2) are the "input," "output," and "error" respectively. Theorem 1 is a generalization of a result of Desoer and Liu (1981), it gives a global parametrization of all achievable input-output maps, and of all stabilizing compensators, under the assumption that P is incr. $\mathcal{S}$-stable. This theorem is an extension to the nonlinear case, the well-known linear Q-parametrization result, proved by Zames (1981) in a very general algebraic context.

Theorem 1. (Global parametrization of stable ${}^1S(P,C)$).
Let $P:\mathcal{L}_e^{n_i}\to\mathcal{L}_e^{n_o}$, $C:\mathcal{L}_e^{n_o}\to\mathcal{L}_e^{n_i}$ be nonlinear causal maps. Assume that P is incr. $\mathcal{S}$-stable. Under these conditions (U.t.c.),
(a) H_{yu} is $\mathcal{S}$-stable $\Leftrightarrow\exists$ some $\mathcal{S}$-stable $Q:\mathcal{L}_e^{n_o}\to\mathcal{L}_e^{n_i}$ s.t.

$$C = Q(I-PQ)^{-1} \tag{3.1}$$

(b) $C = Q(I-PQ)^{-1} \Leftrightarrow Q = C(I+PC)^{-1}$ (3.2)

(c) With $u_2 = 0$ and with $C = Q(I-PQ)^{-1}$, the partial map $H_{y_2u_1}:(u_1,0)\mapsto y_2$ is given by

$$H_{y_2u_1} = PQ \tag{3.3}$$

Comments
(i) Equivalence (b) above requires only that ${}^1S(P,C)$ be well-posed.
(ii) Equivalence (a) gives a global parametrization of $\mathcal{C}(P)$, the family of all compensators that result in an $\mathcal{S}$-stable system ${}^1S(P,C)$; more precisely:

$$\mathcal{C}(P) = \{C \mid C = Q(I-PQ)^{-1},\ \ Q \text{ is } \mathcal{S}\text{-stable}\}\ .$$

(iii) From (a) and (c), $\mathcal{H}_{y_2u_1}$, the class of all achievable I/O maps is given by

$$\mathcal{H}_{y_2u_1}(P) = \{PQ | Q \text{ is } \mathcal{S}\text{-stable}\} .$$

(iv) Practical design considerations such as robustness of stability, disturbance rejection, plant saturation, etc. impose additional restrictions on Q (see e.g., Desoer and Chen 1981, Gustafson and Desoer 1983).

(v) The equation (3.3), $H_{y_2u_1} = PQ$, raises a number of new problems: given a non-linear map P, how can one describe the constraints imposed by P on the achievable I/O map $H_{y_2u_1}$? If we have a desired I/O map $H_{y_2u_1}$ and a given P, how does one find a Q_a such that in some appropriate sense, $PQ_a \approx H_{y_2u_1}$? Then having such a Q_a how does one synthesize C?

Proof:

(I) Proof of (b).

We shall prove only the ($\Rightarrow$) implication, since the ($\Leftarrow$) implication can be shown in the same way. By assumption,

$$C = Q(I-PQ)^{-1} .$$

Composing with P and adding identity we obtain successively,

$$I + PC = I + PQ(I-PQ)^{-1} = (I-PQ)^{-1}$$

By taking the inverse, and composing with C, we obtain

$$C(I+PC)^{-1} = Q(I-PQ)^{-1}(I-PQ) = Q$$

Hence, $\quad Q = C(I+PC)^{-1}$.

(II) Proof of (a).

($\Rightarrow$) Set $u_2 = 0$, the map $H_{y_1u_1} : u_1 \mapsto y_1$ is given by $H_{y_1u_1} = C(I+PC)^{-1}$ which by assumption is $\mathcal{S}$-stable. Let $Q := C(I+PC)^{-1}$, then Q is $\mathcal{S}$-stable and from (b), we have $C = Q(I-PQ)^{-1}$.

($\Leftarrow$) Refer to Fig. 1, write the summing node equations

$$e_1 = u_1 - Pe_2 \qquad e_2 = u_2 + Ce_1 \tag{3.4) (3.5}$$

Define

$$\tilde{u}_1 := PC\, e_1 - P(u_2+Ce_1) \tag{3.6}$$

Using (3.5) and (3.6), rewrite (3.4) as

$$e_1 = u_1 + \tilde{u}_1 - PC\, e_1 \tag{3.7}$$

From equation (3.7)

$$e_1 = (I+PC)^{-1}(u_1+\tilde{u}_1) \tag{3.8}$$

$$y_1 = Ce_1 = C(I+PC)^{-1}(u_1+\tilde{u}_1) = Q(u_1+\tilde{u}_1) \tag{3.9}$$

Now, since P is incr. $\mathcal{S}$-stable, $\exists\, \tilde{\phi}_p \in K_o$ s.t. $\forall^{\dagger}(u_1, u_2) \in \mathcal{L}_e^{n_o} \times \mathcal{L}_e^{n_i}$, $\forall T \in \mathcal{J}$,

$$\|\tilde{u}_1\|_T = \|P(Ce_1)-P(u_2+Ce_1)\|_T \le \tilde{\phi}_p(\|u_2\|_T) \le \tilde{\phi}_p(\|u_1\|_T+\|u_2\|_T) \tag{3.10}$$

