SOJOURN TIMES AND THE OVERTAKING CONDITION

IN JACKSONIAN NETWORKS

by

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ABSTRACT

Consider an open multiclass Jacksonian network in equilibrium and a path such that a customer travelling along it cannot be overtaken by subsequent arrivals. Then the sojourn times of this customer in the nodes constituting the path are all mutually independent and so the total sojourn time is easily calculated. Two examples are given to suggest that the non-overtaking condition may be necessary to insure independence when there is a single customer class.

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1. Introduction

In 1957, Reich [1] proved that, in equilibrium, the sojourn times of a customer in each of two M/M/1 queues in tandem are independent, and in 1963 [2], he extended this result to an arbitrary number of such queues in tandem. This result was very recently extended by Lemoine [3] to the case of Jacksonian networks which are trees. Since trees have no parallel paths, and since the service discipline is FCFS, every path in a tree network has the non-overtaking property: a customer travelling along the path cannot be overtaken by a subsequent arrival.

The main result of this paper is to show that in any open multiclass Jacksonian network, the sojourn times of a customer at the various nodes of a non-overtaking path are all mutually independent. Since the distribution of the sojourn times at each node is known, it is easy to calculate the sojourn times for non-overtaking paths.

The paper shows that in a single-class three node network which has two parallel paths (Figure 3) the sojourn times at the various nodes are not all independent. Thus the non-overtaking condition cannot be generally relaxed. It is also shown that for any network with a single customer class the sojourn times along any path which permits overtaking cannot be independent at least under light traffic.

The paper is organized as follows. In section 2 the notion of a marked customer is made precise and some technical results are recalled. The definition of overtaking is given in section 3 which also contains two basic lemmas. The main result occupies section 4 and the "negative" examples section 5. Some concluding remarks are collected in section 6.

2. The marked customer

It is convenient to first recall some results about Poisson processes.

Let (S,Σ,P_N) be a probability space and $(\mathcal{F}_t)$, $t \in R_+$, an increasing family of sub-σ-fields. For $n = 1,\dots,d$ let $N^n = (N_t^n)$, $t \geq 0$, be independent $(\mathcal{F}_t)$-Poisson processes with rates λ^n. Let $N = (N_t^1,\dots,N_t^d)$, $t \geq 0$. We say that N is a (vector) $(\mathcal{F}_t)$-Poisson process with rate $\lambda = (\lambda^1,\dots,\lambda^d)$. The following statement is the strong Markov property for Poisson processes (see e.g. [4].)

Lemma 2.1. Let T be an a.s. finite $(\mathcal{F}_t)$-stopping time. Then $\hat{N}$ is a $(\hat{\mathcal{F}}_t)$-Poisson process with rate λ, where

$$\hat{N}_t = N_{T+t} - N_T,\ \hat{\mathcal{F}}_t = \mathcal{F}_{T+t},\ t \geq 0.$$

Consequently $\mathcal{F}_T$ and $\mathcal{G}_T = \sigma\{\hat{N}_t, t \geq 0\}$ are independent.

The result will be used below in a slightly different form. Consider the space $(S,\mathcal{F}_\infty^N,P_N)$. Recall that we can always assume that S consists of the space of sample paths of the Poisson process and N_t is just the coordinate map (see e.g. [5, Ch. XIII.]) In the remainder such a canonical representation will always be used. This allows us to define the translation operators $\{\theta_t\}$, $\{\hat{\theta}_t\}$, $t \geq 0$, by

$$N_s \cdot \theta_t(\sigma) = N_{s+t}(\sigma),\ N_s \cdot \hat{\theta}_t(\sigma) = N_{s+t}(\sigma) - N_t(\sigma),\ \sigma \in S,\ s \geq 0.$$

Simple calculations show that $\{\theta_t\}$, $\{\hat{\theta}_t\}$ are semigroups. Now let $(X,\mathcal{X},P_0)$ be another probability space, where X is a countable set. Define

$$(\Omega,\mathcal{X},P) = (X\times S,\mathcal{X}\times\mathcal{F}_\infty^N,P_0\times P_N),\ \mathcal{F}_t = \mathcal{X}\times\mathcal{F}_t^N.$$

Let (X_t), $t \in R_+$, be a $(\mathcal{F}_t)$-adapted, right-continuous process with values in X.

Lemma 2.2. Let T be an a.s. finite $(\mathcal{F}_t)$-stopping time. Then for all bounded $\mathcal{F}$- measurable functions ϕ

$$E\{\phi(X_T, N\cdot\hat{\theta}_T)|\mathcal{F}_T\} = \Phi(X_T),$$

where

$$\Phi(x) = \int_S \phi(x,\sigma)P_N(d\sigma), \quad x \in X.$$

Proof. Under P, $\mathcal{X}$ and $\mathcal{F}^N_\infty$ are independent so that on $(\Omega,\mathcal{F},P)$ N is a $(\mathcal{F}_t, P)$-Poisson process and Lemma 2.1 applies. Now, if ϕ is of the form $\phi(x,\sigma) = 1_A(x)1_B(N_t(\sigma))$ where $A \subset X$, $B \subset \mathbb{N}$, $t \geq 0$ are fixed and 1_A, 1_B are indicator functions, then

$$\begin{aligned} E\{\phi(X_T,N\cdot\hat{\theta}_T)|\mathcal{F}_T\} &= 1_A(X_T)P\{N_t\cdot\hat{\theta}_T\in B|\mathcal{F}_T\}, \text{ since } X_T \text{ is } \mathcal{F}_T\text{-measurable,} \\ &= 1_A(X_T)P\{\hat{N}_t\in B|\hat{\mathcal{F}}_0\} \\ &= 1_A(X_T)P\{N_t\in B\}, \text{ by Lemma 2.1,} \end{aligned}$$

and this is indeed equal to $\Phi(X_T)$ since, in this case,

$$\Phi(x) = \int_S 1_A(x)1_B(N_t)P_N(d\sigma) = 1_A(x)P\{N_t\in B\}$$

The general case now follows by a monotone class argument. ¤

Consider now a Jacksonian network $\mathcal{N}$ with N nodes and L customer types. The arrivals of external customers of class ℓ at node i form an independent Poisson process with rate γ_i^ℓ. Node i is an M/M/1 queuing system with FCFS discipline and service rate μ_i independent of customer class. A class ℓ customer who completes service at i changes into a class m customer and either immediately joins the queue at j with probability $r_{ij}^{\ell m}$ or leaves the network with probability $r_{i0}^{\ell m}$. Naturally $\sum_{j=0}^{N} \sum_{m=1}^{L} r_{ij}^{\ell m} = 1$. Let $\{\lambda_i^\ell\}$ be the solution, assumed unique, to the equations

$$\lambda_i^\ell = \gamma_i^\ell + \sum_{j=1}^{N}\sum_{m=1}^{L} \lambda_j^m r_{ji}^{m\ell}, \quad i = 1,\dots,N, \quad \ell = 1,\dots,L.$$

Set $\lambda_i = \sum_\ell \lambda_i^\ell$, and assume $\rho_i = \lambda_i\mu_i^{-1} < 1$.

