FOUNDATIONS OF FEEBACK THEORY FOR NONLINEAR DYNAMICAL SYSTEMS

by

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ABSTRACT

We study the fundamental properties of feedback for nonlinear, time-varying, multi-input, multi-output, distributed systems. The classical Black formula is generalized to the nonlinear case. Achievable advantages and limitations of feedback in nonlinear dynamical systems are classified and studied in five categories: desensitization, disturbance attenuation, linearizing effect, asymptotic tracking and disturbance rejection, stabilization. Conditions under which feedback is beneficial for nonlinear dynamical systems are derived. Our results show that if the appropriate linearized inverse return difference operator is small, then the nonlinear feedback system has advantages over the open-loop system. Several examples are provided to illustrate the results.

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CONTENTS

I. INTRODUCTION

Feedback is one of the most important engineering inventions.

Historically [1], some third century B.C. water clocks may be viewed as primitive feedback devices. Some more definite feedback systems such as furnace temperature regulators, float regulators, windmills, etc. were invented between 16th and 18th century. However, it is only at the turn of the 19th century, when James Watt invented the steam engine governor, that the concept of feedback began to be appreciated and used by engineers. Attempts to understand and to analyze the associated stability problems brought by feedback were then made by several pioneers, e.g. Airy, Maxwell, Lyapunov, Routh, Hurwitz, Vyschnegradskii, etc. Up to the 1920's, feedback devices were predominantly mechanical regulators whose primary objective was to reduce the regulated error to zero. The need of long distance telephony in the 1920's [2] resulted in the crucial invention of the negative feedback amplifier by H.S. Black [3,4]. Black's major invention was to conceive the benefits of feedback resulting from a high forward-path gain: he fed the output back to the input stage; he showed that by using a high gain in the forward path, one obtains an amplifier which is 1) more linear than the vacuum tubes in the forward path, 2) insensitive to variations in the vacuum tubes in the forward path, and 3) insensitive to noise injected at the output stage. Depending on the applications, the requirements on negative feedback amplifiers and on mechanical regulators may be quite different. Nevertheless, during World War II, the need of very accurate servomechanisms for anti-aircraft defense brought them together. It is our opinion that there is a unified underlying discipline of feedback: different applications emphasize different aspects of that discipline.

In practice, feedback is indispensable in many system designs because of 1) uncertainties: typically, incomplete knowledge of the plant due to plain ignorance or to the inordinate cost of measurements; unpredictable environmental effects; manufacturing tolerances; changes in the characteristics due to ageing, wearing, loading,...; etc., and 2) the use of inherently unstable plants, e.g. rockets, some chemical reactors, nuclear reactors, some advanced design airplanes,..., etc. The effectiveness of feedback in coping with uncertainties was actually illustrated in the process of Black's invention of the negative feedback amplifier [4]: he realized that an "open-loop" cancellation scheme is impractical (because it requires the two "paths" track each other) and he eventually conceived the negative feedback amplifier. Moreover, Black's paper [3] exhibited many of the achievable advantages of feedback such as desensitization and disturbance attenuation.

Even though most of the existing expositions of the effects of feedback are essentially based on transfer functions calculations (thus necessarily restricted to the linear time-invariant case only), we believe that the benefits of feedback are the consequence of two facts: first, a topological structure - the loop; second, an order of magnitude relation (in the context of Black's classical paper [3], it reads $|\beta\mu| >> 1$) which is independent of the linearity requirement. Pursuing this point of view, we derive below the basic properties of feedback in a much more general framework: we make full use of the recent developments in the input-output formulation of nonlinear, distributed, time-varying, multi-input, multi-output systems (see e.g. [5,6,7,8]). Such formulation allows for unstable, continuous-time as well as discrete-time subsystems; this is achieved by using causality and the technique of extended spaces, i.e. considering only the time interval $[0,T]$, with T finite but arbitrary.

After introducing the notation and the general framework, we generalize, in section II, the classical Black formula to nonlinear, distributed, time-varying, multi-input, multi-output systems. Our generalized Black formulas clearly show that the properties of feedback are independent of the linearity assumption. In section III, we demonstrate, for the general nonlinear system described[1] in section I.2, the advantages and limitations of feedback in desensitization, disturbance attenuation, linearizing effect, asymptotic tracking and disturbance rejection, and stabilization.

I.1 NOTATION

Let $\mathbb{R}$ ($\mathbb{C}$) denote the field of real (complex, resp.) numbers. Let $\mathbb{N}$ denote the set of non-negative integers. Let $\mathbb{Q}_+$ denote the set of non-negative rational numbers. Let $\mathbb{R}_+$ denote the non-negative real line $[0,\infty)$. Let $\mathring{\mathbb{C}}_+$ denote the <u>open</u> right-half complex plane. Let $\mathbb{R}[s]$ ($\mathbb{R}(s)$) be the set of all polynomials (rational functions, resp.) in s with real coefficients. Let $\mathbb{R}^{p\times q}$ ($\mathbb{C}^{p\times q}$, $\mathbb{R}[s]^{p\times q}$, $\mathbb{R}(s)^{p\times q}$) denote the set of all $p\times q$ matrices with elements in $\mathbb{R}$ ($\mathbb{C}$, $\mathbb{R}[s]$, $\mathbb{R}(s)$, resp.). Let $\partial p(s)$ denote the degree of $p(s) \in \mathbb{R}[s]$. Let $\mathcal{T} \subset \mathbb{R}_+$ be the set of time instants at which various signals of interest are defined: typically, $\mathcal{T} = \mathbb{R}_+$ for the continuous-time case, $\mathcal{T} = \mathbb{N}$ for the discrete time case. Let $\mathcal{M}$ be a normed (seminormed) space of functions mapping $\mathcal{T}$ into some vector space $\mathcal{V}$, (typically, $\mathcal{V} = \mathbb{R}^n$, $\mathcal{M} = L_2^n$, L_∞^n or ℓ_2^n, ℓ_∞^n, etc.). Associated with the normed (seminormed) space $\mathcal{M}$ is the extended normed (seminormed) space $\mathcal{M}_e$ defined by $\mathcal{M}_e := \{f:\mathcal{T} \to \mathcal{V} \mid \forall T \in \mathcal{T},\ |f|_T < \infty\}$, where $|f|_T := |f_T|$, f_T is obtained from f by a projection map $\underset{\sim}{P}_T$, more

[1] In describing the feedback system under consideration, we adopted the control terminology, i.e. the power stage of the amplifier is called the plant; the preamplifier is called the compensator; etc. We trust that this will cause no great inconvenience to feedback amplifier enthusiasts.

precisely, $f_T := \underset{\sim}{P}_T f$ is defined by $f_T(t) = \begin{cases} f(t), & t \leq T \\ 0 & , t > T \end{cases}$, for $t, T \in \mathcal{T}$. Let $\underset{\sim}{P}_T \mathcal{M}_e$ denote the class $\{\underset{\sim}{P}_T f | f \in \mathcal{M}_e\}$. Let $\underset{\sim}{H}: \mathcal{M}_e \to \tilde{\mathcal{M}}_e$; $\underset{\sim}{H}$ is said to be causal iff $\underset{\sim}{P}_T \underset{\sim}{H} \underset{\sim}{P}_T = \underset{\sim}{P}_T \underset{\sim}{H}$, $\forall T \in \mathcal{T}$ [8, p.38-39]. "Nonlinear" means "not necessarily linear". ":=" means "is defined by". "u.t.c." means "under these conditions". Operators, i.e. maps from $\mathcal{M}_e$ into $\tilde{\mathcal{M}}_e$, are labelled by boldface symbols (e.g. $\underset{\sim}{G}, \underset{\sim}{K}, \underset{\sim}{F}, \ldots$). Let $|\cdot|_2$ denote the ℓ_2-norm on $\mathbb{C}^n$. Let C^1 denote the class of continuously differentiable maps [19, pp. 172].

I.2 GENERAL FRAMEWORK

We will consider the nonlinear, feedback system S shown in Fig. I.1, where

$\underset{\sim}{G}: \mathcal{U}_e \to \mathcal{Y}_e$, is a nonlinear, causal operator representing the plant, (I.1)

$\underset{\sim}{K}: \mathcal{R}_e \to \mathcal{U}_e$, is a nonlinear, causal operator representing the compensator, (I.2)

$\underset{\sim}{F}: \mathcal{Y}_e \to \mathcal{R}_e$, is a nonlinear, causal operator representing the feedback, (I.3)

$r \in \mathcal{R}_e$, is the system input, (I.4)

$u \in \mathcal{U}_e$, is the plant input, (I.5)

$y \in \mathcal{Y}_e$, is the system output, (I.6)

$e \in \mathcal{R}_e$, is the error signal, (I.7)

$\mathcal{R}_e$, $\mathcal{U}_e$, $\mathcal{Y}_e$ are extended normed spaces, unless otherwise stated. (I.8)

We shall assume that

$(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}$ is a well-defined nonlinear, causal operator mapping from $\mathcal{R}_e$ into $\mathcal{R}_e$. (I.9)

Note that the closed-loop input-output map $\underset{\sim}{H}_{yr}: r \to y$ is given by $\underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}$.

II. BLACK'S FORMULA GENERALIZED

H.S. Black's invention of the negative feedback amplifier was based on the following analysis [3]: consider the feedback system S shown in Fig. I.1; let $\underset{\sim}{G}\underset{\sim}{K}$ and $\underset{\sim}{F}$ be specialized into the <u>scalar transfer functions</u> μ and β, respectively, then the closed-loop input-output transfer function is(2)

$$h_{yr} := \frac{\mu}{1+\beta\mu} = \frac{1}{\beta} \cdot \frac{\beta\mu}{1+\beta\mu}$$

$$= \frac{1}{\beta}[1 - \frac{1}{1+\beta\mu}] \qquad \text{(II.1)}$$

$$\simeq \frac{1}{\beta} \qquad \text{(II.2)}$$

<u>for those frequencies where $|\beta\mu| >> 1$</u>. (II.3)

Black's crucial observation is that for those frequencies where $|\beta\mu| >> 1$, or equivalently $|1+\beta\mu| >> 1$, the output $y \simeq \frac{1}{\beta} r$, i.e., the closed-loop input-output transfer function is essentially <u>independent</u> of μ and is essentially <u>specified by</u> β. So the recipe is: β is specified by the desired h_{yr} and the forward path gain μ is made as large as possible to achieve (II.3).

Equations (II.1)-(II.3) summarize Black's fundamental observation. We note that it is valid because 1) there is a loop structure, and 2) the loop gain $|\beta\mu|$ is large. This reasoning can be greatly generalized to the case of nonlinear system S shown in Fig. I.1. Note that in the

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(2) In the single-input single-output, linear, time-invariant case, $\mu\beta = \beta\mu$; however, if any one of these three conditions fails, one <u>must</u> write $\beta\mu$. We do so to be self-consistent.

linear, time-invariant case, we only have to consider the sinusoidal inputs within some frequency band of interest and the corresponding sinusoidal steady-state response. But in the nonlinear case, we have to formulate the condition in terms of inputs of interest, e.g., sinusoids of various frequencies and amplitudes, step, ramp, parabolas, etc.

Theorem II.1: (Black's formula generalized: soft version)

Consider the nonlinear, feedback system S shown in Fig I.1 and described by Equations (I.1)-(I.9). Let $\mathcal{R}_{d,e} \subset \mathcal{R}_e$ be the set of inputs of interest. U.t.c. if, for T sufficiently large,

$$|(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r|_T \ll |r|_T, \qquad \forall r \in \mathcal{R}_{d,e} \tag{II.4}$$

then, asumptotically

$$\underset{\sim}{F}\,\underset{\sim}{H}_{yr} \simeq \underset{\sim}{I} \quad \text{on} \quad \mathcal{R}_{d,e} \tag{II.5}$$

in the sense that, for $T \in \mathcal{J}$ sufficiently large,

$$|r - \underset{\sim}{F}\,\underset{\sim}{H}_{yr}\, r|_T \ll |r|_T, \qquad \forall r \in \mathcal{R}_{d,e} \tag{II.6}$$

Proof:

Since $\underset{\sim}{F}$, $\underset{\sim}{G}$, $\underset{\sim}{K}$ are nonlinear, we have

$$\underset{\sim}{H}_{yr} = \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}.$$

Apply the nonlinear operator $\underset{\sim}{F}$ on the left to both sides of this equation:

$$\begin{aligned}\underset{\sim}{F}\,\underset{\sim}{H}_{yr} &= \underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} \\ &= \underset{\sim}{I} - (\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}\end{aligned}$$

Hence for all $r \in \mathcal{R}_e$

$$r - \underset{\sim}{F}\, \underset{\sim}{H}_{yr} r = (\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} r$$

Now let $r \in \mathcal{R}_{d,e} \subset \mathcal{R}_e$ and let $T \in \mathcal{J}$ be large, then, using (II.4),

$$|\underset{\sim}{r} - \underset{\sim}{F}\, \underset{\sim}{H}_{yr} r|_T = |(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} r|_T \ll |r|_T$$

and (II.5) follows. Q.E.D.

Remarks II.1: a) (II.5) says that the feedback system $\underset{\sim}{H}_{yr}$ followed by $\underset{\sim}{F}$ behaves approximately like an identity operator as far as the inputs of interest are concerned. Equivalently, $\underset{\sim}{F}$ is an approximate left-inverse of $\underset{\sim}{H}_{yr}$ on $\mathcal{R}_{d,e}$; thus, on $\mathcal{R}_{d,e}$, $\underset{\sim}{H}_{yr}$ is essentially independent of $\underset{\sim}{G}$ and is essentially specified by $\underset{\sim}{F}$. (The left inverse is the one of interest because any operator $\underset{\sim}{P}: \mathcal{U} \to \mathcal{V}$ has a right inverse $\underset{\sim}{Q}$ in the sense that there always exists a $\underset{\sim}{Q}$ such that $\underset{\sim}{P}\underset{\sim}{Q} = \underset{\sim}{I}_d$ where $\underset{\sim}{I}_d$ denotes the identity restricted to $\underset{\sim}{P}(\mathcal{U})$).

b) Consider $\underset{\sim}{G}$ perturbed into $\tilde{\underset{\sim}{G}}$; call $\tilde{\underset{\sim}{H}}_{yr}$ the resulting closed-loop input-output map. If $\tilde{\underset{\sim}{G}}$ satisfies (II.4), then $\underset{\sim}{F}\, \tilde{\underset{\sim}{H}}_{yr} \simeq \underset{\sim}{I}$, on $\mathcal{R}_{d,e}$; i.e., on $\mathcal{R}_{d,e}$, $\underset{\sim}{H}_{yr}$ is insensitive to the plant perturbations. This, however, does not assert that the relative change in $\underset{\sim}{H}_{yr}$ will be much less than that in $\underset{\sim}{G}$; it simply asserts that changes in $\underset{\sim}{G}$ have little effect on $\underset{\sim}{H}_{yr}$. The exact relation between the relative change in $\underset{\sim}{H}_{yr}$ and the relative change in $\underset{\sim}{G}$ is given by Equation (III.7) below and discussed in Remarks III.1.

c) (II.5) is a soft version of Black's formula (II.2). To obtain $\underset{\sim}{H}_{yr} \simeq \underset{\sim}{F}^{-1}$ requires some additional assumptions. This is done in Theorems II.2 and II.3 below. ¤

Note that eqn. (II.1) gives the exact relation

$$h_{yr} - \frac{1}{\beta} = -\frac{1}{\beta} \cdot \frac{1}{1+\beta\mu} \tag{II.7}$$

As feedback designers know (see e.g. [9]), it is often advantageous to write this equation in terms of the "inverse loop-gain"

$$h_{yr} - \frac{1}{\beta} = -\frac{1}{\beta} \frac{(\beta\mu)^{-1}}{1+(\beta\mu)^{-1}} \tag{II.8}$$

Theorem II.2 below generalizes Black's result to the nonlinear case: an estimate of the difference $\underset{\sim}{H}_{yr} r - \underset{\sim}{F}^{-1} r$ is obtained under the condition that the "inverse loop-gain" is small for the class of inputs of interest. Note the similarity in form between the right-hand sides of eqn. (II.7) and eqn. (II.9) below.

Theorem II.2 (Generalized Black's formula)

Consider the nonlinear, feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9). Let $\mathcal{R}_{d,e} \subset \mathcal{R}_e$ be the set of inputs of interest. Suppose that

(a1) $\forall T \in \mathcal{T}$, $\underset{\sim}{P}_T \mathcal{R}_e$ is a Banach space;

(a2) $\underset{\sim}{F}^{-1} : \mathcal{R}_e \to \mathcal{Y}_e$ and $(\underset{\sim}{FGK})^{-1} : \mathcal{R}_e \to \mathcal{R}_e$ are well-defined nonlinear, causal maps;

(a3) $(\underset{\sim}{FGK})^{-1}$ is continuous[(3)] on $\mathcal{R}_e$, and for each $r \in \mathcal{R}_{d,e}$, $z_{n+1} := r - (\underset{\sim}{FGK})^{-1} z_n \in \mathcal{N}(\mathcal{R}_{d,e}) \subset \mathcal{R}_e$, where $z_0 = r$, $n \in \mathbb{N}$, and $\mathcal{N}(\mathcal{R}_{d,e})$ denotes a neighborhood of $\mathcal{R}_{d,e}$ in $\mathcal{R}_e$.

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(3) An operator N is continuous on an extended space $\mathcal{R}_e$ iff $\forall T \in \mathcal{T}$, $\underset{\sim}{N}$ is continuous on $\underset{\sim}{P}_T \tilde{\mathcal{R}}_e$.

U.t.c. if

(i)
$$\lambda(F^{-1}) := \sup_{\substack{r \in \mathcal{R}_{d,e},\ T \in \mathcal{T} \\ e := (I+FGK)^{-1} r \\ e_T \neq 0}} \frac{|F^{-1}(r-e)-F^{-1}r|_T}{|e|_T} < \infty \ ;$$

(ii) for each $T \in \mathcal{T}$,

$$\tilde{\gamma}_T[(FGK)^{-1}] := \sup_{\substack{r_1, r_2 \in \mathcal{N}(\mathcal{R}_{d,e}) \\ r_{1,T} \neq r_{2,T}}} \frac{|(FGK)^{-1}r_1-(FGK)^{-1}r_2|_T}{|r_1-r_2|_T} < 1 \ ,$$

then, for each $T \in \mathcal{T}$,

$$|H_{yr}r-F^{-1}r|_T \leq \lambda(F^{-1}) \frac{|(FGK)^{-1}r|_T}{1-\tilde{\gamma}_T[(FGK)^{-1}]} \ , \quad \forall r \in \mathcal{R}_{d,e} \tag{II.9}$$

In particular, if for $T \in \mathcal{T}$ sufficiently large,

$$\tilde{\gamma}_T[(FGK)^{-1}] \ll 1 \tag{II.10}$$

and

$$|(FGK)^{-1}r|_T \ll \frac{|F^{-1}r|_T}{\lambda(F^{-1})} \ , \quad \forall r \in \mathcal{R}_{d,e} \ , \tag{II.11}$$

then asymptotically,

$$H_{yr} \simeq F^{-1} \text{ on } \mathcal{R}_{d,e} \tag{II.12}$$

in the sense that for $T \in \mathcal{T}$ sufficiently large,

$$|H_{yr}r-F^{-1}r|_T \ll |F^{-1}r|_T \ , \quad \forall r \in \mathcal{R}_{d,e} \ . \tag{II.13}$$

Proof of Theorem II.2: see Appendix.

Remark II.2: Note that the classical Black condition that $|\beta\mu| \gg 1$ (which is achieved, in design, with $|\mu| \gg 1$) is a sufficient condition for the approximation (II.2). Thus one may want to pursue the idea of small inverse forward path gain (large $|\mu|$ in the single-input single-output case) as follows: assuming the existence of the required inverses, from

$$\underset{\sim}{H}_{yr} = \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} \tag{II.14}$$

we obtain

$$\begin{aligned} \underset{\sim}{H}_{yr}^{-1} &= (\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})(\underset{\sim}{G}\underset{\sim}{K})^{-1} \\ &= \underset{\sim}{F} + (\underset{\sim}{G}\underset{\sim}{K})^{-1} \end{aligned} \tag{II.15}$$

This formula is the generalization to the <u>nonlinear</u> case of the well-known corresponding relation with matrix transfer functions [9, p. 121]. If we assume that $\forall y \in \mathcal{Y}_{d,e}$, the set of outputs of interest, and for $T \in \mathcal{T}$ sufficiently large

$$|(\underset{\sim}{G}\underset{\sim}{K})^{-1}y|_T \ll |\underset{\sim}{F}y|_T \tag{II.16}$$

then, asymptotically

$$\underset{\sim}{H}_{yr}^{-1} \simeq \underset{\sim}{F}\ , \text{ on } \mathcal{Y}_{d,e} \tag{II.17}$$

in the sense that for $T \in \mathcal{T}$ sufficiently large, $|\underset{\sim}{H}_{yr}^{-1}y-\underset{\sim}{F}y|_T \ll |\underset{\sim}{F}y|_T$, $\forall y \in \mathcal{Y}_{d,e}$. Note, however, since $\underset{\sim}{F}$ and $\underset{\sim}{H}_{yr}$ are <u>nonlinear</u>, eqn. (II.17) does <u>not</u> imply that $\underset{\sim}{H}_{yr} \simeq \underset{\sim}{F}^{-1}$. ¤

Going back to the Black formula (II.1), we note that the approximation (II.2), $h_{yr} \simeq \frac{1}{\beta}$, is valid as long as

$$\left|\frac{1}{\beta}\cdot\frac{1}{1+\beta\mu}\right| \ll \left|\frac{1}{\beta}\right| \tag{II.18}$$

Theorem II.3 below generalizes this condition to the nonlinear case: eqn. (II.18) should be compared with the condition (ii) of Theorem II.3 below.

Theorem II.3:

Consider the nonlinear, feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9). Let $\mathcal{R}_{d,e} \subset \mathcal{R}_e$ be the set of inputs of interest. Suppose that $\underset{\sim}{F}^{-1}: \mathcal{R}_e \to \mathcal{Y}_e$ is a well-defined nonlinear, causal map.

U.t.c. if

(i) $$\lambda(\underset{\sim}{F}^{-1}) := \sup_{\substack{r \in \mathcal{R}_{d,e},\ T \in \mathcal{T} \\ e := (\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} r \\ e_T \neq 0}} \frac{|\underset{\sim}{F}^{-1}(r-e)-\underset{\sim}{F}^{-1}r|_T}{|e|_T} < \infty \ ;$$

(ii) for $T \in \mathcal{T}$ sufficiently large,

$$\lambda(\underset{\sim}{F}^{-1}) \cdot |(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} r|_T << |\underset{\sim}{F}^{-1} r|_T, \ \forall r \in \mathcal{R}_{d,e},$$

then, asymptotically,

$$\underset{\sim}{H}_{yr} \simeq \underset{\sim}{F}^{-1} \text{ on } \mathcal{R}_{d,e} \tag{II.19}$$

in the sense that for $T \in \mathcal{T}$ sufficiently large,

$$|\underset{\sim}{H}_{yr} r - \underset{\sim}{F}^{-1} r|_T << |\underset{\sim}{F}^{-1} r|_T, \ \forall r \in \mathcal{R}_{d,e} \tag{II.20}$$

Proof of Theorem II.3: see Appendix.

Corollary II.3.1 (Linear time-invariant case)

Consider the feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9). Let the operators $\underset{\sim}{G}$, $\underset{\sim}{K}$, and $\underset{\sim}{F}$ be linear, time-invariant and represented by transfer function matrices G(s), K(s) and F(s), respectively. Let $\mathcal{R}_{d,e} \subset \mathcal{R}_e$ consist of all sinusoidal inputs with frequencies in some interval $\Omega \subset \mathbb{R}$. Suppose that

(a1) $\underset{\sim}{F}^{-1}: \mathcal{R}_e \to \mathcal{Y}_e$ is a well-defined causal map;

(a2) the closed-loop system is exp. stable, i.e. the impulse response of the transfer function $\underset{\sim}{H}_{yr}: r \to y$ is bounded by a decaying exponential.