Hence the map $\tilde{\pi} : (u_1,u_2) \mapsto \tilde{u}_1$ is $\mathcal{S}$-stable. Define the projection map $\pi_i : (u_1,u_2) \mapsto u_i$, $i = 1,2$. From (3.9), the map $H_{y_1u} : (u_1,u_2) \mapsto y_1$ is given by

$$H_{y_1u} = Q(\pi_1+\tilde{\pi}) \tag{3.11}$$

Since π_1 and $\tilde{\pi}$ are $\mathcal{S}$-stable, and by assumption Q is $\mathcal{S}$-stable, the map H_{y_1u} is $\mathcal{S}$-stable. From Fig. 1, we have

$$y_2 = P(u_2+y_1) \tag{3.12}$$

Hence the map $H_{y_2u} : (u_1,u_2) \mapsto y_2$ is given by

$$H_{y_2u} = P(\pi_2+H_{y_1u}) \tag{3.13}$$

Now π_2 and H_{y_1u} are $\mathcal{S}$-stable, and by assumption P is $\mathcal{S}$-stable, it follows that H_{y_2u} is $\mathcal{S}$-stable. Therefore H_{yu} is $\mathcal{S}$-stable.

<u>(III) Proof of (c).</u>

Since $C = Q(I-PQ)^{-1}$, from (b) we have $Q = C(I+PC)^{-1}$. With $u_2 = 0$, $H_{y_1u_1} = H_{e_2u_1}$ $= C(I+PC)^{-1} = Q$.

Hence, $\quad H_{y_2u_1} = PQ$. □

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†For $(u_1,u_2) \in \mathcal{L}^{n_o} \times \mathcal{L}^{n_i}$, we define $\|(u_1,u_2)\| := \|u_1\| + \|u_2\|$.

IV. Two-Step Stabilization of Nonlinear Plants

The equivalence (a) of Theorem 1 above requires that the plant be incr. $\mathcal{S}$-stable. In practice, unstable plants do occur (e.g., chemical reactors, high performance airplanes, etc.), it is important to extend this method to include unstable plants.

Theorem 2. (Two-step stabilization of nonlinear plants).

Let $P : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$, $F : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ be nonlinear causal maps such that the system ${}^1S(P,F)$ shown in Fig. 2 is incr. $\mathcal{S}$-stable. Let $P_1 := P[I-F(-P)]^{-1}$.

U.t.c., if

$$C := F + Q(I-P_1Q)^{-1} \text{ for some } \mathcal{S}\text{-stable } Q : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}, \qquad (4.1)$$

then

(a) the system ${}^1S(P,C)$ is $\mathcal{S}$-stable; and

(b) the system ${}^3S(P,F,C-F)$ shown in Fig. 3 is $\mathcal{S}$-stable.

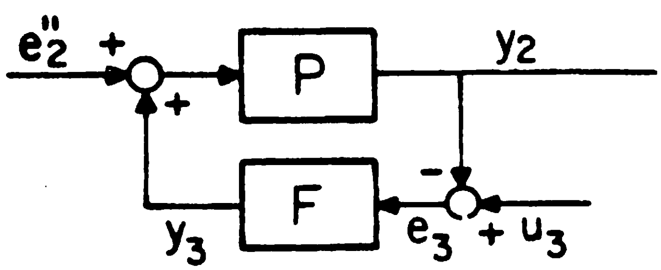

Fig. 2. Shows the system ${}^1S(P,F)$ in which F stabilizes P.

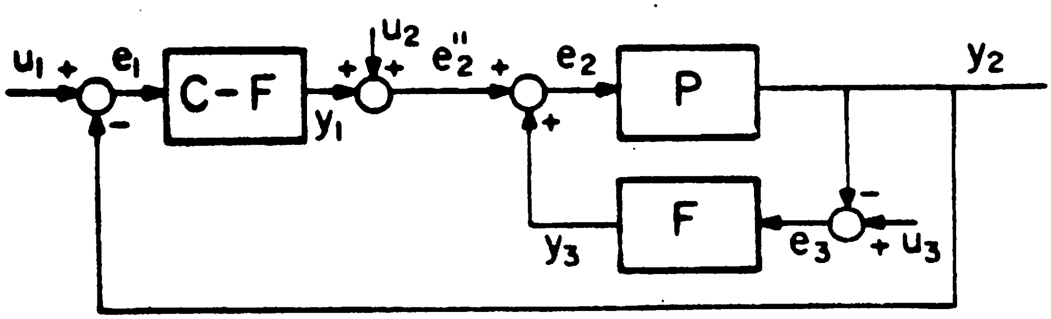

Fig. 3. Shows the system ${}^3S(P,F,C-F)$.

