THE DISCRETE FOURIER TRANSFORM VIA CIRCULANTS

by

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Introduction

The discrete Fourier Transform (DFT), defined below in (2-1), is a valuable tool in many fields from signal processing to partial differential equations. There is strong incentive for computing the transform quickly, and, after two decades of active research, we now know the minimum number of essential multiplications required for the task and have algorithms which use precisely this number. It does not follow that, in the end, these will be the most desirable techniques but they are certainly of interest in their own right. Major credit for this fine work seems to belong to I.J. Good, R.M. Rader, and S. Winograd. This story, and more, is told in the book [5].

At the heart of these methods lie Winograd's algorithms for n point DFT's, where n is a small prime. He used a quick way of forming the product of two polynomials modulo a third one and that theory, in turn, uses the Chinese Remainder Theorem and some abstract algebra. Section 5 exhibits the correspondence between his approach and ours.

This communication points out that these optimal algorithms follow directly from the spectral factorization of certain matrices called circulants. Specifically the real and imaginary parts of the eigenvectors of these circulants take the place of polynomial representations in Winograd's theory. Such an approach is not as strange as it may seem at first glance; if an arbitrary column vector must be multiplied by a matrix C using few multiplications then it is natural to see whether C can be written as $C = X D Y$ where the elements of X and Y are small integers and D is diagonal. All normal matrices can be factored into the canonical form XDX^T and, for some small circulants, the eigenvectors X of C are very simple. Another connection is that I.J. Good, in [3b], showed how to use the DFT to invert large circulants. We are, in some sense, inverting that work.

The canonical factorization of circulants is not as powerful as the number theoretical approach of Winograd. For example, the factorization does not yield the fast algorithm for DFT(11) nor the proofs of minimality. On the other hand it does yield the algorithms which are in use and it does so in a rather simple way. Some people may enjoy a low brow derivation of these valuable schemes.

A recent survey article [2] linked the DFT with an impressive variety of topics in pure and applied mathematics and yet no mention was made of the circulants lurking in the background.

We barely mention the well known FFT and the reader is referred to [5] to see its connection to the algorithms discussed here.

2. The DFT and Cyclic Convolution

The material in this section is standard but must be included if only to establish the notation.

The vector or sequence $\{x_0,\ldots,x_{n-1}\}$ to be transformed may be thought of as data (complex values) given at n equally spaced points. The transformed sequence $\{\hat{x}_0,\ldots,\hat{x}_{n-1}\}$ is defined by

$$\hat{x}_k = \sum_{j=0}^{n-1} \omega^{kj} x_j, \quad k = 0,\ldots,n-1, \tag{2-1}$$

where

$$\omega = \omega_n = \exp(2\pi \sqrt{-1}/n),$$

is the primitive nth root of unity. Different professional groups give somewhat different definitions of DFT but the variations are minor. For

example, signal engineers call (2-1) the inverse DFT. In order to suppress indices we can use matrix notation. Let $\underset{\sim}{x}$ and $\underset{\sim}{\hat{x}}$ denote the column vectors associated with the two sequences and let F denote the Fourier Transform matrix,

$$F(\omega) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{n-2} \\ . & . & . & \dots & . \\ 1 & \omega^{n-1} & . & \dots & \omega \end{bmatrix} = \left[\omega^{ij}\right] . \tag{2-2}$$

Note that the order of F is implied by ω. Note also that we have used the relation $\omega^n = 1$ to reduce all the exponents in F below n. The definition (2-1), commonly called DFT(n), now becomes

$$\underset{\sim}{\hat{x}} = F(\omega_n)\underset{\sim}{x}. \tag{2-3}$$

F is a very special Vandermonde matrix and a well known property is

$$F^4(\omega_n) = n^2 I . \tag{2-4}$$

The letter I denotes the identity matrix.

The inverse DFT is simple

$$\underset{\sim}{x} = \frac{1}{n} F(\omega_n^{-1}) \underset{\sim}{\hat{x}} , \tag{2-5}$$

The eigenvectors of F are too complicated, see [2], to yield a fast method for computing the DFT so we turn to other arrangements.

It is customary to remove the trivial part of the computation of (2-3) as follows. Since

$$\hat{x}_0 = x_0 + x_1 + \ldots + x_{n-1}$$

the essential computation is

$$\hat{x}_k - x_0 = \sum_{j=1}^{n-1} \omega^{kj} x_j , \quad k = 1,\ldots,n-1, \tag{2-6}$$

or, equivalently,

$$\hat{x}_k - \hat{x}_0 = \sum_{j=1}^{n-1} (\omega^{kj} - 1) x_j , \quad k = 1,\ldots,n-1 \tag{2-7}.$$

The remarkable fact is that the $\hat{x}_k$, $k=1,\ldots,n-1$, can be computed using between n and $2n$ multiplications instead of $(n-1)^2$ as suggested by (2-6) or (2-7).

