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Electronics Research Laboratory
University of California
Berkeley, California
Internal Technical Memorandum M-83

OPTIMAL CONTROLS FOR LINEAR SYSTEMS
AND THE MAXIMUM PRINCIPLE

by

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The research herein was supported by the Air Force Office
of Scientific Research under Grant AF-AFOSR-292-63.

July 22, 1964

ACKNOWLEDGMENT

I wish to thank Professor E. I. Jury for being my research advisor and teacher throughout my graduate studies, and also Professors E. Polak and D. A. Ludwig for numerous suggestions which I have used in writing this report.

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I. INTRODUCTION

We shall consider two optimal control problems for linear systems whose behavior may be described by the set of differential equations

$$\dot{x}_i(t) = \sum_{j=1}^n a_{ij}x_j(t) + \sum_{j=1}^r b_{ij}u_j(t) \quad i = 1, 2, \dots, n. \quad (1)$$

The $\dot{}$ indicates differentiation with respect to t , and all of the quantities in (1) are assumed to be real. The vector $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ is called the state (at time t) and the function $t \rightarrow u(t) = (u_1(t), u_2(t), \dots, u_r(t))$, the control.

The first problem is to choose a control over $t_0 \leq t \leq t_1$ which transfers a specified initial state to a preassigned terminal state in such a way that the integral

$$\int_{t_0}^{t_1} f^0(x(t), u(t)) dt \quad (2)$$

is minimized. In (2), f^0 is a convex function and $x(\cdot)$ is the solution of (1) corresponding to the control u . We assume that the magnitude of each component of the control is bounded in absolute value by unity, i. e. $|u_i(t)| \leq 1$, $i = 1, 2, \dots, r$ for all t .

The second problem is to choose, for each initial state, the control over the interval $0 \leq t < \infty$ in such a way that the integral

$$\int_0^\infty f^0(x(t), u(t)) dt$$

is minimized. It is assumed again that f^0 is convex and that each component of the control is bounded in absolute value by unity. This

mathematical model is frequently used to represent the behavior of a regulator system.

Both of these problems will be treated by using Pontryagin's maximum principle. As is well known, the maximum principle specifies a set of necessary conditions for a control to be optimal. Two of the principal results of this report are that the maximum principle is also a sufficient condition for optimality for the first problem and for a particular case of the second. These results are essentially the contents of Theorems 4 and 10, which are similar to results obtained by Gamkrelidze for the time optimal control problem.¹

In Sec. II, we develop both of the problems together. The main results are Theorems 1 and 2 which assert the existence of an optimal control for the problems. These theorems are apparently new although some papers have been devoted to the question of the existence of optimal controls for related problems.^{2, 3}

Sec. III is concerned with the first problem exclusively. The main result is that the maximum principle is a sufficient condition for optimality when the problem is normal (Theorem 4). We show that the problem is normal whenever the specified terminal state is in the interior of K , the set of states reachable from the initial state. K is shown to be convex, and to have a nonempty interior when (3) is controllable. If (3) is controllable, the interior of K is, roughly speaking, almost all of K , and therefore the maximum principle is almost always a sufficient condition for optimality.

We consider a special case of the second problem in Sec. IV. The most important assumption is that

$$f^0(x, u) = \frac{1}{2} \langle x, Qx \rangle + \frac{1}{2} \langle u, Ru \rangle$$

where Q and R are non-negative and positive definite matrices, respectively. We show that the maximum principle, suitably strengthened, is a necessary and sufficient condition for a control to be optimal

(Theorem 10). Because of its engineering importance, the problem of constructing an optimal feedback control (the synthesis problem) is treated in some detail. This problem was discussed by Jen-wei⁴ and Letov,⁵ but it appears that their results are in error.⁶

A. NOTATION AND CONVENTIONS

For the most part we shall use standard mathematical notation. Typical symbols for functions are $x(\cdot)$, $t \rightarrow u(t)$, $g: \mathbb{R}^n \rightarrow \mathbb{R}$, while the values of the functions at particular values of their arguments are denoted $x(t)$, $u(t)$, $g(x)$, respectively. Matrices are denoted by capital letters such as A , B , Q . No distinction is made between row and column vectors; the meaning will always be clear from context. For example, if A is an $n \times n$ matrix and x, ψ are n -vectors ψA and Ax have meaning only if ψ is a row-vector and x is a column vector. The scalar product of $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ is denoted $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. The norm of x , unless specified otherwise, is $\max_{1 \leq i \leq n} |x_i|$. The $O(\epsilon)$ and $o(\epsilon)$ notation have their

usual meaning: $f = o(\epsilon)$ means $\lim_{\epsilon \rightarrow 0} f/\epsilon = 0$ and $g = O(\epsilon)$ means there is an A such that $|g/\epsilon| < A$. $\epsilon \rightarrow 0$

II. TWO OPTIMAL CONTROL PROBLEMS

A. SUMMARY

In Part B the two control problems to be studied are formulated precisely, and the notions of admissible control and transfer are defined. The synthesis problem, i. e., that of finding an optimal feedback control, is stated in Part C. The principal results of the section are in Part D where existence theorems for optimal controls are proved (Theorems 1 and 2). Their proofs, like those of similar theorems in Markus and Lee,² are based on the weak compactness of the set of admissible controls. What is new in the proofs is the exploitation of the convexity of the cost function f^0 and the application of the Banach-Saks theorem to obtain strongly, instead of weakly, converging sequences. Lemmas 1 and 2 are also important because they are used later in Sec. III, where Problem I is treated, and in Sec. IV, where a special case of Problem II is solved.

B. PROBLEM STATEMENT

In this section we formulate the two problems that will be considered. The first is a particular case of those considered by Pontryagin in Ref. 7, with the greatest specialization being to a linear system, and the second is a regulator problem.

The control region Ω , a subset of R^r the real r -dimensional vector space, is assumed to be

$$\Omega = \{(u_1, u_2, \dots, u_r) \mid |u_i| \leq 1, i = 1, 2, \dots, r\}.$$

We say a function $t \rightarrow u(t)$, is an admissible control iff $u(t) \in \Omega$ for each t and it is measurable. For brevity, frequently the adjective "admissible" will be omitted and "control" will mean "admissible control."

Comment: In any engineering application, of course, a control must be at least piece-wise continuous for it to be physically realizable. However, in proving the existence theorems (Theorem 1 and 2) it is necessary

to use measurable controls. Technically, the reason for this is the fact that the class of measurable controls is closed under passages to limits.

We will consider systems whose state at time t is described by the set of differential equations,

$$\dot{x}_i(t) = \sum_{j=1}^n a_{ij} x_j(t) + \sum_{j=1}^r b_{ij} u_j(t), \quad i = 1, 2, \dots, n,$$

the coefficients a_{ij} and b_{ij} being real constants. Introducing the matrices $x = (x_1, x_2, \dots, x_n)$, $u = (u_1, u_2, \dots, u_r)$, $A = (a_{ij})$ and $B = (b_{ij})$, these equations are

$$\dot{x} = Ax + Bu. \quad (3)$$

Eqs. 3 are called the state equations of the system.

For any control $t \rightarrow u(t)$, the solution (trajectory) of (3) satisfying the initial condition $x(t_0, u) = x_0$, is given by

$$x(t, u) = e^{A(t-t_0)} x_0 + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau. \quad (4)$$

For brevity, $x(\cdot, u)$ often will be abbreviated to $x(\cdot)$, when the control intended is clear from context. Occasionally, the notation $x(\cdot, u, x_0)$ will be used whenever the initial state is important.

The control $t \rightarrow u(t)$, $t_0 \leq t \leq t_1$, is said to transfer the state from position x_0 to x_1 iff the solution of (3), $x(\cdot, u)$, satisfies the boundary conditions $x(t_0, u) = x_0$ and $x(t_1, u) = x_1$. The states x_0 and x_1 are called, respectively, the initial and terminal states.

We consider in addition the integral

$$\int_{t_0}^{t_1} f^0(x(t, u), u(t)) dt$$

where the function f^0 satisfies the conditions

1. f^0 and $\frac{\partial f^0}{\partial x_i}$, $i = 1, 2, \dots, n$

are continuous on $R^n \times \Omega$

2. f^0 is convex: For all x_1, x_2, u_1, u_2 and $0 \leq \lambda \leq 1$,

$$f^0(\lambda x_1 + (1 - \lambda)x_2, \lambda u_1 + (1 - \lambda)u_2) \leq \lambda f^0(x_1, u_1) + (1 - \lambda)f^0(x_2, u_2).$$

The first problem we shall study is

Problem I: For the system (3), given states x_0 and x_1 , and times t_0, t_1 among all admissible controls $t \rightarrow u(t)$, $t_0 \leq t \leq t_1$, which transfer x_0 to x_1 , find one for which $\int_{t_0}^{t_1} f^0(x, u) dt$ is a minimum.

Comment: The case when the cost function f^0 depends solely on the control u , is especially important in engineering problems. For example, if $f^0(x, u) = \sum_{i=1}^r |u_i|$, the integral

$$\int_{t_0}^{t_1} \sum_{i=1}^r |u_i(t)| dt$$

might physically represent the fuel expended in control by u .

In Problem I the control is explicitly required to move the state to position x_1 . However, in some engineering problems this requirement is not essential. For example, suppose the process described by (3) is the deviation of a regulator from its equilibrium state, $x = 0$, the normal mode of operation. In other words, $x(t)$ represents the displacement of the state from the desired one at time t . Then, a natural measure of the performance of a control $t \rightarrow u(t)$, $0 \leq t < \infty$, is the integral

$$\int_0^{\infty} f^0(x, u) dt$$

where we might, for example, specify $f^0(x, u) \geq 0$ with equality if $(x, u) = (0, 0)$.

With this motivation, we formulate

Problem II: For the system (3) given the initial condition $x(0) = x_0$, find among all admissible controls $t \rightarrow u(t)$, $0 \leq t < \infty$ one for which the integral

$$\int_0^{\infty} f^0(x, u) dt$$

is a minimum.

A control which is a solution to either of the problems cited above is called an optimal control, and the corresponding trajectory, an optimal trajectory.

C. THE SYNTHESIS PROBLEM

In engineering problems, it is often necessary to consider more than a single initial state. For example, in the regulator problem cited above, no initial state is distinguished (x_0 might represent a displacement from equilibrium due to some load disturbance). In this case, a solution of Problem II for all x_0 is needed.

This requirement leads to the so-called synthesis problem for Problem II. We say a function $x \rightarrow u(x)$ from R^n to Ω is an admissible feedback control iff for every x_0

1. The differential equation

$$\dot{x} = Ax + Bu(x)$$

has a solution, $x(\cdot)$, $0 \leq t < \infty$, satisfying $x(0) = x_0$.

2. The (time) function $t \rightarrow u(x(t))$ is an admissible control in the sense previously defined.

The synthesis problem corresponding to Problem II is

Problem II': For the system (3) among all admissible feedback controls $x \rightarrow u(x)$, find one such that for every initial state the integral

$$\int_0^{\infty} f^0(x, u(x)) dt$$

is a minimum.

A solution to the synthesis problem is called an optimal feedback control. If $x \rightarrow u(x)$ is such a control, the differential equation

$$\dot{x} = Ax + Bu(x)$$

determines all of the optimal trajectories of the system.

Comment 1: The relation between Problems II and II' may be seen as follows:

Suppose $t \rightarrow u(t)$, $0 \leq t < \infty$, is optimal for initial state x_0 . Let the corresponding solution of (3) be $x(\cdot, u)$. Then if $t_0 \leq t' < \infty$ the control $t \rightarrow u'(t)$ defined by $u'(t) = u(t)$, $t' \leq t < \infty$, is optimal for initial state $x(t', u)$. This is clear since once at state $x(t', u)$, the subsequent control must be optimal. Observe that the value of the control depends solely on the state, and the value of t' is immaterial. Thus $t \rightarrow u(t)$ actually determines a value of $u(x)$ for each state on the trajectory $x(\cdot, u)$. Conversely, if $x \rightarrow u(x)$ is a solution of the synthesis problem, the control $t \rightarrow u(x(t))$ (for each initial state) is a solution to Problem II.

Comment 2: For many engineering problems, the solution to the synthesis problems is the desired one. The function $x \rightarrow u(x)$ defines a feedback control; whereas, a solution to Problem II is an open loop control. Generally speaking, feedback control systems are preferable to open-loop systems.

D. EXISTENCE THEOREMS

In this section, we show that the problems formulated above have a solution under suitable hypotheses (Theorems 1, 2 and Corollary 2). The proofs are based on the weak compactness of the set of admissible controls, the convexity of f^0 , and the Banach-Saks Theorem. The actual proofs of the theorems will not be needed in the sequel, although Lemmas 1 and 2 are used later. The reader may therefore omit the proofs of the theorems without loss of continuity, and go on to the next chapter.

Consider t_0 and t_1 fixed (finite) and let U be the set of all admissible controls defined on $[t_0, t_1]$. U is a subset of the Hilbert space L_2 of square integrable vector functions on $[t_0, t_1]$. In proving the

existence theorems, however, it is necessary to consider L_2 with its weak topology. The following lemma plays a crucial role in what follows.

Lemma 1: U is convex and compact in the weak L_2 topology.

