ASYMPTOTIC UNBOUNDED ROOT LOCI-

FORMULAE AND COMPUTATION

by

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ABSTRACT

We study the asymptotic behavior of the closed-loop eigenvalues (root loci) of a strictly proper, linear, time-invariant control system as the loop gain goes to ∞. We develop and use for this purpose the eigen-properties of restricted linear maps of the form $A(\mathrm{mod}\ S_2)\big|_{S_1}$ where A is a map from $\mathbb{C}^m$ to $\mathbb{C}^m$ and S_1, S_2 are subspaces of complimentary dimension. We suggest a numerically robust way of mechanizing the computation of our formulae using the singular value decomposition. Finally, some applications are indicated.

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Section 1. Introduction

It is well known that high loop gain enhances the desirable effects of feedback, e.g. desensitization and disturbance attenuation. It is known also that practical control systems are driven to instability by high gain feedback. This paper presents a new geometric way of calculating and numerically stable way of computing the asymptotic behavior of unbounded root loci of a strictly proper, linear, time invariant control system shown in Figure 1 as loop gain $\to\infty$ $(k\to\infty)$.

The asymptotic behavior of unbounded root loci has been studied extensively by Kouvaritakis and Shaked [3], Kouvaritakis [4], Kouvaritakis and Edmunds [5] and Owens [11,12]. We believe that our (mathematically) new approach leads to more explicit and simpler formulae. For an intrinsic algebraic-geometric picture of multivariable root loci we refer the reader to Brockett and Byrnes [10].

The present paper recognizes that the calculation of the asymptotes of the unbounded root loci is a process of identifying subspaces in the input space and'modding out' subspaces in the output space of the open-loop control system where the effects of the $O(k)$, $O(k^{1/2})$, $O(k^{1/3})$, ... unbounded root loci dominate asymptotically (for this standard notation see [6]). To compute the asymptotes of the unbounded root loci we are led naturally to the use of numerically stable orthogonal projections and the singular value decomposition (see for instance, Golub and Reinsch [17], Stewart [1]).

The organization of the paper is as follows: In Section 2, we develop some properties of restricted linear maps of the form $A(\text{mod } S_2)\Big|_{S_1}$, where A is a map from $\mathbb{C}^n$ to $\mathbb{C}^n$ and S_1, S_2 are subspaces of complimentary dimension. This development was motivated by Wonham [18]. In Section 3, we apply this theory to the computation of the asymptotic values of the multivariable root loci. Using the results of van Dooren, et al [7] and the simple null structure assumption, we relate these asymptotic values to the structure at ∞ of the Smith - McMillan form of the open loop transfer function. In Section 4, we derive a numerically stable method for computing the formulae of Section 3, using the singular value decomposition. We calculate formulae for the pivots and indicate robust computation to obtain them in Section 5. Concluding remarks on relaxing the assumptions of and extending our work are collected in Section 6.

Notation

i. $\mathcal{R}(A)$ stands for the range of a matrix $A \in \mathbb{C}^{n \times m}$ and KerA stands for the kernel of A.

ii. $A^{\dagger}$ stands for (any) generalized (or pseudo-inverse) of A, defined as follows:

Let $f_1, \ldots, f_k \in \mathbb{C}^n$ be a basis for the $\mathcal{R}(A)$ with $e_1, \ldots, e_k$ chosen such that $Ae_i = f_i$, $i = 1, \ldots, k$. Complete the basis $f_1, \ldots, f_k$ to obtain a basis $f_1, \ldots, f_n$ of $\mathbb{C}^n$. Now, define

$$A^{\dagger} f_i = e_i \quad i = 1, \ldots, k$$

$$A^{\dagger} f_i = \text{arbitrary} \quad i = k+1, \ldots, n.$$

Section 2. Restrictions of a Linear Map

2.1 General Theory

Given a linear map A from $\mathbb{C}^n$ to $\mathbb{C}^n$ and a subspace $S_2 \subset \mathbb{C}^n$ of dimension $(n-m)$, the operator A(mod S_2) is the linear map from $\mathbb{C}^n$ to $\mathbb{C}^n/S_2$ defined by the following diagram:

$$
\begin{tikzcd}
\mathbb{C}^n \arrow[r, "A"] \arrow[dr, "(A(\text{mod } S_2) = P\cdot A)"'] & \mathbb{C}^n \arrow[d, "P"] \\
 & \mathbb{C}^n/S_2
\end{tikzcd}
$$

Here, P stands for the canonical projection $\mathbb{C}^n \to \mathbb{C}^n/S_2$. The maps whose structure we will expose here are of the form $A(\text{mod } S_2)\big|_{S_1}$ where $S_1 \subset \mathbb{C}^n$ is a subspace of dimension complimentary to S_2 (namely, m). Pictorially, we have for $A(\text{mod } S_2)\big|_{S_1}$

$$
\begin{tikzcd}
S_1 \arrow[r, hook, "i"] \arrow[drr, "A(\text{mod } S_2)\big|_{S_1} = P\cdot A\cdot i"'] & \mathbb{C}^n \arrow[r, "A"] & \mathbb{C}^n \arrow[d, "P"] \\
 & & \mathbb{C}^n/S_2
\end{tikzcd}
$$

Here i is the (canonical) inclusion map of S_1 in $\mathbb{C}^n$. Note that if $\hat{S}_2$ is <u>any</u> direct summand of S_2 then $\hat{S}_2$ is isomorphic to $\mathbb{C}^n/_{S_2}$. Since $\mathbb{C}^n/_{S_2}$ is an abstract vector space, we will for the purpose of computation identify $\mathbb{C}^n/_{S_2}$ with $\hat{S}_2$. We have then the following representation theorem for $A(\text{mod } S_2)\big|_{S_1}$.

<u>Theorem 2.1</u> (Representation Theorem)

Let the columns of $T_1 \in \mathbb{C}^{n\text{x}m}$ form a basis for S_1, and the columns of $T_2 \in \mathbb{C}^{n\text{x}m}$ form a basis for $\hat{S}_2$, some direct summand of S_2. Then, the matrix representation for $A(\text{mod } S_2)\big|_{S_1}$ with respect to the bases furnished by T_1 and T_2 is

$$(T_2^* T_2)^{-1} T_2^* A T_1 \in \mathbb{C}^{m\text{x}m} \quad . \tag{2.1}$$

<u>Proof</u>: Recall from elementary linear algebra [1, pg. 125] that $T_2(T_2^* T_2)^{-1} T_2^* \in \mathbb{C}^{n\text{x}n}$ is the matrix representation of the projection from $\mathbb{C}^n$ onto $\hat{S}_2$ with the columns of T_2 as basis for $\hat{S}_2$. Since the columns of T_1, T_2 are chosen as bases for S_1, $\hat{S}_2$ the result follows. □

<u>Notes</u>: (i) $(T_2^* T_2)^{-1} T_2^*$ is a left inverse of T_2.