In 1968 C.M. Rader pointed out in [6] that when n is prime the variables can be reordered so that (2-6) becomes a cyclic convolution. Let us illustrate this for $n = 5$, in which case it is only necessary to exchange the last two variables.

$$\begin{pmatrix} \hat{x}_1 - x_0 \\ \hat{x}_2 - x_0 \\ \hat{x}_4 - x_0 \\ \hat{x}_3 - x_0 \end{pmatrix} = \begin{bmatrix} \omega & \omega^2 & \omega^4 & \omega^3 \\ \omega^2 & \omega^4 & \omega^3 & \omega \\ \omega^4 & \omega^3 & \omega & \omega^2 \\ \omega^3 & \omega & \omega^2 & \omega^4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \\ x_3 \end{pmatrix} \qquad (2\text{-}8)$$

Note that each row of the matrix is obtained by shifting left the row above in a cyclic fashion. The new ordering is monotonic when $\{1,2,3,4\}$ is seen as a multiplicative group modulo 5:

$$\begin{array}{cccc} 2^0 & 2^1 & 2^2 & 2^3 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 4 & 3 \end{array} \qquad (\text{mod } 5)$$

Rader's observation reduces the DFT computation to that of cyclic convolution. There are several clever tricks available for doing this quickly but it was Winograd's achievement to determine the minimal number of multiplications that are needed and to exhibit the algorithms which achieve the minimum. The minimum depends quite strongly on n. It also depends on the algebraic field in which the multiplication is taken to act but we will not focus on this aspect of the algorithms which is treated fully in [5].

Finally we remark that the case when n is prime is the important one as explained in Section 4.

3. Circulant Multiplication.

We take n to be prime and rewrite Rader's observation slightly. Instead of taking both the $\{x_i\}$ and the $\{\hat{x}_k\}$ in the same new ordering we reverse it for the $\{x_j\}$. When $n = 5$ this yields

$$\begin{pmatrix} \hat{x}_1 - x_0 \\ \hat{x}_2 - x_0 \\ \hat{x}_4 - x_0 \\ \hat{x}_3 - x_0 \end{pmatrix} = \begin{bmatrix} \omega & \omega^3 & \omega^4 & \omega^2 \\ \omega^2 & \omega & \omega^3 & \omega^4 \\ \omega^4 & \omega^2 & \omega & \omega^3 \\ \omega^3 & \omega^4 & \omega^2 & \omega \end{bmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_4 \\ x_2 \end{pmatrix} \tag{3-1}$$

Of course the only difference from (2-8) is that each row of the matrix in (3-1) is obtained from the row above by shifting <u>right</u> instead of left. The advantage is that the matrix in (3-1) is a <u>circulant</u> matrix and a great deal is known about them. See [1] and [4].

For all prime n Rader's observation reduces DFT(n) to multiplying an arbitrary vector by a special circulant matrix of order $n-1$. We now list the standard facts about circulant matrices.

FACT 1. Every $m \times m$ circulant is a polynomial in the full cycle (or shift) matrix

$$P = P_m = \begin{bmatrix} 0 & 0 & . & . & 1 \\ 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & 0 & 1 & 0 \end{bmatrix} . \tag{3-2}$$

The coefficients of the polynomial are given by column 1 of the circulant. The degree is less than m.

There is nothing to prove here but we will illustrate the property using the matrix in (3-1) which can be written as

$$\phi(P_4) \equiv \omega I + \omega^2 P_4 + \omega^4 P_4{}^2 + \omega^3 P_4{}^3. \tag{3-3}$$

Note that $I = P_4{}^0 = P_4{}^4$ and, in general, note the special pattern of the 1's in the powers of P_m.

FACT 2. P_m is a permutation matrix. Its eigenvalues are the roots of unity and its eigenvectors are mutually orthogonal. Moreover P_m is real and so it has a <u>real</u> canonical form $\Lambda = \Lambda_m$ which is a direct sum of powers of

$$R_m = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta = 2\pi/m,$$

together with 1 and, if m is even, -1.

Corresponding to each complex eigenvalue are the real and imaginary parts of the eigenvector. These two real vectors span the associated invariant plane and, of more importance to us, these two vectors are columns of an orthogonal (real) matrix which reduces P_m to Λ_m. In symbols,

$$P_m = S_m \Lambda_m S_m{}^T, \quad S^T = S^{-1}. \tag{3-4}$$

Here S^T denotes the transpose of S. We call (3-4) the <u>real spectral factorization</u> of P_m and show it for small values of m in Table 1.

[Table 1 could go near here.]

Eigenvectors are only defined up to a constant nonzero factor. Hence there is no loss in generality in writing

$$S = G\Delta \tag{3-5}$$

where Δ is diagonal and positive definite and may be chosen at our convenience.

FACT 3. Let $\phi(P_m)$ be any $m \times m$ circulant and let $P_m = (G\Delta) \wedge (G\Delta)^T$ be the real spectral factorization of the cyclic shift matrix P_m. Then

$$\phi(P_m) = G\Delta\phi(\wedge)(G\Delta)^T \tag{3-6}$$

is the associated spectral factorization of $\phi(P_m)$. In other words, the eigenvectors of P_m are eigenvectors of $\phi(P_m)$ for any polynomial ϕ.

