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Electronics Research Laboratory
University of California
Berkeley, California
Internal Technical Memorandum M-78

RELATIONSHIP BETWEEN THE TOTAL
SQUARE INTEGRAL FORMULA AND THE
STABILITY DETERMINANTS

by

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This research was supported by the Air Force Office of Scientific
Research Contract AF-AFOSR-292-64.

July 7, 1964

In the evaluation of the Total Square Integral for the error or the output of a discrete system, the following formula has been obtained:¹

$$I_n = \frac{1}{2\pi j} \oint_{\text{unit circle}} z^{-1} \mathcal{F}(z) \mathcal{J}(z^{-1}) dz = \frac{|\underline{\Omega}|}{a_0 |\underline{\Omega}|}. \quad (1)$$

The function $\mathcal{F}(z)$ is given by

$$\mathcal{F}(z) = \frac{B(z)}{A(z)} \quad \text{where}$$

$$A(z) = \sum_{r=0}^n a_r z^{n-r} = a_0 \prod_{i=1}^n (z - z_i), \quad a_0 > 0 \quad (2)$$

$$\text{and } B(z) = \sum_{r=0}^n b_r z^{n-r}.$$

For the integral I_n to exist, the zeros of $A(z)$ should lie inside the unit circle in the z -plane. Alternatively, for the integral to exist, $|\underline{\Omega}|$ should never vanish. Hence a certain relationship should exist between the determinant $|\underline{\Omega}|$ and the zeros of $A(z)$. This relationship will be shown to be

$$|\underline{\Omega}| = a_0^{n+1} \prod_{i=1}^n \prod_{j=1}^n (1 - z_i z_j). \quad (3)$$

$i \leq j$

It is noticed from this equation that when all the zeros z_i are inside the unit circle, the determinant $|\underline{\Omega}|$ is strictly positive for all n .

To show the validity of Eq. (3) we will first reformulate it in terms of the stability constants A_k and B_k defined by

$$A_k + B_k = |\underline{X}_k + \underline{Y}_k| \quad (4)$$

$$A_k - B_k = |\underline{X}_k - \underline{Y}_k| \quad (5)$$

where

$$\underline{X}_k = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & a_{n-k+1} \\ 0 & a_n & a_{n-1} & \cdots & a_{n-k+2} \\ 0 & 0 & a_n & & \cdot \\ \vdots & & & \ddots & \cdot \\ \vdots & & & & a_n & a_{n-1} \\ 0 & \cdot & \cdot & \cdot & 0 & a_n \end{bmatrix}$$

and

$$\underline{Y}_k = \begin{bmatrix} a_{k-1} & \cdots & a_2 & a_1 & a_0 \\ a_{k-2} & \cdots & a_1 & a_0 & 0 \\ \vdots & & a_0 & 0 & 0 \\ \vdots & & & \ddots & \cdot \\ a_1 & a_0 & & & 0 \\ a_0 & 0 & \cdot & & \end{bmatrix}$$

Now from (2) $A(1) = a_0 \prod_{i=1}^n (1 - z_i)$

and $A(-1) = (-1)^n a_0 \prod_{i=1}^n (1 + z_i)$.

It can also be shown that [†]

$$A_{n-1} - B_{n-1} = (-1)^{\frac{n(n-1)}{2}} a_0^{n-1} \prod_{i=1}^n \prod_{j=1}^n (1 - z_i z_j) \quad i < j$$

Combining these three equations we can rewrite Eq. (3) as ^{††}

$$|\underline{\Omega}| = (-1)^{\frac{n(n+1)}{2}} A(1) A(-1) (A_{n-1} - B_{n-1}) . \quad (6)$$

To simplify the proof of this equation we use the following relationship: ^{*}

$$A_n + B_n = A(1) (A_{n-1} - B_{n-1}) .$$

Hence Eq. (6) becomes

$$|\underline{\Omega}| = (-1)^{\frac{n(n+1)}{2}} A(-1) (A_n + B_n) \quad (7)$$

where

[†]Ref. 1, p. 95.