Following [6] or [7] we give a precise description of the Markov process describing the evolution of the state of the network. Let $\bar{X} = \{1,\ldots,L\}^*$ be the set of all finite sequences of elements in $\{1,\ldots,L\}$ including the null sequence ϕ. The state space is $X = \bar{X}^n$, so that a state is an n-tuple $x = (x_1,\ldots,x_n)$ where x_i represents the customers in queue at node i with the right-most element in x_i, being the class of the customer in service and the left-most element, the class of the customer who arrived most recently.

We adopt the following notation. $a \cdot b$ is the concatenation of two elements in $\bar{X}$. Also, for $x_i \in \bar{X}$,

$a(x_i)$ = left-most element in x_i, $a(\phi) = 0$,

$d(x_i)$ = right-most element in x_i, $d(\phi) = 0$,

$|x_i|_\ell$ = number of customers in x_i of class ℓ,

$|x_i| = \Sigma|x_i|_\ell$ = number of customers in x_i,

and if $|x_i| > 0$ let $\hat{x}_i$ be obtained from x_i by deleting its right-most elements.

There are three types of possible state transitions and associated Poisson processes:

(i) Internal Transitions. For $1 \le i,j \le N$, $1 \le \ell,m \le L$, let $E_{ij}^{\ell m} = \{x \in X | d(x_i)=\ell\}$. $T_{ij}^{\ell m} : E_{ij}^{\ell m} \to X$ with $T_{ij}^{\ell m}(x_1,\ldots,x_N)$ $= (x_1,\ldots,\hat{x}_i,\ldots,m\, x_j,\ldots,x_N)$, and let $N_{ij}^{\ell m}$ be an independent Poisson process with rate $\mu_i r_{ij}^{\ell m}$.

(ii) External arrivals. For $1 \le i \le N$, $1 \le \ell \le L$, let $U_i^\ell = X$, $A_i^\ell : U_i^\ell \to X$ with $A_i^\ell(x_1,\ldots,x_N) = (x_1,\ldots,\ell \cdot x_i,\ldots,x_N)$, and let N_i^ℓ be an independent Poisson process with rate γ_i^ℓ.

(iii) External departures. For $1 \le i \le N$, $1 \le \ell, m \le L$, let $V_i^{\ell m} = \{x \in X \mid d(x_i) = \ell\}$, $D_i^{\ell m} : V_i^{\ell m} \to X$ with $D_i^{\ell m}(x_1, \ldots, x_N) = (x_1, \ldots, \hat{x}_i, \ldots, x_N)^i$, and let $N_{i0}^{\ell m}$ be an independent Poisson process with rate $\mu_i r_{i0}^{\ell m}$.

Let X_0 be an independent random variable with values in X; X_0 is the initial state. Then the state process (X_t), $t \ge 0$ is the unique right-continuous piecewise constant solution of the differential equation (2.1), (2.2) below. For $E \subset X$ let $\xi_t(E) = 1_E(X_t)$, $\xi_t(x) = 1_{\{x\}}(X_t)$.

$$\begin{aligned} d\xi_t(x) &= \sum_{ij\ell m} [\xi_{t-}((T_{ij}^{\ell m})^{-1}x) - \xi_{t-}(x)]\xi_{t-}(E_{ij}^{\ell m})dN_{ij}^{\ell m}(t) \\ &+ \sum_{i\ell} [\xi_{t-}((A_i^{\ell})^{-1}x) - \xi_{t-}(x)]\xi_{t-}(U_i^{\ell})dN_i^{\ell}(t) \\ &+ \sum_{i\ell m} [\xi_{t-}((D_i^{\ell m})^{-1}x) - \xi_{t-}(x)]\xi_{t-}(V_i^{\ell m})dN_{i0}^{\ell m}(t), \end{aligned} \tag{2.1}$$

$$\xi_0(x) = 1_{\{x\}}(X_0). \tag{2.2}$$

Let $N = (N_t^1, \ldots, N_t^d)$, $t \ge 0$, with rate $\lambda = (\lambda^1, \ldots, \lambda^d)$, be the collection of Poisson processes introduced above. (N_t) is given its canonical representation. The state process (X_t) is then $\sigma(X_0) \vee \mathcal{F}_t^N$ adapted, strong Markov process and $\sigma(X_0)$ is independent of $\mathcal{F}_\infty^N$. Assume that X_0 is given the equilibrium distribution P_0 given by

$$P_0(x_1, \ldots, x_N) = P^1(x_1) \ldots P^N(x_N), \tag{2.3}$$

$$P^i(x_i) = \rho_i^{|x_i|}(1-\rho_i) \prod_{\ell=1}^{L} (p_i^{\ell})^{|x_i|_\ell}, \tag{2.4}$$

where $p_i^{\ell} = \lambda_i^{\ell}\lambda_i^{-1}$. (See [6] or [8]). The state process (X_t) is now stationary.

Suppose now that at time 0 a customer of type ℓ_1, call him C, is introduced at the end of the queue at node 1 and that after he leaves node 1 he proceeds in sequence through nodes $2, 3, \ldots, n$, maintaining class identity ℓ_i while he is at node i. We wish to analyze the sojourn times of C at these nodes. To do this we need to augment the state description of the network from (X_t) given above to $(\tilde{X}_t)$ say, so that at t the position of C in the network is given by $\tilde{X}_t$. We do this simply by increasing the number of customer classes to 2L with zero external arrival rates for the new classes L+1,...,2L, and by agreeing to "mark" the customer C as being of class L+ℓ whenever "unmarked" he would be of class ℓ. The process $(\tilde{X}_t)$ now satisfies a differential equation analogous to (2.2) and with an initial distribution $\tilde{P}_0$ of $\tilde{X}_0$ which is obtained from the steady-state distribution P_0 of $X_0 = (X_{10},\ldots,X_{N0})$, by simply adding of type a customer of type $L+\ell_1$ (namely C) to the left of X_{10}. The process $(\tilde{X}_t)$ is then also a right-continuous, strong Markov process. It is, however, not in equilibrium and so we may not directly apply known equilibrium results of Jacksonian networks. (This point is somewhat overlooked in [1]-[3].)