Proof: If u_1 and $u_2 \in U$, then $\lambda u_1 + (1 - \lambda) u_2 \in U$ for all $0 \leq \lambda \leq 1$, because for any $t \in [t_0, t_1]$, $\lambda u_1(t) + (1 - \lambda) u_2(t) \in \Omega$ since Ω is convex. Hence, U is convex.

U is bounded as a subset of L_2 since Ω is bounded and $t_1 - t_0$ is finite. It is easy to show U is closed in L_2 (consider L_2 with its norm topology). Therefore, U is compact in the weak L_2 topology.

We will also use the Banach-Saks theorem whose proof may be found in Ref. 8.

Theorem (Banach-Saks): Given in L_2 a sequence $\{u_n\}$ which converges weakly to an element u , there is a subsequence $\{u_{n_k}\}$ such that the arithmetic means

$$\frac{u_{n_1} + u_{n_2} + \dots + u_{n_k}}{k} \longrightarrow u$$

in norm.

The next result is simple, but is quite useful and will be needed in Secs. 3 and 4 as well as here.

Lemma 2: Let u_1 and $u_2 \in U$ and suppose $x(\cdot, u_1)$ and $x(\cdot, u_2)$ are the corresponding solutions of (3). Then if $0 \leq \lambda \leq 1$

$$x(\cdot, \lambda u_1 + (1 - \lambda) u_2) = \lambda x(\cdot, u_1) + (1 - \lambda) x(\cdot, u_2) .$$

Proof: From (4) we have

$$\begin{aligned} x(t, \lambda u_1 + (1 - \lambda) u_2) &= e^{A(t - t_0)} x_0 + \int_{t_0}^t e^{A(t - \tau)} B(\lambda u_1(\tau) + (1 - \lambda) u_2(\tau)) d\tau \\ &= \lambda e^{A(t - t_0)} x_0 + \lambda \int_{t_0}^t e^{A(t - \tau)} B u_1(\tau) d\tau \end{aligned}$$

$$\begin{aligned}
& + (1 - \lambda) e^{A(t-t_0)} x_0 + (1 - \lambda) \int_{t_0}^t e^{A(t-\tau)} B u_2(\tau) d\tau \\
& = \lambda x(t, u_1) + (1 - \lambda) x(t, u_2)
\end{aligned}$$

which proves the lemma.

Using Lemma 2, it is easy to prove by induction

Corollary 1: Let $u_1, u_2, \dots, u_k \in U$ and $x(\cdot, u_i)$, $i = 1, 2, \dots, k$ be the corresponding solutions of (3). Then

$$x \left(\cdot, \frac{u_1 + u_2 + \dots + u_k}{k} \right) = \frac{1}{k} \sum_{i=1}^k x(\cdot, u_i).$$

We note also the following property of convex functions:

Lemma 3: Let $x \rightarrow \phi(x)$ be a convex function defined on R^n . Then

$$\phi \left(\frac{x_1 + x_2 + \dots + x_k}{k} \right) \leq \frac{1}{k} \sum_{i=1}^k \phi(x_i).$$

Proof: The lemma is true for $k = 2$ by definition. Assume it to be true for $n = k - 1$. Then,

$$\begin{aligned}
\phi \left(\frac{x_1 + x_2 + \dots + x_k}{k} \right) &= \phi \left(\frac{x_1}{k} + \frac{k-1}{k} \frac{x_2 + \dots + x_k}{k-1} \right) \\
&\leq \frac{1}{k} \phi(x_1) + \frac{k-1}{k} \phi \left(\frac{x_2 + \dots + x_k}{k-1} \right) \\
&\leq \frac{1}{k} \phi(x_1) + \frac{k-1}{k} \frac{1}{k-1} \sum_{i=2}^k \phi(x_i) \\
&= \frac{1}{k} \sum_{i=1}^k \phi(x_i)
\end{aligned}$$

which proves the lemma.

We are now in a position to prove an existence theorem for optimal control in Problem I.

Theorem 1: In Problem I, if there is a control $t \rightarrow u(t)$, $t_0 \leq t \leq t_1$, which transfers x_0 to x_1 , there is an optimal one.

Proof: Let $F \subset U$ be the set of all controls which transfer x_0 to x_1 . F is convex, in view of Lemma 2, and it is nonempty by hypothesis. Consider the function $C: F \rightarrow \mathbb{R}$ defined by

$$C(u) = \int_{t_0}^{t_1} f^0(x(t, u), u(t)) dt$$

and let $d = \inf_{u \in F} C(u)$. We have to show there is a $u \in F$ such that $C(u) = d$.

If F is finite, the theorem is trivial. If F is infinite, choose a sequence $\{u_n\}$ in F such that

$$C(u_n) \leq d + \frac{1}{2^n}. \quad (5)$$

Since U is compact, there is a $u^* \in U$ and a subsequence of $\{u_n\}$ converging weakly to u^* . From (4) and the definition of weak convergence, we see that $u^* \in F$, i.e., u^* transfers x_0 to x_1 . We will show u^* is an optimal control.

The Banach-Saks theorem (renumbering indices if necessary) states there is a sequence of functions $\{v_n\}$ converging to u^* in norm with

$$v_k = \frac{u_1 + u_2 + \dots + u_k}{k}$$

The $v_k \in F$ because F is convex. Denote the solution of (3) for control u_k by $x_k(\cdot)$. Then using Corollary 1, Lemma 3 and Eq. (5) in that order,

$$\begin{aligned}
C(v_k) &= \int_{t_0}^{t_1} f^0(x(t, v_k), v_k) dt \\
&= \int_{t_0}^{t_1} f^0\left(\frac{x_1 + x_2 + \dots + x_k}{k}, \frac{u_1 + u_2 + \dots + u_k}{k}\right) dt \\
&\leq \frac{1}{k} \sum_{i=1}^k \int_{t_0}^{t_1} f^0(x_i, u_i) dt \\
&\leq d + \frac{1}{k} \sum_{i=1}^k \frac{1}{2^i} < d + \frac{1}{k}.
\end{aligned}$$

Since $v_k \rightarrow u^*$ in norm, it is clear that $x(t, v_k) \rightarrow x(t, u^*)$ for each t (See (4)). In addition there is a subsequence of $\{v_k\}$ converging to u^* almost everywhere on $[t_0, t_1]$, so we may assume $v_k \rightarrow u^*$ a.e.. Therefore, since f^0 is continuous

$$\lim_{k \rightarrow \infty} f^0(x(t, v_k), v_k) = f^0(x(t, u^*), u^*) \quad \text{a.e.}$$

The theorem then follows from the inequalities,

$$\begin{aligned}
d &\leq \int_{t_0}^{t_1} f^0(x(t, u^*), u^*) dt \\
&= \int_{t_0}^{t_1} \lim_{k \rightarrow \infty} f^0(x(t, v_k), v_k) dt \\
&= \int_{t_0}^{t_1} \lim_{k \rightarrow \infty} \inf f^0(x(t, v_k), v_k) dt
\end{aligned}$$

$$\leq \lim_{k \rightarrow \infty} \inf \int_{t_0}^{t_1} f^0(x(t, v_k)) dt$$

$$= \lim_{k \rightarrow \infty} \inf C(v_k) = d.$$

The fourth line follows from the Fatou-Lebesgue theorem.⁹

In order to prove the existence of an optimal control for Problem II, two additional assumptions will be made about the function f^0 :

1. $f^0(x, u) \geq 0$, $f^0(0, 0) = 0$ (6a)
2. for every C , if $\int_0^\infty f^0(x, u) dt < C$ then there exist C_1

such that

$$\int_0^\infty \sum_{i=1}^r u_i^2(t) dt < C_1. \quad (6b)$$

Eq. (6a) is natural in engineering problems, and it rules out the possibility of the integral in Problem II diverging to $-\infty$. Eq. (6b) is somewhat more restrictive but is satisfied in the important cases when $f^0(x, u) = \phi(x) + ||u||$ where $||u||$ is either $\sum_{i=1}^r u_i^2$ or $\sum_{i=1}^r |u_i|$. The integral of the latter two terms might represent, physically, energy and fuel, respectively, expended in control by u .

With these assumptions, an existence theorem for an optimal control in Problem II is

Theorem 2: In Problem II, suppose that Eq. (6a) and (6b) are satisfied. Then if there is a control v such that

$$\int_0^\infty f^0(x(t, v), v) dt < \infty$$

there is an optimal control.

Proof: For each admissible control, let

$$C(u) = \int_0^{\infty} f^0(x(t, u), u) dt .$$

From (6a), $C(u) \geq 0$ for all u . By hypothesis, there is an admissible control v such that $C(v) < \infty$. This fact along with (6b) allows us to restrict the search for an optimal control to some bounded set in $L_2[0, \infty)$, the Hilbert space of square integrable function on $[0, \infty)$. The theorem then follows in essentially the same way as Theorem 1 was proven.

If in addition, we make the following assumptions about the system of (3)

1. it is a Lyapunov stable¹⁰ (7a)

2. it is controllable,^{11, 12} (7b)

then the second hypothesis of Theorem 2 is always satisfied. To show this, we will need the following theorem due to LaSalle.¹³

Theorem (LaSalle): Suppose system (3) satisfies conditions (7). Then for any initial state x_0 , there is an admissible control $t \rightarrow u(t)$, $t_0 \leq t \leq t_1$, which transfers x_0 to the origin $x = 0$.

Corollary 2: In Problem II, in Eq. (6) and (7) are satisfied, there exists an optimal control.

Proof: Using LaSalle's theorem, there exists a control $t \rightarrow u(t)$, $t_0 \leq t \leq t_1$, which transfers x_0 to $x = 0$. Define the control $t \rightarrow v(t)$ by

$$v(t) = \begin{cases} u(t) & t_0 \leq t \leq t_1 \\ 0 & t_1 < t < \infty \end{cases}$$

Since $f^0(0, 0) = 0$, $C(v) < \infty$, which, in view of Theorem 2, proves the corollary.

E. NOTES AND REMARKS

The idea of introducing the weak topology was used first by Gamkrelidze in proving an existence theorem for the (linear) time optimal control problem.¹ Existence theorems for optimum control were proven by Markus and Lee² and Roxin³ for certain nonlinear systems, but their hypotheses are not fulfilled for the problems considered here.

III. THE MAXIMUM PRINCIPLE AND PROBLEM I

A. SUMMARY

We shall treat Problem I using the maximum principle in this chapter. In Part B we formulate the maximum principle as it applies to the problem. The main result of the chapter is Theorem 4 which asserts that the maximum principle is a sufficient condition for optimality when the problem is normal. In Part D, we show that except possibly when the terminal state x_1 lies on the boundary of K , the set of states reachable from the initial state x_0 , the problem is normal. We show that K is convex and has nonempty interior when the system (3) is controllable. Therefore, when (3) is controllable "almost all" choices of a terminal state for which the problem makes sense, give rise to a normal problem.

B. THE MAXIMUM PRINCIPLE

In this section, we shall formulate the maximum principle as it applies to Problem I. For the most part, we shall use the notation and conventions in Pontryagin.¹⁴

We introduce the scalar differential equation

$$\dot{x}_0(t) = f^0(x, u), \quad t_0 \leq t \leq t_1 \quad (8)$$

with the initial condition

$$x_0(t_0) = 0.$$

The meaning of f^0 , x , and u is the same as in the last chapter. We observe that the right hand side of (8) does not depend on x_0 , and that if $t \rightarrow u(t)$, $t_0 \leq t \leq t_1$, is any admissible control, the solution of (8) is given by

$$x_0(t) = \int_{t_0}^t f^0(x(t, u), u(t)) dt$$

where $x(\cdot, u)$ is the solution of (3), the state equations, corresponding to u . The value of $x_0(t_1)$ may therefore be interpreted as the cost associated with the control u .

Following Pontryagin, we combine Eqs. (3) and (8) into a single equation. Letting $\underline{x} = (x_0', x_1, \dots, x_n)$, (3) and (8) are equivalent to the

vector equation,

$$\dot{\underline{x}} = \underline{f}(\underline{x}, u) \quad (9)$$

where the i -th component of $\underline{f}(\underline{x}, u)$ is given by

$$f^i(\underline{x}, u) = \begin{cases} f^0(\underline{x}, u) & \text{if } i = 0 \\ \sum_{j=1}^n a_{ij} x_j + \sum_{j=1}^r b_{ij} u_j, & i = 1, 2, \dots, n. \end{cases}$$

We consider in addition to the system of Eqs. (9), the auxiliary set of equations

$$\dot{\psi}_i(t) = -\psi_0 \frac{\partial f^0(\underline{x}, u)}{\partial x_i} - \sum_{j=1}^n a_{ji} \psi_j(t) \quad i = 0, 1, 2, \dots, n \quad (10)$$

In (10), the coefficients a_{ij} are the same as in (3). We do not specify the initial conditions for the ψ_i , and therefore to each control u , there corresponds not just one solution of (10), but a family of solutions. We note that $\psi_0(t)$ is a constant because $(\partial f^0 / \partial x_0) = 0$, and that (10) is a linear equation in the variables $\psi_0, \psi_1, \dots, \psi_n$.