^{††}From the stability determinant form we know that, for stability

$$(-1)^{\frac{n(n-1)}{2}} (A_{n-1} - B_{n-1}) > 0$$

and $(-1)^n A(1) A(-1) > 0$,

therefore $|\underline{\Omega}|$ is strictly positive.

^{*}Ref. 1, p. 87.

$$\underline{\Omega} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_1 & a_0 + a_2 & a_1 + a_3 & \dots & a_{n-1} \\ a_2 & a_3 & a_0 + a_4 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & 0 & \dots & a_0 \end{bmatrix}$$

and

$$A_n + B_n = \begin{bmatrix} a_n + a_{n-1} & a_{n-1} + a_{n-2} & \dots & a_2 + a_1 & a_1 + a_0 \\ a_{n-2} & a_n + a_{n-3} & \dots & a_3 + a_0 & a_2 \\ a_{n-3} & a_{n-4} & \dots & a_4 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_0 & 0 & \dots & 0 & a_n \end{bmatrix}.$$

Proof: The proof will be by demonstration and will involve manipulating $|\underline{\Omega}|$ until it assumes the desired form.

The procedure is demonstrated for $n = 4$ and $n = 5$ and the general procedure for any n is stated.

$$\underline{n = 4} \quad A(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

$$|\underline{\Omega}| = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_0 + a_2 & a_1 + a_3 & a_2 + a_4 & a_3 \\ a_2 & a_3 & a_0 + a_4 & a_1 & a_2 \\ a_3 & a_4 & 0 & a_0 & a_1 \\ a_4 & 0 & 0 & 0 & a_0 \end{bmatrix}.$$

From row 3 subtract rows 2, 4 and to it add rows 1, 5.

$$= \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_0 + a_2 & a_1 + a_3 & a_2 + a_4 & a_3 \\ A(-1) & -A(-1) & A(-1) & -A(-1) & A(-1) \\ a_3 & a_4 & 0 & a_0 & a_1 \\ a_4 & 0 & 0 & 0 & a_0 \end{vmatrix}$$

Add column 3 to columns 2, 4 and subtract it from columns 1, 5. The third row will then be all zeros except for the third member. Factor out $A(-1)$ and we then have a 4-th order determinant

$$= A(-1) \begin{vmatrix} a_0 - a_2 & a_1 + a_2 & a_2 + a_3 & a_4 - a_2 \\ -a_3 & a_0 + a_2 + a_1 + a_3 & a_2 + a_4 + a_1 + a_3 & -a_1 \\ a_3 & a_4 & a_0 & a_1 \\ a_4 & 0 & 0 & a_0 \end{vmatrix}$$

Subtract rows 1 from 2 and 4 from 3.

$$= A(-1) \begin{vmatrix} a_0 - a_2 & a_1 + a_2 & a_2 + a_3 & a_4 - a_2 \\ -a_3 - a_0 + a_2 & a_0 + a_3 & a_1 + a_4 & -a_1 - a_4 + a_2 \\ a_3 - a_4 & a_4 & a_0 & a_1 - a_0 \\ a_4 & 0 & 0 & a_0 \end{vmatrix}$$

Add columns 2 to 1 and 3 to 4

$$= A(-1) \begin{vmatrix} a_1 + a_0 & a_2 + a_1 & a_3 + a_2 & a_4 + a_3 \\ a_2 & a_3 + a_0 & a_4 + a_1 & a_2 \\ a_3 & a_4 & a_0 & a_1 \\ a_4 & 0 & 0 & a_0 \end{vmatrix}$$

Interchange columns 1 and 4, 2 and 3. We have to multiply by $(-1)^2 = +1$.

$$= A(-1) \begin{vmatrix} a_4 + a_3 & a_3 + a_2 & a_2 + a_1 & a_1 + a_0 \\ a_2 & a_4 + a_1 & a_3 + a_0 & a_2 \\ a_1 & a_0 & a_4 & a_3 \\ a_0 & 0 & 0 & a_4 \end{vmatrix}$$

