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NONLINEAR CIRCUIT THEORY

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Many highly nonlinear devices have been invented over the past two decades [1-4] and are becoming increasingly important in the design of modern electronic systems. The behaviors of these systems can be extremely complex and any circuit theory for analyzing such exotic phenomena must necessarily be nonlinear. In particular, some recent research on the modeling of nonlinear devices [5-8] has demonstrated the dire need for developing a *foundation* for nonlinear circuits. Our objective in this short course is to provide a unified exposition of some recent developments in this area. Due to limitation of both time and space, only some aspects of this vast terrain can be covered with a certain degree of depth. Even then, only main results will be presented and only some of them will be proved. Although some of the unproved theorems can be easily worked out by the reader, others are rather long and highly technical (involving sophisticated mathematical machineries). Whenever applicable, references where the proofs, or basic techniques for constructing the proofs, can be found will be given. Some of these references are chosen in view of their clarity of exposition and do not necessarily imply the original source. We have chosen to present the *circuit-theoretic and qualitative* aspects of *lumped nonlinear* networks because no unified exposition of this area is presently available. Some important aspects which are not covered here include the "existence and uniqueness of solutions of nonlinear dc networks" and "computational methods for nonlinear networks." Readers interested in the first area are referred to a collection of relevant literature in [9], as well as to a comprehensive paper on piecewise-linear networks in [10]. Readers interested in the second area are referred to [11].

The following materials are subdivided into 5 major sections. The equations, theorems, and headings for the subsections are numbered consecutively and independently in each section. The following *table of contents* may be used to identify the various topics presented:

I. Classification and Representation of Nonlinear n-Ports

1. Algebraic and Dynamic n-Ports

Let N denote either an *(n+1)-terminal element* (Fig. 1(a)) or an *n-port* (Fig. 1(b)). The four basic network variables associated with each terminal j, or each port j, of N are the *voltage* $v_j(t)$, *current* $i_j(t)$, *flux* $\phi_j(t)$, and *charge* $q_j(t)$, where

$$\phi_j(t) \triangleq \phi_j(t_0) + \int_{t_0}^{t} v_j(\tau)d\tau,\ j = 1,2,\ldots,n \tag{1}$$

$$q_j(t) \triangleq q_j(t_0) + \int_{t_0}^{t} i_j(\tau)d\tau,\ j = 1,2,\ldots,n \tag{2}$$

Observe that whereas $v_j(t)$ and $i_j(t)$ can be uniquely measured at any finite time $t > -\infty$, say by a voltmeter and ammeter, $\phi_j(t)$ and $q_j(t)$ can only be measured to within a constant of integration $\phi_j(t_0) \triangleq \int_{-\infty}^{t_0} v_j(\tau)d\tau$ and $q_j(t_0) \triangleq \int_{-\infty}^{t_0} i_j(\tau)d\tau$ respectively. These two constants must be assumed *a priori* because $v_j(t)$ and $i_j(t)$ can never be measured at $t = -\infty$. Neither can their values be extrapolated to $t = -\infty$ since *nonlinear* n-ports can have multiple equilibrium states (see Sec. II-2-G).

The two variables v_j and ϕ_j (resp., i_j and q_j) are said to be dynamically dependent in the sense that they are related by (1) and (2) for *any* N. Hence, out of the 6 distinct pairwise combinations of 4 basic network variables, only the 4 combinations (v_j,i_j), (i_j,ϕ_j), (v_j,q_j), and (ϕ_j,q_j) are not related by an *a priori* relation independent of N. The 6 pairwise combinations are depicted by the branches of the complete graph in Fig. 1(c), where the pairs connected by solid branches are dynamically independent. More generally, a pair of nx1 vectors (ξ,η) is said to be a

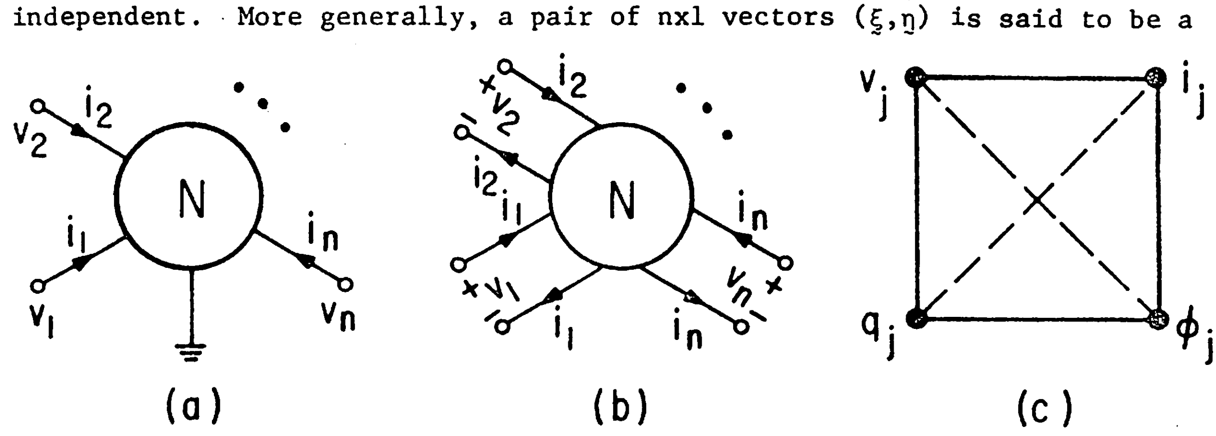

Fig. 1. (a) An (n+1)-terminal element, (b) An n-port, (c) A complete graph of 4 basic network variables.

pair of *dynamically independent terminal or port vectors* if each component

$$\boxed{(\xi_j,\eta_j) \in \left\{(v_j,i_j),(i_j,\phi_j),(v_j,q_j),(\phi_j,q_j)\right\}} \tag{3}$$

To simplify our notation, we do not distinguish the "order" of the variables in each pair so that (ξ_j,η_j) could be defined for example, either as (v_j,i_j) or (i_j,v_j). Of course, once the choice is made, the order must be preserved consistently. Since virtually every definition and property that apply for an (n+1)-terminal element also hold for an n-port, and vice-versa, we will avoid unnecessary repetitions by stating all definitions and results for n-ports in this lecture notes, unless otherwise stated.

A pair of dynamically independent port vector waveforms $\left(\underset{\sim}{\xi}(t),\underset{\sim}{\eta}(t)\right)$ measured from an n-port N over the time interval $[t_0,\infty)$ is said to be an *admissible signal pair* of N. The collection of *all* admissible signal pairs $\left(\underset{\sim}{\xi}(\cdot),\underset{\sim}{\eta}(\cdot)\right)$ of N measured with respect to the same initial time t_0 is said to be a *constitutive relation* of N_1 where $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ is any pair of dynamically independent port vectors.

Definition 1. Linear and nonlinear n-ports

An n-port N is said to be linear if for every two admissible signal pairs $\left(\underset{\sim}{\xi}'(\cdot),\underset{\sim}{\eta}'(\cdot)\right)$ and $\left(\underset{\sim}{\xi}''(\cdot),\underset{\sim}{\eta}''(\cdot)\right)$, and any two scalars α and β, $\left(\hat{\underset{\sim}{\xi}}(\cdot),\hat{\underset{\sim}{\eta}}(\cdot)\right) \triangleq \left(\alpha\underset{\sim}{\xi}'(\cdot)+\beta\underset{\sim}{\xi}''(\cdot),\alpha\underset{\sim}{\eta}'(\cdot)+\beta\underset{\sim}{\eta}''(\cdot)\right)$ is also an admissible signal pair. Otherwise, N is said to be *nonlinear*.

Definition 2. Time-invariant and time-varying n-ports

An n-port N is said to be *time-invariant* if for any admissible signal pair $\left(\underset{\sim}{\xi}(\cdot),\underset{\sim}{\eta}(\cdot)\right)$ and any $T \in [t_0,\infty)$, the *translated* pair $\left(\hat{\underset{\sim}{\xi}}(\cdot),\hat{\underset{\sim}{\eta}}(\cdot)\right)$ defined by $\left(\hat{\underset{\sim}{\xi}}(t),\hat{\underset{\sim}{\eta}}(t)\right) \triangleq \left(\underset{\sim}{\xi}(t-T),\underset{\sim}{\eta}(t-T)\right)$ is also an admissible signal pair. Otherwise, N is said to be *time varying*.

Definition 3. Algebraic and dynamic n-ports

An n-port N characterized by a constitutive relation between a pair of dynamically independent port vectors $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ is said to be an *algebraic n-port* if for any two admissible signal pairs $\left(\underset{\sim}{\xi}'(\cdot),\underset{\sim}{\eta}'(\cdot)\right)$ and $\left(\underset{\sim}{\xi}''(\cdot),\underset{\sim}{\eta}''(\cdot)\right)$, and for any $T \in [t_0,\infty)$, the *concatenated* pair $\left(\hat{\underset{\sim}{\xi}}(\cdot),\hat{\underset{\sim}{\eta}}(\cdot)\right)$ defined by

$$\begin{aligned}\left(\hat{\underset{\sim}{\xi}}(t),\hat{\underset{\sim}{\eta}}(t)\right) &\triangleq \left(\underset{\sim}{\xi}'(t),\underset{\sim}{\eta}'(t)\right)\ , \quad t \le T \\ &\triangleq \left(\underset{\sim}{\xi}''(t),\underset{\sim}{\eta}''(t)\right)\ , \quad t > T\end{aligned} \tag{4}$$

is also an admissible signal pair. Otherwise, N is said to be a *dynamic n-port*.

In less formal terms, we say N is linear, time-invariant, or algebraic if it is *closed* under superposition, translation, or concatenation of admissible signal pairs, respectively. An n-port N is said to be characterized by an *algebraic constitutive relation between* ξ *and* η, or a *dynamic constitutive relation between* ξ *and* η depending on whether N is an algebraic n-port, or a dynamic n-port. Roughly speaking, a constitutive relation between ξ and η is *algebraic* if it can be specified by algebraic equations involving only $\xi(t)$ and $\eta(t)$ at any time t, and not their derivatives or integrals. The following theorem is a direct consequence of this property.

Theorem 1. Algebraic n-port characterization

Every time-invariant algebraic n-port can be characterized by a subset of points in $\mathbb{R}^n \times \mathbb{R}^n$, i.e., a *relation*

$$\mathcal{R} \subset \mathbb{R}^n \times \mathbb{R}^n \tag{5}$$

such that every admissible signal pair $\left(\xi(\cdot),\eta(\cdot)\right)$ of N satisfies the following *inclusion property*:

$$\left(\xi(t),\eta(t)\right) \in \mathcal{R}, \quad \forall t \in [t_0,\infty) \tag{6}$$

Conversely, any pair of waveforms $\left(\xi(\cdot),\eta(\cdot)\right)$ satisfying (6) is an admissible signal pair of N.

Proof. Let $\mathcal{R}$ in (5) be defined by

$$\mathcal{R} = \left\{\left(\xi(t),\eta(t)\right) : \begin{array}{l}\text{over all admissible signal pairs} \\ \left(\xi(\cdot),\eta(\cdot)\right) \text{ of N and over all } t \in [t_0,\infty)\end{array}\right\} \tag{7}$$

Then (6) is satisfied by construction. Conversely, if $\left(\xi(\cdot),\eta(\cdot)\right)$ satisfies (6), then at each time $t = \hat{t} \in [t_0,\infty)$, there exists an admissible signal pair $\left(\hat{\xi}(\cdot),\hat{\eta}(\cdot)\right)$ such that $\left(\hat{\xi}(\hat{t}),\hat{\eta}(\hat{t})\right) = \left(\xi(\hat{t}),\eta(\hat{t})\right)$ in view of (7). It follows from the concatenation and time invariance hypotheses that $\left(\xi(\cdot),\eta(\cdot)\right)$ is also an admissible signal pair. ◻

Corollary 1.

Theorem 1 also holds for *time-varying* algebraic n-ports provided the relation $\mathcal{R}$ in (5) is replaced by a "time" parametrized relation

$$\mathcal{R}_t \in \mathbb{R} \times \mathbb{R}^n, \quad t \in [t_0,\infty) \tag{8}$$

Corollary 2.

Every time-invariant algebraic n-port is *rate independent* in the sense that if $\left(\underset{\sim}{\xi}(\cdot),\underset{\sim}{\eta}(\cdot)\right)$ is an admissible signal pair of N, then for every $\alpha \in (-\infty,\infty)$, the pair $\left(\hat{\underset{\sim}{\xi}}(\cdot),\hat{\underset{\sim}{\eta}}(\cdot)\right)$ defined by

$$\left(\hat{\underset{\sim}{\xi}}(t),\hat{\underset{\sim}{\eta}}(t)\right) \triangleq \left(\underset{\sim}{\xi}(\alpha t),\underset{\sim}{\eta}(\alpha t)\right), \quad \forall t \in [t_0,\infty) \tag{9}$$

is also an admissible signal pair of N.

Observe that Def. 3 applies only to a subclass of n-ports whose constitutive relations involve $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ where (ξ_j,η_j) is defined by (3). One could enlarge the class of algebraic and dynamic n-ports of Def. 3 by allowing $\xi_j(t)$ and $\eta_j(t)$ to become the <u>mth</u> time derivature or integral of $v_j(t)$ and $i_j(t)$, respectively. However, the class of n-ports encompassed by Def. 3 seems to be more than adequate for developing a reasonably general theory of nonlinear n-ports.

To avoid unnecessary repetitions, *all n-ports are henceforth assumed to be lumped and time-invariant,* unless stated otherwise.

A. <u>Four basic circuit elements</u>

Let N be an algebraic n-port where all components of $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ are of the same type. Then $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ can assume only one of the 4 combinations depicted by the solid branches in Fig. 2; namely,

$$(\underset{\sim}{\xi},\underset{\sim}{\eta}) \in \left\{(\underset{\sim}{v},\underset{\sim}{i}),\ (\underset{\sim}{i},\underset{\sim}{\phi}),\ (\underset{\sim}{v},\underset{\sim}{q}),\ (\underset{\sim}{\phi},\underset{\sim}{q})\right\} \tag{10}$$

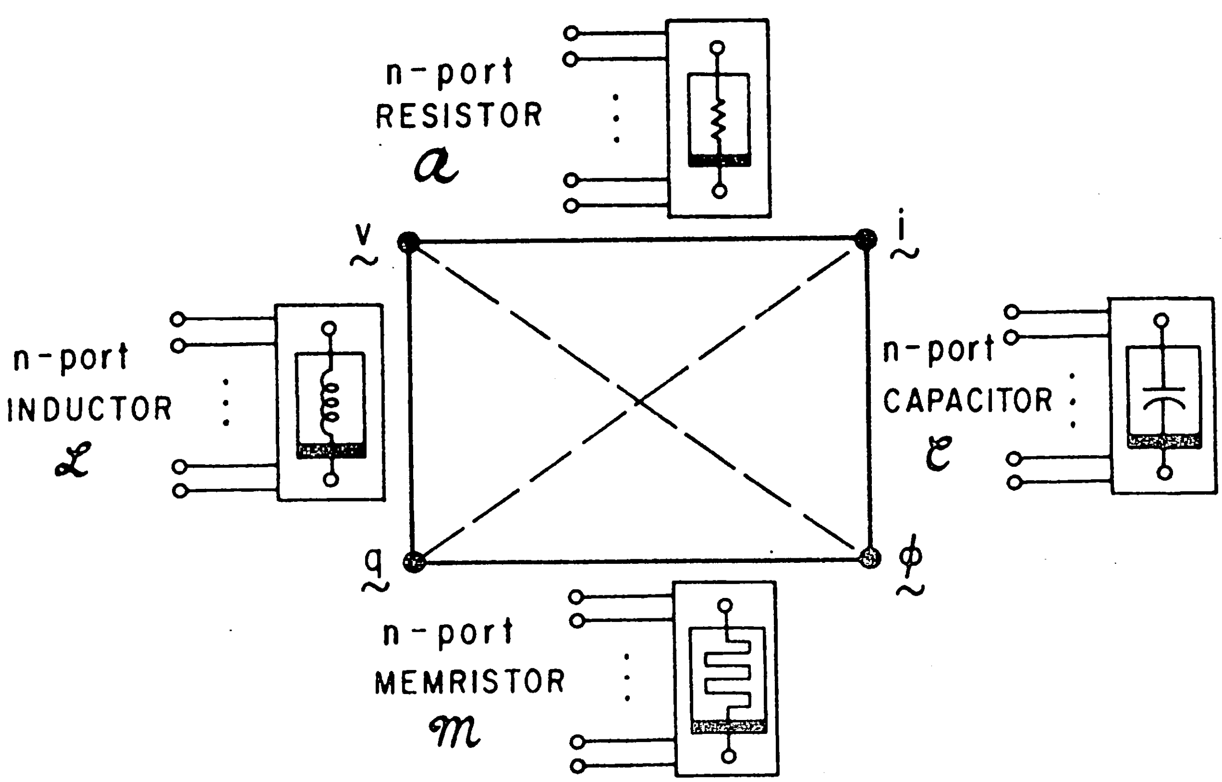

Fig. 2. Each solid branch of the complete graph defines a basic n-port circuit element.

These 4 exhaustive combinations lead naturally to the following axiomatic classifications of basic circuit elements:

Definition 4. n-port resistor, inductor, capacitor, memristor

An *Algebraic* n-port N is said to be an n-port *resistor, inductor, capacitor, memristor* [12], respectively, if N can be characterized by an *algebraic constitutive relation* between $\underset{\sim}{v}$ and $\underset{\sim}{i}$, $\underset{\sim}{\phi}$ and $\underset{\sim}{i}$, $\underset{\sim}{q}$ and $\underset{\sim}{v}$, and $\underset{\sim}{\phi}$ and $\underset{\sim}{q}$, respectively.

The symbols proposed for the 4 basic n-port circuit elements are shown in Fig. 2 corresponding to each pair of *dynamically independent* port vectors. There are many examples of nonlinear devices which can be realistically modeled by one of these 4 basic elements -- at least over some restricted range of operating frequencies.

Examples of nonlinear resistors:

(1) one-port: *pn junction diodes, zener diodes, tunnel diodes, etc.*

(2) two-port: *transistors* (Fig. 3(a)) modeled by the following dc "pnp" Ebers-Moll equation:

$$\begin{aligned} i_1 &= A_1\left[\exp(Kv_1)-1\right] - B_1\left[\exp(Kv_2)-1\right] \\ i_2 &= -A_2\left[\exp(Kv_1)-1\right] + B_2\left[\exp(Kv_2)-1\right] \end{aligned} \tag{11}$$

where $A_1 = I_{ES}$, $B_1 = \alpha_R I_{CS} = \alpha_F I_{ES} = A_2$, $B_2 = I_{CS}$, and $K = q/kT$, where T is the temperature.

(3) three-port: a. *OP AMP* (Fig. 3(b)) modeled by $i_1 = 0$, $i_2 = 0$, $v_3 = f(v_2-v_1)$ as in Fig. 3(b), where $A \to \infty$.

b. *Analog Multiplier* (Fig. 3(c)) modeled by $i_1 = 0$, $i_2 = 0$, and $v_3 = Mv_1v_2$.

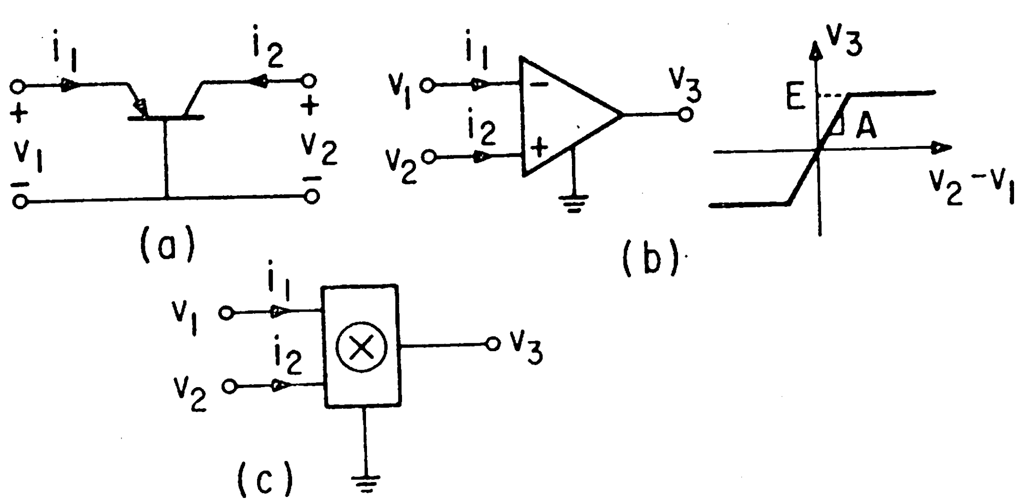

Fig. 3. (a) pnp transistor (b) OP AMP (c) Analog multiplier

c. *Conjunctors* (Fig. 4(a)). There are 6 types

$M = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI}$ [13]:

$$
\begin{array}{ll}
\mathrm{I.} & v_1 = -Ki_2i_3,\ v_2 = Ki_1i_3,\ v_3 = 0 \\
\mathrm{II.} & v_1 = -Ki_2v_3,\ v_2 = Ki_1v_3,\ i_3 = 0 \\
\mathrm{III.} & v_1 = -Kv_2i_3,\ i_2 = Ki_1i_3,\ v_3 = 0 \\
\mathrm{IV.} & v_1 = -Kv_2v_3,\ i_2 = Ki_1v_3,\ i_3 = 0 \\
\mathrm{V.} & i_1 = -Kv_2i_3,\ i_2 = Kv_1i_3,\ v_3 = 0 \\
\mathrm{VI.} & i_1 = -Kv_2v_3,\ i_2 = Kv_1v_3,\ i_3 = 0
\end{array}
\tag{12}
$$

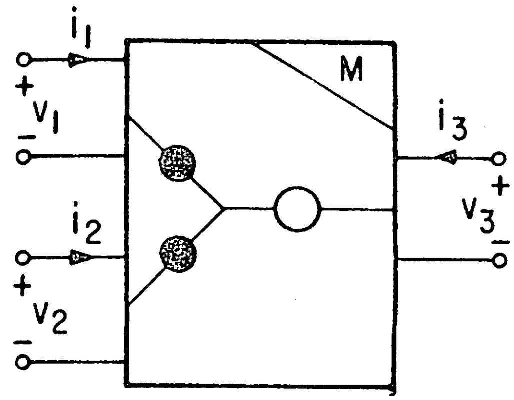

Fig. 4. A type M Conjunctor.

Examples of nonlinear inductors:

(1) one-port: *Josephson junction* [4] modeled by

$$i = I_0 \sin k_0\phi \tag{13}$$

where I_0 and k_0 are constants.

(2) two-port: A pair of *nonlinear coupled coils*.

(3) three-port: A type V *3-port traditor* (Fig. 5) described by [14]:

$$i_1 = -A\phi_2i_3,\ i_2 = -A\phi_1i_3, \phi_3 = A\phi_1\phi_2 \tag{14}$$

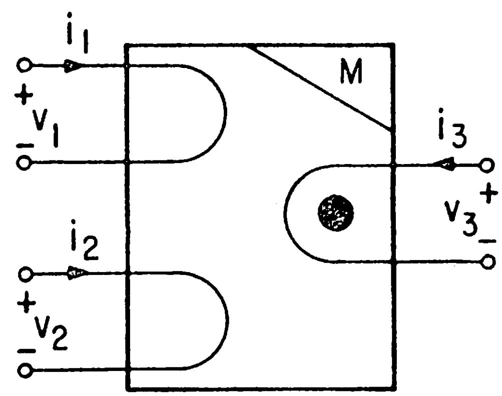

Fig. 5. A type M traditor.

Examples of nonlinear capacitors

(1) one-port: *varactor diode* modeled by $q = Q_0\left[\exp(Kv)-1\right]$.

(2) two-port: A pair of *coupled nonlinear capacitors*.

(3) three-port: A *type II traditor* (Fig. 5) described by [14]:

$$v_1 = -Aq_2v_3,\ v_2 = -Aq_1v_3,\ q_3 = Aq_1q_2 \tag{15}$$

Examples of nonlinear memristors

Consider a "coulomb cell" [12] consisting of a gold anode immersed in an electrolyte in a silver can (cathode) as in Fig. 6(a). Assume an initial amount of silver is previously deposited at the anode. When a battery is connected across the port, silver ions will be transferred from anode back to the cathode and a large current flows so that the element is equivalent to a very small *linear* resistance R_1. At some time $t = T_0$ when most silver has been transferred, very few ions are left so that a very small current flows for $t \geq T_0$ and the element is equivalent

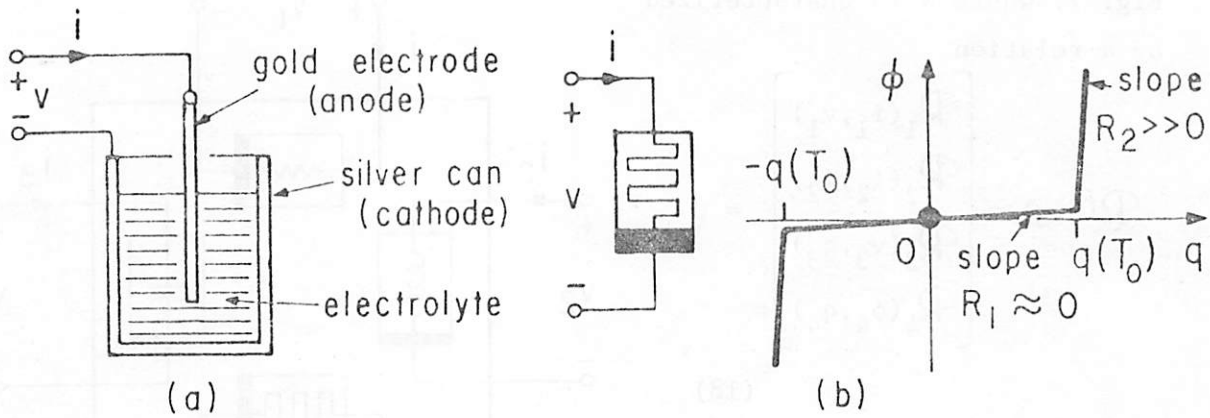

Fig. 6. (a) a coulomb cell (b) a memristor model

to a very large *linear* resistance R_2. This element can be realistically modeled as a memristor described by the ϕ-q curve $\phi = \phi(q)$ shown in Fig. 6(b). Observe that since

$$v(t) = d\phi(t)/dt = \left(d\phi(q)/dq\right)(dq/dt) \triangleq M(q)\, i(t) \tag{16}$$

is just a charge-dependent Ohm's law, a memristor is equivalent to a *charge-controlled linear resistor*. In this case, $M(q) = R_1 \approx 0$ for $|q| < q(T_0)$, and $M(q) = R_2 >> 0$ for $|q| > q(T_0)$.

Other examples of memristors can be found in [15-17].

There exist a small class of *ideal* elements which may assume more than one identities. For example, a dc {resp., ac} *voltage source* can be classified either as a time-invariant {resp., time-varying} one-port resistor or capacitor. Similarly, a dc {resp., ac} *current source* can be classified either as a time-invariant {resp., time-varying} one-port resistor or inductor. A *nullator* characterized $v = 0$ and $i = 0$ can classified as a one-port resistor, inductor, capacitor, or memristor. A (p+q)-port transformer characterized by

$$\begin{bmatrix} \underset{\sim}{v}_a \\ \hline \underset{\sim}{i}_b \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{0} & \underset{\sim}{K}^T \\ \hline -\underset{\sim}{K} & \underset{\sim}{0} \end{array}\right] \begin{bmatrix} \underset{\sim}{i}_a \\ \hline \underset{\sim}{v}_b \end{bmatrix} \tag{17}$$

(where $\underset{\sim}{v}_a$ and $\underset{\sim}{i}_a$ are px1 vectors, $\underset{\sim}{i}_b$ and $\underset{\sim}{v}_b$ are qx1 vectors, and $\underset{\sim}{K}$ is a qxp real matrix) can also be classified as a (p+q)-port resistor, inductor, capacitor, or memristor.

B. Algebraic n-ports

A simple example of an algebraic n-port involving a mixture of distinct pairs of dynamically independent variables is the 4-port shown in

Fig. 7, where N is characterized by a relation

$$\underset{\sim}{\mathcal{R}}(\xi,\eta) = \begin{bmatrix} \mathcal{R}_1(i_1,v_1) \\ \mathcal{R}_2(i_2,\phi_2) \\ \mathcal{R}_3(v_3,q_3) \\ \mathcal{R}_4(\phi_4,q_4) \end{bmatrix} = \underset{\sim}{0} \tag{18}$$

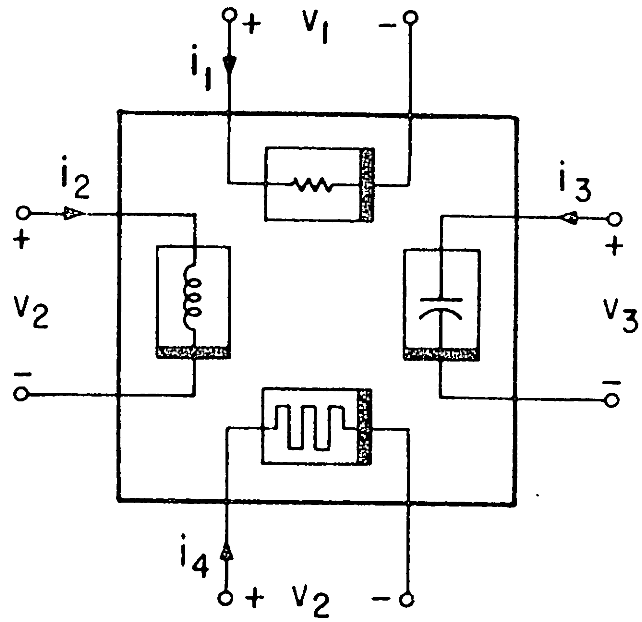

Fig. 7. An algebraic 4-port.

This example can be generalized by allowing all variables in (18) to be *coupled* to each other in a nonlinear way. The resulting algebraic 4-port would then be characterized by a system of 4 implicit equations

$$\mathcal{R}_i(i_1,v_1,i_2,\phi_2,v_3,q_3,\phi_4,q_4) = 0, \quad i = 1,2,3,4 \tag{19}$$

Observe that (19) defines an algebraic 4-port because all variables are dynamically independent of each other. We will now consider two interesting classes of algebraic n-ports.

(1) Mutators [18]

Mutators are generic names for a family of *linear* algebraic 2-ports. A *type 1 L-R mutator* is characterized by the constitutive relation $\phi_1 = v_2$ and $i_1 = -i_2$. Observe that if we terminate port 2 by a resistor having a constitutive relation $i_R = g(v_R)$ as shown in Figure 8(a), the resulting one-port is equivalent to an inductor characterized by an identical constitutive relation $i_1 = g(\phi_1)$. Conversely, if we terminate port 1 by an inductor having a constitutive relation $f(i_L,\phi_L) = 0$ as shown in Fig. 8(b), the resulting one-port is equivalent to a resistor characterized by an identical constitutive relation $f(i_2,v_2) = 0$. These observations follow immediately from the identities $i_1 = -i_2 = i_R$, $\phi_1 = v_2 = v_R$ in Fig. 8(a), and $i_2 = -i_1 = i_L$, $v_2 = \phi_1 = \phi_L$ in Fig. 8(b). Since this 2-port transforms one "elèment specie" X into another "element specie" Y, it is indeed revealing to call it a mutator. A *type 2 L-R mutator* is characterized by the constitutive relation $\phi_1 = -i_2$ and $i_1 = v_2$. The same "mutation" property also holds in this case except that the two variables

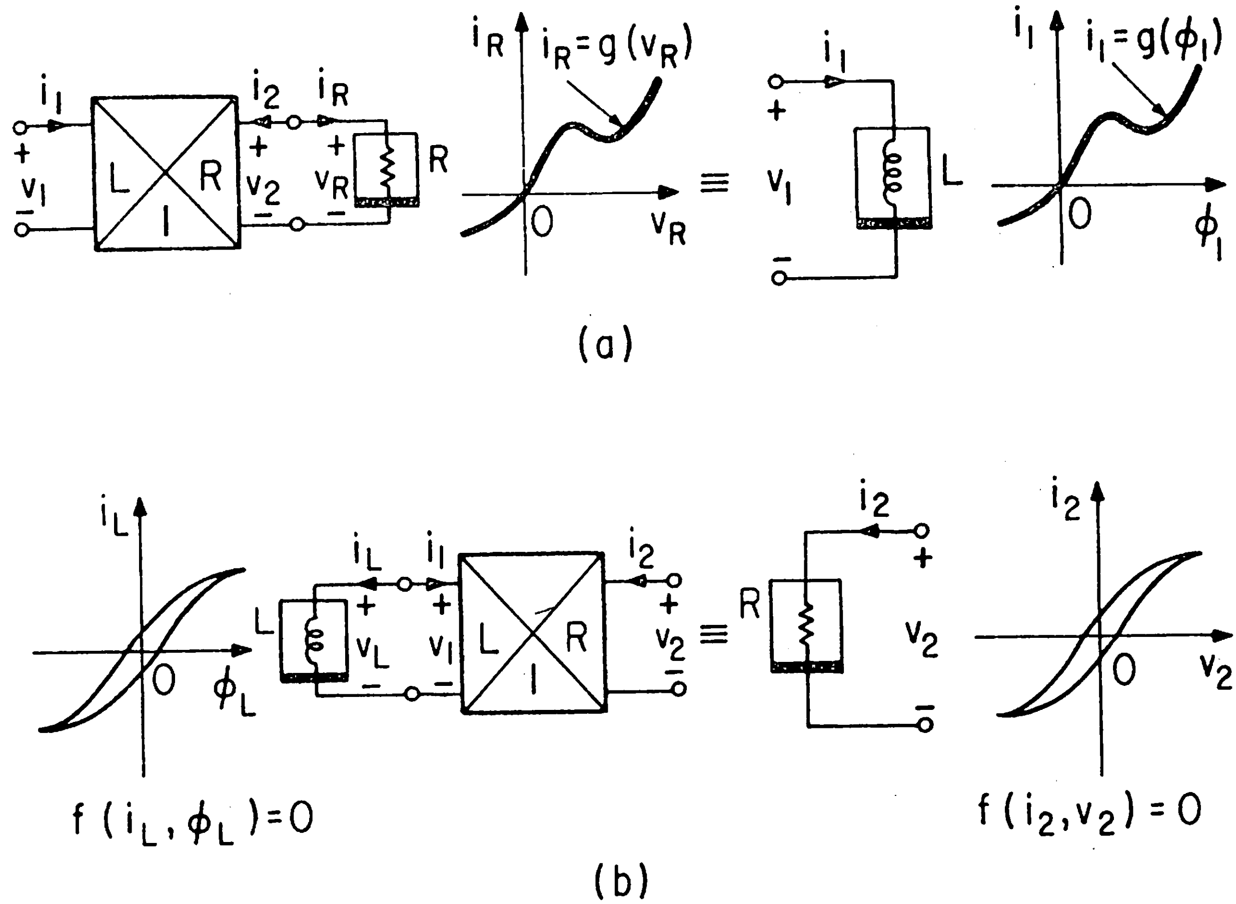

Fig. 8. A type 1 L-R mutator with two possible terminations.

in the resulting element are interchanged. For example, the constitutive relations for the inductor L in Fig. 8(a) and the resistor R in Fig. 8(b) are given respectively by $\phi_1 = g(i_1)$ and $f(v_2,i_2) = 0$.