(ii) If the columns of $\tilde{T}_1 \in \mathbb{C}^{n\times m}$ and $\tilde{T}_2 \in \mathbb{C}^{n\times m}$ form bases for S_1 and $\tilde{S}_2$ (any other direct summand of S_2), then the representations are related by $(\tilde{T}_2^* \tilde{T}_2)^{-1} \tilde{T}_2^* A \tilde{T}_1 = P(T_2^* T_2)^{-1} T_2^* A T_1 Q$. where $P,Q \in \mathbb{C}^{m\times m}$ are nonsingular matrices.

<u>Definition 2.2</u> $\lambda \in \mathbb{C}$ is <u>an eigenvalue</u> of $\underline{A(\text{mod } S_2)|_{S_1}}$ if $\exists$ non-zero $x \in S_1$ such that $(A - \lambda I)x \ (\text{mod } S_2) = 0$; equivalently, $\exists$ $x \in S_1 \ni (A - \lambda I)x \in S_2$.

<u>Proposition 2.3</u> (Generalized eigenvalue problem for eigenvalues of $A(\text{mod } S_2)|_{S_1}$).

Let $B \in \mathbb{C}^{n\times m}$; $C \in \mathbb{C}^{m\times n}$ be chosen so that $R(B) = S_2$, $\text{Ker } C = S_1$. Then, the eigenvalues of $A(\text{mod } S_2)|_{S_1}$ are precisely the solutions, λ, of the generalized eigenvalue problem

$$\det \left[\begin{array}{c|c} A - \lambda I & B \\ \hline C & 0 \end{array}\right] = 0 \qquad (2.2)$$

<u>Proof</u>: Observe that λ is a solution of (2.2) iff $\exists$ $x \in \mathbb{C}$, $u \in \mathbb{C}^m$ not both zero such that

$$(A - \lambda I)x + Bu = 0$$

with

$$x \in \text{Ker } C. \qquad \square$$

In fact solutions of all generalized eigenvalue problems can be obtained from Definition 2.1 as follows:

<u>Proposition 2.4</u> (converse to Proposition 2.3)

The solutions λ of the generalized eigenvalue problem

$$\det \left[\begin{array}{c|c} A - \lambda I & B \\ \hline C & D \end{array}\right] = 0 \qquad (2.3)$$

are the eigenvalues of

$$A - BD^{\dagger}C(\mathrm{mod}\ B(\mathrm{Ker}\ D))\big|_{C^{-1}(\mathcal{R}(D))} \quad . \tag{2.4}$$

(Here, $D^{\dagger}$ stands for <u>any</u> pseudo-inverse of D, $C^{-1}(\mathcal{R}(D))$ stands for the inverse image under C of $\mathcal{R}(D)$ and B(Ker D) stands for the image under B of Ker D).

<u>Proof:</u> λ is a solution of (2.3) iff $\exists$ $x \in \mathbb{C}^n$, $u \in \mathbb{C}^m$ not both zero such that

$$(A - \lambda I)x + Bu = 0 \tag{2.5}$$

$$Cx + Du = 0 \tag{2.6}$$

Note that Cx must belong to the range of D so that $x \in C^{-1}(R(D))$ and the (non-unique) solution of (2.6) is

$$u = -D^{+}Cx + v \tag{2.7}$$

where v is any element of Ker D.

Use (2.7) in (2.5) to obtain

$$(A-BD^{+}C - \lambda I)x + Bv = 0$$

with $x \in C^{-1}(\mathcal{R}(D))$ and $v \in \mathrm{Ker}\ D$.

The converse is similar. $\square$

We now specialize to the case when $S_1 \oplus S_2 = \mathbb{C}^n$. There is then a natural isomorphism $\tilde{I}$ between S_1 and $\mathbb{C}^n/_{S_2}$ as follows

$$\begin{tikzcd} S_1 \arrow[r, hook, "i"] \arrow[drr, "\tilde{I} = P\cdot I\cdot i"'] & \mathbb{C}^n \arrow[r, "I"] & \mathbb{C}^n \arrow[d, "P"] \\ & & \mathbb{C}^n/_{S_2} \end{tikzcd} \qquad \tilde{I} := I(\mathrm{mod}\ S_2)\big|_{S_1}$$

It is clear that in this case there are m eigenvalues of $A(\mathrm{mod}\ S_2)\big|_{S_1}$. We explore their structure:

<u>Definition 2.5</u> $A(\mathrm{mod}\ S_2)\big|_{S_1}$ is said to have <u>simple null structure</u> if

$\exists$ $x \in S_1$ such that

$$A(\text{mod } S_2)x \neq 0 \quad \text{and} \quad A(\text{mod } S_2)\, \tilde{I}^{-1} A(\text{mod } S_2)x = 0.$$

Comments: (i) The definition states that there are no generalized eigenvectors associated with the eigenvalue $\lambda = 0$ of $A(\text{mod } S_2)|_{S_1}$.

(ii) Definition 2.5 is useful for counting the number of non-zero (possibly repeated) eigenvalues of $A(\text{mod } S_2)|_{S_1}$ as follows:

Proposition 2.6 (Number of non zero eigenvalues of $A(\text{mod } S_2)|_{S_1}$).

If $A(\text{mod } S_2)|_{S_1}$ has simple null structure the number (counting nultiplicities) of its non-zero eigenvalues is equal to its rank, namely the dimension of $\mathcal{R}(A(\text{mod } S_2)|_{S_1})$.

Definition 2.7 : $A(\text{mod } S_2)|_{S_1}$ is said to have simple structure associated with an eigenvalue λ if $A - \lambda I(\text{mod } S_2)|_{S_1}$ has simple null structure.

With these definitions on hand, one may state the Jordan canonical form theorem for the operator $A(\text{mod } S_2)|_{S_1}$.

Theorem 2.8 (Jordan Canonical form for $A(\text{mod } S_2)|_{S_1}$)

Assume $S_1 \oplus S_2 = \mathbb{C}^n$ and identify $\mathbb{C}^n/_{S_2}$ with S_1. Then, there exists a choice of basis for S_1 - the columns of $T \in \mathbb{C}^{n\times m}$ such that the matrix representation of $A(\text{mod } S_2)|_{S_1} : S_1 \to \mathbb{C}^n/_{S_2} \cong S_1$ is

$$(T^* T)^{-1} T^* AT = \text{diag } [J_1, \ldots, J_p] \tag{2.8}$$

where $J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$.

Since $(T^* T)^{-1} T^* = T^L$, a left inverse of T, we may write (2.8) as

$$T^L AT = \text{diag } [J_1, \ldots, J_p] \quad .$$

Proof: Follows exactly along the same lines as the regular Jordan canonical form theorem and is omitted. □

2.2. Specialization to Orthogonal Projections

For the purpose of numerically stable computation we specialize the above definitions and propositions to orthogonal bases and orthogonal projections. Let the subspace $\hat{S}_2 \subset \mathbb{C}^n$ be the orthogonal complement of S_2 and the columns of P_1, $P_2 \in \mathbb{C}^{n\times m}$ form orthonormal basis for S_1, $\hat{S}_2$ respectively. We will identify $A(\text{mod } S_2)\big|_{S_1}$ with the representation furnished by P_1, P_2, namely $P_2^* AP_1$(by Theorem 2.1)$\in \mathbb{C}^{m\times m}$. We also denote $P_2^* AP_1$ by $A\big|_{S_1} \to \hat{S}_2$.