Comments.

(i) None of the maps P, C, F, C-F are required to be stable.

(ii) The key assumptions are (a) well-posedness, (b) ${}^{1}S(P,F)$ is incr. $\mathcal{S}$-stable, (c) $C = F + Q(I-P_1Q)^{-1}$ where $P_1 = P[I-F(-P)]^{-1}$ and Q is $\mathcal{S}$-stable.

(iii) It can be easily checked (using the summing node equations) that the system ${}^{3}S(P,F,C-F)$ is $\mathcal{S}$-stable iff the map $(u_1,u_2,u_3) \mapsto (y_1,y_2,y_3)$ is $\mathcal{S}$-stable.

(iv) If P is incr. $\mathcal{S}$-stable, then by choosing F the zero map, we have $P_1 = P$, $C = Q(I-PQ)^{-1}$, and Theorem 2 reduces to Theorem 1.

(v) In the proof we show that (b) implies (a), a simple example shows that (a) does not imply (b). However, if F is incr. $\mathcal{S}$-stable, then (a) and (b) are equivalent (Anantharam and Desoer 1982, Thm. 3).

Proof:

(I) Proof of (b): ${}^{3}S(P,F,C-F)$ is $\mathcal{S}$-stable.

Consider the system ${}^{1}S(P,F)$ of Fig. 2, let $\psi = (\psi_2,\psi_3) : (e_2'',u_3) \mapsto (y_2,y_3)$ be its I/O map. Note that $P_1(\cdot) := P[I-F(-P)]^{-1}(\cdot) = \psi_2(\cdot,0)$. By (A.2), ψ is incr. $\mathcal{S}$-stable, hence P_1 is incr. $\mathcal{S}$-stable, further from assumption (4.1), Q is $\mathcal{S}$-stable and $C-F = Q(I-P_1Q)^{-1}$; hence, by Theorem 1, these three conclusions imply that the system ${}^{1}S(P_1,C-F)$ shown in Fig. 4 is $\mathcal{S}$-stable.

Next consider Fig. 3 which shows the system ${}^{3}S(P,F,C-F)$ with input (u_1,u_2,u_3) and output (y_1,y_2,e_2'',y_3). We claim that the map ${}^{3}H : (u_1,u_2,u_3) \mapsto (y_1,y_2,e_2'',y_3)$ is $\mathcal{S}$-stable. Let

$$\Delta\tilde{y}_2 := \psi_2(e_2'',u_3) - \psi_2(e_2'',0) \ . \tag{4.2}$$

Drive the system ${}^{3}S(P,F,C-F)$ with input $(u_1-\Delta\tilde{y}_2,u_2,0)$, call the corresponding output $(\tilde{y}_1,\tilde{y}_2,\tilde{e}_2'',\tilde{y}_3)$, and note that $\tilde{y}_2 = P[I-F(-P)]^{-1}\tilde{e}_2'' = P_1\tilde{e}_2''$; thus if we ignore $\tilde{y}_3$, the system reduces to ${}^{1}S(P_1,C-F)$, (which has just been shown to be $\mathcal{S}$-stable), with input

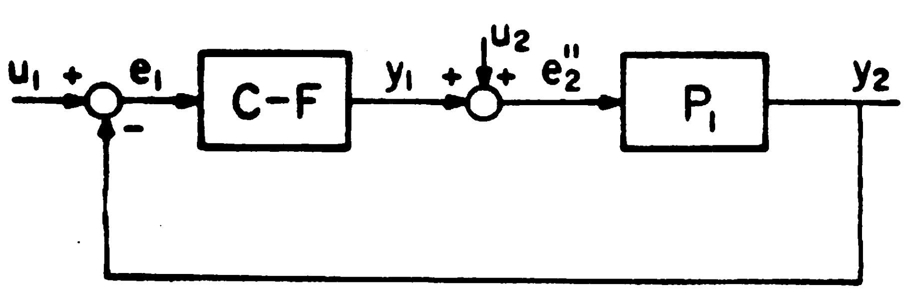

Fig. 4. Shows the system ${}^{1}S(P_1,C-F)$.

$(u_1-\Delta\tilde{y}_2,u_2)$ and output $(\tilde{y}_1,\tilde{y}_2,\tilde{e}''_2)$. Hence, for ${}^3S(P,F,C-F)$, the partial map (with respect to ${}^3H$), ${}^2\tilde{H}:(u_1-\Delta\tilde{y}_2,u_2,0)\mapsto(\tilde{y}_1,\tilde{y}_2,\tilde{e}''_2)$ is $\mathcal{S}$-stable. Since ψ_2 is incr. $\mathcal{S}$-stable, $\exists\, \tilde{\phi}_2\in K_o$ s.t. $\forall e''_2$, $\forall u_3$, $\forall T$,

$$\|\Delta\tilde{y}_2\|_T = \|\psi_2(e''_2,u_3)-\psi_2(e''_2,0)\|_T \le \tilde{\phi}_2(\|u_3\|_T) \le \tilde{\phi}_2(\|u_1\|_T+\|u_2\|_T+\|u_3\|_T) \tag{4.3}$$