By permuting the two pairs of dynamically-independent variables, we can define 6 mutually exclusive classes of X-Y mutators; namely, L-R, C-R, L-C, M-R, M-L, and M-C. Just as in the case of L-R mutators, each class of X-Y mutator may assume two different types. Table 1 contains a list of all mutually exclusive types of mutators along with their constitutive relations. The mutation property of each X-Y mutator is depicted in Figs. 9(a) and (b), where $k = 1$ for a type 1 mutator and $k = 2$ for a type 2 mutator. Observe that a type 1 L-C mutator is just a *gyrator*.

Observe that by using different mutators, it is possible to synthesize any three of the four basic circuit elements R,L,C,M given the fourth element. Moreover, since mutators are linear 2-ports, they can be realized using only linear elements. For example, a type 1 L-R mutator and a type 1 C-R mutator can be synthesized by the circuits shown in Figs. 10(a) and (b), respectively.

Table 1. Twelve Types of Linear X-Y Mutators

Mutator	Variables for port 1	Variables for port 2	Constitutive Relation
L-R Mutator (Type 1)	ϕ_1 , i_1	i_2 , v_2	$\phi_1 = v_2$, $i_1 = -i_2$
L-R Mutator (Type 2)	ϕ_1 , i_1	v_2 , i_2	$\phi_1 = -i_2$, $i_1 = v_2$
C-R Mutator (Type 1)	v_1 , q_1	i_2 , v_2	$v_1 = v_2$, $q_1 = -i_2$
C-R Mutator (Type 2)	v_1 , q_1	v_2 , i_2	$v_1 = -i_2$, $q_1 = v_2$
L-C Mutator (Type 1)	ϕ_1 , i_1	v_2 , q_2	$\phi_1 = -q_2$, $i_1 = v_2$
L-C Mutator (Type 2)	ϕ_1 , i_1	q_2 , v_2	$\phi_1 = v_2$, $i_1 = -q_2$
M-R Mutator (Type 1)	ϕ_1 , q_1	i_2 , v_2	$\phi_1 = v_2$, $q_1 = -i_2$
M-R Mutator (Type 2)	ϕ_1 , q_1	v_2 , i_2	$\phi_1 = -i_2$, $q_1 = v_2$
M-L Mutator (Type 1)	ϕ_1 , q_1	i_2 , ϕ_2	$\phi_1 = \phi_2$, $q_1 = -i_2$
M-L Mutator (Type 2)	ϕ_1 , q_1	ϕ_2 , i_2	$\phi_1 = -i_2$, $q_1 = \phi_2$
M-C Mutator (Type 1)	ϕ_1 , q_1	q_2 , v_2	$\phi_1 = v_2$, $q_1 = -q_2$
M-C Mutator (Type 2)	ϕ_1 , q_1	v_2 , q_2	$\phi_1 = -q_2$, $q_1 = v_2$

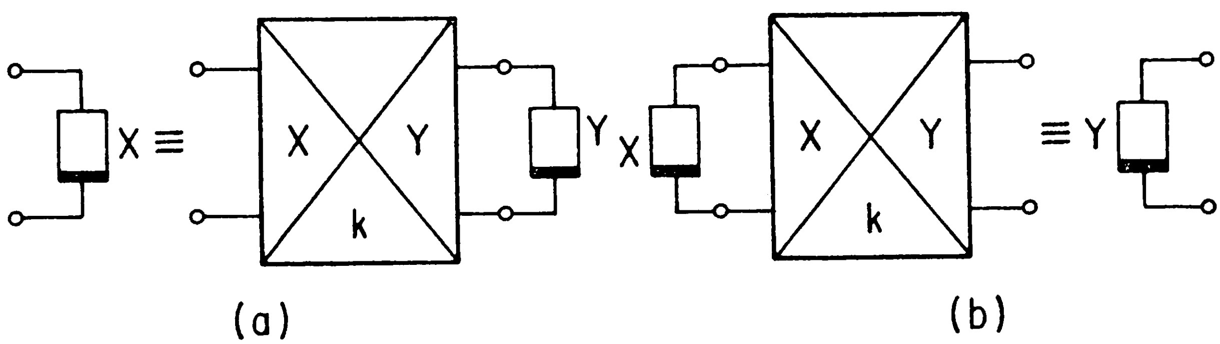

Fig. 9. Mutation property of X-Y mutators, k = 1 or k = 2.

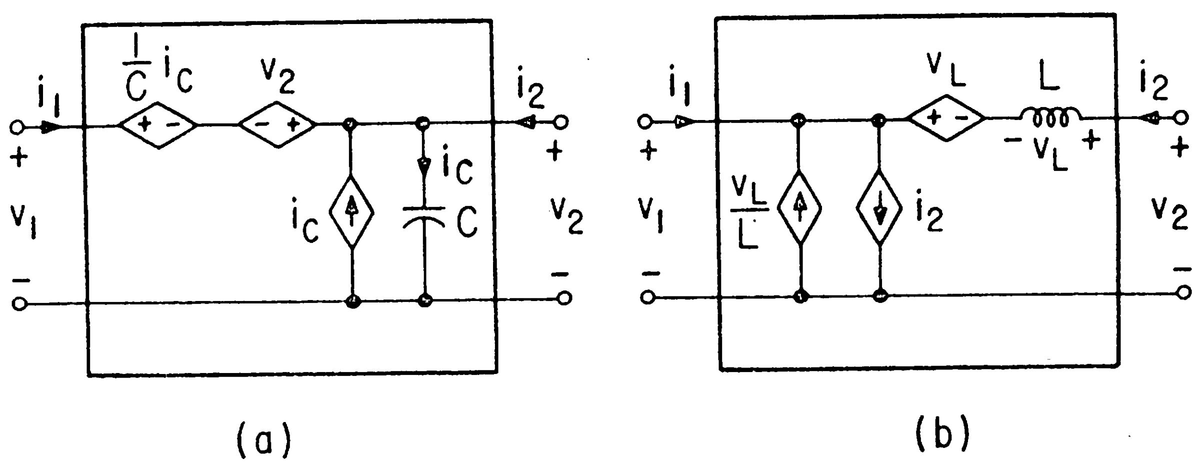

Fig. 10. Linear active circuit realization of a type 1 L-R mutator (a), and a type 1 C-R mutator (b).

(2) Traditors [14].

Let (ξ_k, η_k) denote either (i_k, ϕ_k) or (v_k, q_k). Let $\mathcal{L}$ (called the Lagrangian) be defined by the multilinear form

$$\mathcal{L} \triangleq A\ \eta_1\eta_2 \cdots \eta_k \cdots \eta_{n-1}\dot{\eta}_n \tag{20}$$

where A is any constant real number. *Traditors* are generic names for a family of nonlinear algebraic n-ports characterized by the following constitutive relations:

$$\xi_k = -\frac{\partial\mathcal{L}}{\partial\eta_k}, \quad k = 1,2,\ldots,n-1 \tag{21}$$

$$\xi_n = \frac{d}{dt}\,(A\eta_1\eta_2 \cdots \eta_{n-1}) \tag{22}$$

At first sight, (21) and (22) do not seem to qualify as an algebraic constitutive relation since the rate-independence condition (9) appears to be violated. To show that this is not the case, let us substitute (20) for $\mathcal{L}$ in (21) and obtain the explicit algebraic function

$$\xi_k = -A\eta_1\ \eta_2 \cdots \eta_{k-1}\ \eta_{k+1} \cdots \eta_{n-1}\ \dot{\eta}_n \tag{23}$$

Next, let us integrate both sides of (22) with respect to t to obtain

$$\int_{-\infty}^{t} \xi_n(\tau)d\tau = A\eta_1\eta_2 \cdots \eta_{n-1} \tag{24}$$

Now since $\dot{\eta}_n = i_n$ if $\xi_n = v_n$ and $\dot{\eta}_n = v_n$ if $\xi_n = i_n$, (23) and (24) may assume two distinct forms.

<u>Form 1</u> $\xi_k = -A\eta_1\ \eta_2 \cdots \eta_{k-1}\eta_{k+1} \cdots \eta_{n-1}i_n,\ k = 1,2,\ldots,n-1$ (23')

$$\phi_n = A\eta_1\ \eta_2 \cdots \eta_{n-1} \tag{24'}$$

<u>Form 2</u> $\xi_k = -A\eta_1\ \eta_2 \cdots \eta_{k-1}\eta_{k+1} \cdots \eta_{n-1}v_n,\ k = 1,2,\ldots,n-1$ (23")

$$q_n = A\eta_1\ \eta_2 \cdots \eta_{n-1} \tag{24"}$$

Since in either case, $(\xi_k,\eta_k) \in \left\{(i_k,\phi_k),(v_k,q_k)\right\}$, both (23'), (24') and (23"), (24") define an algebraic n-port. In fact, it is easily seen that the *generalized traditor* defined with

$$\hat{\mathcal{L}} \triangleq A\ \dot{\eta}_n f(\eta_1,\eta_2, \ldots, \eta_{n-1}) \tag{25}$$

is also an algebraic n-port.

As a specific example, consider the class of 3-port traditors (n=3).

Depending on the choice of (ξ_k,η_k), the following 6 mutually exclusive types can be defined:

$$
\begin{array}{llllll}
\text{I.} & v_1 = -Aq_2i_3 & \text{II.} & v_1 = -Aq_2v_3 & \text{III.} & v_1 = -A\phi_2i_3 \\
 & v_2 = -Aq_1i_3 & & v_2 = -Aq_1v_3 & & i_2 = -Aq_1i_3 \\
 & \phi_3 = Aq_1q_2 & & q_3 = Aq_1q_2 & & \phi_3 = Aq_1\phi_2 \\
\text{IV.} & v_1 = -A\phi_2v_3 & \text{V.} & i_1 = -A\phi_2i_3 & \text{VI.} & i_1 = -A\phi_2v_3 \\
 & i_2 = -Aq_1v_3 & & i_2 = -A\phi_1i_3 & & i_2 = -A\phi_1v_3 \\
 & q_3 = Aq_1\phi_2 & & \phi_3 = A\phi_1\phi_2 & & q_3 = A\phi_1\phi_2
\end{array}
\tag{26}
$$

The symbol of a type M traditor is shown in Fig. 5, where M = I,II...VI. Observe that a type V traditor has already been identified in (14) as a 3-port inductor, while a type II traditor has been identified in (15) as a 3-port capacitor. The remaining 4 types, however, can not be identified from among the 4 basic circuit elements.

Arbitrary interconnections of algebraic n-ports will not necessarily result in another algebraic n-port. However, the following theorem can be proved which guarantees closure property under the condition that the port interconnections (series or parallel) are *compatible* in the sense that all ports to be connected must be associated with the same type of dynamically independent variables. For example, with reference to Table 1, port 2 of a type 1 L-C mutator and port 1 of a type 1 C-R mutator are compatible since they both involve "voltage" and "charge" as port variables. On the other hand, port 1 of a type 1 L-C mutator and port 1 of a type 1 C-R mutator are incompatible since the former involves (ϕ_1,i_1) while the latter involve (v_1,q_1).

<u>Theorem 2</u>. <u>Algebraic n-port interconnection closure property</u> [28].

Compatible interconnections among a group of ports belonging to an algebraic n-port, or to two or more algebraic m-ports, always results in another algebraic n-port. Moreover, each port of the resulting n-port inherits the same pair of dynamically-independent variables associated with the original m-ports.

As an application of Theorem 2, recall that a type I traditor involves the port variables (v_1,q_1), (v_2,q_2), and (ϕ_3,i_3). Hence the first two ports are compatible with each other, whereas the third port is compatible with a one-port inductor. Hence theorem 2 guarantees the interconnections shown in Fig. 11(a) are compatible and that the result must be a one-port capacitor. Similarly, theorem 2 guarantees the interconnection between a type II traditor and a type I traditor as shown in Fig. 11(b) must result in an algebraic 4-port. In fact, it is easily verified that the result is a 4-port traditor as defined in (21)-(22). Finally, let us use theorem 2 to prove the following basic result.

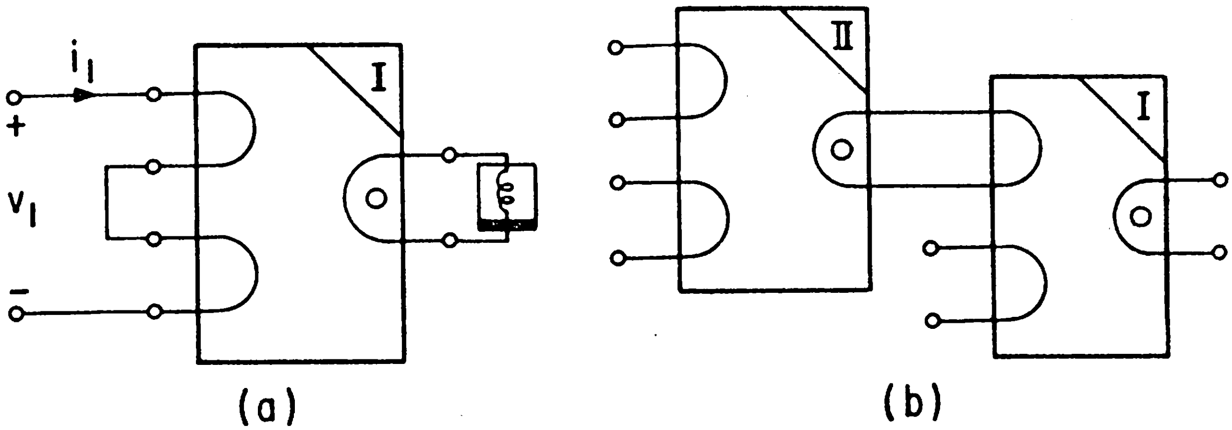

Fig. 11. Compatible algebraic n-port interconnections resulting in (a) one-port capacitor, (b) 4-port traditor.

Theorem 3a. Algebraic n-port realizability theorem.

Every algebraic n-port can be synthesized using only "linear" *mutators* and a "nonlinear" n-port *resistor*.

Proof. We will give a constructive proof. Let $\mathcal{R}$ be the prescribed constitutive relation of an algebraic n-port N. Transform each pair of variables (ϕ_j, i_j), (q_j, v_j), and (ϕ_j, q_j) into (v_j, i_j) using an L-R, C-R, and M-R mutator as depicted in Fig. 12. All mutators are linear since they can be realized by using only linear controlled sources and linear capacitors (see Fig. 10 for example). Theorem 2 then guarantees that the two n-ports shown in Fig. 12 are equivalent. ¤

It follows from theorem 3 that all types of traditors can be synthesized using only linear elements and a nonlinear n-port resistor. It also follows that in studying the qualitative behaviors of networks containing algebraic n-ports, there is little loss of generality to assume that the only nonlinear elements are resistors; i.e., all inductors and capacitors are linear.

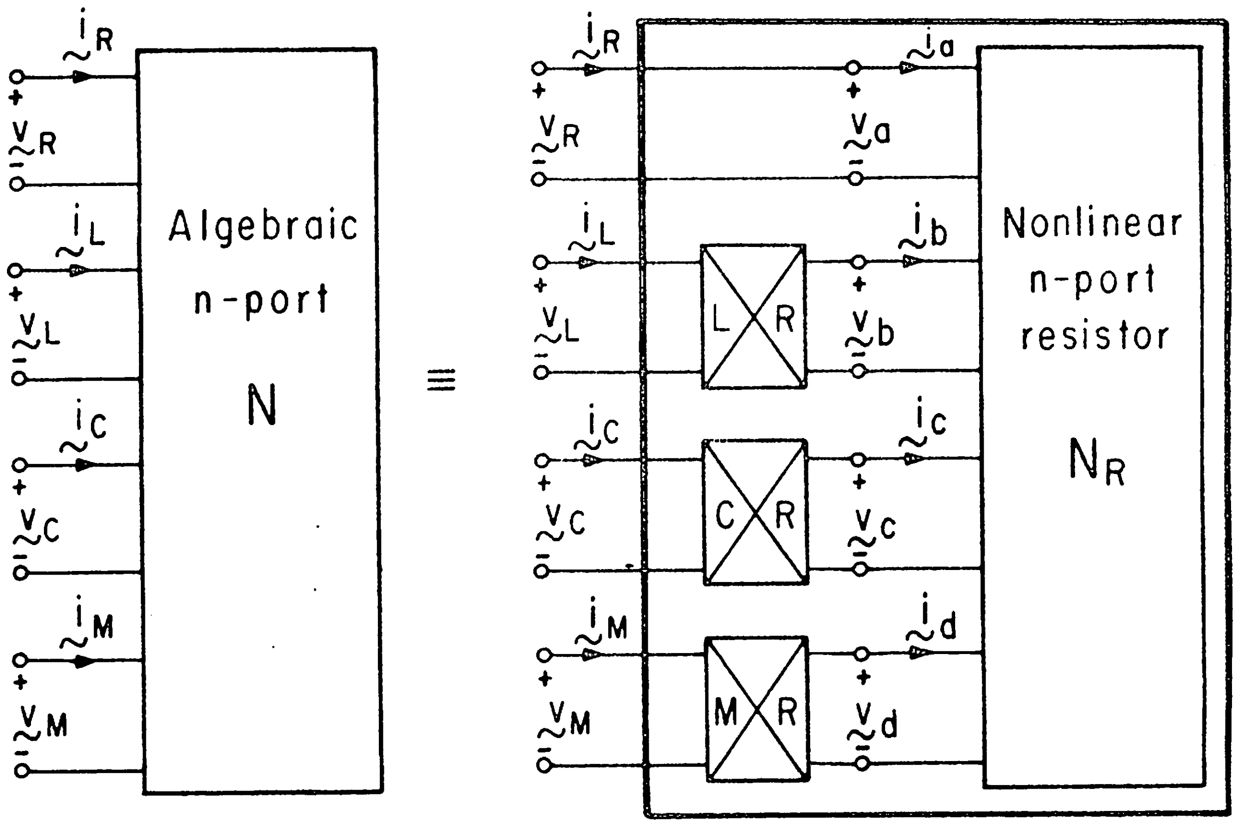

Fig. 12. Realization of algebraic n-ports using only mutators and a nonlinear n-port resistor.

C. Dynamic n-ports.

The class of dynamic n-ports is rather large and little is presently known of its general properties. We will consider here an important subclass of dynamic n-ports whose constitutive relations can be represented by

$$\boxed{\begin{aligned} \dot{\underset{\sim}{x}} &= \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{\eta}) \\ \underset{\sim}{\xi} &= \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{\eta}) \end{aligned}} \tag{27}$$

where $\underset{\sim}{x} \in \mathbb{R}^m$ is any *state* variable, $\underset{\sim}{f} : \mathbb{R}^{m+n} \to \mathbb{R}^m$, $\underset{\sim}{g} : \mathbb{R}^{m+n} \to \mathbb{R}^n$, and where each component of $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ satisfies (3). We can define 4 basic classes of dynamic n-ports completely analogous to the 4 basic classes of algebraic n-ports.

Definition 5. R, L, C, M-dynamic n-ports.

A dynamic n-port described by (27) is said to be an *R-dynamic n-port* {resp.; *L-dynamic n-port*, *C-dynamic n-port*, *M-dynamic n-port*} if each component (ξ_j,η_j) in (27) involves only (v_j,i_j) {resp.; (i_j,ϕ_j), (v_j,q_j), (ϕ_j,q_j)}.

Consider the following simple examples:

(1) R-dynamic n-port

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{i}) \qquad \underset{\sim}{v} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{i}) \tag{28-a}$$

(2) L-dynamic n-port

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{i}) \qquad \underset{\sim}{\phi} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{i}) \tag{28-b}$$

(3) C-dynamic n-port

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{v}) \qquad \underset{\sim}{q} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{v}) \tag{28-c}$$

(4) M-dynamic n-port

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{q}) \qquad \underset{\sim}{\phi} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{q}) \tag{28-d}$$

To show that (28-a)-(28-d) represent 4 distinct families of dynamic n-ports, note that it is generally not possible to recast the equation from one class into another. For example, to show that (28-b) cannot be recast into the form of (28-a), note that if we differentiate $\underset{\sim}{\phi} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{i})$ with respect to t, the resulting expression

$$\underset{\sim}{v} = \underset{\sim}{g}_x(\underset{\sim}{x},\underset{\sim}{i})\underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{i}) + \underset{\sim}{g}_i(\underset{\sim}{x},\underset{\sim}{i})\dot{\underset{\sim}{i}} \triangleq \underset{\sim}{h}(\underset{\sim}{x},\underset{\sim}{i},\dot{\underset{\sim}{i}}) \tag{29}$$

contains a new variable $\dot{\underset{\sim}{i}}$ which is not allowed in (28-a). On the other hand, just as in the case of algebraic n-ports, some dynamic n-ports may assume more than one identities.

For each $\underset{\sim}{\eta} \in \mathbb{R}^n$, let $\hat{\underset{\sim}{x}}(\underset{\sim}{\eta})$ be an *equilibrium point* of (27), i.e.,

$$\underset{\sim}{f}\left(\hat{\underset{\sim}{x}}(\underset{\sim}{\eta}),\underset{\sim}{\eta}\right) = \underset{\sim}{0} \tag{30-a}$$

$$\underset{\sim}{\xi} = \underset{\sim}{g}\left(\hat{\underset{\sim}{x}}(\underset{\sim}{\eta}),\underset{\sim}{\eta}\right) \tag{30-b}$$

Observe that (30-b) can be interpreted as an *algebraic* constitutive relation and hence an algebraic n-port can be considered as a limiting case of a dynamic n-port when $\dot{\underset{\sim}{x}}(t) \to \underset{\sim}{0}$. In particular, an n-port resistor, inductor, capacitor, and memristor can be considered as a limiting case of an R, L, C, M-dynamic n-port, respectively. Consequently, from the modeling point of view, electronic devices can be modeled more realistically using R. L, C, M-dynamic n-ports as building blocks. For example, consider again the Ebers-Moll equation for a pnp transistor. Under high power operations, the temperature T in the exponent of (11) is actually a state variable obeying an appropriate heat balance equation

$$\dot{T} = f(T,v_1,v_2) \tag{31}$$

Observe that (31) and (11) define an R-dynamic 2-port. Now if the temperature T does not change rapidly, then we can approximate T by the *ambient* temperature and the transistor reduces to a 2-port resistor. Since the characteristics of most electronic devices depend on temperature to a greater or lesser extent, a truly realistic circuit model should make use of R, L, C, M-dynamic n-ports as building blocks. For more complex devices, additional state variables will be needed. We will close this section with the following analog of Theorem 3.

Theorem 3b. Dynamic n-port realizability theorem.

Every dynamic n-port described by (27) can be synthesized using only "linear" *mutators*, "linear" 2-terminal *capacitors*, and a "nonlinear" n-port *resistor*.

Proof. We will give a constructive proof for the case where N is described by

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{v}) \qquad \underset{\sim}{i} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{v}) \tag{32}$$

The proof for the other cases follows by a similar procedure and by using mutators whenever appropriate. To synthesize (32), consider the circuit shown in Fig. 13, where the n-port resistor N_R is chosen to be characterized by

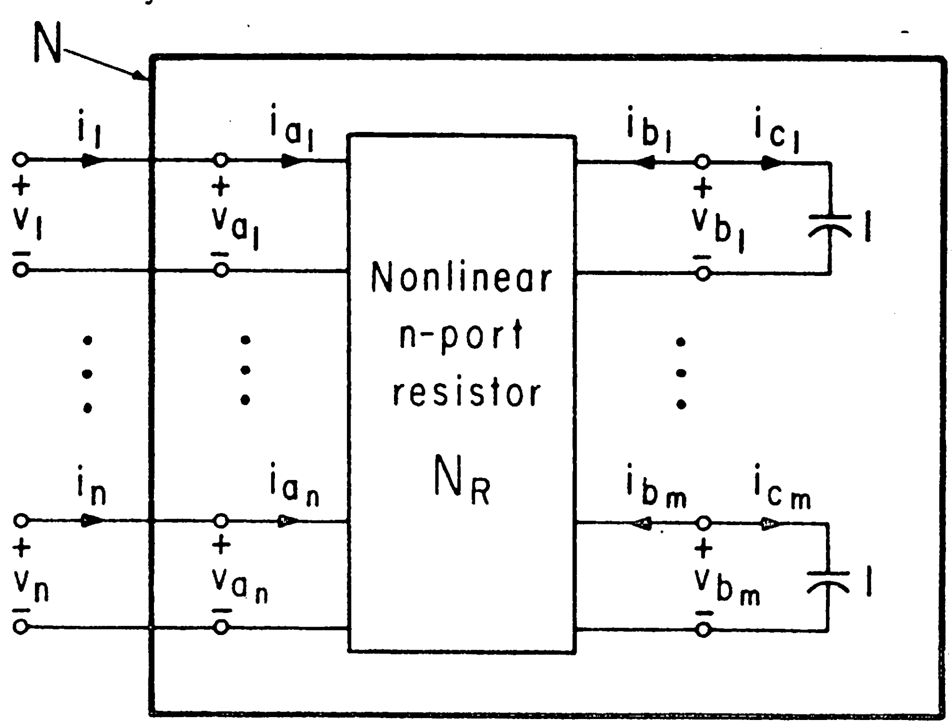

Fig. 13. Realization of R-dynamic n-ports using only linear capacitors, and a nonlinear n-port resistor.

$$\underset{\sim}{i}_a = \underset{\sim}{g}(\underset{\sim}{v}_b, \underset{\sim}{v}_a)$$

$$\underset{\sim}{i}_b = -\underset{\sim}{f}(\underset{\sim}{v}_b, \underset{\sim}{v}_a) \tag{33}$$

where $(\underset{\sim}{v}_a, \underset{\sim}{i}_a)$ and $(\underset{\sim}{v}_b, \underset{\sim}{i}_b)$ are the port voltage and current vectors associated with the ports on the left and right hand side of N_R, respectively. Choosing $\underset{\sim}{x} = \underset{\sim}{v}_b$ and noting that $\underset{\sim}{v} = \underset{\sim}{v}_a$, $\underset{\sim}{i} = \underset{\sim}{i}_a$, $\underset{\sim}{i}_c = -\underset{\sim}{i}_b$ and $\underset{\sim}{v}_b = \underset{\sim}{v}_c$ we find the n-port N in Fig. 13 is precisely characterized by (31). ◻

2. Mathematical Representations of Algebraic n-Ports

Theorem 1 implies that the most general way for representing a time-invariant algebraic n-port is to specify its associated constitutive relation between $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ as a subset of points in $\mathbb{R}^n \times \mathbb{R}^n$. A large class of such point set relations can be represented mathematically either in the *implicit* form

$$\underset{\sim}{f}(\underset{\sim}{\xi}, \underset{\sim}{\eta}) = \underset{\sim}{0} \tag{34}$$

or in the *parametric form*

$$\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{\rho}), \quad \underset{\sim}{\eta} = \underset{\sim}{\eta}(\underset{\sim}{\rho}) \tag{35}$$

$$\underset{\sim}{\rho} \in \mathcal{P} \subset \mathbb{R}^m \tag{36}$$

where $0 \leq m \leq 2n$. The parametric representations (35) is sufficiently general to allow most "singular" and "exotic" elements to be represented analytically. For example, the case m = 0, n = 1 corresponds to an element characterized by an empty set; the case m = 1, n = 1, and $\mathcal{P} = \{0\} \subset \mathbb{R}^1$ corresponds to a nullator (v=0,i=0), and the case m = 2, n = 1, $\mathcal{P} = \mathbb{R}^2$ corresponds to a norator $(v=\rho_1, i=\rho_2)$. The parametric representation also includes all conventional function representations as special cases. For example, if $\underset{\sim}{\eta}(\cdot)$ in (35) is bijective, then N can be characterized by an explicit function $\underset{\sim}{\xi} = \underset{\sim}{\xi} \circ \underset{\sim}{\eta}^{-1}(\underset{\sim}{\eta}) \triangleq \underset{\sim}{f}(\underset{\sim}{\eta})$, where "$\circ$" denotes the composition operation. Similarly if $\underset{\sim}{\xi}(\cdot)$ in (35) is bijective, then N can be characterized by $\underset{\sim}{\eta} = \underset{\sim}{\eta} \circ \underset{\sim}{\xi}^{-1}(\underset{\sim}{\xi}) = \underset{\sim}{g}(\underset{\sim}{\xi})$. In view of its greater generality and flexibility, it is often desirable to choose the parametric representation in formulating and manipulating circuit equations of general nonlinear networks[19]. For the purpose of this short course, however, we will find the *generalized coordinate representation* and the *hybrid representation,* both of which can be reduced from the parametric representation, to be more than adequate. To simplify our notation, these representations will be formulated only for *n-port resistors*. The same results apply of course to any algebraic n-ports.

A. Generalized coordinate representation.

Let N be an n-port resistor with port voltage vector $\underset{\sim}{v}$ and port current vector $\underset{\sim}{i}$. Let $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ be nx1 vectors which are related to $\underset{\sim}{v}$ and $\underset{\sim}{i}$ as follows:[1]

$$\boxed{\begin{aligned} \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix} &= \begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{c} & \underset{\sim}{d} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix} \triangleq \underset{\sim}{\Omega} \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix} \qquad (37) \\ \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix} &= \begin{bmatrix} \underset{\sim}{\alpha} & \underset{\sim}{\beta} \\ \underset{\sim}{\gamma} & \underset{\sim}{\delta} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix} \triangleq \underset{\sim}{\Omega}^{-1} \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix} \qquad (38) \end{aligned}}$$

where $\underset{\sim}{\Omega}$ is any 2nx2n non-singular constant real matrix which we will call the *coordinate transformation matrix*. The vectors $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ are called the *generalized port coordinates*.

If there exists an $\underset{\sim}{\Omega}$ such that the constitutive relation of N can be described explicitly by a function

$$\boxed{\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{\eta}), \quad \underset{\sim}{\eta} \in \mathbb{R}^n} \tag{39}$$

then we say N is globally characterized by $\underset{\sim}{\Omega}$ and $\underset{\sim}{\xi}(\cdot)$. To show that (39) can be transformed into a parametric representation, let us substitute (39) into (37) to obtain

$$\underset{\sim}{v} = \underset{\sim}{a}\underset{\sim}{\xi}(\underset{\sim}{\eta}) + \underset{\sim}{b}\underset{\sim}{\eta} \triangleq \underset{\sim}{v}(\underset{\sim}{\eta}) \tag{40}$$

$$\underset{\sim}{i} = \underset{\sim}{c}\underset{\sim}{\xi}(\underset{\sim}{\eta}) + \underset{\sim}{d}\underset{\sim}{\eta} \triangleq \underset{\sim}{i}(\underset{\sim}{\eta}) \tag{41}$$

where $\underset{\sim}{\eta} \triangleq \underset{\sim}{\rho} \in \mathbb{R}^n$ is the parametric vector.

If $\underset{\sim}{\xi}(\cdot)$ is a C^1 function, then we define the associated *linearized representation* about an operating point Q located at $(\underset{\sim}{\xi}_Q, \underset{\sim}{\eta}_Q)$ by

$$\boxed{\hat{\underset{\sim}{\xi}} = \underset{\sim}{\Lambda}(\underset{\sim}{\eta}_Q)\hat{\underset{\sim}{\eta}}, \quad \hat{\underset{\sim}{\eta}} \in \mathbb{R}^n} \tag{42}$$

where the nxn real *constant* Jacobian matrix

[1] We deliberately choose the same notations ξ and η here as in Sec. 1 because $(\underset{\sim}{\xi}, \underset{\sim}{\eta})$ can be considered as a "generalized" pair of dynamically independent vectors.

$$\underset{\sim}{\Lambda}(\underset{\sim}{\eta}_Q) \triangleq \left.\frac{\partial \underset{\sim}{\xi}(\underset{\sim}{\eta})}{\partial \underset{\sim}{\eta}}\right|_{\underset{\sim}{\eta}=\underset{\sim}{\eta}_Q} \tag{43}$$

is called the incremental *constitutive matrix* associated with $\underset{\sim}{\xi}(\cdot)$. Observe that (42) defines a distinct *linear n-port resistor* $\hat{N}_Q$ associated with N at the operating point Q. It is important to remember that $(\hat{\underset{\sim}{\xi}},\hat{\underset{\sim}{\eta}})$ represents a distinct pair of dynamically independent port vectors in $\mathbb{R}^n$ and that their magnitudes need *not* be small. The usual restriction to an "incremental" signal comes into play only when one tries to approximate the *nonlinear* n-port N by $\hat{N}_Q$ in actual computation. From the circuit-theoretic point of view, it is desirable to consider $\hat{N}_Q$ just like any other linear n-port resistor having no restrictions on its domain of definition. Geometrically, each component $\hat{\xi}_k = \hat{\xi}_k(\underset{\sim}{\eta})$ of $\hat{N}_Q$ can be interpreted as a hyperplane tangent to an n-dimensional surface defined by $\xi_k = \xi_k(\underset{\sim}{\eta})$, $\underset{\sim}{\eta} \in \mathbb{R}^n$. We will henceforth call (39) the *global representation* of N and (42) its associated *linearized representation.* Together they cover virtually all nonlinear n-port representations reported in the literature. Consequently, it is most desirable to formulate theorems in terms of these generalized representations since the corresponding results for any specific representation then falls out as a trivial special case. For ease of future reference, Table 2 contains the coordinate transformation matrix $\underset{\sim}{\Omega}$ and its inverse $\underset{\sim}{\Omega}^{-1}$ for the most common representations found in the literature. Here, $\underset{\sim}{0}_m$ denotes an mxm zero matrix, $\underset{\sim}{1}_m$ denotes an mxm unit matrix, and $\underset{\sim}{r}^{\pm 1/2}$ is a diagonal matrix whose kk<u>th</u> element is $r_k^{\pm r/2}$. Our next theorem allows one to transform one generalized representation into another.