Letting $\underline{\psi} = (\psi_0, \psi_1, \dots, \psi_n)$, we define the real valued function \mathcal{H} by the equation

$$\mathcal{H}(\underline{\psi}, \underline{x}, u) = \langle \underline{\psi}, \underline{f} \rangle \quad (11)$$

using \mathcal{H} , (9) and (10) can be expressed in the form of the following Hamiltonian system:

$$\frac{dx_i}{dt} = \frac{\partial \mathcal{H}}{\partial \psi_i} \quad i = 0, 1, 2, \dots, n \quad (12)$$

$$\frac{d\psi_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i} \quad i = 0, 1, 2, \dots, n. \quad (13)$$

This is the reason why \mathcal{H} is called the Hamiltonian.

For fixed values of $\underline{\psi}$ and \underline{x} , the Hamiltonian is a function of $u \in \Omega$. Let $\mathcal{M}(\underline{\psi}, \underline{x})$ be the maximum of $\mathcal{H}(\underline{\psi}, \underline{x}, u)$ over Ω :

$$\mathcal{M}(\underline{\psi}, \underline{x}) = \max_{u \in \Omega} \mathcal{H}(\underline{\psi}, \underline{x}, u) \quad (14)$$

Since Ω is compact, and \mathcal{H} is continuous, $\mathcal{H}(\underline{\psi}, \underline{x})$ is well defined.

The maximum principle as it applies to Problem I is expressed in

Theorem 3: If $t \rightarrow u(t)$, $t_0 \leq t \leq t_1$, is an optimal control for Problem I, then there is a nontrivial solution of (12), $\underline{\psi}(\cdot)$, corresponding to u such that

1. $\mathcal{H}(\underline{\psi}(t), \underline{x}(t), u(t)) = \mathcal{H}(\underline{\psi}(t), \underline{x}(t))$ almost everywhere on $[t_0, t_1]$.
2. $\mathcal{H}(\underline{\psi}(t), \underline{x}(t)) = \text{constant}$.
3. $\psi_0 \leq 0$.

It will be necessary in the sequel to distinguish between the case when $\psi_0 < 0$ and when $\psi_0 = 0$. If $\psi_0 < 0$, the trajectory $\underline{x}(\cdot, u)$ is said to be normal; if $\psi_0 = 0$, it is said to be abnormal. When $\psi_0 < 0$, since (12) is linear, in the variables $\psi_0, \psi_1, \dots, \psi_n$, we may assume $\psi_0 = -1$. We shall also say the problem is normal or abnormal according to whether $\psi_0 < 0$ or $\psi_0 = 0$. This terminology is also used in the classical calculus of variations.¹⁵

We remark that the maximum principle consists of a set of necessary conditions for a control to be optimal. Somewhat loosely, we shall speak of the maximum principle itself as being a necessary condition for a control to be optimal. Similarly, we shall say that the maximum principle is a sufficient condition for a control to be optimal iff every control which satisfies the conditions 1-3 of Theorem 3 is necessarily an optimal control.

C. SUFFICIENT CONDITIONS FOR OPTIMALITY

We shall show the maximum principle is a sufficient condition for a control to be optimal for Problem I, provided the problem is normal. This will be accomplished in two steps. First, we derive the weaker result that the maximum principle is a sufficient condition for a control to be stationary.[†] Then we shall prove that it is a sufficient condition for a control to be optimal (Theorem 4).

If $t \rightarrow u(t)$ and $t \rightarrow u'(t)$, $t_0 \leq t \leq t_1$, are two admissible controls, it will be convenient to let

[†]A control u is said to be stationary if controls $\mathcal{O}(\epsilon)$ distant from u , in a suitable metric, are at most $\alpha(\epsilon)$ "better than it."

$$d(u, u') = \int_{t_0}^{t_1} ||u(t) - u'(t)|| dt . \quad (15)$$

Although we shall not use the fact that d is a distance in the mathematical sense, it is helpful to keep the intuitive idea of a distance in mind.

In the sequel we need a formula which can be deduced from a more general result by Rozonoer.¹⁶ For convenience and completeness, we have given a proof of this result in Appendix A. The formula in question is contained in

Lemma 4: Let $t \rightarrow u(t)$ and $t \rightarrow u'(t)$ be admissible controls and $\underline{x}(\cdot)$ and $\underline{x}'(\cdot)$ be the solutions of (12) corresponding to u and u' respectively. Suppose there is a solution $\underline{\psi}(\cdot)$ of (13) corresponding to u such that almost everywhere on $[t_0, t_1]$

$$\mathcal{H}(\underline{\psi}(t), \underline{x}(t), u(t)) = \mathcal{M}(\underline{\psi}(t), \underline{x}(t)) .$$

Then

$$\langle \underline{\psi}(t_1), \underline{x}'(t_1) - \underline{x}(t_1) \rangle + o(\epsilon) \leq 0 \quad (16)$$

when $d(u, u') \leq \epsilon$.

Proof: Lemma 4 is proved in Appendix A.

We define the cost associated to a control u by the formula

$$C(u) = \int_{t_0}^{t_1} f^0(x(t, u), u) dt . \quad (17)$$

Using Lemma 4, we now show that the maximum principle is a sufficient condition for a control to be stationary, when the problem is normal.

Lemma 5: Let $t \rightarrow u(t)$ and $t \rightarrow u'(t)$, $t_0 \leq t \leq t_1$, be admissible controls which transfer x_0 to x_1 . Suppose there is a solution of (13) $\underline{\psi}(\cdot)$ such that almost everywhere

$$\mathcal{H}(\underline{\psi}(t), \underline{x}(t), u(t)) = \mathcal{M}(\underline{\psi}(t), \underline{x}(t)) .$$

Then, if $\underline{x}(\cdot, u)$ is normal,

$$C(u) - C(u') \leq o(\epsilon)$$

when $d(u, u') \leq \epsilon$.

Proof: Let $\underline{x}(\cdot)$ and $\underline{x}'(\cdot)$ denote the solutions of (12) corresponding to u and u' , respectively. Since $\underline{x}(\cdot)$ is normal, we may assume $\psi_0 = -1$, and because both u and u' transfer x_0 to x_1 , $x(t_1) = x'(t_1)$. Therefore, from Lemma 4, it follows that

$$x_0(t_1) - x'_0(t_1) + o(\epsilon) \leq 0 .$$

But since $C(u) = x_0(t_1)$ and $C(u') = x'_0(t_1)$, this proves the lemma.

We are now in a position to prove the main result of the chapter:

Theorem 4: In Problem I, let $t \rightarrow u(t)$, $t_0 \leq t \leq t_1$, be an admissible control which transfers x_0 to x_1 . Then a sufficient condition for u to be optimal is that there exists a solution $\underline{\psi}(\cdot)$ of (13) corresponding to u such that

1. $\mathcal{H}(\underline{\psi}(t), \underline{x}(t), u(t)) = \mathcal{M}(\underline{\psi}(t), \underline{x}(t))$ almost everywhere on $[t_0, t_1]$.
2. $\psi_0 < 0$.

Proof: The proof is by contradiction. Suppose that u' is an optimal control (such a control exists by Theorem 1) but that u is not optimal, i.e.

$$C(u) - C(u') = \int_{t_0}^{t_1} f^0(x, u) - f^0(x', u') dt = d > 0$$

where $x(\cdot)$ and $x'(\cdot)$ are the solutions of (3) corresponding to u and u' respectively. For each $0 \leq \lambda \leq 1$, we consider the controls

$$t \rightarrow u_\lambda(t) = (1 - \lambda) u(t) + \lambda u'(t), \quad t_0 \leq t \leq t_1 .$$

In view of Lemma 1, each u_λ is admissible, and we observe that

$$\begin{aligned} d(u, u_\lambda) &= \int_{t_0}^{t_1} \|u(t) - u_\lambda(t)\| dt \\ &= \lambda \int_{t_0}^{t_1} \|u(t) - u'(t)\| dt = \lambda \beta \end{aligned} \quad (18)$$

where we define β by the last equality.

By Lemma 2, the solutions $x_\lambda(\cdot)$ of (3) corresponding to u_λ satisfy the boundary conditions $x_\lambda(t_1) = x_1$, so each u_λ transfers x_0 to x_1 . Then using Lemma 2 and the convexity of f^0 , in that order,

$$\begin{aligned} C(u) - C(u_\lambda) &= \int_{t_0}^{t_1} f^0(x, u) - f^0(x_\lambda, u_\lambda) dt \\ &= \int_{t_0}^{t_1} f^0(x, u) - f^0((1 - \lambda)x + \lambda x', (1 - \lambda)u + \lambda u') dt \\ &\geq \int_{t_0}^{t_1} f^0(x, u) - ((1 - \lambda) f^0(x, u) + \lambda f^0(x', u')) dt \\ &= \lambda \int_{t_0}^{t_1} f^0(x, u) - f^0(x', u') dt = \lambda d. \end{aligned}$$

Putting $\lambda = \epsilon/\beta$ in the last expression and in (18), this contradicts Lemma 5. Therefore u must be an optimal control.

Comment: Theorem 4 shows that Problem I is equivalent to solving a two-point boundary value problem: In the system of Eqs. (12) and (13), the initial and final values of x are known, but the boundary conditions on

$\psi_1, \psi_2, \dots, \psi_n$ are not. When f^0 does not depend on x explicitly, and only on u , special methods for solving the boundary value problem have been derived. See, for example, Neustadt,¹⁷ who without using the maximum principle derives the equivalent of Theorem 4 under these circumstances. In addition, he gives a computational procedure for solving the resulting boundary value problem.

D. NORMALITY

In this section, we shall show that the normality hypothesis in Theorem 4 is essentially unimportant when system (3) is controllable. This will be accomplished by showing that except possibly when the terminal state x_1 lies on the boundary of K , the set of states reachable from the initial state x_0 , the problem is normal. We show K is convex, and that it has nonempty interior when system (3) is controllable. The set of terminal states which can give rise to an abnormal problem is therefore, in a sense, negligible when (3) is controllable.

We first consider the set of states reachable from the initial state x_0 at time $t = t_1$ by admissible controls. Referring to (4), we see that this set is (U is the set of admissible controls)

$$K = \{ y \in R^n \mid y = x(t_1, u) \text{ for some } u \in U \}$$

where

$$x(t_1, u) = e^{A(t_1-t_0)} x_0 + \int_{t_0}^{t_1} e^{A(t-\tau)} B u(\tau) d\tau.$$

In considering the time-optimal control problem, LaSalle proved that K was convex and compact.¹³ We shall give a different proof of the same result.

Lemma 6: K is convex and compact.

Proof: It suffices to prove Lemma 6 for the set

$$K' = \left\{ y \in R^n \mid y = \int_{t_0}^{t_1} e^{-A\tau} B u(\tau) d\tau, \quad d\tau, \quad u \in U \right\}$$

because K may be obtained from K' by multiplying by the matrix e^{At_1} and translating the result by $e^{A(t_1-t_0)}x_0$. These two operations, of course, do not effect the properties of convexity and compactness.

We recall that U is convex and weakly compact (Lemma 1), and we define the linear transformation $L:U \rightarrow R^n$ by

$$Lu = \int_{t_0}^{t_1} e^{-A\tau} Bu(\tau) d\tau.$$

Then if $x_1, x_2 \in K'$, there exists $u_1, u_2 \in U$ such that $x_1 = Lu_1$ and $x_2 = Lu_2$. Since L is linear, for any $0 \leq \lambda \leq 1$,

$$z = \lambda x_1 + (1 - \lambda) x_2 = L(\lambda u_1 + (1 - \lambda) u_2)$$

U being convex, $\lambda u_1 + (1 - \lambda) u_2 \in U$, and therefore $z \in K'$, proving K' is convex.

To prove K' is compact, first we observe that L is continuous when U is given the weak L_2 topology and R^n the usual topology. Since $K' = L(U)$, K' is compact (the continuous image of a compact set is compact), which completes the proof of Lemma 6.

We now show that the set of terminal states x_1 which give rise to abnormal problems is a subset of the boundary of K .

For abnormal problems $\psi^0 = 0$, and the Hamiltonian reduces to (see (11))

$$\mathcal{H}(\underline{\psi}, \underline{x}, u) = \langle \underline{\psi}, Ax + Bu \rangle \quad (19)$$

where we define $\underline{\psi} = (\psi_1, \psi_2, \dots, \psi_n)$. Moreover, with $\psi_0 = 0$, the differential Eqs. (10) which the ψ_i satisfy, simplify to

$$\dot{\psi}_i(t) = - \sum_{j=1}^n a_{ji} \psi_j(t), \quad i = 1, 2, \dots, n$$

or in vector form,

$$\dot{\psi} = -\psi A. \quad (20)$$

Clearly \mathcal{H} , in abnormal problems, attains its maximum as a function of u simultaneously with the function

$$\langle \psi, Bu \rangle$$

We want to consider all possible abnormal optimal trajectories, so in the sequel we assume that u maximizes \mathcal{H} :

$$u(t) = \text{sgn}(\psi(t) \cdot B), \quad t_0 \leq t \leq t_1 \quad (21)$$

where $\text{sgn}: \mathbb{R}^r \rightarrow \mathbb{R}^r$ is the vector valued function whose i th component, $1 \leq i \leq r$, is $(x = (x_1, x_2, \dots, x_r))$

$$(\text{sgn } x)_i = \begin{cases} 1 & \text{if } x_i > 0 \\ -1 & \text{if } x_i < 0 \\ \text{undefined} & \text{if } x_i = 0 \end{cases}.$$

We observe that if x_1 is a terminal state which gives rise to an abnormal problem, any optimal control for x_1 must satisfy (21) for some solutions of (20).