Definition 2.1. Let $T_0 \equiv 0$ and T_i the departure time of C from node i, i = 1,...,n. $S_i = T_i - T_{i-1}$ is the sojourn time of C at i.

Evidently T_i is a stopping time of the process $(\tilde{X}_t)$. We show in section 4 that $S_1,\ldots,S_n$ are all independent if the path (1,...,n) does not permit overtaking.

3. The non-overtaking condition

Recall the routing probabilities $\{r_{ij}^{\ell m}\}$ introduced previously.

Definition 3.4. For $1 \leq i,j \leq N$, write $i \to j$ if for some ℓ,m $r_{ij}^{\ell m} > 0$. Let

$$P_{ij} = \{\pi=(i,i_1,\ldots,i_m,j) \mid i\to i_1, i_1\to i_2,\ldots,i_m\to j, \text{ for some } i_1,\ldots,i_m\}$$

$$P_{ij}^k = \{(\pi,\pi') \mid \pi\in P_{ik}, \pi'\in P_{kj}\}.$$

Thus P_{ij} is the set of all paths in the network going from i to j and P^k_{ij} consists of those paths which in addition go via k. Note that the path (1,...,n) taken by C is in P_{1n}.

Definition 3.2. The path $\pi = (i_1,i_2,\ldots,i_m) \in P_{i_1 i_m}$ permits no overtaking if

$$P_{i_u i_v} \subset P^{i_{n+1}}_{i_n i_v}, \text{ for } 1 \le n < v \le m.$$

The condition means that all paths from i_n to i_v must go through i_{n+1}; hence a customer who traverses $i_1,\ldots,i_m$ cannot be "overtaken" by any customer who enters i_1 after him. In Figure 3, the path (1,3) permits no overtaking, but the path (1,2,3) does because (1,3) does not go through 2.

Assumption. In the remainder of the section and the next section it is assumed that the path (1,2,...,n) permits no overtaking and the nodes i = 1,...,n-1 are without self-loop, i.e., $r^{\ell m}_{ii} = 0$, all ℓ,m.

Definition 3.3. For i = 1,...,n-1, let

$$P_i = \{j \mid P^i_{jk} \subsetneq P_{jk} \text{ for some } k \in \{i+1,\ldots,n\}\}.$$

Thus P_i consists of all nodes j from which there is a path reaching {i+1,...,n} without going through i. See Figure 1.

Lemma 3.1. (i) $j \notin P_i$ for $1 \le j \le i \le n-1$

(ii) $\{i\} \cup P_i \subset P_{i-1}$, $2 \le i \le n-1$

Proof (i) By the assumption every path from j to k for $j \le i < k \le n$ must go through i that is, $P_{jk} = P^i_{jk}$.

(ii) $i \in P_{i-1}$ since (i,i+1) is a path. Next let $j \in P_{i+1}$. We show that $j \in P_i$. Let $k \in \{i+2,\ldots,n\}$ and $\pi \in P_{jk}$ such that $\pi \notin P^{i+1}_{jk}$. It is enough to show that $\pi \in P^i_{jk}$. If $\pi \in P^i_{jk}$ then $\pi' = (\pi',\pi'')$ with

$\pi' \in P_{ji}$ and $\pi'' \in P_{ik}$. But then $\pi'' \in P_{ik}^{i+1}$ since $(1,2,\ldots,n)$ permits no overtaking, and so $\pi \in P_{jk}^{i+1}$ which is a contradiction. ◻

In the remainder of this section and the next we shall denote the augmented state process $\tilde{X}_t$ by X_t. There should be no confusion since we will not need to refer to the unaugmented process. For any subset of nodes $J \subset \{1,\ldots,N\}$ let $(X_t^J) = \{X_{it} | i \in J\}$ denote those components of X_t which correspond to the nodes in J. It is convenient to introduce a new node 0 to which go all the customers who leave the original network. (This is in keeping with the notation $r_{i0}^{\ell m}$, $N_{i0}^{\ell m}$.)

For $i = 1,\ldots,n-1$, let $Q_i = P_i \cup \{i\}$, $P_i^c = \{j | 1 \le j \le N, j \notin P_i\}$,

$$D_i = P_i^c \cup \{0\},$$

For $i = 1,\ldots,n-1$ define the vector processes N_i, $\tilde{N}_i$ as follows:

$$N^i = \{N_{kj}^{\ell m}, N_r^{\ell} | j \in Q_i, r \in P_i, 0 \le j \le N, 1 \le \ell, m \le N\}$$

$$\bar{N}^i = N_{kj}^{\ell m}, \tilde{N}_{k0}^{\ell m} = \sum_{j' \in D_i} N_{kj'}^{\ell m}, N_r | k \in Q_i, r \in P_i, j \in P_i, 1 \le \ell, m \le N\} \tag{34}$$

Thus N^i consists of the Poisson processes associated with internal transitions and departures from nodes in Q_i together with external arrivals into P_i; $\bar{N}^i$ consists of these same processes except that customers leaving P_i are not distinguished by their destination. Later $\bar{N}^i$ will be used to construct a simpler network equivalent to $\mathcal{N}$.

Recall Definition 2.1. The next lemma summarizes the crucial observation that after T_{i-1} the progress of C depends only upon the state of the queues in nodes Q_i at T_{i-1} and the processes $\bar{N}^i$ after T_{i-1}.

<u>Lemma 3.2</u>. For $i = 1,\ldots,n-1$, there is a measurable mapping ϕ_i depending only on P_i such that

$$(S_i, X_{T_i}^{P_i}) = \phi_i(X_{T_{i-1}}^{Q_i}, \bar{N}^i \cdot \hat{\theta}_{T_{i-1}}).$$

("depending only on P_i" means that ϕ_i is the same for all networks with N nodes L classes for which $(1,\ldots,n)$ permits no overtaking and which have the same set P_i.)