<u>Theorem 4</u>. <u>Generalized coordinate transformations</u>.

Let $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ and $(\underset{\sim}{\xi}',\underset{\sim}{\eta}')$ denote two distinct sets of coordinates for an n-port N, and let $\underset{\sim}{\Omega}$ and $\underset{\sim}{\Omega}'$ denote their respective coordinate transformation matrices. Then

(a) $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ and $(\underset{\sim}{\xi}',\underset{\sim}{\eta}')$ are related by

$$\begin{bmatrix} \underset{\sim}{\xi}' \\ \underset{\sim}{\eta}' \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\alpha}' & \underset{\sim}{\beta}' \\ \underset{\sim}{\gamma}' & \underset{\sim}{\delta}' \end{bmatrix} \begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{c} & \underset{\sim}{d} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{C} & \underset{\sim}{D} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix} \tag{44}$$

(b) If $\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{\eta})$ is the global constitutive relation of N $\forall \underset{\sim}{\eta} \in \mathbb{R}^n$, then

$$\underset{\sim}{\xi}' = \underset{\sim}{A}\underset{\sim}{\xi}(\underset{\sim}{\eta}) + \underset{\sim}{B}\underset{\sim}{\eta} \triangleq \underset{\sim}{\xi}'(\underset{\sim}{\eta}) \tag{45}$$

Table 2. Coordinate transformation matrices for common representations

Coordinate Transformation Matrix	Inverse Coordinate Transformation Matrix
$\Omega = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$\Omega^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$
Voltage-Controlled Representation $a = 0_n \quad b = 1_n$ $c = 1_n \quad d = 0_n$	**Voltage-Controlled Representation** $\alpha = 0_n \quad \beta = 1_n$ $\gamma = 1_n \quad \delta = 0_n$
Current-Controlled Representation $a = 1_n \quad b = 0_n$ $c = 0_n \quad d = 1_n$	**Current-Controlled Representation** $\alpha = 1_n \quad \beta = 0_n$ $\gamma = 0_n \quad \delta = 1_n$
Hybrid Representation I $a = \begin{bmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{bmatrix} \quad b = \begin{bmatrix} 0_k & 0 \\ 1_k & 1_{n-k} \end{bmatrix}$ $c = \begin{bmatrix} 0_k & 1 \\ 0 & 1_{n-k} \end{bmatrix} \quad d = \begin{bmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{bmatrix}$	**Hybrid Representation I** $\alpha = \begin{bmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{bmatrix} \quad \beta = \begin{bmatrix} 0_k & 0 \\ 0 & 1_{n-k} \end{bmatrix}$ $\gamma = \begin{bmatrix} 0_k & 0 \\ 0 & 1_{n-k} \end{bmatrix} \quad \delta = \begin{bmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{bmatrix}$
Hybrid Representation II $a = \begin{bmatrix} 0_k & 0 \\ 0 & 1_{n-k} \end{bmatrix} \quad b = \begin{bmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{bmatrix}$ $c = \begin{bmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{bmatrix} \quad d = \begin{bmatrix} 0_k & 0 \\ 0 & 1_{n-k} \end{bmatrix}$	**Hybrid Representation II** $\alpha = \begin{bmatrix} 0_k & 0 \\ 0 & 1_{n-k} \end{bmatrix} \quad \beta = \begin{bmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{bmatrix}$ $\gamma = \begin{bmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{bmatrix} \quad \delta = \begin{bmatrix} 0_k & 0 \\ 0 & 1_{n-k} \end{bmatrix}$
Transmission Representation I $a = \begin{bmatrix} 1_{n/2} & 0_{n/2} \\ 0_{n/2} & 0_{n/2} \end{bmatrix} \quad b = \begin{bmatrix} 0_{n/2} & 0_{n/2} \\ 1_{n/2} & 0_{n/2} \end{bmatrix}$ $c = \begin{bmatrix} 0_{n/2} & 1_{n/2} \\ 0_{n/2} & 0_{n/2} \end{bmatrix} \quad d = \begin{bmatrix} 0_{n/2} & 0_{n/2} \\ 0_{n/2} & -1_{n/2} \end{bmatrix}$	**Transmission Representation I** $\alpha = \begin{bmatrix} 1_{n/2} & 0_{n/2} \\ 0_{n/2} & 0_{n/2} \end{bmatrix} \quad \beta = \begin{bmatrix} 0_{n/2} & 0_{n/2} \\ 1_{n/2} & 0_{n/2} \end{bmatrix}$ $\gamma = \begin{bmatrix} 0_{n/2} & 1_{n/2} \\ 0_{n/2} & 0_{n/2} \end{bmatrix} \quad \delta = \begin{bmatrix} 0_{n/2} & 0_{n/2} \\ 0_{n/2} & -1_{n/2} \end{bmatrix}$
Transmission Representation II $a = \begin{bmatrix} 0_{n/2} & 0_{n/2} \\ 1_{n/2} & 0_{n/2} \end{bmatrix} \quad b = \begin{bmatrix} 1_{n/2} & 0_{n/2} \\ 0_{n/2} & 0_{n/2} \end{bmatrix}$ $c = \begin{bmatrix} 0_{n/2} & 0_{n/2} \\ 0_{n/2} & 1_{n/2} \end{bmatrix} \quad d = \begin{bmatrix} 0_{n/2} & -1_{n/2} \\ 0_{n/2} & 0_{n/2} \end{bmatrix}$	**Transmission Representation II** $\alpha = \begin{bmatrix} 0_{n/2} & 1_{n/2} \\ 0_{n/2} & 0_{n/2} \end{bmatrix} \quad \beta = \begin{bmatrix} 0_{n/2} & 0_{n/2} \\ 0_{n/2} & 1_{n/2} \end{bmatrix}$ $\gamma = \begin{bmatrix} 1_{n/2} & 0_{n/2} \\ 0_{n/2} & 0_{n/2} \end{bmatrix} \quad \delta = \begin{bmatrix} 0_{n/2} & 0_{n/2} \\ -1_{n/2} & 0_{n/2} \end{bmatrix}$
Scattering Representation (with port numbers $r_1, r_2, \ldots, r_n$) $a = r^{1/2}, \quad b = r^{1/2}$ $c = -r^{-1/2}, \quad d = r^{-1/2}$	**Scattering Representation (with port numbers $r_1, r_2, \ldots, r_n$)** $\alpha = 1/2\ r^{-1/2}, \quad \beta = -1/2\ r^{1/2}$ $\gamma = 1/2\ r^{-1/2}, \quad \delta = 1/2\ r^{1/2}$

$$\underset{\sim}{\eta}' = \underset{\sim}{C}\underset{\sim}{\xi}(\underset{\sim}{\eta}) + \underset{\sim}{D}\underset{\sim}{\eta} \triangleq \underset{\sim}{\eta}'(\underset{\sim}{\eta}) \tag{46}$$

is the associated parametric representation in terms of $\underset{\sim}{\xi}'$ and $\underset{\sim}{\eta}'$. In particular, N admits the global representation $\underset{\sim}{\xi}' = \underset{\sim}{\xi}'(\underset{\sim}{\eta}')$ if, and only if, $\underset{\sim}{\eta}'(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ from (46) is *bijective* (i.e., one-to-one and onto).

(c) The associated constitutive matrices $\underset{\sim}{\Lambda}(\underset{\sim}{\eta}_Q)$ and $\underset{\sim}{\Lambda}'(\underset{\sim}{\eta}_Q)$ of the linearized representations are related by:

$$\boxed{\underset{\sim}{\Lambda}'(\underset{\sim}{\eta}_Q') = \left[(\underset{\sim}{\alpha}'\underset{\sim}{a}+\underset{\sim}{\beta}'\underset{\sim}{c})\underset{\sim}{\Lambda}(\underset{\sim}{\eta}_Q) + \underset{\sim}{\alpha}'\underset{\sim}{b}+\underset{\sim}{\beta}'\underset{\sim}{d}\right]\left[(\underset{\sim}{\gamma}'\underset{\sim}{a}+\underset{\sim}{\delta}'\underset{\sim}{c})\underset{\sim}{\Lambda}(\underset{\sim}{\eta}_Q) + (\underset{\sim}{\gamma}'\underset{\sim}{b}+\underset{\sim}{\delta}'\underset{\sim}{d})\right]^{-1}} \tag{47}$$

where the inverse in the second matrix exists if, and only if, the constitutive matrix $\underset{\sim}{\Lambda}'(\underset{\sim}{\eta}_Q')$ exists.

The single formula (47) is extremely useful and contains, among other things, the familiar 2-port parameter conversion formulas listed in many textbooks as special cases. For example, to derive the relationship between the *open-circuit resistance matrix* and the ABCD *chain matrix* (transmission representation I), we simply substitute the appropriate coordinate transformation matrices from Table 2 into (47) and obtain

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix}\begin{bmatrix} C & D \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} A/C & (AD-BC)/C \\ 1/C & D/C \end{bmatrix} \tag{48}$$

B. Hybrid Representation

The hybrid representations I and II in Table 2 are just 2 out of 2^n distinct representations obtained by an arbitrary mixture of port current and voltages. Rather than treating each case separately, it is convenient to define an arbitrary hybrid representation by choosing $a = d = A$ and $b = c = B$ in (37) where A and B are diagonal nxn matrices satisfying the property that either $A_{jj} = 1$, $B_{jj} = 0$, or $A_{jj} = 0$, $B_{jj} = 1$:

$$\begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{A} \end{bmatrix}\begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix} \triangleq \underset{\sim}{M}\begin{bmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \end{bmatrix} \tag{49}$$

We will call $(\underset{\sim}{\xi},\underset{\sim}{\eta}) \triangleq (\underset{\sim}{x},\underset{\sim}{y})$ a *hybrid pair* and the corresponding hybrid constitutive relation for an n-port N will be denoted by

$$\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x}) \tag{50}$$

Observe that the coordinate transformation matrix $\Omega \triangleq M$ in this case satisfies the property that $M = M^T = M^{-1}$. It follows from theorem 4(b) that if N admits a hybrid representation $y = h(x)$, then it also admits another hybrid representation $y' = h'(x')$ (corresponding to A' and B') if, and only if, the function

$$f(x) \triangleq (B'A+A'B)h(x) + (B'B+A'A)x \tag{51}$$

is bijective. Our next theorem gives the conditions on $h(\cdot)$ which guarantee the existence of various hybrid matrices.

Theorem 5. Existence of hybrid representation

Let N be characterized by a C^k hybrid representation $y = h(x)$, $h : \mathbb{R}^n \to \mathbb{R}^n$. Then

(a) $h(\cdot)$ is C^k-diffeomorphic $(k\geq 1)$[2] and hence N admits a C^k inverse

$$x = h^{-1}(y), \qquad \forall y \in \mathbb{R}^n \tag{52}$$

if, and only if, [20,25]

1. $$\det\left(\partial h(x)/\partial x\right) \neq 0, \quad \forall x \in \mathbb{R}^n \tag{53}$$
2. $$\lim_{\|x\|\to\infty} \|h(x)\| = \infty \tag{54}$$

(b) N admits a C^k diffeomorphic $(k\geq 0)$ hybrid representation $y' = h'(x')$ if $h(x)$ in (53)-(54) is replaced by $f(x)$ in (51). In particular, if the components of x and y are rearranged into (x_a, x_b) and $(y_a, y_b) = \left(h_a(x_a,x_b),\ h_b(x_a,x_b)\right)$, where $x_a, y_a \in \mathbb{R}^a$ and $x_b, y_b \in \mathbb{R}^b$, such that $x' = (y_a, x_b)$ and $y' = (x_a, y_b)$, then $y' = h'(x')$ exists if [21]

1. $$\det\left(\partial h_a(x_a,x_b)/\partial x_a\right) \neq 0, \quad \forall x \in \mathbb{R}^n \tag{55}$$
2. $$\lim_{\|x_a\|\to\infty} \|h_a(x_a,x_b)\| = \infty, \qquad \forall x_b \in \mathbb{R}^b \tag{56}$$

---

[2] A function $h : \mathbb{R}^n \to \mathbb{R}^n$ is said to be C^k if $h(\cdot)$ has *continuous derivatives* of all orders up to k. It is said to be *C^k diffeomorphic* if $h(\cdot)$ is C^k bijective and its inverse $h^{-1}(\cdot)$ is also C^k. A C^0 diffeomorphic function is said to be *homeomorphic*. In the more general case where $h(\cdot) : D \subset \mathbb{R}^n \to \mathbb{R}^n$, $h(\cdot)$ is said to be C^k diffeomorphic *from D onto h(D)* if it is C^k bijective and $h^{-1}(\cdot)$ is also C^k on $h(D)$. Unless otherwise stated, we always assume $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$.

(c) All 2^n distinct hybrid representations of an n-port resistor N exist if N has a C^1 hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$ which satisfies the following two conditions [22]:

1. $\underset{\sim}{H}(\underset{\sim}{x}) \triangleq \partial \underset{\sim}{h}(\underset{\sim}{x})/\partial \underset{\sim}{x}$ is a P-matrix,[3] $\forall \underset{\sim}{x} \in \mathbb{R}^n$ (57)

2. $\lim_{\|\underset{\sim}{x}\| \to \infty} |h_j(\underset{\sim}{x})| = \infty, \quad j = 1,2,\ldots,n$ (58)

Theorem 5(a) is sometimes called the *global diffeomorphism theorem*. It is one of the most often used tools in nonlinear circuit theory. To show that both conditions are necessary for theorem 5(a) to hold when $k \geq 1$, consider the following counterexamples:

<u>Counterexample 1</u>.

$$\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x}) \;:\; y_1 = e^{x_1}\cos x_2, \quad y_2 = e^{x_1}\sin x_2 \tag{59}$$

Observe that (59) satisfies (53) $\left(\text{since } \det \underset{\sim}{H}(\underset{\sim}{x}) = e^{2x_1} \neq 0 \;\; \forall \underset{\sim}{x}\right)$ but violates (54) (since $\|\underset{\sim}{y}\| = e^{x_1} = 1 \; \forall \underset{\sim}{x} = [0 \; x_2]^T$. Indeed $\underset{\sim}{h}(\cdot)$ is not injective since both $(0,0)$ and $(0,2\pi)$ map into the same point $(1,0)$.

<u>Counterexample 2</u>.

$$\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x}) \;:\; y_1 = x_1^2 - x_2^2, \quad y_2 = 2x_1x_2 \tag{60}$$

Observe that (60) satisfies (54) $\left(\text{since } \|\underset{\sim}{y}\| = (x_1^2+x_2^2)^2 \to \infty \;\; \forall \; \|\underset{\sim}{x}\| \to \infty\right)$ but violates (53) $\left(\text{since } \det \underset{\sim}{H}(\underset{\sim}{x}) = 4(x_1^2+x_2^2) = 0 \text{ at the single point } \underset{\sim}{x} = (0,0)\right)$. Indeed, $\underset{\sim}{h}(\cdot)$ is not injective since both $(1,1)$ and $(-1,1)$ map into the same point $(0,2)$.

It is interesting to note that condition 1 is *not* necessary for a function to be globally homeomorphic (k=0). For example, the function $i = h(v) = v^3$ is homeomorphic even though $dh(v)/dv = 0$ at $v = 0$. In this case, $h(\cdot)$ is not C^1 since $h^{-1}(\cdot)$ has infinite slope at the origin. Since on many occasion one is only interested in obtaining a global inverse function, the following result is useful.

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[3] An nxn matrix $\underset{\sim}{A}$ is said to be a P-matrix if *all* its principal submatrices of all orders obtained by deleting any k corresponding rows and columns of $\underset{\sim}{A}$, $k = 0,1,\ldots,n-1$, have positive determinants [23]. It follows that if $\underset{\sim}{A}$ is symmetric, then $\underset{\sim}{A}$ is a P-matrix if, and only if, it is positive definite.

Theorem 6. Global inversion theorem [24].

Let $\underset{\sim}{h} : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function, $n \neq 2$. Then $\underset{\sim}{h}(\cdot)$ is homeomorphic and hence bijective if:

(1) $\det \underset{\sim}{H}(\underset{\sim}{x}) \triangleq \partial \underset{\sim}{h}(\underset{\sim}{x})/\partial \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \in \mathbb{R}^n$ *except* possibly on a set of *isolated* points.

(2) $\lim_{\|\underset{\sim}{x}\| \to \infty} \|\underset{\sim}{h}(\underset{\sim}{x})\| = \infty$

It is rather surprising to note that theorem 6 is valid for all n except n = 2 [24]. Needless to say, this theorem also holds if the inequality sign in (1) is reversed.

II. Structural and Circuit Theoretic Properties of Algebraic n-Ports

1. Structural Properties

Consider now the class of all algebraic n-ports which admit at least one hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^n$. Since the qualitative behavior of dynamic nonlinear networks depend strongly on the *mathematical structures* of $\underset{\sim}{h}(\cdot)$, our objective in this section is to classify and study some of these basic structures. At the crudest level, a hybrid representation $\underset{\sim}{h}(\cdot)$ can be classified as *injective* (i.e., one-to-one), *surjective* (i.e., onto), or *bijective* (i.e., one-to-one and onto). If $\underset{\sim}{h}(\cdot)$ is C^0 (i.e., continuous) and bijective, then its inverse exists and is also C^0 [24]. Hence, every C^0 bijective $\underset{\sim}{h}(\cdot)$ is *homeomorphic*. The above classifications are *not* invariant properties of N in the sense that another hybrid representation $\underset{\sim}{y}' = \underset{\sim}{h}'(\underset{\sim}{x}')$ may fail to be injective, surjective, bijective, or homeomorphic even if the original function $\underset{\sim}{h}(\cdot)$ have all these properties. For example, of the following two equivalent hybrid representations,

$$\underset{\sim}{y} \triangleq \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \triangleq \underset{\sim}{h}(\underset{\sim}{x}), \quad \underset{\sim}{y}' \triangleq \begin{bmatrix} v_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 0 \\ -1 & 3 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \\ v_3 \end{bmatrix} \triangleq \underset{\sim}{h}'(\underset{\sim}{x}') \tag{1}$$

$\underset{\sim}{h}(\cdot)$ is homeomorphic (and hence injective, surjective and C^0 bijective) but $\underset{\sim}{h}'(\cdot)$ is *not* injective, surjective, bijective, or homeomorphic since the associated hybrid matrix is singular.

Definition 1. Strongly-uniformly, uniformly, strictly increasing representations.

A hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^n$ is said to be

(a) *increasing* if

$$\langle \underset{\sim}{h}(\underset{\sim}{x}') - \underset{\sim}{h}(\underset{\sim}{x}''), \underset{\sim}{x}'-\underset{\sim}{x}'' \rangle \geq 0, \quad \forall \underset{\sim}{x}', \underset{\sim}{x}'' \in \mathbb{R}^n \tag{2}$$

(b) *strictly increasing* if

$$\langle \underset{\sim}{h}(\underset{\sim}{x}')-\underset{\sim}{h}(\underset{\sim}{x}''), \underset{\sim}{x}'-\underset{\sim}{x}'' \rangle > 0, \quad \forall \underset{\sim}{x}', \underset{\sim}{x}'' \in \mathbb{R}^n, \quad \underset{\sim}{x}' \neq \underset{\sim}{x}'' \tag{3}$$

(c) *uniformly increasing* if there exists a constant $c > 0$ such that

$$\langle \underset{\sim}{h}(\underset{\sim}{x}')-\underset{\sim}{h}(\underset{\sim}{x}''), \underset{\sim}{x}'-\underset{\sim}{x}'' \rangle \geq c\|\underset{\sim}{x}'-\underset{\sim}{x}''\|^2, \quad \forall \underset{\sim}{x}', \underset{\sim}{x}'' \in \mathbb{R}^n \tag{4}$$

(d) *strongly uniformly increasing* if there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|\underset{\sim}{x}'-\underset{\sim}{x}''\|^2 \leq \langle \underset{\sim}{h}(\underset{\sim}{x}')-\underset{\sim}{h}(\underset{\sim}{x}''), \underset{\sim}{x}'-\underset{\sim}{x}'' \rangle \leq c_2\|\underset{\sim}{x}'-\underset{\sim}{x}''\|^2, \quad \forall \underset{\sim}{x}', \underset{\sim}{x}'' \in \mathbb{R}^n \tag{5}$$

where $\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle$ denotes the scalar product between $\underset{\sim}{x}$ and $\underset{\sim}{y}$.

Theorem 1. Invariant structural property [22].

Every hybrid representation of an algebraic n-port N which exists is *increasing* {resp.; *strictly increasing*} if N has a hybrid representation $\underset{\sim}{h}(\cdot)$ which is increasing {resp.; strictly increasing}.

It follows from Theorem 1 that there is no ambiguity in calling an n-port increasing or strictly increasing since these two properties are *invariants* of an n-port in the sense that they do not depend on a particular choice of hybrid representation. In contrast to this, we will give two counterexamples shortly showing uniformly and strongly uniformly increasing properties are not invariant. But first we must present a set of criteria for checking these properties.

Theorem 2. Strongly-uniformly, uniformly, strictly increasing criteria [25].

Let N be characterized by a C^1 hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^n$ and let $\underset{\sim}{H}(\underset{\sim}{x}) \triangleq \partial\underset{\sim}{h}(\underset{\sim}{x})/\partial\underset{\sim}{x}$ denote its Jacobian matrix. Then[4]

(a) N is *increasing* if, and only if, $\underset{\sim}{H}(\underset{\sim}{x})$ is positive-semidefinite (psd) $\forall \underset{\sim}{x} \in \mathbb{R}^n$.

(b) N is *strictly increasing* if $\underset{\sim}{H}(\underset{\sim}{x})$ is *positive definite* (pd) $\forall \underset{\sim}{x} \in \mathbb{R}^n$.

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[4] See bottom of next page.

(c) $\underset{\sim}{h}(\cdot)$ is *uniformly increasing* if, and only if, $\underset{\sim}{H}(\underset{\sim}{x})$ is *uniformly positive definite* (upd) $\forall \underset{\sim}{x} \in \mathbb{R}^n$.

(d) $\underset{\sim}{h}(\cdot)$ is *strongly uniformly increasing* if, and only if, $\underset{\sim}{H}(\underset{\sim}{x})$ is *strongly uniformly positive definite* (supd) $\forall \underset{\sim}{x} \in \mathbb{R}^n$.

Observe that unlike (a), (c), and (d) which provide both necessary and sufficient conditions, (b) gives only a sufficient condition. To show that $\underset{\sim}{H}(\underset{\sim}{x})$ need *not* be pd for $\underset{\sim}{h}(\cdot)$ to be strictly increasing, consider the one-port characterized by $i = h(v) = v^3$. Observe that $h(\cdot)$ is strictly increasing even though $H(0) = dh(0)/dv = 0$. It turns out that the condition in (b) can be relaxed by requiring $\underset{\sim}{H}(\underset{\sim}{x})$ to be pd only $\forall \underset{\sim}{x} \in \mathbb{R}^n$ *except* for at most a set of *isolated* points where $\underset{\sim}{H}(\underset{\sim}{x})$ is psd [24].

Observe also that in the definition of psd and pd matrices in footnote 4, we need to test only those $\underset{\sim}{z} \in \mathbb{R}^n$ having a unit magnitude; i.e., $\|\underset{\sim}{z}\| = 1$. This is clear since any $\underset{\sim}{z} \in \mathbb{R}^n$ can be written in the form $\underset{\sim}{z} = c\hat{\underset{\sim}{z}}$, where $\hat{\underset{\sim}{z}} \triangleq \underset{\sim}{z}/\|\underset{\sim}{z}\|$, and $c \triangleq \|\underset{\sim}{z}\|$. Hence $\underset{\sim}{z}^T\underset{\sim}{H}\underset{\sim}{z} = c^2\hat{\underset{\sim}{z}}^T\underset{\sim}{H}\hat{\underset{\sim}{z}}$ where $\|\hat{\underset{\sim}{z}}\| = 1$. Observe next that if we decompose a matrix H into its symmetric part $\underset{\sim}{H}_s \triangleq \frac{1}{2}(\underset{\sim}{H}+\underset{\sim}{H}^T)$ and *skew-symmetric* part $H_a \triangleq \frac{1}{2}(\underset{\sim}{H}-\underset{\sim}{H}^T)$, then $\underset{\sim}{z}^T\underset{\sim}{H}\underset{\sim}{z} = \underset{\sim}{z}^T\underset{\sim}{H}_s\underset{\sim}{z}$. Hence $\underset{\sim}{H}$ is psd or pd if, and only if, its symmetric part $\underset{\sim}{H}_s$ is psd or pd. This property allows us to apply the standard tests for psd or pd symmetric matrices for determining whether $\underset{\sim}{h}(\cdot)$ is increasing, strictly increasing, uniformly increasing, or strongly uniformly increasing. Now although every strictly increasing function is injective, it need not be bijective (e.g., $i = \tanh v$). Consequently, the inverse of a strictly increasing function need not be defined in all of $\mathbb{R}^n$. Our next theorem guarantees that every uniformly increasing hybrid representation is bijective in $\mathbb{R}^n$.

<u>Theorem 3</u>. <u>Existence of all 2^n hybrid representations</u> [26].

Let N be characterized by a C^1 hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^n$. If $\underset{\sim}{h}(\cdot)$ is *uniformly increasing,* then N admits all 2^n distinct C^1 hybrid representations in $\mathbb{R}^n$.

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[4] An nxn, not necessarily symmetric, real matrix $\underset{\sim}{H}$ is said to be

(a) *positive semi-definite* (psd) if $\underset{\sim}{z}^T\underset{\sim}{H}\underset{\sim}{z} \geq 0$, $\forall \underset{\sim}{z} \in \mathbb{R}^n$.

(b) *positive definite* (pd) if $\underset{\sim}{z}^T\underset{\sim}{H}\underset{\sim}{z} > 0$, $\forall \underset{\sim}{z} \neq \underset{\sim}{0}$, $\underset{\sim}{z} \in \mathbb{R}^n$.

(c) *uniformly positive definite* (upd) if there exists a positive constant c_1 such that $\underset{\sim}{H}-c_1\underset{\sim}{1}$ is pd.

(d) *strongly uniformly positive definite* (supd) if there exist two positive constants c_1 and c_2 such that both $\underset{\sim}{H}-c_1\underset{\sim}{1}$ and $c_2\underset{\sim}{1}-\underset{\sim}{H}$ are pd.

We are now ready to present a counterexample showing that not all hybrid representations in theorem 3 need be uniformly increasing.

Counterexample 1. Consider a one-port N characterized by

$$\begin{aligned} i = h(v) &= \exp(v)-1, \quad v \geq 0 \\ &= v \quad\quad\quad\quad\;\;, \quad v < 0 \end{aligned} \tag{6}$$

Since the slope $H(v) \triangleq dh(v)/dv \geq 1$, $\forall v$, it follows from Theorem 2(c) that $h(v)$ is uniformly increasing. Now consider the inverse representation.

$$\begin{aligned} v = h^{-1}(i) &= \ln(i+1), \; i \geq 0 \\ &= i \quad\quad\quad\;\;, \; i < 0 \end{aligned} \tag{7}$$

Since the slope $H'(i) \triangleq dh^{-1}(i)/di = 1/i+1 \to 0$ as $i \to \infty$, $h^{-1}(i)$ is *not* uniformly increasing. Our next theorem provides the additional condition needed to guarantee this property.

Theorem 4. Uniform-increasing closure condition.

Let N be characterized by a C^1 hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^n$.

(a) If $\underset{\sim}{h}(\cdot)$ is *uniformly increasing* and if its associated hybrid matrix $\underset{\sim}{H}(\underset{\sim}{x}) \triangleq \partial\underset{\sim}{h}(\underset{\sim}{x})/\partial\underset{\sim}{x}$ is *bounded* in the sense that

$$\|\underset{\sim}{H}(\underset{\sim}{x})\| \leq K < \infty, \quad \forall \underset{\sim}{x} \in \mathbb{R}^n \tag{8}$$

where $\|\cdot\|$ denotes any *matrix norm*[5] of $\underset{\sim}{H}(\cdot)$, then *all* 2^n hybrid representation of N are *uniformly increasing* [27].

(b) If both $\underset{\sim}{h}(\cdot)$ and $\underset{\sim}{h}^{-1}(\cdot)$ are *uniformly increasing*, then its associated hybrid matrix $\underset{\sim}{H}(\underset{\sim}{x})$ is *bounded* and *all* 2^n hybrid representation of N are *uniformly increasing* [28].

Since every strongly uniformly increasing $\underset{\sim}{h}(\cdot)$ is unformly increasing, it follows from Theorem 3 that all 2^n hybrid representations exist if $\underset{\sim}{h}(\cdot)$ is strongly uniformly increasing. Again, our next counterexample shows that not all such representations need be strongly uniformly increasing.

Counterexample 2. Consider a 2-port N characterized by

$$\begin{aligned} i_1 &= h_1(v_1,v_2) = \frac{3}{2}\, v_1 + \frac{1}{2}\, v_2 \;\ln(v_1^2+v_2^2+1) \\ i_2 &= h_2(v_1,v_2) = \frac{3}{2}\, v_2 - \frac{1}{2}\, v_1 \;\ln(v_1^2+v_2^2+1) \end{aligned} \tag{9}$$

[5] We define the *norm* of an nxn matrix $\underset{\sim}{H}$ by $\|\underset{\sim}{H}\| \triangleq \max_{\|\underset{\sim}{z}\| \neq 0} \|\underset{\sim}{H}\underset{\sim}{z}\|/\|\underset{\sim}{z}\| = \max_{\|\underset{\sim}{z}\|=1} \|\underset{\sim}{H}\underset{\sim}{z}\|$

It can be shown that the hybrid matrix $\underset{\sim}{H}(\underset{\sim}{v})$ associated with (9) satisfies the inequality

$$\frac{1}{2} \leq \underset{\sim}{z}^T \underset{\sim}{H}(\underset{\sim}{v}) \underset{\sim}{z} \leq \frac{5}{2}, \quad \forall \|\underset{\sim}{z}\| = 1 \tag{10}$$

Hence, $\underset{\sim}{h}(\underset{\sim}{v})$ is strongly uniformly increasing in view of Theorem 2(d). However, if we choose $\underset{\sim}{z} = [1\ 0]^T$, then $\|\underset{\sim}{H}(\underset{\sim}{v})\underset{\sim}{z}\| \to \infty$ as $\|\underset{\sim}{v}\| \to \infty$. Hence $\underset{\sim}{H}(\underset{\sim}{v})$ is *not* bounded and $\underset{\sim}{h}^{-1}(\cdot)$ is *not* uniformly increasing in view of Theorem 4(b). Our next theorem provides the additional condition needed to guarantee this property.

Theorem 5. Strongly-uniformly increasing closure condition [28].

(a) If $\underset{\sim}{h}(\cdot)$ is *strongly uniformly increasing*, and if its associated hybrid matrix $\underset{\sim}{H}(\underset{\sim}{x})$ is *bounded*, then all 2^n hybrid representations of N are *strongly uniformly increasing*.

(b) If both $\underset{\sim}{h}(\cdot)$ and $\underset{\sim}{h}^{-1}(\cdot)$ are *strongly uniformly increasing*, then all 2^n hybrid representations of N are *strongly uniformly increasing*.

The relationships between various classes of functions we have introduced so far are given in the next theorem.

Theorem 6. Relationship between structural properties [24,25].

Let $\underset{\sim}{h} : \mathbb{R}^n \to \mathbb{R}^n$ be a *continuous* function.

(a) If $\underset{\sim}{h}(\cdot)$ is *strictly increasing*, then $\underset{\sim}{h}(\cdot)$ is *injective*.

(b) If $\underset{\sim}{h}(\cdot)$ is *bijective*, then $\underset{\sim}{h}(\cdot)$ is *homeomorphic*.

(c) If $\underset{\sim}{h}(\cdot)$ is *uniformly increasing*, then $\underset{\sim}{h}(\cdot)$ is *homeomorphic*.

(d) If $\underset{\sim}{h}(\cdot)$ is *bijective*, then $\underset{\sim}{h}(\cdot)$ satisfies

$$\lim_{\|\underset{\sim}{x}\| \to \infty} \|\underset{\sim}{h}(\underset{\sim}{x})\| = \infty \tag{11}$$

We will henceforth refer to (11) as the *norm condition*. A function $\underset{\sim}{h} : \mathbb{R}^n \to \mathbb{R}^n$ satisfying (11) need *not* be *surjective* (e.g., let $y_1 = x_1^2$ and $y_2 = x_2^2$). Conversely, a surjective $\underset{\sim}{h}(\cdot)$ need *not* satisfy (11). For example, the function

$$\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x}) : y_1 = e^{x_1} - e^{-x_2}, \quad y_2 = e^{-x_1} - e^{x_2} \tag{12}$$

can be shown to be surjective but $\|\underset{\sim}{h}(\underset{\sim}{x})\| = 0$ along the line $x_1 = -x_2$.

Another structural property of an important subclass of algebraic n-ports is the *path independence* of line integrals of the associated constitutive relations. We will now define the relevant concepts for characterizing this property.