Now we consider the set

$$\Gamma = \{y \in \mathbb{R}^n \mid y = x(t_1, u), \quad u \text{ satisfies (21)}\}.$$

In view of the previous discussion, Γ contains the set of terminal states which give rise to abnormal problems. The relation of Γ and K is given in

Lemma 7: Γ is the boundary of K .

Proof: First we review how the boundary of the convex set K , denoted ∂K , may be characterized.¹⁸ Geometrically, $y \in \partial K$ iff there exists a hyperplane containing y such that K lies entirely on one side of it. Analytically, this is equivalent to the statement: $y \in \partial K$ iff there exists $\psi^0 \in \mathbb{R}^n$ such that for all $z \in K$

$$\langle \psi^0, y \rangle \geq \langle \psi^0, z \rangle .$$

We note that, as in the proof of Lemma 6, it suffices to consider the sets

$$K' = \left\{ y \in \mathbb{R}^n \mid y = \int_{t_0}^{t_1} e^{-A\tau} Bu(\tau) d\tau, u \in U \right\}$$

$$\Gamma' = \left\{ y \in \mathbb{R}^n \mid y = \int_{t_0}^{t_1} e^{-A\tau} Bu(\tau) d\tau, u \text{ satisfies (21)} \right\}$$

and prove that Γ' is the boundary of K' .

We remark that every solution, $\psi(\cdot)$, of (20) is of the form $\psi^0 \cdot e^{-At}$, for some $\psi^0 \in \mathbb{R}^n$. Therefore, if $y \in \Gamma'$, for some ψ^0 ,

$$y = \int_{t_0}^{t_1} e^{-A\tau} B \operatorname{sgn}(\psi^0 \cdot e^{-A\tau}) \cdot B d\tau .$$

We consider

$$\langle \psi^0, y \rangle = \int_{t_0}^{t_1} \langle \psi^0 \cdot e^{-A\tau} B, \operatorname{sgn}(\psi^0 \cdot e^{-A\tau} \cdot B) \rangle d\tau . \quad (22)$$

Note that $\langle \psi^0, y \rangle$ is well defined even though $\operatorname{sgn}(\psi^0 \cdot e^{-A\tau} B)$ may not be.

Now let $z \in K'$. Then for some $u \in U$,

$$z = \int_{t_0}^{t_1} e^{-A\tau} B u(\tau) d\tau$$

and

$$\langle \psi^0, z \rangle = \left\langle \int_{t_0}^{t_1} \psi^0 e^{-A\tau} B, u(\tau) \right\rangle dt.$$

Comparing (22) with the last expression, since each component of u is bounded in absolute value by unity,

$$\langle \psi^0, y \rangle \geq \langle \psi^0, z \rangle.$$

The last inequality, in view of the previous discussion, shows that $y \in \partial K'$, which proves the lemma.

Recalling the definitions of K and Γ , we have immediately from Lemma 7

Corollary 3: If the terminal state x_1 belongs to the interior of the set of states reachable from x_0 by admissible controls, Problem I is normal.

Corollary 3 is correct but meaningless if K , the set of states reachable from x_0 by admissible controls, has empty interior. A necessary and sufficient condition for the interior of K to be nonempty is for system (3) to be controllable. To see this, recall that by definition, (3) is controllable iff given any states x_0 to x_1 , there exists a bounded function $t \rightarrow u(t) \in R^r$, $t_0 \leq t \leq t_1$, which transfers x_0 to x_1 . It follows that without the restriction $u(t) \in \Omega$, the set of states reachable from x_0 is all of R^n iff (3) is controllable. Because (3) is linear, it is then clear that the interior of K is nonempty iff (3) is controllable.

We summarize the main results of this chapter by combining Theorems 3 and 4 and Corollary 3 in

Theorem 5: In Problem I suppose x_1 belongs to the interior of the set of states reachable from x_0 by admissible controls, and suppose $t \rightarrow u(t)$, $t_0 \leq t \leq t_1$, transfers x_0 to x_1 . Then a necessary and sufficient condition for u to be optimal is that there exist a solution $\underline{\psi}(\cdot)$ of (13) corresponding to u such that

1. $\mathcal{H}(\underline{\psi}(t), \underline{x}(t), u(t)) = \mathcal{M}(\underline{\psi}(t), \underline{x}(t), u(t))$ almost everywhere on $[t_0, t_1]$.
2. $\mathcal{M}(\underline{\psi}(t), \underline{x}(t)) = \text{constant}$.
3. $\psi_0 < 0$.

Remark: Theorem 3, like Corollary 3, is meaningful only when the system (3) is controllable.

E. NOTES AND REMARKS

LaSalle proved Lemma 6 using Liapunoff's theorem on the range of a vector measure.¹³ We have preferred to give an independent proof of the lemma because our proof, perhaps, relies on more well known tools. The idea used in proving Lemma 7 is contained in Neustadt.¹⁷

IV. A SPECIAL CASE OF PROBLEM II

A. SUMMARY

We shall consider a special case of Problem II in this section. The most important assumption is that

$$f^0(x, u) = \frac{1}{2} \langle x, Qx \rangle + \frac{1}{2} \langle u, Ru \rangle$$

where Q and R are non-negative and positive definite matrices, respectively. Special emphasis is placed on the problem of constructing an optimal feedback control because of its engineering importance.

As the problem is formulated, there is no form of the maximum principle which is applicable, the difficulty being in the boundary conditions at ∞ . In Part C we show how the problem can be stated as one for which the maximum principle applies without actually changing the problem. The principal results in part C are Theorems 6 and 7 which state, respectively, that the optimal control is unique and that the optimally controlled system is asymptotically stable in the large.

In Part D the maximum principle as it applies to the reformulated problem is stated. We strengthen the maximum principle in Part E to a form in which we can show it is both a necessary and sufficient condition for optimality. The sufficiency proof is carried out in Part F, and the final result is stated in Theorem 10.

We consider the synthesis problem in Parts G and H. The optimal feedback control is shown to be a linear function of the state in a neighborhood of the origin in state space (Proposition 1), but no closed form expression for it is derived for all states. However, it may be computed, roughly speaking, by running system (3) backwards in time, in a way similar to that proposed by LaSalle for the time optimal control problem.¹³ Part H deals with some computational aspects of the synthesis problem.

B. PROBLEM STATEMENT

In this chapter we shall consider a special case of Problem II (see Sec. II, Part B). The following special assumptions are made:

1. $f^0(x, u) = \frac{1}{2} \langle x, Qx \rangle + \frac{1}{2} \langle u, Ru \rangle$ where Q and R are non-negative and positive definite matrices, respectively.
 2. System (3) is controllable and Lyapunov stable.
 3. The only element of the set
- $$M = \{x \in R^n \mid \langle e^{At} x, Q e^{At} x \rangle = 0 \quad \forall t \geq 0\}$$
- is $x = 0$.
- (23)

We note that the first two assumptions fulfill the hypotheses of Corollary 2, and therefore an optimal control for the problem exists. The reason for the third assumption will be explained later.

C. PROPERTIES OF THE OPTIMAL SYSTEM

The purpose of this section is to convert the problem into one for which the maximum principle is applicable. The difficulty is in the form of the boundary conditions at infinity. We first prove the optimal control is unique.

Theorem 6: If u_1 and u_2 are optimal controls, then $u_1(t) = u_2(t)$ almost everywhere on $0 \leq t < \infty$.

Proof: We observe that the function $u \longrightarrow \langle u, Ru \rangle$ is strictly convex, i.e., for $0 < \lambda < 1$,

$$\langle \lambda u_1 + (1 - \lambda) u_2, R(\lambda u_1 + (1 - \lambda) u_2) \rangle \leq \lambda \langle u_1, Ru_1 \rangle + (1 - \lambda) \langle u_2, Ru_2 \rangle$$

with equality only if $u_1 = u_2$. Therefore f^0 is also strictly convex. Consider the (admissible) control,

$$t \longrightarrow u(t) = \frac{u_1(t) + u_2(t)}{2}, \quad 0 \leq t < \infty.$$

Let $x_1(\cdot)$, $x_2(\cdot)$, and $x(\cdot)$ be the solutions of (3) corresponding to u_1 , u_2 , and u respectively. Then utilizing Lemma 2, we have

$$\begin{aligned} \int_0^{\infty} f^0(x, u) dt &= \int_0^{\infty} f^0 \left(\frac{x_1 + x_2}{2}, \frac{u_1 + u_2}{2} \right) dt \\ &\leq \frac{1}{2} \int_0^{\infty} f^0(x_1, u_1) + f^0(x_2, u_2) dt . \end{aligned}$$

Since f^0 is strictly convex, the above inequality holds with equality only if $u_1 = u_2$ almost everywhere. But since u_1 and u_2 are optimal, we must have equality, which proves the theorem.

We now show that the optimally controlled system is asymptotically stable in the large (a. s. i. l.). This enables us to adjoin to the problem the requirement that the control transfer the initial state to the origin, in the limit at $t \rightarrow \infty$, without changing the problem. With this additional specification, the maximum principle is applicable.¹⁹

We let $x \rightarrow u(x)$ be the optimal feedback control for the problem (see Sec. 11, Part B). Such a control exists and is unique by Corollary 2 and Theorem 6, respectively. Consider the system

$$\dot{x} = Ax + Bu(x) \quad t \geq 0. \quad (24)$$

System (24) describes the behavior of the optimally controlled system.

We define the function $V: R^n \rightarrow R$ by the equation,

$$V(x_0) = \int_0^{\infty} \langle x, Qx \rangle + \langle u, Ru \rangle dt \quad (25)$$

where $x(\cdot)$ is the solution of (24) for the initial condition $x(0) = x_0$, and u is the associated control. Clearly, an equivalent definition of $V(x_0)$ is

$$V(x_0) = \inf_{u \in U} \int_0^{\infty} \langle x, Qx \rangle + \langle u, Ru \rangle dt$$

where U is the set of all admissible controls, and $x(\cdot)$ is the solution of (3) corresponding to the control u and initial state x_0 . Since an optimal control exists for each initial state x_0 , we see that V is well defined.

The assumption $M = \{0\}$ (see (23)) is needed to prove the first statement in the next lemma, and this lemma will be used in proving system (24) is a.s.i.l..

Lemma 8: If $M = \{0\}$, then V has the following properties:

1. $V(x_0) = 0$ iff $x_0 = 0$.
2. If $a \geq 1$ then $V(ax_0) \geq V(x_0)$.
3. V is continuous.

Proof: Lemma 8 is proved in Appendix 2.

We now are in a position to prove (24) is a.s.i.l..

Theorem 7: If $M = \{0\}$, then system (24) is a.s.i.l..

Proof: We have to prove that (24) is stable and that for any initial state $\lim_{t \rightarrow \infty} x(t) = 0$. We first observe that for any x_0 ,

$$V(x_0) = V(x(t_1)) + \int_0^{t_1} \langle x, Qx \rangle + \langle u, Ru \rangle dt. \quad (26)$$

Eq. (26) expresses the fact that the control $t \rightarrow u'(t) = u(t)$, $t_1 \leq t < \infty$ is the optimal control for initial state $x(t_1)$. From (26), since the integrand in (26) is non-negative, it follows that

$$V(x_0) \leq V(x(t_1)). \quad (27)$$

In other words, V as a function of time is nonincreasing.

We prove the stability of (24) first. Given $\epsilon > 0$, let

$$S_\epsilon = \{x \in R^n \mid \|x\| = \epsilon\}.$$

Then since V is continuous, $V(x) > 0$ for each $x \in S_\epsilon$, and S_ϵ is compact,

$$\inf_{x \in S_\epsilon} V(x) = b > 0 \quad (28)$$

Since $V(0) = 0$ and V is continuous, we can find $0 < \delta < \epsilon$ such that if $\|x_0\| < \delta$, then $V(x_0) < b$. Hence for any initial state x_0 satisfying $\|x_0\| < \delta$, in view of (27) and (28), the solution of (24) remains inside the set S_ϵ . This shows (24) is stable.

From 2 of Lemma 8 and (28), it follows that if $\|x\| \geq \epsilon$, then $V(x) \geq b$. Since

$$V(x_0) = \lim_{t_1 \rightarrow \infty} \int_0^{t_1} \langle x, Qx \rangle + \langle u, Ru \rangle dt$$

it follows from (26) that there exists T such that for all $t_1 > T$

$$V(x(t_1)) < b$$

which implies $\|x(t_1)\| \leq \epsilon$. Since ϵ was arbitrary, this shows $\lim_{t \rightarrow \infty} x(t) = 0$, which completes the proof of the theorem.

D. THE MAXIMUM PRINCIPLE

In view of the previous section, the maximum principle can now be applied to the problem. We shall formulate the maximum principle for it in this section. With a few changes all of the discussion and definitions of Sec. III, Part A, where we formulated the maximum principle for Problem I, carries over in this case. For this reason, the presentation here will be somewhat abbreviated.