Proof. Evidently, there is a function f_i, depending only on P_i, such that

$$X_{T_i}^{P_i} = f_i(S_i, X_{T_i}^{P_i}, \bar{N}^i \cdot \hat{\theta}_{T_{i-1}}, 1_{[0,S_i]}(\cdot) F^i \cdot \hat{\theta}_{T_{i-1}}), \tag{3.2}$$

where F_i is the set of flows of customers going from P_i^c to P_i and $1_{[0,S_i]}(\cdot)$ is the indicator of the random time interval $[0,S_i]$. By Definition 3.3, F_i is simply the flows from i to P_i (See Figure 1). Hence there exist functions g_i and h_i, depending only on P_i, such that

$$S_i = g_i(X_{T_{i-1}}^i, \bar{N}^i \cdot \hat{\theta}_{T_{i-1}}) \tag{3.3}$$

$$1_{[0,S_i]} F^i \cdot \hat{\theta}_{T_{i-1}} = h_i(S_i, X_{T_{i-1}}^i, \bar{N}^i \cdot \hat{\theta}_{T_{i-1}}) \tag{3.4}$$

The assertion follows from (3.2), (3.3), (3.4). ◻

Corollary 3.1. There is a mapping ϕ, depending only on $(P_1,\ldots,P_{n-1})$, such that

$$(S_1,\ldots,S_n) = \phi(X_0^{Q_1}, \bar{N}^1)$$

Proof. Assume, as induction hypothesis, that for some f_{i-1} depending only on $(P_1,\ldots,P_{i-1})$,

$$(S_1,\ldots,S_{i-1}, X_{T_{i-1}}^{P_{i-1}}) = f_{i-1}(X_0^{Q_1}, \bar{N}^1) \tag{3.5}$$

By Lemma 3.2, for $i \leq n-1$,

$$(S_i, X_{T_i}^{P_i}) = \phi_i(X_{T_{i-1}}^{Q_i}, \bar{N}^i \cdot \hat{\theta}_{T_{i-1}}).$$

By Lemma 3.1, $Q_i \subset P_{i-1}$ and $\bar{N}^i$ is a subvector of $\bar{N}^1$. Hence (3.5) is proved for $i = 2,\ldots,n$. For S_n, observe that

$$S_n = \phi_n(X^n_{T_{n-1}}, \tilde{N}^n \cdot \hat{\theta}_{T_{n-1}})$$

where $\tilde{N}^n$ is obtained as in (3.1) by setting $P_n = \phi$, $Q_n = \{n\}$, $D_n = \{0,1,\ldots,N\}$. The assertion now follows since $n \in P_{n-1}$ and $\tilde{N}^n$ is a subvector of $\bar{N}^1$. ¤

4. The main result.

We show here that the sojourn times $S_1,\ldots,S_n$ of C are all independent. Moreover S_i has the same distribution as the sojourn time of a customer in an M/M/1 queue (in equilibrium) with arrival rate λ_i and service rate μ_i. Recall that λ_i is the total average arrival rate into node i.

We first give an outline of the proof (see Figure 2). The idea is to reduce the problem to the path $(2,\ldots,n)$. To do this it will be shown that in $\mathcal{N}$.

a) S_1 and $(S_2,\ldots,S_n)$ are independent;

b) $(S_2,\ldots,S_n)$ has the same distribution as that of the sojourn times of C if C were introduced into the network at node 2 with $\mathcal{N}$ being in equilibrium.

To prove a) we introduce a simpler network $\tilde{\mathcal{N}}$ (Definition 4.1) such that $(S_1,\ldots,S_n)$ have the same distribution in $\mathcal{N}$ and $\tilde{\mathcal{N}}$ (Lemma 4.2) and for which S_1 and $(S_2,\ldots,S_n)$ are independent (Lemma 4.4). To prove b) we show that in $\mathcal{N}$, $X^{P_1}_{T_1-}$ and $X^{P_1^c}_{T_1}$ both have the equilibrium distribution and then b) follows from the fact that (X_t) is a strong Markov process and $T_1 = S_1$ is a stopping time of (X_t). ($X^{P_1}_{T_1-}$ and $X^{P_1^c}_{T_1}$ do not contain C so that the equilibrium distribution is well defined and given by (2.3), (2.4).)

Definition 4.1. (See Fig. 2) Let $\tilde{\mathcal{N}}$ be the network obtained from $\mathcal{N}$ by changing the Poisson processes associated with the latter in the following way:

For $1 \le i \le N$, $0 \le j \le N$, $1 \le \ell, m \le L$,

$$\tilde{N}_{ij}^{\ell m} = \begin{cases} 0 & \text{if } (i,j) \in Q_1 \times P_1^c \\ \sum_{j' \in D_1} N_{ij'}^{\ell m}, & \text{if } (i,j) \in Q_1 \times \{0\} \\ N_{ij}^{\ell m}, & \text{otherwise,} \end{cases}$$

and for $1 \le i \le N$, $1 \le \ell \le L$,

$$\tilde{N}_i^{\ell} = \begin{cases} N_i^{\ell} & \text{if } i \in P_1 \\ \text{an independent Poisson process with rate } \lambda_i^{\ell}, & \text{if } i \in P_1^c. \end{cases}$$

Essentially, $\tilde{\mathcal{N}}$ is obtained by forcing customers who, in $\mathcal{N}$, moved from Q_1 and P_1^c to leave the network and then "compensating" the nodes in P_1^c by external Poisson arrivals with the same average rate. Thus in $\tilde{\mathcal{N}}$ the average rate of arrivals of any class at any node is the same as the corresponding rate in $\mathcal{N}$.

Lemma 4.1. The following elements in $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are the same:

(i) the vector process $\bar{N}^1$ (and hence $\bar{N}^i, \ldots, i=2,\ldots,n-1$).

(ii) the subsets $P_1, \ldots, P_{n-1}$

(iii) the equilibrium distribution of the unmarked state.

Proof. (ii) is immediate and (i) follows by checking that (3.1) yields the same process $\bar{N}^1$ whether it is obtained from N^1 or $\tilde{N}^1$. (iii) follows from (2.3), (2.4) since the λ_i^{ℓ} are the same for $\mathcal{N}, \tilde{\mathcal{N}}$. □

Suppose now that at time 0- the unmarked state of $\tilde{\mathcal{N}}$, which we continue to denote by X_{0-}, has the equilibrium distribution given by (2.3), (2.4). At time 0 we introduce customer C at the end of the queue in node 1, i.e., to the left of X_{0-}^1. Let T_i, S_i continue to denote the various departure and sojourn times of C in $\tilde{\mathcal{N}}$.

Lemma 4.2. $\{S_i, X_{T_i}^{P_i} | i=1,\ldots,n-1\}$ and $\{S_1,\ldots,S_n\}$ have the same distribution in $\mathcal{N}$ and $\tilde{\mathcal{N}}$.

Proof Follows from Corollary 3.1, and Lemma 4.1. ◻

Lemma 4.3. Let z^1, z^2 be independent random variables. Then

$$E\{h(f(z^1),z^2)|f(z^1),g(z^1)\} = E\{h(f(z^1),z^2)|f(z^1)\}$$

for any bounded measurable function h and measurable functions f, g.

Proof. This is obvious if $h(f(z^1),z^2) = h_1(f(z^1))h_2(z^2)$. The result follows from this using a monotone class argument. ◻

Lemma 4.4. In $\tilde{\mathcal{N}}$, S_1 and $(S_2,\ldots,S_n)$ are independent.