U.t.c., if $\forall\omega \in \Omega$, $\forall y \in \text{range}[F(j\omega)^{-1}] \subset \mathbb{C}^{n_0}$

$$|[(I+GKF)(j\omega)]^{-1}y| \ll |y| , \tag{II.21}$$

then

$$H_{yr}(j\omega) \simeq F(j\omega)^{-1}, \; \forall\omega \in \Omega \tag{II.22}$$

in the sense that $\forall r \in \mathbb{C}^n$

$$|H_{yr}(j\omega)r - F(j\omega)^{-1}r| \ll |F(j\omega)^{-1}r|, \; \forall\omega \in \Omega \tag{II.23}$$

Proof of Corollary II.3.1: see Appendix

Remark II.3.1: If we use the ℓ_2-norm in $\mathbb{C}^n$, condition (II.21) is satisfied if the largest singular value(4) of $[(I+GKF)(j\omega)]^{-1}$ is much smaller than 1, for all $\omega \in \Omega$.

Comments on Theorems II.2 and II.3:

(a) Theorems II.2 and II.3 conclude that, under suitable conditions, the output $y = \underset{\sim}{H}_{yr} r$ is, asymptotically (i.e. for large T), approximately equal to $\underset{\sim}{F}^{-1} r$ over the inputs of interest within small relative error. Thus eqns. (II.12) and (II.19) are complete generalizations of the Black formula (II.2) to the nonlinear, time-varying, multi-input, multi-output, distributed systems S shown in Fig. I.1 and described by eqns. (I.1)-(I.9).

(b) Typically, $\mathcal{R}_{d,e}$, the set of inputs of interest, consists of sinusoids of various frequencies and amplitudes, or steps, ramps, parabolas, etc., of various magnitudes.

(c) Note that the extended spaces framework allows us to treat the case

(4) If $A \in \mathbb{C}^{n \times n}$, the largest singular value of A is the square root of the largest eigenvalue of A*A, where A* denotes the complex conjugate of A; it is also the ℓ_2-induced norm of the linear map $\underset{\sim}{A}: \mathbb{C}^n \to \mathbb{C}^n$.

where some of the operators $\underset{\sim}{G}$, $\underset{\sim}{K}$, $\underset{\sim}{F}$ may be unstable and to state asymptotic conditions such as eqns. (II.10), (II.11).

(d) It is the nonlinearities of the maps $\underset{\sim}{G}$, $\underset{\sim}{K}$, $\underset{\sim}{F}$ which forces us to use the incremental gain (e.g. $\tilde{\gamma}_T[(\underset{\sim}{FGK})^{-1}]$ in theorem II.2), or Lipschitz constants (e.g. $\lambda(\underset{\sim}{F}^{-1})$ in theorems II.2 and II.3), over appropriate sets, to obtain our estimates. In the linear case, one would use the induced norms of the corresponding maps over appropriate sets.

(e) Theorems II.2 and II.3 have important design implications: Given a plant $\underset{\sim}{G}$, we first choose $\underset{\sim}{F}$ such that, over the inputs of interest, $\underset{\sim}{F}^{-1}$ is asympotically the desired input-output map. Next we choose the compensator $\underset{\sim}{K}$ so that the conditions of theorem II.2 (or of theorem II.3) are satisfied. Then, asymptotically, the closed-loop input-output map $\underset{\sim}{H}_{yr}$ is close to $\underset{\sim}{F}^{-1}$ over the inputs of interest as we desired.

(f) Note that $\underset{\sim}{F}^{-1}$ can be nonlinear. A simple well-known example of realizing a nonlinear map by a feedback system (with large forward-path gain) is the logarithmic amplifier shown in Fig. II.1 Recall that node 2 is a virtual ground, and that the diode operates at currents much larger than its saturating current I_s, thus $\underset{\sim}{F}^{-1}: i_D \mapsto v_0$ is given by $v_0 = -v_D \simeq -v_T \ln(i_D/I_s)$. Hence $v_0 \simeq -v_T \ln[v_i/(R_i I_s)]$.

Examples:

To illustrate the implication of the generalized Black formula on nonlinear dynamical systems, we present the following two examples:

Example II.1 (Nonlinear, single-input single-output dynamical system)

Consider the nonlinear, feedback system S shown in Fig. I.1, where $\underset{\sim}{G}$ is characterized by a rational transfer function $\dfrac{5\times 10^8}{(s+1)(s+10^3)(s+10^4)}$ followed by a nonlinear memoryless map $\phi(\cdot)$ with $\phi(\cdot) \in C^1$ described by

$$\phi(z) = \begin{cases} \frac{11}{30} + \sqrt{(\frac{11}{30})^2 + \frac{z-3}{18.75}} & , \ z \geq 0.5 \\ 0.8z & , \ |z| \leq 0.5 \\ 0.8e^{z+0.5} - 1.2 & , \ z \leq -0.5 \end{cases} \tag{II.24}$$

$\underset{\sim}{K}$ and $\underset{\sim}{F}$ are characterized by constants k and 1, respectively. The closed-loop system and the characteristics of the nonlinearity $\phi(\cdot)$ are shown in Fig. II.2 and Fig. II.3, respectively. By theorem II.2 (or theorem II.3), if k becomes large, then, asymptotically, the output y of the closed-loop system will be approximately equal to the reference signal $r(\cdot)$ (since $\underset{\sim}{F}^{-1} = 1$ in this case). Fig. II.4 - Fig. II.6 show the system output $y(\cdot)$, the error signal $e(\cdot)$, and $z(\cdot)$, the input to the nonlinearity $\phi(\cdot)$, in the "steady state" for different values of k while the closed-loop system is driven by $r(t) = \sin 10t$. The effect due to high forward-path gain in a feedback system is clearly illustrated by Fig. II.4. Note that the high forward-path gain distorts $z(\cdot)$, the input to the nonlinearity $\phi(\cdot)$, so that asymptotically, the output $y(\cdot)$ is approximately equal to $\sin 10t$.

Example II.2 (Nonlinear, multi-input, multi-output, dynamical system)

Consider the nonlinear, feedback system S shown in Fig. I.1, where $\underset{\sim}{G}$ is characterized by a rational function matrix

$$L(s) = \begin{bmatrix} \frac{5\times 10^8}{(s+1)(s+10^3)(s+10^4)} & \frac{1\times 10^8}{(s+1)(s+10^3)(s+10^4)} \\ \frac{5\times 10^7}{(s+1)(s+10^3)(s+10^4)} & \frac{5\times 10^8}{(s+1)(s+10^3)(s+10^4)} \end{bmatrix} \tag{II.25}$$

followed by a nonlinear memoryless C^1 map $\Phi(\cdot)$ described by

$$\Phi(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}) = \begin{bmatrix} (1+0.2\tanh z_2)\cdot\nu(z_1) \\ (1+0.2\tanh z_1)\cdot\nu(z_2) \end{bmatrix} \tag{II.26}$$

with

$$\nu(z) := \begin{cases} z & , \text{ if } |z| \le 0.5 \\ \operatorname{sgn} z \cdot [\frac{3}{7} + \sqrt{\frac{|z|-3/7}{14}}] & , \text{ if } |z| \ge 0.5 \end{cases} , \qquad \text{(II.27)}$$

$\underset{\sim}{K}$ and $\underset{\sim}{F}$ are represented by the constant matrices kI and I, resp., both in $\mathbb{R}^{2x2}$.

The closed-loop system, the characteristics of $\nu(z)$, and the characteristics of $1+0.2\tanh z$ are shown in Fig. II.7, Fig. II.8, and Fig. II.9, respectively. By theorem II.2 (or theorem II.3), if k is sufficiently large, then, asymptotically, the output y of the closed-loop system will be approximately equal to the reference signal r (since $\underset{\sim}{F}^{-1} = \underset{\sim}{I}$ in this case). Fig. II.10 - Fig. II.13 show the system output components $y_1(\cdot)$, $y_2(\cdot)$ and the error signal components $e_1(\cdot)$, $e_2(\cdot)$, respectively, for different values of $k \in \mathbb{R}$ while the closed-loop system is driven by the reference signal $r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \sin 10t \\ 0.8 \sin 15t \end{bmatrix}$. Fig. II.10 and Fig. II.11 show that as we increase the compensator gain k, the system output (vector) function approaches to the reference signal r as if the closed-loop system was an identity map despite the complicated couplings in the nonlinear plant $\underset{\sim}{G}$. Fig. II.14 and II.15 show, for k = 40, a period of the steady-state trajectories of the system outputs, $y(\cdot)$, and of the nonlinearity inputs, $z(\cdot)$, on the y-plane and z-plane, respectively. Note that the greatly distorted trajectory of $z(\cdot)$ (due to the coupling and saturation effects of $\Phi(\cdot)$) produces a system output $y(\cdot)$ very close to the reference signal $\begin{bmatrix} \sin 10t \\ 0.8 \sin 15t \end{bmatrix}$. Consider the three large irregular lobes on the $z(\cdot)$ trajectory in the 2nd, 3rd and 4th quadrant of Fig. II.15 which reach their peaks at time instants t = 4.54, 4.90, 5.55 respectively. Observe that at those time instants, at least one of the desired plant output component ($y_1^* = \sin 10t$, $y_2^* = 0.8 \sin 15t$) reaches the peak of the negative cycle of sinusoidal waves (see Fig. II.10 and Fig. II.11). Further observe that y_1 and z_1, y_2 and z_2 are of same sign for all t since $1+0.2\tanh z > 0$ and $\nu(z)$ is an odd function. Now at time $t = 4.54$, the desired plant output $y_1^*(t) \simeq .98$, $y_2^*(t) \simeq -.70$, thus $\nu(z_1)$ $(\nu(z_2))$ is required to operate in its positive (negative, resp.) "saturation" region. Due to the <u>negative</u> value of z_2, $1+0.2\tanh z_2 \simeq 0.8$. Consequently, $(1+0.2\tanh z_2)\nu(z_1)$ "saturates" earlier than $\nu(z_1)$ itself and z_1 is required to be a large positive number so that $y_1 = (1+0.2\tanh z_2)\nu(z_1)$ will be approximately equal to the desired value 0.98. This explains the large lobe on the trajectory of $z(\cdot)$ in the 4th quadrant. Similar reasoning explains the other two large lobes in the 2nd and 3rd quadrant.

III. ADVANTAGES AND LIMITATIONS OF FEEDBACK

Consider the nonlinear, feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9) which satisfies the conditions stated in theorem II.2 (or theorem II.3), then asymptotically, the closed-loop system input-output map $\underset{\sim}{H}_{yr}$ is approximately $\underset{\sim}{F}^{-1}$. Thus we should expect that the closed-loop system input-output map is <u>insensitive</u> to the variations in the forward path map $\underset{\sim}{G}\underset{\sim}{K}$ and that, if $\underset{\sim}{F}$ is <u>linear</u>, the closed-loop system is close to a linear system even though the forward path map $\underset{\sim}{G}\underset{\sim}{K}$ is <u>highly</u> <u>nonlinear</u>.

In the following, we show the advantages and limitations of feedback for the nonlinear, feedback system S shown in Fig. I.1: section III.1 establishes the <u>exact</u> effect the plant perturbations on a closed-loop input-output map and demonstrates the relations between desensitization and the feedback structure, the perturbation on the feedback map $\underset{\sim}{F}$, and the closed-loop stability; section III.2 establishes the <u>exact</u> effect of various <u>additive</u> external disturbances on the closed-loop system output; section III.3 defines a nonlinearity measure and then shows precisely that feedback has a linearizing effect on a nonlinear plant; sections

III.4 and III.5 briefly review the idea that feedback can achieve asymptotic tracking and disturbance rejection, and stabilize unstable systems.

III.1 DESENSITIZATION

One of the major reasons for using feedback in design is that feedback can reduce the effect of the plant perturbations on the input-output map. One way to quantitatively demonstrate the desensitization effect of feedback is to compare a feedback design with a corresponding open-loop design [10]: consider the nonlinear, feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9). Note that the closed-loop input-output map $H_{yr}: r \mapsto y$ is given by $GK(I+FGK)^{-1}$. Also consider a comparison open-loop system (shown in Fig. III.1) consisting of the same plant G preceeded by a compensator K_0. Thus the open-loop input-output map $H_{y_0r}: r \mapsto y_0$ is given by GK_0. Now if we select

$$K_0 = K(I+FGK)^{-1} , \tag{III.1}$$

then for all system inputs r, $y = y_0$, i.e. the (nominal) open-loop input-output map $H_{y_0r}: r \mapsto y_0$ is identical to the (nominal) closed-loop input-output map $H_{yr}: r \mapsto y$. Consider now an arbitrary, not necessarily small, perturbation ΔG on the plant G, then the plant G becomes $\tilde{G} := G + \Delta G$; the closed-loop (open-loop) system input-output map H_{yr} (H_{y_0r}) becomes $\tilde{H}_{yr} := H_{yr} + \Delta H_{yr} = \tilde{G}K(I+F\tilde{G}K)^{-1}$, ($\tilde{H}_{y_0r} := H_{y_0r} + \Delta H_{y_0r} = \tilde{G}K_0 = \tilde{G}K(I+FGK)^{-1}$, resp.). The perturbed closed-loop (open-loop) system is shown in Fig. III.2 (Fig. III.3, respectively).

Note that the changes of the closed-loop, and the open-loop system input-output maps due to the plant perturbation ΔG are given by

$$\Delta H_{yr} := \tilde{H}_{yr} - H_{yr} = \tilde{G}K(I+F\tilde{G}K)^{-1} - GK(I+FGK)^{-1} \tag{III.2}$$

$$\Delta H_{y_0r} := \tilde{H}_{y_0r} - H_{y_0r} = \tilde{G}K_0 - GK_0 = \Delta G \cdot K_0 = \Delta G \cdot K(I+FGK)^{-1} \tag{III.3}$$

respectively.

Theorem III.1 below generalizes some of the results in [10,11,12] and establishes the <u>exact</u> relation between $\Delta \underset{\sim}{H}_{yr}$ and $\Delta \underset{\sim}{H}_{y_0 r}$, and thus makes precise the desensitization effect of feedback for <u>nonlinear</u> systems.

<u>Theorem III.1</u> (<u>Desensitization effect of feedback</u>)

Consider the nonlinear, feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9). Also consider the comparison open-loop system shown in Fig. III.1. Let $\Delta \underset{\sim}{H}_{yr}$ and $\Delta \underset{\sim}{H}_{y_0 r}$ denote the changes of the closed-loop, and the open-loop system input-output maps due to the plant perturbation $\Delta \underset{\sim}{G}$, respectively. Assume that

(a1) $\underset{\sim}{F}: \mathcal{Y}_e \to \mathcal{R}_e$ is <u>linear</u>;

(a2) the perturbed plant $\tilde{\underset{\sim}{G}}$ satisfies (I.9), i.e. $(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}$ is a well-defined nonlinear, causal map mapping $\mathcal{R}_e$ into $\mathcal{R}_e$;

(a3) $\tilde{\underset{\sim}{G}}\underset{\sim}{K}: \mathcal{R}_e \to \mathcal{Y}_e$ and $(\underset{\sim}{I}+\tilde{\underset{\sim}{G}}\underset{\sim}{K}\underset{\sim}{F})^{-1}: \mathcal{Y}_e \to \mathcal{Y}_e$ are C^1 maps,

then

$$\Delta \underset{\sim}{H}_{yr} = \int_0^1 [\underset{\sim}{I}+D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot \underset{\sim}{F}]^{-1} d\alpha \cdot \Delta \underset{\sim}{H}_{y_0 r} \quad , \text{ on } \mathcal{R}_e \tag{III.4}$$

where the Fréchet derivative [13, p. 32] $D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})$ is evaluated at $(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r+\alpha\Delta r)$ with $\Delta r := \underset{\sim}{F}\cdot \Delta \underset{\sim}{H}_{y_0 r}(r)$, $r \in \mathcal{R}_e$, and $\alpha \in [0,1]$.

<u>Proof of Theorem III.1</u>: see Appendix.

When the map $\tilde{\underset{\sim}{G}}\underset{\sim}{K}$ is <u>linear</u>, theorem III.1 reduces to the following well-known result [10; 11, p. 24-26].

<u>Corollary III.1.1</u> (<u>Linear case</u>):

Under the conditions stated in theorem III.1, if in addition, $\tilde{\underset{\sim}{G}}\underset{\sim}{K}$ is <u>linear</u>, then

$$\underset{\sim}{\Delta H}_{yr} = (\underset{\sim}{I}+\underset{\sim\sim\sim}{\tilde{G}KF})^{-1}\cdot \underset{\sim}{\Delta H}_{y_0 r} \quad , \text{ on } \mathcal{R}_e \tag{III.5}$$

Proof of Corollary III.1.1: Follows directly from the fact that $D(\underset{\sim\sim}{\tilde{G}K}) = \underset{\sim\sim}{\tilde{G}K}$, when $\underset{\sim\sim}{\tilde{G}K}$ is linear.

Remarks III.1:

(a) Theorem III.1 indicates that for a class of plant perturbations $\underset{\sim}{\Delta G}$, if $\underset{\sim}{K}$ and $\underset{\sim}{F}$ are chosen such that $\forall r \in \mathcal{R}_{d,e}$ $(\subset \mathcal{R}_e)$, the class of inputs of interest,

$$\left|\int_0^1 [\underset{\sim}{I}+D(\underset{\sim\sim}{\tilde{G}K})\cdot \underset{\sim}{F}]^{-1} d\alpha \cdot \underset{\sim}{\Delta H}_{y_0 r}(r)\right| \ll \left|\underset{\sim}{\Delta H}_{y_0 r}(r)\right| \tag{III.6}$$

then, for such inputs $r(\cdot)$, the change of output $(\underset{\sim}{\Delta H}_{yr}(r))$ in the feedback system S caused by the plant perturbation $\underset{\sim}{\Delta G}$ is much smaller than the corresponding change in the open-loop system. Thus, with appropriate feedback design, the nonlinear closed-loop system can be made less vulnerable to the perturbations on the plant and hence performs more closely to the desired input-output map.

(b) Equation (III.4) makes precise the concept (built upon <u>linear</u> cases) that if one makes the (linearized) inverse return difference small, then the closed-loop system is insensitive to the plant perturbations. Note that eqn. (III.4) states precisely where $D(\underset{\sim\sim}{\tilde{G}K})$ has to be evaluated and along what path the linearized inverse return difference map should be integrated.

(c) <u>Differential sensitivity</u>: suppose that $\underset{\sim}{G}$, $\underset{\sim}{H}_{yr}$ are invertible, then eqn. (III.4) implies that, since $\underset{\sim}{\Delta H}_{y_0 r} = \underset{\sim}{\Delta G}\cdot\underset{\sim}{G}^{-1}\cdot \underset{\sim\sim}{GK}(\underset{\sim}{I}+\underset{\sim\sim\sim}{FGK})^{-1}$,

$$\underset{\sim}{\Delta H}_{yr}\cdot \underset{\sim}{H}_{yr}^{-1} = \int_0^1 [\underset{\sim}{I}+D(\underset{\sim\sim}{\tilde{G}K})\cdot \underset{\sim}{F}]^{-1} d\alpha \cdot \underset{\sim}{\Delta G}\cdot \underset{\sim}{G}^{-1} \tag{III.7}$$

For $\underset{\sim}{\Delta G}$, hence Δr, sufficiently small, (III.7) can be approximated by

$$\Delta \underset{\sim}{H}_{yr} \cdot \underset{\sim}{H}_{yr}^{-1} \simeq [\underset{\sim}{I} + D(\underset{\sim}{G}\underset{\sim}{K}) \cdot \underset{\sim}{F}]^{-1} \cdot \Delta \underset{\sim}{G} \cdot \underset{\sim}{G}^{-1} \tag{III.8}$$

The map $[\underset{\sim}{I} + D(\underset{\sim}{G}\underset{\sim}{K}) \cdot \underset{\sim}{F}]^{-1}$ is thus a complete generalization of the classical <u>differential sensitivity function</u> (for linear time-invariant case, see, e.g. [14,15] for single-input single-output case, [10] for multi-input multi-output case; for some nonlinear case, see e.g. [11]).

(d) Consider the special case where $\underset{\sim}{G}$, $\underset{\sim}{K}$, $\underset{\sim}{F}$ are represented by some transfer function matrices G(s), K(s), F(s), respectively. To achieve desensitization with respect to the given plant G(s) by feedback, one may design K(s) and F(s) so that the <u>maximum singular value</u> of the matrix $[I + \tilde{G}(j\omega)K(j\omega)F(j\omega)]^{-1}$ be much less than 1 <u>over the frequency band of interest</u>.[5] Then, by Corollary III.1, $|(\Delta \underset{\sim}{H}_{yr} r)(j\omega)|_2 \ll |(\Delta \underset{\sim}{H}_{y_0 r} r)(j\omega)|_2$, for <u>any</u> $(\Delta \underset{\sim}{H}_{y_0 r} r)(j\omega) \in \mathbb{C}^{n_0}$ over the frequency band of interest. Note that this requirement is <u>not</u> equivalent to the following: "over the frequency band of interest, $|\lambda_i(j\omega)| \gg 1$, $\forall i$, where $\lambda_i(j\omega)$ is the i-th <u>eigenvalue</u> of $I + \tilde{G}(j\omega)K(j\omega)F(j\omega)$". Hence, in the linear, time-invariant, multi-input, multi-output case, plotting the eigenvalue loci of $I + \tilde{G}(j\omega)K(j\omega)F(j\omega)$ with ω as a parameter, although useful for stability studies [16,17], does <u>not</u> have the same <u>desensitization interpretation</u> as in the single-input, single-output case (see e.g. [14; 15, Chap. 11]).

<u>Discussion:</u>

A. <u>Desensitization and Feedback Structure</u>: We note that one feedback structure is not necessarily superior to another one in terms of sensitivity with respect to the plant. We compare the nonlinear, feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9) with the nonlinear, multi-loop, feedback system shown in Fig. III.4 which consists of the

(5) Note that for any physical system, $[I + \tilde{G}(j\omega)K(j\omega)F(j\omega)]^{-1} \to I$ as $|\omega| \to \infty$. Hence it is <u>impossible</u> to fulfill this requirement for <u>all</u> $\omega \in \mathbb{R}$.

same plant G and nonlinear, causal operators K_1, K_2, F_1 and F_2.

Suppose that the (nominal) closed-loop system input-output maps of these two nonlinear, feedback systems are identical, i.e.

$$GK(I+FGK)^{-1} = GK_2(I+F_2GK_2)^{-1}K_1[I+F_1GK_2(I+F_2GK_2)^{-1}K_1]^{-1} \tag{III.9}$$

Now we have the following result.

Proposition III.2:

If GK, GK_2, K_1 are linear, then eqn. (III.9) becomes

$$(I+GKF)^{-1}GK = [I+GK_2(F_2+K_1F_1)]^{-1}\cdot GK_2K_1 \tag{III.10}$$

Proof of Proposition III.2: see Appendix.

With eqn. (III.10), the relation of the (differential) sensitivities of the two feedback structures shown in Fig. I.1 and Fig. III.4 is made clear in the following remarks.

Remarks III.2;

(a) Suppose that, in addition, the maps F, F_1 and F_2 are also linear; then $(I+GKF)^{-1}$ and $[I+GK_2(F_2+K_1F_1)]^{-1}$ are the differential sensitivity functions (see equn. (III.8)) of the feedback systems shown in Fig. I.1 and Fig. III.4, respectively. Thus eqn. (III.10) exhibits a relation between these two differential sensitivity functions.