In these coordinates, the eigenvalues of $A(\text{mod } S_2)\big|_{S_1}$ are the zeros of the polynomial

$$\det(\lambda P_2^* P_1 - P_2^* AP_1) = 0 \tag{2.9}$$

Definition 2.9 The adjoint of $A\big|_{S_1} \to \hat{S}_2$ is defined to be the linear map $A^*\big|_{\hat{S}_2} \to S_1$.

The following proposition is now obvious:

Proposition 2.10 (Orthogonal decomposition of domain and range)

$$\mathcal{R}(A\big|_{S_1} \to \hat{S}_2) \overset{\perp}{\oplus} \eta(A^*\big|_{\hat{S}_2} \to S_1) = \hat{S}_2$$

$$\eta(A\big|_{S_1} \to \hat{S}_2) \overset{\perp}{\oplus} \mathcal{R}(A^*\big|_{\hat{S}_2} \to S_1) = S_1 .$$

Section 3. System Description, Assumptions and Main Formulae

The system under study is the system of Figures, where $G(s)$ is the mxm transfer function matrix of a linear, time-invariant, strictly proper control system assumed to have Taylor expansion about $s = \infty$ (convergent $\forall\ |s| > M$):

$$G(s) = \frac{G_1}{s} + \frac{G_2}{s^2} + \frac{G_3}{s^3} + \ldots.. \tag{3.1}$$

with G_1, G_2,... $\in \mathbb{R}^{mxm}$; k real and positive. We consider the case when G(s) is a strictly proper rational transfer function matrix, i.e. $G(s) \in \mathbb{R}(s)^{mxm}$ (Formally, all the results of Section 3.1 go through for strictly proper irrational transfer functions with convergent Taylor series at $s = \infty$). We study the closed loop poles of the system of Figure 1 as $k \to \infty$. The motivation is that G(s) represents the composition of a linear, time-invariant plant and controller and k represents high gain feedback; as $k \to \infty$ the gain tends to ∞ in all control channels. The one parameter curves traced on an appropriately defined Riemann surface [2] by the closed-loop eigenvalues (parametrized by k)are referred to as the multivariable root loci. As $k \to \infty$ some of the root loci tend to finite points in (copies of) the complex plane located at the (McMillan) zeros of the system (see for e.g. [3,4,5]), the others go to ∞ as $k \to \infty$ and are referred to as the unbounded root loci of the system. We classify the unbounded root loci by the velocity (with k) with which they tend to ∞:

Definition 3.1 An unbounded multivariable root locus $s_n(k)$ is said to be an nth order unbounded root locus (n = 1,2,3,...) if asymptotically

$$s_n(k) = \mu_n(k)^{1/n} + O(k^o) \tag{3.2}$$

where $|\mu_n| < \infty$ and $O(k^o)$ is a term of order 1.

We identify an nth order unbounded root locus with μ_n, the coefficient of its asymptotic value.

Theorem 3.2 (Generalized eigenvalue problem for the nth order unbounded root locus)

$\mu_n = (-\lambda)^{1/n} \in \mathbb{C}$ is the coefficient of the asymptotic value of an nth order unbounded root locus iff λ is a solution of the generalized eigenvalue problem

$$\det \left[\begin{array}{c:c:c:c} G_n - \lambda I & G_{n-1} & \cdot & G_1 \\ \hdashline G_{n-1} & G_{n-2} & \cdot & 0 \\ \hdashline \cdot & \cdot & \cdot & \cdot \\ \hdashline G_2 & G_1 & \cdot & 0 \\ \hdashline G_1 & 0 & \cdot & 0 \end{array}\right] = 0 \qquad (3.3)$$

provided (3.3) is a polynomial equation in λ.

Proof: Assume, asymptotically, that the value s of the unbounded root locus is given by

$$s = \mu k^{1/n} + 0(1) \qquad (3.4)$$

with $\mu \neq 0$.

Using the standard method to find the terms in the asymptotic expansion of an implicitly defined variable [6, Chap. 3 esp. article 8] we rewrite $\det(I + kG(s)) = 0$ asymptotically as $: \exists\ e_1, e_2, \ldots \in \mathbb{R}^m$ such that

$$[I + \frac{kG_1}{s} + \frac{kG_2}{s^2} \quad \ldots]\ [e_1 + \frac{e_2}{k^{1/n}} + \ldots + \frac{e_n}{k^{\frac{n-1}{n}}} + \ldots] = 0 \qquad (3.5)$$

or using (3.4)

$$[I + \frac{k^{\frac{n-1}{n}} G_1}{\mu} + \frac{k^{\frac{n-2}{n}} G_2}{\mu^2} \ \ldots]\ [e_1 + \frac{e_2}{k^{1/n}} + \ldots + \frac{e_n}{k^{\frac{n-1}{n}}} \ldots] = 0 \qquad (3.6)$$

Equating terms of $0(k^{\frac{n-1}{n}})$, $0(k^{\frac{n-2}{n}})$, . . . , $0(1)$ we obtain from (3.6)

$$\det \left[\begin{array}{c:c:c:c} G_n + \mu^n I & G_{n-1} & & G_1 \\ \hdashline G_{n-1} & G_{n-2} & & 0 \\ \hdashline \cdot & \cdot & & \cdot \\ \hdashline G_2 & G_1 & & 0 \\ \hdashline G_1 & 0 & & 0 \end{array}\right] \left[\begin{array}{c} e_1 \\ \hdashline \mu e_2 \\ \hdashline \cdot \\ \hdashline \cdot \\ \hdashline \mu^{n-1} e_n \end{array}\right] = 0 \qquad (3.7)$$

(3.3)now follows readily from (3.7), provided of course that (3.3) is non-degenerate (i.e. depends on λ). $\square$

Comments: (i) There are n separate nth order root loci corresponding to the distinct nth roots of 1 associated with each solution of (3.3); so that these root loci constitute at ∞, an n-cycle of the Riemann surface of the system (see [8, pg. 32], [2, pg. 112]). (ii) The matrix of (3.4) is a triangular, block Toeplitz matrix so that (3.4) must admit of simplification. We take this up next:

3.1 Formulae for the asymptotic values of the unbounded root loci

3.1.1 First-Order

Clearly, these are the negatives of the non-zero eigenvalues of G_1

$$s_{i,1} = -\lambda_{i,1} k + 0(k^o).$$

3.1.2 Second Order

These are given by

$$s_{i,2} = (-\lambda_{i,2} k)^{1/2} + 0(k^o)$$

where $\lambda_{i,2}$ is a non zero solution of

$$\det \left[\begin{array}{c|c} G_2 - \lambda I & G_1 \\ \hline G_1 & 0 \end{array}\right] = 0 \quad .$$

From Proposition 2.3, then $\lambda_{i,2}$ is a non-zero eigenvalue of

$$G_2(\text{mod } \mathcal{R}(G_1))\Big|_{\eta(G_1)} \triangleq: \hat{G}_2$$

3.1.3 Third Order

These are given by

$$s_{i,3} = (-\lambda_{i,3} k)^{1/3} + 0(k^o)$$

where $\lambda_{1,3}$ is a nonzero solution of

$$\det \left[\begin{array}{c|c|c} G_3 - \lambda I & G_2 & G_1 \\ \hline G_2 & G_1 & 0 \\ \hline G_1 & 0 & 0 \end{array}\right] = 0. \qquad (3.8)$$

<u>Proposition 3.3</u> (Third order eigenvalue formula)

$\lambda_{i,3}$ is a <u>non-zero</u> eigenvalue of

$$(G_3 - G_2\, G_1^{\dagger}\, G_2)(\text{mod } R(G_1)) \quad (\text{mod } R(\hat{G}_2)) \Big|_{\eta(\hat{G}_2)}$$

where

$$\hat{G}_2 := G_2(\text{mod } R(G_1)) \Big|_{\eta(G_1)}$$

and $G_1^{\dagger}$ is <u>any</u> pseudo inverse of G_1 .

