NONLINEAR FREQUENCY SHIFT INDUCED BY THE LOWER-HYBRID DRIFT INSTABILITY

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ABSTRACT

It is found that a finite perturbation of the ion orbits leads to a nonlinear frequency shift that reduces the mode frequency and has a weak stabilizing effect on the lower-hybrid drift instability. This result is obtained from a self-consistent solution of the Vlasov-Poisson equations using perturbation theory in which the nonlinear dielectric function and the nonlinear temporal evolution of a single unstable mode in the low drift velocity regime are calculated analytically.

I. INTRODUCTION

The linear theory of the lower-hybrid drift instability is well understood and has been discussed in detail by Davidson et al.[1] When the amplitude of the wave is small but finite after a time equal to many multiples of the growth time, further evolution will be different from exponential growth at very small amplitudes. In order to analyze this, the nonlinear dielectric response function and the nonlinear temporal evolution of a single unstable mode are derived self-consistently by using perturbation theory to solve the Vlasov and Poisson equations. The single-mode approximation is valid for the instability when the plasma parameters are close to those for linear marginal stability. This requires $v_E \ll v_{ti}$ (the low drift velocity regime) for the lower hybrid drift instability, where $v_E = cE/B$ is the equilibrium $E \times B$ drift velocity and $v_{ti}^2 = T_i/M$ is the ion thermal speed. M and T_i are the ion mass and temperature, respectively. Similar single mode studies include: modulation of the Langmuir wave due to weak nonlinearity[2,3]; nonlinear evolution of drift-cyclotron and drift-cone instabilities both in theory[4-7] and simulation.[8]

In this paper we demonstrate that a finite amplitude perturbation of the ion orbits develops during growth which leads to a weakly stabilizing nonlinear shift of the lower-hybrid drift mode frequency. The largest shifts in the mode frequency occur for modes with wavelengths much longer than that of the most unstable mode ($k \ll k_m$), at which wavenumbers the lower-hybrid drift instability converts into the drift cyclotron instability.

For simplicity, we use a one-dimensional slab configuration shown in Fig. 1; wave propagation is in the x direction; the magnetic field is uniform and in the z direction; the density gradient is in the y direction.

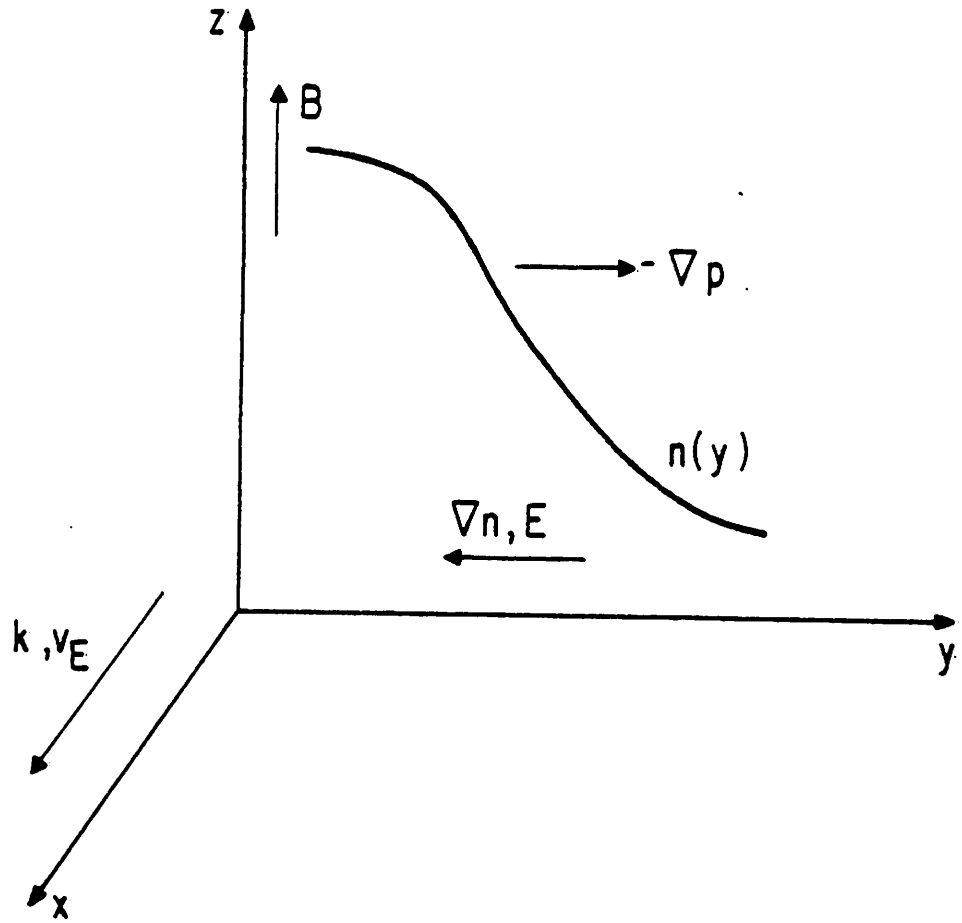

FIG. 1 Slab coordinates for lower-hybrid drift instability.

The ions are treated as unmagnetized because the wave frequency and growth rate are much greater than the ion cyclotron frequency. The ions are in force balance; that is, the force due to the ion pressure gradient in y cancels that of the equilibrium electric field. The lower-hybrid-drift instability is analyzed in the electrostatic limit; electromagnetic effects are assumed to be small.

In Sec. II, the nonlinear dielectric response is calculated by solving the coupled Vlasov-Poisson equations, and a nonlinear dispersion relation is obtained. Section III is devoted to a derivation of the time evolution of the lower-hybrid drift instability. The field energy level at which saturation might occur and the frequency shift due to the finite amplitude of the wave are also determined. Finally, conclusions and a comparative discussion of several saturation mechanisms are given in Sec. IV.