Hence the map $(u_1,u_2,u_3)\mapsto\Delta\tilde{y}_2$ is $\mathcal{S}$-stable. Therefore, the map ${}^2\pi:(u_1,u_2,u_3)\mapsto(u_1-\Delta\tilde{y}_2,u_2,0)$ is $\mathcal{S}$-stable. Considering the composition ${}^2\tilde{H}\,{}^2\pi$ we see that, for ${}^3S(P,F,C-F)$, the map $(u_1,u_2,u_3)\mapsto(\tilde{y}_1,\tilde{y}_2,\tilde{e}''_2)$ is $\mathcal{S}$-stable.

Now, we claim that $y_1=\tilde{y}_1$, $y_2=\tilde{y}_2+\Delta\tilde{y}_2$, $e''_2=\tilde{e}''_2$, and hence the map $(u_1,u_2,u_3)\mapsto(y_1,y_2,e''_2)$ is $\mathcal{S}$-stable. To prove this, write the equations for ${}^3S(P,F,C-F)$ with input (u_1,u_2,u_3) and with input $(u_1-\Delta\tilde{y}_2,u_2,0)$, respectively:

$$y_1 = (C-F)(u_1-y_2) \tag{4.4a}$$

$$y_2 = \psi_2(e''_2,u_3) \tag{4.4b}$$

$$e''_2 = y_1 + u_2 \tag{4.4c}$$

$$\tilde{y}_1 = (C-F)(u_1-\Delta\tilde{y}_2-\tilde{y}_2) \tag{4.5a}$$

$$\tilde{y}_2 = \psi_2(\tilde{e}''_2,0) \tag{4.5b}$$

$$\tilde{e}''_2 = \tilde{y}_1 + u_2 \tag{4.5c}$$

Using (4.2), rewrite the equations (4.4) as

$$y_1 = (C-F)[u_1-\Delta\tilde{y}_2-(y_2-\Delta\tilde{y}_2)] \tag{4.6a}$$

$$y_2 - \Delta\tilde{y}_2 = \psi_2(e''_2,0) \tag{4.6b}$$

$$e''_2 = y_1 + u_2 \tag{4.6c}$$

From Eqs. (4.5) and (4.6), we see that $(y_1,y_2-\Delta\tilde{y}_2,e''_2)$ and $(\tilde{y}_1,\tilde{y}_2,\tilde{e}''_2)$ satisfy the same equations, by the well-posedness assumption (A.3), Eqs. (4.5) and (4.6) both have a unique solution, hence $y_1=\tilde{y}_1$, $y_2=\tilde{y}_2+\Delta\tilde{y}_2$, $e''_2=\tilde{e}''_2$. Since $y_3=\psi_3(e''_2,u_3)$ and ψ_3 is $\mathcal{S}$-stable, the map $(u_1,u_2,u_3)\mapsto y_3$ is $\mathcal{S}$-stable. Consequently, the map ${}^3H:(u_1,u_2,u_3)\mapsto(y_1,y_2,e''_2,y_3)$ is $\mathcal{S}$-stable and (b) is established.

<u>(II) Proof of (a): ${}^1S(P,C)$ is $\mathcal{S}$-stable.</u>

Write the summing node equations for ${}^3S(P,F,C-F)$ in terms of e_1, e_2, e_3, and e''_2: (see Fig. 3),

$$e_1 = u_1 - Pe_2 \tag{4.7a}$$

$$e''_2 = u_2 + (C-F)e_1 \tag{4.7b}$$

$$e_2 = e''_2 + Fe_3 \tag{4.7c}$$

$$e_3 = u_3 - Pe_2 \tag{4.7d}$$

Let $u_1 = u_3$, then, by (4.7a) and (4.7d), $e_1 = e_3$; thus by adding (4.7c) and (4.7b) we have

$$e_1 = u_1 - Pe_2 \tag{4.8a}$$

$$e_2 = u_2 + Ce_1 \, . \tag{4.8b}$$

The equations (4.8) describe ${}^1S(P,C)$. Since ${}^3S(P,F,C-F)$ is $\mathcal{S}$-stable, the map $(u_1,u_2,u_1) \to (e_1,e_2)$ defined by (4.8) is $\mathcal{S}$-stable. Hence ${}^1S(P,C)$ is $\mathcal{S}$-stable. ¤

V. Simultaneous Stabilization of Nonlinear Plants

The main result is Theorem 4: a necessary and sufficient condition for given two nonlinear plants be simultaneously stabilized by <u>one</u> compensator.

<u>Theorem 3</u>.

Let $\bar{P} : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ and $C,\ F : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ be nonlinear causal maps. Let $P := \bar{P}[I-F(-\bar{P})]^{-1}$.