Definition 2. Line integral

Let $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$ be a C^0 vector-valued function mapping an open set $X \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and let $\Gamma[\underset{\sim}{x}_a, \underset{\sim}{x}_b]$ be a space curve from $\underset{\sim}{x}_a$ to $\underset{\sim}{x}_b$, represented parametrically by a *continuous and piecewise C^1 function*[6] (henceforth called a *piecewise C^1 path*) $\underset{\sim}{\gamma} : [a,b] \to \mathbb{R}^n$; namely, $\underset{\sim}{x}_a = \underset{\sim}{\gamma}(a)$, $\underset{\sim}{x}_b = \underset{\sim}{\gamma}(b)$, and $x_j = \gamma_j(\rho)$, $j = 1,2,\ldots,n$, $a \leq \rho \leq b$. The *line integral* of $\underset{\sim}{f}(\cdot)$ along $\Gamma[\underset{\sim}{x}_a, \underset{\sim}{x}_b]$ is defined by the following sum of n Riemann integrals:

$$\int_{\Gamma[\underset{\sim}{x}_a, \underset{\sim}{x}_b]} \underset{\sim}{f}(\underset{\sim}{x}) \cdot d\underset{\sim}{x} \triangleq \int_a^b \langle \underset{\sim}{f}\ \underset{\sim}{\gamma}(\rho)\ , \underset{\sim}{\gamma}'(\rho) \rangle\, d\rho = \sum_{j=1}^{n} \int_a^b f_j\big(\underset{\sim}{\gamma}(\rho)\big)\, \gamma_j'(\rho) d\rho \qquad (13)$$

Notice that we call (13) the line integral with respect to the space curve Γ, and *not* with respect to the parametric function $\gamma(\cdot)$ even though Γ have infinitely many distinct parametric representations. This is because (13) gives the same value no matter which $\gamma(\cdot)$ is chosen [30]. In general, the line integral of $\underset{\sim}{f}(\cdot)$ along two distinct paths Γ_1 and Γ_2 having identical end points are different. Hence, we must specify both the path Γ and its endpoints as in (13). If the two endpoints coincide with each other, then Γ is a *closed path* and we will sometimes denote the line integral along Γ by $\oint_\Gamma \underset{\sim}{f}(\underset{\sim}{x}) \cdot d\underset{\sim}{x}$.

Definition 3. State function.

A C^0 vector-valued function $\underset{\sim}{f}$ mapping an *open* set $X \subset \mathbb{R}^n$ into $\mathbb{R}^n$ is said to be a *state function* if the associated line integral (13) is independent of any piecewise C^1 path $\Gamma \subset X$ having identical endpoints. Such line integrals will henceforth be denoted by:

$$\int_{\Gamma[\underset{\sim}{x}_a, \underset{\sim}{x}_b]} \underset{\sim}{f}(\underset{\sim}{x}) \cdot d\underset{\sim}{x} = \int_{\underset{\sim}{x}_a}^{\underset{\sim}{x}_b} \underset{\sim}{f}(\underset{\sim}{x}) \cdot d\underset{\sim}{x} \qquad (14)$$

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[6] A function is said to be *piecewise C^1* if it is C^1 everywhere except for a *finite* number of points. Since the parametric representation $\underset{\sim}{\gamma}(\cdot)$ in Def. 2 is both *continuous* and *piecewise* C^1, it must have *finite* left and right-hand derivatives on [a,b]. Hence, each integrand $f_j(\cdot)\ \gamma_j'(\cdot)$ in (13) is *bounded* on [a,b] and is continuous except possibly for a finite number of discontinuities. Therefore the line integral defined by (13) always exists. Infact, (13) still exists even if $\gamma(\cdot)$ is continuous on [a,b] but only C^1 *almost everywhere* on [a,b]; i.e., C^1 except on a set of measure zero [29,30].

Definition 4. Gradient map

A vector-valued function $\underset{\sim}{f}$ mapping an open set $X \subset \mathbb{R}^n$ into $\mathbb{R}^n$ is said to be a *gradient map* (or *exact function*) if there exists a C^1 *scalar* function $F : X \to \mathbb{R}^1$, henceforth called a *potential function*, such that:

$$\underset{\sim}{f}(\underset{\sim}{x}) = \nabla F(\underset{\sim}{x}), \quad \forall \underset{\sim}{x} \in X \tag{15}$$

where $\nabla F(\underset{\sim}{x})$ denotes the *gradient* of $\underset{\sim}{f}(\cdot)$

Theorem 7. State function and graidnet map criteria [30]

Let $\underset{\sim}{f}$ be a C^0 vector-valued function mapping an open set $X \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Then

1) $\underset{\sim}{f}$ is a *state function* if, and only if, $\underset{\sim}{f}$ is *gradient map*.
2) $\underset{\sim}{f}$ is a *gradient map* if, and only if, the associated line integral $\oint_\Gamma \underset{\sim}{f}(\underset{\sim}{x}) \cdot d\underset{\sim}{x} = 0$ along any piecewise C^1 closed path $\Gamma \subset X$.

To check whether a line integral is *path independent* using Theorem 7, we must produce a C^1 scalar function $F(\underset{\sim}{x})$ such that $\underset{\sim}{f}(\underset{\sim}{x}) = \nabla F(\underset{\sim}{x})$. This is far from a trivial task. Hence it would be advantageous to develop a systematic method for finding $F(\cdot)$. First of all, let us observe that if $F(\cdot)$ is C^2, then $\underset{\sim}{f}(\cdot)$ is C^1 and its Jacobian matrix is just the *Hessian* matrix of $F(\underset{\sim}{x})$. Now since mixed second partial derivatives of C^2 functions are equal to each other, we have:

$$\frac{\partial f_j(x)}{\partial x_k} = \frac{\partial f_k(x)}{\partial x_j} \qquad j,k = 1,2, \ldots, n \tag{16}$$

for all $\underset{\sim}{x} \in X$. Thus we obtain the useful property that the Jacobian matrix of any C^1 graident map is symmetric.[7] It follows from Theorem 7 that a necessary condition for the line integral of $\underset{\sim}{f}(\cdot)$ to be path independent is that the Jacobian matrix of $\underset{\sim}{f}(\cdot)$ must be symmetric. To show that this condition is not sufficient to guarantee path independence, consider the C^1 function $\underset{\sim}{f} : X \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$\underset{\sim}{f}(\underset{\sim}{x}) = [-x_2 \quad x_1]^T/(x_1^2+x_2^2) \tag{17}$$

where $X \triangleq \mathbb{R}^2 - \{0\}$ is just the "punctured" $\mathbb{R}^2$ plane; i.e., without the *origin*. It is easy to verify that the Jacobian matrix of $\underset{\sim}{f}(\cdot)$ is symmetric.

[7] In *Vector Calculus* a C^1 function $\underset{\sim}{f} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be *closed* if it has a symmetric Jacobian. Using this terminology, we have shown that *every* C^1 *exact function is closed* [30].

Yet there does not exist a C^2 scalar function $F(\underset{\sim}{x})$ such that (15) holds [30]. Hence, *the line integral of a* C^1 *function* $\underset{\sim}{f}$, having a symmetric Jacobian matrix, *need not be path independent.*[8] The problem with (17) is that the domain X is not *simply connected* [30]. In $\mathbb{R}^2$, X is said to be simply connected if every C^1 closed path $\subset X$ can be continuously shrunk to a single point belonging to X. For example, any region with a "hole" in it is not simply connected. Our next theorem is of fundamental importance in nonlinear network theory.

Theorem 8. The symmetry principle [30].

A C^1 function $\underset{\sim}{f}: X \subset \mathbb{R}^n \to \mathbb{R}^n$ defined in a simply connected open set X is a state function if and only if, its Jacobian matrix is symmetric.

Corollary. If $\underset{\sim}{f}: X \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^0 state function, then:

$$F(\underset{\sim}{x}) \triangleq \int_{\underset{\sim}{x}_0}^{\underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x}) \cdot d\underset{\sim}{x} + F(\underset{\sim}{x}_0) \tag{18}$$

is a C^1 potential function of $\underset{\sim}{f}(\cdot)$, where $\underset{\sim}{x}_0$ is any convenient fixed point in X and $F(\underset{\sim}{x}_0)$ is a scalar constant depending on $\underset{\sim}{x}_0$. In particular, (18) assumes the explicit form

$$F(\underset{\sim}{x}) = \int_0^1 \left\langle (\underset{\sim}{x}-\underset{\sim}{x}_0),\ \underset{\sim}{f}\left(\underset{\sim}{x}_0 + (\underset{\sim}{x}-\underset{\sim}{x}_0)\right)\right\rangle d\rho + F(\underset{\sim}{x}_0) \tag{19}$$

for a *straight-line path* from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}$, or

$$F(\underset{\sim}{x}) = \int_0^{x_1} f_1(\rho_1, 0, \ldots, 0)d\rho_1 + \int_0^{x_2} f_2(x_1, \rho_2, 0, \ldots, 0)d\rho_2$$

$$+ \ldots \int_0^{x_n} f_n(x_1, x_2, \ldots, x_{n-1}, \rho_n)d\rho_n + F(x_0) \tag{20}$$

for a *polyonal path* along the coordinate axes.

2. Circuit-Theoretic Properties

The foundations of *linear* n-ports and linear network theory [13] are built upon a few basic circuit-theoretic concepts; namely, *reciprocity, anti-reciprocity, non-energicness, losslessness, passivity,* and *activity.* Our objective in this section is to generalize these concepts for algebraic n-ports and to derive their characteristic properties. To avoid unnecessary repetitions, we make the *standing assumption* that all

[8] In terms of Vector Calculus terminology, this example shows that *not every* C^1 *closed function is exact.*

definitions are formulated for *time-invariant lumped* n-ports. These definitions however, can be easily generalized for the time-varying case.

A. Reciprocity and Anti-reciprocity

Definition 5. Reciprocal and anti-reciprocal n-ports

An *algebraic* n-port N characterized by a C^1 constitutive relation $\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{\eta})$ is said to be *reciprocal* (resp., *anti-reciprocal*) *at an operating point Q* if its associated *linearized n-port* $\hat{N}_Q$ characterized by (42) of Sec. I is reciprocal (resp., anti-reciprocal). N is said to be reciprocal (resp., anti-reciprocal) if it is reciprocal (resp., anti-reciprocal) at *all* operating points of N. It is said to be *non-reciprocal* if there exists an operating point Q where N is not reciprocal at Q.

Recall that each admissible signal pair $\left(\hat{\underset{\sim}{\xi}}(t),\hat{\underset{\sim}{\eta}}(t)\right)$ of $\hat{N}_Q$ must lie on the hyper plane tangent to $\xi_j = \xi_j(\underset{\sim}{\eta})$ at $\underset{\sim}{\eta} = \underset{\sim}{\eta}_Q$, $j = 1,2,\ldots,n$. Since ξ_j or η_j could represent v_j, i_j, ϕ_j, or q_j, $\xi_j(t),\eta_j(t)$ need not be a voltage-current pair. However, by differentiating either $\xi_j(t)$, or $\eta_j(t)$, or both, $j = 1,2,\ldots,n$, each admissible signal pair $\left(\hat{\underset{\sim}{\xi}}(t),\hat{\underset{\sim}{\eta}}(t)\right)$ of an algebraic n-port induces a corresponding voltage-current signal pair $\left(\hat{\underset{\sim}{v}}(t),\hat{\underset{\sim}{i}}(t)\right)$ -- henceforth called a *tangent v-i signal pair*. It follows from Def. 5 that an algebraic n-port N is *reciprocal* (resp., *anti-reciprocal*) at an operating point Q if, and only if, for *any two* tangent v-i signal pairs $\left(\hat{\underset{\sim}{v}}'(t),\hat{\underset{\sim}{i}}'(t)\right)$ and $\left(\hat{\underset{\sim}{v}}''(t),\hat{\underset{\sim}{i}}''(t)\right)$ associated with $\hat{N}_Q$,

$$\langle \hat{\underset{\sim}{V}}'(s),\hat{\underset{\sim}{I}}''(s)\rangle = \langle \hat{\underset{\sim}{V}}''(s),\hat{\underset{\sim}{I}}'(s)\rangle \tag{21}$$

$$\left\{\text{resp.,}\quad \langle \hat{\underset{\sim}{V}}'(s),\hat{\underset{\sim}{I}}''(s)\rangle = -\langle \hat{\underset{\sim}{V}}''(s),\hat{\underset{\sim}{I}}'(s)\rangle\right\} \tag{22}$$

where $\hat{\underset{\sim}{V}}(s)$ and $\hat{\underset{\sim}{I}}(s)$ denote the *single-sided Laplace transforms* of $\hat{\underset{\sim}{v}}(t)$ and $\hat{\underset{\sim}{i}}(t)$, respectively. Using this definition, the following necessary and sufficient conditions can be easily derived.

Theorem 9. Reciprocity and antireciprocity criteria [28][9].

(a) An n-port *resistor* is *reciprocal* (resp., *anti-reciprocal*) if, and only if, the *incremental resistance matrix* $\underset{\sim}{R}(\underset{\sim}{i}_Q)$ or *conductance matrix* $\underset{\sim}{G}(\underset{\sim}{v}_Q)$ is symmetric {resp.; skew-symmetric}.

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[9] Let the *linearized* n-port *resistor* {resp., *inductor, capacitor, memristor*} be characterized by either $\hat{\underset{\sim}{v}} = \underset{\sim}{R}(\underset{\sim}{i}_Q)\hat{\underset{\sim}{i}}$ or $\hat{\underset{\sim}{i}} = \underset{\sim}{G}(\underset{\sim}{v}_Q)\hat{\underset{\sim}{v}}$ {resp., $\hat{\underset{\sim}{\phi}} = \underset{\sim}{L}(\underset{\sim}{i}_Q)\hat{\underset{\sim}{i}}$ or $\hat{\underset{\sim}{i}} = \underset{\sim}{\Gamma}(\underset{\sim}{\phi}_Q)\hat{\underset{\sim}{\phi}}$, $\hat{\underset{\sim}{q}} = \underset{\sim}{C}(\underset{\sim}{v}_Q)\hat{\underset{\sim}{v}}$ or $\hat{\underset{\sim}{v}} = \underset{\sim}{S}(\underset{\sim}{q}_Q)\hat{\underset{\sim}{q}}$, $\hat{\underset{\sim}{\phi}} = \underset{\sim}{M}(\underset{\sim}{q}_Q)\hat{\underset{\sim}{q}}$ or $\hat{\underset{\sim}{q}} = \underset{\sim}{W}(\underset{\sim}{\phi}_Q)\hat{\underset{\sim}{\phi}}$}.

(b) An n-port *inductor* is *reciprocal* {resp., *anti-reciprocal*} if, and only if, the *incremental inductance matrix* $\underset{\sim}{L}(\underset{\sim}{i}_Q)$ or *reciprocal inductance matrix* $\underset{\sim}{\Gamma}(\underset{\sim}{\phi}_Q)$ is symmetric {resp., skew-symmetric}.

(c) An n-port *capacitor* is *reciprocal* {resp., *anti-reciprocal*} if, and only if, the *incremental capacitance matrix* $\underset{\sim}{C}(\underset{\sim}{v}_Q)$ or *reciproacl capacitance* matrix $\underset{\sim}{S}(\underset{\sim}{q}_Q)$ is symmetric {resp., skew-symmetric}.

(d) An n-port *memristor* is *reciprocal* {resp., *anti-recirpocal*} if, and only if, the *incremental memristance matrix* $\underset{\sim}{M}(\underset{\sim}{q}_Q)$ or *memductance matrix* $\underset{\sim}{W}(\underset{\sim}{\phi}_q)$ is symmetric {resp., skew-symmetric}.

Theorem 9 is valid only for the choice of coordinates indicated in footnote 9. The following theorem is *coordinate independent* and is applicable to all n-port resistors characterized by a generalized coordinate representation as defined in (37)-(38) of Sec. I. The same theorem also applies, *mutatis mutandis,* to n-port inductors, capacitors, and memristors.

<u>Theorem 10</u>. <u>Generalized reciprocity and anti-reciprocity criteria</u> [28].

A C^1 n-port *resistor* characterized by a linearized representation $\hat{\underset{\sim}{\xi}} = \underset{\sim}{\Lambda}(\underset{\sim}{\eta}_Q)\hat{\underset{\sim}{\eta}}$ about an operating point Q is *reciprocal at Q* if, and only if, the *characteristic matrix* defined by

$$\boxed{\underset{\sim}{\mathcal{K}}(\underset{\sim}{\eta}_Q) \triangleq \left[\underset{\sim}{c}\underset{\sim}{\Lambda}(\underset{\sim}{\eta}_Q) + \underset{\sim}{d}\right]^T \left[\underset{\sim}{a}\underset{\sim}{\Lambda}(\underset{\sim}{\eta}_Q) + \underset{\sim}{b}\right]} \tag{23}$$

is *symmetric*. It is *anti-reciprocal at Q* if, and only if, $\underset{\sim}{\mathcal{K}}(\underset{\sim}{\eta}_Q)$ is *skew-symmetric*.

To illustrate the application of Theorem 10, let us derive the reciprocity criteria for a 2-port resistor N characterized by an ABCD chain matrix by substituting the $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{d}$ matrices (for transmission representation I) from Table 2 into (23):

$$\underset{\sim}{\mathcal{K}}(\underset{\sim}{\eta}) = \left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right\}^T \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\} = \begin{bmatrix} AC & BC \\ AD-1 & BD \end{bmatrix} \tag{24}$$

It follows from (24) and Theorem 10 that N is reciprocal if, and only if, AD-BC = 1. Theorem 10 is extremely general and includes all known reciprocity criteria as special cases. In particular, for the important case where N is characterized by either hybrid representation I or II with an *incremental hybrid matrix* $\underset{\sim}{H}(\underset{\sim}{\eta}_Q)$ defined by

$$\hat{\underset{\sim}{\xi}} = \begin{bmatrix} \hat{\underset{\sim}{\xi}}_a \\ \hat{\underset{\sim}{\xi}}_b \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H}_{aa}(\underset{\sim}{\eta}_Q) & \underset{\sim}{H}_{ab}(\underset{\sim}{\eta}_Q) \\ \underset{\sim}{H}_{ba}(\underset{\sim}{\eta}_Q) & \underset{\sim}{H}_{bb}(\underset{\sim}{\eta}_Q) \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{\eta}}_a \\ \hat{\underset{\sim}{\eta}}_b \end{bmatrix} = \underset{\sim}{H}(\underset{\sim}{\eta}_Q)\hat{\underset{\sim}{\eta}} \tag{25}$$

we obtain the following useful corollary to Theorem 10.

Corollary.

(a) N is *reciprocal at Q* if, and only if, $\underset{\sim}{H}_{aa}(\underset{\sim}{\eta}_Q) = \underset{\sim}{H}^T_{aa}(\underset{\sim}{\eta}_Q)$, $\underset{\sim}{H}_{bb}(\underset{\sim}{\eta}_Q) = \underset{\sim}{H}^T_{bb}(\underset{\sim}{\eta}_Q)$, and $\underset{\sim}{H}_{ab}(\underset{\sim}{\eta}_Q) = -\underset{\sim}{H}^T_{ba}(\underset{\sim}{\eta}_Q)$.

(b) N is *anti-reciprocal at Q* if, and only if, $\underset{\sim}{H}(\underset{\sim}{\eta}_Q)$ is skew-symmetric.

Observe that the (p+q)-port transformer defined in (17) of Sec. I is both reciprocal and anti-reciprocal in view of the above corollary. Conversely, it can be proved that *the only n-port resistor which is both reciprocal and anti-reciprocal is a (p+q)-port transformer*.

It follows from Theorem 8 and 9 that the *potential functions* listed in Table 3 for an n-port resistor, inductor, capacitor, and memristor are *path-independent line integrals* if, and only if, the associated n-port is *reciprocal* (assuming the constitutive relation is defined in a *simply-connected* open subset in $\mathbb{R}^n$). Even if the domain is not simply connected, these potential functions are still well defined so long as the associated constitutive relation is a *gradient map* in view of Theorem 7.

Table 3. Potential functions associated with the 4 basic reciprocal n-ports.

Reciprocal n-port	Potential Function	
Resistor R	Content $\mathcal{G}(\underset{\sim}{i}) \triangleq \int_{\underset{\sim}{0}}^{\underset{\sim}{i}} \underset{\sim}{v}(\underset{\sim}{i})\cdot d\underset{\sim}{i}$	Co-content $\hat{\mathcal{G}}(\underset{\sim}{v}) \triangleq \int_{\underset{\sim}{0}}^{\underset{\sim}{v}} \underset{\sim}{i}(\underset{\sim}{v})\cdot d\underset{\sim}{v}$
Inductor L	Inductor Energy $\mathcal{W}_L(\underset{\sim}{\phi}) \triangleq \int_{\underset{\sim}{0}}^{\underset{\sim}{\phi}} \underset{\sim}{i}(\underset{\sim}{\phi})\cdot d\underset{\sim}{\phi}$	Inductor Co-Energy $\hat{\mathcal{W}}_L(\underset{\sim}{i}) \triangleq \int_{\underset{\sim}{0}}^{\underset{\sim}{i}} \underset{\sim}{\phi}(\underset{\sim}{i})\cdot d\underset{\sim}{i}$
Capacitor C	Capacitor Energy $\mathcal{W}_C(\underset{\sim}{q}) \triangleq \int_{0}^{\underset{\sim}{q}} \underset{\sim}{v}(\underset{\sim}{q})\cdot d\underset{\sim}{q}$	Capacitor Co-Energy $\hat{\mathcal{W}}_C(\underset{\sim}{v}) \triangleq \int_{\underset{\sim}{0}}^{\underset{\sim}{v}} \underset{\sim}{q}(\underset{\sim}{v})\cdot d\underset{\sim}{v}$
Memristor M	Action $\mathcal{M}(\underset{\sim}{q}) \triangleq \int_{\underset{\sim}{0}}^{\underset{\sim}{q}} \underset{\sim}{\phi}(\underset{\sim}{q})\cdot d\underset{\sim}{q}$	Co-Action $\hat{\mathcal{M}}(\underset{\sim}{\phi}) \triangleq \int_{\underset{\sim}{0}}^{\underset{\sim}{\phi}} \underset{\sim}{q}(\underset{\sim}{\phi})\cdot d\underset{\sim}{\phi}$

If we let $F(\underset{\sim}{x})$ and $\hat{F}(\underset{\sim}{y})$ denote the corresponding potential functions in the left and right columns in Table 3, and if the constitutive relations $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$ are bijective, then it follows from the integration-by-parts formula for line integrals that the following identity holds:

$$\underset{\sim}{F}(\underset{\sim}{x}) + \hat{F}(\underset{\sim}{y}) = \langle \underset{\sim}{x},\underset{\sim}{y} \rangle \tag{26}$$

B. Non-Energicness

Definition 6. Non-energic n-ports

An *algebraic* n-port N is said to be *non-energic* if for *any* admissible signal pair $\left(\underset{\sim}{v}(t),\underset{\sim}{i}(t)\right)$, the total instantaneous power

$$\langle \underset{\sim}{v}(t),\underset{\sim}{i}(t)\rangle = \sum_{j=1}^{n} v_j(t)i_j(t) = 0, \quad \forall t \in [t_0,\infty) \tag{27}$$

Otherwise, N is said to be *energic*.

The (p+q)-port transformer, gyrator, conjunctor and traditor defined in Sec. I are all non-energic. The following theorems provide necessary and sufficient conditions for various types of non-energic algebraic n-ports.

Theorem 11. Non-energic linear n-port criteria [32]

A *linear algebraic*[10] n-port N is *non-energic* if, and only if, N is an *anti-reciprocal n-port resistor*. Conversely, every anti-reciprocal n-port resistor N characterized by a C^1 hybrid representation $\underset{\sim}{h}(\cdot)$ is non-energic if, and only if, $\underset{\sim}{h}(\cdot)$ is an affine function.

Theorem 12. Non-energic n-port resistor criteria [32]

(a) An n-port *resistor* characterized by a hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$ is *non-energic* if, and only if, $\underset{\sim}{h}(\cdot)$ assumes the form

$$\underset{\sim}{h}(\underset{\sim}{x}) = \underset{\sim}{H}_a(\underset{\sim}{x})\underset{\sim}{x} \tag{28}$$

where $\underset{\sim}{H}_a(\underset{\sim}{x})$ is a *skew-symmetric* matrix for all $\underset{\sim}{x}$.

(b) A *reciprocal* n-port *resistor* characterized by[11] $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{i})$ $\forall \underset{\sim}{i}$ in a *cone* X_c is *non-energic* if, and only if, its *content* $\mathcal{G}(i)$ is *0-order homogeneous*.

(c) A *reciprocal* n-port *resistor* characterized by $\underset{\sim}{i} = \underset{\sim}{i}(\underset{\sim}{v})$ $\forall \underset{\sim}{v}$ in a *cone* X_c is *non-energic* if, and only if, its *co-content* $\hat{\mathcal{G}}(\underset{\sim}{v})$ is

[10] Def. 6 and Theorem 11 are also applicable for *dynamic* n-ports.

[11] See bottom of next page.

0-order homogeneous.

Observe that even though a cone may not be simply connected, the content and co-content are still well defined so long as the associated constitutive relation are *gradient maps* in view of Theorem 7. Observe also that (28) implies N is anti-reciprocal only if $\underset{\sim}{h}(\cdot)$ is a *linear* function since $\underset{\sim}{H}_a(\cdot)$ is not the associated Jacobian matrix if $\underset{\sim}{h}(\cdot)$ is nonlinear. Hence a non-energic n-port resistor characterized by a hybrid representation must be nonlinear if it is not anti-reciprocal and linear if it is anti-reciprocal. Our next theorem is expressed in terms of the generalized coordinate representation and is therefore completely general.

Theorem 13. Generalized non-energic n-port resistor criteria [28].

An n-port resistor characterized by (39) of Sec. I is *non-energic* if, and only if, the *characteristic function*

$$\boxed{\kappa(\underset{\sim}{\eta}) \triangleq \langle \underset{\sim}{c}\underset{\sim}{\xi}(\underset{\sim}{\eta}) + \underset{\sim}{d}\underset{\sim}{\eta},\ \underset{\sim}{a}\underset{\sim}{\xi}(\underset{\sim}{\eta}) + \underset{\sim}{b}\underset{\sim}{\eta}\rangle} \qquad (29)$$

vanishes identically.

The non-energic criteria for n-port inductors and capacitors are dual of each other. Hence, we will consider only the capacitor case:

Theorem 14. Non-energic n-port capacitor criteria [32].

(a) A charge-controlled n-port *capacitor* characterized by $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{q})$, $\underset{\sim}{q} \in X \subset \mathbb{R}^n$ is *non-energic* if, and only if, $\underset{\sim}{v}(\underset{\sim}{q}) \equiv \underset{\sim}{0}$ (i.e., each port is equivalent to a short circuit).

(b) A voltage-controlled n-port *capacitor* N characterized by $\underset{\sim}{q} = \underset{\sim}{q}(\underset{\sim}{v})$, $\forall \underset{\sim}{v}$ in a *cone* $X_c \subset \mathbb{R}^n$, is *non-energic* if, and only if, N is *reciprocal* and $\underset{\sim}{q}(\underset{\sim}{v})$ is *0-order homogeneous.*

(c) A voltage-controlled n-port *capacitor* N defined in a *cone* $X_c \subset \mathbb{R}^n$ is *non-energic* if, and only if, its *co-energy* $\hat{\mathcal{W}}_c(\underset{\sim}{v})$ is (to within an additive constant) a C^2 *1st order homogeneous function.*

Since an n-port can not store energy if it is non-energic, Theorem 14(a) is intuitively reasonable. What is surprising is Theorem 14(b) which shows there exists a large class of non-trivial n-port capacitors which are

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[11] A subset $X_c \subset \mathbb{R}^n$ is said to be a *cone* if $\underset{\sim}{x} \in X_c$ implies $\lambda\underset{\sim}{x} \in X_c$ $\forall\lambda > 0$. A scalar function $\phi : X_c \subset \mathbb{R}^n \to \mathbb{R}^1$ is said to be *k-order homogeneous* if $\phi(\lambda\underset{\sim}{x}) = \lambda^k\phi(\underset{\sim}{x})$ $\forall\lambda > 0$, $\underset{\sim}{x} \neq \underset{\sim}{0}$. A vector-valued function $\underset{\sim}{f} : X_c \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be *k-order homogeneous* if all components of $\underset{\sim}{f}$ are k-order homogeneous.

incapable ov energy storage. For example, the 2-port shown in Fig. 14 is non-energic since its constitutive relation $q_1 = \ln\left[(v_1-v_2)/v_1\right]$ and $q_2 = \ln\left[v_2/(v_1-v_2)\right]$ is 0-order homogeneous. Since the energy $\mathcal{W}_c(q) \equiv 0$ for a non-energic capacitor, it can be shown that the *co-energy* of every voltage-controlled non-energic capacitor is given explicitly by $\hat{\mathcal{W}}_c(\mathbf{v}) = \langle \mathbf{v}, \mathbf{q}(\mathbf{v})\rangle$. This follows immediately from (26) if $\mathbf{q}(\cdot)$ is bijective. For the above example, it is easily verified that $\hat{\mathcal{W}}_c(\mathbf{v})$ is a C^2 <u>1st</u> order homogeneous function, as it should be in view of Theorem 14(c).

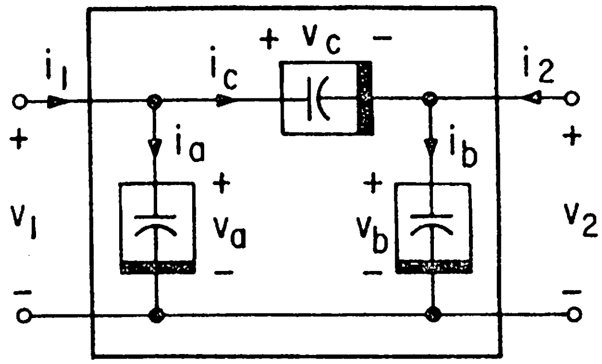

Fig. 14. A non-energic 2-port capacitor, where $v_a = \exp(-q_a)$, $v_b = \exp(q_b)$, and $q_c = \ln(v_c)$.

<u>Theorem 15</u>. <u>Non-energic n-port memristor criteria</u> [28].

A C^1 charge-controlled or flux-controlled n-port *memristor* is *non-energic* if, and only if, it is *anti-reciprocal.*

C. <u>Losslessness</u>

Roughly speaking, an n-port is lossless if whatever energy that enters it is stored and can be recovered later. This intuitive definition is adequate for *linear* n-ports but needs to be refined for algebraic n-ports. Let $Q_1(\boldsymbol{\xi}_{Q_1}, \boldsymbol{\eta}_{Q_1})$ and $Q_2(\boldsymbol{\xi}_{Q_2}, \boldsymbol{\eta}_{Q_2})$ denote any two not necessarily distinct operating points in the $\boldsymbol{\xi}$-$\boldsymbol{\eta}$ space associated with an algebraic n-port N. An admissible signal pair $\left(\boldsymbol{\xi}(t), \boldsymbol{\eta}(t)\right)$ is said to be an *admissible piecewise C^1 path between Q_1 and Q_2*, henceforth denoted by $\Gamma(Q_1,Q_2)$ if:

1. $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t)$ are *continuous* and *piecewise C^1* $\forall t \in [t_1,t_2]$, where t_1 and t_2 are finite numbers.
2. $\left(\boldsymbol{\xi}(t_1), \boldsymbol{\eta}(t_1)\right) = (\boldsymbol{\xi}_{Q_1}, \boldsymbol{\eta}_{Q_1})$ and $\left(\boldsymbol{\xi}(t_2), \boldsymbol{\eta}(t_2)\right) = (\boldsymbol{\xi}_{Q_2}, \boldsymbol{\eta}_{Q_2})$.
3. The associated admissible signal pair $\left(\mathbf{v}(t), \mathbf{i}(t)\right)$ (obtained by differentiating either $\xi_j(t)$ or $\eta_j(t)$, or both) are *continuous* and *piecewise C^1* on $[t_1,t_2]$.

Observe that if N-is an n-port resistor, then condition 3 is redundant. However, if N is an inductor {resp., capacitor}, then

$\xi(t) = \phi(t)$ {resp., $\xi(t) = q(t)$} must be C^2 in view of condition 3.

Definition 7. Lossless n-ports

An algebraic n-port N is said to be *lossless* if, for *any* two *admissible piecewise C^1 paths* $\Gamma'(Q_1,Q_2)$ and $\Gamma''(Q_1,Q_2)$ between two operating points Q_1 and Q_2, the energy $E(Q_1,Q_2)$ entering N while the signal traverses from Q_1 to Q_2 is the same in each case. Otherwise, N is said to be *not lossless.*

Stated mathematically, Def. 7 implies that if

$$\left(\xi'(t'_{Q_1}),\eta'(t'_{Q_1})\right) = \left(\xi''(t''_{Q_1}),\eta''(t''_{Q_1})\right) = Q_1 \text{ and}$$

$$\left(\xi'(t_{Q_2}),\eta'(t'_{Q_2})\right) = \left(\xi''(t''_{Q_2}),\eta''(t''_{Q_2})\right) = Q_2, \text{ then}$$

$$E(Q_1,Q_2) = \int_{t'_{Q_1}}^{t'_{Q_2}} \langle v'(t),i(t)\rangle dt = \int_{t''_{Q_2}}^{t''_{Q_2}} \langle v''(t),i''(t)\rangle dt \tag{30}$$

Notice that Def. 7 only requires the two admissible signal pairs to go through the same end points Q_1 and Q_2. The times where this occur are completely arbitrary. For example, the first admissible signal pair may start at Q_1 at $t = 0$ and arrives at Q_2 at $t = 1$, while the second may start at Q_1 at $t = 0.5$ and arrives at Q_2 at $t = 2$. Observe also that (30) must hold over all possible admissible signal pairs from Q_1 to Q_2.

Our reason for defining losslessness via admissible paths between two points, rather than around *closed* paths was to allow its generalization for *dynamic* n-ports where a return path may not exist. For example, consider a dynamic 2-port capacitor with two ideal diodes in series with each port such that the port currents $i_1(t) \geq 0$ and $i_2(t) \geq 0$ for all $t \geq t_0$. It is clear that any admissible path Γ in the q_1-q_2 plane for this 2-port must be a monotone increasing curve and hence Γ can never be a closed path.