(b) In the special case where G, K, F are represented by some scalar transfer functions, eqn. (III.10) reduces to

$$\frac{[1+g(s)k(s)f(s)]^{-1}}{[1+g(s)k_2(s)\cdot\left(f_2(s)+k_1(s)f_1(s)\right)]^{-1}} = \frac{k_2(s)k_1(s)}{k(s)} \tag{III.11}$$

Hence, by appropriately designing $k(s)$, $k_1(s)$, $k_2(s)$, consistent with other requirements, we can make the feedback system shown in Fig. I.1 either more, or less sensitive (to plant perturbations, over the frequency band of interest) than the one shown in Fig. III.4.

(c) For a recent discussion of using local feedback to design an audio power amplifier, see [42].

B. Desensitization and Feedback Perturbations

Proposition III.3 below derives the exact relation between the relative change in the closed-loop system input-output map (due to perturbations on the feedback $\underset{\sim}{F}$) and the relative change in the feedback $\underset{\sim}{F}$, thus makes clear the tradeoff between the sensitivities of the closed-loop system with respect to the plant and to the feedback.

Proposition III.3 (Desensitization and feedback perturbation)

Consider the nonlinear, feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9), where the plant $\underset{\sim}{G}$ is perturbed and becomes $\tilde{\underset{\sim}{G}}$. Let the feedback map $\underset{\sim}{F}$ be perturbed and become $\tilde{\underset{\sim}{F}} := \underset{\sim}{F} + \Delta\underset{\sim}{F}$. Let $\underset{\sim}{H}_{yr} := \tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}$: $\mathcal{R}_e \to \mathcal{Y}_e$ and $\tilde{\underset{\sim}{H}}_{yr} := \tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\tilde{\underset{\sim}{F}}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1} = \underset{\sim}{H}_{yr} + \Delta\underset{\sim}{H}_{yr}: \mathcal{R}_e \to \mathcal{Y}_e$ be well-defined nonlinear, causal maps (thus $\Delta\underset{\sim}{H}_{yr}$ includes the effect of plant and feedback perturbations). Suppose that

(a1) $\underset{\sim}{F}: \mathcal{Y}_e \to \mathcal{R}_e$ is linear;

(a2) $\underset{\sim}{F}^{-1}: \mathcal{R}_e \to \mathcal{Y}_e$ and $\tilde{\underset{\sim}{H}}_{yr}^{-1}: \mathcal{Y}_e \to \mathcal{R}_e$ are well-defined, causal maps;

(a3) $\tilde{\underset{\sim}{G}}\underset{\sim}{K}$ and $(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}$ are C^1 maps.

Then

$$\Delta\underset{\sim}{H}_{yr}\cdot\tilde{\underset{\sim}{H}}_{yr}^{-1} = \{\int_0^1 [\underset{\sim}{I}+D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1}d\alpha-\underset{\sim}{I}\}\cdot\underset{\sim}{F}^{-1}\cdot\Delta\underset{\sim}{F}, \text{ on } \mathcal{Y}_e \qquad \text{(III.12)}$$

where the Fréchet derivative $D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})$ is evaluated at $(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r+\alpha\Delta r)$ with $\Delta r := -\Delta\underset{\sim}{F}\cdot\tilde{\underset{\sim}{H}}_{yr}r$, $r \in \mathcal{R}_e$ and $\alpha \in [0,1]$.

Proof of Proposition III.3: see Appendix.

Remarks III.3:

(a) Note that if we choose to desensitize the closed-loop system with respect to the plant $\underset{\sim}{G}$ by making the inverse linearized return difference $[\underset{\sim}{I}+D(\underset{\sim}{\tilde{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1}$ "small" over the neighborhood of $\mathcal{R}_{d,e}$, the class of inputs of interest, as is suggested by eqn. (III.4), then $\int_0^1[\underset{\sim}{I}+D(\underset{\sim}{\tilde{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1}d\alpha-\underset{\sim}{I}\simeq-\underset{\sim}{I}$ and by eqn. (III.12), $\Delta\underset{\sim}{H}_{yr}\cdot\underset{\sim}{\tilde{H}}_{yr}^{-1}\simeq-\underset{\sim}{F}^{-1}\cdot\Delta\underset{\sim}{F}$ on $\mathcal{Y}_{d,e}:=\underset{\sim}{\tilde{H}}_{yr}\mathcal{R}_{d,e}$. Thus, the relative change in $\underset{\sim}{H}_{yr}$ is approximately equal to the relative change in the feedback $\underset{\sim}{F}$; consequently, the closed-loop system is insensitive to the plant perturbations but sensitive to the feedback perturbations.

(b) In the special case where $\underset{\sim}{\tilde{G}}$, $\underset{\sim}{K}$, $\underset{\sim}{F}$ are represented by some scalar transfer functions, eqn. (III.12) reduces to the classical result: over the frequency band of interest, if $|1+\tilde{g}(j\omega)k(j\omega)f(j\omega)|\gg 1$, then $\frac{\Delta h_{yr}/\tilde{h}_{yr}}{\Delta f/f}\simeq-1$.

(c) It is often advantageous to trade the insensitivity with respect to the feedback map $\underset{\sim}{F}$ for the insensitivity with respect to the plant $\underset{\sim}{G}$, since the feedback $\underset{\sim}{F}$ is usually operated at a low power level and hence can be built with inexpensive, high quality components.

C. Desensitization and Instability

It is well-known (see e.g. [14, p. 141-143]) that, for most linear, time-invariant, single-input, single-output feedback systems, the closed-loop system stability requirement imposes an upper bound on the system loop gain, thus the stability requirement limits the benefit of desensitization by feedback. We show below that such a constraint still holds for a large class of linear, time-invariant, multi-input multi-output systems.

Consider the feedback system S shown in Fig. I.1 where $\underset{\sim}{K}$, $\underset{\sim}{G}$ and $\underset{\sim}{F}$ are represented by $kM\in\mathbb{R}^{n_i\times n_o}$, $G(s)\in\mathbb{R}(s)^{n_o\times n_i}$, $I_{n_o}\in\mathbb{R}^{n_o\times n_o}$, respectively where $k\in\mathbb{R}_+$. To achieve desensitization with respect to the given plant G(s) by feedback, we may choose $k \in \mathbb{R}_+$ as large as possible so that the maximum singular value of the matrix $[I+k\tilde{G}(j\omega)M]^{-1}$ be much less than 1 over the frequency band of interest. However, stability considerations often impose an upper bound on the allowable k's. More precisely, we have the following proposition.

Proposition III.4 (Desensitization and instability)

Consider the feedback system S shown in Fig. I.1, where $\underset{\sim}{K}$, $\underset{\sim}{G}$, $\underset{\sim}{F}$ are represented by $kM \in \mathbb{R}^{n_i \times n_o}$, $G(s) \in \mathbb{R}(s)^{n_o \times n_i}$, $I \in \mathbb{R}^{n_o \times n_o}$, respectively, with $k > 0$, and $s \mapsto \det[I+k\,G(s)M] \not\equiv$ constant. Assume that[6] $\forall i = 1,2,\ldots,n_o$, and $\forall j = 1,2,\ldots,n_i$,

$$\partial[d_{ij}(s)] - \partial[n_{ij}(s)] \geq 3 \qquad \text{(III.14)}$$

where $\frac{n_{ij}}{d_{ij}}$ is the $(i,j)^{th}$ element of $G(s) \in \mathbb{R}(s)^{n_o \times n_i}$. Then, for $k \in \mathbb{R}_+$ sufficiently large, $\det[I+kG(s)\cdot M]$ has $\mathring{\mathbb{C}}_+$-zeros with real parts which tends to $+\infty$ as $k \to \infty$.

Proof of Proposition III.4: see Appendix.

Remarks III.4:

(a) Since $\det[I+kG(s)M]$ is equal to the ratio of the closed-loop sytsem characteristic polynomial to the open-loop system characteristic polynomial (see e.g. [18]), Proposition III.4 states a condition under which the closed-loop system becomes unstable for k sufficiently large.

(b) When $n_i = n_o = 1$, i.e., single-input single-output case, Proposition III.4 reduces to the classical result which can be easily proved by, e.g., the root locus method (see e.g. [14, p. 141-143]).

(6) Recall that if for some (i,j), $n_{ij}(s) \equiv 0$, then $\partial n_{ij} := -\infty$.

III.2 DISTURBANCE ATTENUATION

All physical systems operate in some environment where they are subjected to some "uncontrollable" disturbances. If we knew exactly these disturbances, then we could program (in advance) the system inputs such that the effect of these disturbances be cancelled out. However, in most real systems, there is either no complete knowledge of such disturbances (temperature, wind, wear, load changes, etc.) or the cost of measuring them and compensating for them is prohibitive; hence such "open-loop" design based on cancellation is not practical and we have to resort to feedback. The analysis below shows exactly what feedback can achieve for disturbances attenuation.

Consider the nonlinear, feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9) but subjected to some additive external disturbances as shown in Fig. III.5 where

$d_i(\cdot)$ is the system-input disturbance,

$d_g(\cdot)$ is the plant-input disturbance,

$d_o(\cdot)$ is the system-output disturbance,

$d_f(\cdot)$ is the feedback-path disturbance.

It is intuitively clear that, in general, an error-driven feedback system such as the one shown in Fig. III.5 cannot attenuate the input disturbances $d_i(\cdot)$ and the feedback-path disturbance $d_f(\cdot)$, since such feedback systems cannot distinguish the system-input disturbance $d_i(\cdot)$ from the system input $r(\cdot)$ and the feedback path disturbance $d_f(\cdot)$ from the system output $y(\cdot)$. As seen from Fig. III.5, the error signal $\tilde{e}(\cdot)$ is affected by the corrupting signals d_i and d_f; hence $\tilde{e}(\cdot)$ cannot drive the plant as desired (in some cases, judicious filtering may alleviate such problems). We expect that feedback can reduce the effect of plant-input and system-output disturbances on the system output;

indeed such effects could be modeled by some appropriate plant perturbations, and their effect on the system output has been shown, in sec. III.1, to be reducible by feedback.

The propositions below evaluate exactly the effects of the disturbances d_i, d_o, d_f, d_g on the system output $y(\cdot)$. Note that unlike the <u>linear</u> case, the effect of the disturbance d_α ($\alpha = i,o,f,g$) on the system output $y(\cdot)$ is <u>not</u> given by $\underset{\sim}{H}_{yd_\alpha}(d_\alpha)$, where $\underset{\sim}{H}_{yd_\alpha} : d_\alpha \longmapsto y$ is calculated when r and all the other disturbances are set to <u>zero</u>.

<u>Proposition III.5</u> (<u>System-output disturbance, feedback-path disturbance and feedback</u>)

Consider the nonlinear, feedback system shown in Fig. III.5 and described by eqns. (I.1)-(I.9). Let $\tilde{\underset{\sim}{G}}u := \underset{\sim}{G}\tilde{u}+d_o$ and $\tilde{\underset{\sim}{F}}y := \underset{\sim}{F}(\tilde{y}+d_f)$. Suppose that

(a1) $\underset{\sim}{F}: \mathcal{Y}_e \to \mathcal{R}_e$ is <u>linear</u>;

(a2) $\underset{\sim}{G}\underset{\sim}{K}$ and $(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}$ are C^1 maps.

U.t.c.

(i) if $d_o \neq 0$ and $d_i = d_g = d_f = 0$, then $\forall r \in \mathcal{R}_e$,

$$\begin{aligned}\Delta y &:= \tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r) - \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}(r) \\ &= \int_0^1 [\underset{\sim}{I}+D(\underset{\sim}{G}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1} d\alpha \cdot d_o \end{aligned} \tag{III.15}$$

where the Fréchet derivative $D(\underset{\sim}{G}\underset{\sim}{K})$ is evaluated at $(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r+\alpha\Delta r)$ with $\Delta r = \underset{\sim}{F}\cdot d_o$ and $\alpha \in [0,1]$.

(ii) if $d_o \neq 0$, $d_f \neq 0$ and $d_i = d_g = 0$, then $\forall r \in \mathcal{R}_e$

$$\begin{aligned}\Delta y &:= \tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\tilde{\underset{\sim}{F}}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r) - \tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r) \\ &= \{\int_0^1 [\underset{\sim}{I}+D(\underset{\sim}{G}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1} d\alpha - \underset{\sim}{I}\}\cdot d_f \end{aligned} \tag{III.16}$$

where the Fréchet derivative $D(\underset{\sim}{G}\underset{\sim}{K})$ is evaluated at $(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r+\alpha\Delta r)$ with $\Delta r = -\underset{\sim}{F}\cdot d_f$ and $\alpha \in [0,1]$.

Proof of Proposition III.5: see Appendix.

Proposition III.6: (Plant-input disturbance and feedback)

Consider the nonlinear, feedback system shown in Fig. III.5 and described by eqns. (I.1)-(I.9), where $d_i = d_o = d_f = 0$. Let $\tilde{G}\tilde{u} := G(\tilde{u}+d_g)$. Suppose that

(a1) $F: \mathcal{Y}_e \to \mathcal{R}_e$ is linear;

(a2) $\tilde{G}K$, $(I+F\tilde{G}K)^{-1}$ are C^1 maps.

Then, $\forall r \in \mathcal{R}_e$

$$\begin{aligned}\Delta y &:= \tilde{G}K(I+F\tilde{G}K)^{-1}(r) - GK(I+FGK)^{-1}(r) \\ &= \{\int_0^1 [I+D(\tilde{G}K)\cdot F]^{-1} d\alpha\}\cdot[\int_0^1 DG(u+\beta d_g)d\beta]\cdot d_g\end{aligned} \tag{III.17}$$

where the Fréchet derivative $D(\tilde{G}K)$ is evaluated at $(I+F\tilde{G}K)^{-1}(r+\alpha\Delta r)$ with $\Delta r = F[G(u+d_g)-G(u)]$, $u := K(I+FGK)^{-1}r$, and $\alpha \in [0,1]$.

Proof of Proposition III.6: see Appendix.

Proposition III.7: (System-input disturbance and feedback)

Consider the nonlinear, feedback system shown in Fig. III.5 and described by eqns. (I.1)-(I.9), where $d_g = d_o = d_f = 0$. Suppose that F, GK and $(I+FGK)^{-1}$ are C^1 maps, then $\forall r \in \mathcal{R}_e$,

$$\begin{aligned}\Delta y &:= GK(I+FGK)^{-1}(r+d_i) - GK(I+FGK)^{-1}(r) \\ &= \int_0^1 D(GK)[I+DF\cdot D(GK)]^{-1} d\alpha\cdot d_i\end{aligned} \tag{III.18}$$

where the Fréchet derivative $D(GK)$ is evaluated at $(I+FGK)^{-1}(r+\alpha d_i)$ and DF is evaluated at $GK[I+FGK]^{-1}(r+\alpha d_i)$ with $\alpha \in [0,1]$.

Proof of Proposition III.7: Follows directly from Taylor's expansion theorem [19, p. 190].

Comments on Propositions III.5-III.7:

(a) Eqns. (III.15)-(III.18) show exactly how feedback can reduce the effects of various external disturbances on the system output. Note that, by eqns. (III.15) and (III.16), simultaneous disturbance attenuation of d_o and d_f is, in general, impossible.

(b) In the special case that G, K and F are linear, the effects of the disturbances d_o, d_f, d_g, d_i on the system output reduce to $(I+GKF)^{-1}d_o$, $[(I+GKF)^{-1}-I]d_f$, $(I+GKF)^{-1}Gd_g$ and $GK(I+FGK)^{-1}d_i$, respectively. Note that in this case, those disturbance-output maps are related by, with obvious notation

$$H_{yd_o} = I + H_{yd_f} = I - H_{yd_g}KF = I - H_{yd_i}F \tag{III.19}$$

III.3 LINEARIZING EFFECT [20](7)

It is often required that the map from the system input to the system output is as linear as possible, e.g. HiFi amplifiers, telephone repeaters, measuring instruments, pen recorders, etc. How to design such a system which uses some inherently nonlinear plant is an important problem. From the discussion in section II, we know that if the feedback map F is linear and if the inverse loop gain is small, then the closed-loop system input-output map will be close to a linear map. Thus we expect that feedback has a linearizing effect on an otherwise nonlinear system. To make this idea precise, we first introduce the concept of nonlinearity measure.

A Nonlinearity Measure

Let $\mathcal{U}_e$ be an extended normed (input) space. Let $\mathcal{Y}_e$ be an extended semi-semi-normed (output) space. Let $\mathcal{N} = \{N: \mathcal{U}_e \to \mathcal{Y}_e |$, N is causal, nonlinear$\}$.

(7) The results of this section were obtained with the collaboration of A. N. Payne.

Let $\mathcal{L} = \{\underset{\sim}{L}: \mathcal{U}_e \to \mathcal{Y}_e | \underset{\sim}{L}$ is causal, linear$\}$. Now consider $\underset{\sim}{N} \in \mathcal{N}$ and $\mathcal{V} \subset \mathcal{U}_e$, a set of inputs of interest. Intuitively, the degree of nonlinearity of $\underset{\sim}{N}$, when $\underset{\sim}{N}$ is driven by $u \in \mathcal{V}$, may be measured by the error $|\underset{\sim}{N}u - \underset{\sim}{L}u|$ for $u \in \mathcal{V}$, where $\underset{\sim}{L} \in \mathcal{L}$ is a "best" linear approximation of $\underset{\sim}{N}$ over $\mathcal{V}$. More precisely, we introduce the following definition.

<u>Definition III.8</u> (<u>Nonlinearity measure</u>)

Let $\underset{\sim}{N} \in \mathcal{N}$, $\mathcal{V} \subset \mathcal{U}_e$ and $T \in \mathcal{T}$. The <u>nonlinearity measure</u> of $\underset{\sim}{N}$ <u>over</u> $\mathcal{V}$ <u>with respect to</u> T is the non-negative real number defined by

$$\delta_T(\underset{\sim}{N},\mathcal{V}) := \inf_{\underset{\sim}{L}\in\mathcal{L}} \sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}u|_T \; . \tag{III.20}$$

<u>Remarks III.8</u>:

(a) $\underset{\sim}{L}^* \in \mathcal{L}$ is thus said to be a <u>best linear approximation</u> of $\underset{\sim}{N}$ over $\mathcal{V}$ iff $\underset{\sim}{L}^*$ is a minimizer of (III.20), i.e., $\delta_T(\underset{\sim}{N},\mathcal{V}) = \sup_{u\in\mathcal{V}} |\underset{\sim}{N}u - \underset{\sim}{L}^*u|_T$.

(b) In the case where $\mathcal{Y}_e$ is a <u>seminormed</u> space, we then have the nonlinearity measure of $\underset{\sim}{N}$ over $\mathcal{V}$ <u>with respect to</u> $\sup\mathcal{T}$ (typically, $\sup\mathcal{T} = \infty$) and eqn. (III.20) becomes $\delta(\underset{\sim}{N},\mathcal{V}) = \inf_{\underset{\sim}{L}\in\mathcal{L}} \sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}u|$.

(c) The well-known describing function (see e.g. [21,22]) is the best linear approximation of a nonlinear operator with respect to our nonlinearity measure (III.20) provided that $\mathcal{V}$, the class of inputs, is suitably defined. Recall that the criterion which the describing function method uses to find a best linear approximation $\underset{\sim}{L}$ of a nonlinear system $\underset{\sim}{N}$ is to minimize the mean square error $\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau [(\underset{\sim}{N}u)(t)-(\underset{\sim}{L}u)(t)]^2 dt$ over a class of inputs $u(\cdot)$ (usually $u(t) = a\sin\omega t$, $a > 0$, $\omega > 0$, and thus $\underset{\sim}{L}$ depends on the parameters a, ω). To see the relation between the describing function and our nonlinearity measure, let $a > 0$, $\omega > 0$ be given, let $\mathcal{V}$ be the <u>singleton</u> $\{a \sin \omega t\}$ and $\mathcal{Y}_e = \{y(\cdot): \mathbb{R}_+ \to \mathbb{R}^{n_0} | y(\cdot)$ is asymptotically $\frac{2\pi}{\omega}$-periodic[8]$\}$ be equipped with

[8] A function $y(\cdot): \mathbb{R}_+ \to \mathbb{R}^n$ is said to be <u>asymptotically</u> <u>T-periodic</u> iff $y(\cdot) = y_T(\cdot) + y_o(\cdot)$, where $y_T(\cdot)$ is a T-periodic function and $y_o(t)$ tends to θ_n as $t \to \infty$.

the semi-norm $|y| := [\lim_{T\to\infty} \frac{1}{T}\int_0^T |y(t)|^2 dt]^{1/2}$, then a best linear approximation to the nonlinear system $\underset{\sim}{N}$ according to our definition III.8 is a minimizer of $\lim_{T\to\infty} \frac{1}{T}\int_0^T [(\underset{\sim}{N}u)(t)-(\underset{\sim}{L}u)(t)]^2 dt$ which is precisely the describing function of $\underset{\sim}{N}$ with respect to the inputs $u(t) = a\sin\omega t$. Note that in this case, the minimizer of (III.20) (i.e. the describing function of $\underset{\sim}{N}$ with respect to $u(\cdot)$) is parametrized by a and ω.

(d) With the framework of extended spaces, we can discuss the nonlinearity measure of a nonlinear system over a bounded time interval, say, $[0,T]$. Note that a nonlinear system $\underset{\sim}{N}$ may have its nonlinearity measure $\delta_T(\underset{\sim}{N},\mathscr{V}) = 0$, $\forall T \leq T^* \in \mathscr{T}$, but $\delta_T(\underset{\sim}{N},\mathscr{V}) \neq 0$ for $T > T^*$, simply because $\underset{\sim}{N}$ is operating within the linear range of its characteristics before time T^*.

(e) At the cost of some complication, the class of nonlinear operators under consideration can be extended to include the nonlinear dynamical relations.

(f) Other nonlinearity measures may be defined, e.g., we can define

$$\delta_T(\underset{\sim}{N},\mathscr{V}) = \inf_{\underset{\sim}{L}\in\mathscr{L}} \sup_{\substack{u\in\mathscr{V}\\ u_T\neq 0}} \frac{|\underset{\sim}{N}u-\underset{\sim}{L}u|_T}{|u|_T}$$

. Note that such nonlinearity measure does satisfy all the remarks mentioned above and all the properties stated below. However, we have not been able to obtain results similar to the Theorem III.14 below.

Properties of the Nonlinearity Measure $\delta_T(\underset{\sim}{N},\mathscr{V})$

Proposition III.9:

If $\underset{\sim}{N}_2 = \underset{\sim}{N}_1 + \underset{\sim}{L}_1$ for some $\underset{\sim}{L}_1 \in \mathscr{L}$, then $\delta_T(\underset{\sim}{N}_1,\mathscr{V}) = \delta_T(\underset{\sim}{N}_2,\mathscr{V})$, $\forall T \in \mathscr{T}$, $\forall \mathscr{V} \subset \mathscr{U}_e$.

Proposition III.10:

If $\mathscr{V}_1 \subset \mathscr{V}_2 \subset \mathscr{U}_e$, then $\delta_T(\underset{\sim}{N},\mathscr{V}_1) \leq \delta_T(\underset{\sim}{N},\mathscr{V}_2)$, $\forall T \in \mathscr{T}$.

Proposition III.11:

Suppose that $\forall T \in \mathcal{T}$, $P_T\mathcal{Y}_e$ is a normed space.and that $N0 = 0$. U.t.c. if N is Fréchet differentiable[9] at 0, then $\forall T \in \mathcal{T}$,

$$0 \leq \delta_T(N,\bar{B}_T(0;\beta)) \leq \sup_{u\in\bar{B}_T(0;\beta)} |Nu-DN(0)\cdot u|_T \to 0, \text{ as } \beta \to 0 \qquad \text{(III.21)}$$

where $\bar{B}_T(0;\beta) := \{u\in\mathcal{U}_e \,|\, |u|_T \leq \beta\}$ and $DN(0)$ denotes the Fréchet derivative of N at 0.

Proposition III.12:

Let $\mathcal{V}\subset\mathcal{U}_e$ be the set of inputs of interest. If for some $L \in \mathcal{L}$, $Nu = Lu$, $\forall u \in \mathcal{V}$, then $\delta_T(N,\mathcal{V}) = 0$, $\forall T \in \mathcal{T}$. In particular, if $N \in \mathcal{L}$, then $\delta_T(N,\mathcal{V}) = 0$, $\forall T \in \mathcal{T}$, $\forall \mathcal{V}\subset\mathcal{U}_e$.