Whenever G's elements are small integers then (3-6) provides a minimal multiplication algorithm for forming the product $\phi(P_m)\underset{\sim}{x}$. In symbols,

$$\begin{aligned}\phi(P_m)\underset{\sim}{x} &= G\Delta\phi(P_m)\Delta G^T\underset{\sim}{x} \\ &= G(D(G^T\underset{\sim}{x})),\end{aligned} \tag{3-7}$$

where

$$D = \Delta\phi(P_m)\Delta$$

is block diagonal with 2×2 and 1×1 blocks. Only the application of

D involves genuine multiplications, G and G^T act via additions and subtractions.

[Table 2 could go near here.]

The only way that the DFT affects the circulant product (3-7) is through the complex polynomial ϕ. In Table 2 we list ϕ and $\phi(R_m^k)$ for several small values of m. Recall from Fact 2 that R_m is the matrix representing rotation through an angle $2\pi/m$. It turns out that $\phi(R^k)$ is also of the form $\begin{pmatrix} \beta & -\gamma \\ \gamma & \beta \end{pmatrix}$ and, as a bonus, β and γ are either both real or both pure imaginary. Multiplication of a real vector by such a matrix requires 3 real multiplications and 3 additions. This holds even when β and γ are matrices. One implementation follows from the matrix identity

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix} = \begin{pmatrix} I & 0 & I \\ I & I & 0 \end{pmatrix} \begin{pmatrix} C & 0 & 0 \\ 0 & B+C & 0 \\ 0 & 0 & B-C \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \\ I & 0 \end{pmatrix} \tag{3-8}$$

In general (3-8) is preferable to

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix} = \begin{pmatrix} I & I \\ -iI & iI \end{pmatrix} \begin{pmatrix} \tfrac{1}{2}(B+iC) & 0 \\ 0 & \tfrac{1}{2}(B-iC) \end{pmatrix} \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix} \tag{3-9}$$

Another useful identity is

$$\begin{pmatrix} B & C \\ C & B \end{pmatrix} = \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} \tfrac{1}{2}(B+C) & 0 \\ 0 & \tfrac{1}{2}(B-C) \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \tag{3-10}$$

which is just a block version of (3-7) when $m = 2$. We shall use (3-8) and (3-10) later.

For completeness we give the nonzero elements of D in Table 3. These numbers are the values, at the (n-1)st roots of unity, of a polynomial of which the coefficients are nth roots of unity.

[Table 3 could go near here.]

Our interest is in the number of real multiplications required to compute DFT(n). For reasons that appear in the next section we make the peculiar definition

$$\text{mult}(n) \equiv 1 + \text{no. of multiplications required for DFT}(n).$$

If n is prime and G's elements are small integers then

$$\text{mult}(n) = 3(n-1)/2 = 2n - 1 - (n-1)/2$$

when $\underset{\sim}{x}$ is real. For complex data the counts are only doubled. On binary computers the count can be reduced by 1 when $n = 3,5$ because multiplication by $\frac{1}{2}$ or $\frac{1}{4}$ can be accomplished by a shift.

It is clear that DFT(2) and DFT(4) require no multiplications. However in dealing with larger values of n we must define

$$\text{mult}(2) = 2, \quad \text{mult}(4) = 4.$$

More precisely

$$\text{mult}(n) = \max\{n, 1 + \text{no. of multiplications required for DFT}(n)\}.$$

In order to obtain a systematic development of the G matrices for larger values of m it is worth noting that the factorization developed in this section extends immediately to matrices of the form

$$\begin{bmatrix} z_1 & -z_4 & -z_3 & -z_2 \\ z_2 & z_1 & -z_4 & -z_3 \\ z_3 & z_2 & z_1 & -z_4 \\ z_4 & z_3 & z_2 & z_1 \end{bmatrix}$$

Such matrices are polynomials in the orthogonal matrix

$$\tilde{P}_4 \equiv \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

The only difference is that the eigenvalues of $\tilde{P}_m$ are in the roots of -1 (instead of +1). The factorization (3-10) reduces a circulant of order 2m to a direct sum of a circulant of order m and one of these "improper" circulants of order m.

In order for G to have small integer elements it is necessary that x^m-1 have no irreducible factors (over the rationals) of degree exceeding 2.

4. The DFT Factorization

In order to deal nicely with DFT(n) for large n it is convenient to recast the results of the previous section as a factorization of the DFT matrix itself. The task is to compute $F(\omega_n)\underset{\sim}{x} = \underset{\sim}{\hat{x}}$.