<u>Remark</u>: Pictorially, we have

$$\begin{tikzcd} \eta(\hat{G}_2) \arrow[r, hook, "i_2"] & \eta(G_1) \arrow[r, hook, "i_1"] & C^m \arrow[r, "G_3 - G_2 G_1^{\dagger} G_2"] & C^m \arrow[d, "P_1"] \\ & & & C^m/R(G_1) \arrow[d, "P_2"] \\ & & & (C^m/R(G_1))/R(\hat{G}_2). \end{tikzcd}$$

<u>Proof</u>: Let $v_1, v_2, v_3 \in \mathbb{R}^m$ not all zero such that

$$(G_3 - \lambda I)v_1 + G_2v_2 + G_1v_3 = 0 \tag{3.9}$$

$$G_2v_1 + G_1v_2 = 0 \tag{3.10}$$

$$G_1v_1 = 0 \tag{3.11}$$

(3.11)yields that $v_1 \in \eta(G_1)$. Next (3.10) yields that

$v_1 \in \eta(G_2 \ (\text{mod } R(G_1)\big|_{\eta(G_1)})$ i.e. $\eta(\hat{G}_2)$.

Further from (3.10), we obtain

$$v_2 = -G_1^{\dagger}\, G_2\, v_1 + u_1 \tag{3.12}$$

where u_1 is (any) vector belonging to $\eta(G_1)$ and $G_1^{\dagger}$ is a pseudo-inverse of G_1. Using (3.12) in (3.9) we obtain

$$(G_3 - G_2G_1^{\dagger}G_2 - \lambda I)v_1 + G_2u_1 + G_1v_3 = 0.$$

i.e. $(G_3 - G_2G_1^{\dagger}G_2 - \lambda I)v_1 \bmod R(G_1) \bmod R(\hat{G}_2) = 0.$

This proves the proposition.

3.1.4 Fourth Order

These are of the form

$$s_{i,4} = (-\lambda_{i,4}k)^{1/4} + O(k^o)$$

Proposition 3.4 (Fourth order eigenlocus formula) $\lambda_{1,4}$ is an eigenvalue of

$$(G_4 - G_3G_1^{\dagger}G_2 - G_2G_1^{\dagger}G_3 + G_2G_1^{\dagger}G_2G_1^{\dagger}G_2)(\mathrm{mod}R(G_1))(\mathrm{mod}R(\hat{G}_2))(\mathrm{mod}R(\hat{G}_3))\Big|_{\eta(\hat{G}_3)}$$

where

$$\hat{G}_3 \overset{\Delta}{:=} (G_3 - G_2G_1^{\dagger}G_2)(\mathrm{mod}R(G_1))(\mathrm{mod}R(\hat{G}_2))\Big|_{\eta(\hat{G}_2)}$$

and $G_1^{\dagger}$ is any pseudo-inverse of G_1.

Proof: From Theorem 3.2 , $\lambda_{i,4}$ is the coefficient of a fourth order root locus if $\exists$ $v_1,v_2,v_3,v_4 \in \mathbb{R}^m$ not all zero such that

$$(G_4 - \lambda I)v_1 + G_3v_2 + G_2v_3 + G_1v_4 = 0 \tag{3.13}$$

$$G_3v_1 + G_2v_2 + G_1v_3 = 0 \tag{3.14}$$

$$G_2v_1 + G_1v_2 = 0 \tag{3.15}$$

$$G_1v_1 = 0 \tag{3.16}$$

As before from (3.15), (3.16) we have $v_1 \in \eta(\hat{G}_2)$ and $v_2 = -G_1^{\dagger}G_2v_1 + u_1$ for some $u_1 \in \eta(G_1)$. Using this in (3.14) we get

$$(G_3 - G_2G_1^{\dagger}G_2)v_1 + G_2u_1 + G_1v_3 = 0 \quad . \tag{3.17}$$

Then, $v_1 \in \eta(\hat{G}_3)$, $u_1 \in \eta(\hat{G}_2)$ and

$$v_3 = -G_1^\dagger (G_3 - G_2 G_1^\dagger G_2) v_1 - G_1^\dagger G_2 u_1 + u_2 \tag{3.18}$$

for some $u_2 \in \eta(G_1)$.

Using (3.18) in (3.13) we obtain

$$(G_4 - G_3 G_1^\dagger G_2 - G_2 G_1^\dagger G_3 + G_2 G_1^\dagger G_2 G_1^\dagger G_2 - \lambda I) v_1 + (G_3 - G_2 G_1^\dagger G_2) u_1 + G_2 u_2 + G_1 v_4 = 0$$

or

$$(G_4 - G_3 G_1^\dagger G_2 - G_2 G_1^\dagger G_3 + G_2 G_1^\dagger G_2 G_1^\dagger G_2 - \lambda I) v_1 (\mathrm{mod} \mathcal{R}(G_1)) (\mathrm{mod} \mathcal{R}(\hat{G}_2)) (\mathrm{mod} \mathcal{R}(\hat{G}_3)) = 0$$

This proves the proposition.

3.1.5 Higher Order

We invite the reader to write formulae for the higher order root loci. The basic idea is to solve the triangular algebraic equations as if G_1 were inverible using any pseudo-inverse , $G_1^\dagger$. The non-uniqueness in this process is kept track off by succesively restricting in the domain to $\eta(G_1) \supset \eta(\hat{G}_2) \supset \ldots$ Also the conditions for the algebraic equations to have solutions are kept track of by succesively modding out in the range $\mathcal{R}(G_1)$, $\mathcal{R}(\hat{G}_2)$, ...

3.2 Simple Null Structure and Integer Order for All Unbounded Root Loci

In general, the branches of the algebraic function obtained from det $(I + kG(s)) = 0$ at $s = \infty$ have asymptotic expansion (see [8])

$$s = \lambda(k)^{m/n} + 0(k^o)$$

showing possible non-integral order, unbounded root loci.