II. DERIVATION OF THE NONLINEAR DIELECTRIC FUNCTION

We follow the method of reductive perturbation theory.[2] We assume that the distribution functions $F^s(y,\underline{v},t)$ for species s and the electric potential $\phi(x,t)$ can be expanded as

$$F^s(y,\underline{v},t) = \tilde{F}_0^s(y,\underline{v}) + \sum_{n=1}^{\infty} \varepsilon^n \tilde{F}_n^s(y,\underline{v},t)e^{in\theta} + \text{c.c.} \qquad (1)$$

and

$$\phi(x,t) = \sum_{n=1}^{\infty} \varepsilon^n \tilde{\phi}_n(x,t)e^{in\theta} + \text{c.c.} \qquad (2)$$

where

$$\tilde{F}_n^s(y,\underline{v},t) = \sum_{j=0}^{\infty} \varepsilon^j F_{nj}^s(y,\underline{v},t) \quad , \qquad n=0,1,... \qquad (3)$$

$$\tilde{\phi}_n(x,t) = \sum_{j=0}^{\infty} \varepsilon^j \phi_{nj}(x,t) \quad , \qquad n = 1,2,\ldots \tag{4}$$

$$\theta \equiv kx - \omega t . \tag{5}$$

The small parameters is ε, which is on the order of $e\phi/T_i \ll 1$. k and ω are the wave number and frequency of a single mode. Quasilinear analysis indicates that current relaxation (the relative drift between the electrons and ions goes to zero) can cause saturation for $v_E < v_{ti}$.[9] However, the effect of current relaxation is small for $v_E \ll v_{ti}$. Therefore, we shall specialize to the case $v_E \ll v_{ti}$ and treat the density gradient and v_E as constant in our derivation. The distribution function $F_{nj}(y,\underline{v},t)$ can then be expressed as

$$F_{nj}(y,\underline{v},t) = n_o(y) f_{nj}(\underline{v},t) . \tag{6}$$

The Poisson equation of the system is

$$-\nabla^2\phi = 4\pi n_o e \int d\underline{v}\, (f^i - f^e) . \tag{7}$$

Substituting Eqs. (1) and (2) into Eq. (7) yields

$$(nk)^2\tilde{\phi}_n = 4\pi n_o e \int d\underline{v}\, (\tilde{f}^i_n - \tilde{f}^e_n) . \tag{8}$$

Since the characteristic frequency of the lower-hybrid drift instability is much less than the electron plasma and cyclotron frequencies, it is assumed that electrons respond to the wave linearly, i.e.,

$$-4\pi n_o e \int d\underline{v}\, \tilde{f}^e_n = -\chi_e(nk,n\omega)(nk)^2\tilde{\phi}_n \tag{9}$$

where χ_e is the linear electron susceptibility. To justify further the assumption of a linear electron response, we also assume a value of zero for the plasma beta (plasma pressure $\ll$ magnetic pressure) and $T_e = 0$, so that electron resonance broadening can be neglected. For finite electron temperature and plasma beta, electron resonance broadening can stabilize the instability.[10,11] Using Eqs. (3), (4) and (9), Eqs. (8) reduces to

$$[1 + \chi_e(nk,n\omega)](nk)^2\phi_{nj} = 4\pi n_o e \int f^i_{nj}\, d\underline{v} \ . \tag{10}$$

The Vlasov equation for the ion distribution in one dimension is

$$\frac{\partial f^i}{\partial t} + v\frac{\partial f^i}{\partial x} - \frac{e}{M}\frac{\partial \phi}{\partial x}\frac{\partial f^i}{\partial v} = 0 \ , \tag{11}$$

which can be rewritten as

$$\frac{df^i}{dt} \equiv \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)f^i = \frac{e}{M}\frac{\partial \phi}{\partial x}\frac{\partial f^i}{\partial v} \ , \tag{12}$$

so that f^i can be integrated over the characteristics corresponding to the unperturbed orbits, i.e., $x = vt$. We obtain, as usual,

$$f^i - f^i_{00} = \frac{e}{M}\int_{-\infty}^{t} \frac{\partial \phi}{\partial x}\frac{\partial f^i}{\partial v}\, dt' \ . \tag{13}$$

Expansion of Eq. (13) in the series given by Eqs. (1) and (2) yields

$$\sum_{j=1}^{\infty} \varepsilon^j \left[f_{0j} + \sum_{n=1}^{j} (f_{n,j-n} e^{in\theta} + \text{c.c.}) \right]$$

$$= \sum_{j=1}^{\infty} \varepsilon^j \frac{e}{M} \int_{-\infty}^{t} \sum_{\substack{n=1 \\ \ell=0}}^{\infty} \left\{ (ink\phi_{n\ell} e^{in\theta'} + \text{c.c.}) \right.$$

$$\left. \cdot \left[\frac{\partial f_{0,j-n-\ell}}{\partial v} + \sum_{n'=1}^{\ell} \left(\frac{\partial f_{n',j-n-\ell-n'}}{\partial v} e^{in'\theta'} + \text{c.c.} \right) \right] \right\} dt' \ . \tag{14}$$