Proof. It is convenient to denote

$$P = P_1,\ Q = Q_1 = P_1 \cup \{1\},\ R = \{i|1\leq i\leq N, i\notin Q\}.$$

The proof is divided into several steps.

Step 1. $P(X_{S_1}^1 | X_{S_1}^P, S_1) = P(X_{S_1}^1 | S_1)$. (4.1)

To see this define the following vector processes:

α = flows of customers from R to 1,

β = external arrivals into 1,

γ = external arrivals and service processes in R.

By Lemma 3.2, since $T_1 = S_1$,

$$(S_1, X_{S_1}^P) = \phi_1(X_{0-}^1, X_0^P, \bar{N}^1). \tag{4.2}$$

By definition of S_1, we have (see Fig. 2), $X_{S_1}^1$ is measurable with respect to $\sigma(S_1,\alpha,\beta)$. Also $\sigma(\alpha) \subset \sigma(X_0^R,\gamma)$. Hence

$$X_{S_1}^1 \text{ is measurable with respect to } \sigma(S_1, X_0^R, \beta, \gamma). \tag{4.3}$$

But

$$\sigma(X_0^R,\beta,\gamma) \text{ and } \sigma(X_{0-}^1, X_0^P, \bar{N}^1) \text{ are independent} \tag{4.4}$$

and so (4.1) follows by applying Lemma 4.3 to (4.2), (4.3), (4.4).

Step 2. $X^1_{S_1}$ and $X^P_{S_1}$ are independent (4.5)

By construction of $\tilde{\mathcal{N}}$ the link joining nodes 1 and 2 is not part of any loop. Hence, by the output theorem for Jacksonian networks (see e.g. [6] or [7]), for the unmarked state $X^1_{S_1}$, $X^P_{S_1}$ are independent. Now at time $S_1 = T_1$, the position of C is known as a function of $X^P_{S_1}$ since C is at the end of queue at node 2. Hence $X^1_{S_1}$, $X^P_{S_1}$ are independent for the marked state as well.

Step 3. S_1 and $X^P_{S_1}$ are independent (4.6)

This is proved by using a technique of Reich [1]. Let $W = X^P_{S_1}$ and let V be the number of customers in node 1 at S_1. For any complex number z, we find

$$E\{z^V|W=w\} = \int_0^\infty E\{z^V|W=w,S_1=t\}dP\{S_1\le t|W=w\}$$

$$= \int_0^\infty E\{z^V|S_1=t\}dP\{S_1\le t|W=w\}, \text{ by (4.1)}.$$

Now, again by the output theorem, the arrivals into node 1 form a Poisson process with rate λ_1 and so

$$E\{z^V|S_1=t\} = \int_0^\infty z^v \exp(-\lambda_1 t)\frac{(\lambda t)^v}{v!} = \exp(z-1)\lambda_1 t.$$

Also, by (4.5), z^V and W are independent. Hence

$$E\{z^V\} = \int_0^\infty \exp(z-1)\lambda_1 t \; dP\{S_1\le t|W=w\}$$

Since the left-hand side does not depend on w it follows that S_1 and W are independent, and (4.6) is proved.

Step 4. S_1 and $(S_2,\ldots,S_n)$ are independent.

First observe that by the output theorem for Jacksonian networks, the arrivals into node 1 form independent Poisson processes. Hence (see Figure 2) (X_t^Q) is a strong Markov process and S_1 is its stopping time. Therefore, by the strong Markov property,

$$P\{S_2,\ldots,S_n|S_1,X_{S_1}^Q\} = P\{S_2,\ldots,S_n|X_{S_1}^Q\}. \tag{4.7}$$

Next we claim that

$$P\{S_2,\ldots,S_n|X_{S_1}^1,X_{S_1}^P\} = P\{S_2,\ldots,S_n|X_{S_1}^P\}. \tag{4.8}$$

To prove this it is evidently enough to show that

$$X_{S_1}^1 \text{ and } \{S_2,\ldots,S_n,X_{S_1}^P\} \text{ are independent.} \tag{4.9}$$

But by repeated use of Lemma 3.2 and the fact that $P_2 \cup \{2\} \subset P_1 = P$ we see that

$$(S_2,\ldots,S_n) \text{ measurable with respect to } \sigma(X_{S_1}^P,\bar{N}^2\cdot\hat{\theta}_{S_1}); \tag{4.10}$$

Moreover

$$X_{S_1} \text{ and } \bar{N}^2 \cdot \hat{\theta}_{S_1} \text{ and independent.} \tag{4.11}$$

From (4.5), (4.10), (4.11) it is easy to conclude (4.9). From (4.7), (4.8)

$$P\{S_2,\ldots,S_n|S_1,X_{S_1}^Q\} = P\{S_2,\ldots,S_n|X_{S_1}^P\},$$

and conditioning both sides with respect to $(S_1,X_{S_1}^P)$ gives

$$P\{S_2,\ldots,S_n|S_1,X_{S_1}^P\} = P\{S_2,\ldots,S_n|X_{S_1}^P\}. \tag{4.12}$$

Finally, denoting $W = X_{S_1}^P$,

$$P(S_1,\dots,S_n) = \sum_w P(S_1,\dots,S_n,W{=}w) = \sum_w P(S_2,\dots,S_n|S_1,W{=}w)P(S_1,W{=}w)$$

$$= \sum_w P(S_2,\dots,S_n|W{=}w)P(S_1,W{=}w), \text{ by } (4.12)$$

$$= \sum_w P(S_2,\dots,S_n|W{=}w)P(S_1)P(W{=}w), \text{ by } (4.6)$$

$$= P(S_2,\dots,S_n)P(S_1).$$

The lemma is proved. □

Theorem 4.1. In $\mathcal{N}$ the sojourn times $S_1,\dots,S_n$ are independent. Moreover S_i has the same distribution as the sojourn time in an M/M/1 queue with arrival rate λ_i and service rate μ_i.

Proof. By Lemmas 4.2, 4.4 S_1 and $\{S_2,\dots,S_n\}$ are independent. Moreover X^1_{0-} has the equilibrium distribution given by (2.4) and so S_1 is distributed as asserted. Now in the network $X^P_{S_1-} = X^P_{T_1-}$ also has the equilibrium distribution given by (2.3), (2.4) [9]. We can apply the argument above to $\mathcal{N}$ and construct an equivalent network $\tilde{\tilde{\mathcal{N}}}$ to conclude that S_2 and $\{S_3,\dots,S_n\}$ are independent with S_2 distributed as asserted. The result follows by successive repetitions. □

5. Paths which permit overtaking

Consider the network of Figure 3 with 3 nodes and only one class of customers. In terms of the notation of Section 2, and dropping the superscript since there is only class, we have the following parameters:

arrival rates: $\gamma_1 = \lambda$, $\gamma_2 = 0$, $\gamma_3 = 0$,

routing probabilities: $r_{12} = p = 1-r_{13} = 1-q$; $r_{23} = 1$; $r_{30} = 1$,

service rate at node i is μ_i, $i = 1,2,3$.