We introduce the scalar differential equation

$$\dot{x}_0(t) = \frac{1}{2} \langle x, Qx \rangle + \frac{1}{2} \langle u, Ru \rangle \quad t \geq 0 \quad (29)$$

with the initial condition

$$x_0(0) = 0$$

and consider the set of auxiliary equations,

$$\dot{\psi}_i(t) = -\frac{\psi_0}{2} \frac{\partial \langle x, Qx \rangle}{\partial x_i} - \sum_{j=1}^n a_{ji} \psi_j(t) \quad i = 0, 1, 2, \dots, n. \quad (30)$$

The coefficients a_{ij} are the same as in (3), the state equations. We do not specify the initial conditions for the ψ_i , and therefore to each control u there corresponds not one solution of (30), but a family of solutions.

We observe that $\psi_0(t)$ is a constant because $(\partial/\partial x_0) \langle x, Qx \rangle = 0$.

The Hamiltonian for the problem is

$$\mathcal{H}(\underline{\psi}, \underline{x}, u) = \psi_0 f^0 + \langle \underline{\psi}, A\underline{x} + B\underline{u} \rangle \quad (31)$$

where $\underline{\psi} = (\psi_1, \psi_2, \dots, \psi_n)$ and $\underline{\psi}$ and \underline{x} are, as in Chapter II, $(\psi_0, \psi_1, \dots, \psi_n)$ and (x_0, x_1, \dots, x_n) , respectively. In terms of \mathcal{H} Eqs. (3), (29) and (30) may be expressed in the form of the following Hamiltonian system:

$$\dot{x}_i(t) = \frac{\partial \mathcal{H}}{\partial \psi_i} \quad i = 0, 1, \dots, n \quad (32)$$

$$\dot{\psi}_i(t) = -\frac{\partial \mathcal{H}}{\partial x_i} \quad i = 0, 1, \dots, n \quad (33)$$

Considering $\underline{\psi}$ and \underline{x} fixed, the Hamiltonian is a function of $u \in \Omega$. Let $\mathcal{M}(\underline{\psi}, \underline{x})$ denote the maximum of \mathcal{H} over Ω :

$$\mathcal{M}(\underline{\psi}, \underline{x}) = \max_{u \in \Omega} \mathcal{H}(\underline{\psi}, \underline{x}, u) \quad (34)$$

$\mathcal{M}(\underline{\psi}, \underline{x})$ is well defined because Ω is compact and \mathcal{H} is continuous.

The maximum principle as it applied to the problem is expressed in

Theorem 8: If $t \rightarrow u(t)$, $0 \leq t < \infty$, is an optimal control, then there is a nontrivial solution of (33), $\underline{\psi}(\cdot)$, corresponding to u such that

1. $\mathcal{H}(\underline{\psi}(t), \underline{x}(t), u(t)) = \mathcal{M}(\underline{\psi}(t), \underline{x}(t))$ almost everywhere on $[0, \infty)$.
2. $\mathcal{H}(\underline{\psi}(t), \underline{x}(t)) = 0$.
3. $\psi_0 \leq 0$.

We need to distinguish between the cases when $\psi_0 < 0$ and $\psi_0 = 0$. The problem is called normal when $\psi_0 < 0$, and abnormal when $\psi_0 = 0$. When the problem is normal, since (33) is linear and homogeneous in $\psi_0, \psi_1, \dots, \psi_n$, we may assume $\psi_0 = -1$.

E. A SPECIAL FORM OF THE MAXIMUM PRINCIPLE

In this section we shall strengthen Theorem 8, the maximum principle as it applies to our problem, in two ways. First, we show that the problem is always normal, and secondly that the vector $\underline{\psi}(t)$ in Theorem 8 must satisfy the boundary condition $\lim_{t \rightarrow \infty} \underline{\psi}(t) = (-1, 0, 0, \dots, 0)$. The final result is stated in Theorem 9. In this form, the maximum principle will later be shown to be a sufficient condition for optimality.

Lemma 9, whose proof is based on a construction that has been used in studying the time optimal control problem,¹ states that the problem is normal.

Lemma 9: For any initial state, the problem is normal.

Proof: Suppose u is an optimal control, and $\psi^0 = 0$ in Theorem 8. In this case we obtain from (31) and (33) the equations

$$\mathcal{H}(\underline{\psi}, \underline{x}, u) = \langle \underline{\psi}, A\underline{x} + B\underline{u} \rangle$$

$$\dot{\underline{\psi}} = -\underline{\psi}A$$

where $\underline{\psi} = (\psi_1, \psi_2, \dots, \psi_n)$. We observe that these equations are identical to (19) and (20). Then by exactly the same argument that follows (20),

$$u(t) = \text{sgn}(\underline{\psi}(t) \cdot B) \quad t \geq 0.$$

We shall obtain a contradiction by showing that

$$\int_0^{\infty} \langle u, Ru \rangle dt = \infty \quad (34)$$

First we observe that $\psi(t) \neq 0$, since $\psi_0 = 0$ and $\underline{\psi}(\cdot)$ is a non-trivial solution of (33). To establish (34), it suffices to show that the set of points, t , for which $\psi(t) \cdot B = 0$ is a discrete set. Observe that $\psi(t) \cdot B$ is an analytic function since $\psi(t) = \psi^0 e^{-At}$ for some $0 \neq \psi^0 \in \mathbb{R}^n$. Thus if $\psi(t)B$ vanishes on a nondiscrete set, it vanishes identically,

$$\psi^0 e^{-At} B = 0 \quad t \geq 0 \quad (35)$$

Differentiating (35) successively with respect to t and putting $t = 0$, we have

$$\begin{aligned} \psi^0 \cdot B &= 0 \\ \psi^0 \cdot AB &= 0 \\ &\vdots \\ \psi^0 \cdot A^{n-1}B &= 0 \end{aligned} \quad (36)$$

Since by hypothesis (3) is controllable (see Ref. 20),

$$\text{Rank } \{B, AB, \dots, A^{n-1}B\} = n$$

and therefore, since $\psi^0 \neq 0$, (36) can not hold, proving the lemma.

In view of Lemma 9, we may put $\psi^0 = -1$. Then if u maximizes the Hamiltonian, from (31) we have

$$u(t) = \text{sat } (R^{-1} B^* \psi(t)) \quad t \geq 0 \quad (37)$$

where B^* is the transpose of B , and $\text{sat } R^r \rightarrow R^r$ is the function whose i th component, $i = 1 \leq r$, is $(x = (x_1, x_2, \dots, x_r))$

$$(\text{sat } x)_i = \begin{cases} 1 & \text{if } x_i \geq 1 \\ x_i & \text{if } |x_i| < 1 \\ -1 & \text{if } x_i \leq -1 \end{cases}$$

In addition, with $\psi_0 = -1$, the differential Eqs. (30) (or equivalently (33)) which the ψ_i satisfy become

$$\dot{\psi}_i(t) = \sum_{j=1}^n q_{ij} x_j - \sum_{j=1}^n a_{ji} \psi_j(t) \quad i = 1, 2, \dots, n$$

where $Q = (q_{ij})$, or in vector notation

$$\dot{\psi} = -A^* \psi + Qx \quad (38)$$

We observe that to each control u , there corresponds a family of solutions of (38) which depends on the initial conditions for the ψ_i .

Lemma 10: Let $u(\cdot)$ be an optimal control and $\psi(\cdot)$ a solution of (38) corresponding to u such that

$$u(t) = \text{sat } (R^{-1} B^* \psi(t)) .$$

Then $\lim_{t \rightarrow \infty} \psi(t) = 0$.

Proof: Lemma 10 is proved in Appendix 2.

Combining Theorems 8 and Lemmas 9 and 10, we have the following form of the maximum principle,

Theorem 9: If $t \rightarrow u(t)$, $0 \leq t < \infty$, is an optimal control, then there is a solution of (38), $\psi(\cdot)$, corresponding to u such that

1. $u(t) = \text{sat} (R^{-1} B^* \psi(t)).$
2. $\lim_{t \rightarrow \infty} \psi(t) = 0 .$

F. SUFFICIENT CONDITIONS FOR OPTIMALITY

We show that the maximum principle as stated in Theorem 9 is also a sufficient condition for a control to be optimal (Theorem 10). The proof of this result is similar to the proof of Theorem 4, the analogous statement for Problem I, and it relies on Lemma 4. For brevity in notation, if u and u' are admissible controls defined over $[0, \infty)$, we let

$$C(u, t_1) = \int_0^{t_1} \langle \dot{x}(t, u), Qx(t, u) \rangle + \langle u, Ru \rangle dt$$

$$d(u, u', t_1) = \int_0^{t_1} \|u(t) - u'(t)\| dt .$$
(39)

We first establish a preliminary result.

Lemma 11: Let $t \rightarrow u(t)$, $0 \leq t < \infty$, satisfy conditions (1) and (2) in Theorem 9. Let $t \rightarrow u'(t)$, $0 \leq t < \infty$, be any admissible control. Then for any $\delta > 0$ and $N > 0$, there exists a $t_1 > N$ such that

$$C(u, t_1) - C(u', t_1) \leq \delta \|x(t_1, u) - x(t_1, u')\| + o(\epsilon) \quad (40)$$

when $d(u, u', t_1) \leq \epsilon$.

Proof: We observe that by putting $t_0 = 0$ in Lemma 4, it becomes valid here, and we consider the end time t_1 as a variable (but always finite). Then since $\psi_0 = -1$, we have from Lemma 4,

$$\begin{aligned} & \langle \psi(t_1), \underline{x}'(t_1, u') - \underline{x}(t_1, u) \rangle + o(\epsilon) \\ &= C(u, t_1) - C(u', t_1) + \langle \psi(t_1), x(t_1, u') - x(t_1, u) \rangle + o(\epsilon) \leq 0 . \end{aligned} \quad (41)$$

Eq. (41) is valid for any finite t_1 .

Now we use the hypothesis $\lim_{t \rightarrow \infty} \psi(t) = 0$. Given N and $\delta > 0$, we can find $t_1 > N$ such that

$$\langle \psi(t_1), x(t_1, u') - x(t_1, u) \rangle \leq \delta \|x(t_1, u') - x(t_1, u)\|.$$

The last expression together with (41) proves the lemma.

We are now in a position to prove the principal result of this section.

Theorem 10: Let $t \rightarrow u(t)$, $0 \leq t < \infty$, transfer x_0 to the origin $x = 0$. Then a necessary and sufficient condition for u to be optimal is that there exists a solution $\psi(\cdot)$ of (38) such that

1. $u(t) = \text{sat}(R^{-1} B^* \psi(t)) \quad t \geq 0$.
2. $\lim_{t \rightarrow \infty} \psi(t) = 0$.

Proof: The necessity part of the theorem is just a restatement of Theorem 9.

To prove the sufficiency, suppose u' is the optimal control but u is nonoptimal. Then for some α ,

$$\int_0^\infty f^0(x, u) - f^0(x', u') dt \geq 2\alpha > 0$$

where $x(\cdot)$ and $x'(\cdot)$ are the solutions of (3) corresponding to u and u' , respectively. We can then find an $N > 0$ such that for all $t_1 > N$

$$\int_0^{t_1} f^0(x, u) - f^0(x', u') dt \geq \alpha. \quad (42)$$

For $0 < \lambda < 1$, consider the controls

$$t \rightarrow u_\lambda(t) = (1 - \lambda) u(t) + \lambda u'(t), \quad t \geq 0. \quad (43)$$

Let $x_\lambda(\cdot)$ be the solution of (3) corresponding to u_λ . Then using Lemma 2, the convexity of f^0 , and (42),

$$\begin{aligned}
& \int_0^{t_1} f^0(x, u) - f^0(x_\lambda, u_\lambda) \, dt \\
&= \int_0^{t_1} f^0(x, u) - f^0((1 - \lambda)x + \lambda x', (1 - \lambda)u + \lambda u') \, dt \\
&\geq \int_0^{t_1} f^0(x, u) - \{(1 - \lambda)f^0(x, u) + \lambda f^0(x', u')\} \, dt \\
&= \lambda \int_0^{t_1} f^0(x, u) - f^0(x', u') \, dt \geq \alpha \lambda > 0 .
\end{aligned}$$

Restating the last inequality, we have,

$$C(u, t_1) - C(u_\lambda, t_1) \geq \alpha \lambda > 0 . \quad (44)$$

Now we use Lemma 11 to obtain a contradiction. From Lemma 2,

$$x_\lambda(t) = (1 - \lambda)x(t) + \lambda x'(t)$$

or

$$x_\lambda(t) - x(t) = \lambda(x'(t) - x(t)).$$

Because u' is optimal it transfers x_0 to $x = 0$, and therefore $x'(t)$ is bounded. Similarly, $x(t)$ is bounded. Therefore, there is a constant β such that for all t_1

$$\|x(t_1) - x_\lambda(t_1)\| \leq \beta \lambda .$$

We observe that for finite t_1 ,

$$\begin{aligned} d(u, u', t_1) &= \int_0^{t_1} \|u(t) - u_\lambda(t)\| dt = \lambda \int_0^{t_1} \|u(t) - u'(t)\| dt \\ &= \lambda \gamma \end{aligned}$$

with $\gamma < \infty$.