Our reason for requiring the admissible signal pairs in Def. 7 to be *continuous* and *piecewise C^1* is rather subtle. To show that relaxing this to allow *discontinuous* signals would lead to a contradictory classification, consider a capacitor C characterized by the non-monotonic charge-controlled q-v curve shown in Fig. 15(a). It follows from Def. 7 that C is lossless. Observe that any continuous and piecewise C^1 signal which traverses between Q_1 and Q_2, such as the signal shown in Fig. 15(b), must follow the q-v curve continuously. Now suppose we drive this

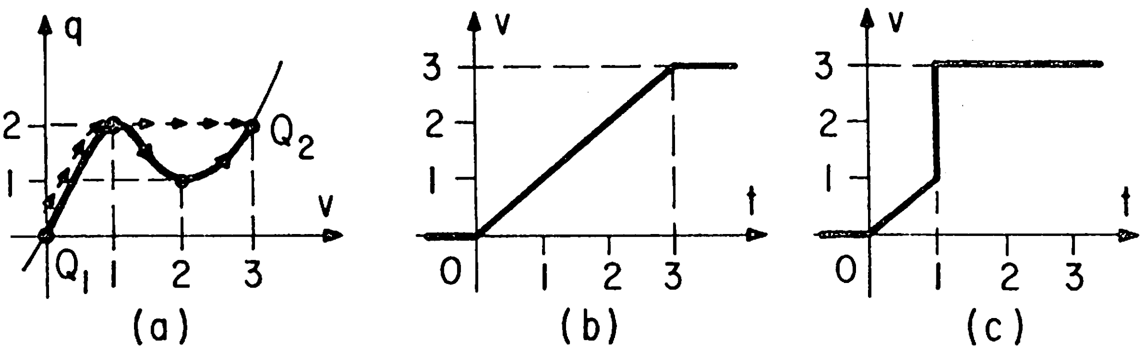

Fig. 15. Two distinct paths in the q-v plane due to two signals having identical end points.

capacitor with the discontinuous signal shown in Fig. 15(c). The resulting path is shown by the dotted lines in Fig. 15(a), where the horizontal path represents an *instantaneous* jump. It is easy to verify that the energy $E(Q_1,Q_2)$ corresponding to the 2 waveforms shown in Figs. 15(b) and (c) are different and hence C is *not lossless* if we enlarge the class of admissible signals to include discontinuous waveforms.

<u>Theorem 16</u>. <u>Implications of losslessness</u> [28].

Let N be a *lossless algebraic* n-port. Then

(a) For any admissible piecewise C^1 *closed* path $\Gamma(Q_1,Q_1)$, the energy $E(Q_1,Q_1) = 0$.

(b) If $\Gamma_{12}(Q_1,Q_2)$ is any admissible piecewise C^1 path from Q_1 to Q_2, and $\Gamma_{21}(Q_2,Q_1)$ is any admissible piecewise C^1 path from Q_2 to Q_1, then $E(Q_1,Q_2) = -E(Q_2,Q_1)$.

<u>Theorem 17</u>. <u>n-port resistor losslessness criteria</u> [28].

(a) Every *non-energic algebraic* n-port is *lossless*.

(b) An n-port *resistor* N is *lossless* if, and only if, it is *non-energic*.

<u>Theorem 18</u>. <u>n-port inductor and capacitor losslessness criteria</u> [28].

(a) A C^1 flux-controlled or current-controlled n-port *inductor* is *lossless* if, and only if, it is *reciprocal*.

(b) A C^1 charge-controlled or voltage-controlled n-port *capacitor* is *lossless* if, and only if, it is *reciprocal*.

<u>Theorem 19</u>. <u>n-port memristor losslessness criteria</u> [28].

A C^1 charge-controlled or flux-controlled n-port *memristor* is *lossless* if, and only if, it is *non-energic*.

Our next theorem provides a relationship between *losslessness* and the *average power*

$$P_{av} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t)\rangle dt \tag{31}$$

dissipated corresponding to each admissible signal pair $\left(\underset{\sim}{v}(t), \underset{\sim}{i}(t)\right)$.

Theorem 20. Implications of losslessness on average power [28].

Let N be a *lossless* n-port *capacitor* and let $\left(\underset{\sim}{v}(t), \underset{\sim}{i}(t)\right)$ be any continuous and piecewise C^1 admissible signal pair.

(a) If N is C^0 charge-controlled and $\underset{\sim}{q}(t)$ is *bounded* $\forall t \in [0,\infty)$, then $P_{av} = 0$.

(b) If N is C^1 voltage-controlled and $\underset{\sim}{v}(t)$ is *bounded* $\forall t \in [0,\infty)$, then $P_{av} = 0$.

To show that the *boundedness* hypothesis is necessary for Theorem 20 to hold, consider the dc voltage source $v_c = E$ as a charge-controlled capacitor. If we connect this capacitor across a 1 ohm resistor at $t = 0$, then $i_c(t) = -E$, $t \geq 0$ and $q_c(t) \to -\infty$ as $t \to \infty$. Hence $P_{av} \neq 0$. Similarly, if we apply a unit step current source $i_s(t) = u(t)$ across a 1 F capacitor at $t = 0$, then again $q_c(t) \to \infty$ and hence $P_{av} \neq 0$. On the other hand, boundedness is *not* a necessary condition for average power to vanish. For example, if we apply the unit step current source across a capacitor characterized by $v = \sin q$, then $q(t) \to \infty$ and yet $P_{av} = 0$.

The dual of Theorem 20 obviously applies for n-port inductors. Using some rather delicate mathematical analysis, the following converse of Theorem 20 can be proved to hold for *all algebraic* n-ports.

Theorem 21. Implications of zero average power on losslessness [28].

Let N be an *algebraic* n-port characterized by a continuous constitutive relation $\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{\eta})$. If the average power $P_{av} = 0$ for *every* admissible continuous and piecewise C^1 signal pairs $\left(\underset{\sim}{v}(t), \underset{\sim}{i}(t)\right)$ associated with a *bounded* $\underset{\sim}{\eta}(t)$, then N is *lossless*.

D. Passivity and Activity

Although the classical definition of passivity [31] is adequate for *linear* n-ports, it is flawed with serious conceptual difficulties and inconsistencies for *nonlinear* n-ports. The difficulty stems from the standard assumption of the *zero state* $\underset{\sim}{x} = \underset{\sim}{0}$ as the unique relaxed state where the energy storage is zero [31.33]. For example, a linear capacitor or inductor is said to be relaxed at $t = t_0$ if $q_c(t_0) = 0$ or $\phi_L(t_0) = 0$, respectively. In particular, one usually assumes $q_c(-\infty) = 0$ and $\phi_L(-\infty) = 0$

without questioning their physical significance. It turns out that for nonlinear n-ports, this assumption is untenable because it is possible for a nonlinear n-port to have either no equilibrium point, or multiple equilibrium points. In the former there exists no state of zero initial storage while in the latter there is no justification to prefer one state over several other equally valid states of zero energy storage. Consequently, any *general* definition of passivity for nonlinear n-ports must not involve the concept of an initial state of zero energy storage.

Although the passivity definition for *algebraic* n-ports to be proposed in this section is unconventional, it is entirely self-consistent and, when generalized to *dynamic* nonlinear n-ports having a state representation [34-36], it can be shown to be equivalent to the definition proposed by Rohrer [35] and Willems [36].

Let $Q(\underset{\sim}{\xi}_Q, \underset{\sim}{\eta}_Q)$ be an operating point of an algebraic n-port N. An admissible signal pair $\left(\underset{\sim}{\xi}(t), \underset{\sim}{\eta}(t)\right)$ is sadd to be an *admissible piecewise* C^1 *path through* Q, henceforth denoted by Γ_Q if:

1. $\underset{\sim}{\xi}(t)$ and $\underset{\sim}{\eta}(t)$ are continuous and piecewise C^1 functions of t $\forall t \in [0,T]$, $0 \leq T < \infty$.
2. $\underset{\sim}{\xi}(0) = \underset{\sim}{\xi}_Q$ and $\underset{\sim}{\eta}(0) = \underset{\sim}{\eta}_Q$
3. The associated admissible signal pair $\left(\underset{\sim}{v}(t), \underset{\sim}{i}(t)\right)$ is continuous and piecewise C^1 $\forall t \in [0,T]$.

Definition 8. Available energy at Q.

We define the *available energy* $E_A(Q)$ at an operating point Q of an algebraic n-port N by

$$\boxed{E_A(Q) \triangleq \sup \int_0^T -\langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle \, dt} \tag{32}$$

where the "supremum" is taken over *all* admissible piecewise C^1 paths through Q, and over *all* $T \geq 0$.

Observe that in view of our associated reference convention, energy enters N whenever $\langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle > 0$. Conversely, energy is being extracted from N whenever $\langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle < 0$. Observe that $E_A(Q) \geq 0$ since we can choose $T = 0$ in (32) if necessary. Physically, $E_A(Q)$ is the maximum energy that can be extracted over all time $t \geq 0$, when the n-port is initially operating at Q at $t = 0$. Observe that we use "sup" instead of "max" in (32) since the latter may not exist while the former does.

Definition 9. Passive and active n-ports.

An *algebraic* n-port N is said to be *passive* if the available energy $E_A(Q)$ at *each* operating point of N is *bounded*. N is said to be *active* if it is not passive.

It is important to distinguish between the *value* $E_A(Q)$ of the available energy function at Q, and the function $E_A(\cdot)$ itself in Def. 9. Passivity requires only the former to be bounded. For example, the available energy of a 1 Farad capacitor at the operating point $v = v_Q$ is given by $E_A(Q) = \frac{1}{2} v_Q^2$. It follows from Def. 9 that this capacitor is passive since $E_A(Q) < \infty \; \forall v_Q \in \mathbb{R}^1$ even though $E_A \to \infty$ as $|v_Q| \to \infty$ (remember ∞ is *not* a point in $\mathbb{R}^1$). For complicated n-ports, it is far from a trivial task to calculate $E_A(\cdot)$. Consequently, the following theorems for testing passivity are extremely useful.

Theorem 22. n-port resistor passivity criteria [28].

An n-port *resistor* N characterized by a generalized coordinate representation is *passive* if, and only if, $\kappa(\underset{\sim}{\eta}) \geq 0 \; \forall \underset{\sim}{\eta} \in \mathbb{R}^n$, where $\kappa(\underset{\sim}{\eta})$ is the characteristic function defined in (29). In particular, N is *passive* if, and only if, $\langle \underset{\sim}{v}, \underset{\sim}{i} \rangle \geq 0 \; \forall \; (\underset{\sim}{v}, \underset{\sim}{i})$ satisfying the constitutive relation of N.

Corollary.

1. A one-port *resistor* is *passive* if, and only if, its v-i curve lies in the first and third quadrants only.
2. An n-port *resistor* N characterized by a continuous hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$ is *passive* if, and only if, $\underset{\sim}{h}(\underset{\sim}{0}) = \underset{\sim}{0}$.
3. Every non-energic n-port resistor is passive.

Theorem 23. n-port resistor activity criteria [28].

An n-port *resistor* characterized by a hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$ is *active* if $\underset{\sim}{h}(\cdot)$ does not depend on some port variable x_k; i.e.,

$$y_j = h_j(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \; j = 1,2,\ldots,n \tag{33}$$

Corollary.

Every linear or nonlinear *controlled source* (which is not controlled by its associated port current or voltage) is active.

Theorem 24. n-port inductor passivity criteria [28].

(a) A reciprocal C^1 flux-controlled n-port *inductor* is *passive* if, and only if,

$$\inf_{\underset{\sim}{\phi}\in\mathbb{R}^n} \int_{\underset{\sim}{0}}^{\underset{\sim}{\phi}} \underset{\sim}{i}(\underset{\sim}{\phi})\cdot d\underset{\sim}{q} > -\infty \tag{34}$$

(b) A reciprocal C^1 current-controlled n-port *inductor* is *passive* if, and only if,

$$\inf_{\underset{\sim}{i}\in\mathbb{R}^n} \int_{\underset{\sim}{0}}^{\underset{\sim}{q}} \underset{\sim}{\phi}(\underset{\sim}{i})\cdot d\underset{\sim}{i} > -\infty \tag{35}$$

<u>Theorem 25</u>. <u>n-port capacitor passivity criteria</u> [28].

(a) A reciprocal C^1 charge-controlled n-port *capacitor* is *passive* if, and only if,

$$\inf_{\underset{\sim}{q}\in\mathbb{R}^n} \int_{\underset{\sim}{0}}^{\underset{\sim}{q}} \underset{\sim}{v}(\underset{\sim}{q})\cdot d\underset{\sim}{q} > -\infty \tag{36}$$

(b) A reciprocal C^1 voltage-controlled n-port *capacitor* is *passive* if, and only if,

$$\inf_{\underset{\sim}{v}\in\mathbb{R}^n} \int_{\underset{\sim}{0}}^{\underset{\sim}{v}} \underset{\sim}{q}(\underset{\sim}{v})\cdot d\underset{\sim}{v} > -\infty \tag{37}$$

<u>Corollary</u>.

(a) A C^1 flux-controlled or current-controlled one-port *inductor* is *passive* if the area under the i-ϕ curve in the 2<u>nd</u> and 4<u>th</u> quadrants is bounded.

(b) A C^1 charge-controlled or voltage-controlled one-port *capacitor* is *passive* if the area under the v-q curve in the 2<u>nd</u> and 4<u>th</u> quadrants is bounded.

To illustrate the application of this corollary, consider the 4 v-q curves shown in Fig. 16. Since the area under the 2<u>nd</u> and 4<u>th</u> quadrants (shown shaded) is bounded in Figs. 16(a), (b), and (c), it follows that the associated one-port capacitors are passive in view of the above corollary. Since this corollary provides only a sufficient condition for passivity, it is not applicable in Fig. 16(d) since the area is infinite. However, we can use Theorem 25 to conclude that this capacitor is active. It is interesting to observe that the classical definition of passivity would have classified all 4 capacitors as active [37]. Our definition classifies the first 3 capacitors as passive since only a finite amount of energy can be extracted in each case. To show that our classification is more reasonable, observe that the v-q curve in Fig. 16(a) is

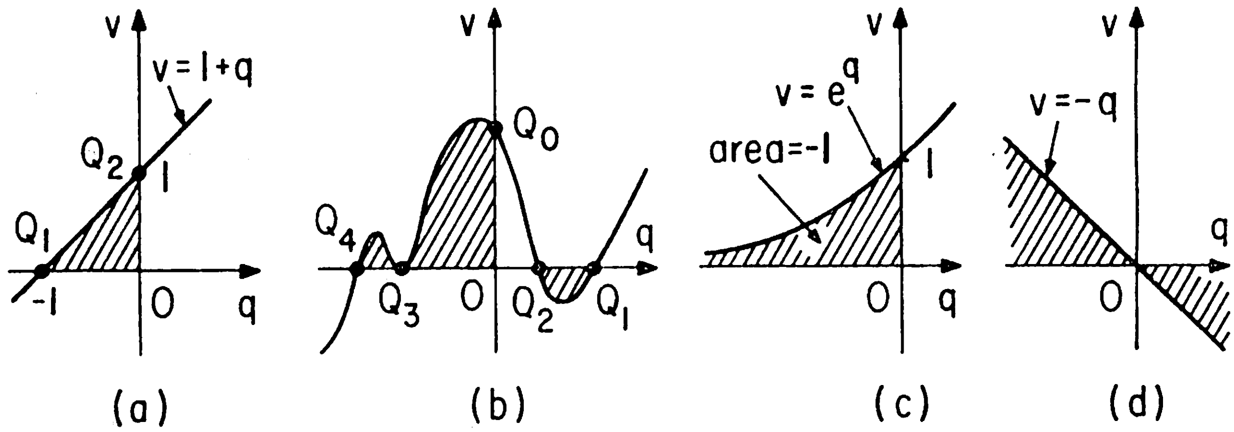

Fig. 16. The v-q curves associated with 4 one-port capacitors.

indistinguishable from that of a 1-Farad capacitor having an initial charge of 1 coulomb. Such a capacitor is clearly passive since passivity should be defined as an inherent property of an element, and should not depend upon initial conditions. Moreover, since *linear circuit theory* implies that every "active" one-port is "potentially" unstable under passive embeddings, the capacitor in Fig. 16(a) can never be unstable (in the sense of Lyapunov) when connected to a passive network and therefore should be classified as passive.

The passivity criteria in Theorems 24 and 25 require that the n-port inductors and capacitors be reciprocal. Our next theorem shows that this condition is *necessary* for passivity.

Theorem 26. n-port inductor and capacitor activity criteria [28].

(a) Every *non-reciprocal* C^1 flux-controlled or current-controlled n-port *inductor* is *active*.

(b) Every *non-reciprocal* C^1 charge-controlled or voltage-controlled n-port *capacitor* is *active*.

Theorem 27. n-port memristor passivity criteria [28].

A C^1 charge-controlled or flux-controlled n-port *memristor* is *passive* if, and only if, its incremental memristance matrix $\underset{\sim}{M}(\underset{\sim}{q})$ or memductance matrix $\underset{\sim}{W}(\underset{\sim}{\phi})$ is positive-semi-definite.

Finally, since a "lossless" or "not lossless" n-port may be either passive or active, we propose the following definition to distinguish the various possibilities.

Definition 10. Lossy and Generative n-ports.

An *algebraic* n-port is said to be *lossy* if it is *passive and not lossless*. It is said to be *generative* if it is *active and lossless*.

E. Local Passivity and Local Activity

All physically realizable n-ports having no internal power supplies must be *passive* since at most only a finite amount of energy can be extracted. Therefore, any *realistic* algebraic n-port circuit model of a multi-terminal or multiport device must be passive. From the applications point of view, it is important to know whether a given device is capable of power amplification or oscillation when operating over some "dynamic range." An important characterization of a device's dynamic range is given by the next definition.

Definition 11. Locally passive and locally active n-ports.

Let N be an *algebraic* n-port characterized by a global representation $\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{\eta})$. Let $Q(\underset{\sim}{\xi}_Q, \underset{\sim}{\eta}_Q)$ be an operating point and let $\underset{\sim}{\xi}(\underset{\sim}{\eta})$ be differentiable at $\underset{\sim}{\eta} = \underset{\sim}{\eta}_Q$ so that its associated *linearized n-port* $\hat{N}_Q$ is defined by (42) of Sec. I. We say N is *locally passive at Q* if $\hat{N}_Q$ is *passive*. Otherwise, N is said to be *locally active at Q*. N is said to be locally passive if it is locally passive at all operating points of N. It is said to be locally active if there exists at least one operating point where N is locally active.

Theorem 28. Local passivity criteria [28].

(a) An n-port *resistor* is *locally passive* if, and only if, its incremental resistance or conductance matrix is positive semi-definite.

(b) An n-port *inductor* is *locally passive* if, and only if, its incremental inductance or reciprocal inductance matrix is positive semi-definite.

(c) An n-port *capacitor* is *locally passive* if, and only if, its incremental capacitance or reciprocal capacitance matrix is positive semi-definite.

(d) An n-port *memristor* is *locally passive* if, and only if, its incremental memristance or memductance matrix is positive semi-definite.

The following corollary provides a relationship between locally passive and increasing n-ports (Theorem 2).

Corollary.

An n-port *resistor, inductor, capacitor,* or *memristor* is *locally passive* if, and only if, its constitutive relation (associated with the representations in Theorem 28) is an *increasing* function.

Theorem 28 can be generalized to a coordinate independent form. We will state the result for n-port resistors.

__Theorem 29.__ __Generalized local passivity criteria for n-port resistors__ [28].

An n-port *resistor* characterized by a generalized coordinate representation is *locally passive* if, and only if, it *characteristic matrix* $\mathcal{K}(\eta)$ as defined by (23) is positive semi-definite.

The following corollary is useful in the synthesis of 2-port resistors [18,38-39].

__Corollary.__ __Local passivity criteria via the chain matrix.__

(a) A 2-port *resistor* N characterized by hybrid representation I (A B C D chain matrix) is *locally passive* if, and only if, AC $\geq$ 0 and 4ABCD $\geq$ $(AD+BC-1)^2$ at each operating point.

(b) If N is *reciprocal*, then N is *locally passive* if, and only if, AC $\geq$ 0.

(c) If N is *anti-reciprocal*, then N is *locally passive*.

(d) If N is *locally passive*, then all 4 parameters A, B, C, and D must be either all non-negative, or all non-positive. Moreover, if N is also reciprocal, then A and D can not be zero.

An n-port may be passive but not locally passive, or vice-versa. Our next theorem provides a relationship between these two properties.

__Theorem 30.__ __Passivity and local passivity criteria__ [28].

A *locally passive* n-port *resistor* characterized by a C^1 hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$ is *passive* if, and only if, $\underset{\sim}{h}(\underset{\sim}{0}) = \underset{\sim}{0}$.

F. Local Non-energicness and Local Losslessness

__Definition 12.__ __Locally non-energic and locally lossless n-ports__

An algebraic n-port N is said to be *locally non-energic at Q* (resp., *locally lossless at Q*) if its linearized n-port $\hat{N}_Q$ is *non-energic* (resp., *lossless*). N is locally non-energic (resp., locally lossless) if it is locally non-energic (resp., locally lossless) at all operating points of N.

__Theorem 31.__ __Local non-energicness criteria__ [28].

(a) An n-port *resistor* is *locally non-energic* if, and only if, it is *anti-reciprocal*.

(b) An n-port *inductor* is *locally non-energic* if, and only if, its incremental inductance or reciprocal inductance matrix is a zero matrix (i.e., $\underset{\sim}{\phi}(\underset{\sim}{i}) \equiv \underset{\sim}{\phi}_0$ or $\underset{\sim}{i}(\underset{\sim}{\phi}) \equiv \underset{\sim}{i}_0$).

(c) An n-port *capacitor* is *locally non-energic* if, and only if, its incremental capacitance or reciprocal capacitance matrix is a zero matrix (i.e., $\underset{\sim}{q}(\underset{\sim}{v}) \equiv \underset{\sim}{q}_0$ or $\underset{\sim}{v}(\underset{\sim}{q}) \equiv \underset{\sim}{v}_0$).

(d) An n-port *memristor* is *locally non-energic* if, and only if, it is *anti-reciprocal.*

Theorem 32. Local losslessness criteria [28].

(a) An n-port *resistor* is *locally lossless* if, and only if, it is *anti-reciprocal.*

(b) An n-port *inductor* is *locally lossless* if, and only if, it is *reciprocal.*

(c) An n-port *capacitor* is *locally lossless* if, and only if, it is *reciprocal.*

(d) An n-port *memristor* is *locally lossless* if, and only if, it is *anti-reciprocal.*

Corollary.

(a) An n-port *resistor* or *memristor* is *locally lossless* if, and only if, it is *locally non-energic.*

(b) An n-port *inductor* or *capacitor* is *locally lossless* if, and only if, it is *lossless.*

G. Relaxed Algebraic n-ports

Roughly speaking, an operating point on an n-port capacitor or inductor is said to be *relaxed* if it does not discharge energy when connected to an external resistor. For linear n-ports, the *origin* is the relaxed point and there is no ambiguity when we say N has *zero* initial condition or N is *initially relaxed.* To show that this notion is too crude for nonlinear n-ports, consider the v-q curve shown in Fig. 16(b). There are 5 operating points $(Q_0, Q_1, Q_2, Q_3, Q_4)$ which can be said to have a zero initial condition. A careful analysis will reveal that no *net* energy can be extracted from this capacitor if, and only if, its initial operating point is at Q_4 and hence only Q_4 qualifies as a relaxed point. This observation justifies our next definition.

Definition 13. Relaxed operating point.

An *algebraic n-port* is said to be *relaxed* at an operating point Q if its *available energy* is zero at Q; i.e., $E_A(Q) = 0$.

A careful analysis of the v-q curves in Fig. 16 shows that the first two capacitors have exactly one relaxed operating point at Q_1 and Q_4, respectively, whereas the last two capacitors do *not* have any. For the Josephson junction iductor i-ϕ curve shown in Fig. 17(a), there are *infinitely many* relaxed operating points; namely, $\phi = \pm 2\pi(n/k)$, n = 1,2,... . Observe that if we displace this i-ϕ curve vertically as shown in Fig. 17(b) (this is equivalent to connecting a current-source

$i_s = I_1$ across the inductor), the resulting inductor does *not* have any relaxed operating points. Since it is far from a trivial task to determine whether an operating point is relaxed or not, the following theorems can be used for this purpose.

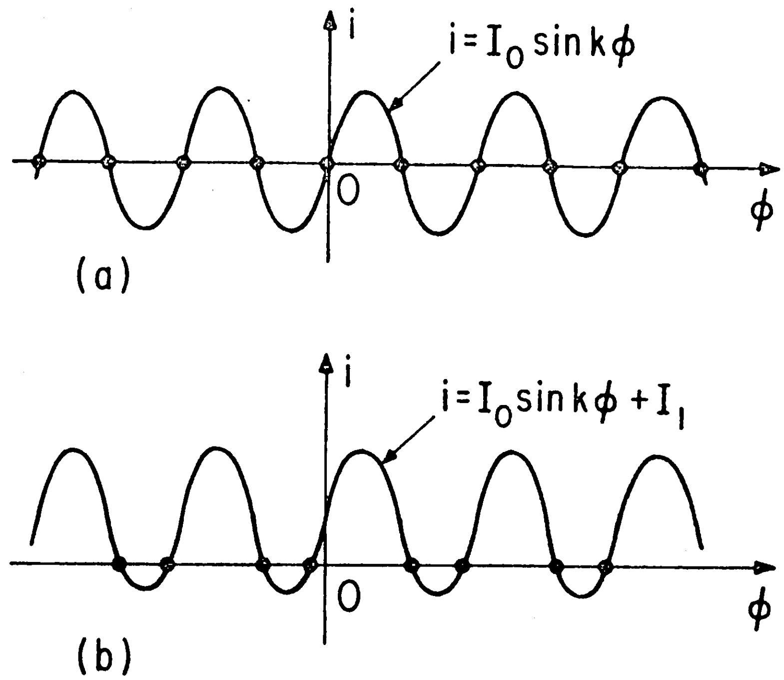

Fig. 17. Two nonlinear one-port inductors.

Theorem 33. Relaxed operating point criteria [28].

(a) Every operating point of a *passive* n-port *resistor* is *relaxed*.

(b) An operating point Q of a *reciprocal* C^0 flux-controlled n-port *inductor* is *relaxed* if, and only if,

$$\int_{\underset{\sim}{\phi}_Q}^{\underset{\sim}{\phi}} \underset{\sim}{i}(\underset{\sim}{\phi}) \cdot d\underset{\sim}{\phi} \geq 0, \qquad \forall \underset{\sim}{\phi} \in \mathbb{R}^n \tag{38}$$

(c) An operating point Q of a *reciprocal* C^0 charge-controlled n-port *capacitor* is *relaxed* if, and only if,

$$\int_{\underset{\sim}{q}_Q}^{\underset{\sim}{q}} \underset{\sim}{v}(\underset{\sim}{q}) \cdot d\underset{\sim}{q} \geq 0, \qquad \forall \underset{\sim}{q} \in \mathbb{R}^n \tag{39}$$

(d) Every operating point of a *passive* n-port *memristor* is *relaxed*.

Corollary.

A reciprocal C^0 flux-controlled n-port *inductor* or charge-controlled n-port *capacitor* is *passive* if it has at least one *relaxed* operating point.

Theorem 34. Relaxed n-port implies passivity [28].

Every *active algebraic* n-port characterized by a continuous constitutive relation has *no relaxed* operating points.

It follows from Theorems 33 and 34 that the inductor in Fig. 17(a) is passive while that in Fig. 17(b) is active.

Theorem 35. Necessary conditions for relaxed operating point [28].

(a) If a reciprocal C^0 flux-controlled or current-controlled n-port *inductor* N has a *relaxed* operating point Q, then $\underset{\sim}{i}_L(Q) = \underset{\sim}{0}$ and N is *locally passive* at Q.

(b) If a reciprocal C^0 charge-controlled or voltage-controlled n-port *capacitor* N has a *relaxed* operating point Q, then $\underset{\sim}{v}_C(Q) = \underset{\sim}{0}$ and N is *locally passive* at Q.

Theorem 36. Necessary conditions for multiple relaxed operating points [28].

(a) Let Q_0 be a *relaxed* operating point of a reciprocal C^0 flux-controlled n-port *inductor*. Then Q_1 is also a relaxed operating point *only if*

$$\int_{\underset{\sim}{\phi}_{Q_0}}^{\underset{\sim}{\phi}_{Q_1}} \underset{\sim}{i}(\underset{\sim}{\phi})\cdot d\underset{\sim}{\phi} = 0 \qquad (40)$$

(b) Let Q_0 be a *relaxed* operating point of a reciprocal C^0 charge-controlled n-port *capacitor*. Then Q_1 is also a relaxed operating point *only if*

$$\int_{\underset{\sim}{q}_{Q_0}}^{\underset{\sim}{q}_{Q_1}} \underset{\sim}{v}(\underset{\sim}{q})\cdot d\underset{\sim}{q} = 0 \qquad (41)$$

It follows from Theorem 36 that the existence of more than one relaxed operating points for inductors and capacitors is somewhat rare since (40) and (41) are rather stringent conditions.

3. Invariance Properties Relative to Representation and Datum of Multi-terminal Elements.

Since the circuit-theoretic properties defined in Sec. 2 are independent of the choice of representation of the n-port's constitutive relation, it is clear that they are invariant properties of an *algebraic* n-port or an (n+1)-terminal element with respect to some datum terminal. Our objective in this section is to investigate what happens if a different datum terminal is chosen. First of all, observe that an (n+1)-terminal element N (other than R, L, C, M) may be an *algebraic* n-port with respect to one datum terminal but becomes a *dynamic* n-port with respect to another datum terminal. For example, the 3-terminal element N characterized by $i_1 = f_1(v_1,i_2)$ and $\phi_2 = f_2(v_1,i_2)$ with terminal 3 chosen as datum is an algebraic 2-port. However, N is *not* an algebraic 2-port if terminal 1 or 3 is chosen as datum. The following theorems show what properties are truly *invariant* in the sense that they do not depend on the choice of representation or the choice of datum.

Theorem 37. Power invariance theorem [28].

The total instantaneous power $p(t) = \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle$ entering an (n+1)-terminal *resistor, inductor, capacitor,* or *memristor* is independent of the choice of the datum terminal.

Theorem 38. Invariant structural and circuit-theoretic properties [28].

The following properties are *invariants* of an (n+1)-terminal *algebraic* element N:

(a) *Structural properties:* increasing, strictly increasing, uniformly increasing, and strongly uniformly increasing.

(b) *Circuit theoretic properties:* reciprocal, anti-reciprocal, non-energic, lossless, passive, locally non-energic, locally lossless, and locally passive.

It follows from Theorem 38 that the properties listed under (a) and (b) are truly inherent attributes of an algebraic n-port or (n+1)-terminal element.

III. Synthesis of Nonlinear Resistive n-ports

Theorems 3a and 3b of Sec. I assert that any nonlinear algebraic or dynamic n-port can be realized using only *linear elements* and *one nonlinear n-port resistor*. Consequently, the basic nonlinear n-port synthesis problem reduces to that of realizing a prescribed constitutive relation $\underset{\sim}{f}(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0}$ using a minimal set of practical building blocks. Since only 2-terminal resistors characterized by monotone increasing v-i curves can be *accurately* realized in practice without running into *instability* problems [18,39,40], only such elements qualify as basic *nonlinear* building blocks. Our objective in this section is to investigate the state-of-the-art of the following yet unsolved problem:

Basic n-port resistor synthesis problem. Given a constitutive relation $\underset{\sim}{f}(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0}$, synthesize an n-port using only *independent sources, linear controlled sources,* and *2-terminal resistors* (characterized by monotone increasing v-i curves passing through the origin) as the *basic building blocks*.

Any n-port realization N can be decomposed as shown in Fig. 18 where N_1 is an (n+m)-port containing only *linear resistors* and *linear controlled sources,* and N_2 is an m-port containing only nonlinear resistors and

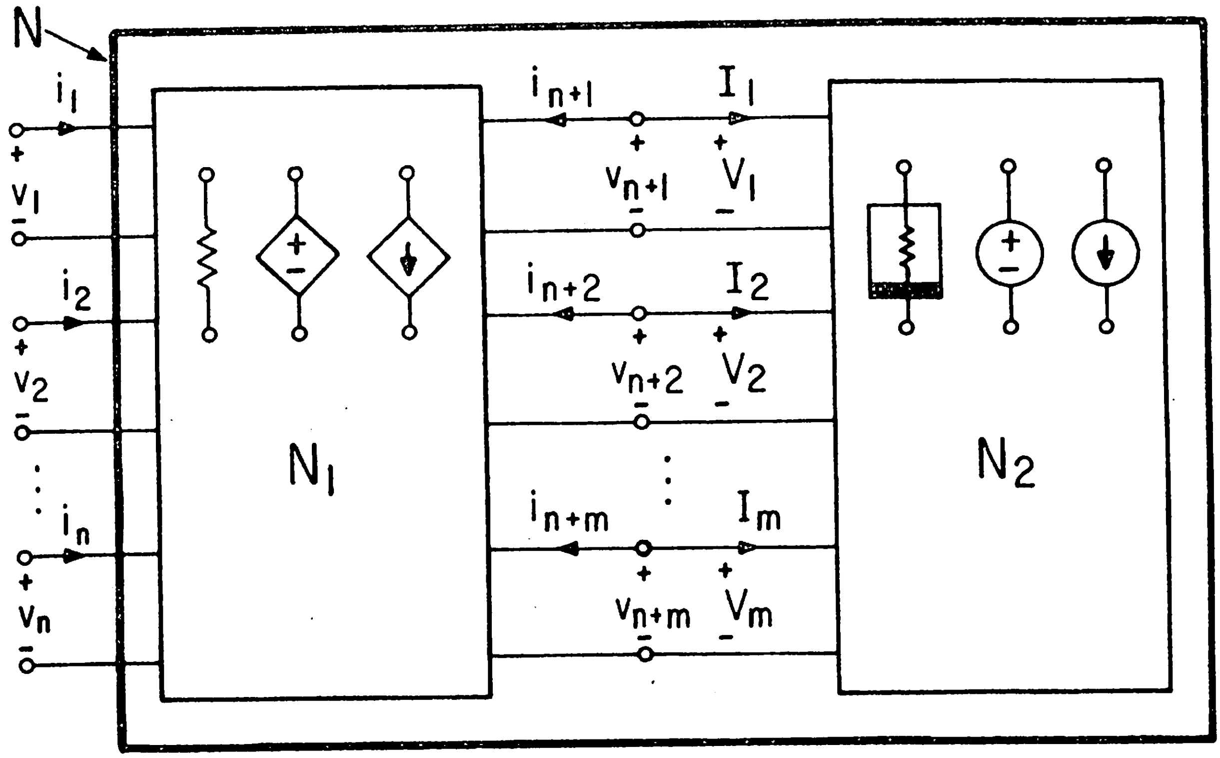

Fig. 18. N can be decomposed into a linear (n+m)-port N_1 and a nonlinear m-port N_2.

independent sources. Since N_2 contains only 2-terminal monotone increasing resistors and independent sources, it is easy to show that N_2 is a *reciprocal* and *locally passive* (*increasing*) m-port [24,41]. It follows from this observation that the linear elements alone must be responsible for any *non-reciprocity* or *non-monotonicity* in the resulting realization. Hence, the difficult problem of synthesizing an *arbitrary* constitutive relation would be greatly simplified if we can decompose it into reciprocal and locally passive components. Methods for accomplishing these decompositions will be presented in the next two subsections. For simplicity, we will consider only the case where the constitutive relation is *voltage-controlled;* i.e., $\underset{\sim}{i} = \underset{\sim}{g}(\underset{\sim}{v})$. The dual results clearly hold also for the current-controlled case. In fact, the methods can also be generalized for hybrid representations.