However, we show now that under some simple assumptions the <u>only</u> unbounded root loci are those of integral order. First, some preliminaries

<u>Notation:</u> Define $T_n \in \mathbb{R}^{nmxnm}$ to be the block Toeplitz matrix:

$$T_n = \begin{pmatrix} G_n & G_{n-1} & \cdots & G_1 \\ G_{n-1} & & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\ G_1 & 0 & \cdots & 0 \end{pmatrix} \tag{3.19}$$

<u>Assumption 1</u> (Non-degeneracy assumption)

There exists some n_o such that

$$\begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R(T_{n_o}) \quad . \tag{3.20}$$

<u>Comments</u>: (i) If (3.20) is satisfied for some n_o it is satisfied for all $n \geq n_o$. Hence, in the sequel we will understand that n_o is the smallest integer so that (3.20) is satisfied.

(ii) (3.20) implies in particular that no linear combination of outputs is identically zero and that the inputs are linearly independent. Further insight into the nature of this assumption follows from Proposition 3.5.

To study the behavior at $s = \infty$ of G(s) perform the change of variables, $w = 1/s$. Then,

$$G(w) = G_1 w + G_2 w^2 + \ldots \tag{3.21}$$

Recall that G(w) admits of a unique Smith-McMillan form $\Lambda(w)$ given by

$$G(w) = M(w)\ \Lambda(w)\ N(w) \tag{3.22}$$

where M(w) and N(w) are unimodular matrices and

$$\Lambda(w) = \text{diag}\left(\frac{e_1(w)}{f_1(w)}, \ldots, \frac{e_m(w)}{f_m(w)}\right) \quad . \tag{3.23}$$

with the e_i and f_i monic coprime polynomials, with e_i dividing e_{i+1} and f_{i+1} dividing f_i for all i. Further

<u>Proposition 3.5</u> (Explication of Assumption 1)

Assumption 1 $\Leftrightarrow$ G(w) has normal rank m.

<u>Proof:</u> Is straighforward, see for e.g. [7] . $\square$

For any $\alpha \in C$, which is either a pole or zero of G(w)

$$\Lambda(w) = \Lambda_\alpha(w) \cdot \tilde{\Lambda}(w)$$

where

$$\Lambda_\alpha = \begin{bmatrix} (w-\alpha)^{\sigma_1} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & (w-\alpha)^{\sigma_m} \end{bmatrix} \tag{3.24}$$

with $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_m$.

The matrix $\Lambda_\alpha(w)$ contains information about the order $\omega_p(\omega_z)$ and degree $\delta_p(\delta_z)$ of the pole (zero) at $w = \alpha$ as follows

$$\omega_p = -\sigma_1 \text{ if } \sigma_1 \leq 0 \qquad \delta_p = -\sum_{\sigma_i<0} \sigma_i \quad . \tag{3.25}$$

$$\omega_z = \sigma_m \text{ if } \sigma_m \geq 0 \qquad \delta_z = \sum_{\sigma_i>0} \sigma_i \quad . \tag{3.26}$$

We are interested in the order and degree of the zero of G(w) at w = 0. A theorem of [7] relates ω_z, δ_z for the zero at $w = 0$ to the ranks of the block Toeplitz matrices T_n, defined by (3.19). Define, for $i \geq 1$ (with the understanding that $T_o = 0$)

$$\rho_i = \text{rank } T_i - \text{rank } T_{i-1} \quad .$$

Then, we have

<u>Theorem 3.6</u> [7] (Order and degree of zeros related to rank T_n)

$$\omega_z = \min \{i | \rho_i = m\} \quad . \tag{3.27}$$

$$\delta_z = \sum_{i=1}^{\omega_z} (m-\rho_i) + m \; . \tag{3.28}$$

Corollary 3.7 (Further explication of Assumption 1) [7].

Let n_o be the smallest integer so that (3.20) is satisfied. Then,

$$n_o = \omega_z.$$

Comment: The number of unbounded root loci of the system is δ_z. We will show under Assumption 2, that δ_z unbounded roots of the 1st, 2nd, ...,ω_zth order are obtained. First, a preliminary proposition.

Proposition 3.8 (Connection between rank $\hat{G}_n$ and rank T_n)

$$\begin{aligned}
\text{rank } G_1 &= \rho_1 \\
\text{rank } \hat{G}_2 &:= \dim (R(\hat{G}_2)) = \rho_2 - \rho_1 \\
\text{rank } \hat{G}_3 &:= \dim (R(\hat{G}_3)) = \rho_3 - \rho_2
\end{aligned}$$

and so on.

Proof: We leave it to the reader to verify that:

$$\begin{aligned}
\text{rank } T_1 &= \text{rank } G_1 = \rho_1 \\
\text{rank } T_2 - \text{rank } T_1 &= \text{rank } G_1 + \text{rank } \hat{G}_2 = \rho_2 \\
\text{rank } T_3 - \text{rank } T_2 &= \text{rank } G_1 + \text{rank } \hat{G}_2 + \text{rank } \hat{G}_3 = \rho_3
\end{aligned}$$

and so on. □

Recall from Proposition 2.6 that the connection between rank and number of non-zero eigenvalues is simple-null structure. Hence, we assume

Assumption 2 (Simple null structure)

Assume that

$$\begin{aligned}
&G_1 \\
&\hat{G}_2 := G_2 \bmod R(G_1)\Big|_{\eta(G_1)} \\
&\hat{G}_3 := G_3 - G_2 G_1^{\dagger} G_2 \,(\mathrm{mod} R(\hat{G}_2))(\mathrm{mod} R(G_1))\Big|_{\eta(\hat{G}_2)}
\end{aligned}$$

etc. have simple null structure.

The key observation to make is that Assumption 2 guarantees that the operators $G_1, \hat{G}_2, \hat{G}_3, \ldots,$ are of the form studied in the previous section (i.e. of the form $A(\text{mod } S_2)\big|_{S_1}$ with $S_1 + S_2 = C^m$). Precisely,

<u>Proposition 3.9</u>

G_1 has simple null structure $\Longleftrightarrow \mathcal{R}(G_1) + \eta(G_1) = \mathbb{R}^m$.

<u>Proof:</u> Follows from Jordan canonical form.

Let I_1 now be the (natural) isomorphism between $\eta(G_1)$ and $\mathbb{R}^m/\mathcal{R}(G_1)$. Then, we have

<u>Proposition 3.10</u>

$\hat{G}_2$ has simple null structure $\Longleftrightarrow \eta(G_2) + I_1^{-1}\,\mathcal{R}(G_2) = \eta(G_1)$. □

<u>Proof</u>: exactly as in Proposition 3.9

Similar considerations hold for $\hat{G}_3$, $\hat{G}_4$, etc.. Pictorially, we have the condition that the operators $I_1, I_2, I_3, \ldots,$ etc. defined so that the diagram below commutes are all isomorphisms.

$$\begin{tikzcd}
\eta(\hat{G}_3) \arrow[r, hook] \arrow[dddrrrr, "I_3"'] & \eta(\hat{G}_2) \arrow[r, "i_2"] \arrow[ddrrr, "I_2"'] & \eta(G_1) \arrow[r, hook, "i_1"] \arrow[drr, "I_1"'] & \mathbb{R}^m \arrow[r, "I"] & \mathbb{R}^m \arrow[d, "P_1"] \\
 & & & & \mathbb{R}^m/\mathcal{R}(G_1) \arrow[d, "P_2"] \\
 & & & & (\mathbb{R}^m/\mathcal{R}(G_1))/\mathcal{R}(\hat{G}_2) \arrow[d, "P_3"] \\
 & & & & ((\mathbb{R}^m/\mathcal{R}(G_1))/\mathcal{R}(\hat{G}_2)/\mathcal{R}(\hat{G}_3)
\end{tikzcd}$$

Proposition (2.6) then assures us that the number of non-zero eigenvalues of G_1, $\hat{G}_2$, $\hat{G}_3$,... is the same as rank G_1 , rank $\hat{G}_2$, rank $\hat{G}_3$, ... Using this, we obtain

<u>Theorem 3.11</u> (Asymptotic Unbounded Root Loci)

Under Assumptions 1 and 2 the <u>only</u> unbounded root loci of the system of Figure 1 are the 1st, 2nd, ..., n_oth order unbounded root loci specified by Theorem 3.2.