By inspection

$$= A(-1) (A_4 + B_4).$$

Here $\frac{n(n+1)}{2} = +1$, so we have verified Eq. (7) for $n = 4$.

$$\underline{n = 5} \quad A(z) = a_0 z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5$$

$$|\underline{\Omega}| = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_0 + a_2 & a_1 + a_3 & a_2 + a_4 & a_3 + a_5 & a_4 \\ a_2 & a_3 & a_0 + a_4 & a_1 + a_5 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_0 & a_1 & a_2 \\ a_4 & a_5 & 0 & 0 & a_0 & a_1 \\ a_5 & 0 & 0 & 0 & 0 & a_0 \end{vmatrix}$$

From row 4 subtract rows 1, 3, 5 and to it add rows 2, 6.

$$= \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_0 + a_2 & a_1 + a_3 & a_2 + a_4 & a_3 + a_5 & a_4 \\ a_2 & a_3 & a_0 + a_4 & a_1 + a_5 & a_2 & a_3 \\ A(-1) & -A(-1) & A(-1) & -A(-1) & A(-1) & -A(-1) \\ a_4 & a_5 & 0 & 0 & a_0 & a_1 \\ a_5 & 0 & 0 & 0 & 0 & a_0 \end{vmatrix}$$

Add column 4 to columns 1, 3, 5 and subtract it from columns 2, 6. The fourth row will then be all zeros except for the fourth member. Factor out $-A(-1)$ and we then have a 5-th order determinant

$$= -A(-1) \begin{vmatrix} a_0 + a_3 & a_1 - a_3 & a_2 + a_3 & a_4 + a_3 & a_5 - a_3 \\ a_1 + a_2 + a_4 & a_0 - a_4 & a_1 + a_3 + a_2 + a_4 & a_3 + a_5 + a_2 + a_4 & -a_2 \\ a_2 + a_1 + a_5 & a_3 - a_1 - a_5 & a_0 + a_4 + a_1 + a_5 & a_2 + a_1 + a_5 & a_3 - a_1 - a_5 \\ a_4 & a_5 & 0 & a_0 & a_1 \\ 0 & 0 & 0 & 0 & a_0 \end{vmatrix}$$

From row 3 subtract row 2 and to it add row 1. Subtract rows 1 from 2 and 5 from 4.

$$= -A(-1) \begin{vmatrix} a_0 + a_3 & a_1 - a_3 & a_2 + a_3 & a_4 + a_3 & a_5 - a_3 \\ a_1 + a_2 + a_4 - a_0 - a_3 & a_0 - a_4 - a_1 + a_3 & a_1 + a_4 & a_5 + a_2 & a_3 - a_5 - a_2 \\ a_5 - a_4 + a_0 + a_3 & -a_5 - a_0 + a_4 & a_0 + a_5 & a_1 & -a_1 + a_2 \\ a_4 - a_5 & a_5 & 0 & a_0 & a_1 - a_0 \\ a_5 & 0 & 0 & 0 & a_0 \end{vmatrix}$$

Add columns 2 to 1 and 4 to 5

Add column 3 to column 2

$$= -A(-1) \begin{vmatrix} a_1 + a_0 & a_2 + a_1 & a_3 + a_2 & a_4 + a_3 & a_5 + a_4 \\ a_2 & a_3 + a_0 & a_4 + a_1 & a_5 + a_2 & a_3 \\ a_3 & a_4 & a_5 + a_0 & a_1 & a_2 \\ a_4 & a_5 & 0 & a_0 & a_1 \\ a_5 & 0 & 0 & 0 & a_0 \end{vmatrix}$$

Interchange columns 1 and 5, 2 and 4. We have to multiply by $(-1)^2 = +1$.

$$= -A(-1) \begin{vmatrix} a_5 + a_4 & a_4 + a_3 & a_3 + a_2 & a_2 + a_1 & a_1 + a_0 \\ a_3 & a_5 + a_2 & a_4 + a_1 & a_3 + a_0 & a_2 \\ a_2 & a_1 & a_5 + a_0 & a_4 & a_3 \\ a_1 & a_0 & 0 & a_5 & a_4 \\ a_0 & 0 & 0 & 0 & a_5 \end{vmatrix}$$

By inspection

$$= -A(-1) (A_5 + B_5).$$

Here $\frac{n(n+1)}{2} = -1$, so we have verified Eq. (7) for $n = 5$.