Convergence of the series solution for the perturbed distribution function requires that the electric potential cause only a small perturbation to the unperturbed orbits. Hence, ion trapping is excluded by this assumption. One should also note that the superscript i for ions has been dropped from Eq. (14). Equating coefficients of $\exp(in\theta)$ at the same order in ε in Eq. (14) yields

$$f_{01} = 0 \ , \tag{15}$$

and

$$f_{10} = \frac{e\phi_{10}}{M} \cdot \frac{\partial f_{00}/\partial v}{v - V} \tag{16}$$

where $V = \omega/k$ and f_{00} is the equilibrium ion distribution function in the absence of perturbation. We then substitute Eq. (16) into Eq. (10) to obtain

$$\left[1 + \chi_e(k,\omega)\right] k^2 \phi_{10} = \frac{4\pi n_o e^2}{M} \phi_{10} \int \frac{\partial f_{00}/\partial v}{v - V} dv$$

$$= - k^2 \chi_i(k,\omega) \phi_{10} , \qquad (17)$$

which yields the linear dielectric function $D(k,\omega)$,

$$D(k,\omega) = 1 + \chi_e(k,\omega) + \chi_i(k,\omega) . \qquad (18)$$

We now proceed to calculate the second order components. Assuming $\omega = \omega_r + i\delta$ and $\delta \to 0$, and equating coefficients of the constant terms in Eq. (14) gives

$$f_{02} = \lim_{\delta \to 0} \frac{e}{M} \frac{ik}{2\delta} \left(-\phi_{10}^* \frac{\partial f_{10}}{\partial v} + \phi_{10} \frac{\partial f_{10}^*}{\partial v} \right) . \qquad (19)$$

Substituting f_{10} by using Eq. (16), we obtain the quasilinear modification to the distribution function,

$$f_{02} = \left| \frac{e\phi_{10}}{M} \right|^2 \frac{\partial}{\partial v} \left(\frac{\partial f_{00}/\partial v}{(v-V)^2} \right) . \qquad (20)$$

The component f_{11} results from the second order terms of Eq. (14) for $\exp(i\theta)$,

$$f_{11} = \frac{e\phi_{11}}{M} \cdot \frac{\partial f_{00}/\partial v}{v - V} . \qquad (21)$$

Then, following the steps used in deriving Eq. (17) and Eq. (18), we obtain from Eq. (10)

$$D(k,\omega)\phi_{11} = 0 . \tag{22}$$

The components f_{11} and ϕ_{11} are uncoupled from the lower order components and are irrelevant to the rest of the calculation. Therefore, we set $\phi_{11} = f_{11} = 0$ in the rest of the analysis.

Equating the coefficients of the second harmonic terms $(2\omega, 2k)$ in Eq. (14) give

$$f_{02} = \frac{1}{2}\left(\frac{e\phi_{10}}{M}\right)^2 \frac{1}{v-V} \frac{\partial}{\partial v}\left(\frac{\partial f_{00}/\partial v}{v-V}\right) + \frac{e\phi_{20}}{M} \frac{\partial f_{00}/\partial v}{v-V} . \tag{23}$$

The first term appearing on the right side of Eq. (23) is the modification due to the unshielded second harmonic oscillation of a single wave, and the second term represents the shielded effect. Similarly, Eqs. (10) and (23) yield

$$\phi_{20} = \frac{\omega_{pi}^2}{8k^2 D(2k,2\omega)} \frac{e\phi_{10}^2}{M} \int \frac{1}{v-V} \frac{\partial}{\partial v}\left(\frac{\partial f_{00}/\partial v}{v-V}\right) dv . \tag{24}$$

For the third order component f_{12}, Eq. (14) gives

$$f_{12} = \frac{e}{M}\left(\phi_{10} \frac{\partial f_{02}/\partial v}{v-V} - \phi_{10}^* \frac{\partial f_{20}/\partial v}{v-V} + \phi_{12} \frac{\partial f_{00}/\partial v}{v-V} + 2\phi_{20} \frac{\partial f_{10}^*/\partial v}{v-V}\right) . \tag{25}$$

Using f_{10} , f_{02} and f_{20} from Eqs. (16), (20) and (23), we find that

$$f_{12} = \left|\frac{e\phi_{10}}{M}\right|^2 \cdot \left(\frac{e\phi_{10}}{M}\right)\left\{\frac{1}{v-V}\frac{\partial^2}{\partial v^2}\left(\frac{\partial f_{00}\ \partial v}{(v-V)^2}\right)\right.$$
$$-\frac{1}{2}\frac{1}{v-V}\frac{\partial}{\partial v}\left[\frac{1}{v-V}\frac{\partial}{\partial v}\left(\frac{\partial f_{00}/\partial v}{v-V}\right)\right]$$
$$\left.+\frac{\omega_{pi}^2}{8k^2D(2k,2\omega)}\left[\int\frac{1}{v-V}\frac{\partial}{\partial v}\left(\frac{\partial f_{00}/\partial v}{v-V}\right)dv\right]\cdot\frac{1}{v-V}\frac{\partial}{\partial v}\left(\frac{\partial f_{00}/\partial v}{v-V}\right)\right\} \cdot$$
$$+\frac{e}{M}\phi_{12}\frac{\partial f_{00}/\partial v}{v-V} \ . \tag{26}$$

Substituting Eq. (26) into Eq. (10) for ϕ_{12}, combining to obtain $\tilde{\phi}_1 = \phi_{10} + \varepsilon^2\phi_{12}$, and then using Eqs. (2), (17) and (22) yields the nonlinear dispersion relation

$$D(k,\omega)\tilde{\phi}_1 = \frac{\omega_{pi}^2}{k^2}\left\{\int\frac{1}{v-V}\frac{\partial^2}{\partial v^2}\left(\frac{\partial f_{00}/\partial v}{(v-V)^2}\right)dv\right.$$
$$-\frac{1}{2}\int\frac{1}{v-V}\frac{\partial}{\partial v}\left[\frac{1}{v-V}\frac{\partial}{\partial v}\left(\frac{\partial f_{00}/\partial v}{v-V}\right)\right]dv$$
$$\left.+\frac{\omega_{pi}^2}{8k^2D(2k,2\omega)}\left[\int\frac{1}{v-V}\frac{\partial}{\partial v}\left(\frac{\partial f_{00}/\partial v}{v-V}\right)dv\right]^2\right\}\cdot\left|\frac{e\tilde{\phi}_1}{M}\right|^2\cdot\tilde{\phi}_1$$
$$+\mathcal{O}(\varepsilon^4) \ . \tag{27}$$

The first term on the right side is the nonlinear coupling of the potential with the quasilinear perturbation. The last two terms arise from the nonlinear coupling of second harmonic variations in f^i and ϕ with the perturbations at the fundamental, with and without plasma shielding effects.