1. Decomposition into Reciprocal and Linear n-ports

Given a C^1 constitutive relation

$$\underset{\sim}{i} = \underset{\sim}{g}(\underset{\sim}{v}), \quad \underset{\sim}{i}, \underset{\sim}{v} \in \mathbb{R}^n \tag{1}$$

with an incremental conductance matrix

$$\underset{\sim}{G}(\underset{\sim}{v}) = \partial \underset{\sim}{g}(\underset{\sim}{v}) / \partial \underset{\sim}{v} \tag{2}$$

one might be tempted to decompose $\underset{\sim}{G}(\underset{\sim}{v})$ into a symmetric and a skew-symmetric part as in *linear synthesis* [31,33]; namely,

$$\underset{\sim}{G}(\underset{\sim}{v}) = \frac{1}{2}\left[\underset{\sim}{G}(\underset{\sim}{v}) + \underset{\sim}{G}^T(\underset{\sim}{v})\right] + \frac{1}{2}\left[\underset{\sim}{G}(\underset{\sim}{v}) - \underset{\sim}{G}^T(\underset{\sim}{v})\right] \triangleq \underset{\sim}{G}_s(\underset{\sim}{v}) + \underset{\sim}{G}_a(\underset{\sim}{v}) \tag{3}$$

Unfortunately, such a direct generalization is not valid for *nonlinear* n-ports because not every nxn matrix function is the Jacobian of some vector-valued function. In fact, our next theorem identifies the class of realizable skew-symmetric matrices.

Theorem 1. Anti-reciprocal n-port realizability criterion [22].

Every *anti-reciprocal* n-port resistor characterized by a C^2 hybrid representation is *affine* in the sense that

$$\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x}) = \underset{\sim}{H}\underset{\sim}{x} + \underset{\sim}{c} \tag{4}$$

where $\underset{\sim}{H}$ is an nxn constant skew-symmetric matrix and $\underset{\sim}{c}$ is an nx1 constant vector.

It follows from Theorem 1 that a necessary (but not sufficient) condition for (3) to be realizable is for $\underset{\sim}{G}_a(\underset{\sim}{v})$ to be a constant matrix. This observation dealt a severe blow to any hope of synthesizing nonlinear n-ports using only *reciprocal* and *anti-reciprocal* elements. To overcome this problem, we must identify some more general classes of realizable non-reciprocal n-ports which include anti-reciprocal n-ports as a proper subclass. Two useful generalizations have been identified [42-43]:

Definition 1. Quasi-antireciprocal n-ports

A C^1 voltage-controlled n-port resistor is said to be *quasi-antireciprocal* if its incremental conductance matrix $\underset{\sim}{G}(\underset{\sim}{v})$ can be decomposed into a *skew-symmetric* matrix $\underset{\sim}{G}_a(\underset{\sim}{v})$ and a *diagonal* matrix $\underset{\sim}{G}_d(\underset{\sim}{v})$; i.e.,

$$\underset{\sim}{G}(\underset{\sim}{v}) = \underset{\sim}{G}_a(\underset{\sim}{v}) + \underset{\sim}{G}_d(\underset{\sim}{v}) \tag{5}$$

Definition 2. Solenoidal n-ports

A C^1 voltage-controlled n-port resistor is said to be *solenoidal* if $\underset{\sim}{g}(\underset{\sim}{v})$ has *zero divergence;* i.e.,

$$\text{div } \underset{\sim}{g}(\underset{\sim}{v}) \triangleq \sum_{i=1}^{n} \underset{\sim}{G}_{ii}(\underset{\sim}{v}) \equiv 0 \tag{6}$$

Every anti-reciprocal n-port is quasi-antireciprocal and solenoidal. The converse is of course not true. Moreover, unlike anti-reciprocity (which is an invariant proeprty), it is easy to find examples of quasi-antireciprocal or solenoidal n-ports whose inverse representations fail to possess either property. Observe that a quasi-antireciprocal n-port need not be solenoidal, or vice-versa. However, since condition (6) is much weaker than condition (5), the class of solenoidal n-ports is, roughly speaking, much larger than the class of quasi-antireciprocal n-ports. Our motivation for introducing these two classes of n-ports is given by the following theorems.

Theorem 1. Reciprocal-quasi-antireciprocal decomposition [42]
(a) Every C^1 voltage-controlled 2-port resistor can be decomposed into a parallel connection of a *reciprocal* and a *quasi-antireciprocal* voltage-controlled 2-port resistor.
(b) Every C^1 voltage-controlled n-port resistor characterized by a *pairwise-coupled* constitutive relation

$$i_j = g_j(v_1,v_2,\ldots,v_n) = \sum_{k=1}^{n} g_{jk}(v_j,v_k), \quad j = 1,2,\ldots,n \tag{7}$$

can be decomposed into a parallel connection of a *reciprocal* and a *quasi-antireciprocal* voltage-controlled n-port resistor.

Theorem 2. Synthesis of quasi-antireciprocal n-ports [42]
(a) Every C^1 voltage-controlled quasi-antireciprocal 2-port resistor can be synthesized by cascading a *reciprocal* voltage-controlled 2-port resistor with a *gyrator* (see Fig. 19).

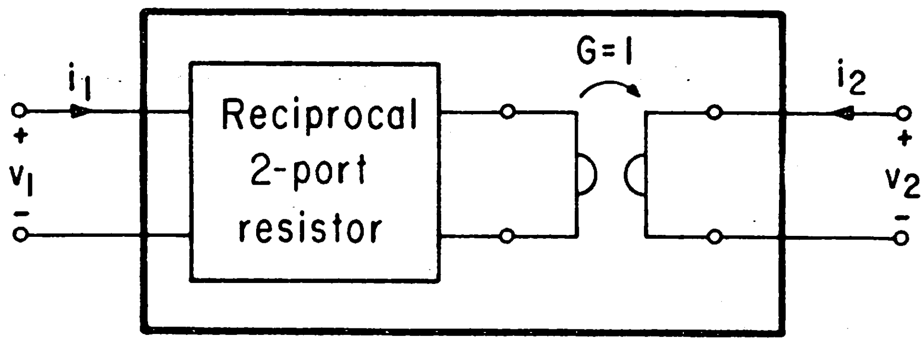

Fig. 19. Decomposition of a quasi-antireciprocal 2-port resistor.

(b) Every C^2 voltage-controlled *quasi-antireciprocal n-port* resistor can be synthesized using at most $\frac{1}{2}$ n(n-1) voltage-controlled *quasi-antireciprocal 2-port* resistors, and at most n 2-terminal voltage-controlled resistors. (See Fig. 20 for the case n=4).

Corollary.

Every C^1 voltage-controlled *quasi-antireciprocal n-port* resistor can

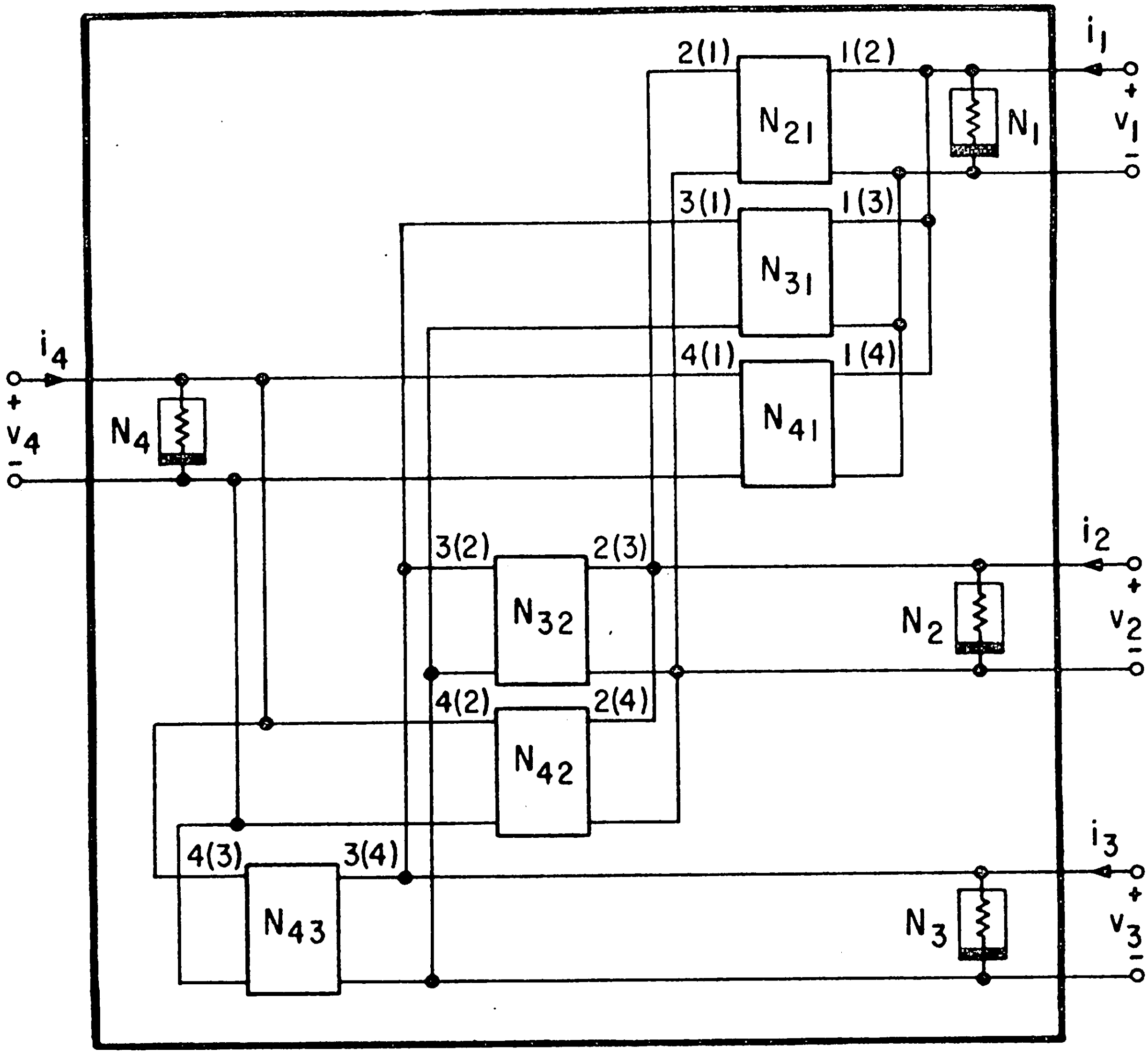

Fig. 20. Decomposition of a quasi-antireciprocal n-port resistor.

be synthesized using only *2-terminal* voltage-controlled resistors, *reciprocal* voltage-controlled 2-port resistors, and *gyrators*.

Since Theorem 1(a) holds for *all* C^1 voltage-controlled 2-port resistors, we have obtained a complete generalization of (3) for n=2. However, Theorem 1(b) gives only a partial generalization since it is valid only for "pairwise-coupled" constitutive relations. In both cases, *explicit* formulas are given in [42] for specifying the constitutive relations of the component n-ports. A complete generalization for arbitrary n-ports is given by our next theorem.

Theorem 3. Reciprocal-solenoidal decomposition [43].

Every C^1 voltage-controlled n-port resistor can be decomposed into a parallel connection of a *reciprocal* and a *solenoidal* voltage-controlled n-port resistor.

Since this theorem is valid for *all* C^1 voltage-controlled n-port resistors, it can be considered as a nonlinear generalization of the well-known decomposition of a linear n-port into a reciprocal and an anti-reciprocal n-port. Unlike theorem 1, however, the constitutive relations of the reciprocal and the solenoidal n-ports cannot usually be expressed in explicit form because the above decomposition requires solving a nonlinear *Poisson's equation* whose solution, though always exists, may not be obtained in closed form except in special cases. Our next result shows how solenoidal n-ports may be further decomposed into simpler building blocks.

Theorem 4. Synthesis of solenoidal n-ports [43]

Every C^1 *solenoidal* voltage-controlled n-port resistor can be synthesized using at most $\frac{1}{2}$ n(n-1) *reciprocal* voltage-controlled n-port resistors, and at most $\frac{1}{2}$ n(n-1) *linear* 2n-port resistors. (see Fig. 21 for the 2-port case).

Corollary.

Every C^1 voltage-controlled n-port resistor can be synthesized using only *reciprocal* voltage-controlled n-port resistors and *linear* elements.

2. Decomposition into Locally Passive and Linear n-ports.

Our objective in this section is to show that under rather mild assumptions, every reciprocal n-port resistor can be synthesized by terminating a linear 2n-port -- called a linear transformation converter (LTC) -- by a *reciprocal* and *locally passive n-port*.

Definition 3. Linear transformation converter (LTC) [44]

A *linear* 2n-port is called an LTC if it is characterized by a *non-singular* transmission matrix; namely,

$$\begin{bmatrix} v_a \\ i_a \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_b \\ -i_b \end{bmatrix} \tag{8}$$

There are three basic types of LTC building blocks which we define next.

Definition 4. 2n-port rotator, reflector, scalar

(1) An LTC is said to be a *2n-port rotator* if it is characterized by a 2nx2n *orthogonal* transmission matrix having a positive determinant.

(2) An LTC is said to be a *2n-port reflector* if it is characterized by a

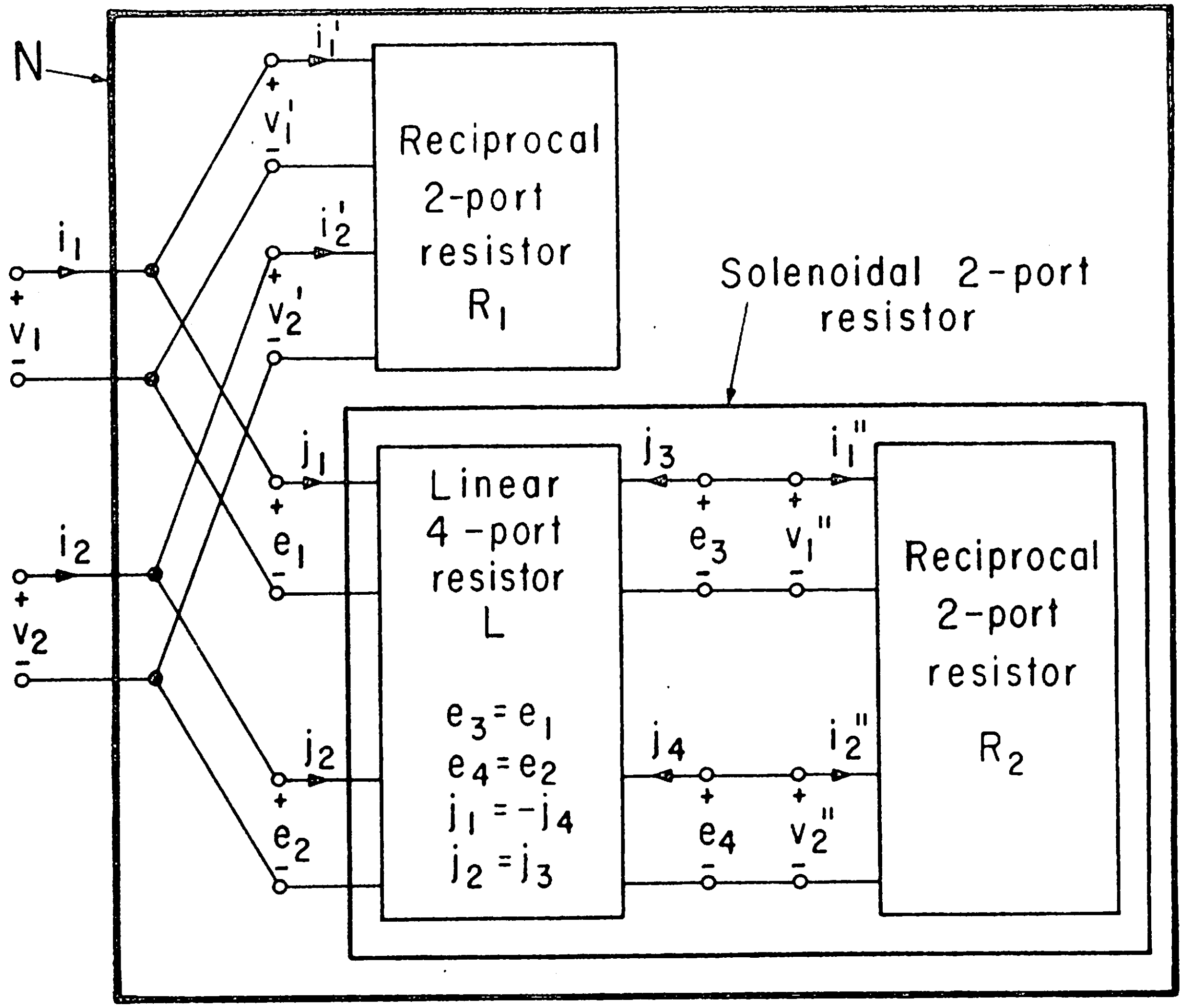

Fig. 21. Decomposition of N into two reciprocal 2-port resistors and a linear 4-port resistor.

2nx2n *orthogonal* transmission matrix having a negative determinant.

(3) An LTC is said to be a *2n-port scalar* if it is characterized by a non-singular *diagonal* 2nx2n transmission matrix.

The symbol and terminal characterization for these three elements are shown in Fig. 22. The significance of these elements is given by our next theorem.

<u>Theorem 5.</u> <u>LTC decomposition theorem</u> [44]

A 2n-port LTC with an prescribed transmission matrix T_R can be realized by any one of the four cascade configurations shown in Figs. 23(a)-(d). If only scalors with *positive* diagonal transmission matrices are used, then configurations (a) and (b) correspond to the case

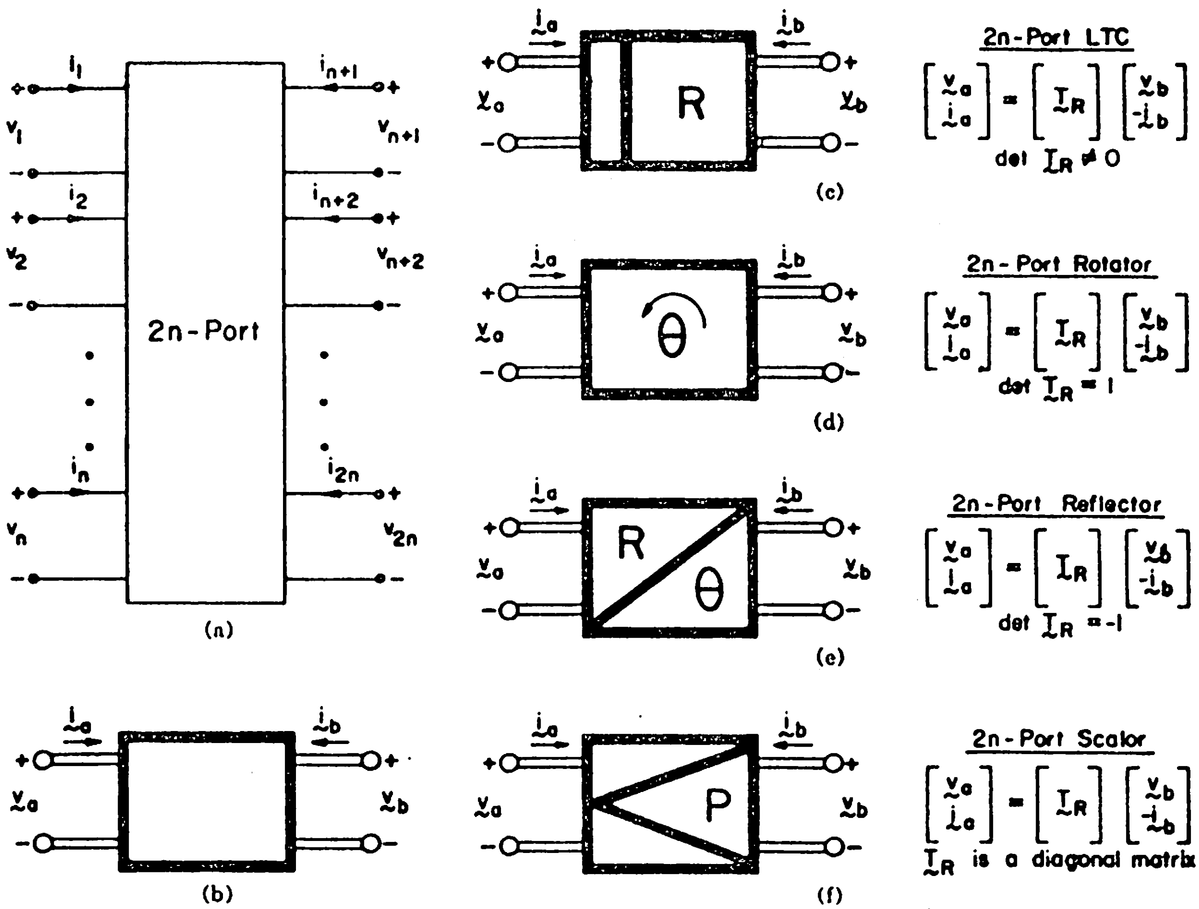

Fig. 22. The symbol and terminal characterization for a 2n-port LTC, rotator, reflector, and scalor.

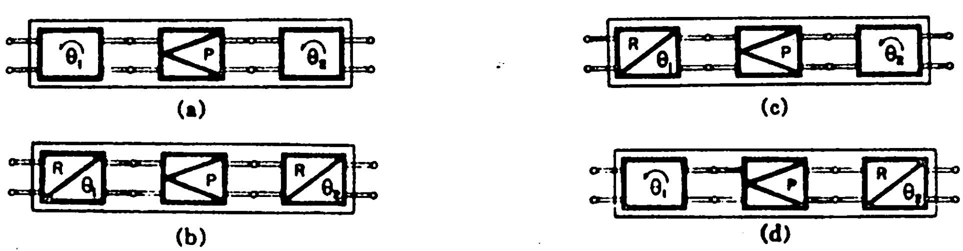

Fig. 23. Four basic cascade configurations for realizing a 2n-port LTC.

where det $T_R > 0$, whereas configurations (c) and (d) correspond to the case where det $T_R < 0$.

Theorem 5 shows that every LTC can be realized using only 3 basic building blocks. The significance of the LTC lies in its capability of transforming a *locally active* n-port resistor into a locally passive n-port resistor.

Theorem 6. LTC-locally passive n-port resistor decomposition [28].

Every C^1 and *reciprocal* voltage-controlled n-port resistor characterized by a constitutive relation $i = \hat{i}(v)$ satisfying

$$\left|\frac{\partial \hat{i}_j(\underset{\sim}{v})}{\partial v_k}\right| < M < \infty, \quad \forall v, \ \forall j,k = 1,2,\ldots,n \tag{9}$$

can be synthesized by connecting a *reciprocal* and *locally passive* n-port resistor across a *reciprocal* 2n-port LTC. (See Fig. 24). In particular, the LTC can be synthesized using n identical negative resistors with a conductance $G = -(nM+\varepsilon)$, where ε is any positive number.

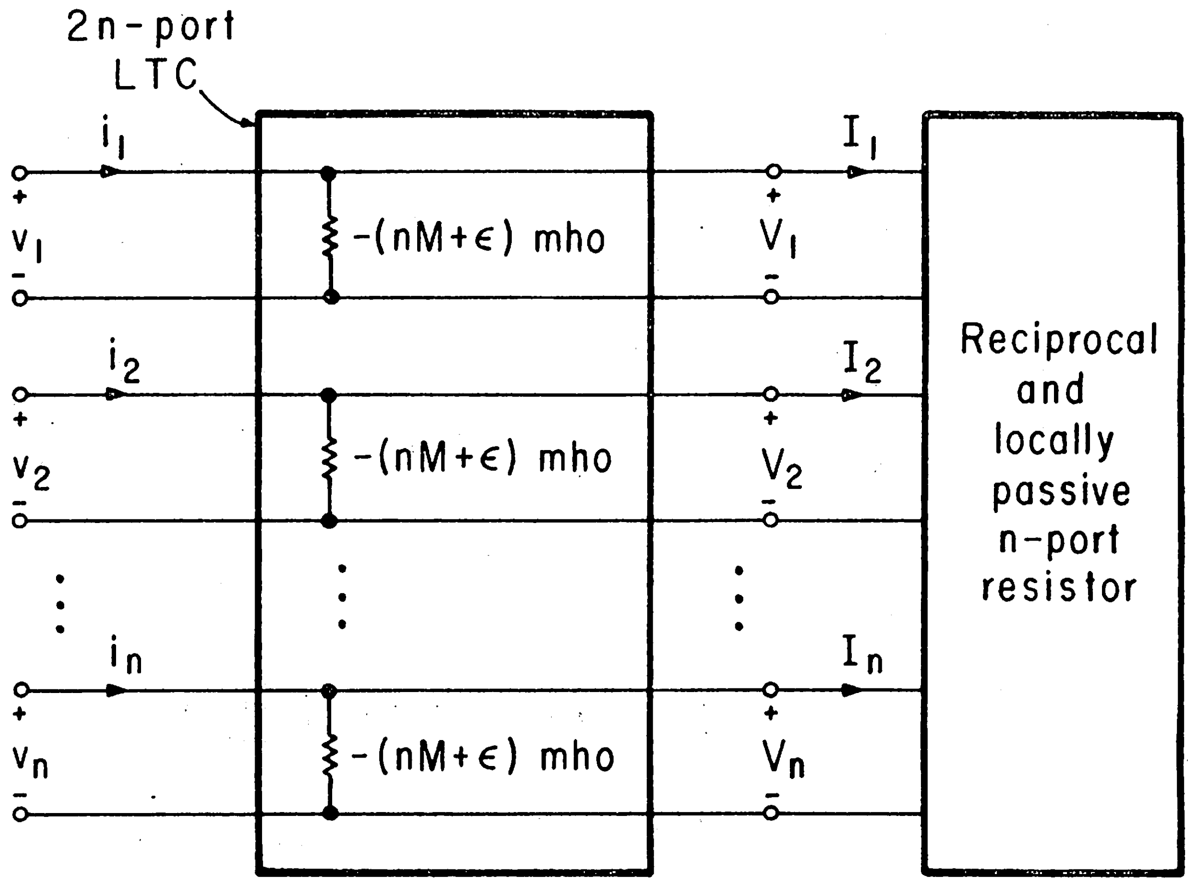

Fig. 24. Synthesis of reciprocal n-port resistor.

Observe that since the constitutive relation $\hat{\underset{\sim}{i}}(\underset{\sim}{v})$ is C^1, (9) is always satisfied on a *compact domain*. Since all physical n-ports must have a bounded dynamic range, it is clear that (9) represents very little loss of generality.

3. The Last Synthesis Hurdle

The last two sections show that the basic n-port resistor synthesis problem reduces to that of realizing a *reciprocal* and *locally passive* n-port resistor. If we allow only *locally passive* 2-terminal resistors as building blocks, then this problem includes the classic "linear resistor n-port synthesis problem" [45] as a special case.[1] Consequently, we must allow at least negative resistors, or ideal transformers as additional building blocks. We therefore close this section with the question: "can any *reciprocal* and *locally passive* n-port resistor be realized using only locally passive 2-terminal resistors and ideal transformers, or negative resistors?"

IV. Qualitative Properties of Dynamic Nonlinear Networks.

The dynamic behaviors of linear networks are simple and a complete theory has been developed. In contrast to this, the dynamic behaviors of nonlinear networks can be extremely complex and unpredictable, even for simple networks. For *autonomous* networks, i.e., networks containing only time-invariant lumped elements and dc sources, there may be several *equilibrium points*, such as those associated with a flip-flop circuit having different stability properties [39]. For "completely stable" circuits each solution must tend to an equilibrium point determined by the initial condition. A non-completely stable circuit, on the other hand, could display a great variety of exotic qualitative phenomena [46]. The simplest behavior consists of periodic oscillations. Even then, depending on the initial condition, a circuit may support several distinct periodic oscillations having distinct frequencies. Much more complicated behaviors are possible, however. For example, an autonomous network could support one or more *almost periodic oscillations* [47]. In fact, even more bizarre behaviors resembling stochastic processes have been observed [48].

For *non-autonomous* networks, i.e., networks containing time-varying elements, or ac sources, a periodic input may give a periodic or non-periodic output [46-47]. In the former case, the period of the output waveform need not coincide with that of the input signal. Moreover, various modes of *ferroresonance* and *jump phenomena* could occur. In the latter case, *subharmonic* oscillations and various *synchronization phenomena* have been widely observed.

Both autonomous and non-autonomous networks may also exhibit *finite escape time* solutions [49], i.e., solutions tending to ∞ in a finite time.

[1] For *linear* n-ports [45], a *necessary* condition for a symmetric conductance matrix $\hat{\underset{\sim}{G}}$ to be realizable using only linear positive resistors is for $\underset{\sim}{G}$ to be a *paramount matrix*. A *sufficient* realizability condition is for $\underset{\sim}{G}$ to be a *dominant* matrix. However, if *negative* resistors or ideal transformers are also allowed as building blocks, then any symmetric $\underset{\sim}{G}$ can always be realized.

Even if a solution is bounded in finite time, it could still become unbounded as $t \to \infty$.

The *qualitative theory* of nonlinear networks is concerned with the analysis of the various nonlinear behaviors described in the preceding paragraphs. The objective is to derive conditions under which certain nonlinear phenomenon may, or may not occur, without actually solving the differential equations describing the network. In particular, the theory is aimed at identifying various subclasses of nonlinear networks displaying similar qualitative behaviors.

Although a large body of mathematical results relevant to this study is available, a direct application of these results often lead to circuit conditions which are either too restrictive or artificial. The goal of "qualitative theory of nonlinear networks" is to derive useful theorems involving physically meaningful conditions. In particular, it would be most desirable that the conditions be expressed in terms of the *network topology* and the elements' *constitutive relations*. Any additional conditions of a mathematical nature should be readily verifiable, preferably by engineers having only rudimentary mathematical trainings. Although many more years of research will be needed to develop such a qualitative theory of dynamic nonlinear networks, research in this area over the past three decades have provided a firm foundation for building such a theory [50-78]. Our objective in this lecture is to focus on some of the recent developments in this area.

1. Formulation of State Equations.

Let $\mathcal{N}$ be an RLC network containing multi-terminal and multiport resistors, inductors, capacitors and independent voltage and current sources. Linear and nonlinear controlled sources are also included since they can be considered as multiport resistors. Without loss of generality, each (n+1)-terminal element or n-port can be modeled by "n" "coupled" 2-terminal elements. The reason for doing this is to allow an (n+1)-terminal element or an n-port to be represented by an element graph so that topological results from graph theory may be brought to bear. For example, the graphs corresponding to the elements in Figs. 1(a) and (b) are shown in Figs. 25(a) and (b), respectively.

Our goal in this section is to show how the state equations for $\mathcal{N}$ may be formulated. To do this, it is convenient to regard all multi-terminal and multiport capacitors and inductors, as "coupled" 2-terminal capacitors and "coupled" 2-terminal inductors, and to connect them across

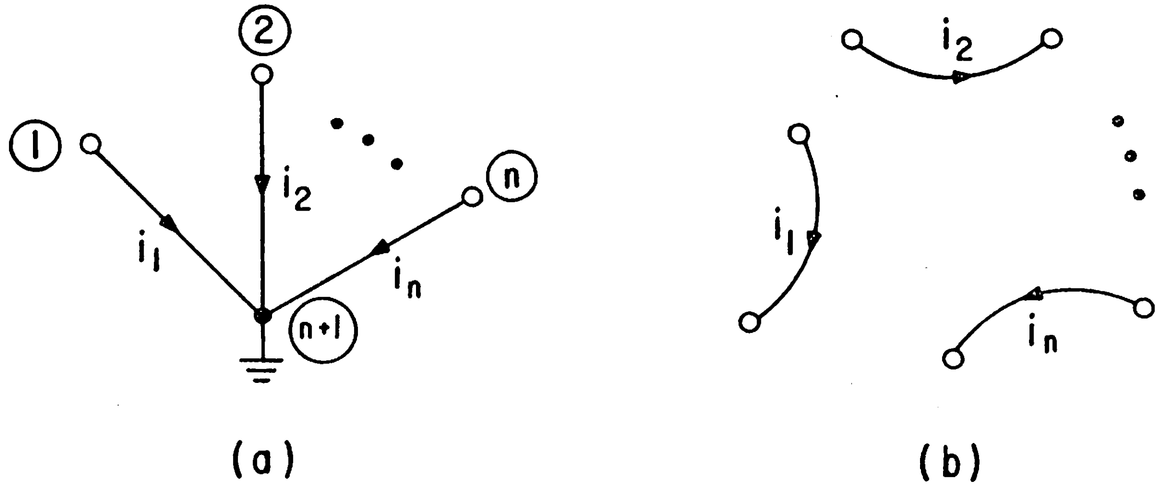

Fig. 25. (a) Graph of an (n+1)-terminal element. (b) Graph of an n-port.

an n-port resistor N as shown in Fig. 26, where $n = n_C + n_L$, n_C being the number of capacitor-terminated ports, and n_L being the number of inductor terminated ports. Without loss of generality, we can always choose the polarity as shown in Fig. 26 so that the capacitor voltage is equal to the port voltage, and the inductor current is equal to the port current. We assume that the n-port resistor N admits the following *hybrid* representation:[1]

$$\mathbf{i}_a = \mathbf{h}_a(\mathbf{v}_a, \mathbf{i}_b; \mathbf{u}_S) \tag{1}$$

$$\mathbf{v}_b = \mathbf{h}_b(\mathbf{v}_a, \mathbf{i}_b; \mathbf{u}_S) \tag{2}$$

where

$$\mathbf{u}_S \triangleq [\mathbf{E}_S \ \mathbf{I}_S]^T \tag{3}$$

denotes the *source vector* with $\mathbf{E}_S$ and $\mathbf{I}_S$ representing the voltages and currents of all voltage and current sources in N, respectively. Now assume the constitutive relations of the capacitors and inductors can be represented either in the "voltage and current-controlled" form

$$\mathbf{q}_C = \mathbf{q}_C(\mathbf{v}_C) \tag{4}$$

$$\boldsymbol{\phi}_L = \boldsymbol{\phi}_L(\mathbf{i}_L) \tag{5}$$

or in the "charge and flux-controlled" form

$$\mathbf{v}_C = \mathbf{v}_C(\mathbf{q}_C) \tag{6}$$

[1]This corresponds to the hybrid representation 2 in Sec. I.