Proposition III.13:

Let $\mathcal{V}\subset\mathcal{U}_e$ be the set of inputs of interest. Let $\mathcal{L}$ be specialized into the class of continuous,[10] linear, causal operators mapping $\mathcal{U}_e$ into $\mathcal{Y}_e$. Suppose that

(a1) $\forall T \in \mathcal{T}$, $P_T\mathcal{Y}_e$ is a Banach space;

(a2) $\forall T \in \mathcal{T}$, $\mathcal{V}\subset\mathcal{U}_e$ is bounded, i.e. $\sup_{u\in\mathcal{V}} |u|_T < \infty$;

(a3) $\forall T \in \mathcal{T}$, $\exists\beta > 0$ such that $\mathcal{V}\supset \bar{B}_T(0;\beta) := \{u\in\mathcal{U}_e \,|\, |u|_T \leq \beta\}$.

U.t.c. if for some $T \in \mathcal{T}$, $\delta_T(N,\mathcal{V}) = 0$, then, $\exists L^* \in \mathcal{L}$ such that

$$|Nu-L^*u|_T = 0, \ \forall u \in \mathcal{V}. \qquad \text{(III.22)}$$

Proofs of Propositions III.9-III.13: see Appendix.

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(9) $N \in \mathcal{N}$ is said to be Fréchet differentiable at x iff $\forall T \in \mathcal{T}$, P_TN is Fréchet differentiable at x.

(10) $L \in \mathcal{L}$ is said to be continuous iff $\forall T \in \mathcal{T}$, P_TL is continuous, i.e. $\forall T \in \mathcal{T}$, $|L|_T := \sup_{\substack{u\in\mathcal{U}_e \\ |u|_T\neq 0}} \frac{|Lu|_T}{|u|_T} < \infty$.

Comments on Propositions III.9-III.13:

(a) Proposition III.9 states the obvious fact that if two nonlinear, causal operators differ by a linear causal operator, then they must have the same nonlinearity measure. It is also intuitively clear, from a perturbational viewpoint, that if a linear, causal operator is subject to some nonlinear causal perturbation, then the nonlinearity measure of the perturbed nonlinear, causal operator must be the same as that of the nonlinear perturbation.

(b) Proposition III.10 emphasizes the fact that the nonlinearity measure depends on the class of inputs we are considering: the larger the class of inputs we consider, the greater the nonlinearity measure of operator $\underset{\sim}{N}$.

(c) Proposition III.11 is another way of stating the well-known fact that (since $\underset{\sim}{N}0 = 0$) the best local linear approximation of a Fréchet differentiable nonlinear operator $\underset{\sim}{N}$ at the operating point 0 is the Fréchet derivative of $\underset{\sim}{N}$ at 0. Note that by eqn. (III.21), $\delta_T(\underset{\sim}{N},\bar{B}_T(0;\beta)) \to 0$ as $\beta \to 0$, i.e. $\underset{\sim}{N}$ behaves locally like a linear operator as we expected.

(d) Proposition III.12 states that $\delta_T(\underset{\sim}{N},\mathcal{V})$ satisfies the natural requirement for a nonlinearity measure, namely, if $\underset{\sim}{N}$ behaves as a linear causal operator over the class of inputs $\mathcal{V}$ in the time interval $[0,T] \subset \mathcal{T}$, then $\delta_T(\underset{\sim}{N},\mathcal{V}) = 0$.

(e) With some mild technical assumptions, proposition III.13 establishes the following desirable property of $\delta_T(\underset{\sim}{N},\mathcal{V})$: if $\delta_T(\underset{\sim}{N},\mathcal{V}) = 0$, then $\underset{\sim}{N}$ behaves like a linear, causal operator over $\mathcal{V}$ in the time interval $[0,T] \subset \mathcal{T}$. Note that if $\delta_T(\underset{\sim}{N},\mathcal{V}) = 0$, then $\delta_{T'}(\underset{\sim}{N},\mathcal{V}) = 0$, $\forall T' \leq T$.

Linearizing Effect of Feedback

With the nonlinearity measure defined in eqn. (III.20), we now can make precise the idea that feedback has a linearizing effect on an otherwise nonlinear system.

Note that the nonlinearity measure defined in (III.20) allows us to compare nonlinear systems by their degree of nonlinearity. However, a meaningful comparison requires careful choice of the sets of inputs since the nonlinearity measure depends on the set of inputs we are considering. From an engineering point of view, we are interested in comparing systems which produce desired outputs (e.g., signals within certain frequency band or dynamical range). Hence in the following discussion of the linearizing effect of feedback, we shall compare the nonlinearity of measure of a nonlinear plant and of a feedback system which includes such a plant; we shall choose a set of inputs for each system so that both systems produce the same set of desired outputs.

Consider the nonlinear feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9), except now that

$\mathcal{Y}_e$ is an extended seminormed space (III.23)

$$\left.\begin{array}{l} \text{Let } \mathcal{Y}_{d,e} \subset \mathcal{Y}_e \text{ be the set of desired outputs. Let } \mathcal{R}_{d,e} \subset \mathcal{R}_e \\ \text{be the set of system-inputs } r(\cdot) \text{ such that } \underset{\sim}{H}_{yr}\, \mathcal{R}_{d,e} = \mathcal{Y}_{d,e}. \\ \text{Let } \mathcal{U}_{d,e} \subset \mathcal{U}_e \text{ be the set of plant-inputs } u(\cdot) \text{ such that} \\ \underset{\sim}{G}\,\mathcal{U}_{d,e} = \mathcal{Y}_{d,e}. \end{array}\right\} \qquad \text{(III.24)}$$

Now we have the following theorem:

Theorem III.14 (Linearizing effect of feedback)

Consider the nonlinear, feedback system S shown in Fig. I.1 and described by eqns. (I.1)-(I.9) and (III.23)-(III.24). For some $T \in \mathcal{T}$, let $\underset{\sim}{L}^*_G \in \mathcal{L}$ be a best linear approximation to $\underset{\sim}{G}$, i.e.

$$\delta_T(\underset{\sim}{G}, \mathcal{U}_{d,e}) = \sup_{u \in \mathcal{U}_{d,e}} |\underset{\sim}{G}u - \underset{\sim}{L}^*_G u|_T \qquad \text{(III.25)}$$

Assume that $\underset{\sim}{F}: \mathcal{Y}_e \to \mathcal{R}_e$ and $\underset{\sim}{K}: \mathcal{R}_e \to \mathcal{U}_e$ are linear, causal and that the linear map $(\underset{\sim}{I} + \underset{\sim}{L}^*_G \underset{\sim}{K}\underset{\sim}{F})$ has a causal inverse, then

$$\delta_T(\underset{\sim}{H}_{yr},\mathcal{R}_{d,e}) \le \rho\cdot\delta_T(\underset{\sim}{G},\mathcal{U}_{d,e}) \tag{III.26}$$

where $\underset{\sim}{H}_{yr} := \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}$ is the closed-loop input-output map and

$$\rho := \sup_{\substack{y\in\tilde{\mathcal{Y}}_e \\ y_T\neq 0}} \frac{|(\underset{\sim}{I}+\underset{\sim}{L}^*_G\underset{\sim}{K}\underset{\sim}{F})^{-1}y|_T}{|y|_T} \tag{III.27}$$

with $\tilde{\mathcal{Y}}_e := (\underset{\sim}{G}-\underset{\sim}{L}^*_G)\mathcal{U}_{d,e}$.

<u>Proof of Theorem III.14</u>: see Appendix.

<u>Remarks III.14</u>:

(a) In a design problem, given some $\underset{\sim}{G}\in\mathcal{N}$ together with its best linear approximation $\underset{\sim}{L}^*_G$ over $\mathcal{U}_{d,e}$ with respect to some $T\in\mathcal{T}$, if one designs $\underset{\sim}{K}$, $\underset{\sim}{F}$ such that ρ be much less than 1, consistent with other requirements, then by eqn. (III.26), $\delta_T(\underset{\sim}{H}_{yr},\mathcal{R}_{d,e}) \ll \delta_T(\underset{\sim}{G},\mathcal{U}_{d,e})$, i.e. for the class of inputs under consideration and for the time interval of interest, the closed-loop system is much closer to a linear system than $\underset{\sim}{G}$ itself. This result clearly exhibits the linearizing effect of feedback.

(b) Note that ρ is defined via the <u>inverse</u> linearized return difference operator $(\underset{\sim}{I}+\underset{\sim}{L}^*_G\underset{\sim}{K}\underset{\sim}{F})^{-1}$ (when we break the loop <u>after the plant</u> $\underset{\sim}{G}$): since the nonlinear plant $\underset{\sim}{G}$ can be thought as a <u>linear</u> plant $\underset{\sim}{L}^*_G$ being subject to some <u>nonlinear</u> perturbation $\underset{\sim}{G}-\underset{\sim}{L}^*_G$, and we know that (see eqn. (III.8) or [43]) as a first order approximation, the effect of a <u>nonlinear</u> perturbation on the otherwise <u>linear</u> closed-loop system is reduced by the factor $(\underset{\sim}{I}+\underset{\sim}{L}^*_G\underset{\sim}{K}\underset{\sim}{F})^{-1}$ by feedback.

(c) If $\underset{\sim}{L}^*_G$, $\underset{\sim}{K}$, $\underset{\sim}{F}$ are linear and <u>time-invariant</u>, thus represented by transfer function matrices $L^*_G(j\omega)$, $K(j\omega)$, $F(j\omega)$, respectively), then $\rho \ll 1$ if the maximum singular value of $[I+L^*_G(j\omega)K(j\omega)F(j\omega)]^{-1}$ is small <u>over the frequencies of interest</u>.

Example III.1 (Single-input single-output memoryless system)

Consider the nonlinear, feedback system S shown in Fig. I.1, where $\underset{\sim}{G}$ is characterized by the piecewise-linear function shown in Fig. III.6, $\underset{\sim}{K}$ and $\underset{\sim}{F}$ are represetned by constant gains 10 and 1, respectively. It is easy to show that the closed-loop input-output map $\underset{\sim}{H}_{yr}$ is characterized by the piecewise-linear function shown in Fig. III.7. Now let us consider the case where $\mathcal{Y}_{d,e} = \{y(\cdot): \mathbb{R}_+ \to \mathbb{R} \,\big|\, |y|_\infty \le 0.8\}$, then the corresponding $\mathcal{U}_{d,e} = \{u(\cdot): \mathbb{R}_+ \to \mathbb{R} \,\big|\, |u|_\infty \le 1.2\}$ and $\mathcal{R}_{d,e} = \{r: \mathbb{R}_+ \to \mathbb{R} \,\big|\, |r|_\infty \le 0.92\}$. A straightforward minmax calculation shows that the best linear approximation $\underset{\sim}{L}^*_G$ of $\underset{\sim}{G}$ is a constant gain of 0.6 and the nonlinearity measure of $\underset{\sim}{G}$ is $\delta_T(\underset{\sim}{G},\mathcal{U}_{d,e}) = 0.12$, $\forall T \in \mathcal{T}$; more precisely, $\delta_T(\underset{\sim}{G},\mathcal{U}_{d,e}) = \sup_{u\in\mathcal{U}_{d,e}} |\underset{\sim}{G}u-0.6u|_{\infty,T} = 0.12$. Similarly, $\delta_T(\underset{\sim}{H}_{yr},\mathcal{R}_{d,e}) = \sup_{r\in\mathcal{R}_{d,e}} |\underset{\sim}{H}_{yr}r - \frac{6}{7}r|_{\infty,T} = \frac{0.12}{7}$, $\forall T \in \mathcal{T}$. Thus the nonlinearity measure of $\underset{\sim}{G}$ has been reduced by 7 by feedback. Note that $\rho = \dfrac{1}{1+0.6\times 10} = \dfrac{1}{7}$, i.e. for this example, the equality holds in eqn. (III.26). The best linear approximations of $\underset{\sim}{G}$ and $\underset{\sim}{H}_{yr}$ are shown, by the broken lines, in Fig. III.6 and Fig. III.7, respectively. To further illustrate the linearizing effect of feedback, we drive the nonlinear plant $\underset{\sim}{G}$ with $u = 1.2\sin\omega t$ and the closed-loop system $\underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}$ with $r = 0.92\sin\omega t$. The corresponding (open-loop system) output y_0 and the (closed-loop) output y are shown in Fig. III.8.

In general, it is quite difficult to calculate the nonlinearity measure δ_T of a nonlinear dynamical system and to obtain the best linear approximation of such a system. However, for a given nonlinear plant $\underset{\sim}{G}$, we may illustrate the linearizing effect of feedback by computing the closed-loop system output with respect to several different compensator gains while the closed-loop system is driven by some test signals. Examples II.1 and II.2 in section II clearly exhibit the linearizing effect of feedback on nonlinear dynamical systems. Note that the higher the compensator gain is, the more linear the closed-loop system appears to be as we expected from the result of theorem III.14 (since ρ defined in eqn. (III.27) decreases as the gain of $\underset{\sim}{K}$ increases).

III.4 ASYMPTOTIC TRACKING AND DISTURBANCE REJECTION

One important application of feedback in control is the servomechanism design which aims at asymptotic tracking and asymptotic disturbance rejection. Let us consider the asymptotic tracking problem. From the discussion of generalized Black's formula in sec. II, we know that if we let $\underset{\sim}{F} = \underset{\sim}{I}$ in the nonlinear, feedback system S shown in Fig. I.1 and if we make the "forward-path gain" sufficiently large, then, asymptotically, the output $y(\cdot)$ will be approximately equal to the system input $r(\cdot)$. Thus we might intuitively guess that we can obtain <u>perfect</u> asymptotic tracking, i.e. <u>zero</u> steady state error, by requiring the "forward-path gain" be infinite <u>at the frequency of the system inputs</u>. This turns out to be correct. Indeed in the classical servomechanism design [23], an <u>integrator</u> is required in the compensator in order that the system output track <u>step</u> signals with zero steady-state error. For <u>multi-input</u>, <u>multi-output</u> systems, such a design principle has also been proven to be correct for linear (see e.g. [24,25,26]) as well as <u>nonlinear</u> cases (see e.g. [27]).

III.5 STABILIZATION

Stability is a primary concern of engineers since an unstable system is obviously useless. However, there are many inherently unstable systems such as rocket booster systems, nuclear reactors, some chemical reactors, etc. which are useful in practice and hence must be stabilized. Note that any open-loop stabilization scheme is doomed to failure in practice because it is based on some kind of cancellation which will eventually fail as a result of changes in element characteristics, effects of environment, etc. Hence feedback seems to be the only way out.

Many researchers have studied the use of feedback in stabilizing unstable systems. For lumped, linear, time-invariant systems, it has been shown that a constant state feedback (see e.g. [28,29]) or a dynamical output feedback (see e.g. [30]) can stabilize an unstable system; recently, Youla et. al. [31] gave a characterization of all stabilizing feedback controllers. For lumped, linear, time-varying systems, a time-varying state feedback can be obtained (see e.g. [32,38,39,40,41]) to stabilize an unstable system. For distributed, linear, time-invariant systems, state feedback can also stabilize unstable systems (see e.g. [33;34, chap. 14]). In contrast to linear cases, little is known about the nonlinear case except for some limiting cases. It should also be pointed out that little is known about how to proceed with the design of a, say, state feedback, stabilization scheme so that the resulting closed-loop system stability is very <u>robust</u> with respect to changes in the plant and/or the feedback map. In this aspect, for the linear time-invariant case, singular value analysis has provided some valuable information (see e.g. [44]).

IV. CONCLUSION

This paper has treated the fundamental properties of feedback for nonlinear, time-varying, multi-input, multi-output, distributed systems. We observed that the classical Black formula does <u>not</u> depend on the <u>linearity</u> nor the time-invariance assumptions; we used the input-output description of nonlinear systems to actually generalize Black's formula to the <u>nonlinear</u> case (Theorems II.1 to II.3). Our analysis then established achievable advantages of feedback, familiar to feedback engineers, for <u>nonlinear</u> systems (section III): theorem III.1 showed the exact relation between the changes in the open-loop and closed-loop input-output maps caused by nonlinear, not necessarily small, plant perturbations; propositions III.5-III.6 calculated the exact effect of various additive external disturbances on the output of a nonlinear system; theorem III.14 related the nonlinearity measure of a nonlinear plant and that of a feedback system including such a plant; sections III.4 and III.5 briefly reviewed the use of feedback to achieve asymptotic tracking and disturbance rejection, and to stabilize unstable plants, while references are given for more detailed discussion. These results showed precisely how to achieve desensitization, disturbance attenuation, linearizing, asymptotic tracking and disturbance rejection by feedback in nonlinear systems.

The benefits of feedback do not come without limitations or tradeoffs as propositions III.2-III.5 showed: proposition III.2 showed the relation between desensitization and feedback structure; proposition III.3 showed the tradeoff between the sensitivities of a nonlinear, feedback system with respect to the perturbations on the plant and on the feedback map; proposition III.4 showed that stability requirements restrict the achievable desensitization effect by feedback; proposition III.5 showed the tradeoff between the output disturbance attenuation and the feedback-path disturbance attenuation. Note that, due to the lack of appropriate language and tools, we did not discuss the tradeoff between the gain and bandwidth. Consequently, we did not explore the limitations on the benefits achievable by feedback imposed by the plant with fixed gain and and bandwidth (in the context of the Bode design method [45], the gain-bandwidth of a given active device imposes an upper bound on the return difference over a specified bandwidth).

Also note that we have only treated deterministic systems, i.e. no stochastic models were introduced for noise, perturbations, element variations, etc. Thus, in particular, we did not mention the well-known limitation on compensator gain caused by noise.

In clarifying the features of <u>nonlinear</u> systems that are required for feedback to be advantageous, this paper will help engineers obtain better understanding of nonlinear, feedback systems.

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APPENDIX

Proof of Theorem II.2:

Note that

$$
\begin{aligned}
\underset{\sim}{H}_{yr} &:= \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} \\
&= \underset{\sim}{F}^{-1}\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} && \text{(since } \underset{\sim}{F} \text{ is invertible)} \\
&= \underset{\sim}{F}^{-1}\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K}[(\underset{\sim}{I}+(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1})(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})]^{-1} && \text{(since } \underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K} \text{ is invertible)} \\
&= \underset{\sim}{F}^{-1}[\underset{\sim}{I}+(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}]^{-1} && \text{(A.1)}
\end{aligned}
$$

To estimate $\underset{\sim}{H}_{yr}r$ for $r \in \mathcal{R}_{d,e}$, we consider first $z := [\underset{\sim}{I}+(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}]^{-1}r$. To obtain for any $T \in \mathcal{T}$, z_T, note that $r = [I+(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}]z$, hence $z_T = r_T - (\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}z_T$. Now the Lipschitz constant [13, p. 63] of the right hand side, over $\mathcal{N}(\mathcal{R}_{d,e})$, is $\tilde{\gamma}_T[(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}] < 1$. By assumption (a3), the successive approximations starting with $z_T = r$ remain in $\mathcal{N}(\mathcal{R}_{d,e})$ forever; since the contraction constant is < 1, we have that

$$
|z-r|_T \leq \frac{|(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r|_T}{1-\tilde{\gamma}_T[(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}]} \tag{A.2}
$$

Thus, for each $T \in \mathcal{T}$,

$$
\begin{aligned}
|\underset{\sim}{H}_{yr}r-\underset{\sim}{F}^{-1}r|_T &= |\underset{\sim}{F}^{-1}(r-e)-\underset{\sim}{F}^{-1}r|_T \\
&= |\underset{\sim}{F}^{-1}[\underset{\sim}{I}+(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}]^{-1}r-\underset{\sim}{F}^{-1}r|_T \\
&\leq \lambda(\underset{\sim}{F}^{-1})\,|[\underset{\sim}{I}+(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}]^{-1}r-r|_T && \text{(by assumption (i))} \\
&\leq \lambda(\underset{\sim}{F}^{-1})\frac{|(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r|_T}{1-\tilde{\gamma}_T[(\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}]} && \text{(by (A.2))}
\end{aligned}
$$

In particular, if eqns. (II.10) and (II.11) hold, i.e. for $T \in \mathcal{T}$ sufficiently large,

$$|(\underset{\sim\sim\sim}{FGK})^{-1}r| << \frac{|\underset{\sim}{F}^{-1}r|_T}{\lambda(\underset{\sim}{F}^{-1})} \quad \text{and} \quad \tilde{\gamma}_T[(\underset{\sim\sim\sim}{FGK})^{-1}] << 1 ,$$

then for $T \in \mathcal{T}$ sufficiently large,

$$|\underset{\sim}{H}_{yr}r-\underset{\sim}{F}^{-1}r|_T << \frac{|\underset{\sim}{F}^{-1}r|_T}{1-\tilde{\gamma}_T[(\underset{\sim\sim\sim}{FGK})^{-1}]} \simeq |\underset{\sim}{F}^{-1}r|_T , \; \forall r \in \mathcal{R}_{d,e} .$$ Q.E.D.

Proof of Theorem II.3:

Since $\underset{\sim}{F}$ is invertible, we have, from Fig. I.1,

$$y = \underset{\sim}{H}_{yr}r = \underset{\sim}{F}^{-1}(r-e)$$

Hence, for $T \in \mathcal{T}$ sufficiently large, $\forall r \in \mathcal{R}_{d,e}$,

$$\begin{aligned} |\underset{\sim}{H}_{yr}r-\underset{\sim}{F}^{-1}r|_T &= |\underset{\sim}{F}^{-1}(r-e)-\underset{\sim}{F}^{-1}r|_T \\ &\leq \lambda(\underset{\sim}{F}^{-1})|e|_T && \text{(by assumption (i))} \\ &= \lambda(\underset{\sim}{F}^{-1})|(\underset{\sim}{I}+\underset{\sim\sim\sim}{FGK})^{-1}r|_T \\ &<< |\underset{\sim}{F}^{-1}r|_T && \text{(by assumption (ii))} \end{aligned}$$ Q.E.D.

Proof of Corollary II.3.1:

Consider the system S in the sinusoidal steady state (since the closed-loop system is exp. stable by assumption (a2)) with input $r\cdot\exp(j\omega t)$ and error $e\cdot\exp(j\omega t)$, where $r, e \in \mathbb{C}^n$. Then, by linearity of $F(j\omega)$,

$$H_{yr}(j\omega)r = F(j\omega)^{-1}(r-e) = F(j\omega)^{-1}r - F(j\omega)^{-1}e$$

Thus

$$\begin{aligned} H_{yr}(j\omega)r - F(j\omega)^{-1}r &= -F(j\omega)^{-1}e \\ &= -F(j\omega)^{-1}[(I+FGK)(j\omega)]^{-1}r \\ &= -[(I+FGK)F](j\omega)^{-1}r \end{aligned}$$

$$= -[F(j\omega)(I+GKF)(j\omega)]^{-1}r \qquad \text{(by linearity of } F(j\omega))$$
$$= -[(I+GKF)(j\omega)]^{-1}F(j\omega)^{-1}r$$

Hence

$$|H_{yr}(j\omega)r-F(j\omega)^{-1}r| = |[(I+GKF)(j\omega)]^{-1}F(j\omega)^{-1}r|$$
$$\ll |F(j\omega)^{-1}r| \quad \text{(by assumption (II.21))} \qquad \text{Q.E.D.}$$

<u>Proof of Theorem III.1</u>:

$$\begin{aligned}
\Delta \underset{\sim}{H}_{yr} &:= \underset{\sim}{\tilde{H}}_{yr} - \underset{\sim}{H}_{yr} \\
&= \underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1} - \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} \\
&= \underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1} - \underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} + \underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} - \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} \\
&= \underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1} - \underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}[\underset{\sim}{I}-\underset{\sim}{F}\cdot\Delta\underset{\sim}{G}\cdot\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}]^{-1} \\
&\quad + \Delta\underset{\sim}{G}\cdot\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} \quad \text{(since } \underset{\sim}{F} \text{ is linear)}
\end{aligned} \tag{A.3}$$

Evaluating eqn. (A.3) at $r \in \mathcal{R}_e$, we have

$$\Delta \underset{\sim}{H}_{yr}(r) = \underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}(r) - \underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}(r+\Delta r) + \Delta \underset{\sim}{H}_{y_0 r}(r) \tag{A.4}$$

where

$$r+\Delta r := [\underset{\sim}{I}-\underset{\sim}{F}\cdot\Delta\underset{\sim}{G}\cdot\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}]^{-1}(r) \tag{A.5}$$

Since, by assumption, $\underset{\sim}{\tilde{H}}_{yr} = \underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}$ is a C^1 map, we can evaluate $\Delta \underset{\sim}{H}_{yr}(r)$ by the Taylor's expansion theorem [19, p. 190] and obtain

$$\begin{aligned}
\Delta \underset{\sim}{H}_{yr}(r) &= -\int_0^1 D[\underset{\sim}{\tilde{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}](r+\alpha\Delta r)\cdot\Delta r d\alpha + \Delta \underset{\sim}{H}_{y_0 r}(r) \\
&= -\int_0^1 D(\underset{\sim}{\tilde{G}}\underset{\sim}{K})\cdot[\underset{\sim}{I}+\underset{\sim}{F}\cdot D(\underset{\sim}{\tilde{G}}\underset{\sim}{K})]^{-1}\cdot\Delta r d\alpha + \Delta \underset{\sim}{H}_{y_0 r}(r)
\end{aligned} \tag{A.6}$$

where the Fréchet derivative $D(\underset{\sim}{\tilde{G}}\underset{\sim}{K})$ is evaluated at $(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}(r+\alpha\Delta r)$.