We define $W(z)$ by

$$W(z) = v_{ti}^2 \int \frac{\partial f_{00}/\partial v}{v - V}\, dv \tag{28}$$

where $z = V/v_{ti} = \omega/kv_{ti}$. The quasilinear term gives

$$\int \frac{1}{v-V} \frac{\partial^2}{\partial v^2}\left(\frac{\partial f_{00}/\partial v}{(v-V)^2}\right) dv = \frac{1}{12 v_{ti}^6} \frac{d^4 W(z)}{dz^4} . \tag{29}$$

The unshielded second harmonic effect becomes

$$-\frac{1}{2}\int \frac{1}{v-V}\frac{\partial}{\partial v}\left[\frac{1}{v-V}\frac{\partial}{\partial v}\left(\frac{\partial f_{00}/\partial v}{v-V}\right)\right] dv = -\frac{1}{16 v_{ti}^6}\frac{d^4 W(z)}{dz^4} , \tag{30}$$

and the term associated with the shielded second harmonic oscillation is

$$\frac{\omega_{pi}^2}{8k^2 D(2k,2\omega)}\left[\int \frac{1}{v-V}\frac{\partial}{\partial v}\left(\frac{\partial f_{00}/\partial v}{v-V}\right) dv\right]^2$$

$$= \frac{1}{32 k^2 \lambda_D^2\, v_{ti}^6 D(2k,2\omega)}\left(\frac{d^2 W(z)}{dz^2}\right)^2 . \tag{31}$$

By using Eqs. (29), (30) and (31), we rewrite Eq. (27) as

$$D(k,\omega)\psi = \frac{1}{16k^2\lambda_D^2}\left[\frac{1}{3}\frac{d^4W(z)}{dz^4} + \frac{1}{2k^2\lambda_D^2 D(2k,2\omega)}\left(\frac{d^2W(z)}{dz^2}\right)^2\right]|\psi|^2\psi \qquad (32)$$

where $\psi = e\tilde{\phi}_1/T_i$ and λ_D is the ion Debye length. The quasilinear term and the unshielded second harmonic effect are combined in the first term on the right side of Eq. (32).

III. NONLINEAR EVOLUTION AND FREQUENCY SHIFT

In this section, we estimate the field energy at saturation caused by a finite nonlinear frequency shift by solving the nonlinear dispersion relation. When the wave amplitude is very small, Eq. (32) reduces to the usual linear dispersion relation

$$D(k,\omega) = D_R(k,\omega) + iD_I(k,\omega) . \qquad (33)$$

Let us examine Eq. (33) in the lower drift velocity regime characterized by

$$|\gamma/\omega_r| \ll 1 \quad , \quad V \quad \text{and} \quad v_E \ll v_{ti} , \qquad (34)$$

where ω_r and γ are the real and imaginary parts of frequency, respectively. The dielectric function for cold electrons and Maxwellian ions is expressed as

$$D(k,\omega) = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} + \frac{1}{k^2\lambda_D^2}\frac{\omega}{\omega - kv_E} + i\sqrt{\frac{\pi}{2}}\;\frac{1}{k^2\lambda_D^2}\frac{\omega}{|k|v_{ti}} \quad . \tag{35}$$

The real part of the frequency is determined to zeroth order in $|\gamma/\omega_r|$ by

$$D_R(k,\omega_r) = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} + \frac{1}{k^2\lambda_D^2}\frac{\omega_r}{\omega_r - kv_E} = 0 \quad . \tag{36}$$

The solution is

$$\omega_r = \frac{k^2}{k^2 + k_m^2}\,kv_E \equiv \omega_o \tag{37}$$

and the growth rate $\gamma = -D_I/(\partial D_R/\partial\omega)_o$ is given by

$$\gamma = \sqrt{\frac{\pi}{2}}\;\frac{k^2/k_m^2}{(1 + k^2/k_m^2)^3}\;\frac{|k|}{k_m}\left(\frac{v_E}{v_{ti}}\right)^2\omega_{\ell h} \tag{38}$$

where

$$k_m = \left[\frac{1}{\lambda_D^2}\cdot\frac{1}{1 + \omega_{pe}^2/\omega_{ce}^2}\right]^{\frac{1}{2}} \tag{39}$$

is the wave number of the most unstable mode, and

$$\omega_{\ell h} = \frac{\omega_{pi}}{\sqrt{1 + \omega_{pe}^2/\omega_{ce}^2}} \tag{40}$$

is the lower hybrid frequency.[9]

When the amplitude of the wave is small but finite, we expand Eq. (32) around ω_o by replacing ω with $\omega_o + i(\partial/\partial t)$ and use Eqs. (35) and (36), to obtain

$$i[(1 + i\alpha)\frac{\partial}{\partial t} - \gamma]\psi = (A + iB)|\psi|^2\psi \tag{41}$$

where

$$\alpha \equiv (\partial D_I/\partial\omega)_o/(\partial D_R/\partial\omega)_o \approx -\gamma/\omega_o \ , \tag{42}$$

$$A = -\frac{(\omega_o - kv_E)^2}{8kv_E}\left(\frac{4}{3} + \frac{1}{k^2\lambda_D^2 D_R(2k, 2\omega_o)}\right) \ , \tag{43}$$

and

$$B = -\frac{\gamma}{16}\left(5 + \frac{6}{k^2\lambda_D^2 D_R(2k, 2\omega_o)}\right) . \tag{44}$$

By using Eqs. (36) and (39), we get

$$\left[k^2\lambda_D^2 D_R(2k, 2\omega_o)\right]^{-1} = \frac{4k_m^2}{3k^2} . \tag{45}$$