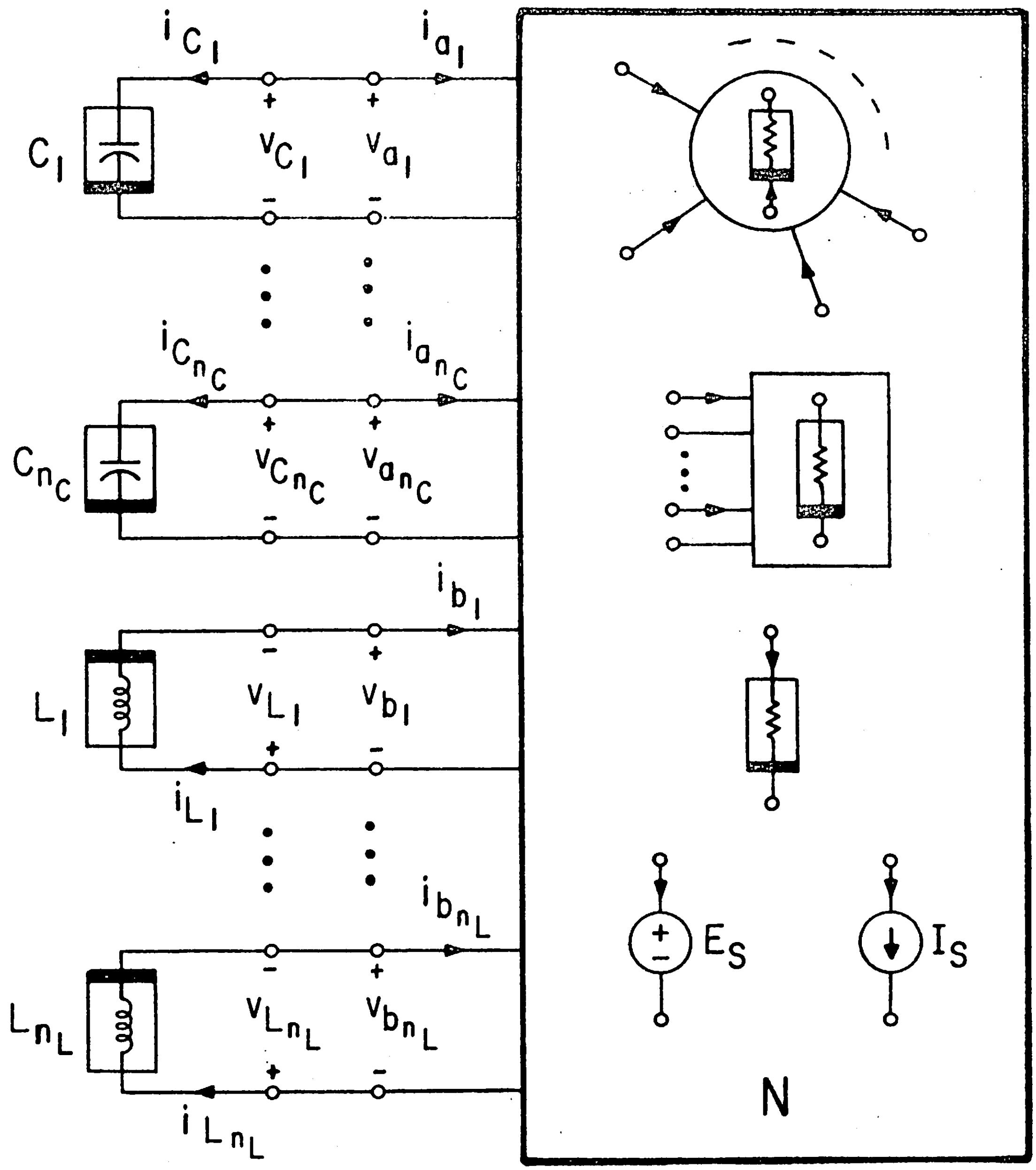

Fig. 26. Any RLC nonlinear network $\mathcal{N}$ can be represented by an n-port resistor N terminated by coupled 2-terminal capacitors and inductors.

$$\underset{\sim}{i}_L = \underset{\sim}{i}_L(\underset{\sim}{\phi}_L) \tag{7}$$

From Fig. 26, we obtain $\underset{\sim}{i}_C = -\underset{\sim}{i}_a$, $\underset{\sim}{v}_C = \underset{\sim}{v}_a$, $\underset{\sim}{i}_L = \underset{\sim}{i}_b$ and $\underset{\sim}{v}_L = -\underset{\sim}{v}_b$. The *state equations* of N corresponding to (4) and (5) are therefore given by:

$$\dot{\underset{\sim}{v}}_C = -\underset{\sim}{C}^{-1}(\underset{\sim}{v}_C)\underset{\sim}{h}_a(\underset{\sim}{v}_C,\underset{\sim}{i}_L;\underset{\sim}{u}_S) \triangleq \underset{\sim}{f}_C(\underset{\sim}{v}_C,\underset{\sim}{i}_L;\underset{\sim}{u}_S) \tag{8}$$

$$\dot{\underset{\sim}{i}}_L = -\underset{\sim}{L}^{-1}(\underset{\sim}{i}_L)\underset{\sim}{h}_b(\underset{\sim}{v}_C,\underset{\sim}{i}_L;\underset{\sim}{u}_S) \triangleq \underset{\sim}{f}_L(\underset{\sim}{v}_C,\underset{\sim}{i}_L;\underset{\sim}{u}_S) \tag{9}$$

where $\underset{\sim}{v}_C$ and $\underset{\sim}{i}_L$ are the *state variables*, and where $\underset{\sim}{C}(\underset{\sim}{v}_C) \triangleq \partial \underset{\sim}{q}_C(\underset{\sim}{v}_C)/\partial \underset{\sim}{v}_C$ denotes the $n_C \times n_C$ *incremental capacitance matrix* of the capacitor, and where $\underset{\sim}{L}(\underset{\sim}{i}_L) \triangleq \partial \underset{\sim}{\phi}_L(\underset{\sim}{i}_L)/\partial \underset{\sim}{i}_L$ denotes the $n_L \times n_L$ *incremental inductance matrix* of the of the inductors. If we choose $\underset{\sim}{q}_C$ and $\underset{\sim}{\phi}_L$ instead as state variables, then the state equations of N corresponding to (6) and (7) are given by:

$$\boxed{\begin{aligned} \dot{\underset{\sim}{q}}_C &= -\underset{\sim}{h}_a\left(\underset{\sim}{v}_C(\underset{\sim}{q}_C), \underset{\sim}{i}_L(\underset{\sim}{\phi}_L); \underset{\sim}{u}_S\right) \triangleq \underset{\sim}{f}_C(\underset{\sim}{q}_C, \underset{\sim}{\phi}_L; \underset{\sim}{u}_S) && (10) \\ \dot{\underset{\sim}{\phi}}_L &= -\underset{\sim}{h}_b\left(\underset{\sim}{v}_C(\underset{\sim}{q}_C), \underset{\sim}{i}_L(\underset{\sim}{\phi}_L); \underset{\sim}{u}_S\right) = \underset{\sim}{f}_L(\underset{\sim}{q}_C, \underset{\sim}{\phi}_L, \underset{\sim}{u}_S) && (11) \end{aligned}}$$

Now if we define

$$\underset{\sim}{x} \triangleq \begin{bmatrix} \underset{\sim}{v}_a \\ \underset{\sim}{i}_b \end{bmatrix} = \begin{bmatrix} \underset{\sim}{v}_C \\ \underset{\sim}{i}_L \end{bmatrix}, \quad \underset{\sim}{y} \triangleq \begin{bmatrix} \underset{\sim}{i}_a \\ \underset{\sim}{v}_b \end{bmatrix} = -\begin{bmatrix} \underset{\sim}{i}_C \\ \underset{\sim}{v}_L \end{bmatrix}, \quad \underset{\sim}{z} \triangleq \begin{bmatrix} \underset{\sim}{q}_C \\ \underset{\sim}{\phi}_L \end{bmatrix} \tag{12}$$

then (1)-(2) describing the n-port resistor N may be recast into the compact form

$$\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x}; \underset{\sim}{u}_S) \tag{13}$$

where

$$\underset{\sim}{h} \triangleq [\underset{\sim}{h}_a \ \underset{\sim}{h}_b]^T \tag{14}$$

The state equations (8)-(9) now assume the form

$$\boxed{\dot{\underset{\sim}{x}} = -\underset{\sim}{D}^{-1}(\underset{\sim}{x})\underset{\sim}{h}(\underset{\sim}{x}; \underset{\sim}{u}_S) = \underset{\sim}{f}(\underset{\sim}{x}; \underset{\sim}{u}_S) \qquad (15)}$$

where

$$\underset{\sim}{D}(\underset{\sim}{x}) \triangleq \left[\begin{array}{c|c} \underset{\sim}{C}(\underset{\sim}{x}) & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{L}(\underset{\sim}{x}) \end{array}\right] \tag{16}$$

Similarly, (10)-(11) may be recast into the compact form[2]

$$\boxed{\dot{\underset{\sim}{z}} \triangleq -\underset{\sim}{h}\left(\underset{\sim}{g}(\underset{\sim}{z}); \underset{\sim}{u}_S\right) = \underset{\sim}{f}(\underset{\sim}{z}; \underset{\sim}{u}_S) \qquad (17)}$$

---

[2] The reader is cautioned that our notations here differ from those in [69-72]. Specifically, the vectors $\underset{\sim}{x}$, $\underset{\sim}{y}$, $\underset{\sim}{z}$, $\underset{\sim}{h}(\underset{\sim}{x})$, and $\underset{\sim}{g}(\underset{\sim}{z})$ in this section correspond to $\underset{\sim}{x}_p$, $-\underset{\sim}{y}_p$, $\underset{\sim}{z}_p$, $\underset{\sim}{g}_p(\underset{\sim}{x}_p)$, and $\underset{\sim}{h}_p(\underset{\sim}{z}_p)$ in [69-72].

where

$$g(z) \triangleq \left[v_C(q_C) \quad i_L(\phi_L)\right]^T \tag{18}$$

If all sources in N are dc, then u_S is a constant vector and the network $\mathcal{N}$ is said to be *autonomous*. Otherwise, $u_S = u_S(t)$ and $\mathcal{N}$ is said to be *non-autonomous*.

The preceding state equation formulation is deceptively simple because it assumes that the hybrid representation of N is given explicitly by (1)-(2), and that the capacitors and inductors are characterized by either (4)-(5), or (6)-(7). While the latter assumption is usually satisfied by most nonlinear networks, the former is not. It is easy to find examples where N cannot be described by (1)-(2) [11,57]. Indeed, this hybrid representation is defined if, and only if, i_a and v_b are *uniquely* determined by any port voltage v_a and any port current i_b, and for any value of the source vector u_S. Hence the basic issue here is to determine the *existence and uniqueness* of the solution of the *resistive* network obtained by replacing all capacitors by voltage sources and all inductors by current sources, and then letting all external and internal sources assume all possible values. This subject has been studied extensively over the past decade and a large collection of results are now available in the literature [9,57-68,74-76].

It is clear from Fig. 26 that the hybrid representation (1)-(2) cannot be defined if there exists at least one loop formed exclusively by capacitors and voltage sources (C-E loop), or a cut set formed exclusively by inductors and current sources (L-J cut set). This is because the voltage v_a cannot be arbitrarily prescribed if $\mathcal{N}$ contains a C-E loop, while the current i_b cannot be arbitrarily prescribed if $\mathcal{N}$ contains an L-J cut set. This is the reason why many papers on dynamic nonlinear networks assume apriori that $\mathcal{N}$ *contains neither C-E loops nor L-J cut sets* since realistic models of many semiconductor devices contain capacitor loops (representing stray capacitances) and inductor cut sets (representing parasitic inductance)[11], specially for high-frequency operations, this restriction appears to be overly stringent. Fortunately, our next theorem shows how this condition can always be satisfied by an appropriate transformation.

<u>Theorem 1</u>. <u>C-E loop and L-J cut set elimination theorem</u> [11,70,78]

(a) Every C-E loop in $\mathcal{N}$ may be eliminated by *open-circuiting any* one capacitor in the loop, and by modifying the constitutive relations of the remaining elements in the loop, without altering the solutions of $\mathcal{N}$.

(b) Every L-J cut set in $\mathcal{N}$ may be eliminated by *short-circuiting any* one inductor in the cut set, and by modifying the constitutive relations of the remaining elements in the cut set, without altering the solutions of $\mathcal{N}$.

(c) The *modified* constitutive relations of the capacitors and the inductors can be derived *explicitly* from the original constitutive relations [11,70,78]. Moreover, if the original elements are *reciprocal, increasing, strictly-increasing,* or *uniformly increasing,* then these properties are preserved and therefore remain invariant in the transformed constitutive relations.

Theorem 1 shows that in so far as the *qualitative properties* of dynamic RLC networks are concerned, there is little loss of generality by assuming apriori that $\mathcal{N}$ contains neither C-E loops nor L-J cut sets. This fact greatly simplifies many research problems on nonlinear RLC networks. For example, the procedures for formulating state equations are drastically simplified when there are no C-E loops and L-J cut sets [59-67]. Under this assumption, the associated state equations can usually be handled easily even if it may not be possible to express the equations in *explicit* analytical form. By introducing additional assumptions, it is possible to derive the state equations explicitly for various subclasses of nonlinear networks. One important subclass studied extensively by Brayton and Moser is often referred to as the class of *complete networks*]54]. Another subclass that also admits an explicit formulation corresponds to those networks having an explicit *Lagrangian* or *Hamiltonian* function [79].

A. State Equations of Complete Networks

Definition 1. Topologically complete n-ports

An n-port N is said to be *topologically complete* if either the *voltage*, or the *current*, of each internal branch[3] is determined *topologically* by the port voltages across the voltage-driven ports (capacitor-terminated ports in Fig. 26) via KVL, and by the port currents in the current-driven ports (inductor-terminated ports in Fig. 26) via KCL, *without* invoking the constitutive relations of the internal elements.

[3] As always, multiterminal and multiport elements inside N are represented by "coupled" 2-terminal elements, so that topologically, each terminal pair, or each port, is represented by a *branch* of the element graph.

Definition 2. Complete n-ports

A topologically complete n-port N is said to be a *complete n-port* if each multiterminal or multiport element R of N is characterized by a *hybrid representation* $\underset{\sim}{y}_R = \underset{\sim}{h}_R(\underset{\sim}{x}_R)$ such that the independent variable $\underset{\sim}{x}_R$ is determined *topologically* by the external port variables ($\underset{\sim}{v}_C$ and $\underset{\sim}{i}_L$ in Fig. 26) via KVL and KCL alone.

Definition 3. Complete RLC networks

An RLC network $\mathcal{N}$ is said to be *complete* if its associated n-port resistor N is complete.

It follows from Def. 3 that given the values of the external port variables ($\underset{\sim}{v}_C$ and $\underset{\sim}{i}_L$ in Fig. 26), all branch currents and voltages associated with the internal elements of a *complete* network can be uniquely determined by using merely KVL, KCL, and *direct substitution* into the element's constitutive relations. In other words, *no equations need be solved at all.* Clearly, the "completeness" requirement is a very strong condition and the class of complete networks is rather small. However, by augmenting a non-complete network with parasitic capacitors and inductors, it is always possible to derive a complete network from it. We will now show how the state equations of such networks can be formulated explicitly.

Let $\mathcal{N}$ be a *complete* and *connected* network containing "coupled" 2-terminal capacitors, 2-terminal inductors, 2-terminal resistors, and dc voltage and current sources. Assume that N contains neither C-E loops nor L-J cut sets. For simplicity, let us first assume that the resistors are uncoupled. Let $\mathcal{T}_1$ be a *subtree* made up of "composite" branches each of which consists of a capacitor and all voltage-controlled resistors (possibly none) connected in parallel with it, and let $\mathcal{L}_2$ be a *subcotree* made up of "composite" branches, each of which consists of an inductor and all current-controlled resistors (possibly none) connected in series with it. The composite branches are shown in Figs. 27(a) and (b), respectively. If we extract all elements in $\mathcal{T}_1$ and $\mathcal{L}_2$ and consider them as loads connected across an n-port N, then it is easy to show that N is *complete* iff there is a *subtree* $\mathcal{T}_2$ made up of current-controlled resistors such that $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ form a tree of $\mathcal{N}$, and if all remaining elements are voltage-controlled resistors forming closed loops *exclusively* with branches in $\mathcal{T}_1$. If we denote these voltage-controlled resistors by the subcotree $\mathcal{L}_1$, then $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ is the cotree associated with $\mathcal{T}$. It follows in this case that the fundamental loops associated with branches in $\mathcal{L}_1$

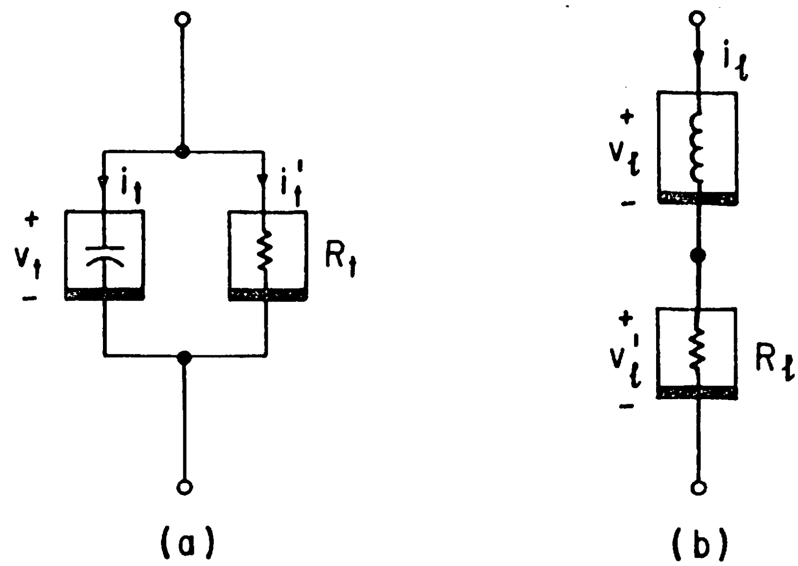

Fig. 27. (a) A typical capacitor composite branch" in $\mathcal{T}_1$. The resistor R_t represents the parallel combination of all voltage-controlled resistors (including dc current sources) connected across the capacitor. Note the current into R_t is denoted by i_t'; (b) A typical "inductor composite branch" in $\mathcal{L}_2$. The resistor R_ℓ represents the series combination of all current-controlled resistors (including dc voltage sources) connected in series with the inductor. Note the voltage across R_ℓ is denoted by v_ℓ'.

contain branches from $\mathcal{T}_1$ only, i.e., $\underset{\sim}{v}_{\mathcal{L}_1} + \underset{\sim}{B}_{\mathcal{L}_1\mathcal{T}_1}\underset{\sim}{v}_{\mathcal{T}_1} = \underset{\sim}{0}$, where $\underset{\sim}{v}_{\mathcal{L}_1}$ and $\underset{\sim}{v}_{\mathcal{T}_1}$ denote the branch voltage of the elements in $\mathcal{L}_1$ and $\mathcal{T}_1$, respectively, and $\underset{\sim}{B}_{\mathcal{L}_1\mathcal{T}_1}$ denotes an appropriate submatrix of the following *fundamental loop matrix* $\underset{\sim}{B}$:

$$\underset{\sim}{B} = \begin{array}{c} \begin{array}{cccc} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{T}_1 & \mathcal{T}_2 \end{array} \\ \left[\begin{array}{cc|cc} \underset{\sim}{1}_{\mathcal{L}_1\mathcal{L}_1} & \underset{\sim}{0}_{\mathcal{L}_1\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_1\mathcal{T}_1} & \underset{\sim}{0}_{\mathcal{L}_1\mathcal{T}_2} \\ \underset{\sim}{0}_{\mathcal{L}_2\mathcal{L}_1} & \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2} \end{array}\right] \end{array} \begin{array}{c} \mathcal{L}_1 \\ \mathcal{L}_2 \end{array}$$

where the upper right-hand corner submatrix is always a *zero matrix*. If we let $\underset{\sim}{i}_{\mathcal{L}_j}$, $\underset{\sim}{v}_{\mathcal{L}_j}$, $\underset{\sim}{i}_{\mathcal{T}_j}$, and $\underset{\sim}{v}_{\mathcal{T}_j}$ denote the current and voltage vectors for elements in $\mathcal{L}_j$ and $\mathcal{T}_j$, respectively, then the voltage-controlled resistors in $\mathcal{L}_1$ and $\mathcal{T}_1$ can be represented by $\underset{\sim}{i}_{\mathcal{L}_1}\left(\underset{\sim}{v}_{\mathcal{L}_1}\right)$ and $\underset{\sim}{i}_{\mathcal{T}_1}\left(\underset{\sim}{v}_{\mathcal{T}_1}\right)$, respectively. Similarly, the current-controlled resistors in $\mathcal{L}_2$ and $\mathcal{T}_2$ can be represented by $\underset{\sim}{v}_{\mathcal{L}_2}\left(\underset{\sim}{i}_{\mathcal{L}_2}\right)$ and $\underset{\sim}{v}_{\mathcal{T}_2}\left(\underset{\sim}{i}_{\mathcal{T}_2}\right)$, respectively. Hence, we can write:

$$\text{KCL:}\quad -\underset{\sim}{C}\left(\underset{\sim}{v}_{\mathcal{T}_1}\right)\frac{d}{dt}\underset{\sim}{v}_{\mathcal{T}_1} = -\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{T}_1}\,\underset{\sim}{i}_{\mathcal{L}_1}\circ\left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{T}_1}\underset{\sim}{v}_{\mathcal{T}_1}\right) + \underset{\sim}{i}_{\mathcal{T}_1}\left(\underset{\sim}{v}_{\mathcal{T}_1}\right) - \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{T}_1}\,\underset{\sim}{i}_{\mathcal{L}_2} \tag{19}$$

$$\text{KVL:}\quad -\underset{\sim}{L}\left(\underset{\sim}{i}_{\mathcal{L}_2}\right)\frac{d}{dt}\underset{\sim}{i}_{\mathcal{L}_2} = \underset{\sim}{v}_{\mathcal{L}_2}\left(\underset{\sim}{i}_{\mathcal{L}_2}\right) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2}\,\underset{\sim}{v}_{\mathcal{T}_2}\circ\left(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{T}_2}\underset{\sim}{i}_{\mathcal{L}_2}\right) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}\,\underset{\sim}{v}_{\mathcal{T}_1} \tag{20}$$

where $C\left(\underset{\sim}{v}_{\mathcal{T}_1}\right)$ and $L\left(\underset{\sim}{i}_{\mathcal{L}_2}\right)$ denote the incremental capacitance and inductance matrix, respectively, and where the symbol "$\circ$" denotes the "composition" operation. It follows from (19)-(20) that the state equations of any *complete network* can be written explicitly with $\underset{\sim}{v}_{\mathcal{T}_1}$ and $\underset{\sim}{i}_{\mathcal{L}_2}$ as the state variables; namely,

$$\boxed{\begin{aligned}\dot{\underset{\sim}{v}}_{\mathcal{T}_1} &= -\underset{\sim}{C}^{-1}\left(\underset{\sim}{v}_{\mathcal{T}_1}\right)\left\{-\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{T}_1}\,\underset{\sim}{i}_{\mathcal{L}_1}\circ\left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{T}_1}\underset{\sim}{v}_{\mathcal{T}_1}\right) + \underset{\sim}{i}_{\mathcal{T}_1}\left(\underset{\sim}{v}_{\mathcal{T}_1}\right) - \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{T}_1}\underset{\sim}{i}_{\mathcal{L}_2}\right\} &(21)\\ \dot{\underset{\sim}{i}}_{\mathcal{L}_2} &= -\underset{\sim}{L}^{-1}\left(\underset{\sim}{i}_{\mathcal{L}_2}\right)\left\{\underset{\sim}{v}_{\mathcal{L}_2}\left(\underset{\sim}{i}_{\mathcal{L}_2}\right) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2}\,\underset{\sim}{v}_{\mathcal{T}_2}\circ\left(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{T}_2}\underset{\sim}{i}_{\mathcal{L}_2}\right) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}\,\underset{\sim}{v}_{\mathcal{T}_1}\right\} &(22)\end{aligned}}$$

An examination of (21)-(22) shows that elements belonging to the same group may be coupled to each other. For example, all capacitors (resp., inductors) may be coupled to each other, and all resistors belonging to $\mathcal{L}_1$ $\left\{\text{resp., } \mathcal{L}_2, \mathcal{T}_1, \mathcal{T}_2\right\}$ may be coupled to each other. Moreover, the couplings need not even be *reciprocal* in the sense that the associated multi-terminal element or multiport resistors inside N need not be reciprocal. However, if all resistors inside N are reciprocal, then it is possible to express the state equations in terms of a *scalar* potential function. To derive this, let us observe that if we extract all capacitors inductors from the complete network $\mathcal{N}$, the resulting n-port resistor N has the following explicit hybrid representation in view of (1), (2), (8), (9), (21), and (22):

$$\underset{\sim}{i}_a = -\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{J}_1}\ \underset{\sim}{i}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_a\right) + \underset{\sim}{i}_{\mathcal{J}_1}(\underset{\sim}{v}_a) - \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1}\ \underset{\sim}{i}_b \triangleq \underset{\sim}{h}_a(\underset{\sim}{v}_a,\underset{\sim}{i}_b) \tag{23}$$

$$\underset{\sim}{v}_b = \underset{\sim}{v}_{\mathcal{L}_2}(\underset{\sim}{i}_b) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}\ \underset{\sim}{v}_{\mathcal{J}_2} \circ \left(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}_b\right) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\ \underset{\sim}{v}_a \triangleq \underset{\sim}{h}_b(\underset{\sim}{v}_a,\underset{\sim}{i}_b) \tag{24}$$

where we have defined $\underset{\sim}{i}_a \triangleq \underset{\sim}{i}_{\mathcal{J}_1}$, $\underset{\sim}{v}_a \triangleq \underset{\sim}{v}_{\mathcal{J}_1}$, $\underset{\sim}{i}_b \triangleq \underset{\sim}{i}_{\mathcal{L}_2}$ and $\underset{\sim}{v}_b \triangleq \underset{\sim}{v}_{\mathcal{L}_2}$. The *incremental hybrid matrix* associated with N is the Jacobian matrix of (23)-(24):

$$\underset{\sim}{H}(\underset{\sim}{v}_a,\underset{\sim}{i}_b) = \begin{bmatrix} \dfrac{\partial \underset{\sim}{i}_{\mathcal{J}_1}}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} + \underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{J}_1}\dfrac{\partial \underset{\sim}{i}_{\mathcal{L}_1}}{\partial \underset{\sim}{v}_{\mathcal{L}_1}}\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1} & -\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1} \\ \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} & \dfrac{\partial \underset{\sim}{v}_{\mathcal{L}_2}}{\partial \underset{\sim}{i}_{\mathcal{L}_2}} + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}\dfrac{\partial \underset{\sim}{v}_{\mathcal{J}_2}}{\partial \underset{\sim}{i}_{\mathcal{J}_2}}\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2} \end{bmatrix} \tag{25}$$

Notice that $\underset{\sim}{H}(\underset{\sim}{v}_a,\underset{\sim}{i}_b)$ is *not* symmetric since the off-diagonal blocks are negative transpose of each other, as expected in view of the Corollary of Theorem 10 in Sec. II-2. Hence, the hybrid representation $\underset{\sim}{h}(\cdot)$ of N is *not* a state function and we cannot define a potential function via its line integral. However, since the only thing that prevents $\underset{\sim}{H}(\underset{\sim}{v}_a,\underset{\sim}{i}_b)$ in (25) from being symmetric is the "negative sign" attached to $\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1}$, we can introduce a new variable $\underset{\sim}{i}^*_b = -\underset{\sim}{i}_b$ so that (23)-(24) becomes

$$\underset{\sim}{i}_a = -\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{J}_1}\ \underset{\sim}{i}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_a\right) + \underset{\sim}{i}_{\mathcal{J}_1}(\underset{\sim}{v}_a) + \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1}\ \underset{\sim}{i}^*_b = \underset{\sim}{h}^*_a(\underset{\sim}{v}_a,\underset{\sim}{i}^*_b) \tag{26}$$

$$\underset{\sim}{v}_b = \underset{\sim}{v}_{\mathcal{L}_2}(-\underset{\sim}{i}^*_b) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}\ \underset{\sim}{v}_{\mathcal{J}_2} \circ \left(-\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}^*_b\right) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\ \underset{\sim}{v}_a = \underset{\sim}{h}^*_b(\underset{\sim}{v}_a,\underset{\sim}{i}^*_b) \tag{27}$$

Observe that (26)-(27) now defines a *state function* $\underset{\sim}{h}^*(\underset{\sim}{v}_a,\underset{\sim}{i}^*_b)$ and we can now define a potential function via its line integral. To derive the explicit form of this potential function, let us consider first the simpler case where all elements in N are 2-terminal uncoupled resistors and define the following 4 scalar functions:[4]

$$\hat{\mathcal{G}}_{\mathcal{L}_1}\left(\underset{\sim}{v}_{\mathcal{L}_1}\right) \triangleq \sum_{\mathcal{L}_1}\int_0^{v_j} \hat{i}_j(x)\,dx \tag{28}$$

[4] Recall the definitions of *content* and *co-content* in Table 3 of sec. II-2.

$$\mathcal{G}^*_{\mathcal{L}_2}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right) \triangleq \sum_{\mathcal{L}_2}\int_0^{i_j^*} \hat{v}_j(-x)dx \tag{29}$$

$$\hat{\mathcal{G}}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) \triangleq \sum_{\mathcal{J}_1}\int_0^{v_j} \hat{i}_j(x)dx \tag{30}$$

$$\mathcal{G}^*_{\mathcal{J}_2}\left(\underset{\sim}{i}^*_{\mathcal{J}_2}\right) \triangleq \sum_{\mathcal{J}_2}\int_0^{i_j^*} \hat{v}_j(-x)dx \tag{31}$$

Observe that $\hat{\mathcal{G}}_{\mathcal{L}_1}\left(\underset{\sim}{v}_{\mathcal{L}_1}\right)$ and $\hat{\mathcal{G}}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)$ are just the sum of all *co-contents* of the resistors in $\mathcal{L}_1$ and $\mathcal{J}_1$ respectively. Similarly, except for the negative sign in (29) and (31), $\mathcal{G}^*_{\mathcal{L}_2}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ and $\mathcal{G}^*_{\mathcal{J}_2}\left(\underset{\sim}{i}^*_{\mathcal{J}_2}\right)$ are just the sum of all "conjugate" contents of the resistors in $\mathcal{L}_2$ and $\mathcal{J}_2$, respectively, where

$$G(i_j^*) \triangleq \int_0^{i_j^*} \hat{v}_j(-x)dx \tag{32}$$

is defined as the conjugate content associated with the constitutive relation $v_j = \hat{v}_j(i_j)$ of the jth resistor in $\mathcal{L}_2$ or $\mathcal{J}_2$. We are now ready to define the potential function associated with (26)-(27):

<u>Definition 4.</u> <u>Hybrid content of N</u>

We define the *hybrid content* of a *complete n-port resistor* N by

$$\boxed{\begin{aligned}\mathcal{H}\left(\underset{\sim}{v}_{\mathcal{J}_1},\underset{\sim}{i}^*_{\mathcal{L}_2}\right) \triangleq{}& \hat{\mathcal{G}}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}\right) + \hat{\mathcal{G}}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) + \mathcal{G}^*_{\mathcal{L}_2}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right) \\ &+ \mathcal{G}^*_{\mathcal{J}_2} \circ \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}^*_{\mathcal{L}_2}\Big) + \underset{\sim}{i}^{*T}_{\mathcal{L}_2}\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1} \qquad (33)\end{aligned}}$$

<u>Theorem 2.</u> <u>State Equations Via the Hybrid Content</u> [64].[5]

The hybrid content $\mathcal{H}\left(\underset{\sim}{v}_{\mathcal{J}_1},\underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ is a potential function associated <u>with the state function</u> $\underset{\sim}{h}^*\left(\underset{\sim}{v}_{\mathcal{J}_1},\underset{\sim}{i}^*_{\mathcal{L}_1}\right)$ defined in (26)-(27).[6] Moreover, the

[5] Comparing (34)-(35) with the state equation derived by Brayton and Moser [54], we can identify the hybrid content to be their *mixed potential*.