Note that eqn. (A.5) implies that

$$\begin{aligned}
\Delta r &= \{[\underset{\sim}{I}-\underset{\sim}{F}\cdot\Delta\underset{\sim}{G}\cdot\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}]^{-1}-\underset{\sim}{I}\}(r) \\
&= \underset{\sim}{F}\cdot\Delta\underset{\sim}{G}\cdot\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}[\underset{\sim}{I}-\underset{\sim}{F}\cdot\Delta\underset{\sim}{G}\cdot\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}}\underset{\sim}{K})^{-1}]^{-1}(r)
\end{aligned}$$

$$= \underset{\sim}{F}\cdot\Delta\underset{\sim}{G}\cdot\underset{\sim}{K}\cdot(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}(r)$$

$$= \underset{\sim}{F}\cdot\Delta \underset{\sim}{H}_{y_0 r}(r)$$

Thus eqn. (A.6) becomes

$$\begin{aligned}
\Delta \underset{\sim}{H}_{yr}(r) &= -\int_0^1 D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot[\underset{\sim}{I}+\underset{\sim}{F}\cdot D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})]^{-1}\cdot\underset{\sim}{F}\cdot\Delta \underset{\sim}{H}_{y_0 r}(r)\,d\alpha + \Delta \underset{\sim}{H}_{y_0 r}(r) \\
&= -\int_0^1 D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}[\underset{\sim}{I}+D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1}d\alpha\cdot\Delta \underset{\sim}{H}_{y_0 r}(r) + \Delta \underset{\sim}{H}_{y_0 r}(r) \\
&\qquad\qquad \text{(since } \underset{\sim}{F} \text{ is linear)} \\
&= \int_0^1 \{\underset{\sim}{I}-D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}[\underset{\sim}{I}+D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1}\}d\alpha\cdot\Delta \underset{\sim}{H}_{y_0 r}(r) \\
&= \int_0^1 [\underset{\sim}{I}+D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1}d\alpha\cdot\Delta \underset{\sim}{H}_{y_0 r}(r) \qquad \text{(A.7)}
\end{aligned}$$

Eqn. (A.7) is true, $\forall r \in \mathcal{R}_e$, thus eqn. (III.4) follows. Q.E.D.

Proof of Proposition III.2:

Note that

$$\underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} = (\underset{\sim}{I}+\underset{\sim}{G}\underset{\sim}{K}\underset{\sim}{F})^{-1}\underset{\sim}{G}\underset{\sim}{K} \quad \text{(since } \underset{\sim}{G}\underset{\sim}{K} \text{ is linear)}$$

and that

$$\begin{aligned}
&\underset{\sim}{G}\underset{\sim}{K}_2(\underset{\sim}{I}+\underset{\sim}{F}_2\underset{\sim}{G}\underset{\sim}{K}_2)^{-1}\underset{\sim}{K}_1[\underset{\sim}{I}+\underset{\sim}{F}_1\underset{\sim}{G}\underset{\sim}{K}_2(\underset{\sim}{I}+\underset{\sim}{F}_2\underset{\sim}{G}\underset{\sim}{K}_2)^{-1}\underset{\sim}{K}_1]^{-1} \\
&\quad = \underset{\sim}{G}\underset{\sim}{K}_2(\underset{\sim}{I}+\underset{\sim}{F}_2\underset{\sim}{G}\underset{\sim}{K}_2)^{-1}[\underset{\sim}{I}+\underset{\sim}{K}_1\underset{\sim}{F}_1\underset{\sim}{G}\underset{\sim}{K}_2(\underset{\sim}{I}+\underset{\sim}{F}_2\underset{\sim}{G}\underset{\sim}{K}_2)^{-1}]^{-1}\underset{\sim}{K}_1 \quad \text{(since } \underset{\sim}{K}_1 \text{ is linear)} \\
&\quad = \underset{\sim}{G}\underset{\sim}{K}_2[\underset{\sim}{I}+(\underset{\sim}{F}_2+\underset{\sim}{K}_1\underset{\sim}{F}_1)\underset{\sim}{G}\underset{\sim}{K}_2]^{-1}\underset{\sim}{K}_1 \\
&\quad = [\underset{\sim}{I}+\underset{\sim}{G}\underset{\sim}{K}_2(\underset{\sim}{F}_2+\underset{\sim}{K}_1\underset{\sim}{F}_1)]^{-1}\underset{\sim}{G}\underset{\sim}{K}_2\underset{\sim}{K}_1 \quad \text{(since } \underset{\sim}{G}\underset{\sim}{K}_2 \text{ is linear)}
\end{aligned}$$

Thus eqn. (III.10) follows from eqn. (III.9). Q.E.D.

Proof of Proposition III.3:

$$\begin{aligned}
\Delta \underset{\sim}{H}_{yr} &:= \tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\tilde{\underset{\sim}{F}}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1} - \tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1} \\
&= \tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}[\underset{\sim}{I}+\Delta\underset{\sim}{F}\cdot\tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}]^{-1} - \tilde{\underset{\sim}{G}}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1} \qquad \text{(A.8)}
\end{aligned}$$

Evaluating eqn. (A.8) at $r \in \mathcal{R}_e$, we have

$$\Delta \underset{\sim}{H}_{yr} r = \underset{\sim}{\tilde{G}K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}K})^{-1}(r+\Delta r) - \underset{\sim}{\tilde{G}K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}K})^{-1}(r) \tag{A.9}$$

where

$$r+\Delta r := [\underset{\sim}{I}+\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{G}K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}K})^{-1}]^{-1} r \tag{A.10}$$

Since, by assumption (a3), $\underset{\sim}{\tilde{G}K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}K})^{-1}$ is a C^1 map, we can evaluate $\Delta \underset{\sim}{H}_{yr} r$ by the Taylor's expansion theorem and obtain

$$\Delta \underset{\sim}{H}_{yr} r = \int_0^1 D(\underset{\sim}{\tilde{G}K})\cdot[\underset{\sim}{I}+\underset{\sim}{F}\cdot D(\underset{\sim}{\tilde{G}K})]^{-1} d\alpha\cdot\Delta r \tag{A.11}$$

where the Fréchet derivative $D(\underset{\sim}{\tilde{G}K})$ is evaluated at $(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}K})^{-1}(r+\alpha\Delta r)$, and $D\underset{\sim}{F} = \underset{\sim}{F}$ since $\underset{\sim}{F}$ is linear.

Note that eqn. (A.10) implies that

$$\begin{aligned}
\Delta r &= \{[\underset{\sim}{I}+\Delta\underset{\sim}{F}\cdot\underset{\sim}{GK}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{GK})^{-1}]^{-1}-\underset{\sim}{I}\} r \\
&= -\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{G}K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}K})^{-1}[\underset{\sim}{I}+\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{G}K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{\tilde{G}K})^{-1}]^{-1} r \\
&= -\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{G}K}(\underset{\sim}{I}+\underset{\sim}{\tilde{F}}\underset{\sim}{\tilde{G}K})^{-1} r \\
&= -\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{H}}_{yr} r
\end{aligned}$$

Thus eqn. (A.11) becomes

$$\begin{aligned}
\Delta \underset{\sim}{H}_{yr} r &= -\int_0^1 D(\underset{\sim}{\tilde{G}K})\cdot[\underset{\sim}{I}+\underset{\sim}{F}\cdot D(\underset{\sim}{\tilde{G}K})]^{-1} d\alpha\cdot\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{H}}_{yr} r \quad \text{(since } \underset{\sim}{F} \text{ is linear)} \quad \text{(A.12)} \\
&= -\int_0^1 D(\underset{\sim}{\tilde{G}K})\cdot[\underset{\sim}{I}+\underset{\sim}{F}\cdot D(\underset{\sim}{\tilde{G}K})]^{-1}\underset{\sim}{F} d\alpha\cdot\underset{\sim}{F}^{-1}\cdot\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{H}}_{yr} r \quad \text{(since } \underset{\sim}{F} \text{ is invertible)} \\
&= -\int_0^1 D(\underset{\sim}{\tilde{G}K})\underset{\sim}{F}\cdot[\underset{\sim}{I}+D(\underset{\sim}{\tilde{G}K})\cdot\underset{\sim}{F}]^{-1} d\alpha\cdot\underset{\sim}{F}^{-1}\cdot\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{H}}_{yr} r \quad \text{(since } \underset{\sim}{F} \text{ is linear)} \\
&= \{\int_0^1 [\underset{\sim}{I}+D(\underset{\sim}{\tilde{G}K})\cdot\underset{\sim}{F}]^{-1} d\alpha-\underset{\sim}{I}\}\cdot\underset{\sim}{F}^{-1}\cdot\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{H}}_{yr} r
\end{aligned}$$

i.e.

$$\Delta \underset{\sim}{H}_{yr} = \{\int_0^1 [\underset{\sim}{I}+D(\underset{\sim}{\tilde{G}}\underset{\sim}{K})\underset{\sim}{F}]^{-1}d\alpha-\underset{\sim}{I}\}\cdot\underset{\sim}{F}^{-1}\cdot\Delta\underset{\sim}{F}\cdot\underset{\sim}{\tilde{H}}_{yr}\ , \quad \text{on } \mathcal{R}_e$$

Since $\underset{\sim}{\tilde{H}}_{yr}$ is invertible, we have

$$\Delta \underset{\sim}{H}_{yr}\cdot\underset{\sim}{\tilde{H}}_{yr}^{-1} = \{\int_0^1 [\underset{\sim}{I}+D(\underset{\sim}{\tilde{G}}\underset{\sim}{K})\underset{\sim}{F}]^{-1}d\alpha-\underset{\sim}{I}\}\cdot\underset{\sim}{F}^{-1}\cdot\Delta\underset{\sim}{F}\ , \quad \text{on } \mathcal{Y}_e$$

Q.E.D.

Proof of Proposition III.4:

For completeness, we first state an algorithm [36] which determines the asymptotic behavior of the zeros of a polynomial. This algorithm is a direct application of the Newton's diagram (or known as the method of Puiseux, see e.g. [35, p. 105]).

Algorithm:

Data: Polynomial $P(s,k) = \sum_{\ell=0}^{n}\alpha_\ell(k)s^\ell \in \mathbb{R}[s]$

where, for $\ell = 0,1,2,\ldots,n$

$$\alpha_\ell(k) = \sum_{j=0}^{m_\ell}\alpha_{\ell j}k^j\ , \quad k,\ \alpha_{\ell j}\text{'s} \in \mathbb{R}$$

$$\alpha_{nm_n} \neq 0\ , \text{ and } \alpha_{\ell m_\ell} \neq 0\ , \ \forall 0 \leq \ell \leq n-1 \text{ such that } m_\ell > 0$$

Step 1: Find $i \in \mathbb{N}$, and τ_p, $q_p \in \mathbb{Q}_+$, $0 \leq p \leq i$, where i, τ_p's, q_p's are such that

(i) i is the largest integer such that $0 = \tau_0 < \tau_1 < \cdots < \tau_i$;

(ii) $q_o = \max\{m_0,m_1,\ldots,m_n\}$;

(iii) for $0 \leq p \leq i$,

$$m_\ell \leq q_p-\ell\cdot\tau_p\ , \ \forall 0 \leq \ell \leq n$$

with equality holds for at least two ℓ's;

(iv) if $\underline{\ell}_p$ ($\bar{\ell}_p$) is the smallest (largest) ℓ such that $m_\ell = q_p-\ell\cdot\tau_p$, then $\underline{\ell}_{p+1} = \bar{\ell}_p$, for $p = 0,1,\ldots,i-1$. (The procedure of finding i, τ_p's, q_p's can be best illustrated graphically by the modified Newton's diagram shown on Fig. A.1.)

Step 2: For each $0 \le p \le i$, form the polynomial

$$\phi_p(z) = \sum_{\ell \in \{\ell \mid \ell\tau_p + m_\ell = q_p\}} \alpha_{\ell m_\ell} z^{\ell} \tag{A.13}$$

Step 3: Calculate the zeros of ϕ_0 and denote them by z_{0j}, $j = 1,2,\ldots,n_0$. Calculate the nonzero zeros of ϕ_p, $1 \le p \le i$, and denote them by z_{pj}, $j = 1,2,\ldots,n_p$.

Step 4: As $|k| \to \infty$, the n zeros of the polynomial $P(s,k)$ behaves as

$$z_{pj}k^{\tau_p}\ , \quad j = 1,2,\ldots,n_p\ , \quad p = 0,1,2,\ldots,i$$

where $n_p \ge 1$, for $1 \le p \le i$, and $\sum_{p=0}^{i} n_p = n$.

Now we can apply this algorithm to prove Proposition III.4. Without loss of generality, we only have to prove the case where $n_o = n_i$ and $M = I_{n_o}$. Note that

$$\begin{aligned}\det[I+kG] &= 1 + k[\text{trace } G(s)] \\ &\quad + k^2[\Sigma \text{ principal minors of } G(s) \text{ of order } 2] \\ &\quad + \cdots + k^m \det G \\ &= \frac{1}{\prod_{i,j} d_{ij}}[\prod_{i,j} d_{ij} + k\sum_{i=1}^{m} n_{ii}(\prod_{j\neq i} d_{ij}) + \cdots + k^m \cdot \prod_{i,j} d_{ij} \cdot \det G] \\ &= \frac{1}{\prod_{i,j} d_{ij}}[\prod_{i,j} d_{ij} + k\alpha_1(s) + k^2\alpha_2(s) + \cdots + k^m\alpha_m(s)]\end{aligned}$$

where $\alpha_j(s) \in \mathbb{R}[s]$, $j = 1,2,\ldots,m$.

Let $\partial[\prod_{i,j} d_{ij}] = n$. Since, by assumption, $\partial d_{ij} - \partial n_{ij} \geq 3$, $\forall i,j = 1,2,\ldots,m$, we have that $\partial[\alpha_j(s)] \leq n-3j$, $j = 1,2,\ldots,m$. Hence with i defined in Step 1 of the algorithm above,

$$\phi_i(z) = z^n + \alpha_{1\beta} z^{n-\beta} + \cdots , \quad \beta \geq 3 ,$$

where $\phi_i(z)$ is defined in (A.13).

Now we claim that $\phi_i(z)$ has $\mathring{\mathbb{C}}_+$-zeros. To see this, consider some $\varepsilon > 0$ sufficiently small; apply the Routh test (see e.g. [37]) to the polynomial $\phi_i(z+\varepsilon)$. Since $\beta \geq 3$, the first column from the left in the Routh array contains strictly negative numbers, thus $\phi_i(z)$ has $\mathring{\mathbb{C}}_+$-zeros. Hence as $k \to \infty$, det[I+kG] has zero behaves as $z_i k^{\tau_i}$ with $z_i \in \mathring{\mathbb{C}}_+$ and $\tau_i > 0$. Q.E.D.

<u>Proof of Proposition III.5</u>:

(i) By definition, $\tilde{\underset{\sim}{G}}\tilde{u} := \underset{\sim}{G}\tilde{u} + d_0$. Then, by eqn. (III.4) (of Theorem III.1), we have that

$$\begin{aligned}\Delta y = \Delta H_{\underset{\sim}{y}r}(r) &= \int_0^1 [\underset{\sim}{I} + D(\tilde{\underset{\sim}{G}}\underset{\sim}{K}) \cdot \underset{\sim}{F}]^{-1} d\alpha \cdot d_0 \quad (\text{since } \Delta H_{\underset{\sim}{y}_0 r}(r) = d_0) \\ &= \int_0^1 [\underset{\sim}{I} + D(\underset{\sim}{G}\underset{\sim}{K}) \cdot \underset{\sim}{F}]^{-1} d\alpha \cdot d_0 \quad (\text{since } D(\tilde{\underset{\sim}{G}}\underset{\sim}{K}) = D(\underset{\sim}{G}\underset{\sim}{K}))\end{aligned}$$

where the Fréchet derivative $D(\underset{\sim}{G}\underset{\sim}{K})$ is evaluated at $(\underset{\sim}{I} + \underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r+\alpha\Delta r)$ with $\Delta r = \underset{\sim}{F} \cdot \Delta H_{\underset{\sim}{y}_0 r}(r) = \underset{\sim}{F} \cdot d_0$.

(ii) By definition, $\tilde{\underset{\sim}{F}}\tilde{y} := \underset{\sim}{F}(d_f + \tilde{y}) = \underset{\sim}{F}d_f + \underset{\sim}{F}\tilde{y}$ (since $\underset{\sim}{F}$ is linear). Then

$$\Delta \underset{\sim}{F} \cdot \tilde{y} = (\tilde{\underset{\sim}{F}} - \underset{\sim}{F})\tilde{y} = \underset{\sim}{F} \cdot d_f \qquad \text{(A.14)}$$

Thus, following the proof of Proposition III.3, in particular, eqn. (A.12) we have that

$$\Delta y = -\int_0^1 D(\underset{\sim}{G}\underset{\sim}{K})[\underset{\sim}{I}+\underset{\sim}{F}\cdot D(\underset{\sim}{G}\underset{\sim}{K})]^{-1} d\alpha\cdot\Delta\underset{\sim}{F}\cdot\tilde{y} \quad (\text{since } D(\tilde{\underset{\sim}{G}}\underset{\sim}{K}) = D(\underset{\sim}{G}\underset{\sim}{K}))$$

$$= -\int_0^1 D(\underset{\sim}{G}\underset{\sim}{K})[\underset{\sim}{I}+\underset{\sim}{F}\cdot D(\underset{\sim}{G}\underset{\sim}{K})]^{-1}\underset{\sim}{F}d\alpha\cdot d_f \quad (\text{by eqn. (A.14)})$$

$$= -\int_0^1 D(\underset{\sim}{G}\underset{\sim}{K})\cdot\underset{\sim}{F}[\underset{\sim}{I}+D(\underset{\sim}{G}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1} d\alpha\cdot d_f \quad (\text{since } \underset{\sim}{F} \text{ is linear})$$

$$= \{\int_0^1 [\underset{\sim}{I}+D(\underset{\sim}{G}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1} d\alpha - \underset{\sim}{I}\}\cdot d_f$$

where the Fréchet derivative $D(\underset{\sim}{G}\underset{\sim}{K})$ is evaluated at $(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r+\alpha\Delta r)$ with $\Delta r = -\Delta\underset{\sim}{F}\cdot\tilde{\underset{\sim}{H}}_{yr}r = -\Delta\underset{\sim}{F}\cdot\tilde{y} = -\underset{\sim}{F}d_f$ and $\alpha \in [0,1]$. Q.E.D.

Proof of Proposition III.6:

By definition, $\tilde{\underset{\sim}{G}}\tilde{u} := \underset{\sim}{G}(\tilde{u}+d_g)$. Then by eqn. (III.4) (of Theorem III.1) we have that

$$\Delta y = \int_0^1 [\underset{\sim}{I}+D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1} d\alpha\cdot[\underset{\sim}{G}(u+d_g)-\underset{\sim}{G}(u)]$$

$$(\text{since } \Delta\underset{\sim}{H}_{y_0 r}(r) = \underset{\sim}{G}(u+d_g)-\underset{\sim}{G}(u), \text{ where } u = \underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r)$$

$$= \{\int_0^1 [\underset{\sim}{I}+D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})\cdot\underset{\sim}{F}]^{-1} d\alpha\}\cdot\{\int_0^1 D\underset{\sim}{G}(u+\beta d_g)d\beta\}\cdot d_g$$

where the Fréchet derivative $D(\tilde{\underset{\sim}{G}}\underset{\sim}{K})$ is evaluated at $(\underset{\sim}{I}+\underset{\sim}{F}\tilde{\underset{\sim}{G}}\underset{\sim}{K})^{-1}(r+\alpha\Delta r)$ with $\Delta r = \underset{\sim}{F}[\underset{\sim}{G}(u+d_g)-\underset{\sim}{G}(u)]$, $u := \underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r$, and $\alpha \in [0,1]$. Q.E.D.

Proof of Proposition III.9:

$$\delta_T(N_2,\mathcal{V}) := \inf_{\underset{\sim}{L}\in\mathcal{L}} \sup_{u\in\mathcal{V}} |\underset{\sim}{N}_2 u - \underset{\sim}{L}u|_T$$

$$= \inf_{\underset{\sim}{L}\in\mathcal{L}} \sup_{u\in\mathcal{V}} |\underset{\sim}{N}_1 u + \underset{\sim}{L}_1 u - \underset{\sim}{L}u|_T$$

$$= \inf_{L'\in\mathcal{L}} \sup_{u\in\mathcal{V}} |\underset{\sim}{N}_1 u - \underset{\sim}{L}'u|_T$$

$$=: \delta_T(\underset{\sim}{N}_1,\mathcal{V}), \quad \forall T\in\mathcal{T}, \ \forall\mathcal{V}\subset\mathcal{U}_e. \qquad \text{Q.E.D.}$$

Proof of Proposition III.10:

$$\delta_T(\underset{\sim}{N},\mathcal{V}_1) := \inf_{\underset{\sim}{L}\in\mathcal{L}} \sup_{u\in\mathcal{V}_1} |\underset{\sim}{N}u-\underset{\sim}{L}u|_T$$

$$\leq \inf_{\underset{\sim}{L}\in\mathcal{L}} \sup_{u\in\mathcal{V}_2} |\underset{\sim}{N}u-\underset{\sim}{L}u|_T \quad (\text{since } \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{U}_e)$$

$$=: \delta_T(\underset{\sim}{N},\mathcal{V}_2) \ , \quad \forall T \in \mathcal{T} \ .$$

Q.E.D.

Proof of Proposition III.11:

Note that $DN(0) \in \mathcal{L}$. Hence

$$0 \leq \delta_T(\underset{\sim}{N},\bar{B}_T(0;\beta)) \leq \sup_{u\in\bar{B}_T(0;\beta)} |\underset{\sim}{N}u-D\underset{\sim}{N}(0)u|_T \tag{A.15}$$

By the definition of Fréchet derivative, we know that for any $\varepsilon > 0$, $\exists\delta > 0$ such that $|\underset{\sim}{N}u-D\underset{\sim}{N}(0)\cdot u|_T \leq \varepsilon|u|_T$, $\forall|u|_T \leq \delta$. Hence as $\beta \to 0$, the right-hand side of eqn. (A.15) tends to zero and $\delta_T(\underset{\sim}{N},\bar{B}_T(0;\beta)) \to 0$.