Substituting Eq. (45) into Eqs. (43) and (44), and using Eqs. (29) through (32), the relative strengths of the nonlinear contributions from the quasi-linear modification (ql), the bare second harmonic oscillation (b) and its shielded effect (s) are given as the ratios,

$$A_{ql} : A_{(2k,2\omega_o)_b} : A_{(2k,2\omega_o)s} = 4 : -3 : \frac{k_m^2}{k^2} \tag{46}$$

and

$$B_{ql} : B_{(2k,2\omega_o)_b} : B_{(2k,2\omega_o)_s} = 20 : -15 : 8 \frac{k_m^2}{k^2} \ . \tag{47}$$

For the most unstable mode, $k = k_m$, the three nonlinear contributions to the ion distribution function are comparable. For $k/k_m \ll 1$ the shielded second harmonic contribution dominates the nonlinear modification of the ion response. This is because the second harmonic perturbation is very close to satisfying the linear dispersion relation for $k \ll k_m$, i.e., $D(2k,2\omega_o)$ nearly vanishes.

In order to obtain the time evolution of the wave amplitude and frequency shift, we define $\psi = r \exp(-is)$, where both r and s are real. Eq. (42) becomes

$$\alpha\dot{r} - r\dot{s} = - Ar^3 , \tag{48}$$

and

$$\dot{r} + \alpha r\dot{s} - \gamma r = Br^3 . \tag{49}$$

Eliminating $\dot{r}$, we obtain

$$\dot{s} = \frac{\alpha\gamma}{1+\alpha^2} + \frac{A+\alpha B}{1+\alpha^2} r^2 \tag{50}$$

where the first term is the linear correction to the frequency in the presence of growth, and the second term is the nonlinear frequency shift which grows in time with r^2 (i.e., $|e\tilde{\phi}_1/T_i|^2$). Eliminating $r\dot{s}$, we obtain

$$\dot{r} = \left(\frac{\gamma}{1+\alpha^2} + \frac{B - \alpha A}{1+\alpha^2}\, r^2\right) r \quad . \tag{51}$$

Integration of Eq. (51) yields

$$r^2 = r_\infty^2 \, \frac{c e^{2\gamma t/(1+\alpha^2)}}{1 + c e^{2\gamma t/(1+\alpha^2)}} \tag{52}$$

where

$$r_\infty \equiv \left|\frac{e\phi}{T_i}\right| (t \to \infty) = \left(\frac{\gamma}{\alpha A - B}\right)^{\frac{1}{2}} \tag{53}$$

is the field energy level at saturation.

From Eqs. (42) through (47), it is obvious that the quasilinear effect and the shielded second harmonic oscillation stabilize a single lower-hybrid wave, i.e., they nonlinearly reduce growth. The nonlinearity due to the bare second harmonic oscillation enhances growth and raises the saturation level. If $r_\infty \gg r(t=0) = r_o$, Eq. (52) gives

$$r^2(t) = \frac{r_o^2 \, e^{\frac{2\gamma t}{1+\alpha^2}}}{1 + \frac{r_o^2}{r_\infty^2} \, e^{\frac{2\gamma t}{1+\alpha^2}}} \tag{54}$$

and Eq. (50) becomes

$$\dot{s}(t) = \frac{\alpha\gamma}{1+\alpha^2} + \frac{A + \alpha B}{1+\alpha^2} \, \frac{r_o^2 \, e^{\frac{2\gamma t}{1+\alpha^2}}}{1 + \frac{r_o^2}{r_\infty^2} \, e^{\frac{2\gamma t}{1+\alpha^2}}} \quad . \tag{55}$$

We identify the second term on the right side of Eq. (55) as a nonlinear shift of the mode frequency.

IV. CONCLUSION

A nonlinear dispersion relation for the lower-hybrid drift instability was derived. We obtained the saturation field energy and the nonlinear frequency shift at saturation. With use of Eqs. (35) through (44), Eq. (53) gives

$$\left|\frac{e\tilde{\phi}_1}{T_i}\right|^2_{sat} = 6\left(\frac{k}{k_m}\right)^4 \Big/ \left(1 + 3\,\frac{k^2}{k_m^2} + \frac{15}{8}\,\frac{k^4}{k_m^4}\right) , \tag{56}$$

Extracting the nonlinear part of s in Eq. (55) gives

$$\Delta\omega_{sat} = \frac{1}{1+3\,\frac{k^2}{k_m^2}+\frac{15}{8}\,\frac{k^4}{k_m^4}}\,\omega_o \tag{57}$$

at saturation in the limit $v_E/v_{ti} \ll 1$.

In Fig. 2 we present $|e\tilde{\phi}_1/T_i|_{sat}$, the frequency shift $\Delta\omega_{sat}$, linear frequency ω_o and growth rate γ versus k/k_m for $v_E/v_{ti} = 0.3$. We note that the saturation amplitude $|e\tilde{\phi}_1/T_i|_{sat}$ is much less than unity only if $k \ll k_m$. However, from Eq. (57) we observe that $\Delta\omega_{sat}$ approaches $-\omega_o$ for $k \ll k_m$. This violates the assumptions of our perturbation theory. Nevertheless, it is true that $|\Delta\omega/\omega_o|$ is largest for long wavelength modes $k \ll k_m$, and that Eqs. (50) and (55) are valid only for $|e\tilde{\phi}_1/T_i|$, $|\Delta\omega/\omega_o| \ll 1$. Also from Eqs. (37), (39) and (40), we notice that ω_o becomes smaller than the