The first step of eliminating x_0 and $\hat{x}_0$ can be seen as performing one step of triangular factorization of $F(\omega_n)$. This leads to (2-7) rather than (2-6). For example,

$$F(\omega_5) = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 1 & & & & \\ 1 & & I_4 & & \\ 1 & & & & \\ 1 & & & & \end{array}\right] \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & (\omega^{\mu\nu}-1) & & \\ 0 & & & & \\ 0 & & \mu,\nu=1,\ldots,4 & & \end{array}\right] \left[\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 1 \\ \hline 0 & & & & \\ 0 & & I_4 & & \\ 0 & & & & \\ 0 & & & & \end{array}\right]$$

Section 3 showed that, when n is prime, there are permutation matrices Π_1 and Π_2 such that

$$\Pi_1 F(\omega_n)\Pi_2 = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 1 & & & & \\ 1 & & G & & \\ 1 & & & & \\ 1 & & & & \end{array}\right] \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & D & & \\ 0 & & & & \\ 0 & & & & \end{array}\right] \left[\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 1 \\ \hline 0 & & & & \\ 0 & & G^T & & \\ 0 & & & & \\ 0 & & & & \end{array}\right] \qquad (4\text{-}1)$$

When G's elements are small integers this leads to an algorithm with minimal number of multiplications. The D matrices exhibited in Section 3 were based on (2-6) but the only modification needed to conform to (2-7) is to change the top element of D from $-1/(n-1)$ to $-n/(n-1)$. Verification of this assertion is left to the reader.

With a slight abuse of notation we will write (4-1) as

$$\Pi_1 F(\omega_n)\Pi_2 = GDG^T, \qquad (4\text{-}2)$$

or

$$F(\omega_n) \sim GDG^T, \tag{4-3}$$

and we call this the DFT factorization of $F(\omega_n)$. The context will make it clear whether $F(\omega_n)$ or a circulant matrix of order $n-1$ is being factored.

Note that $\sim$ (equivalence under permutations) is a true equivalence relation.

The important observation of I.J. Good in [3a] may be summarized as

> Theorem. If ℓ and m are relatively prime then
>
> $$F(\omega_{\ell m}) \sim F(\omega_\ell) \otimes F(\omega_m) \tag{4-4}$$

Here $\otimes$ denotes the direct (or Kronecker or tensor) product of matrices, namely

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdot \\ a_{21}B & a_{22}B & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} .$$

Example: $F(w_6) \sim F(w_2) \otimes F(w_3)$

$$\begin{pmatrix} \hat{x}_0 \\ \hat{x}_4 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_1 \\ x_5 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & \omega^2 & \omega^4 & -1 & \omega^5 & \omega \\ 1 & \omega^4 & \omega^2 & -1 & \omega & \omega^5 \end{bmatrix} \begin{pmatrix} x_0 \\ x_2 \\ x_4 \\ x_3 \\ x_5 \\ x_1 \end{pmatrix} , \quad \omega = \omega_6,$$

$$= [F(\omega_2) \otimes F(\omega_3)] \begin{pmatrix} \underset{\sim}{x}_+ \\ \underset{\sim}{x}_- \end{pmatrix} \quad ; \quad \underset{\sim}{x}_+ = \begin{pmatrix} x_0 \\ x_2 \\ x_4 \end{pmatrix} , \quad \underset{\sim}{x}_- = \begin{pmatrix} x_3 \\ x_5 \\ x_1 \end{pmatrix}$$

Algorithm: $\hat{\underset{\sim}{x}}_+ = F(\omega_3)\, \underset{\sim}{x}_+ ,$

$\hat{\underset{\sim}{x}}_- = F(\omega_3)\, \underset{\sim}{x}_- ,$

$$\begin{pmatrix} \hat{x}_0 \\ \hat{x}_4 \\ \hat{x}_2 \end{pmatrix} = \hat{\underset{\sim}{x}}_+ + \hat{\underset{\sim}{x}}_- \quad ; \quad \begin{pmatrix} \hat{x}_3 \\ \hat{x}_1 \\ \hat{x}_5 \end{pmatrix} = \hat{\underset{\sim}{x}}_+ - \hat{\underset{\sim}{x}}_- .$$

In order to emphasize the theorem's hypothesis we point out that

$F(\omega_9) \neq F(\omega_3) \otimes F(\omega_3)$.

Thus a 2 dimensional DFT on 3×3 points is not equivalent to a 1 dimensional DFT on 9 points.

There is a routine procedure for changing the DFT factorization (4-3) into an enlarged form $G_2 \hat{D} G_1$ where $\hat{D}$ is diagonal (not just block diagonal) and has an order not less than D, namely mult(n). It uses the identity (3-8) to replace $\begin{pmatrix} \beta & -\gamma \\ \gamma & \beta \end{pmatrix}$ by $\mathrm{diag}(\gamma, \beta+\gamma, \beta-\gamma)$. Note the G_2 is not the transpose of G_1. We illustrate the procedure when $n = 5$.