Finally, applying Lemma 11, we get

$$C(u, t_1) - C(u_\lambda, t_1) \leq \delta \beta \lambda + o(\gamma \lambda). \quad (45)$$

Since δ can be made arbitrarily small, the last expression contradicts (44). Therefore u must have been optimal.

G. THE SYNTHESIS PROBLEM

In this section we shall show how the optimal feedback control may be determined. It is shown that the optimal control is a linear function of the state in a neighborhood of the origin (Proposition 1), but no closed form expression for it is derived which is valid for all states (a numerical example for a simple second order system is given in Appendix C, and the results indicate that the feedback control is not a simple function of the state). However, it may be computed, roughly speaking, by running the system (3) backwards in time, in a way similar to that proposed by LaSalle for the time optimal control problem.¹³

It is expedient to combine the state equations (3), the auxiliary system (38), and the equation defining the optimal control in terms of ψ , (37) into a single equation. The equivalent equation reads

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} A & 0 \\ Q & -A^* \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \text{sat}(R^{-1}B^* \psi) \quad (46)$$

where $(x, \psi) = (x_1, x_2, \dots, x_n, \psi_1, \psi_2, \dots, \psi_n)$. The significance of (46) is this: for any initial state x_0 , in view of Theorem 10, there is a ψ^0 such that if $x(0) = x_0$ and $\psi(0) = \psi^0$, then the x component of the solution of (46) is the optimal trajectory, while the ψ component and (37) define the optimal control. According to Theorems 7 and 10, $x(t) \rightarrow 0$ and $\psi(t) \rightarrow 0$ as $t \rightarrow \infty$. Conversely, from Theorem 10, if we can find a ψ^0 such that both $x(t)$ and $\psi(t)$ converge to 0, then the optimal control is determined by (37) and $\psi(t)$. Thus the problem of finding an optimal control is one of choosing an appropriate ψ for each x .

Unfortunately, because (46) is nonlinear, there is no simple relation between x and a proper ψ in general. However, for all states sufficiently close to the origin we shall show there is a linear relation between ψ and x of the form

$$\psi = Px$$

where P is an $n \times n$ real matrix.

We first observe that if we replace the control region Ω by R^r , i. e., the constraints on the control are removed, but otherwise keep the problem the same, then everything done so far is valid if everywhere we substitute $R^{-1} B^* \psi$ for $\text{sat}(R^{-1} B^* \psi)$. In fact, the only places where we used the fact that the control region was Ω are in Eq. (37), Lemma 9, and the existence Theorem 1. When the control region is R^r , (37) is valid with the above substitution, and the proofs of Lemma 9 and Theorem 1 carry over essentially without change.

Assuming the control region is R^r , (46) becomes the linear equation

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} A & BR^{-1}B^* \\ Q & -A^* \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} \quad (47)$$

For brevity, let

$$D = \begin{pmatrix} A & BR^{-1}B^* \\ Q & -A^* \end{pmatrix}$$

Then, taking the Laplace transform of (47)

$$S \begin{pmatrix} \tilde{x}(s) \\ \tilde{\psi}(s) \end{pmatrix} - \begin{pmatrix} x(0) \\ \psi(0) \end{pmatrix} = D \begin{pmatrix} \tilde{x}(s) \\ \tilde{\psi}(s) \end{pmatrix} \quad (48)$$

where $\tilde{x}(s)$ and $\tilde{\psi}(s)$ are the Laplace transform of $x(\cdot)$ and $\psi(\cdot)$, respectively. From (48),

$$\begin{pmatrix} \tilde{x}(s) \\ \tilde{\psi}(s) \end{pmatrix} = (S I - D)^{-1} \begin{pmatrix} x(0) \\ \psi(0) \end{pmatrix} \quad (49)$$

The right hand side of (49) is a column vector of rational functions in s . Given $x(0)$, in order to satisfy the desired boundary conditions at ∞ ($x(t) \rightarrow 0$ and $\psi(t) \rightarrow 0$), it is necessary to choose $\psi(0)$ so that the poles at each of these rational functions have negative real parts. Such a choice is possible because of Theorem 10 and the existence Theorem 2. If we let $x(0)$ take on the values $e_j = (0, 0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in the j th coordinate, $1 \leq j \leq n$, we get corresponding to each e_j for an appropriate value of $\psi(0)$, a vector $p_j = (p_{1j}, p_{2j}, \dots, p_{nj})$. Now from the linearity of (49), it is clear that if $x(0) = \sum_{j=1}^n \alpha_j e_j$, then an appropriate choice for $\psi(0)$ is $\sum_{j=1}^n \alpha_j p_j$. In matrix notation, this can be expressed

$$\psi(0) = P x(0), \quad P = (p_{ij}) \quad (50)$$

It follows that in the case when control region is R^r , the optimal feedback control, is, in view of (37) and (50),[†]

$$x \longrightarrow u(x) = R^{-1} B^* P x . \quad (51)$$

Now we show that for states sufficiently near the origin the optimal feedback control is given by (51) even when the control region is Ω .

Consider the system

$$\dot{x} = Ax + B R^{-1} B^* P x . \quad (52)$$

In view of Theorem 7, it is a.s.i.l.. Let

$$G = \{x \in R^n \mid \|R^{-1} B^* P x\| \leq 1\} .$$

G is a neighborhood of the origin. Since (52) is a.s.i.l., we can find a neighborhood S of the origin such that if $x_0 \in S$, then the whole trajectory of (52) starting at x_0 remains within G . It is then clear that the optimal feedback control for states in S is the same regardless of whether the control region is Ω or R^r . We state this conclusion in

Proposition 1: There is a neighborhood S of $x = 0$ such that if $x \in S$, the optimal feedback control is

$$x \longrightarrow u(x) = R^{-1} B^* P x$$

for some $n \times n$ matrix. P .^{††}

[†] This result was obtained by Kalman.²¹ In order to maintain the continuity of the discussion, and also give a way of computing P , we have preferred to give our own development.

^{††} Later, by Lemma 12 we show P is unique.

We observe that for $x(0) \in S$ an appropriate choice for $\psi(0)$ in (46) to satisfy the boundary conditions at ∞ is given by (50). The next lemma shows that this is the only proper choice.

Lemma 12: For each x_0 , there is a unique ψ^0 such that if $x(0) = x_0$ and $\psi(0) = \psi^0$, then the solution of (46) satisfies

$$\lim_{t \rightarrow \infty} (x(t), \psi(t)) = 0.$$

Proof: Suppose $(x(\cdot), \psi(\cdot))$ and $(x'(\cdot), \psi'(\cdot))$ are two solutions of (46) satisfying $\lim_{t \rightarrow \infty} (x(t), \psi(t)) = \lim_{t \rightarrow \infty} (x'(t), \psi'(t)) = 0$ and $x(0) = x'(0) = x_0$. We have to show $\psi(0) = \psi'(0)$. From Theorem 10, it follows that the controls

$$t \rightarrow u(t) = \text{sat} (R^{-1} B^* \psi(t)) \quad t \geq 0$$

$$t \rightarrow u'(t) = \text{sat} (R^{-1} B^* \psi'(t)) \quad t \geq 0$$

are both optimal for initial state x_0 . By Theorem 6, $u(t) = u'(t)$ on $[0, \infty)$, and therefore $x(t) = x'(t)$ on the same interval. Then from (46)

$$\frac{d}{dt} \begin{pmatrix} 0 \\ \psi - \psi' \end{pmatrix} = \begin{pmatrix} A & 0 \\ Q & -A^* \end{pmatrix} \begin{pmatrix} 0 \\ \psi - \psi' \end{pmatrix}$$

or

$$\frac{d}{dt} (\psi - \psi') = -A^* (\psi - \psi').$$

Solving the last equation

$$\psi(t) - \psi'(t) = e^{-A^* t} (\psi(0) - \psi'(0)).$$

Since $u(t) = u'(t)$ on $[0, \infty)$, and both $\psi(t)$ and $\psi'(t)$ converge to 0, for some T ,

$$R^{-1} B^* \psi(t) = R^{-1} B^* \psi'(t) \quad t \geq T .$$

Hence, after transposing,

$$(\psi(0) - \psi'(0)) e^{-At} B = 0 .$$

We showed in proving Lemma 9 that the last relation holds iff $\psi(0) = \psi'(0)$, which proves the lemma.

Lemma 12 implies that the matrix P in Proposition 1 is unique. Combining Lemma 12 and Eq. (50), we have

Lemma 13: There is a neighborhood S of the origin and a unique matrix P such that if $x(0) \in S$ and $\psi(0) = Px(0)$ then the solution of (46) satisfies the boundary conditions

$$\lim_{t \rightarrow \infty} (x(t), \psi(t)) = 0 .$$

We now are in a position to show how the optimal feedback control may be computed. For states in S , it is given by Proposition 1. For states not in S , we have no closed form solution. However, using Lemma 13 to determine the appropriate initial conditions, we can trace out a family of optimal trajectories from S by integrating (46) backwards in time. By making the family sufficiently large, we can determine the feedback control on an arbitrarily "dense" set of states. This procedure is analogous to the method proposed by LaSalle for the time optimal control problem,¹³ and is explained more precisely in the next two paragraphs.

For each $x \in S$, let $\psi = Px$. The differential Eq. (46) has a unique solution on $-\infty < t \leq 0$ which satisfies the initial condition $\psi(0) = Px$ and $x(0) = x$. Each trajectory generated in this way determines an optimal trajectory. For if we trace out these trajectories in the direction of increasing t , both $x(t)$ and $\psi(t)$ converge to 0, and therefore, in view of Theorem 10, are optimal.

Now we show that the set of optimal trajectories generated in the above way fills the entire state space R^n . Theorem 7 implies that every optimal trajectory eventually enters S . Let t_1 be the time $x(t_1) \in S$. From time t_1 on, the trajectory must be optimal, so if $\psi(\cdot)$ is the corresponding ψ component of the solution of (46), we must have $\psi(t_1) = Px(t_1)$. Hence the trajectory can be traced out by integrating (46) "backwards" in time with initial conditions $x(0) = x(t_1)$ and $\psi(0) = Px(t_1)$.

Since in the "backward tracing" method of determining $u(x)$, the value of $u(x)$ can only be determined in practice on a subset of state space, we need

Proposition 2: The optimal feedback control is a continuous function.

Proof: Proposition 2 is proven in Appendix B.

H. SOME COMPUTATIONAL ASPECTS OF THE SYNTHESIS PROBLEM

In this section we give alternative methods for computing the matrix P of the last section, and we show how a neighborhood of the origin S having the required properties may be determined.

As stated in the footnote on page 43, Kalman derived the result expressed by Eq. (51). Kalman also gave two methods for determining the matrix P which can be more convenient than the method we gave. We shall indicate his results in the next two paragraphs.

We recall that the matrix P has the property that if we put $\psi(0) = Px(0)$ in (47), the solution of (47) converges to 0 as $t \rightarrow \infty$. But the relation $\psi(0) = Px(0)$ holds not only for $t = 0$ but for all t , and therefore $\psi(t) = Px(t)$. Using this in (47), we get the two equations

$$\dot{x} = Ax + B R^{-1} B^* Px$$

$$P \dot{x} = Qx - A^* Px$$

multiplying the top equation by P and equating with the second,

$$P A x + P B R^{-1} B^* P x = Q x - A^* P x$$

and since this relation holds for all x ,

$$P A + A^* P + P B R^{-1} B^* P - Q = 0 . \quad (53)$$

Considering P as an unknown, (53) is a system of nonlinear algebraic equations. In general (53) does not have a unique solution. However, Kalman²² showed that the desired P is the unique symmetric negative definite solution of (53). Therefore, P may be determined by solving (53).

A convenient numerical method for computing P with a digital computer is the following method proposed by Kalman. Consider the $n \times n$ matrix Riccati differential equation

$$\frac{d\pi}{dt} = \pi A + A^* \pi + \pi B R^{-1} B^* \pi - Q.$$

Let $\pi(t, 0)$ be the solution of (54) which satisfies the initial condition $\pi(t, 0) = 0$. Then the matrix P is given by

$$P = \lim_{t \rightarrow \infty} \pi(t, 0) .$$

We observe that the matrix P is a solution of (54) since the right hand side vanishes when $\pi = P$. Kalman showed that the solution $\pi = P$ is a.s.i.l.; given any symmetric initial condition for (54), the solution of (54) converges to P as $t \rightarrow \infty$. This stability property of the solution $\pi = P$ means that the numerical computation is also stable.

Now we consider how a set S satisfying the conditions in Proposition 1 may be computed. Consider the convex set

$$G = \{x \in R^n \mid \|R^{-1} B^* P x\| \leq 1\}.$$

We must choose S so that if $x(0) \in S$ then the whole trajectory of

$$\dot{x} = Ax + BR^{-1} B^* Px$$

remains within G . Letting $D = A + BR^{-1} B^* P$, we compute $e^{Dt} e_j$, where $e_j = (0, 0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in the j^{th} coordinate, for $j = 1, 2, \dots, n$. Then we find the largest a_j , $j = 1, 2, \dots, n$, such that for all $t \geq 0$, $a_j e^{Dt} e_j \in G$. Since G is convex, the convex hull, H , of $\{a_j e_j \mid j = 1, 2, \dots, n\}$ is contained in G , and furthermore if $x(0) \in H$, then $e^{Dt} x(0) \in G$ for all $t \geq 0$. Therefore, an appropriate choice is $S = H$.