We assume that $\lambda < \mu_1$, $\lambda < \mu_3$, $\lambda \cdot p < \mu_2$. Observe that the path (1,2,3). permits overtaking since the path (1,3) does not go through node 2.

As before let $X_{0-} = (X^1_{0-}, X^2_{0-}, X^3_{0-})$ be given the equilibrium distribution defined by (2.3), (2.4) and obtain X_0 by adding C to the left of X^1_{0-}. Suppose C takes the path (1,2,3) and let T_i, S_i be the corresponding departure and sojourn time.

Theorem 5.1. (i) S_1 and S_2 are independent; S_2 and S_3 are independent (ii) S_1 and S_3 are not independent.

Proof. (i) The paths (1,2) and (2,3) do not permit overtaking and so the result follows from Theorem 4.1.

(ii) The idea behind the proof is this. If S_1 is large, then C is likely to leave behind him many customers in node 1. Therefore it is likely that some of these will overtake C, by using the path (1,3), and arrive at node 3 before C. Thus at node 3 C will find a larger queue, thereby increasing S_3. This reasoning suggests that S_1 and S_3 are positively correlated. We now give a formal proof.

By the strong Markov property (applied to the augmented state process and its stopping time S_1) it follows that

$$a(n;x_2,x_3) = E\{S_3 | X^2_{S_1} = x_2, X^3_{S_1} = x_3, X^1_{S_1} = n\}$$

is well-defined. We claim that for all (x_2,x_3,n)

$$a(n+1,x_2,x_3) > a(n;x_2,x_3) \tag{5.1}$$

To see this consider Figure 4 and denote by ω a realization of all the independent Poisson processes involved in the description of the system and of (X^2_{0-}, X^3_{0-}). Let v denote a realization of X^1_{0-}. Now introduce C to the left of X^2_{0-}. Let P denote the probability obtained by giving X_{0-} the equilibrium distribution. Then it should be clear that $S_3(\omega,v+1) \geq S_3(\omega,v)$ for all ω,v. (Indeed, since $S_2(\omega,v+1) \equiv S_2(\omega,v)$ it suffices to show that $X^3_t(\omega,v+1) \geq X^3_t(\omega,v)$, $t \geq 0$ and this obvious.)

Moreover, it is easy to exhibit a set of realizations with positive probability for which $S_3(\omega,v+1) > S_3(\omega,v)$; and then (5.1) follows using the strong Markov property.

Now it is known [9] that the unmarked state at time S_1 also has the equilibrium distribution. Hence $X^1_{S_1}$ and $(X^2_{S_1}, X^3_{S_1})$ are independent. Also

$$\begin{aligned} a(n) &= E\{S_3 | X^1_{S_1}=n\} \\ &= \sum_{x_2,x_3} E\{S_3 | X^2_{S_1}=x_2, X^3_{S_1}=x_3, X^1_{S_1}=n\} P(X^2_{S_1}=x_2, X^3_{S_1}=x_3) \\ &= \sum_{x_2,x_3} a(n;x_2,x_3)p(x_2,x_3), \text{ say} \end{aligned}$$

Since $p(x_2,x_3) > 0$ for all $x_2 \geq 0$, $x_3 \geq 0$, it follows from (5.1) that

$$a(n+1) > a(n), \quad n \geq 0.$$

Next observe that

$$b(n|t) = P\{X^1_{S_1} \geq n | S_1=t\} = \sum_{m=n}^{\infty} \frac{(\lambda t)^m}{m!} \exp(-\lambda t),$$

is such that

$$b(n|t') > b(n|t) > 0, \quad n \geq 0, \quad t' > t > 0.$$

Finally, let

$$\begin{aligned} c(t) &= E\{S_3|S_1=t\} = \sum_{n=0}^{\infty} E\{S_3 | X^1_{S_1}=n, S_1=t\} P\{X^1_{S_1}=n | S_1=t\} \\ &= \sum_{n=0}^{\infty} E\{S_3 | X^1_{S_1}=n\} P\{X^1_{S_1}=n | S_1=t\}, \text{ by the strong Markov property,} \\ &= \sum_{n=0}^{\infty} a(n)[b(n|t)-b(n+1|t)] \end{aligned}$$

We claim that

$$c(t') > c(t), \quad t' > t > 0$$

To prove this observe that $\sum_{n=0}^{\infty} a(n) = ES_3 < \infty$ and $\sum_{n=0}^{\infty} b(n|t)$ $= E\{X^1_{S_1} | S_1=t\} = \lambda t < \infty$ so that the following calculations are justified. Let $t' > t > 0$. Then

$$c(t') - c(t) = \sum_{n=0}^{\infty} a(n)\{[b(n|t')-b(n+1)|t')]-[b(n|t)-b(n+1|t)]\}$$

$$= \sum_{n=0}^{\infty} \{a(n)[b(n|t')-b(n|t)]-a(n)[b(n+1|t')-b(n+1|t]\}$$

$$\geq \sum_{n=0}^{\infty} \{a(n)[b(n|t')-b(n|t)]-a(n+1)[b(n+1|t')-b(n+1|t)]\}$$

$$= a(0)[b(0|t')-b(0|t)] > 0.$$

This shows that

$$E\{S_3|S_1=t'\} > E\{S_3|S_1=t\}, \; t' > t > 0$$

and so S_1, S_3 are not independent. □

For our second example we return to the N node network discussed earlier. Suppose there is a single customer class. Let $\varepsilon\gamma = \varepsilon(\gamma_1,\ldots,\gamma_N)$, $0 < \varepsilon < 1$, be the external arrival rates, $\{r_{ij}\}$ the routing probabilities and $\varepsilon\lambda = \varepsilon(\lambda_1,\ldots,\lambda_N)$ the average total arrival rates.

Suppose the path $(1,\ldots,n)$ permits overtaking. Let $i \geq 2$ be the largest integer such that $(1,\ldots,i)$ permits no overtaking but $(1,\ldots,i+1)$ does. From now on let $i+1 = n$. And let $\pi = (1'=1,2',\ldots,m',n)$ be a path "parallel" to $(1,\ldots,n)$ as in Figure 6.