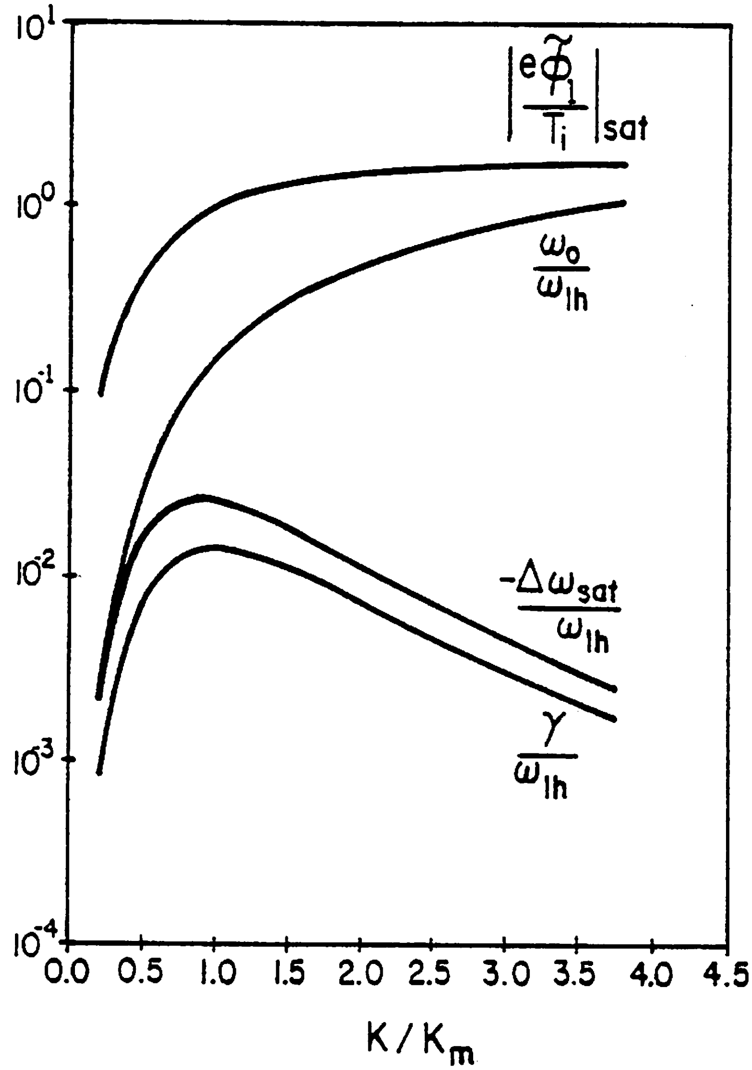

FIG. 2 Saturated lower-hybrid drift mode amplitude $|e\tilde{\phi}_1/T_i|_{sat}$ due to non-linear frequency shift mechanism, normalized frequency $\omega_r/\omega_{\ell h}$, growth rate $\gamma/\omega_{\ell h}$, and nonlinear frequency shift $\Delta\omega_{sat}/\omega_{\ell h}$ as functions of k/k_m for $v_E/v_{ti} = 0.3$. $\omega_{\ell h}$ is the lower hybrid frequency, and k_m is the wave number of the most unstable mode. Note that $|e\tilde{\phi}_1/T_i| \ll 1$ only for $k/k_m \ll 1$, and that $-\Delta\omega_{sat} \ll \omega_r$ only for $k/k_m \gtrsim 1$.

ion cyclotron frequency for $k \lesssim 0.4k_m$ when $v_E/v_{ti} = 0.3$. Therefore, our assumption of unmagnetized ions breaks down, and these long wavelength lower-hybrid drift modes convert into ion cyclotron drift modes. Many studies[4-8] have shown that a nonlinear ion orbit perturbation induces a frequency shift which can stabilize the ion cyclotron drift instability. For modes with $k \gtrsim k_m$, which includes the most unstable mode, $k = k_m$, $|e\tilde{\phi}_1/T_i| = \mathcal{O}(1)$, which is too large and thus also invalidates our perturbation theory. For these short wavelength modes, we expect that other nonlinear effects will be the dominant saturation mechanisms, for example, trapping or quasilinear diffusion.

The nonlinear frequency shift does not appear to be an efficient mechanism for saturating the lower-hybrid drift instability. However, it is important to compare some of the more promising saturation mechanisms and ascertain whether the nonlinear frequency shift effect should have been included in their descriptions.

Let us restrict our discussion to the low drift regime $v_E \ll v_{ti}$ and $T_e \ll T_i$. By using Eqs. (39), (43) and (50), the field energy, $\mathcal{E} = \langle E^2/8\pi \rangle$, for which $\Delta\omega = -0.1\omega$ (nominal value) is

$$\left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}\right)\left(\frac{\mathcal{E}}{nT_i}\right)_{\Delta\omega} = 0.6\left(\frac{k}{k_m}\right)^6 \Bigg/ \left[1 - \frac{3\pi}{16}\frac{\left(8\frac{k^2}{k_m^2} + 5\frac{k^4}{k_m^4}\right)}{\left(1 + \frac{k^2}{k_m^2}\right)^4}\left(\frac{v_E}{v_{ti}}\right)^2\right] \quad (58)$$

where $\Delta\omega$ stands for the nonlinear frequency shift and $\omega = \omega_o/(1+\alpha^2)$ is the real part of mode frequency including the linear correction term. Note that the right side of Eq. (58) is a function of k/k_m and v_E/v_{ti}, and not a function of T_e/T_i.