<u>Proof</u>: The proof is by counting. By Theorem 3.2 and the observations made above the number of 1st, 2nd,...,n_oth order unbounded root loci is

$$\dim R(G_1) + 2 \dim R(\hat{G}_2) \; .... + n_o \dim R(\hat{G}_{n_o}) \quad .$$

Using Proposition 3.8, this is rewritten as

$$\rho_1 + 2(\rho_2 - \rho_1) + ... + n_o(\rho_{n_o} - \rho_{n_o-1}) \tag{3.29}$$

with $\rho_{n_o} = m$ and $n_o = \omega_z$ (by Theorem 3.6 and Corollary 3.7).

Simplifying (3.29) we obtain the number of 1st,...,n_oth order unbounded root loci to be

$$m + \sum_{i=1}^{n_o} (m - \rho_i) = \delta_z .$$

This proves the theorem. □

<u>Comments on dropping the simple null structure assumption</u>

Consider the following example of the failure to Theorem 3.11 without the simple null structure assumption:

$$G_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad G_k = 0 \quad k \geq 3.$$

By Theorem 3.2 there are no first, second, ... order unbounded root loci. Also, unbounded root loci. Also, by Theorem 3.6, $\omega_z = 2$ and $\delta_z = 3$. Note that $\det(I + kG(s)) = 0$ yields $1 + \frac{k^2}{s^3} = 0$ so that there are three $0(k^{2/3})$ root loci. To indicate the nature of the unbounded root loci in the absence of Assumption 2, consider

Proposition 3.12

Let $G_i = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & 1 \\ 0 & \cdots & & 0 \end{bmatrix}$ [$\in \mathbb{R}^{m \times m}$ and let the (m,m)th element of]

G_ℓ, $g_{m,m}^{(\ell)} = 0$ for $\ell = i+1,\ldots,i+p$ and $g_{m,m}^{i+p+1} \neq 0$. Further, assume $p < m-1$.

Then the $im + p + 1$ unbounded root loci for this system are of $0(k^{m/im+p+1})$ with asymptotic values given by $(-g_{m,m}^{(i+p+1)})^{\frac{1}{im+p+1}}$.

Proof: Follows from the result in numerical analysis (see [9]) that assymptotically the eigenvalues of

$$\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} + B(\varepsilon) \text{ where } B(\varepsilon) \text{ is a mxm matrix}$$

of $0(\varepsilon)$ with $\lim_{\varepsilon \to 0} \frac{b_{mm}(\varepsilon)}{\varepsilon} \neq 0$ are of $0(\varepsilon^{1/m})$ with asymptotic values given by $\varepsilon^{1/m} \left[\lim_{\varepsilon \to 0} \frac{b_{mm}(\varepsilon)}{\varepsilon}\right]^{1/m}$. The details are messy and are omitted. □

Section 4. Robust computation of asymptotic values

For robust computation we use orthogonal projections and the singular value decomposition (see for e.g. [17]), which we restate. The reader is assumed to be familiar with the many desirable properties of the singular value decomposition (s.v.d) in numerical computation.

Proposition 4.1 (The Singular Value Decomposition) [17]

A matrix $A \in \mathbb{C}^{mxm}$ of rank r may be decomposed as

$$A = [U_1 \,\vdots\, U_2] \left[\begin{array}{c:c} \Sigma_1 & 0 \\ \hdashline 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^* \\ \hdashline V_2^* \end{array}\right] \tag{4.2}$$

where $U = [U_1 \,\vdots\, U_2] \in \mathbb{C}^{mxm}$ with $U_1 \in \mathbb{C}^{mxr}$; $U_2 \in \mathbb{C}^{mx(m-r)}$

and $V = [V_1 \,\vdots\, V_2] \in \mathbb{C}^{mxm}$ with $V_1 \in \mathbb{C}^{mxr}$; $V_2 \in \mathbb{C}^{mx(m-r)}$

are unitary matrices and $\Sigma_1 \in \mathbb{R}_+^{rxr}$ is a diagonal matrix of positive real numbers.

Comment: The columns of V_1, U_1 represent orthogonal bases for the range spaces of A^*, A respectively. The columns of V_2, U_2 represent orthogonal bases for the null spaces of A, A^* respectively.

Notation: Denote the S.V.D. of $G_1 \in \mathbb{R}^{mxm}$ by

$$G_1 = [U_1^1 \,\vdots\, U_2^1] \left[\begin{array}{c:c} \Sigma_1^1 & 0 \\ \hdashline 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^{1*} \\ \hdashline V_2^{1*} \end{array}\right] \tag{4.2}$$

where $\Sigma_1^1 \in \mathbb{R}_+^{m_1 x m_1}$ and the other matrices are real and of conformal dimensions. Further let the S.V.D. of $U_2^{1*} G_2 V_2^1 \in \mathbb{R}^{(m-m_1) \times (m-m_1)}$ be given by

$$U_2^{1*} G_2 V_2^1 = [U_1^2 \,\vdots\, U_2^2] \left[\begin{array}{c:c} \Sigma_1^2 & 0 \\ \hdashline 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^{2*} \\ \hdashline V_2^{2*} \end{array}\right] \tag{4.3}$$

where $\Sigma_1^2 \in \mathbb{R}_+^{m_2 x m_2}$

denote the S.V.D. of $U_2^{2*} U_2^{1*}(G_3 - G_2 G_1^{\dagger} G_2) V_2^1 V_2^2 \in \mathbb{R}^{(m-m_1-m_2)\times(m-m_1-m_2)}$ by

$$U_2^{2*} U_2^{1*}(G_3 - G_2 G_1^{\dagger} G_2) V_2^1 V_2^2 = [U_1^3 \,\vdots\, U_2^3] \left[\begin{array}{c|c} \Sigma_1^3 & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^{3*} \\ \hline V_2^{3*} \end{array}\right] \tag{4.4}$$

choosing for $G_1^{\dagger}$, the Moore-Penrose inverse,

$$[V_1^1 \quad V_2^1] \left[\begin{array}{c|c} (\Sigma_1^1)^{-1} & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} U_1^{1*} \\ \hline U_2^{1*} \end{array}\right] .$$

Then, using (2.9) to compute the asymptotes of the integral order asymptotic root loci computed in Section 3.1 we obtain

<u>Theorem 4.2</u> (Polynomial equations for the asymptotes of integral order unbounded root loci).