Q.E.D.

Proof of Proposition III.12:

Let $\mathcal{V} \subset \mathcal{U}_e$. If, for some $\underset{\sim}{L} \in \mathcal{L}$, $\underset{\sim}{N}u = \underset{\sim}{L}u$, $\forall u \in \mathcal{V}$, then $\underset{\sim}{L}$ is a minimizer of $\sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}u|_T$, $\forall T \in \mathcal{T}$, and $\delta_T(\underset{\sim}{N},\mathcal{V}) = 0$, $\forall T \in \mathcal{T}$. In particular, if $\underset{\sim}{N} \in \mathcal{L}$, then $\underset{\sim}{N}$ is a minimizer of $\sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}u|_T$, $\forall T \in \mathcal{T}$, $\forall\mathcal{V} \subset \mathcal{U}_e$, hence $\delta_T(\underset{\sim}{N},\mathcal{V}) = 0$, $\forall T \in \mathcal{T}$, $\forall\mathcal{V} \subset \mathcal{U}_e$.

Q.E.D.

Proof of Proposition III.13:

By assumption, for some $T \in \mathcal{T}$,

$$\delta_T(\underset{\sim}{N},\mathcal{V}) := \inf_{\underset{\sim}{L}\in\mathcal{L}} \sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}u|_T = 0 \tag{A.16}$$

Thus for this T, there exists a sequence $(\underset{\sim}{L}_i)_{i=0}^{\infty} \subset \mathcal{L}$ such that

$$\sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}_i u|_T \to 0, \text{ as } i \to \infty \tag{A.17}$$

or equivalently, for any $\varepsilon_1 > 0$, there exists $m_1 > 0$ such that

$$\sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}_i u|_T < \varepsilon_1 \ , \ \forall i > m_1 \tag{A.18}$$

Now for any $\varepsilon > 0$, if we let $\varepsilon_1 = \frac{\varepsilon}{2\beta}$ and choose the corresponding $m_1 > 0$ such that (A.18) holds, then

$$\begin{aligned}
|\underset{\sim}{L}_i-\underset{\sim}{L}_j|_T &= \frac{1}{\beta} \sup_{|u|_T=\beta} |\underset{\sim}{L}_i u-\underset{\sim}{L}_j u|_T \quad \text{(by definition of the induced norm, with } \beta \text{ defined in assumption (a3))} \\
&\leq \frac{1}{\beta} \sup_{u\in\mathcal{V}} |\underset{\sim}{L}_i u-\underset{\sim}{L}_j u|_T \qquad \text{(by assumption (a3))} \\
&\leq \frac{1}{\beta}[\sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}_i u|_T + \sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}_j u|_T] \\
&\leq \frac{1}{\beta}\cdot 2\varepsilon_1 = \varepsilon \ , \quad \forall i,j \geq m_1 \quad \text{(by (A.18) and the choice of } m_1)
\end{aligned}$$

Thus $(\underset{\sim}{L}_i)_{i=0}^{\infty}$ is a Cauchy sequence in $(\underset{\sim}{P}_T\mathcal{L}, |\cdot|_T)$.

Note that for each $T \in \mathcal{T}$, $(\underset{\sim}{P}_T\mathcal{L}, |\cdot|_T)$ is a Banach space with the usual induced norm since, by assumption (a1), $\underset{\sim}{P}_T\mathcal{Y}_e$ is a Banach space. Hence there exists $\underset{\sim}{L}^* \in \mathcal{L}$ such that

$$\underset{\sim}{P}_T\underset{\sim}{L}_i \xrightarrow{|\cdot|_T} \underset{\sim}{P}_T\underset{\sim}{L}^* \tag{A.19}$$

Note

$$\begin{aligned}
|\underset{\sim}{N}u-\underset{\sim}{L}^*u|_T &\leq |\underset{\sim}{N}u-\underset{\sim}{L}_i u|_T + |\underset{\sim}{L}_i u-\underset{\sim}{L}^*u|_T \\
&\leq \sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}_i u|_T + \sup_{u\in\mathcal{V}} |\underset{\sim}{L}_i u-\underset{\sim}{L}^*u|_T \\
&\leq \sup_{u\in\mathcal{V}} |\underset{\sim}{N}u-\underset{\sim}{L}_i u|_T + |\underset{\sim}{L}_i-\underset{\sim}{L}^*|_T \cdot \sup_{u\in\mathcal{V}} |u|_T \ , \quad \forall i
\end{aligned} \tag{A.20}$$

By (A.18), (A.19) and the assumption (a2) that $\sup_{u\in\mathcal{V}} |u|_T < \infty$, the right-hand side of eqn. (A.20) tends to zero as $i \to \infty$. Hence $|\underset{\sim}{N}u-\underset{\sim}{L}^*u|_T = 0$, $\forall u \in \mathcal{V}$. Q.E.D.

Proof of Theorem III.14:

Let $\underset{\sim}{L}_G$ be a linear, causal operator such that the linear operator $\underset{\sim}{L}_{yr} := \underset{\sim}{L}_G \underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{L}_G\underset{\sim}{K})^{-1}$ is well defined and causal. Then

$$\begin{aligned}
\underset{\sim}{H}_{yr}r - \underset{\sim}{L}_{yr}r &= \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r - \underset{\sim}{L}_G\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{L}_G\underset{\sim}{K})^{-1}r \\
&= \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r - \underset{\sim}{L}_G\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r \\
&\quad + \underset{\sim}{L}_G\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r - \underset{\sim}{L}_G\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{L}_G\underset{\sim}{K})^{-1}r \\
&= (\underset{\sim}{G}-\underset{\sim}{L}_G)u \\
&\quad + \underset{\sim}{L}_G\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{L}_G\underset{\sim}{K})^{-1}[(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{L}_G\underset{\sim}{K}) - (\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})](\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r \\
&\qquad \text{(since } \underset{\sim}{F}, \underset{\sim}{L}_G, \underset{\sim}{K} \text{ are linear and } u = \underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r) \\
&= (\underset{\sim}{G}-\underset{\sim}{L}_G)u + \underset{\sim}{L}_G\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{L}_G\underset{\sim}{K})^{-1}\underset{\sim}{F}(\underset{\sim}{L}_G-\underset{\sim}{G})\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1}r \text{ (since } \underset{\sim}{F} \text{ is linear)} \\
&= (\underset{\sim}{G}-\underset{\sim}{L}_G)u - \underset{\sim}{L}_G\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{L}_G\underset{\sim}{K})^{-1}\underset{\sim}{F}(\underset{\sim}{G}-\underset{\sim}{L}_G)u \text{ (since } \underset{\sim}{F},\underset{\sim}{L}_G,\underset{\sim}{K} \text{ are linear)} \\
&= [\underset{\sim}{I}-\underset{\sim}{L}_G\underset{\sim}{K}\underset{\sim}{F}(\underset{\sim}{I}+\underset{\sim}{L}_G\underset{\sim}{K}\underset{\sim}{F})^{-1}](\underset{\sim}{G}-\underset{\sim}{L}_G)u \text{ (since } \underset{\sim}{F} \text{ is linear)} \\
&= (\underset{\sim}{I}+\underset{\sim}{L}_G\underset{\sim}{K}\underset{\sim}{F})^{-1}(\underset{\sim}{G}-\underset{\sim}{L}_G)u \qquad \text{(A.21)}
\end{aligned}$$

Thus

$$\begin{aligned}
|\underset{\sim}{H}_{yr}r - \underset{\sim}{L}_{yr}r|_T &= |(\underset{\sim}{I}+\underset{\sim}{L}_G\underset{\sim}{K}\underset{\sim}{F})^{-1}(\underset{\sim}{G}-\underset{\sim}{L}_G)u|_T \\
&= \frac{|(\underset{\sim}{I}+\underset{\sim}{L}_G\underset{\sim}{K}\underset{\sim}{F})^{-1}(\underset{\sim}{G}-\underset{\sim}{L}_G)u|_T}{|(\underset{\sim}{G}-\underset{\sim}{L}_G)u|_T} \; |(\underset{\sim}{G}-\underset{\sim}{L}_G)u|_T \qquad \text{(A.22)}
\end{aligned}$$

provided that $\underset{\sim}{P}_T(\underset{\sim}{G}-\underset{\sim}{L}_G)u \neq 0$.

On letting $\underset{\sim}{L}^*_{yr} := \underset{\sim}{L}^*_G\underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{L}^*_G\underset{\sim}{K})^{-1}$, where $\underset{\sim}{L}^*_G$ is defined in eqn. (III.25), we have, from eqn. (A.22)

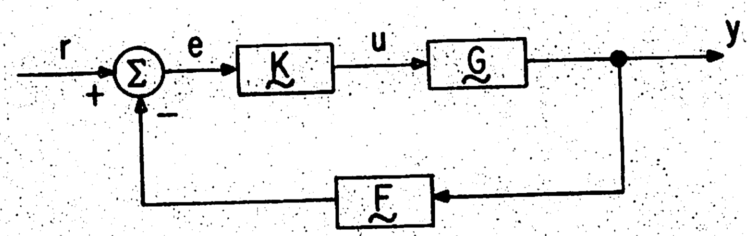
$$\begin{aligned}
\delta_T(\underset{\sim}{H}_{yr},\mathcal{R}_{d,e}) &:= \inf_{\underset{\sim}{L}\in\mathcal{L}} \sup_{r\in\mathcal{R}_{d,e}} |\underset{\sim}{H}_{yr}r - \underset{\sim}{L}r|_T \\
&\leq \sup_{r\in\mathcal{R}_{d,e}} |\underset{\sim}{H}_{yr}r - \underset{\sim}{L}^*_{yr}r|_T
\end{aligned}$$

$$\leq \sup_{\substack{y \in \tilde{\mathcal{Y}}_e \\ y_T \neq 0}} \frac{|(\underset{\sim}{I}+\underset{\sim}{L}_G^* \underset{\sim}{K}\underset{\sim}{F})^{-1} y|_T}{|y|_T} \sup_{u \in \mathcal{U}_{d,e}} |\underset{\sim}{G}u - \underset{\sim}{L}_G^* u|_T$$

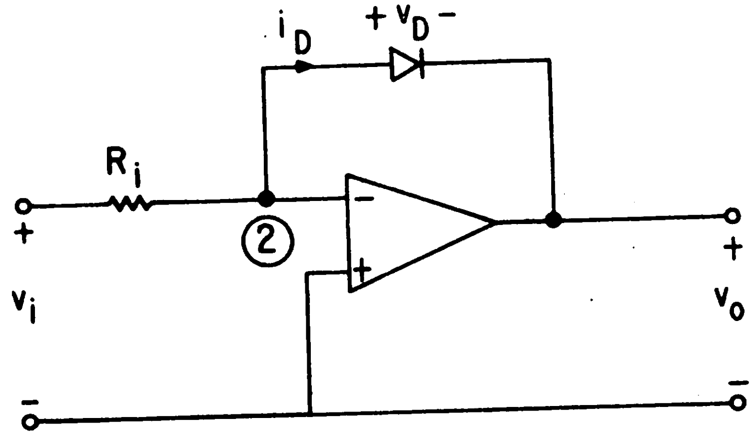
$$= \rho \cdot \delta_T(\underset{\sim}{G}, \mathcal{U}_{d,e}) \tag{A.23}$$

where $\tilde{\mathcal{Y}}_e := (\underset{\sim}{G}-\underset{\sim}{L}_G^*)\mathcal{U}_{d,e}$ and ρ is defined in eqn. (III.27). Note the last inequality follows since when $r \in \mathcal{R}_{d,e}$, the corresponding $u := \underset{\sim}{K}(\underset{\sim}{I}+\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{K})^{-1} r \in \mathcal{U}_{d,e}$. Q.E.D.

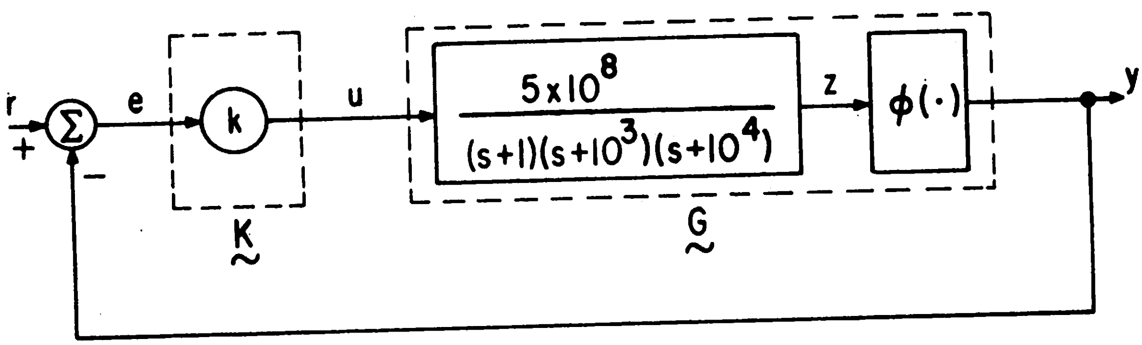
FIGURE CAPTIONS



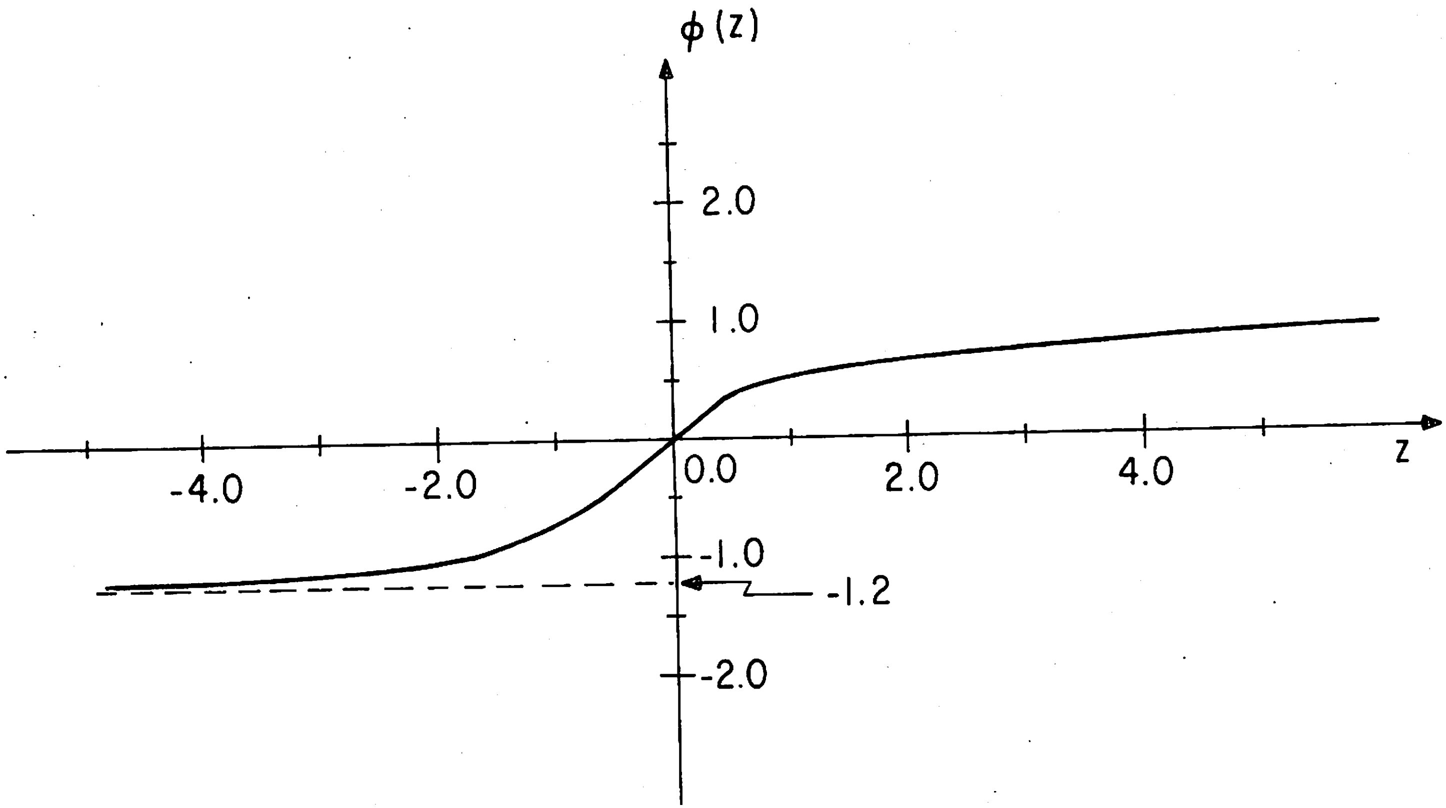
r
+
e
K
u
G
y
F



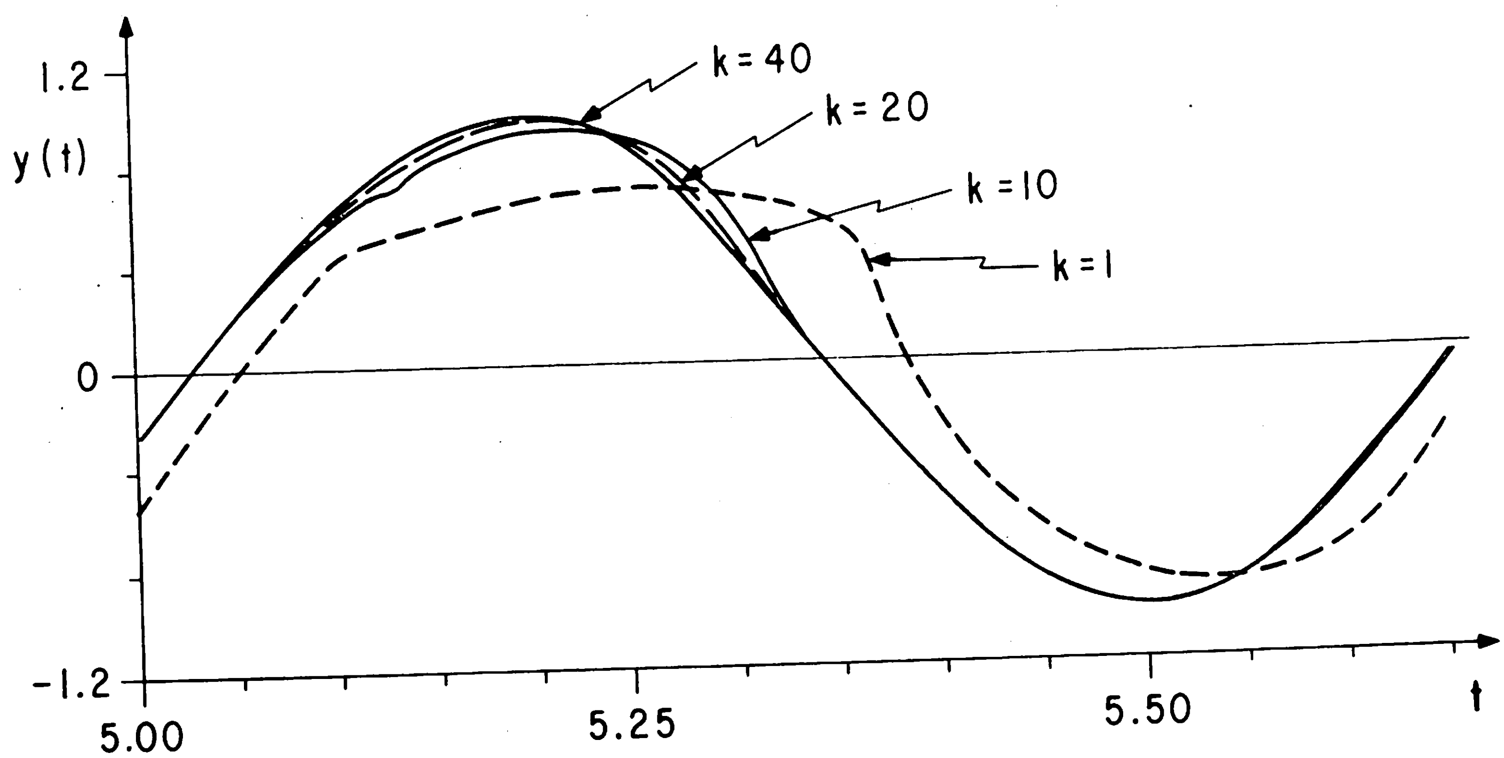
i_D
$+ \; v_D \; -$
R_i
2
+
v_i
−
+
v_o
−



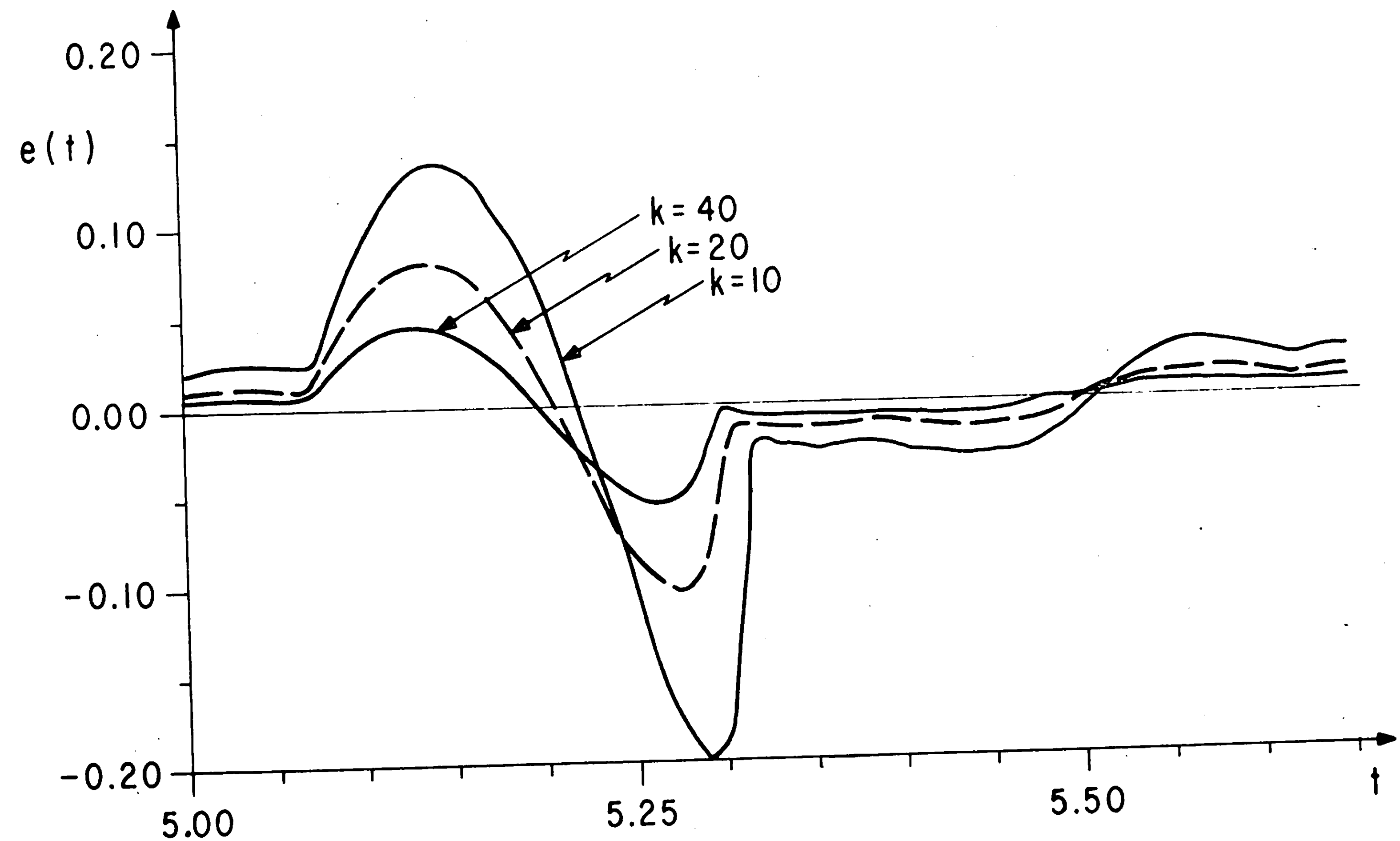
r
+
Σ
−
e
k
u
$\dfrac{5 \times 10^8}{(s+1)(s+10^3)(s+10^4)}$
z
$\phi(\cdot)$
y
K
G