An oft cited saturation mechanism is based on ion trapping. This is also a single-wave effect. For $v_E \gg v_{ti}$, Winske et al[12] estimated that ion trapping requires

$$\left(\frac{e\phi}{T_i}\right)_{trap} \geq C\left(1 + \frac{v_{ph}^2}{v_{ti}^2}\right) \qquad (59)$$

where C is order of 0.1 to fit their simulation data. For the low drift regime, i.e., $v_E < v_{ti}$, Chen et al[13] showed that above the threshold given in Eq. (59), saturation occurs when

$$(1 + \omega_{pe}^2/\omega_{ce}^2)\left(\frac{\mathcal{E}_s}{nT_i}\right)_{trap} = \frac{1}{45\sqrt{\pi}}\,\frac{1}{1 + \frac{k^2}{k_m^2}}\left(\frac{v_E}{v_{ti}}\right)^5 . \qquad (60)$$

This estimate was calculated on the basis of energy conservation; as the ions trap, the velocity distribution flattens in the neighborhood of the wave phase velocity $v \sim \omega/k$ liberating kinetic energy that is then converted into wave energy.[9,13] Simulations described in Ref. 13 demonstrated that the lower-hybrid drift instability was stabilized by ion trapping at amplitude consistent with Eq. (60) when v_E was kept constant in time.

Another possible saturation mechanism is stabilization via current relaxation,[9] which gives

$$(1 + \omega_{pe}^2/\omega_{ce}^2)\left(\frac{\mathcal{E}_s}{nT_i}\right)_{cr} = \frac{1}{8}\,\frac{m}{M}\left(\frac{v_E}{v_{ti}}\right)^2\left(1 + \frac{T_e}{T_i}\right)^2 . \qquad (61)$$

This mechanism can apply to single wave cases or turbulent conditions.

Finally, saturation due to electron resonance broadening[10,11,14] yields

$$(1+\omega_{pe}^2/\omega_{ce}^2)\left(\frac{\mathcal{E}_s}{nT_i}\right)_{rb} = \frac{1}{10}\frac{m}{M}\left(\frac{v_E}{v_{ti}}\right)^2\Big/\left(\frac{T_e}{T_i}\right)^{\frac{1}{4}} . \qquad (62)$$

for plasmas with nonzero T_e. This equation is given in Ref. 11 without derivation. This mechanism requires the presence of turbulence.

Figure 3 is a plot of $(\mathcal{E}/nT_i)_{\Delta\omega}$, $(\mathcal{E}_s/nT_i)_{trap}$ and $(\mathcal{E}_s/nT_i)_{cr}$ versus k/k_m for $M/m = 3672$, $v_E/v_{ti} = 0.3$, and $T_e/T_i = 0.25$. m is the electron mass. It is seen, except for $k \ll k_m$, that $(\mathcal{E}/nT_i)_{\Delta\omega}$ is much larger than $(\mathcal{E}_s/nT_i)_{trap}$ and $(\mathcal{E}_s/nT_i)_{cr}$; and thus the frequency shift is important only for $k < 0.1k_m$ at which the lower-hybrid drift mode is converted into a drift cyclotron mode.

In Fig. 4 we present $(\mathcal{E}/nT_i)_{\Delta\omega}$, $(\mathcal{E}_s/nT_i)_{trap}$, $(\mathcal{E}_s/nT_i)_{cr}$ and $(\mathcal{E}_s/nT_i)_{rb}$ versus v_E/v_{ti} for $M/m = 3672$, $T_e/T_i = 0.25$. For the parameters chosen, it can be seen that for $v_E/v_{ti} < 0.2$ the lowest saturation amplitudes are achieved by ion trapping. Again it shows that for finite k/k_m the nonlinear frequency shift is insignificant at the saturation amplitudes suggested by the other proposed saturation mechanisms. It is also noted that the saturation level due to current relaxation is only slightly higher than that of electron resonance broadening for these parameters; for $T_e/T_i = 0.25$, $(\mathcal{E}_s/nT_i)_{cr} \simeq 1.38(\mathcal{E}_s/nT_i)_{rb}$.

In conclusion, we have shown that a nonlinear frequency shift has a weakly stabilizing influence on the lower-hybrid drift instability. Ion trapping, current relaxation, and electron resonance broadening are more likely saturation mechanisms; which of these actually accounts for saturation depends on the plasma parameters.