Figure 1. Transition to strictly diagonal form.

$$\underset{G}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 & -1 \end{pmatrix}} \underset{D}{\begin{pmatrix} 1 & & & & \\ & -5/4 & & & \\ & & \alpha & & \\ & & & \beta & -\gamma \\ & & & \gamma & \beta \end{pmatrix}} \underset{G^T}{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}}$$

$$= \underset{G_2}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & -1 & -1 & 0 \end{pmatrix}} \underset{\hat{D}}{\begin{pmatrix} 1 & & & & & \\ & -5/4 & & & & \\ & & \alpha & & & \\ & & & \gamma & & \\ & & & & \beta+\gamma & \\ & & & & & \beta-\gamma \end{pmatrix}} \underset{G_1}{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}}$$

The implication of Good's theorem is clear. Given fast algorithms for DFT(ℓ) and DFT(m), i.e.

$$F(\omega_\ell) \sim G_\ell D_\ell G_\ell{}^T$$
$$F(\omega_m) \sim G_m D_m G_m{}^T$$

Then, provided ℓ and m have no common factors,

$$\begin{aligned} F(\omega_{\ell m}) &\sim F(\omega_\ell) \otimes F(\omega_m), \\ &\sim (G_\ell \otimes G_m)\ (D_\ell \otimes D_m)\ (G_\ell \otimes G_m)^T, \end{aligned} \qquad (4\text{-}5)$$

yields a fast algorithm for DFT(ℓm). Formula (4-5) invokes a valuable property of direct products, see [4].

The number of multiplications required is best seen from the full representation

$$F(\omega_{\ell m}) \sim (G_\ell^{(2)} \otimes G_m^{(2)})(\hat{D}_\ell \otimes \hat{D}_m)(G_m^{(1)} \otimes G_\ell^{(1)}) \qquad (4\text{-}6)$$

Since $\hat{D}_\ell \otimes \hat{D}_m$ is diagonal of order $\text{mult}(\ell)\cdot\text{mult}(m)$ we have

$$\text{mult}(\ell m) = \text{mult}(\ell)\cdot\text{mult}(m). \qquad (4\text{-}7)$$

For example, $\text{mult}(210) = \text{mult}(2\cdot 3\cdot 5\cdot 7) = 324$.

We say that (4-5), or (4-6), is the DFT factorization of $F(\omega_{\ell m})$.

In order to get a fast DFT algorithm for <u>any</u> natural number n it is necessary to exhibit a DFT factorization for prime powers p^r. Unfortunately this becomes very messy as r increases. Circulants do appear but they have to be combined with subblocks which involve DFT(p^k) for $k < r$. We content ourselves with exhibiting the DFT factorization for $n = 9$ and $n = 8$ in Table 4.

[Table 4 could go near here.]

One consequence of this messiness and our ignorance of $\text{mult}(p^r)$ is that the favorite composite numbers for the DFT are those with a variety of prime factors. This is in stark contrast to the FFT of Cooley and Tukey which favors $n = 2^k$.

This concludes our elementary presentation of Winograd's DFT. There is far more to this subject than we have indicated here.

5. Relation of Circulant Multiplication to Winograd's Formulation

It is instructive to see in detail the correspondence between our factorization of the circulant and Winograd's derivation of cyclic convolution using polynomials.

Let u be the indeterminate in the polynomials.

Cyclic convolution on n points can be rephrased as the formation of the product of two arbitrary polynomials modulo the polynomial $u^{n-1} - 1$. This can be done rapidly with the aid of the Chinese Remainder Theorem.

For simplicity, we take $n = 5$ and define

$$X(u) = x_1 + x_3 u + x_4 u^2 + x_2 u^3,$$
$$\Omega(u) = \omega + \omega^2 u + \omega^4 u^2 + \omega^3 u^3,$$
$$\hat{X}(u) = \hat{x}_1 + \hat{x}_2 u + \hat{x}_4 u^2 + \hat{x}_3 u^3.$$

The object is to find $\hat{X}$ given any X and a fixed Ω. It turns out that

$$\hat{X} = \Omega X \bmod (u^4 - 1).$$

The fast algorithm requires knowledge of the irreducible factors of $u^{n-1} - 1$. In our case

$$u^4 - 1 = (u-1)(u+1)(u^2+1),$$
$$\equiv Q_1(u)\, Q_2(u)\, Q_3(u),$$

defining Q_1, Q_2, and Q_3. Also needed are another set of polynomials, of degree less than 4, which are defined, for $k = 1,2,3$, by

$$S_k(u) = \begin{cases} 1 \bmod Q_k, \\ 0 \bmod Q_j,\ j \neq k. \end{cases}$$

The S_k are analogous to the fundamental polynomials for Lagrangian interpolation in that they serve to reconstruct a polynomial given only its values "at" the Q_k. The word "at" should be construed as "modulo" in the previous sentence. The existence of the S_k is the essence of the Chinese Remainder Theorem.

Finally define $\Omega_k(u) = \Omega(u) \bmod Q_k$, $k = 1,2,3$.

Now we can formulate the algorithm succinctly.

Phase 1 (called the preweave by signal engineers):

Form $X_k = X \bmod Q_k$, $k = 1,2,3$.

Phase 2 (multiply):

Form $\hat{X}_k = \Omega_k \cdot X_k \bmod Q_k$, $k = 1,2,3$.

Phase 3 (called the postweave or recovery):

Form $\hat{X} = \hat{X}_1 S_1 + \hat{X}_2 S_2 + \hat{X}_3 S_3$.

When the S_k have $0,\pm 1$ coefficients then Phase 3 needs no multiplications. By the Remainder theorem $X_1 = X(1)$, $X_2 = X(-1)$, $X_3 = X_3^{(0)} + X_3^{(1)} u$, $X_3^{(0)} = x_1 - x_4$, etc.