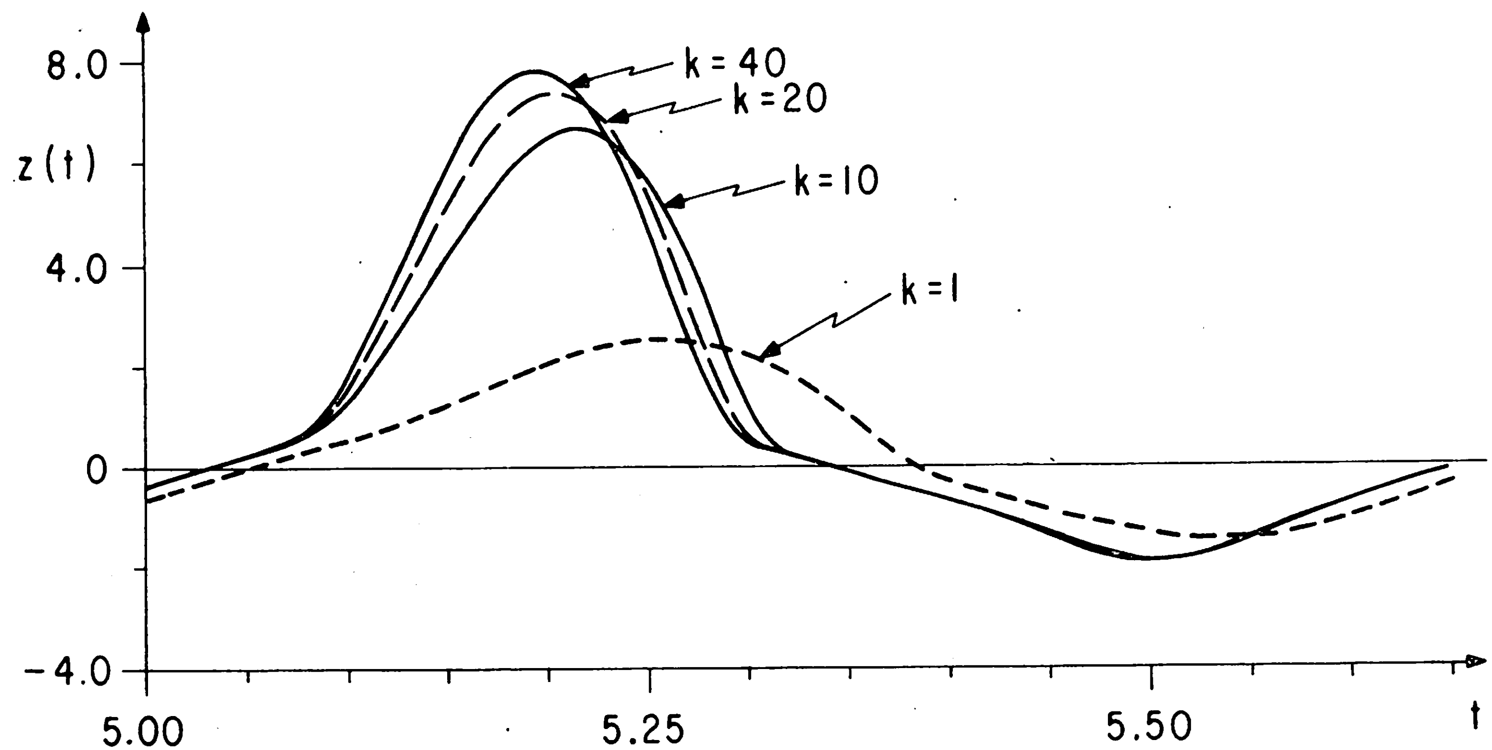
φ (z)
2.0
1.0
0.0
-1.0
-1.2
-2.0
-4.0
-2.0
2.0
4.0
z



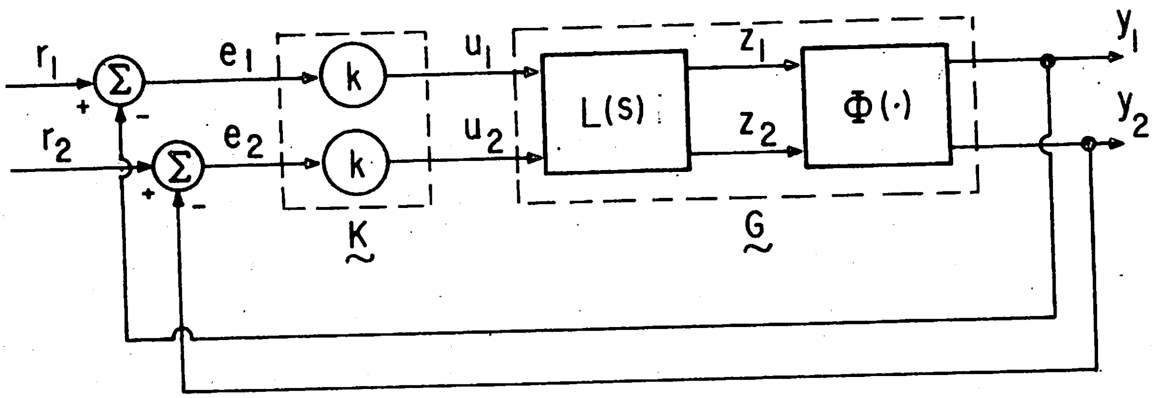
y (t)
1.2
0
-1.2
5.00
5.25
5.50
t
k = 40
k = 20
k = 10
k = 1



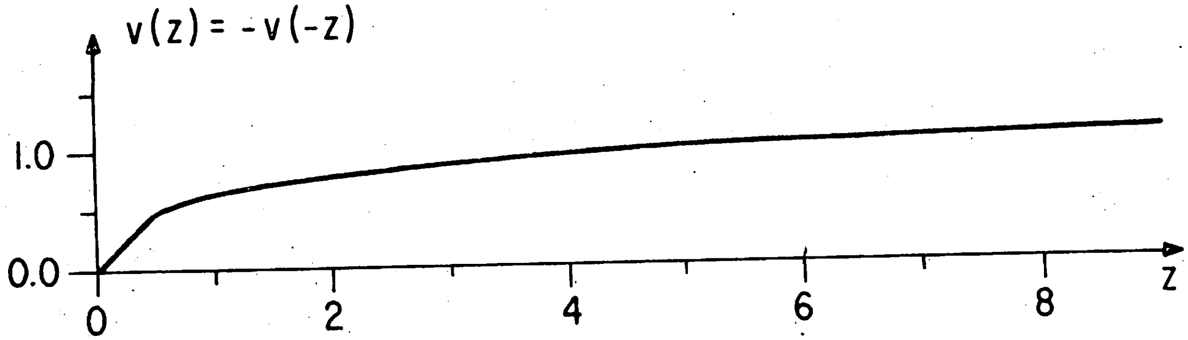
e(t)
0.20
0.10
0.00
-0.10
-0.20
k=40
k=20
k=10
5.00
5.25
5.50
t

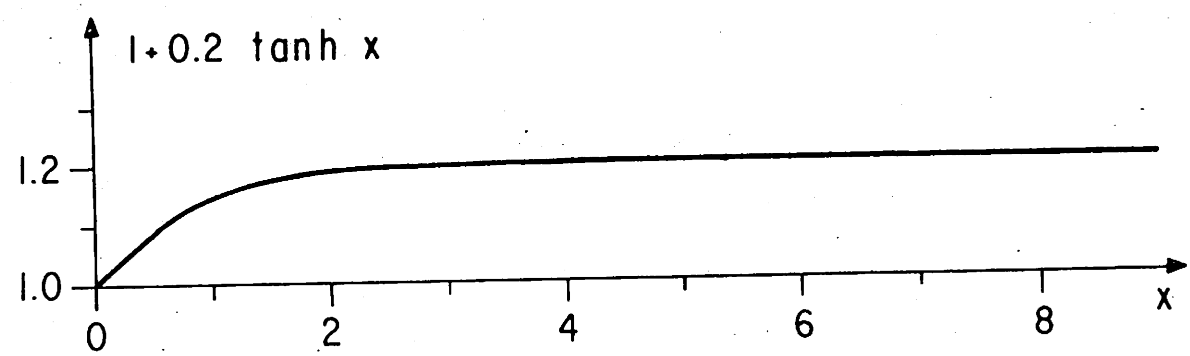


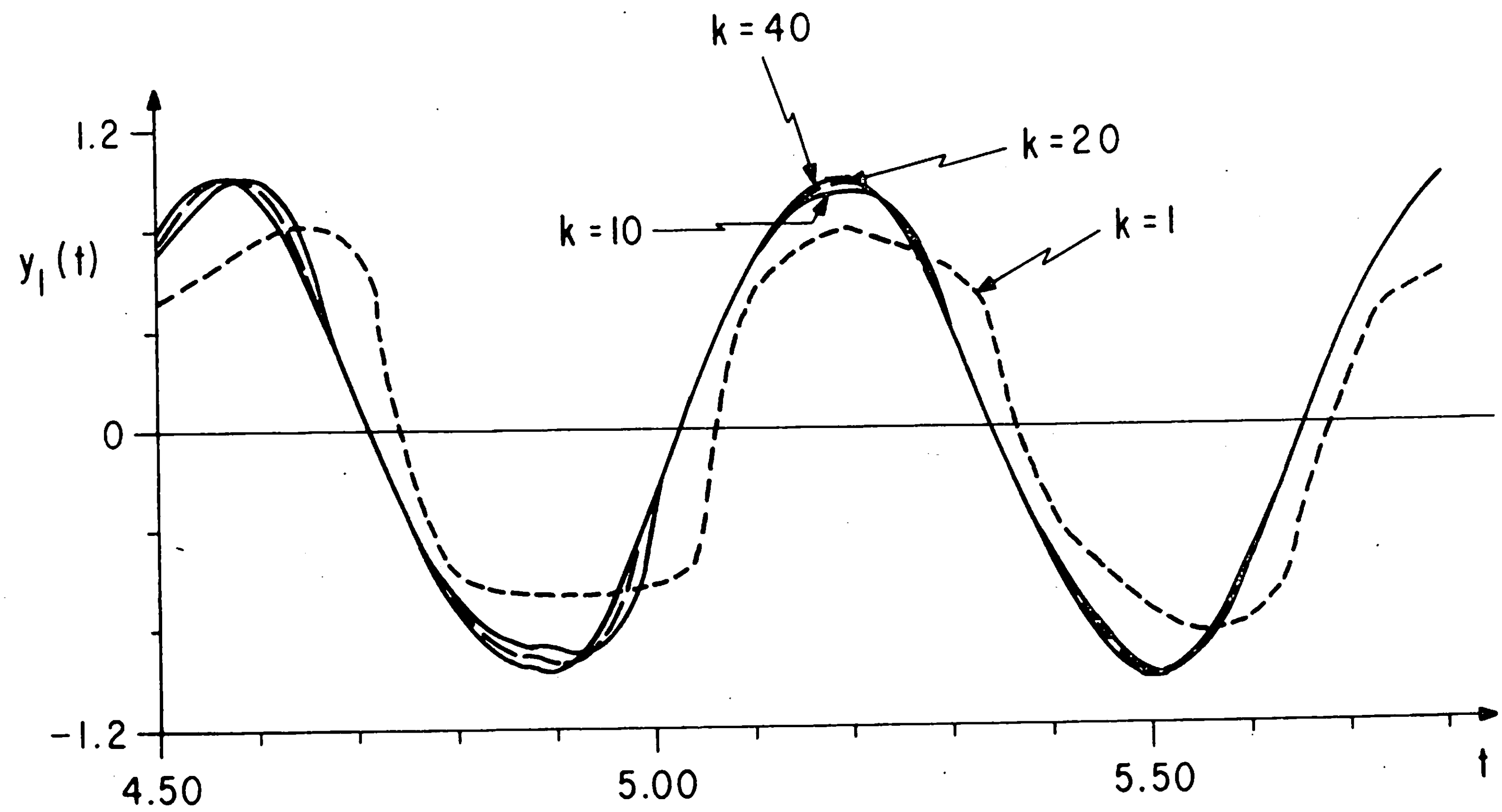
8.0
4.0
0
-4.0
z(t)
5.00
5.25
5.50
t
k = 40
k = 20
k = 10
k = 1



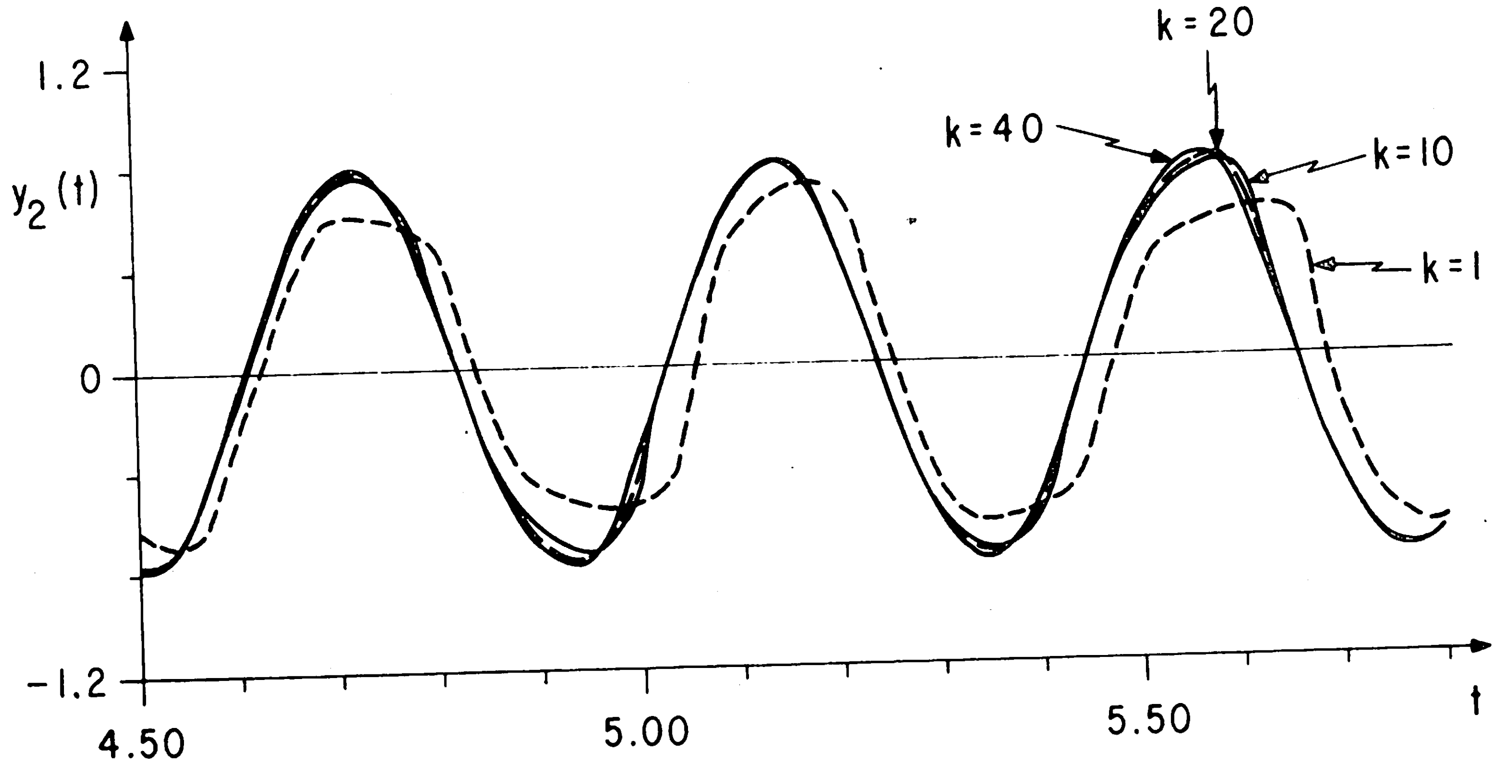
r_1
r_2
e_1
e_2
k
k
u_1
u_2
L(s)
z_1
z_2
$\Phi(\cdot)$
y_1
y_2
$\underset{\sim}{K}$
$\underset{\sim}{G}$