U.t.c., if F is <u>incr. $\mathcal{S}$-stable</u>, then the system ${}^1S(\bar{P},C+F)$ of Fig. 5 is $\mathcal{S}$-stable

$\Leftrightarrow$

the system ${}^1S(P,C)$ is $\mathcal{S}$-stable.

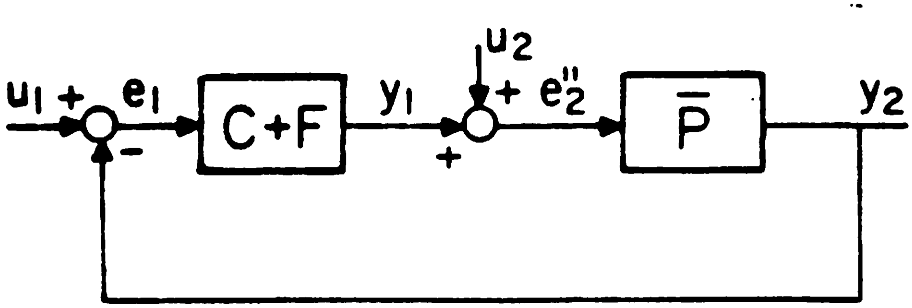

Fig. 5. Shows the system ${}^1S(\bar{P},C+F)$.

Comments.

(i) None of the maps $\overline{P}$, P, and C are required to be stable.

(ii) The Theorem is false if F is not incr. $\mathcal{S}$-stable. Consider the following example: let $\overline{P} = (s-1)/(s+3) =: \overline{n}/\overline{d}$, $F = 3/(s-1)$, and $C = 3/1 =: n_c/d_c$. By calculation, $C + F = 3s/(s-1) =: n_{c+f}/d_{c+f}$, and $P = \overline{P}[I-F(-\overline{P})]^{-1} = (s-1)/(s+6) =: n/d$. The system ${}^1S(P,C)$ is $\mathcal{S}$-stable, since its characteristic polynomial is $nn_c + dd_c = 4s + 3$. However, the system ${}^1S(\overline{P},C+F)$ is unstable, since its characteristic polynomial is $\overline{n}n_{c+f} + \overline{d}d_{c+f} = (s-1)(4s+3)$.

(iii) Traditionally the loop transformation theorem (see e.g., Desoer and Vidyasagar 1975) requires that F be linear, so Theorem 3 is a generalization of the usual stability results obtainable from the loop transformation theorem.

(iv) Roughly speaking, Theorem 3 says given that F is incr. $\mathcal{S}$-stable, and that the system ${}^1S(P,C)$ (${}^1S(\overline{P},C+F)$) is $\mathcal{S}$-stable, if we apply the feedback F (-F, resp.) around the plant and apply the feedforward F (-F, resp.) in parallel with the compensator, then the resulting closed-loop system ${}^1S(\overline{P},C+F)$ (${}^1S(P,C)$, resp.) remains $\mathcal{S}$-stable. This is also the case when the roles of the plant and the compensator are interchanged. The precise statement is given in the following corollary, whose proof can be constructed using the same techniques as those in the proof of Theorem 3.

Corollary 3.1: Let $P : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$, and $\overline{C}$, $F : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ be nonlinear casual maps. Let $C := \overline{C}(I+F\overline{C})^{-1}$.

U.t.c., if F is incr. $\mathcal{S}$-stable, then

$${}^1S(P+F,\overline{C}) \text{ is } \mathcal{S}\text{-stable} \Leftrightarrow {}^1S(P,C) \text{ is } \mathcal{S}\text{-stable}.$$

In order to prove Theorem 3, it is convenient to start by exhibiting the following lemma, whose proof is similar to that of (3.2).

Lemma: Let $\overline{P} : \mathcal{L}_e^{n_i} \mapsto \mathcal{L}_e^{n_o}$ and $F : \mathcal{L}_e^{n_o} \mapsto \mathcal{L}_e^{n_i}$. If $P := \overline{P}[I-F(-\overline{P})]^{-1}$, then $\overline{P} = P[I+F(-P)]^{-1}$.

Comments:

(i) By using relation $P = \overline{P}[I-F(-\overline{P})]^{-1}$, ($\overline{P}=P[I+F(-P)]^{-1}$), the system ${}^1S(P,C)$ of Fig. 1 (${}^1S(\overline{P},C+F)$ of Fig. 5, resp.) can be redrawn as the system of Fig. 6 (Fig. 7, resp.).