Let us determine the S_k and Ω_k explicitly. It is easy to verify that

$$S_1 = \tfrac{1}{4}Q_2Q_3 = \tfrac{1}{4}(1 + u + u^2 + u^3),$$
$$S_2 = -\tfrac{1}{4}Q_1Q_3 = \tfrac{1}{4}(1 - u + u^2 - u^3),$$
$$S_3 = -\tfrac{1}{2}Q_1Q_2 = \tfrac{1}{2}(1 \quad - u^2 \quad).$$

Since $\omega = \exp(2\pi i/5)$ a little algebra yields

$$\Omega_1(u) = \omega + \omega^2 + \omega^4 + \omega^3 = -1, \quad (\text{since } \omega^5 = 1),$$
$$\Omega_2(u) = \omega - \omega^2 + \omega^4 - \omega^3 = 2(\cos \tfrac{2\pi}{5} - \cos \tfrac{4\pi}{5}),$$
$$\Omega_3(u) = (\omega - \omega^4) + (\omega^2 - \omega^3)u$$
$$= 2i(\sin \tfrac{2\pi}{5} + \sin \tfrac{4\pi}{5}\, u),$$
$$= \Omega_3^{(0)} + \Omega_3^{(1)}\, u.$$

These values define the matrix D in Table 3. Note that the polynomial product $\Omega_3 \cdot X_3 \bmod Q_3$ involves more than one scalar multiplication. In fact

$$\hat{X}_3(u) = (1 \; u \; u^2) \begin{pmatrix} \Omega_3^{(0)} \cdot X_3^{(0)} \\ \Omega_3^{(0)} \cdot X_3^{(1)} + \Omega_3^{(1)} \cdot X_3^{(0)} \\ \Omega_3^{(1)} \cdot X_3^{(1)} \end{pmatrix},$$

$$= (1 \; u) \begin{pmatrix} \Omega_3^{(0)} X_3^{(0)} - \Omega_3^{(1)} \cdot X_3^{(1)} \\ \Omega_3^{(1)} X_3^{(0)} + \Omega_3^{(0)} \cdot X_3^{(1)} \end{pmatrix} \bmod Q_3,$$

$$= (1 \; u) \begin{pmatrix} \Omega_3^{(0)} & -\Omega_3^{(1)} \\ \Omega_3^{(1)} & \Omega_3^{(0)} \end{pmatrix} \begin{pmatrix} X_3^{(0)} \\ X_3^{(1)} \end{pmatrix}.$$

When the steps are laid out in matrix form the connection with circulant factorization is apparent. (See Table 1.)

Phase 1.

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3(0) \\ X_3(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_4 \\ x_2 \end{pmatrix}$$

Phase 2.

$$\begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3(0) \\ \hat{X}_3(1) \end{bmatrix} = \begin{bmatrix} \Omega_1 & 0 & 0 & 0 \\ 0 & \Omega_2 & 0 & 0 \\ 0 & 0 & \Omega_3(0) & -\Omega_3(1) \\ 0 & 0 & \Omega_3(1) & \Omega_3(0) \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3(0) \\ X_3(1) \end{pmatrix}$$

Phase 3.

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_4 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3(0) \\ \hat{X}_3(1) \end{pmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ S_1 & S_2 & S_3 & S_3u \end{matrix}$$

To recover the polynomial $\hat{X}$ in Phase 3 it is only necessary to pre-multiply by the row vector $(1\ u\ u^2\ u^3)$. Of course the constants $\frac{1}{4}$, $\frac{1}{2}$ are absorbed into the matrix in Phase 2.

Table 1

$$"P_m = (G\Delta) \wedge (G\Delta)^T"$$

"Real spectral factorization of the cyclic shift matrix"

Notation: $A \oplus B = \mathrm{diag}(A,B)$ = direct sum of A and B.

$$m = 2: \quad G = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Delta^{-2} = 2I, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$m = 3: \quad G = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad \Delta^{-2} = \mathrm{diag}(3,6,2), \quad \Lambda = 1 \oplus \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}$$

$$m = 4: \quad G = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}, \quad \Delta^{-2} = \mathrm{diag}(4,4,2,2), \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$m = 6: \quad G = \begin{pmatrix} 1 & 1 & 2 & 0 & 2 & 0 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -2 & 0 & 2 & 0 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix}, \quad \Delta^{-2} = \mathrm{diag}(6,6,12,4,12,4),$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus R \oplus R^2,$$

$$R = \begin{pmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{pmatrix}.$$

Table 2

"The circulant polynomial arising from DFT(m+1)."

$$\omega_j = \exp(2\pi\sqrt{-1}/j). \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$m = 2$: $\phi(t) = \omega_3 + \omega_3{}^2 t$

$\phi(\wedge) = \phi(1) + \phi(-1) = -1 \oplus \sqrt{-3}$

$m = 4$: $\phi(t) = \omega_5 + \omega_5{}^2 t + \omega_5{}^4 t^2 + \omega_5{}^3 t^3$