v(z) = -v(-z)
1.0
0.0
0
2
4
6
8
z

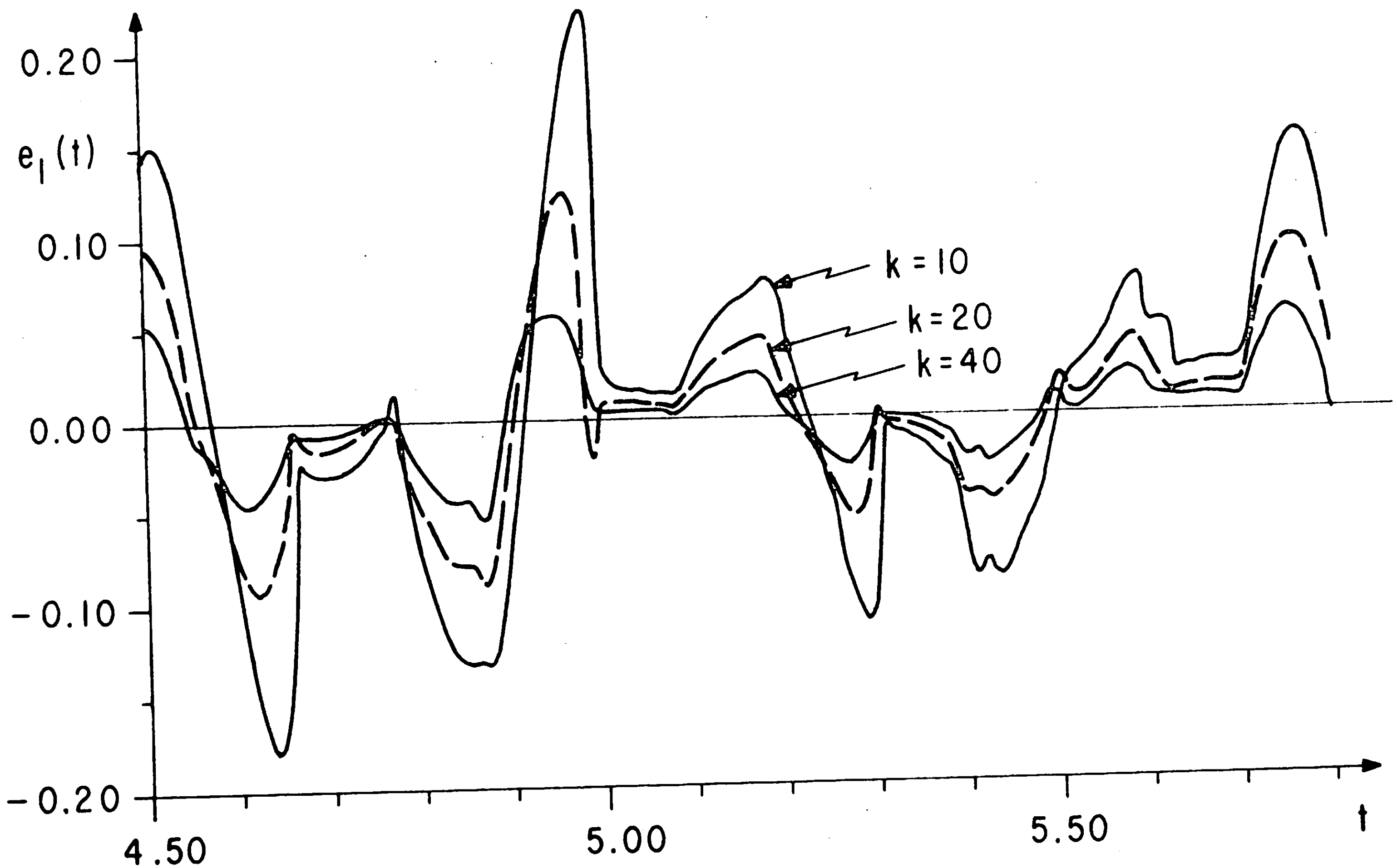

1+0.2 tanh x
1.2
1.0
0
2
4
6
8
x

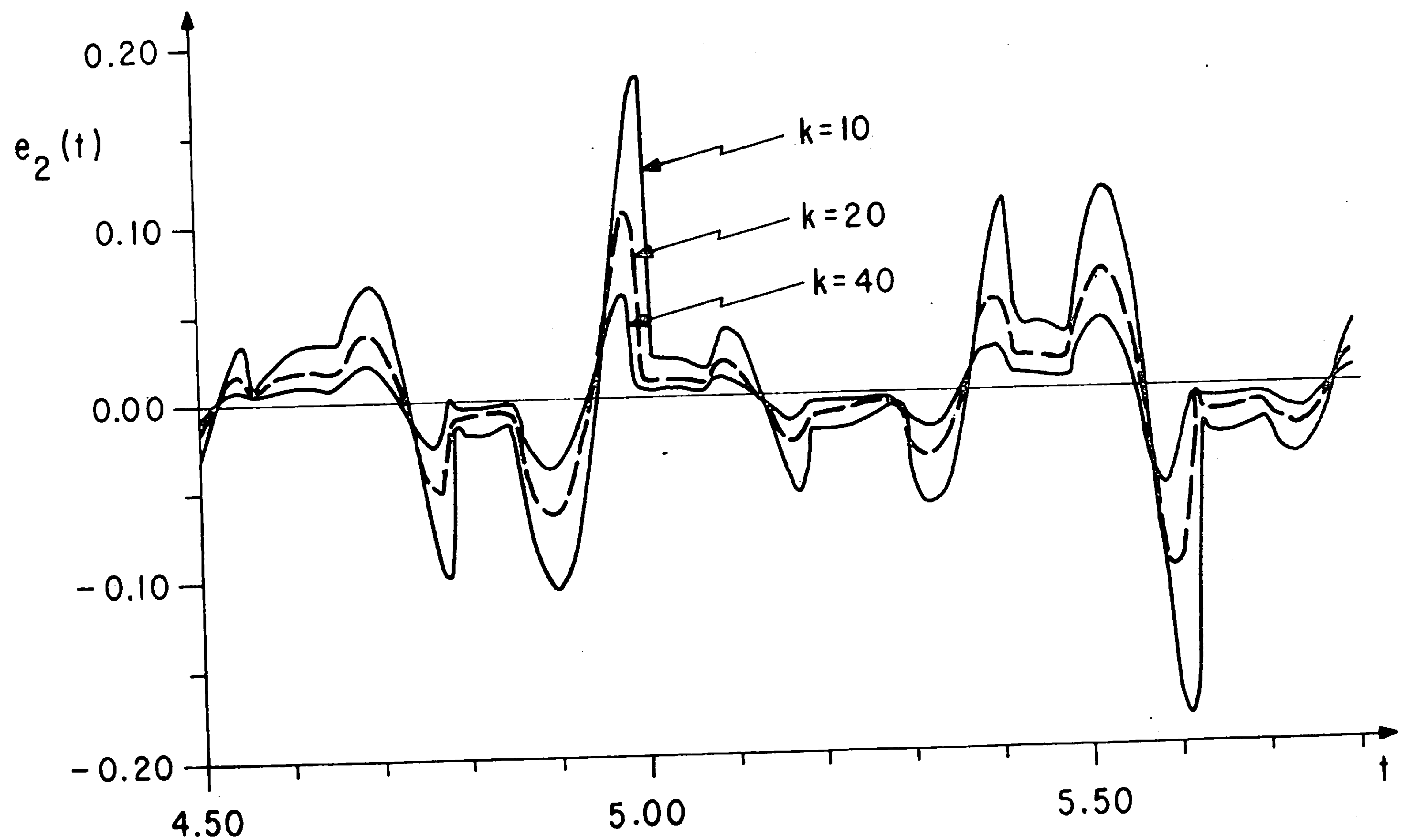


k = 40
k = 20
k = 10
k = 1
1.2
0
−1.2
$y_1(t)$
4.50
5.00
5.50
t

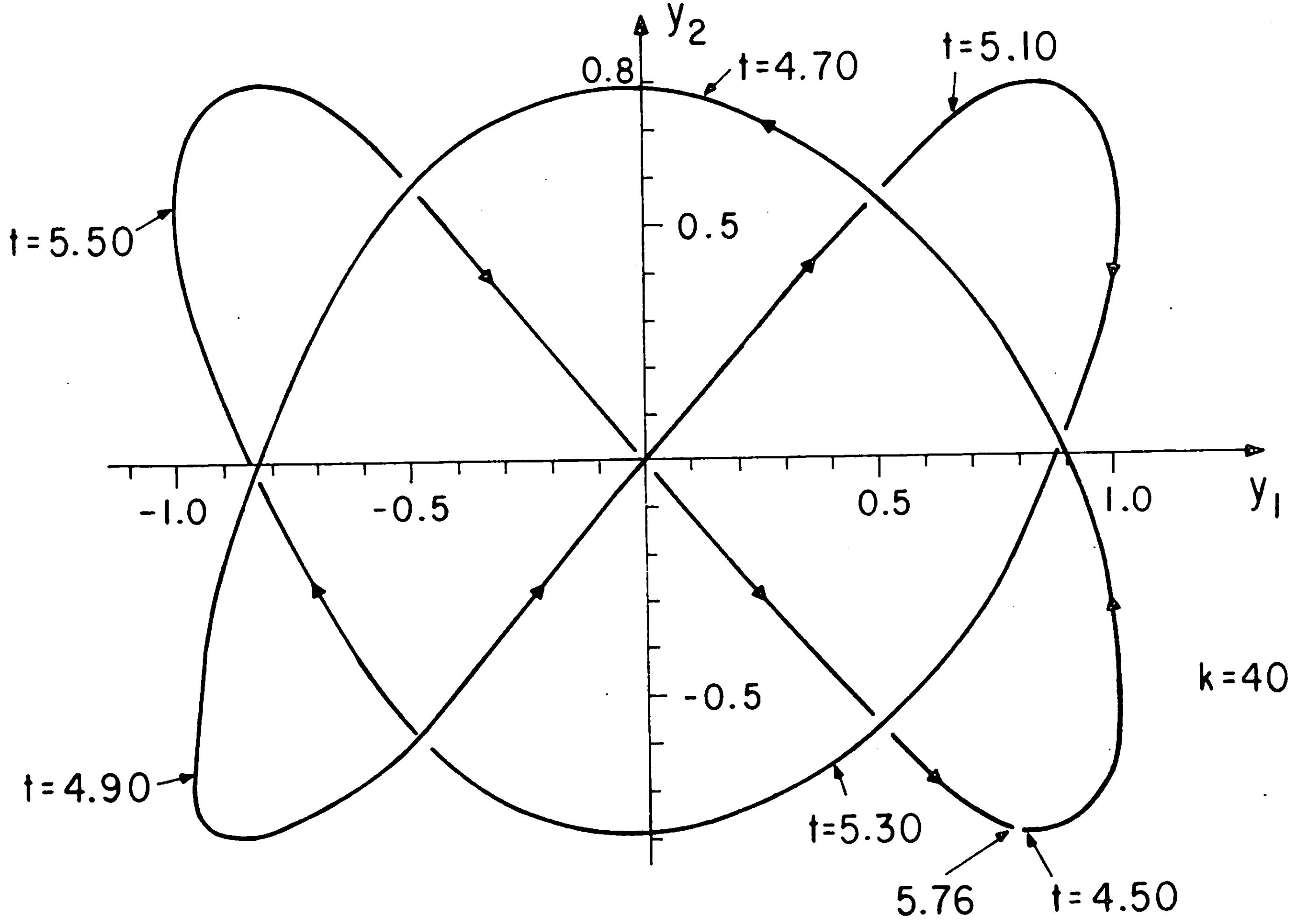


1.2
$y_2(t)$
0
-1.2
4.50
5.00
5.50
t
k = 20
k = 40
k = 10
k = 1

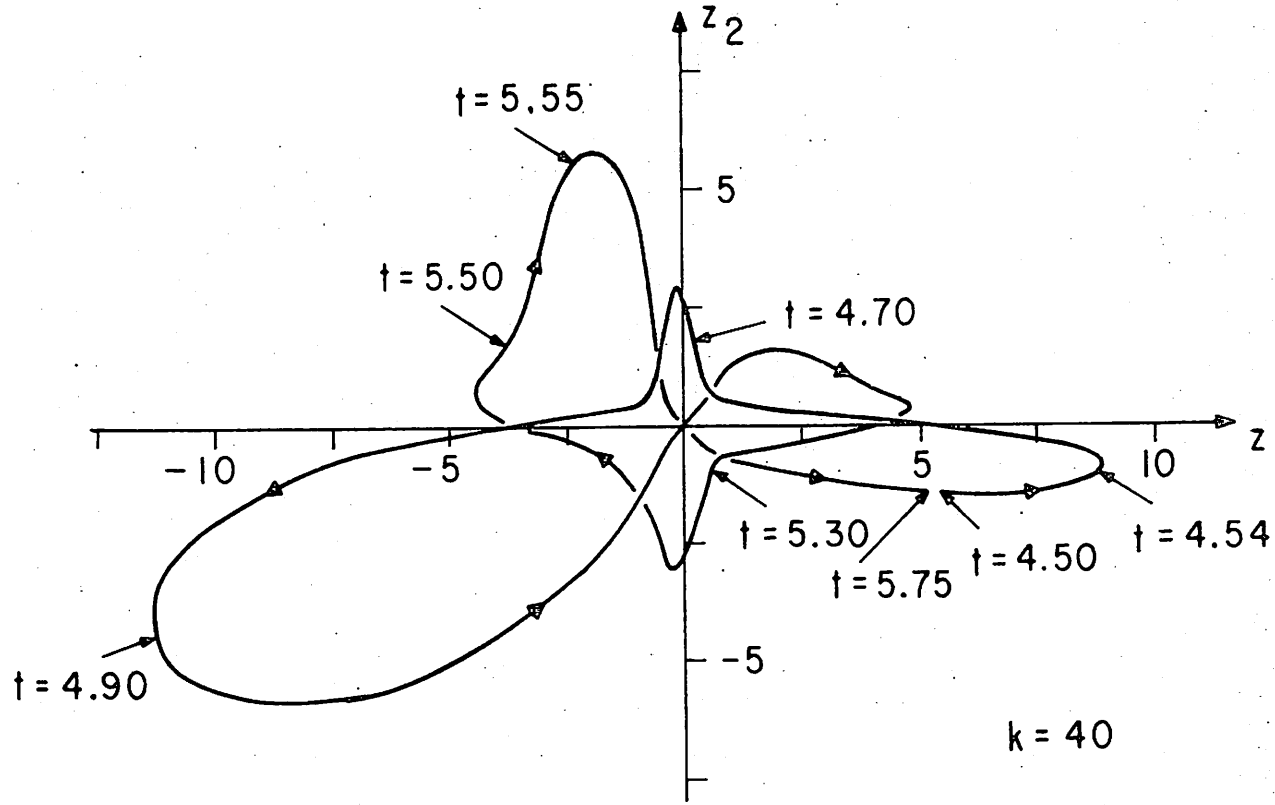

0.20
$e_1(t)$
0.10
0.00
-0.10
-0.20
4.50
5.00
5.50
t
k = 10
k = 20
k = 40



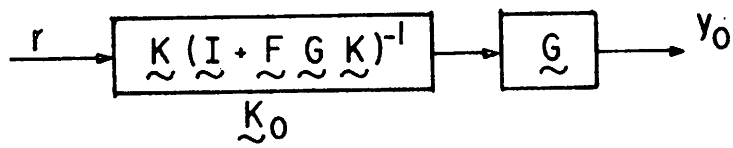
0.20
0.10
0.00
-0.10
-0.20
$e_2(t)$
k=10
k=20
k=40
4.50
5.00
5.50
t



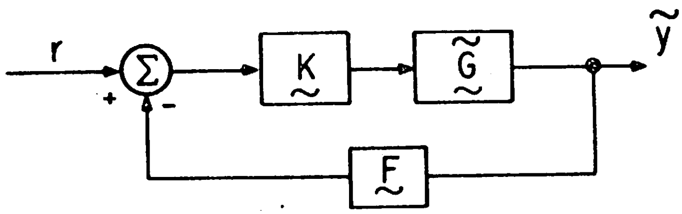
y_2
0.8
0.5
-0.5
y_1
-1.0
-0.5
0.5
1.0
t=4.70
t=5.10
t=5.50
t=4.90
t=5.30
5.76
t=4.50
k=40



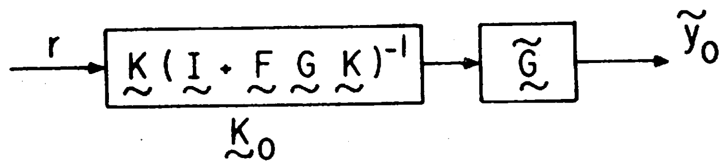
z_2
z_1
5
-5
-10
10
t = 5.55
t = 5.50
t = 4.70
t = 5.30
t = 5.75
t = 4.50
t = 4.54
t = 4.90
k = 40



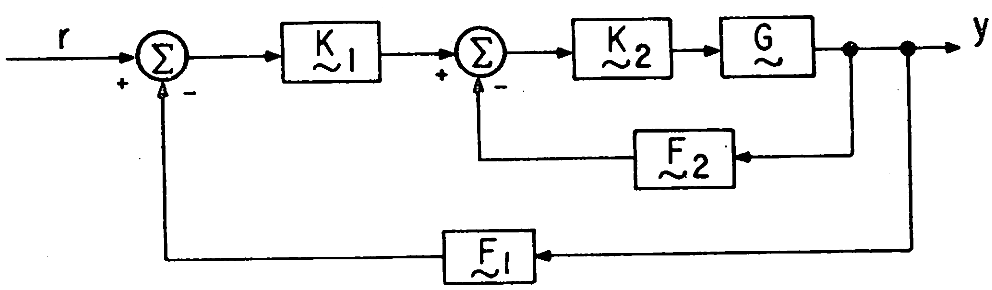
r
K (I + F G K)^-1
K0
G
y0



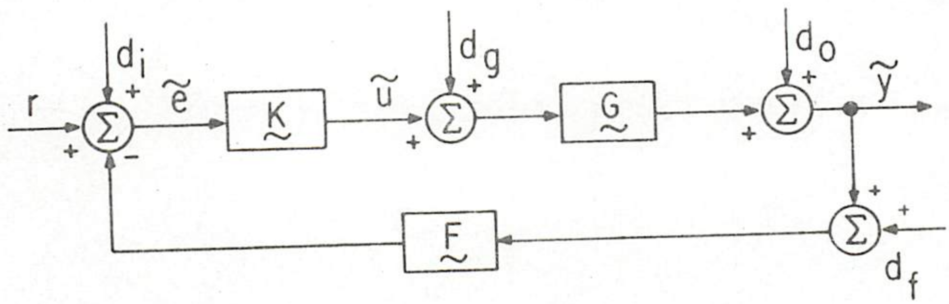
r
Σ
+
-
K
G̃
ỹ
F



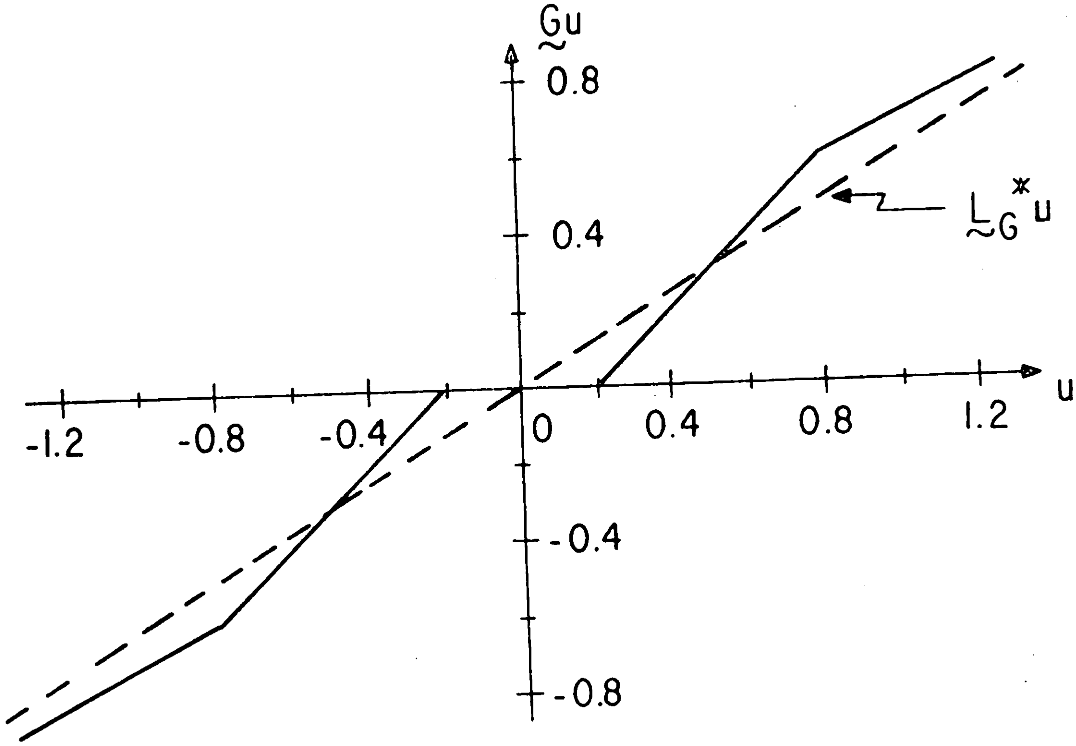
r
K (I + F G K)^-1
K0
G̃
ỹ0



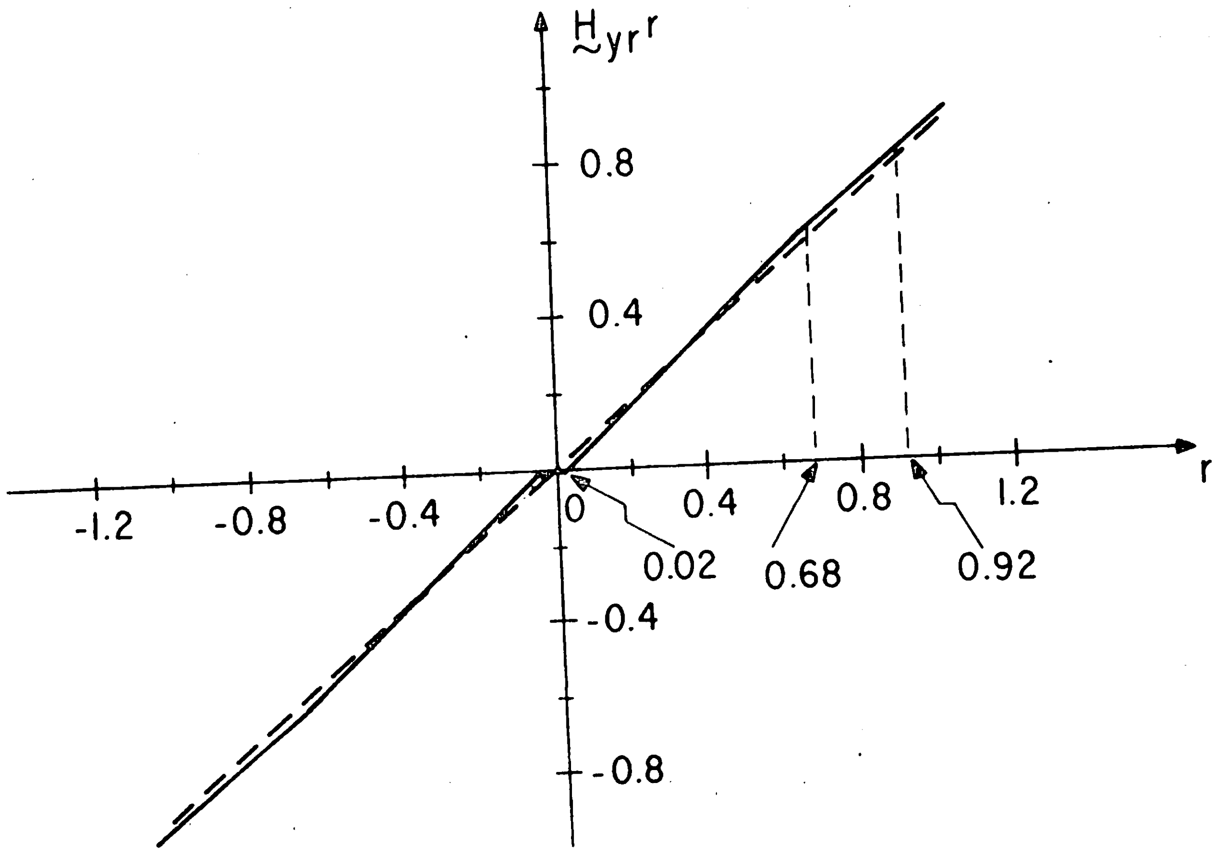
r
Σ
+
-
K1
Σ
+
-
K2
G
y
F2
F1



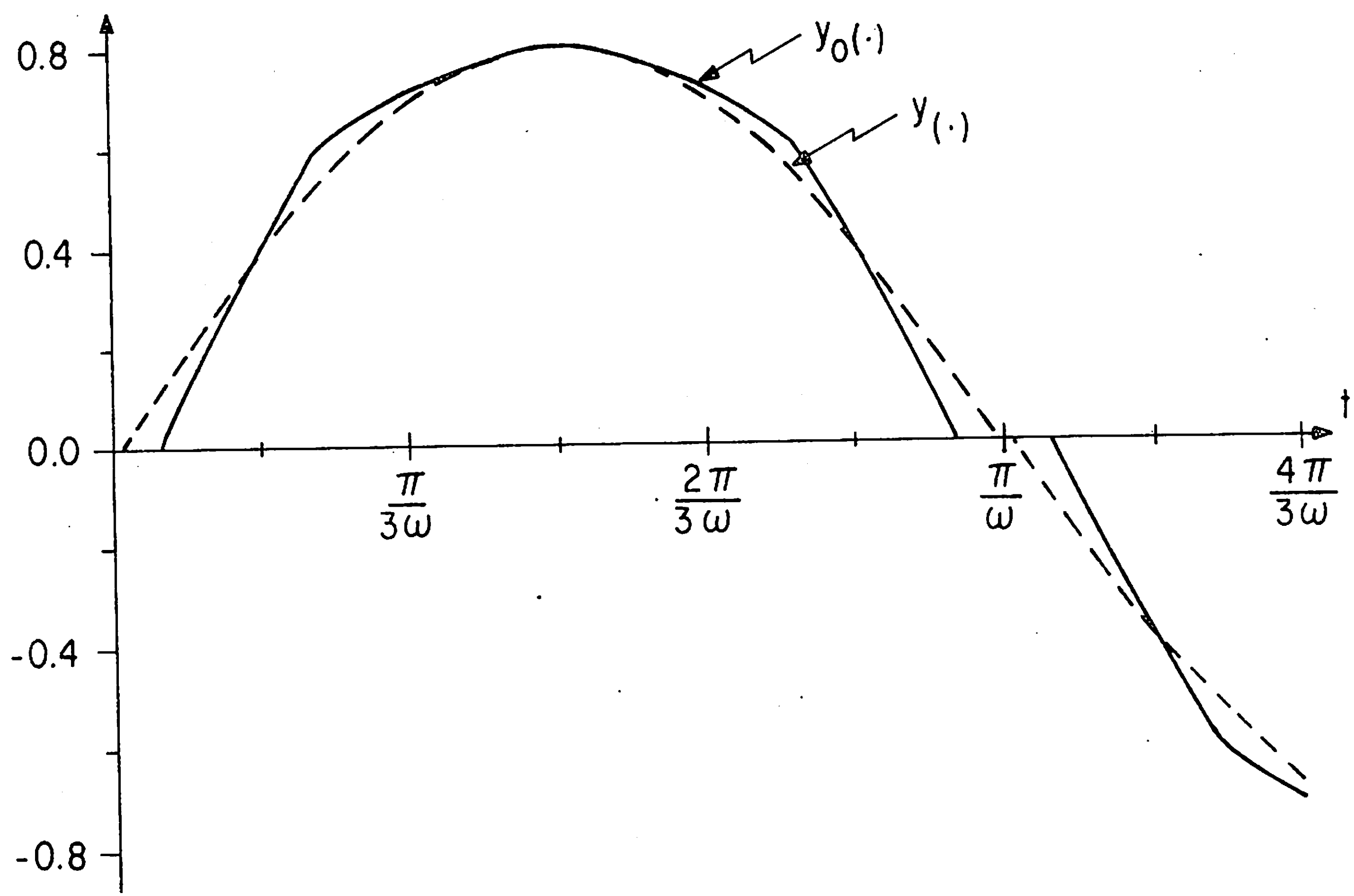
r
d_i
$\tilde{e}$
$\underset{\sim}{K}$
$\tilde{u}$
d_g
$\underset{\sim}{G}$
d_o
$\tilde{y}$
$\underset{\sim}{F}$
d_f



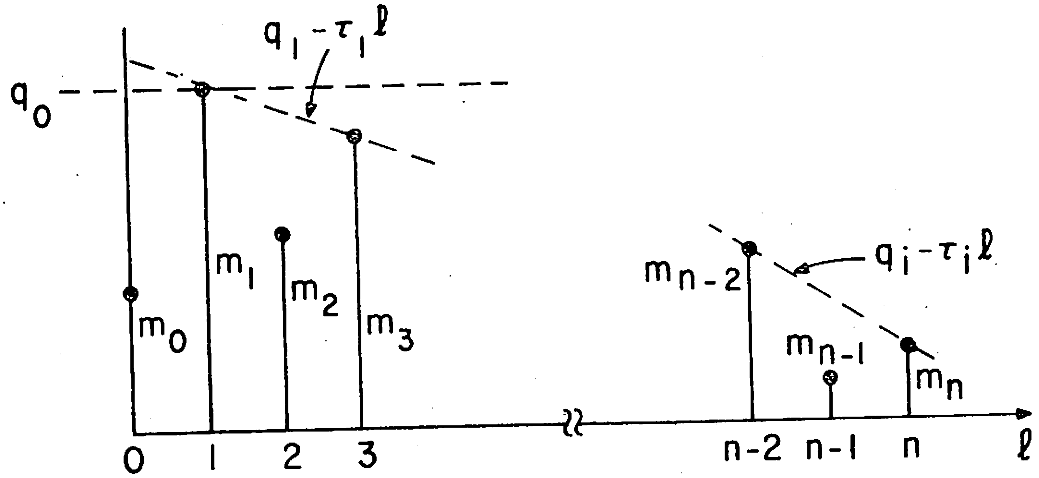
$\tilde{G}u$
0.8
0.4
-0.4
-0.8
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
u
$\tilde{L}_G^* u$



$\tilde{H}_{yr} r$
0.8
0.4
-0.4
-0.8
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
r
0.02
0.68
0.92



0.8
0.4
0.0
-0.4
-0.8
$\frac{\pi}{3\omega}$
$\frac{2\pi}{3\omega}$
$\frac{\pi}{\omega}$
$\frac{4\pi}{3\omega}$
t
$y_0(\cdot)$
$y(\cdot)$


$q_1 - \tau_1 \ell$
q_0
m_1
m_2
m_3
m_0
m_{n-2}
$q_i - \tau_i \ell$
m_{n-1}
m_n
0
1
2
3
n-2
n-1
n
ℓ