As before let C enter node 1 at $T_0 = 0$ when the network is in equilibrium. Suppose C leaves i at T_i and let $S_i = T_i-T_{i-1}$. S_i has the same distribution as the sojourn time for a M/M/1 queue with input rate $\varepsilon\lambda_i$ and service rate μ_i, and so

$$P\{S_i \leq s\} = 1 - \exp\{- \mu_i(1-\rho_i)s\}, \quad s \geq 0 \tag{5.2}$$

where $\rho_i = \varepsilon\lambda_i\mu_i^{-1}$. Also

$$P\{T_{n-1}-T_1 \leq s\} = P\ S_2+\ldots S_{n-1} \leq s\}, \tag{5.3}$$

where the S_i are distributed as in (5.2) and moreover they are independent by Theorem 4.1 since $(1,\ldots,n-1)$ permits no overtaking.

We shall show that S_1 and S_n are dependent, at least for $\varepsilon > 0$ sufficiently small. We adapt an argument of Burke [10].

Let E be the event that there is a customer, say C', in service at node $1 = 1'$ at time T_1 which is when C leaves 1. The probability that C' takes path π' is

$$r' = r_{1'2'}r_{2'3'}\ldots r_{(m-1)'m'}r_{m'n}$$

$r' > 0$ since π' is a path (Definition 3.1).

For each node n let M_n be an independent random variable with the service time distribution,

$$P\{M_n \leq y\} = 1 - \exp\{-\mu_n y\} \tag{5.4}$$

and let

$$Q = M_{1'}+\ldots+M_{m'}.$$

The probability that C will encounter a busy server at node n conditioned on the event E is $P\{X^n_{T_{n-1}^-}>0|E\}$.

<u>Lemma 5.1</u>. $P\{X^n_{T_{n-1}^-}>0|E\} \geq r'P\{Q<T_{n-1}-T_1<Q+M_n\}$.

<u>Proof</u>. C' is in service at 1' at T_1, and so will complete service at $T_1 + M_{1'}$, and enter 2' with probability $r_{1'2'}$. If C' finds 2' empty, he will immediately enter service. If C' finds another customer in service call the latter C'. Then C' will complete service at 2' at time $T_1 + M_{1'} + M_{2'}$ and move to 3' with probability $r_{2'3'}$. In any case,

conditioned on E, C' will enter 3' at $T + M_{1'} + M_{2'}$ with probability $r_{1'2'}r_{2'3'}$. Continuing in this way, and renaming customers if necessary, C' will enter node n at time $T_1+M_{1'}+\ldots+M_{m'} = T_1+Q$ with probability r'. If C' encounters a customer in service at n call the latter C'. Then C' will leave n at $T_1 + Q + M_n$ so that C' is in service at n during the random interval (T_1+Q, T_1+Q+M_n) and so he will block C if the latter's arrival time at n, T_{n-1}, falls in this interval. The assertion is proved. ⌑

<u>Lemma 5.2</u>. There is $\delta > 0$ such that for all $0 \le \varepsilon \le 1$,

$$b(\varepsilon) = r'P\{Q<T_{n-1}-T_1<Q+M_n\} > \delta.$$

<u>Proof</u>. The "blocking" probability is given by

$$b(\varepsilon) = r'P\{\sum_1^m M_{k'} < \sum_2^{m-1} S_i < \sum_1^m M_{k'}+M_n\}$$

where $M_{k'}$ is distributed as in (5.4), S_i as in (5.2), and all of them are independent. For any ε $b(\varepsilon) > 0$, since, for instance, for fixed numbers $0 < a < b$,

$$P\{\sum_1^m M_{k'} < \sum_2^{m-1} S_i < \sum_1^m M_{k'}+M_n\} > P\{\sum_1^m M_{k'}<a\}P\{a < \sum_2^{m-1} S_i<b\}P\{M_n>b\} > 0.$$

Moreover $b(\varepsilon)$ varies continuously with ε and so the assertion follows. ⌑

Customers arrive into node 1 at an average rate $\varepsilon\lambda_1 > 0$. They need not arrive in a Poisson stream, but since the unmarked process is stationary, there is $\tau(\varepsilon) < \infty$ such that with probability at least one-half a customer arrives at node 1 if $S_1 = T_1 > \tau(\varepsilon)$, i.e.,

$$P(E|S_1>\tau(\varepsilon)) = P(X^1_{T_1}>0|S_1>\tau(\varepsilon)) \ge 1/2.$$

Combining this with Lemmas 5.1, 5.2 gives the estimate

$$P(X^n_{T_{n-1}^-}>0|S_1>\tau(\varepsilon)) > 1/2\ \delta. \tag{5.5}$$

Now S_n is the service time for $1 + X^n_{T_{n-1}^-}$ customers. Hence from (5.5)

$$E\{S_n|S_1>\tau(\varepsilon)\} > (1/2\ \delta+1)\mu_n^{-1}.$$

On the other hand the unconditional distribution of $X^n_{T_n^-}$ is just its equilibrium distribution [9] and so,

$$ES_n = (1-\rho_n)^{-1}\mu_n^{-1},$$

and so for ε, equivalently ρ_n, sufficiently small

$$E\ S_n|S_1>\tau(\varepsilon)\} > ES_n$$

so that S_1 and S_n are dependent.

Theorem 5.2. Let be a Jacksonian network with a single class. Along any path which permits overtaking the various sojourn times are not all independent for sufficiently low traffic intensities.

6. Concluding Remarks

In networks which are trees every path permits no overtaking and so the sojourn times at the various nodes are independent. Thus the results of [3] follow from Theorem 4.1. One interesting example to consider is the so-called "full duplex" system of Figure 6 in which there are two classes of customers, the first travelling right to left and the second travelling in the opposite direction. By Theorem 4.1 the sojourn times of each customer class at various nodes are independent. It may be worth recalling here that by the output thereom [6,7] customers of each class leave the system in a Poisson stream, although by the example in [7] the flow of customers between any two adjacent nodes is not Poisson.

Theorem 5.2 establishes a strong presumption in favor of the conjecture that the independence holds only along paths which permit no overtaking. Suppose that of a node in the network has an M/M/m queuing system with m $\geq$ 2. The existence of parallel servers clearly permits overtaking and this suggests that independence will not hold along a path containing such a node. This has been shown by Burke [10] for light traffic in a tradem connection of 3 nodes in which the middle node is M/M/m, m $\geq$ 2 and the extreme nodes are M/M/1. It should be possible to extend this result.

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FIGURE CAPTIONS

Figure 1. (1,2,3,4) permits no overtaking.

Figure 2. Sojourn times in $\mathcal{N}, \tilde{\mathcal{N}}$ are identically distributed.

Figure 3. (1,2,3) permits overtaking

Figure 4. Equivalent network for example.