$\phi(1) = -1, \quad \phi(-1) = 2(\cos 2\pi/5 - \cos 4\pi/5) = 2(\cos \frac{2\pi}{5} + \cos \frac{\pi}{5})$

$$\phi(R) = 2\sqrt{-1} \begin{pmatrix} \sin 2\pi/5 & -\sin 4\pi/5 \\ \sin 4\pi/5 & \sin 2\pi/5 \end{pmatrix}$$

$m = 6$: $\phi(t) = \omega_7 + \omega_7{}^3 t + \omega_7{}^2 t^2 + \omega_7{}^6 t^3 + \omega_7{}^4 t^4 + \omega_7{}^5 t^5$

$\phi(1) = -1, \ \phi(-1) = 2\sqrt{-1}\ (\sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} - \sin \frac{6\pi}{7}),$

$\phi(R) = (2\omega + \omega^3 - \omega^2 - 2\omega^6 - \omega^4 + \omega^5)I + \sqrt{3}(\omega^3 + \omega^2 - \omega^4 - \omega^5)J,$

$= \sqrt{-1}\{(2 \sin \frac{2\pi}{7} - \sin \frac{4\pi}{7} + \sin \frac{6\pi}{7})I + \sqrt{3}(\sin \frac{4\pi}{7} + \sin \frac{6\pi}{7})J\},$

$= \sqrt{-1}\{(\sin \frac{\pi}{7} + 2 \sin \frac{2\pi}{7} - \sin \frac{3\pi}{7})I + \sqrt{3}(\sin \frac{\pi}{7} + \sin \frac{3\pi}{7})J\}.$

$\phi(R^2) = \omega_7 I + \omega_7{}^3 R^2 - \omega_7{}^2 R + \omega_7{}^6 I + \omega_7{}^4 R^2 - \omega_7{}^5 R$

$= \frac{1}{2}(2\omega - \omega^3 - \omega^2 + 2\omega^6 - \omega^4 - \omega^5)I + \frac{\sqrt{3}}{2}(\omega^3 - \omega^2 + \omega^4 - \omega^5)J,$

$= (2 \cos \frac{2\pi}{7} - \cos \frac{4\pi}{7} - \cos \frac{6\pi}{7})I + \sqrt{3}(-\cos \frac{4\pi}{7} + \cos \frac{6\pi}{7})J,$

$= (\cos \frac{\pi}{7} + 2 \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7})I + \sqrt{3}(-\cos \frac{\pi}{7} + \cos \frac{3\pi}{7})J.$

Table 3. "$D = \Delta\phi(\Lambda)\Delta$"

$A \oplus B = \mathrm{diag}(A,B)$, $i = \sqrt{-1}$.

$n = 3$: $D = -\frac{1}{2} \oplus (1/2)\sqrt{3}\, i$

$n = 5$: $D = -\frac{1}{4} \oplus \frac{1}{2}(\cos\frac{\pi}{5} + \cos\frac{2\pi}{5}) \oplus \begin{pmatrix} \sin\frac{2\pi}{5} & -\sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} \end{pmatrix} i$

$n = 7$: $D = -\frac{1}{6} \oplus \frac{1}{3}(-\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7})i \oplus \begin{pmatrix} \beta & -\gamma \\ \gamma & \beta \end{pmatrix} i \oplus \begin{pmatrix} \delta & -\eta \\ \eta & \delta \end{pmatrix}$,

$$\beta = \frac{1}{12}(\sin\frac{\pi}{7} + 2\sin\frac{2\pi}{7} - \sin\frac{3\pi}{7}),$$
$$\gamma = \frac{1}{4}(\sin\frac{\pi}{7} + \sin\frac{3\pi}{7}),$$
$$\delta = \frac{1}{12}(\cos\frac{\pi}{7} + 2\cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}),$$
$$\eta = \frac{1}{4}(-\cos\frac{\pi}{7} + \cos\frac{3\pi}{7}).$$
$$\beta+\gamma = \frac{1}{6}(2\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7})$$
$$\beta-\gamma = \frac{1}{6}(-\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} - 2\sin\frac{3\pi}{7})$$
$$\delta+\eta = \frac{1}{6}(-\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + 2\cos\frac{3\pi}{7})$$
$$\delta-\eta = \frac{1}{6}(2\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7})$$

Note 1. Some minor modifications will be given in Section 4.

Note 2. All angles have been reduced to the range $(0,\pi/2)$.