(i) $\mu_1 = (-\lambda) \neq 0 \in \mathbb{C}$ is the coefficient of the asymtotic value of a 1st order unbounded root locus iff

$$\det\,(-G_1 + \lambda I) = 0 \tag{4.5}$$

(ii) $\mu_2 = (-\lambda)^{1/2} \neq 0 \in \mathbb{C}$ is the coefficient of the asymptotic value of a 2nd order unbounded root locus iff

$$\det\,(\lambda U_2^{1*} V_2^1 - U_2^{1*} G_2 V_2^1) = 0 \tag{4.6}$$

(iii) $\mu_3 = (-\lambda)^{1/3} \neq 0 \in \mathbb{C}$ is the coefficient of the asymptotic value of a 3rd order unbounded root locus iff

$$\det\,(\lambda U_2^{2*} U_2^{1*} V_2^1 V_2^2 - U_2^{2*} U_2^{1*}(G_3 - G_2 G_1^{\dagger} G_2) V_2^1 V_2^2) \neq 0, \tag{4.7}$$

and so on.

Comments: (i) Note that equations (4.5), (4.6), (4.7) are solved by setting up generalized eigenvalue problems of dimension m, $m-m_1$, $m-m_1-m_2$ respectively. They can in fact be set up as ordinary eigenvalue problems, since by Assumption 2, $U_2^{1*}V_2^1$ and $U_2^{2*}U_2^{1*}\ V_2^1\ V_2^2$ are invertible.

(ii) Some geometrical insight into the computation procedures is obtained by expressing the solutions of (4.5), (4.6), (4.7) as the non-zero eigenvalues of

$$G_1$$

$$G_2\big|_{\eta(G_1)\to\eta(G_1^*)} =: \hat{G}_2 \quad ;$$

$$G_3 - G_2\, G_1^{\dagger} G_2\big|_{\eta(\hat{G}_2)\to\eta(\hat{G}_2^*)} =: \hat{G}_3 \quad ; \qquad \text{respectively.}$$

Theorem (4.2) then identifies orthogonal subspaces of the input space ($\mathcal{R}^m$) and the output space ($\mathcal{R}^m$) for the computation of the integral order unbounded root loci. $\mathcal{R}(V_1^1)$, $\mathcal{R}(V_2^1\ V_1^2)$, $\mathcal{R}(V_2^1\ V_2^2\ V_1^3)$, ... are subspaces of the input space and $\mathcal{R}(U_1^1)$, $\mathcal{R}(U_2^1\ U_1^2)$, $\mathcal{R}(U_2^1\ U_2^2\ U_1^3)$, are subspaces of the output space associated with the 1st, 2nd, 3rd,... order unbounded root loci respectively.

Section 5: Pivots for the asymptotic root loci

For the integral order root loci the asymptotic series have the form given by (see [8])

$$s_{i,n} = (-k\lambda_i)^{1/n} + c_i + d_i k^{-1/n} + \dots \qquad n = 1,2,3,\dots n_o \tag{5.1}$$

with $\lambda_1 \neq 0$, c_i $d_i \in \mathbb{C}$. By the pivot of the asymptotic root locus is meant the coefficient of the $0(1)$ term of the asymptotic expansion of (5.1) i.e. c_i. Each cycle of the multivariable root locus at ∞ has the same pivot. To make the calculation we need

<u>Assumption 3</u> (Simple Structure)

Assume that $\hat{G}_1, \hat{G}_2, \hat{G}_3, \ldots, \hat{G}_{n_0}$ have simple structure associated with each of their eigenvalues.

<u>Theorem 5.1</u> (Expression for the pivots)

Under assumptions 1,2,3 the nth order asymptotic unbounded root loci for the system of Figure 1 have the form (5.1) with c_i given by the solution of

$$\det \left[\begin{array}{c|c|c|c|c} G_{n+1} - c_i G_n & G_n - \lambda_i I & & & G_1 \\ \hline G_n - \lambda_i I & G_{n-1} & & G_1 & \\ \hline & & & & \\ \hline G_2 & G_1 & & 0 & 0 \\ \hline G_1 & 0 & & 0 & 0 \end{array}\right] = 0. \qquad (5.2)$$

<u>Proof:</u> Using the same technique as in the proof of Theorem 3.2 we get,

with $\frac{k}{s^n} = \frac{1}{\mu^n}\left(1 - \frac{nc}{\mu k^{1/n}} + O(k^{-2/n})\right)$

$$\left[I + \frac{k^{\frac{n-1}{n}}}{\mu} G_1 + \frac{k^{\frac{n-2}{n}}}{\mu^2} G_2 + \ldots + \frac{G_n}{\mu^n} - \frac{nG_n c}{\mu^{n+1} k^{1/n}} + \frac{G_{n+1}}{k^{1/n} \mu^{n+1}} + \ldots\right]$$

$$\left[e_1 + \frac{e_2}{k^{1/n}} + \ldots \frac{e_n}{k^{\frac{n-1}{n}}} + e_{n+1} + \ldots\right] = 0. \qquad (5.3)$$

Equating terms of $O(k^{\frac{n-1}{n}}), \ldots, O(1), O(k^{\frac{-1}{n}})$, one obtains equation (5.2) with $-\mu^n = \lambda_{i,n}$. □

<u>Comments</u>: (i) For each of the nth root of $\lambda_{i,n}$ the same value of c_i occurs from equation (5.2), justifying the term "pivot of the n- cycle for c_i.

(ii) Equation (5.2)is triangular block Toeplitz and so admits of simplification. We take this up next.

5.1 Formulae for the pivots of the unbounded root loci

5.1.1 First Order: $s = -\lambda_i k + c_i + 0(k^{-1})$.

c_i are the solutions of

$$\det \left[\begin{array}{c|c} G_2 - c_i\, G_1 & G_1 - \lambda_i I \\ \hline G_1 - \lambda_i I & 0 \end{array}\right] = 0 \tag{5.4}$$

where λ_i is a non-zero eigenvalue of G_1. This may be rewritten after row operations as

$$\det \left[\begin{array}{c|c} G_2 - c_i\, \lambda_i I & G_1 - \lambda_i I \\ \hline G_1 - \lambda_i I & 0 \end{array}\right] = 0$$

so that by Proposition 2.3, $c_i \lambda_i$ is an eigenvalue of

$$G_2 \bmod R(G_1 - \lambda_i I)\Big|_{\eta(G_1 - \lambda_i I)} \tag{5.5}$$

By Assumption 3, in analogy to Proposition (3.9) we have

$$R(G_1 - \lambda_i I) + \eta(G_1 - \lambda_i I) = \mathbb{R}^m$$

so that there are as many eigenvalues to the operator in (5.5) as the dimension of $\eta(G_1 - \lambda_i I)$.