I. NOTES AND REMARKS

The only place we needed the Lyapunov stability assumption was in proving existence of optimal controls (see Corollary 2). In fact, the stability was used only to show that any initial state could be transferred to the origin by an admissible control. We give an example of a system which is not stable, but for which there still is an optimal control for arbitrary initial states, in Appendix C. The results of this chapter are valid for this example.

For regulating systems describable by Eqs. (3) having admissible control region Ω , the most widely studied design criterion has been that of time optimality. We will compare briefly, time optimal regulators with ones design according to our performance index. The time optimal feedback control (for so called normal systems) assumes only the values $+1$ and -1 and may be realized with relays, for example. However, in general, no formulas are known for the optimal control as a function of the state. The same lack of knowledge prevails in our case. The optimal control in our case has a simple realization only in some neighborhood of the origin in state space. In terms of performance, there may be some reason to believe our regulator may perform better

in some cases: complaints have been made that some regulators designed for time optimal behavior have excessive overshoots during transient motions. By adjusting the values of the matrices Q and R , we can alter the performance. When Q is "large" compared to R , one would expect performance approaching time optimal behavior.

APPENDIX A

We shall prove Lemma 4. First we observe that for some $M < \infty$, the following inequality is valid;

$$\max_{t_0 \leq t \leq t_1} \|x(t, u) - x(t, u')\| \leq M d(u, u') \quad (A-1)$$

where

$$d(u, u') = \int_{t_0}^{t_1} \|u(t) - u'(t)\| dt$$

(A-1) follows easily from Eq. (4). We also shall need

Lemma A-1: If $f: \Omega \times [t_0, t_1] \rightarrow \mathbb{R}$ is a continuous bounded function, then the function $F: U \rightarrow \mathbb{R}$ defined by

$$F(u) = \int_{t_0}^{t_1} f(u(t), t) dt$$

is continuous with respect to the distance d .

Proof: Given $\epsilon > 0$, we can find δ' such that if $\|u(t) - u'(t)\| < \delta'$ then $|f(u(t), t) - f(u'(t), t)| < \epsilon/2(t_1 - t_0)$. This is possible since f is continuous. We then can find δ such that if $d(u, u') < \delta$ then the set D on which $\|u(t) - u'(t)\| \geq \delta'$ has measure less than $\epsilon/4M$, where M is the maximum of f . Then $d(u, u') < \delta$ implies (D^c denotes the complement of D in $[t_0, t_1]$),

$$\begin{aligned}
|F(u) - F(u')| &= \left| \int_{t_0}^{t_1} f(u) - f(u') \, dt \right| \\
&\leq \int_{t_0}^{t_1} |f(u) - f(u')| \, dt \\
&= \int_D + \int_{D^c} \\
&\leq 2M \frac{\epsilon}{4M} + \frac{\epsilon}{2(t_1 - t_0)} (t_1 - t_0) = \epsilon
\end{aligned}$$

which proves F is continuous.

We now prove Lemma 4, which we restate for convenience.

Lemma 4: Let $t \rightarrow u(t)$ and $t \rightarrow u'(t)$ be admissible controls and $\underline{x}(\cdot)$ and $\underline{x}'(\cdot)$ the solutions of (12) corresponding to u and u' respectively. Suppose there is a solution $\underline{\psi}(\cdot)$ of (13) corresponding to u such that almost everywhere on $[t_0, t_1]$,

$$\mathcal{H}(\underline{\psi}(t), \underline{x}(t), u(t)) = \mathcal{M}(\underline{\psi}(t), \underline{x}(t)) .$$

Then

$$\langle \underline{\psi}(t_1), \underline{x}'(t_1) - \underline{x}(t_1) \rangle + o(\epsilon) \leq 0$$

when $d(u, u') \leq \epsilon$.

Proof: Consider the integral

$$I = \int_{t_0}^{t_1} \langle \underline{\psi}, \underline{f}(\underline{x}, u) - \underline{f}(\underline{x}', u') \rangle \, dt$$

where $\underline{\psi} = (\psi_0, \psi_1, \dots, \psi_n)$ and \underline{f} is defined in (9). Since ψ_0 is a constant,

$$I = \psi_0 I_1 + I_2 \quad (\text{A-2})$$

where

$$I_1 = \int_{t_0}^{t_1} f^0(\underline{x}, u) - f^0(\underline{x}', u') dt$$

and

$$I_2 = \int_{t_0}^{t_1} \langle \psi, A(x - x') + B(u - u') \rangle dt$$

in which $\psi = (\psi_1, \psi_2, \dots, \psi_n)$. I_1 may be evaluated directly:

$$I_1 = x_0(t_1) - x'_0(t_1)$$

so we obtain for I the value

$$I = \psi_0(x_0(t_1) - x'_0(t_1)) + I_2 \quad (\text{A-3})$$

We obtain another expression for I by expanding $f^0(\underline{x}', u')$ in a Taylor series. Recalling that f^0 does not depend on x_0 ,

$$f^0(\underline{x}', u') = f^0(\underline{x}, u) + \nabla f^0(\underline{x}, u) \cdot (\underline{x}' - \underline{x}) + o(\epsilon) \quad .$$

The remainder is $o(\epsilon)$ because of (A-1) .

Substituting in the expression for I_1 ,

$$I_1 = \int_{t_0}^{t_1} f^0(\underline{x}, u) - f^0(\underline{x}, u') dt + \int_{t_0}^{t_1} \nabla f^0(\underline{x}, u') \cdot (\underline{x} - \underline{x}') dt$$

$$+ o(\epsilon) . \quad (A-4)$$

The second integral, I_3 , differs by at most $o(\epsilon)$ from

$$I_4 = \int_{t_0}^{t_1} \nabla f^0(\underline{x}, u) \cdot (\underline{x} - \underline{x}') dt .$$

To see this, first note that

$$|I_3 - I_4| \leq \max_{t_0 \leq t \leq t_1} ||\underline{x}(t) - \underline{x}'(t)|| \int_{t_0}^{t_1} ||\nabla f^0(\underline{x}, u) - \nabla f^0(\underline{x}, u')|| dt .$$

The function ∇f^0 considered as a function on $\Omega \times [t_0, t_1]$ satisfies the conditions in Lemma 1. Therefore the assertion follows.

Substituting (A-4), after replacing $\nabla f^0(\underline{x}, u')$ by $\nabla f^0(\underline{x}, u)$, in (A-2) and rearranging terms,

$$I = \int_{t_0}^{t_1} \psi^0(f^0(\underline{x}, u) - f^0(\underline{x}, u')) + \langle \psi, B(u - u') \rangle dt$$

$$+ \int_{t_0}^{t_1} \langle A^* \psi + \psi_0 \nabla f^0(\underline{x}, u), \underline{x} - \underline{x}' \rangle dt + o(\epsilon) .$$

The integrand in the first integral is

$$\mathcal{H}(\psi, \underline{x}, u) - \mathcal{H}(\psi, \underline{x}, u')$$

which is non-negative by hypothesis. Therefore,

$$I \leq \int_{t_0}^{t_1} \langle A^* \psi + \psi^0 \nabla f^0(x, u), x - x' \rangle dt + o(\epsilon) . \quad (A-5)$$

Referring to (10) the integral in (A-5) is identical to

$$I_5 = \int_{t_0}^{t_1} \langle \dot{\psi}, x' - x \rangle dt$$

and integrating I_5 by parts,

$$I_5 = \langle \psi(t_1), x'(t_1) - x(t_1) \rangle + I_2 .$$

Therefore,

$$I \leq \langle \psi(t_1), x'(t_1) - x(t_1) \rangle + I_2 + o(\epsilon) . \quad (A-6)$$

Finally, subtracting (A-3) from (A-6) we get the desired result:

$$\langle \underline{\psi}(t_1), \underline{x}'(t_1) - \underline{x}(t_1) \rangle + o(\epsilon) \leq 0 .$$

APPENDIX B

Lemma 8: If $M = \{0\}$, then V has the following properties.

- a. $V(x_0) = 0$ iff $x_0 = 0$.
- b. If $a \geq 1$, $V(ax_0) \geq V(x_0)$.
- c. V is continuous.

Proof: (1) If $V(x_0) = 0$, then $u = 0$, because $\langle x, Qx \rangle \geq 0$.
Therefore,

$$V(x_0) = \int_0^\infty \langle e^{At} x_0, Qe^{At} x_0 \rangle dt.$$

By assumption $M = \{0\}$ and therefore $x_0 = 0$. Conversely, $V(0) = 0$.

(2) Let $t \rightarrow u(t)$ be the optimal control for the initial state ax_0 . For initial state x_0 , consider the control $t \rightarrow u'(t) = \frac{1}{a} u(t)$. Denote by $x(\cdot, \omega, z)$ the solution of (3) corresponding to control ω and initial z . Clearly,

$$x(t, u', x_0) = \frac{1}{a} x(t, u, ax_0), \quad t \geq 0.$$

Hence

$$\begin{aligned} V(x_0) &\leq \int_0^\infty \langle x(t, u', x_0), Qx(t, u', x_0) \rangle + \langle u', Ru' \rangle dt \\ &= \frac{1}{a^2} V(ax_0) \end{aligned}$$

which proves b.

(3) We will show V is convex. The continuity of V follows because a convex function defined on R^n is necessarily continuous.²³

Let u_1 and u_2 be optimal controls associated with initial states x^1 and x^2 . For $0 < \lambda < 1$, it follows from (4) that

$$x(t) = \lambda x_1(t) + (1 - \lambda) x_2(t)$$

where

$$\begin{aligned}
x(t) &= x(t, \lambda u_1 + (1 - \lambda) u_2, \lambda x^1 + (1 - \lambda) x^2) \\
x_1(t) &= x(t, u_1, x^1) \\
x_2(t) &= x(t, u_2, x^2)
\end{aligned}$$

Then using the definition of V ,

$$\begin{aligned}
V(\lambda x^1 + (1 - \lambda) x^2) &\leq \int_0^\infty \langle x, Qx \rangle + \langle (\lambda u_1 + (1 - \lambda) u_2, R(\lambda u_1 + (1 - \lambda) u_2)) \rangle dt \\
&\leq \lambda \int_0^\infty \langle x_1, Qx_1 \rangle + \langle u_1, Ru_1 \rangle dt \\
&\quad + (1 - \lambda) \int_0^\infty \langle x_2, Qx_2 \rangle + \langle u_2, Ru_2 \rangle dt \\
&= \lambda V(x^1) + (1 - \lambda) V(x^2)
\end{aligned}$$

which proves c.

Lemma 10: Let $u(\cdot)$ be an optimal control and $\psi(\cdot)$ a solution of (38) corresponding to u such that

$$u(t) = \text{sat} (R^{-1} B^* \psi(t))$$

Then $\lim_{t \rightarrow \infty} \psi(t) = 0$.

Proof: Since u is optimal

$$\int_0^\infty \langle u, Ru \rangle dt < \infty \quad (B-1)$$

Solving Eq. (38), we have for any $T \geq 0$,

$$\psi(t+T) = e^{-A^*(t-T)} \psi(T) + \int_T^t e^{-A^*(t-\tau)} Qx(\tau) d\tau.$$

From Theorem 7, $\lim_{T \rightarrow \infty} x(\tau) = 0$, and therefore given $\epsilon > 0$, there is a T_0 such that if $T > T_0$,

$$\|\psi(t+T) - e^{-A^*(t-T)} \psi(T)\| < \epsilon \quad T \leq t \leq T+1.$$

Therefore, for any $T \geq T_0$, $T \leq t \leq T+1$, from (37),

$$\|u(t+T)\| + \epsilon \|R^{-1} B^*\| \tag{B-2}$$

$$\geq \min(1, \|R^{-1} B^* e^{A(t-T)} \psi(T)\|).$$

Note that the right hand side of (B-2) is uniformly continuous in t (i.e. the continuity is independent of T). From (B-1) and (B-2), it then follows that $\lim_{t \rightarrow \infty} u(t) = 0$.

Since $\lim_{t \rightarrow \infty} u(t) = 0$, from (B-2) it follows that given any $\delta > 0$ there is a T_1 such that for $T > T_1$

$$\delta \geq \max_{T \leq t \leq T+1} \min(1, \|R^{-1} B^* e^{A(t-T)} \psi(T)\|)$$

The proof of the lemma will be completed by showing for some $\alpha > 0$

$$\min_{\psi \in R^n} \max_{0 \leq t \leq 1} \|R^{-1} B^* e^{-At} \psi\| = \alpha \|\psi\|.$$

We first observe that the expression

$$\|R^{-1} B^* e^{-At} \psi\|$$

is homogeneous with respect to ψ and that R^{-1} is invertible, so it suffices to show

$$\min_{\|\psi\| = 1} \max_{0 \leq t \leq 1} \|B^* e^{-At} \psi\| = \beta > 0.$$

Consider the function

$$g(\psi) = \max_{0 \leq t \leq 1} \|B^* e^{-At} \psi\|.$$

In proving Lemma 9 it was shown that there is a $0 \leq t' \leq 1$ such that $B^* e^{-At'} \psi = 0$, and therefore $g(\psi) > 0$. Clearly, g is continuous. Therefore,

$$\min_{\|\psi\| = 1} g(\psi) = \beta > 0$$

which completes the proof of the lemma.