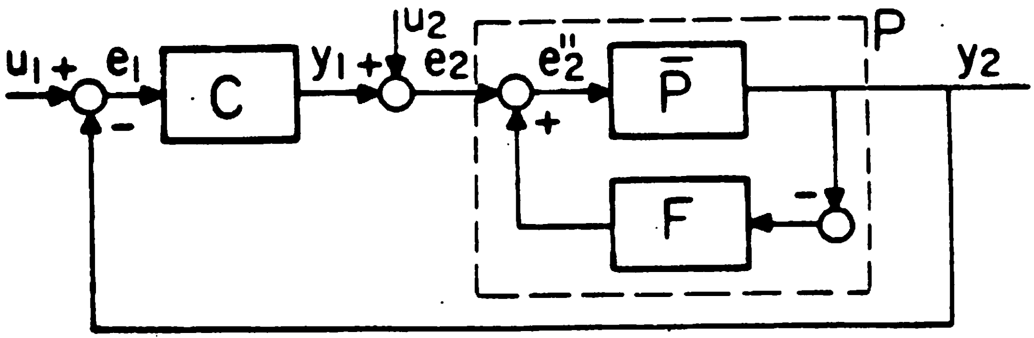

Fig. 6. Shows the system ${}^3S(\bar{P},F,C)$ with $u_3 \equiv 0$.

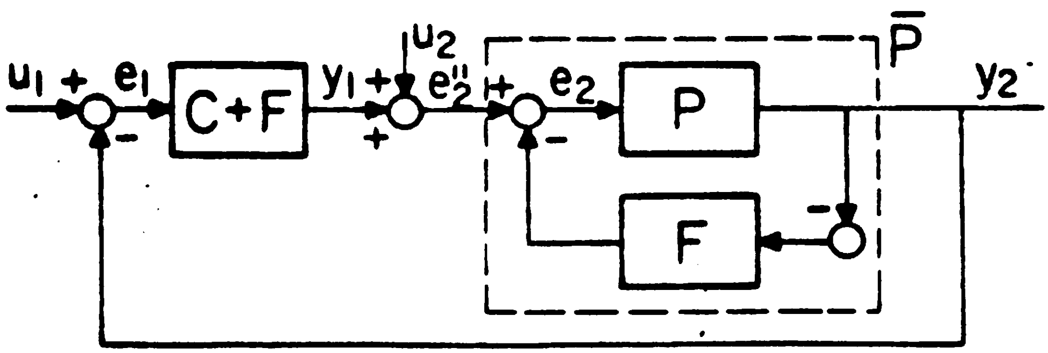

Fig. 7. Shows the system ${}^3S(P,-F,C+F)$ with $u_3 \equiv 0$.

(ii) Note that the system in Fig. 6 (Fig. 7) and the system ${}^1S(P,C)$ (${}^1S(\bar{P},C+F)$, resp.) have the same I/O map $\psi_C : (u_1,u_2) \mapsto (e_1,e_2)$ ($\psi''_{C+F} : (u_1,u_2) \mapsto (e_1,e''_2)$, resp.).

Proof of Theorem 3:

($\Rightarrow$) We show that for the system ${}^1S(P,C)$, the map $\psi_C : (u_1,u_2) \mapsto (e_1,e_2)$ is $\mathcal{S}$-stable. For the system shown in Fig. 6, write the equations defining e_1 and e''_2:

$$e_1 = u_1 - \bar{P}e''_2 \tag{5.1a}$$

$$e''_2 = u_2 + Ce_1 + F(-\bar{P}e''_2)$$

$$= u_2 + Ce_1 + F(e_1 - u_1) \tag{5.1b}$$

Rewrite (5.1b) as

$$e_2'' = u_2 + (C+F)e_1 + [F(e_1-u_1)-Fe_1] \tag{5.2}$$

Let

$$\tilde{u}_1 := u_1 \tag{5.3a}$$

$$\tilde{u}_2 := u_2 + [F(e_1-u_1)-Fe_1] \tag{5.3b}$$

Then, Eqs. (5.1) read

$$e_1 = \tilde{u}_1 - \overline{P}e_2'' \tag{5.4a}$$

$$e_2'' = \tilde{u}_2 + (C+F)e_1 \tag{5.4b}$$

Note that equations (5.4) describe ${}^1S(\overline{P},C+F)$ with input $(\tilde{u}_1,\tilde{u}_2)$; by assumption ${}^1S(\overline{P},C+F)$ is $\mathcal{S}$-stable. Hence the map $\tilde{\psi}_{C+F} : (\tilde{u}_1,\tilde{u}_2) \to (e_1,e_2'')$, specified by (5.4), is $\mathcal{S}$-stable. Since F is incr. $\mathcal{S}$-stable, $\exists\ \tilde{\phi}_F \in K_o$, s.t. $\forall u_1$, $\forall e_1$, $\forall T$,

$$\|F(e_1-u_1)-Fe_1\|_T \le \tilde{\phi}_F(\|u_1\|_T) \le \tilde{\phi}_F(\|u_1\|_T+\|u_2\|_T) \tag{5.5}$$

Hence the map $\tilde{\pi} : (u_1,u_2) \mapsto (\tilde{u}_1,\tilde{u}_2)$ defined by (5.1) and (5.3) is $\mathcal{S}$-stable. Define $\psi_C'' = \tilde{\psi}_{C+F}\tilde{\pi}$, since both $\tilde{\psi}_{C+F}$ and $\tilde{\pi}$ are $\mathcal{S}$-stable, so is $\psi_C'' : (u_1,u_2) \mapsto (e_1,e_2'')$; hence for the system of Fig. 6 the map $(u_1,u_2) \mapsto (e_1,e_2'')$ is $\mathcal{S}$-stable.