Table 4a. The DFT Factorization for $n = 3^2$.

$$\begin{pmatrix} x_0 \\ x_3 \\ x_6 \\ x_1 \\ x_2 \\ x_4 \\ x_8 \\ x_7 \\ x_5 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \omega^3 & \omega^6 & \omega^3 & \omega^6 & \omega^3 & \omega^6 \\ 1 & 1 & 1 & \omega^6 & \omega^3 & \omega^6 & \omega^3 & \omega^6 & \omega^3 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^5 & \omega^7 & \omega^8 & \omega^4 & \omega^2 \\ 1 & \omega^6 & \omega^3 & \omega^2 & \omega & \omega^5 & \omega^7 & \omega^8 & \omega^4 \\ 1 & \omega^3 & \omega^6 & \omega^4 & \omega^2 & \omega & \omega^5 & \omega^7 & \omega^8 \\ 1 & \omega^6 & \omega^3 & \omega^8 & \omega^4 & \omega^2 & \omega & \omega^5 & \omega^7 \\ 1 & \omega^3 & \omega^6 & \omega^7 & \omega^8 & \omega^4 & \omega^2 & \omega & \omega^5 \\ 1 & \omega^6 & \omega^3 & \omega^5 & \omega^7 & \omega^8 & \omega^4 & \omega^2 & \omega \end{bmatrix} \begin{pmatrix} x_0 \\ x_3 \\ x_6 \\ x_1 \\ x_5 \\ x_7 \\ x_8 \\ x_4 \\ x_2 \end{pmatrix}$$

$$= \left[\begin{array}{c|c} I_3 & 0 \\ \hline \begin{matrix} r \\ s \\ r \\ s \\ r \\ s \end{matrix} & G_6 \end{array}\right] \left[\begin{array}{c|c} F(\omega_3) & 0 \\ \hline 0 & D_6 \end{array}\right] \left[\begin{array}{c|c} \begin{matrix} 111 & 000000 \\ 000 & 101010 \\ 000 & 010101 \end{matrix} & \\ \hline & 0 \quad G_6^T \end{array}\right] \left(\begin{array}{c} \underset{\sim}{x}^{(0)} \\ \hline \underset{\sim}{x}^{(1)} \end{array}\right),$$

$$= \left[\begin{array}{c|c|c} I_3 & 0 & 0 \\ \hline 0 & G_6 & \begin{matrix} 010 \\ 001 \\ 010 \\ 001 \\ 010 \\ 001 \end{matrix} \end{array}\right] \left[\begin{array}{c|c|c} F(\omega_3) & 0 & 0 \\ \hline 0 & D_6 & \\ \hline 0 & 0 & F(\omega_3) \end{array}\right] \left[\begin{array}{c|c} \begin{matrix} 111 & 000000 \\ 000 & 101010 \\ 000 & 010101 \end{matrix} & \\ \hline & 0 \quad G_6^T \\ \hline I_3 & 0 \end{array}\right] \left(\begin{array}{c} \underset{\sim}{x}^{(0)} \\ \hline \underset{\sim}{x}^{(1)} \end{array}\right).$$

$\psi(t) = \omega + \omega_5 t + \omega_7 t^2 + \omega_8 t^3 + \omega_4 t^4 + \omega_2 t^5$, $\omega = \exp(2\pi\sqrt{-1}/9)$,

$\Lambda_6 = -1 \oplus \alpha\sqrt{-1} \oplus \begin{pmatrix} \beta & -\gamma \\ \gamma & \beta \end{pmatrix}\sqrt{-1} \oplus \begin{pmatrix} \delta & -\varepsilon \\ \varepsilon & \delta \end{pmatrix}$, see Table 1.

mult(9) = 3 + 8 + 3 = 14. $D_6 = \psi(\Lambda_6)$.

Table 4b. The DFT Factorization for $n = 2^3$.

$$\begin{pmatrix} \hat{x}_0 \\ \hat{x}_4 \\ \hat{x}_2 \\ \hat{x}_6 \\ \hat{x}_1 \\ \hat{x}_5 \\ \hat{x}_7 \\ \hat{x}_3 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & i & i & -i & -i \\ 1 & 1 & -1 & -1 & -i & -i & i & i \\ 1 & -1 & i & -i & \omega & -\omega & -i\omega & i\omega \\ 1 & -1 & i & -i & -\omega & \omega & i\omega & i\omega \\ 1 & -1 & -i & i & -i\omega & i\omega & \omega & -\omega \\ 1 & -1 & -i & i & i\omega & -i\omega & -\omega & \omega \end{bmatrix} \begin{pmatrix} x_0 \\ x_4 \\ x_2 \\ x_6 \\ x_1 \\ x_5 \\ x_7 \\ x_3 \end{pmatrix}$$

$$= \left[\begin{array}{c|c|cc|cccc} 1 & & & & & & & \\ \hline & 1 & & & & & & \\ \hline & & 1 & 1 & & & & \\ & & 1 & -1 & & & & \\ \hline & & & & 1 & 1 & 1 & 1 \\ & 0 & & & 1 & 1 & -1 & -1 \\ & & & & 1 & -1 & 1 & -1 \\ & & & & 1 & -1 & -1 & 1 \end{array}\right] \overset{\text{diag}}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ i \\ 1 \\ i \\ \alpha \\ \alpha i \end{pmatrix}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix} \underset{\sim}{x},$$

where $\alpha = 1/\sqrt{2} = \cos \pi/4 = \sin \pi/4$, $i = \sqrt{-1}$.

The cyclic convolution on $\{x_1, x_3, x_5, x_7\}$ does not lead to an optimal scheme here.

mult(8) = max {8,2} = 8.

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