Proposition 2: The optimal feedback control is a continuous function.

Proof: For brevity, we denote by $x(\cdot, x_0)$ and $\psi(\cdot, \psi^0)$ the x and ψ components of the solution of (46) which satisfy the boundary conditions $x(0, x_0) = x_0$ and $\psi(0, \psi^0) = \psi^0$. By Lemma 9 there is assigned to each x_0 a unique ψ^0 such that $x(t, x_0) \rightarrow 0$ and $\psi(t, \psi^0) \rightarrow 0$ as $t \rightarrow \infty$. Let F be the function defined by this assignment: $F(x_0) = \psi^0$. In view of (37), to prove the proposition, it suffices to show F is continuous. We observe that for some $a > 0$, $S \supset S_a$ = $\{x \in R^n \mid \|x\| \leq a\}$, and that if $x \in S_a$ then $F(x) = Px$, so F is continuous on S_a .

Given x_0 , assuming $\psi^0 = F(x_0)$, there is a $t_1 > 0$ such that $x(t_1) \in \text{int } S_a$ (interior of S_a). We consider the map $G: S_a \rightarrow R^n$ defined by

$$G(y) = x(-t_1, y)$$

where $x(-t_1, y)$ is the first component of $(x(-t_1, y), \psi(-t_1, Py))$. G is 1:1 because of Lemma 12. Moreover G is continuous because the solution of (46) depends continuously on initial conditions. Since S_a is compact,

G^{-1} , the inverse of G , is continuous.²⁴ Observe that $G^{-1}(x_0) = x(t_1, x_0)$. Hence since $x(t_1, x_0) \in \text{int } S_a$, there is a neighborhood W of x_0 such that $G^{-1}(W) \subset \text{int } S_a$. Then if $x \in W$,

$$\begin{aligned} F(x) &= \psi(-t_1, F(G^{-1}(x))) \\ &= \psi(-t_1, P \cdot G^{-1}(x)) . \end{aligned}$$

Since $\psi(-t_1, \psi^0)$ depends continuously on ψ^0 , the last equation proves F is continuous, which proves the proposition.

APPENDIX C

An example of the problem treated in Sec. IV was studied using a digital computer. The system considered was

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= u\end{aligned}\tag{C-1}$$

and the matrices Q and R were

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = 1.\tag{C-2}$$

Although (C-1) is not Lyapunov stable, for the reasons indicated in Sec. IV, Part I, the results of Sec. IV are valid. The matrix P , which defines the optimal feedback control in a neighborhood of the origin, was computed by solving Eq. (53).

$$P = \begin{pmatrix} -\sqrt{2} & -1 \\ -1 & -\sqrt{2} \end{pmatrix}\tag{C-3}$$

It was found that the optimal feedback control, u^0 , is not a simple function of the state. A few optimal trajectories and the value of u^0 at some points on them is shown in Fig. 1.

If it is desired to realize the optimal control in a physical problem, there are various possibilities. Since u^0 is a continuous function of the state, u^0 can be approximated as accurately as desired by computing u^0 on a sufficiently dense set of states. This can be done by the reverse time mapping procedure explained in Sec. IV, Part G. The values of the control as a function of state can then be arranged in a "look-up"

table and stored in some kind of memory device. Another possibility is to approximate u^0 by simpler functions. Because u^0 is continuous, this is possible in principle, and the approximating functions can be chosen as piecewise linear or polynomial functions, for example.

The practicality of either of the above methods was not investigated in detail. However, a comparison was made between u^0 and the control u' defined by

$$\begin{aligned} u'(x) &= \text{sat} (R^{-1} B^* P x) \\ &= - \text{sat} (x_1 + \sqrt{2} x_2). \end{aligned}$$

The control u' is simply the saturation of the optimal control for the case when the control region is the whole real line.

For this problem, the performance index is

$$C(u) = \int_0^{\infty} x_1^2(t, x_0) + u^2(t) dt.$$

It is natural to consider u' as a good approximation to u^0 for initial state x_0 if $C(u)$ and $C(u')$ are about equal. In this sense, u' is a good approximation to u^0 for states in the disk

$$O = \{ (x_1, x_2) \mid x_1^2 + x_2^2 < 16 \}$$

The value of $C(u)$ and $C(u')$ was computed for different values of x_0 . For states in O , the values of $C(u)$ and $C(u')$ were about equal. Some values of C are given in Table I. The last few entries in Table I show that u' is not as good as u^0 for some states not very far from O . For $x_0 = (16.6, -6.11)$, the difference between u' and u^0 is considerable and it is shown in Fig. 2.

TABLE I
A COMPARISON OF CONTROLS u AND u'

x_1	x_2	$C(u^0)$	$C(u')$
0.040	3.24	107.	107.
0.085	2.66	43.4	43.4
0.088	1.97	12.5	12.5
0.98	2.94	106.	106.
1.06	-3.06	43.9	43.9
1.09	1.37	11.6	11.6
2.07	2.55	104.	104.
2.12	-1.88	3.1	3.1
2.12	-3.36	45.0	45.0
3.01	2.15	102.	102.
3.08	-2.60	7.09	7.54
3.96	-2.46	11.3	12.7
4.19	-2.95	12.7	15.8
4.89	-2.11	19.6	21.8

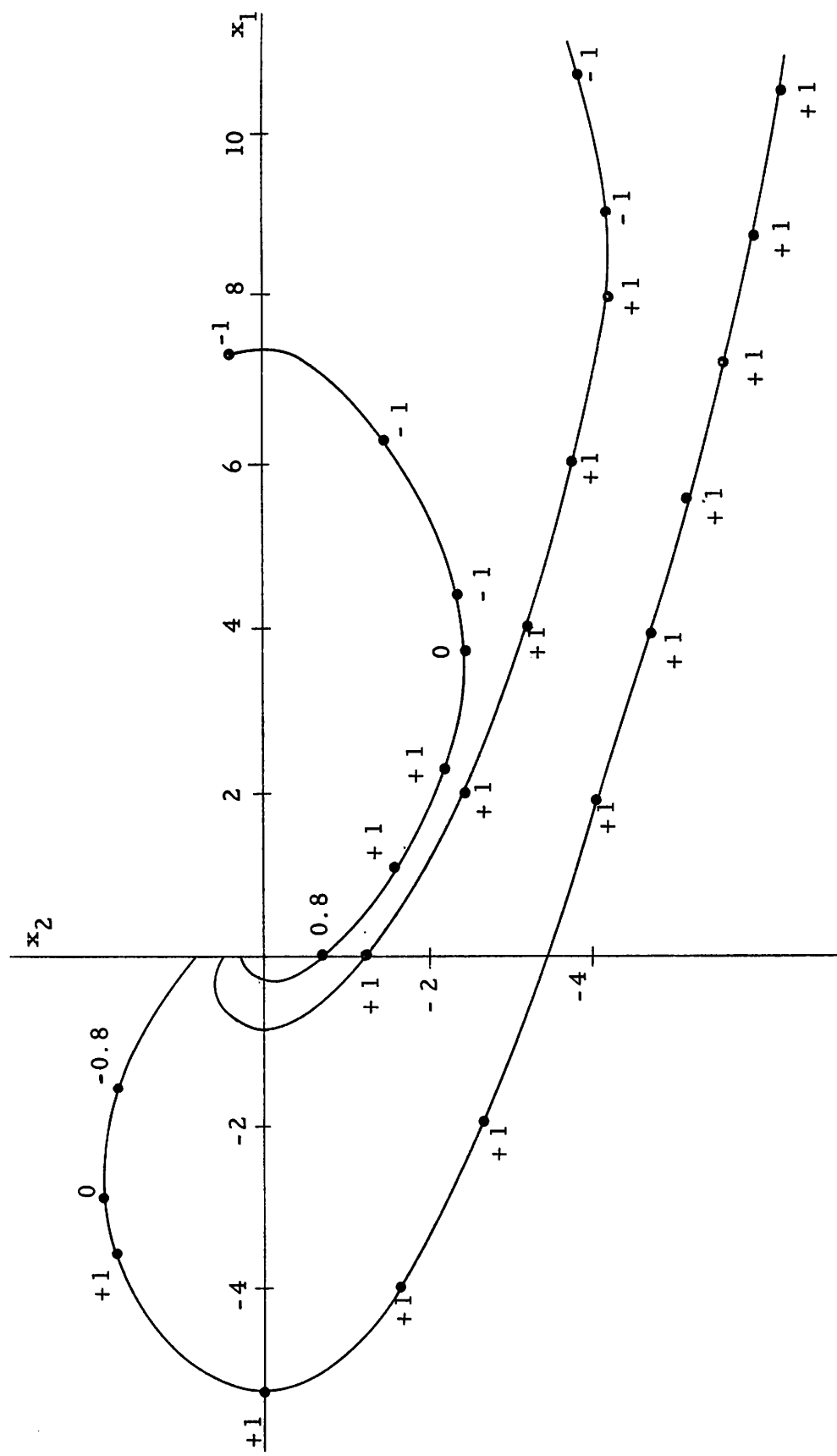


Fig. 1. Some optimal trajectories and typical values of $u^0(x)$.

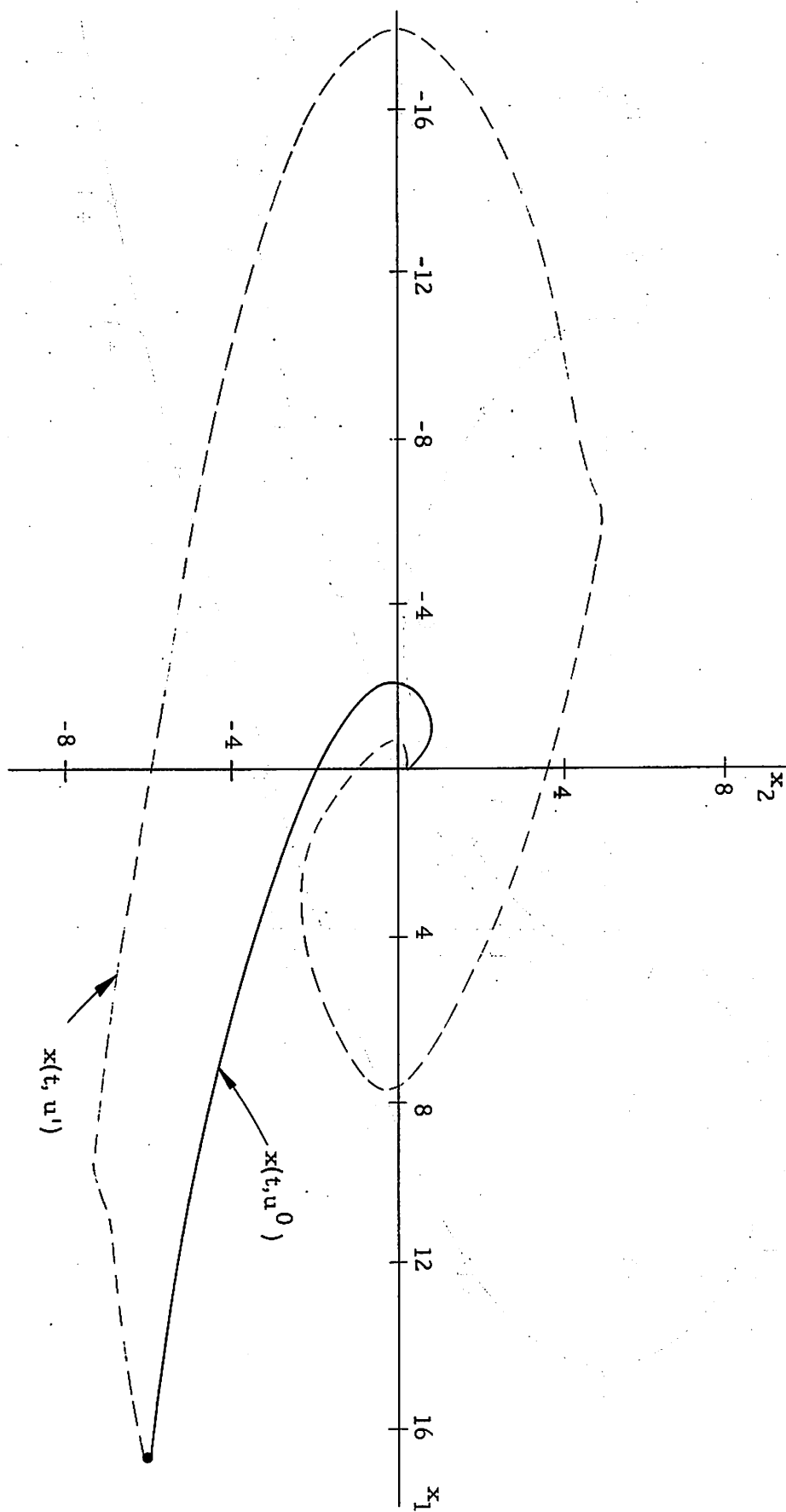


Fig. 2. A comparison of controls u^0 and u^1 . $C(u^0) = 306$; $C(u^1) = 2450$.

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