Now from Fig. 6,

$$e_2 = e_2'' - F(e_1-u_1) \tag{5.6}$$

Since ψ_C'' and F are $\mathcal{S}$-stable, the map $(u_1,u_2) \mapsto e_2$ is $\mathcal{S}$-stable. It then follows that, for the system ${}^1S(P,C)$, $\psi_C : (u_1,u_2) \mapsto (e_1,e_2)$ is $\mathcal{S}$-stable.

($\Leftarrow$) We show that, for the system ${}^1S(\overline{P},C+F)$, the map $\psi_{C+F}'' : (u_1,u_2) \mapsto (e_1,e_2'')$ is $\mathcal{S}$-stable.

Using the Lemma $\overline{P} = P[I+F(-P)]^{-1}$ and redraw ${}^1S(\overline{P},C+F)$ as in Fig. 7. Write the equations defining (e_1,e_2) in Fig. 7.

$$e_1 = u_1 - Pe_2 \tag{5.7a}$$

$$\begin{aligned} e_2 &= u_2 + (C+F)e_1 - F(-Pe_2) \\ &= u_2 + Fe_1 - F(e_1-u_1) + Ce_1 \end{aligned} \tag{5.7b}$$

Let

$$\bar{u}_1 := u_1 \tag{5.8a}$$

$$\bar{u}_2 := u_2 + Fe_1 - F(e_1-u_1) \tag{5.8b}$$

Since F is incr. $\mathcal{S}$-stable, the map $\bar{\pi} : (u_1,u_2) \to (\bar{u}_1,\bar{u}_2)$ defined by (5.7) and (5.8) is $\mathcal{S}$-stable. Now, with (5.8), equations (5.7) read

$$e_1 = \bar{u}_1 - Pe_2 \tag{5.9a}$$

$$e_2 = \bar{u}_2 + Ce_1 \tag{5.9b}$$

Note that equations (5.9) describe ${}^1S(P,C)$ with input $(\bar{u}_1,\bar{u}_2)$. By assumption ${}^1S(P,C)$ is $\mathcal{S}$-stable, hence the map $\bar{\psi}_c : (\bar{u}_1,\bar{u}_2) \to (e_1,e_2)$, specified by (5.9), is $\mathcal{S}$-stable. Define $\psi_{C+F} = \bar{\psi}_C\bar{\pi}$, since both $\bar{\psi}_C$ and $\bar{\pi}$ are $\mathcal{S}$-stable, so is $\psi_{C+F} : (u_1,u_2) \to (e_1,e_2)$; hence for the system of Fig. 7, the map $(u_1,u_2) \to (e_1,e_2)$ is $\mathcal{S}$-stable.

Now from Fig. 7,

$$e_2'' = e_2 + F(e_1-u_1) \tag{5.10}$$

Since ψ_{C+F} and F are $\mathcal{S}$-stable, equation (5.10) implies that the map $(u_1,u_2) \to e_2''$ is $\mathcal{S}$-stable. Consequently, we have shown that for the system ${}^1S(\bar{P},C+F)$, $\psi''_{C+F} : (u_1,u_2) \to (e_1,e_2'')$ is $\mathcal{S}$-stable. □

<u>Theorem 4</u>. (Simultaneous Stabilization)

Let $\bar{P}_1, \bar{P}_2 : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ and $F : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ be nonlinear causal maps. Assume that F is <u>incr. $\mathcal{S}$-stable</u> and is such that $P_1 := \bar{P}_1[I-F(-\bar{P}_1)]^{-1}$ is <u>incr. $\mathcal{S}$-stable</u>. Let $P_2 := \bar{P}_2[I-F(-\bar{P}_2)]^{-1}$. For any $C : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$, let

$$Q := C(I+P_1C)^{-1} \tag{5.11}$$

U.t.c.

${}^1S(\bar{P}_1,C+F)$ and ${}^1S(\bar{P}_2,C+F)$ are $\mathcal{S}$-stable

$\Leftrightarrow$

Q is $\mathcal{S}$-stable and ${}^1S(P_2-P_1,Q)$ is $\mathcal{S}$-stable (see Fig. 8).