General Procedure

Given $|\underline{\Omega}|$, an $(n + 1)$ -th order determinant,

let $q = \frac{n+2}{2}$ for n even

$$= \frac{n+3}{2} \text{ for } n \text{ odd.}$$

- From row q subtract rows $q - 1, q - 3, \dots; q + 1, q + 3, \dots$
and to it add rows $q - 2, q - 4, \dots; q + 2, q + 4, \dots$

Then i -th member of row q will be $(-1)^{n-q+i} A(-1)$ and in particular, q -th member will be $(-1)^n A(-1)$.

- Add column q to columns $q - 1, q - 3, \dots; q + 1, q + 3, \dots$
and subtract it from columns $q - 2, q - 4, \dots; q + 2, q + 4, \dots$

Then row q will be all zeros except for the q -th member which will be $(-1)^n A(-1)$.

- Remove this factor and there will result an n -th order determinant J .
So

$$|\underline{\Omega}| = (-1)^n A(-1) J .$$

- Manipulate the rows of J in the following order:

$\left\{ \begin{array}{l} \text{from row } q - 1 \text{ subtract rows } q - 2, q - 4, \dots \text{ and to it add rows } q - 3, q - 5, \dots \\ \text{from row } q \text{ subtract rows } q + 1, q + 3, \dots \text{ and to it add rows } q + 2, q + 4, \dots \\ \text{from row } q - 2 \text{ subtract rows } q - 3, q - 5, \dots \text{ and to it add rows } q - 4, q - 6, \dots \\ \text{from row } q + 1 \text{ subtract rows } q + 2, q + 4, \dots \text{ and to it add rows } q + 3, q + 5, \dots \\ \dots \dots \dots \\ \dots \dots \dots \\ \dots \dots \dots \end{array} \right.$

$\left\{ \begin{array}{l} \text{from row 3 subtract row 2 and to it add row 1} \\ \text{from row } n - 2 \text{ subtract row } n - 1 \text{ and to it add row } n \\ \text{from row 2 subtract row 1 } (n \geq 3) \\ \text{from row } n - 1 \text{ subtract row } n } (n \geq 4) . \end{array} \right.$

- Manipulate the columns of J in the following order:

add columns 2 to 1 and $n - 1$ to n

add columns 3 to 2 and $n - 2$ to $n - 1$.

.....

.....

If n is even, the final addition of columns will be

$$\frac{n}{2} \text{ to } \frac{n}{2} - 1 \text{ and } \frac{n}{2} + 1 \text{ to } \frac{n}{2} + 2.$$

If n is odd, the final addition of columns will be

$$\frac{n+1}{2} \text{ to } \frac{n-1}{2}.$$

6. Now interchange columns 1 and n , 2 and $n-1$, 3 and $n-2$, ... etc.
The new determinant will be equal to $(-1)^{(n(n-1)/2)} J$ and is identical
to $(A_n + B_n)$.

$$\begin{aligned}\text{Hence } |\underline{\Omega}| &= (-1)^n A(-1) (-1)^{\frac{n(n-1)}{2}} (A_n + B_n) \\ &= (-1)^{\frac{n(n+1)}{2}} A(-1) (A_n + B_n)\end{aligned}$$

So we have verified Eq. (7) and consequently Eq. (3).

REFERENCE

1. E. I. Jury, Theory and Application of the z-transform Method, Wiley and Sons, New York; 1964.