Figure 5. (1,...,n) permits overtaking.

Figure 6. The full-duplex system.

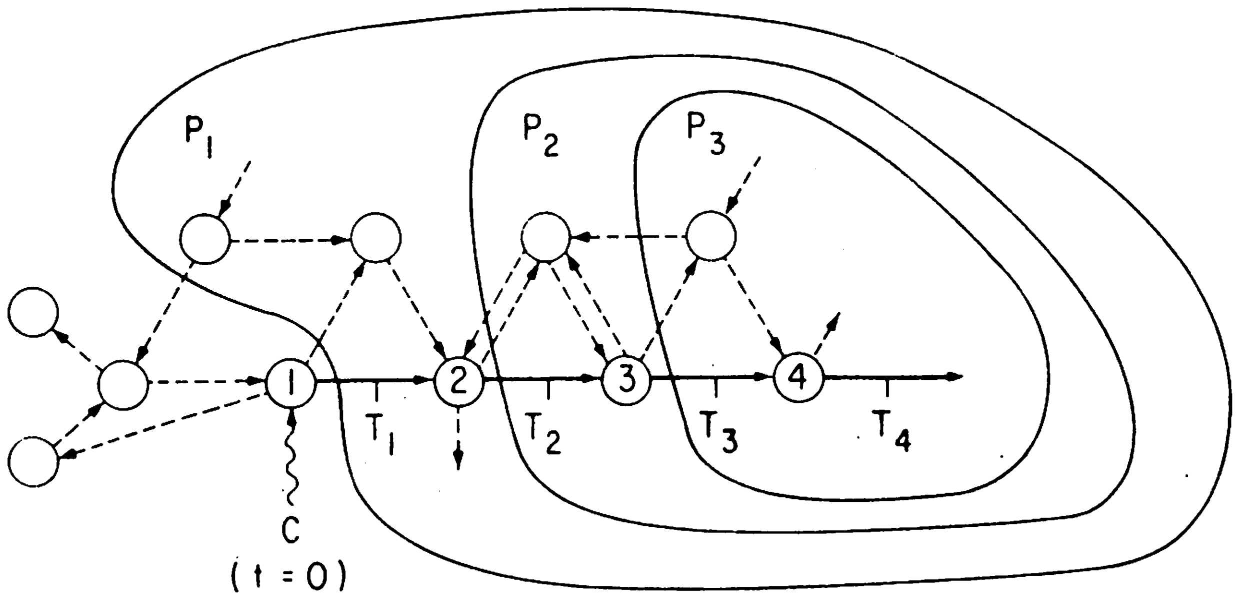
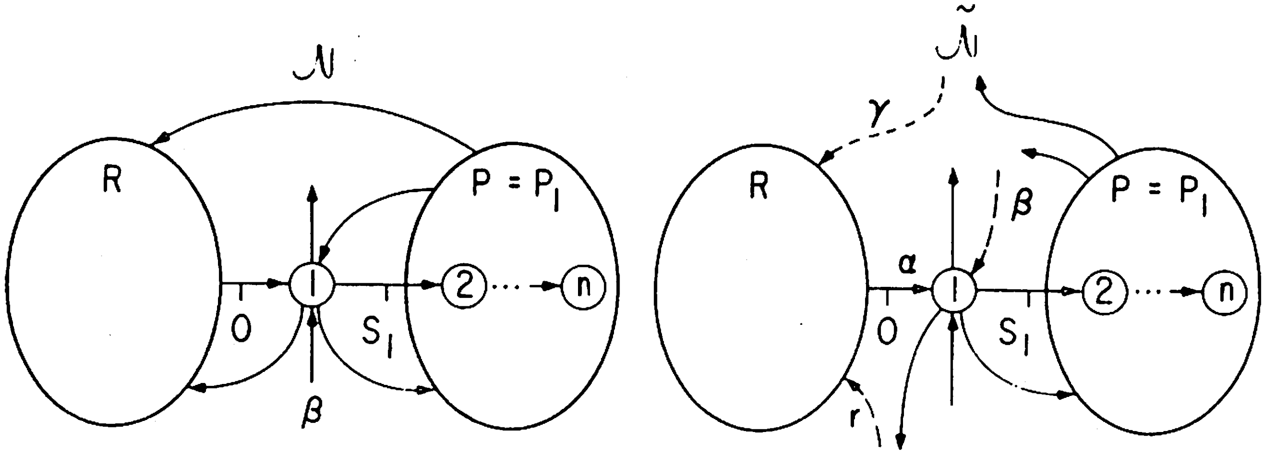


Figure 1



$Q = P_1 \cup \{1\}$, $R = \{1, \ldots, N\}/Q$

Figure 2

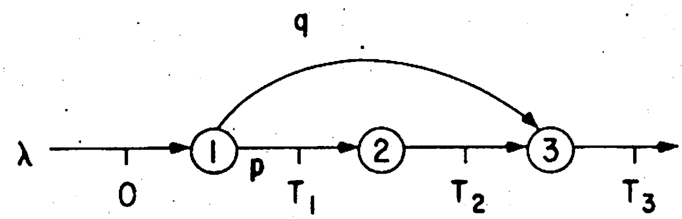


Figure 3

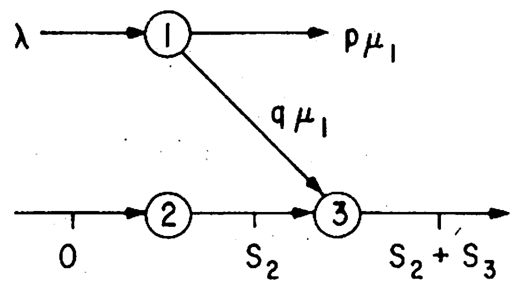


Figure 4

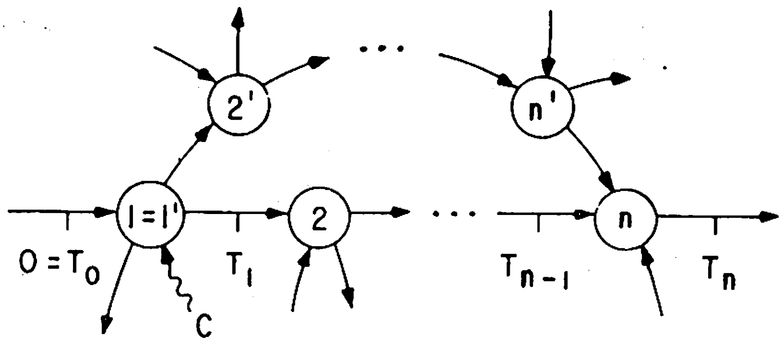


Figure 5

Figure 6