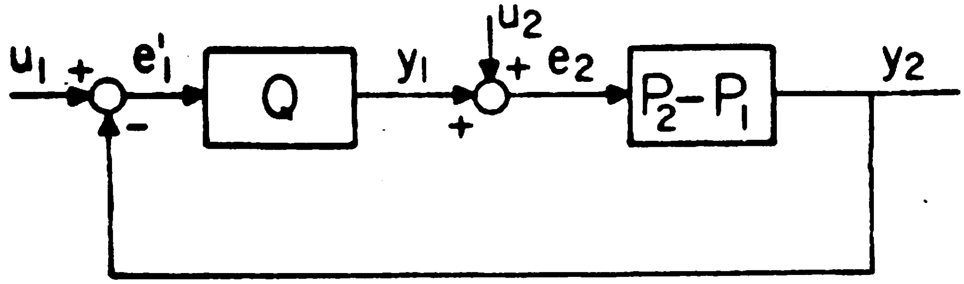

Fig. 8. Shows the system ${}^1S(P_2-P_1,Q)$.

Comments

(i) By Theorem 1, Eq. (5.11) is equivalent to that $C = Q(I-P_1Q)^{-1}$.

(ii) None of the maps $\bar{P}_1$, $\bar{P}_2$, P_2, and C are required to be stable.

(iii) The meaning of the theorem is the following: given two nonlinear, not necessarily stable, plants $\bar{P}_1$ and $\bar{P}_2$, if by applying an incr. $\mathcal{S}$-stable feedback F around $\bar{P}_1$ (see Fig. 6), the resulting closed-loop I/O map $P_1 := \bar{P}_1[I-F(-\bar{P}_1)]^{-1}$ is incr. $\mathcal{S}$-stable, then any compensator of the form $Q(I-P_1Q)^{-1} + F$, for some $\mathcal{S}$-stable Q such that ${}^1S(P_2-P_1,Q)$ is $\mathcal{S}$-stable, will stabilize both $\bar{P}_1$ and $\bar{P}_2$.

(iv) If $\bar{P}_1$ is incr. $\mathcal{S}$-stable, take $F = \Theta$, the zero map from $\mathcal{L}_e^{n_o} \leftrightarrow \mathcal{L}_e^{n_i}$, then $P_1 = \bar{P}_1$ and $P_2 = \bar{P}_2$. The theorem shows, for this special case, that given two nonlinear plants $\bar{P}_1$ and $\bar{P}_2$, with $\bar{P}_1$ incr. $\mathcal{S}$-stable, then the problem of finding a compensator to stabilize both $\bar{P}_1$ and $\bar{P}_2$ is equivalent to that of finding an $\mathcal{S}$-stable compensator to stabilize $\bar{P}_2 - \bar{P}_1$. This result was proven for the linear case in (Vidyasagar and Viswanadham 1982, Corollary 3.1.1).

(v) Suppose that we have n nonlinear plants $\bar{P}_1, \bar{P}_2, \cdots, \bar{P}_n$, then we may apply successively the theorem to the pairs $(\bar{P}_i,\bar{P}_1)$, $i = 2, 3, \cdots n$, thus ${}^1S(\bar{P}_i,C+F)$ is $\mathcal{S}$-stable for $i = 1, 2, \cdots, n$ iff $Q := C(I+P_1C)^{-1}$ is $\mathcal{S}$-stable, and ${}^1S(P_i-P_1,Q)$ is $\mathcal{S}$-stable for $i = 2, 3, \cdots, n$.

(vi) To the best of the authors' knowledge, there are no known general conditions under which a general nonlinear plant is stabilizable by a compensator, incr. $\mathcal{S}$-stable or not.

Proof:

Since by assumption F is incr. $\mathcal{S}$-stable, by Theorem 3 we have ${}^1S(\bar{P}_1,C+F)$ and ${}^1S(\bar{P}_2,C+F)$ are $\mathcal{S}$-stable

$\Leftrightarrow$

${}^1S(P_1,C)$ and ${}^1S(P_2,C)$ are $\mathcal{S}$-stable.

Since P_1 is incr. $\mathcal{S}$-stable and $Q := C(I+P_1C)^{-1}$, by Theorem 1 we have

$$^1S(P_1,C) \text{ is } \mathcal{S}\text{-stable} \Leftrightarrow Q \text{ is } \mathcal{S}\text{-stable}.$$

Using Corollary 3.1 with P replaced by P_2-P_1, F replaced by P_1, $\overline{C}$ replaced by C, and C replaced by Q, we conclude that

$$^1S(P_2,C) \text{ is } \mathcal{S}\text{-stable} \Leftrightarrow {}^1S(P_2-P_1,Q) \text{ is } \mathcal{S}\text{-stable}.$$

The assertion follows. ◻

VI. SUMMARY

In this paper, we introduce a generalized concept of stability: $\mathcal{S}$-stability and incremental $\mathcal{S}$-stability, both applicable to nonlinear systems. Theorem 1 generalizes to the nonlinear case the Q-parametrization results established by Zames (1981). Theorem 2 extends Theorem 1 to include unstable plants. Finally, in Theorem 4, we give a necessary and sufficient condition for the existence of a fixed compensator that stabilizes two given nonlinear plants. It is surprising that these three theorems generalize the linear theory to the nonlinear case and the general formulas of the theory are almost unchanged in form.

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