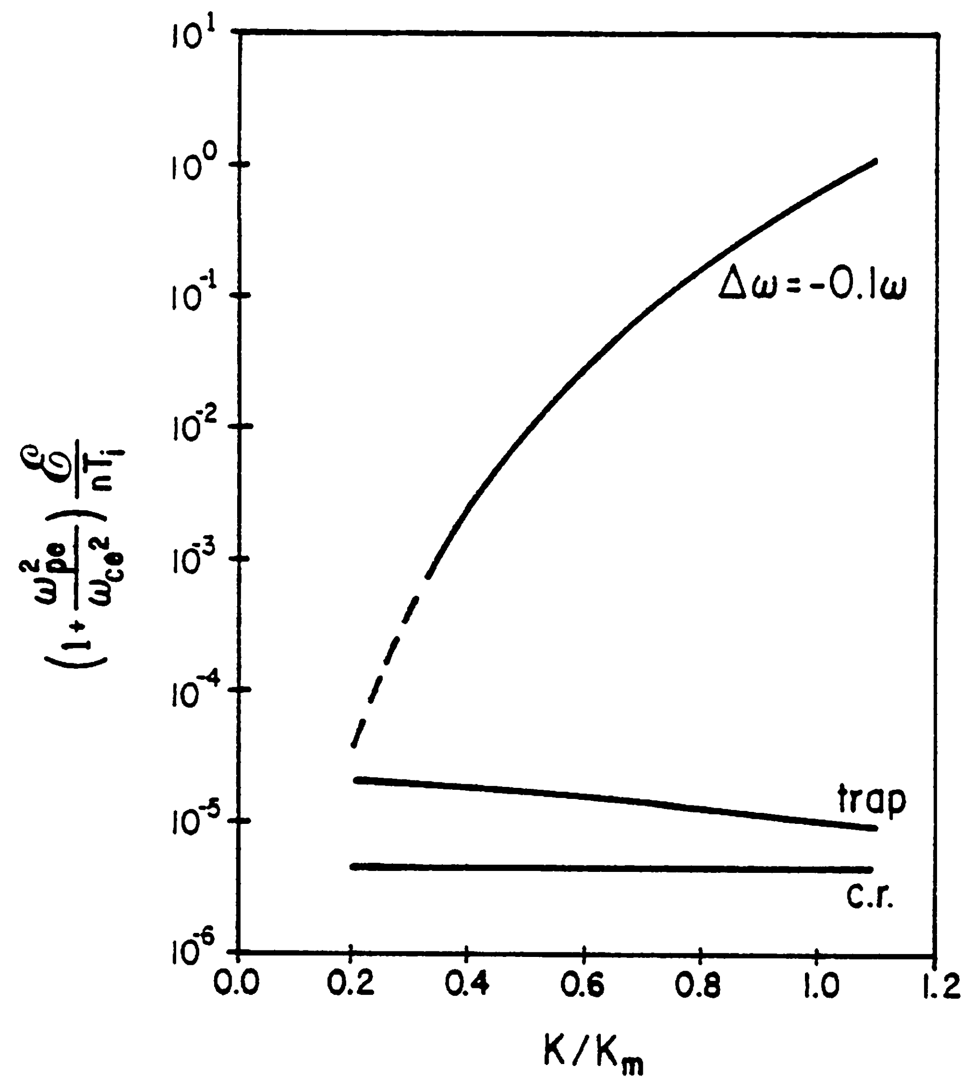

FIG. 3 Saturated lower-hybrid drift mode amplitudes $\mathcal{E}_s/nT_i = \langle E^2/8\pi nT_i\rangle$ as functions of k/k_m for $M/m = 3672$, $v_E/v_{ti} = 0.3$, and $T_e/T_i = 0.25$. Two saturation mechanisms are compared: current relaxation (c.r.) and ion trapping (trap). Also shown is the amplitude at which $\Delta\omega = -0.1\omega$. Only for $k/k_m < 0.1$ is $-\Delta\omega \geq 0.1\omega$ for $(\mathcal{E}_s/nT_i)_{cr}$ and $(\mathcal{E}_s/nT_i)_{trap}$.

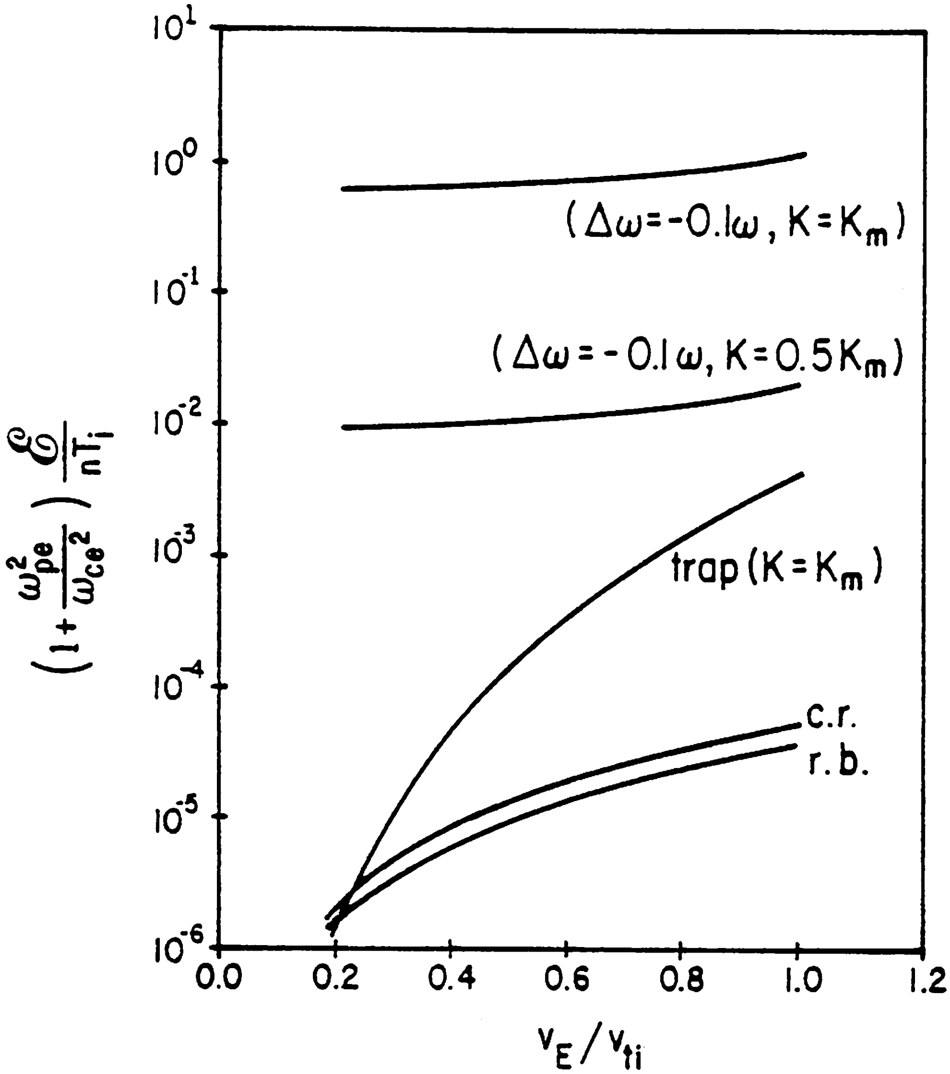

FIG. 4 Saturated lower-hybrid drift mode amplitudes $\mathcal{E}_s/nT_i$ as functions of v_E/v_{ti} for $M/m = 3672$ and $T_e/T_i = 0.25$. The amplitudes corresponding to three saturation mechanisms [ion traping (trap), current relaxation (c.r.), and electron resonance broadening (r.b.)] are compared to that for which $\Delta\omega=-0.1\omega$ for various values of k. Only for $k < 0.1k_m$ and moderate values of v_E/v_{ti} are the frequency shifts significant at the saturated amplitudes given.

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