5.1.2 Second Order: $s = \sqrt{-\lambda_i k} + c_i + 0(k^{-1/2})$.

c_i are the solutions of

$$\det\left[\begin{array}{c:c:c} G_3 - 2c_iG_2 & G_2 - \lambda_iI & G_1 \\ \hdashline G_2 - \lambda_iI & G_1 & 0 \\ \hdashline G_1 & 0 & 0 \end{array}\right] = 0. \tag{5.6}$$

<u>Proposition 5.1</u> (Formula for second order pivots)

The second order pivots c_i corresponding to λ_i of (5.1), the coefficient of a second order unbounded root locus are $\frac{1}{2\lambda_i}$ times the eigenvalues of

$$(G_3 - (G_2 - \lambda_iI)\, G_1^{\dagger}\, (G_2 - \lambda_iI)) \bmod(R(\hat{G}_2 - \lambda_i\hat{I}))\ \bmod R(G_1)\Big|_{\eta(\hat{G}_2 - \lambda_i\hat{I})} \tag{5.7}$$

where $(\hat{G}_2 - \lambda_i\hat{I}) := G_2 - \lambda I \bmod R(G_1)\Big|_{\eta(G_1)}$.

<u>Proof:</u> (5.6) may be rewritten as $:\exists\ v_1, v_2, v_3 \in \mathbb{R}^m$ not all zero such that

$$\left[\begin{array}{c:c:c} G_3 - 2c_i\lambda_iI - 2c_i(G_2 - \lambda_iI) & G_2 - \lambda_iI & G_1 \\ \hdashline G_2 - \lambda_iI & G_1 & 0 \\ \hdashline G_1 & 0 & 0 \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \tag{5.8}$$

Adding $2c_i$ times the second row of (5.8) to the first row of (5.8) and then subtracting $2c_i$ times the third column of (5.8) from the second column of (5.8) we have

$$\det\left[\begin{array}{c:c:c} G_3 - 2c_i\lambda_iI & G_2 - \lambda_iI & G_1 \\ \hdashline G_2 - \lambda_iI & G_1 & 0 \\ \hdashline G_1 & 0 & 0 \end{array}\right] = 0 \tag{5.9}$$

(5.7) now follows readily from (5.9). □

Comment: As before by Assumption 3, there are as many pivots as there are λ_i's for the second order unbounded root loci.

5.1.3. Third order $s = (-\lambda_i k)^{1/3} + c_i + 0(k^{-1/3})$

c_i are the solutions of

$$\det \left[\begin{array}{c|c|c|c} G_4 - 3c_i G_3 & G_3 - \lambda_i I & G_2 & G_1 \\ \hline G_3 - \lambda_i I & G_2 & G_1 & 0_1 \\ \hline G_2 & G_1 & 0 & 0 \\ \hline G_1 & 0 & 0 & 0 \end{array}\right] = 0 \tag{5.10}$$

Proposition 5.2 (Formula for third order pivots)

The third order pivots c_i corresponding to λ_i of (5.1), the coefficient of a third order unbounded root locus are $\frac{1}{3\lambda_i}$ times the eigenvalues of

$$G_4 - (G_3 - \lambda_i I)\, G_1^\dagger\, G_2 - G_2\, G_1^\dagger (G_3 - \lambda_i I) + G_2 G_1^\dagger G_2 G_1^\dagger G_2$$

$$\text{mod}R(G_1)\ \text{mod}R(\hat{G}_2)\ \text{mod}\ R(\hat{G}_3-\lambda_i\hat{I})\Big|_{\eta\,(\hat{G}_3 - \lambda_i\hat{I})} \tag{5.11}$$

where $\hat{G}_3 - \lambda_i\hat{I} := G_3 - \lambda I\ \text{mod}R(G_1)\text{mod}R(\hat{G}_2)\Big|_{\eta(\hat{G}_2)}$.

Proof: Exactly as in Proposition 5.1. □

5.1.4 Higher order pivots $s = (-\lambda_i k)^{1/n} + c_i + 0(k^{-1/n})$

The extension of the foregoing procedure to higher order pivots is exactly as in Section 3.1 and is omitted.

5.2 Computation of the Pivots

The machinery introduced in Section 4 can be used to set up a procedure for the computation of the pivots involving essentially multiplying G_2, $G_3 - (G_2 - \lambda_i I)G_1^{\dagger}(G_2 - \lambda_i I)$, ... by U_2^{1*} , $U_2^{2*}\, U_2^{1*}$, ... on the left and V_2^1, $V_2^1\, V_2^2$, ... on the right. The details are omitted.

Section 6. Concluding Remarks

The above calculations of the asymptotes of the unbounded root loci may be applied to state a necessary and sufficient condition for the closed loop exponential stability of a strictly proper linear time-invariant system under arbitrarily high gain feedback $k \geq k_o$ as follows:

Theorem 6.1 (High gain stability)

If the strictly proper, linear time-invariant plant G(s) satisfies Assumptions 1,2,3; then the closed loop system of Figure 1 is exponentially stable for all $k \geq k_o$ with all closed loop eigenvalues uniformly (for $k \in [k_o,\infty]$) bounded away from the $j\omega$-axis for $k \geq k_o$, iff

(i) the McMillan zeros of G(s) are in the $\mathring{\mathbb{C}}_-$.

(ii) the non zero eigenvalues of G_1 are in $\mathring{\mathbb{C}}_+$

(iii) the eigenvalues of $G_2(\text{mod}\mathcal{R}(G_1))\Big|_{\eta(G_1)}$ are <u>real</u> and <u>positive</u>

(iv) the pivots associated with each eigenvalue of $G_2(\text{mod}\mathcal{R}(G_1))\Big|_{\eta(G_1)}$ have negative real part

(v) $\mathbb{R}^m = \mathcal{R}(G_1) + \mathcal{R}(G_2\Big|_{\eta(G_1)})$.

<u>Comments:</u> (i) Condition (v) of the theorem guarantees that only first and second order unbounded root loci exist.

(ii) Theorem 6.1 is the generalization to multi input - multi output of a well known theorem for single-input, single output systems (see [13]).

(iii) The proof is straight-forward from the preceding calculations.

The results of this paper are easily generalized to the case of proper rather than strictly proper plants. The Taylor series about $s = \infty$ then is

$$G(s) = G_0 + G_1 \frac{1}{s} + \dots .$$

and the calculations would begin with restriction in domain to $\eta(G_o)$ and $\mathrm{mod}\,R(G_o)$ in the range.

The generalization to the case of proper irrational transfer functions analytic outside a compact disc is also immediate. Note, however, that there is no counterpart of the Smith McMillan theory of Section 3.2.

We have not investigated in our set-up the specialization of our computations to asymptotic Linear Quadratic regulators (see for e.g. [14], [15], [16]).

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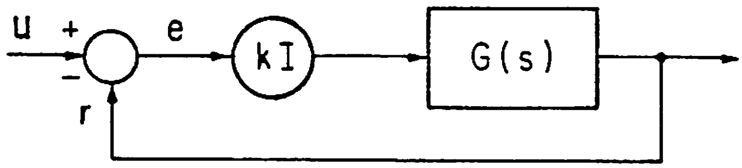

Figure 1. System configuration.