[6] Recall that $\underset{\sim}{v}_{\mathcal{J}_1} = \underset{\sim}{v}_a$ and $\underset{\sim}{i}^*_{\mathcal{L}_2} = \underset{\sim}{i}^*_b$.

state equations of any *complete* RLC network containing uncoupled 2-terminal resistors can be formulated explicitly via the hybrid content as follow:

$$\dot{v}_{\mathcal{J}_1} = -C^{-1}\left(v_{\mathcal{J}_1}\right) \frac{\partial \mathcal{H}\left(v_{\mathcal{J}_1}, i^*_{\mathcal{L}_2}\right)}{\partial v_{\mathcal{J}_1}} \tag{34}$$

$$\dot{i}^*_{\mathcal{L}_2} = L^{-1}\left(i^*_{\mathcal{L}_2}\right) \frac{\partial \mathcal{H}\left(v_{\mathcal{J}_1}, i^*_{\mathcal{L}_2}\right)}{\partial i^*_{\mathcal{L}_2}} \tag{35}$$

<u>Proof</u>. Taking the gradient of $\mathcal{H}\left(v_{\mathcal{J}_1}, i^*_{\mathcal{L}_2}\right)$, we obtain

$$\frac{\partial \mathcal{H}\left(v_{\mathcal{J}_1}, i^*_{\mathcal{L}_2}\right)}{\partial v_{\mathcal{J}_1}} = -B^T_{\mathcal{L}_1\mathcal{J}_1} i_{\mathcal{L}_1} \circ \left(-B_{\mathcal{L}_1\mathcal{J}_1} v_{\mathcal{J}_1}\right) + i_{\mathcal{J}_1}\left(v_{\mathcal{J}_1}\right) + B^T_{\mathcal{L}_2\mathcal{J}_1} i^*_{\mathcal{L}_2} \tag{36}$$

$$\frac{\partial \mathcal{H}\left(v_{\mathcal{J}_1}, i^*_{\mathcal{L}_2}\right)}{\partial i^*_{\mathcal{L}_2}} = v_{\mathcal{L}_2}\left(-i^*_{\mathcal{L}_2}\right) + B_{\mathcal{L}_2\mathcal{J}_2} v_{\mathcal{J}_2} \circ \left(-B^T_{\mathcal{L}_2\mathcal{J}_2} i^*_{\mathcal{L}_2}\right) + B_{\mathcal{L}_2\mathcal{J}_1} v_{\mathcal{J}_1} \tag{37}$$

Substituting (36) and (37) into (34) and (35) and replacing $i^*_{\mathcal{L}_2}$ with $-i_{\mathcal{L}_2}$, we obtain the state equations derived earlier in (21)-(22). ¤

Now consider the general case where the resistors are coupled to each other. Observe that so long as the couplings are reciprocal; i.e., all multi-terminal elements and multiports inside N are reciprocal, then the scalar functions defined in (28)-(31) can be generalized via *line integrals* of the respective constitutive relations.

It is interesting to observe that although the first 4 terms in the hybrid content $\mathcal{H}\left(v_{\mathcal{J}_1}, i^*_{\mathcal{L}_2}\right)$ are defined via the elements' constitutive relations, the last term is strictly a *topological* quantity determined only by the submatrix $B_{\mathcal{L}_2\mathcal{J}_1}$. This peculiar term has the following physical interpretation: partitioning the network $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ where $\mathcal{N}_1$ contains all branches in $\mathcal{L}_1 \cup \mathcal{J}_1$ and $\mathcal{N}_2$ contains all branches in

$\mathcal{L}_2 \cup \mathcal{T}_2$, then *the term* $\underset{\sim}{i}^{*T}_{\mathcal{L}_2} \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1} \underset{\sim}{v}_{\mathcal{T}_1}$ *is equal to the instantaneous power delivered from* $\mathcal{N}_1$ *to* $\mathcal{N}_2$ [77,81,82].

Finally, observe that (34)-(35) implies that the *stationary points* of the *hybrid content* $\mathcal{H}\left(\underset{\sim}{v}_{\mathcal{T}_1}, \underset{\sim}{i}^{*}_{\mathcal{L}_2}\right)$ are *equilibrium points* of the associated complete network $\mathcal{N}$. The significance of the hybrid content is that it plays a crucial role in determining whether all solutions of $\mathcal{N}$ will eventually settle at some equilibrium points [54,77].

B. State Equations Via the Lagrangian and Hamiltonian

Consider a connected network $\mathcal{N}$ containing uncoupled 2-terminal resistors, inductors, capacitors, dc voltage and current sources, and two types of controlled sources;[7] namely, voltage-controlled current sources (VCCS) and current-controlled voltage sources (CCVS). Assume that $\mathcal{N}$ contains no C-E loops and no L-J cut sets. Assume that there exists a *true* $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and an associated *cotree* $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ having the following properties:

1. All branches in $\mathcal{T}_1$ are voltage-controlled capacitors and all branches in $\mathcal{L}_2$ are current-controlled inductors.
2. Each branch in $\mathcal{L}_1$ forms a loop exclusively with branches in $\mathcal{T}_1$. Moreover, branches in $\mathcal{L}_1$ are restricted to *voltage-controlled resistors*, *flux-controlled inductors*, *dc current sources*, and *voltage-controlled current sources* whose controlling voltages are associated with elements in $\mathcal{T}_1 \cup \mathcal{L}_1$.
3. Branches in $\mathcal{T}_2$ are restricted to *current-controlled resistors*, *charge-controlled capacitors*, *dc voltage sources*, and *current-controlled voltage sources* whose controlling currents are associated with elements in $\mathcal{T}_2 \cup \mathcal{L}_2$.

Let us label the branches in $\mathcal{N}$ consecutively as follow:

1. Label the elements in $\mathcal{L}_1$ first. Start with the resistors ($R\mathcal{L}_1$), followed by the inductors ($L\mathcal{L}_1$) dc current sources ($J\mathcal{L}_1$), and voltage-controlled current sources ($J_c\mathcal{L}_1$).
2. Label the elements in $\mathcal{L}_2$ next. By assumption, all these elements are inductors ($L\mathcal{L}_2$).
3. Label the elements in $\mathcal{T}_1$ next. By assumption, all these elements are

[7] The other two types of controlled sources can also be included by modeling them with VCCS and CCVS.

capacitors ($C\mathcal{T}_1$).

4. Finally, label the elements in $\mathcal{T}_2$. Start with the resistors ($R\mathcal{T}_2$), followed by the capacitors ($C\mathcal{T}_2$), dc voltage sources ($E\mathcal{T}_2$), and current-controlled voltage sources ($E_C\mathcal{T}_2$).

Using the above labelling convention and notation, the branch voltage vector $\underset{\sim}{v}$ and the branch current vector $\underset{\sim}{i}$ of $\mathcal{N}$ may be partitioned as follow:

$$\underset{\sim}{v} = \left[\underset{\sim}{v}_{R\mathcal{L}_1} \; \underset{\sim}{v}_{L\mathcal{L}_1} \; \underset{\sim}{v}_{J\mathcal{L}_1} \; \underset{\sim}{v}_{J_C\mathcal{L}_1} \,\vdots\, \underset{\sim}{v}_{L\mathcal{L}_2} \,\vdots\, \underset{\sim}{v}_{C\mathcal{T}_1} \,\vdots\, \underset{\sim}{v}_{R\mathcal{T}_2} \; \underset{\sim}{v}_{C\mathcal{T}_2} \; \underset{\sim}{v}_{E\mathcal{T}_2} \; \underset{\sim}{v}_{E_C\mathcal{T}_2}\right]^T \tag{38}$$

$$\underset{\sim}{i} = \left[\underset{\sim}{i}_{R\mathcal{L}_1} \; \underset{\sim}{i}_{L\mathcal{L}_1} \; \underset{\sim}{i}_{J\mathcal{L}_1} \; \underset{\sim}{i}_{J_C\mathcal{L}_1} \,\vdots\, \underset{\sim}{i}_{L\mathcal{L}_2} \,\vdots\, \underset{\sim}{i}_{C\mathcal{T}_1} \,\vdots\, \underset{\sim}{i}_{R\mathcal{T}_2} \; \underset{\sim}{i}_{C\mathcal{T}_2} \; \underset{\sim}{i}_{E\mathcal{T}_2} \; \underset{\sim}{i}_{E_C\mathcal{T}_2}\right]^T \tag{39}$$

It follows from the preceding assumptions and labelling convention that the *fundamental loop matrix* with respect to the tree $\mathcal{T}$ always assumes the following special form:

$$\begin{array}{c} \begin{array}{ccccc|ccccc} R\mathcal{L}_1 & L\mathcal{L}_1 & J\mathcal{L}_1 & J_C\mathcal{L}_1 & L\mathcal{L}_2 & C\mathcal{T}_1 & R\mathcal{T}_2 & C\mathcal{T}_2 & E\mathcal{T}_2 & E_C\mathcal{T}_2 \end{array} \\ \underset{\sim}{B} = \left[\begin{array}{ccccc|ccccc} \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_{R_1C_1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_{L_1C_1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_{JC_1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{B}_{J_CC_1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{L_2C_1} & \underset{\sim}{B}_{L_2R_2} & \underset{\sim}{B}_{L_2C_2} & \underset{\sim}{B}_{L_2E_2} & \underset{\sim}{B}_{L_2E_C} \end{array}\right] \end{array} \tag{40}$$

<u>Definition 5</u>. <u>Energy and co-energy function</u>[8]

We define the *energy function* $\mathcal{E}$ and *co-energy function* $\hat{\mathcal{E}}$ as follow:

$$\mathcal{E} \triangleq W_{L_1}\left(\underset{\sim}{\phi}_{L\mathcal{L}_1}\right) + W_{C_2}\left(\underset{\sim}{q}_{C\mathcal{T}_2}\right) + \underset{\sim}{i}^T_{J\mathcal{L}_1}\,\underset{\sim}{\phi}_{J\mathcal{L}_1} + \underset{\sim}{q}^T_{E\mathcal{T}_2}\,\underset{\sim}{v}_{E\mathcal{T}_2} \tag{41}$$

$$\hat{\mathcal{E}} \triangleq \hat{W}_{L_2}\left(\underset{\sim}{i}_{L\mathcal{L}_2}\right) + W_{C_1}\left(v_{C\mathcal{T}_1}\right) \tag{42}$$

where $W_{L_1}\left(\phi_{L\mathcal{L}_2}\right)$ and $W_{C_2}\left(q_{C\mathcal{T}_2}\right)$ denote respectively the sum of the *inductor*

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[8] Recall the definitions of energy and co-energy in Table 3 of Section II-2.

energy of all inductors in $\mathcal{L}_1$ and the sum of the *capacitor energy* of all capacitors in $\mathcal{T}_2$ and where $\hat{W}_L\left(\underset{\sim}{i}_{L\mathcal{L}_2}\right)$ and $\hat{W}_{C_2}\left(q_{C\mathcal{T}_2}\right)$ denote respectively the sum of the inductor *co-energy* of all inductors in $\mathcal{L}_2$ and the sum of the *capacitor co-energy* of all capacitors in $\mathcal{T}_1$.

Using the special structure of the fundamental loop matrix $\underset{\sim}{B}$, it can be shown that the energy function $\mathcal{E}$ is a function of $\phi_{C\mathcal{T}_1}$ and $q_{L\mathcal{L}_2}$ only, and that the co-energy function $\hat{\mathcal{E}}$ is a function of $\underset{\sim}{v}_{C\mathcal{T}_1}$ and $\underset{\sim}{i}_{L\mathcal{L}_2}$ only [79].

<u>Definition 5.</u> <u>Lagrangian and General Force</u>

We define the *Lagrangian* $\mathcal{L}$ associated with $\mathcal{N}$ as follow:

$$\boxed{\begin{aligned}\mathcal{L}\left(\phi_{C\mathcal{T}_1},q_{L\mathcal{L}_2},\underset{\sim}{i}_{L\mathcal{L}_2},\underset{\sim}{v}_{C\mathcal{T}_1}\right) &\triangleq \hat{\mathcal{E}}\left(\underset{\sim}{i}_{L\mathcal{L}_2},\underset{\sim}{v}_{C\mathcal{T}_1}\right) - \mathcal{E}\left(\phi_{C\mathcal{T}_1},q_{L\mathcal{L}_2}\right) + \underset{\sim}{i}^T_{L\mathcal{L}_2}\,\underset{\sim}{B}_{L_2C_1}\,\phi_{C\mathcal{T}_1}\\
&= \hat{W}_{L_2}\left(\underset{\sim}{i}_{L\mathcal{L}_2}\right) + \hat{W}_{C_1}\left(\underset{\sim}{v}_{C\mathcal{T}_1}\right) - W_{L_1}\circ\left(-\underset{\sim}{B}_{L_1C_1}\phi_{C\mathcal{T}_1}\right) - W_{C_2}\circ\left(\underset{\sim}{B}^T_{L_2C_2}q_{L\mathcal{L}_2}\right)\\
&\quad + \underset{\sim}{i}^T_{J\mathcal{L}_1}\,B_{JC_1}\,\phi_{C\mathcal{T}_1} - q^T_{L\mathcal{L}_2}\,\underset{\sim}{B}_{L_2E}\,\underset{\sim}{v}_{E\mathcal{T}_2} + \underset{\sim}{i}_{L\mathcal{L}_2}\,\underset{\sim}{B}_{L_2C_1}\,\phi_{C\mathcal{T}_1} \qquad (43)\end{aligned}}$$

We define the *generalized force* $\underset{\sim}{F}$ associated with $\mathcal{N}$ as follow:

$$\underset{\sim}{F} \triangleq \begin{bmatrix}\underset{\sim}{B}^T_{R_1C_1} & \underset{\sim}{0}\\ \underset{\sim}{0} & -\underset{\sim}{B}_{L_2R_2}\end{bmatrix}\begin{bmatrix}\underset{\sim}{i}_{R\mathcal{L}_1}\\ \underset{\sim}{v}_{R\mathcal{T}_2}\end{bmatrix} + \begin{bmatrix}\underset{\sim}{B}^T_{J_CC_1} & \underset{\sim}{0}\\ \underset{\sim}{0} & -\underset{\sim}{B}_{L_2E_C}\end{bmatrix}\begin{bmatrix}\underset{\sim}{i}_{J_C\mathcal{L}_1}\\ \underset{\sim}{v}_{E_C\mathcal{T}_2}\end{bmatrix} \qquad (44)$$

Again, in view of the special structure of the fundamental loop matrix $\underset{\sim}{B}$, it can be shown that the generalized force $\underset{\sim}{F}$ is also a function of $v_{C\mathcal{T}_1}$ and $i_{L\mathcal{L}_2}$ only [79]. Hence, we can write $\underset{\sim}{F} = \underset{\sim}{F}\left(\underset{\sim}{v}_{C\mathcal{T}_1},\underset{\sim}{i}_{L\mathcal{L}_2}\right)$. We are now ready to formulate the Lagrangian equation associated with $\mathcal{N}$.

<u>Theorem 3.</u> <u>Lagrangian Equations of Motion [79]</u>

The solution of the nonlinear RLC network $\mathcal{N}$ satisfying the assumptions stipulated earlier is identical to the solution of the following Lagrangian equations of motion associated with $\mathcal{N}$:

$$\boxed{\frac{d}{dt}\left(\frac{\partial\mathcal{L}(\underset{\sim}{x},\dot{\underset{\sim}{x}})}{\partial\dot{\underset{\sim}{x}}} - \frac{\partial\mathcal{L}(\underset{\sim}{x},\dot{\underset{\sim}{x}})}{\partial\underset{\sim}{x}}\right) = \underset{\sim}{F}(\dot{\underset{\sim}{x}}) \qquad (45)}$$

where

$$\underset{\sim}{x} \triangleq \left[\underset{\sim}{\phi}_{C\mathcal{J}_1} \;\; \underset{\sim}{q}_{L\mathcal{L}_2}\right] \tag{46}$$

and

$$\dot{\underset{\sim}{x}} \triangleq \left[\underset{\sim}{v}_{C\mathcal{J}_1} \;\; \underset{\sim}{i}_{L\mathcal{L}_2}\right] \tag{47}$$

Under various simplifying assumptions, the Lagrangian and the generalized force take on particularly simple forms. Some of these special cases are:

1. <u>$\mathcal{N}$ contains no controlled sources.</u> In this case, we have

$$\underset{\sim}{F} = -\frac{\partial P(\dot{\underset{\sim}{x}})}{\partial \dot{\underset{\sim}{x}}} \tag{48}$$

where

$$P(\dot{\underset{\sim}{x}}) = \sum_{R\mathcal{L}_1} \int_0^{v_j} i_j(v_j)dv_j + \sum_{R\mathcal{J}_2} \int_0^{i_j} v_j(i_j)di_j \tag{49}$$

represents the sum of the *co-contents* of all resistors in $\mathcal{L}_1$, and the sum of the *contents* of all resistors in $\mathcal{T}_2$. Substituting (48) into (45), we obtain the following Lagrangian equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(\underset{\sim}{x},\dot{\underset{\sim}{x}})}{\partial \dot{\underset{\sim}{x}}}\right) - \frac{\partial \mathcal{L}(\underset{\sim}{x},\dot{\underset{\sim}{x}})}{\partial \underset{\sim}{x}} = -\frac{\partial P(\dot{\underset{\sim}{x}})}{\partial \dot{\underset{\sim}{x}}} \tag{50}$$

2. <u>Lossless networks</u>: <u>$\mathcal{N}$ contains no controlled sources and no resistors.</u> In this case, the right side of (50) is identically zero. Moreover, if $\mathcal{N}$ has a tree $\mathcal{T}$ made up of voltage-controlled capacitors and an associated cotree $\mathcal{L}$ made up of flux-controlled inductors, then (50) can be written explicitly by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \underset{\sim}{v}_{C\mathcal{J}_1}}\right) - \frac{\partial \mathcal{L}}{\partial \underset{\sim}{\phi}_{C\mathcal{J}_1}} = \underset{\sim}{0} \tag{51}$$

where the Lagrangian $\mathcal{L}$ is defined by

$$\mathcal{L}\left(\underset{\sim}{\phi}_{C\mathcal{J}_1}, \underset{\sim}{v}_{C\mathcal{J}_1}\right) = \hat{W}_{C_1}\left(\underset{\sim}{v}_{C\mathcal{J}_1}\right) - W_{L_1} \circ \left(-\underset{\sim}{B}_{L_1C_1}\underset{\sim}{\phi}_{C\mathcal{J}_1}\right) \tag{52}$$

On the other hand, if $\mathcal{N}$ has a tree $\mathcal{T}$ made up of charge-controlled capacitors and an associated cotree $\mathcal{L}$ made up of current-controlled inductors, then (50) assumes the form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \underset{\sim}{i}_{L\mathcal{L}_2}}\right) - \frac{\partial \mathcal{L}}{\partial q_{L\mathcal{L}_2}} = \underset{\sim}{0} \tag{53}$$

where the Lagrangian $\mathcal{L}$ is defined by

$$\mathcal{L}\left(\underset{\sim}{q}_{L\mathcal{L}_2}, \underset{\sim}{i}_{L\mathcal{L}_2}\right) = \hat{W}_{L_2}\left(\underset{\sim}{i}_{\mathcal{L}_2}\right) - W_{C_2} \circ \left(B^T_{L_2 C_2}\, q_{L\mathcal{L}_2}\right) \tag{54}$$

The Lagrangian equations of motion consist of a system of *second order* nonlinear differential equations. They can be transformed into state equations by a number of well-known methods. If the network is *lossless*, then by defining the *generalized moments* $\underset{\sim}{y}$ by

$$\underset{\sim}{y} \triangleq \frac{\partial \mathcal{L}(\underset{\sim}{x}\ \dot{\underset{\sim}{x}})}{\partial \dot{\underset{\sim}{x}}} \tag{55}$$

and assuming that (55) can be solved for $\dot{\underset{\sim}{x}}$ uniquely as a function of $\underset{\sim}{x}$ and $\underset{\sim}{y}$, we can define the following useful scalar function:

<u>Definition 6.</u> <u>Hamiltonian function</u>

We define the *Hamiltonian function* $H(\underset{\sim}{x},\underset{\sim}{y})$ associated with a *lossless network* $\mathcal{N}$ by

$$H(\underset{\sim}{x},\underset{\sim}{y}) \triangleq \underset{\sim}{y}^T\dot{\underset{\sim}{x}} - \mathcal{L}(\underset{\sim}{x},\dot{\underset{\sim}{x}}) \tag{56}$$

It can be shown that if all constitutive relations are bijective, then $\dot{\underset{\sim}{x}}$ in (56) can always be expressed as a unique function of $\underset{\sim}{x}$ and $\underset{\sim}{y}$. In this case, we can write

$$\underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{\phi}_{C\mathcal{J}_1} \\ \underset{\sim}{q}_{L\mathcal{L}_2} \end{bmatrix}, \quad \underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{q}_{C\mathcal{J}_1} \\ \underset{\sim}{\phi}_{L\mathcal{L}_2} + \underset{\sim}{B}_{L_2C_1}\underset{\sim}{\phi}_{C\mathcal{J}_1} \end{bmatrix} \triangleq \begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \end{bmatrix} \tag{57}$$

Hence

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} \underset{\sim}{v}_{C\mathcal{J}_1} \\ \underset{\sim}{i}_{L\mathcal{L}_2} \end{bmatrix} \triangleq \begin{bmatrix} \underset{\sim}{f}_1\left(\underset{\sim}{q}_{C\mathcal{J}_1}\right) \\ \underset{\sim}{f}_2(\underset{\sim}{\phi}_{L\mathcal{L}_2}) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_1(\underset{\sim}{y}_1) \\ \underset{\sim}{f}_2 \circ \left(\underset{\sim}{y}_2 - \underset{\sim}{B}_{L_2C_1}\underset{\sim}{x}_1\right) \end{bmatrix} \tag{58}$$

We are now ready to formulate the Hamiltonian equations of motion.

<u>Theorem 4.</u> <u>Hamiltanian Equations of Motion</u> [79]

The solution of a *lossless* network $\mathcal{N}$ satisfying the preceding assumptions is identical to the solution of the following Hamiltonian equations of motion associated with $\mathcal{N}$:

$$\boxed{\begin{aligned} \dot{\underset{\sim}{x}} &= \frac{\partial H(\underset{\sim}{x},\underset{\sim}{y})}{\partial \underset{\sim}{y}} && (59) \\ \dot{\underset{\sim}{y}} &= -\frac{\partial H(\underset{\sim}{x},\underset{\sim}{y})}{\partial \underset{\sim}{x}} && (60) \end{aligned}}$$

where the Hamiltonian function is defined explicitly by:

$$\boxed{\begin{aligned} H(\underset{\sim}{x},\underset{\sim}{y}) = {} & \underset{\sim}{y}_1^T\underset{\sim}{f}_1(\underset{\sim}{y}_1) + \underset{\sim}{y}_2^T\underset{\sim}{f}_2\circ\left(\underset{\sim}{y}_2-\underset{\sim}{B}_{L_2C_1}\underset{\sim}{x}_1\right) - \hat{W}_{L_2}\circ\left(\underset{\sim}{f}_2\left(\underset{\sim}{y}_2-\underset{\sim}{B}_{L_2C_1}\underset{\sim}{x}_1\right)\right) \\ & - \hat{W}_{C_1}\circ\left(\underset{\sim}{f}_1\left(\underset{\sim}{y}_1\right)\right) + W_{L_1}\circ\left(-\underset{\sim}{B}_{L_1C_1}\underset{\sim}{x}_1\right) + W_{C_2}\circ\left(\underset{\sim}{B}^T_{L_2C_2}\underset{\sim}{x}_2\right) \\ & - \underset{\sim}{i}^T_{J\mathcal{L}_1}\underset{\sim}{B}_{JC_1}\underset{\sim}{x}_1 + \underset{\sim}{x}_2^T\underset{\sim}{B}_{L_2E}\underset{\sim}{v}_{E\mathcal{J}_2} - \underset{\sim}{f}_2^T\circ\left(\underset{\sim}{y}_2-\underset{\sim}{B}_{L_2C_1}\underset{\sim}{x}_1\right)\underset{\sim}{B}_{L_2C_1}\underset{\sim}{x}_1 \end{aligned}} \tag{61}$$

The above expression can be further simplified under additional assumptions. In particular, if $\mathcal{N}$ contains no dc sources and if all constitutive relations are bijective, then applying the integration-by-parts-formula, we obtain the following simplified expression:

$$H(\underset{\sim}{x},\underset{\sim}{y}) = W_{C_1}(\underset{\sim}{y}_1) + W_{C_2}\circ\left(\underset{\sim}{B}^T_{L_2C_2}\underset{\sim}{x}_2\right) + W_{L_1}\circ\left(-\underset{\sim}{B}_{L_1C_1}\underset{\sim}{x}_1\right) + W_{L_2}\circ\left(\underset{\sim}{y}_2-\underset{\sim}{B}_{L_2C_1}\underset{\sim}{x}_1\right) \tag{62}$$

Observe that the Hamiltonian in this case is just the sum of the *energy* of all inductors and capacitors in $\mathcal{N}$. The significance of the Hamiltonian is that it plays a crucial role in determining the qualitative properties of nonlinear *lossless networks*.

We close this section with the remark that the expression for the *Lagrangian* in (43) and the *Hamiltonian* in (61) are derived for the case where there are no loop of inductors in $L\mathcal{L}_1$ and no cut set of capacitors in $C\mathcal{J}_2$. In the general case, the expressions will involve also *initial conditions* as in [79].

2. Qualitative Properties of Autonomous Networks.

Our objective in this section, as well as in the next section on *non-autonomous* networks, is to identify various classes of nonlinear RLC networds which shared certain common qualitative properties. Two types of results will be presented: the first involves only the two functions defining the state equations $\left(\underset{\sim}{D}^{-1}(\underset{\sim}{x})\right.$ and $\left.\underset{\sim}{h}(\underset{\sim}{x};\underset{\sim}{u}_S)\right)$ in (15), or $\underset{\sim}{h}(\underset{\sim}{x};\underset{\sim}{u}_S)$ and $\underset{\sim}{g}(\underset{\sim}{z})$, in (17) whereas the second involves only the constitutive relations of the internal elements and their interconnections. The first type of results are more general but requires the state equations be formulated first before the hypotheses could be checked. The second type of results are slightly less general but they are *explicit* in the sense that they usually involve only graph and circuit-theoretic conditions which can often be checked by inspection. Due to limitation of space, only the more basic results will be presented. The reader is referred to a series of recent papers [69-68,77] for the proofs of most of these results, as well as for many additional theorems on qualitative properties.

Since only *autonomous* networks are considered in this section, the state equations that concerned us here are given by (8)-(9) and (10)-(11), or more compactly, by (15) and (17), where the source vector $\underset{\sim}{u}_S$ is a *constant* vector. For simplicity, we will henceforth suppress $\underset{\sim}{u}_S$ and simply write (15) and (17) as follows:[9]

$$\boxed{\dot{\mathbf{x}} = -\mathbf{D}^{-1}(\mathbf{x})\mathbf{h}(\mathbf{x}) \triangleq \mathbf{f}(\mathbf{x})} \qquad (63)$$

$$\boxed{\dot{\mathbf{z}} = -\mathbf{h}\left(\mathbf{g}(\mathbf{z})\right) \triangleq \mathbf{f}(\mathbf{z})} \qquad (64)$$

where $\mathbf{x} = \mathbf{g}(\mathbf{z})$ denotes the capacitor — inductor constitutive relation, $\mathbf{x} \triangleq [\mathbf{v}_C\ \mathbf{i}_L]^T = [\mathbf{v}_a\ \mathbf{i}_b]^T$ and $\mathbf{z} \triangleq [\mathbf{q}_C\ \boldsymbol{\phi}_L]^T$, $\mathbf{y} = \mathbf{h}(\mathbf{x})$ denotes the constitutive relation of the n-port resistor N in Fig. 26, $\mathbf{y} \triangleq [\mathbf{i}_a\ \mathbf{v}_b]^T$.

A. Local Asymptotic Stability at Equilibrium Points

Let $\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right)$ be an equilibrium point of the autonomous system (63). The Jacobian matrix of $\mathbf{f}$ evaluated at this point is given by:[10]

$$\mathbf{A}\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right) = -\mathbf{D}^{-1}\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right)\mathbf{H}\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right) \qquad (65)$$

where $\mathbf{D}\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right)$ is defined by (16), and $\mathbf{H}\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right)$ is the *incremental hybrid matrix* associated with $\mathbf{h}(\cdot)$. To investigate the local stability property of this equilibrium point, we make use of the following lemma:

Lemma [83]

If $\mathbf{D}$ is a real symmetric positive definite matrix and if $\mathbf{H}$ is a real positive definite matrix, then the *real parts* of all eigenvalues of $\mathbf{DH}$ are positive.

Observe that $\mathbf{H}$ in this lemma need not be symmetric. Even if $\mathbf{H}$ is symmetric, $\mathbf{DH}$ may still not be symmetric. Observe also that $\mathbf{DH}$ need not be positive definite if H is not symmetric. We are now ready to state our next theorem.

Theorem 5. Local Asymptotic Stability Criterion [83].

Assuming all partial derivatives of the autonomous system (63) are continuous in a neighborhood of an equilibrium point Q : $\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right)$, then Q is *locally asymptotically stable* if $\mathbf{D}\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right)$ is symmetric and positive definite and if $\mathbf{H}\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right)$ is positive definite.

Proof. Since $\mathbf{D}\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right)$ is symmetric positive definite, its inverse exists and is also positive definite. The theorem then follows immediately from the above lemma and a standard result on local asymptptic stability [84].

[9] For simplicity, we use $\mathbf{f}(\cdot)$ in both (63) and (64) for two different functions. No confusion should arise, however, since only one euation will be used in any given context.

[10] Note that the Jacobian matrix of $\mathbf{f}(\cdot)$ has a second term which vanishes at $\left(\mathbf{v}_{C_Q}, \mathbf{i}_{L_Q}\right)$.

Corollary 1.

Assuming all partial derivatives of the autonomous system (63) are continuous in a neighborhood of an equilibrium point Q, then Q is *locally asymptotically stable* if the inductors and capacitors are *reciprocal* and *strictly locally passive* at Q, and if the n-port resistor N is Fig. 26 is *strictly locally passive* at Q.

Corollary 2 [64].

If the n-port resistor N in Fig. 26 is *reciprocal* then N is *strictly locally passive at Q* if, and only if, the associated n_C-port resistor N_C and the n_L-port resistor N_L in Fig. 28 are *strictly locally passive at Q.*

To apply Corollary 2, one must first solve for the location of the equilibrium points and then test whether the following two matrices are positive definite:

1. $n_C x n_C$ *incremental conductance matrix of* N_C

$$\underset{\sim}{G}(\underset{\sim}{v}_a) \triangleq \left.\frac{\partial \underset{\sim}{i}_a}{\partial \underset{\sim}{v}_a}\right|_{\underset{\sim}{v}_a=\underset{\sim}{v}_{a_Q},\ \underset{\sim}{i}_b=\underset{\sim}{i}_{b_Q}} \qquad (66)$$

2. $n_L x n_L$ *incremental resistance matrix of* N_L

$$\underset{\sim}{R}(\underset{\sim}{i}_b) = \left.\frac{\partial \underset{\sim}{v}_b}{\partial \underset{\sim}{i}_b}\right|_{\underset{\sim}{v}_a=\underset{\sim}{v}_{a_Q},\ \underset{\sim}{i}_b=\underset{\sim}{i}_{b_Q}} \qquad (67)$$

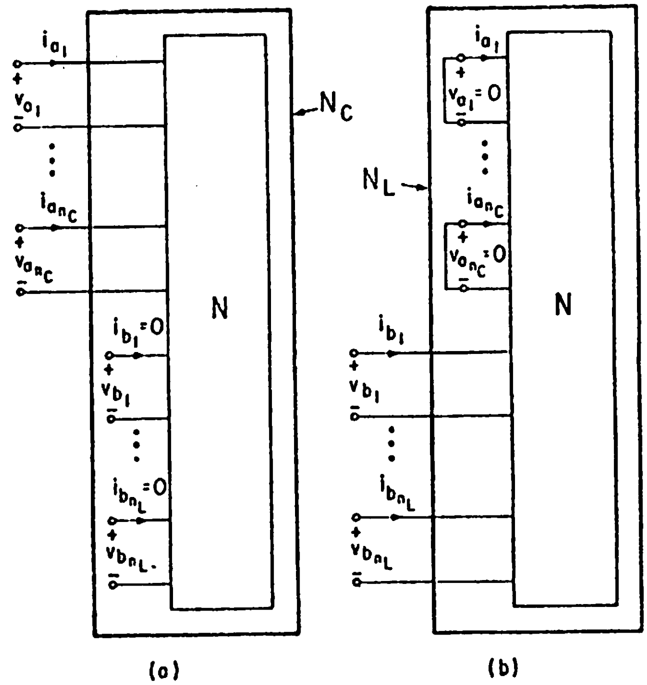

Fig. 28. (a) An n_C-port resistor N_C obtained by open-circuiting all inductor ports in Fig. 26.
(b) An n_L-port resistor N_L obtained by short-circuiting all capacitor ports in Fig. 26.

In the special case where *all* elements inside N are strictly locally passive, one could obviate the above test if the overall n-port N is also strictly locally passive. Unfortunately, it is easy to find counter-examples showing that N need not be strictly locally passive even if all elements inside N are strictly locally passive [70]; i.e., strict local passivity does not possess the closure property. However, by introducing a simple topological condition, we can prove the following result.

Theorem 6. Explicit Local Asymptotic Stability Criterion [28]

Assuming all partial derivatives of the autonomous system (63) are continuous in a neighborhood of an equilibrium point $Q: \left(\underset{\sim}{v}_{C_Q}, \underset{\sim}{i}_{L_Q}\right)$, then Q is *locally asymptotically stable* if the following hold:

(1) Fundamental topological hypothesis: *There is no loop (resp. no cut set) formed exclusively by capacitors, inductors, and/or dc voltage sources (resp., current sources).*

(2) All multi-terminal and multi-port capacitors and inductors are *reciprocal* and *strictly locally passive at Q*.

(3) All multi-terminal and multi-port resistors are *strictly locally passive at Q*.

B. Global Asymptotic Stability at Equilibrium Point

The preceding asymptotic stability criteria are *local* results and the circuit could still oscillate or display other complex motions even if all equilibrium points are locally asymptotically stable. Our next theorem is a *global* result which guarantees that all trajectories must eventually tend to a unique equilibrium point.

Theorem 7a. Global Asymptotic Stability Criterion [71]

Let an RLC network $\mathcal{N}$ be described by the autonomous state equation (64) and assume that the capacitor-inductor constitutive relation $g(\cdot)$ is a C^1 *strictly increasing state function* mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$. Then we have the following properties:

1. If the constitutive relation $\underset{\sim}{h}(\cdot)$ of the n-port resistor N is *strictly passive with respect to some* $\underset{\sim}{x}^* \in \mathbb{R}^n$ in the sense that

$$(\underset{\sim}{x}-\underset{\sim}{x}^*)^T \underset{\sim}{h}(\underset{\sim}{x}) > 0, \qquad \forall \underset{\sim}{x} \in \mathbb{R}^n, \quad \underset{\sim}{x} \neq \underset{\sim}{x}^* \tag{68}$$

then $\mathcal{N}$ has a *unique* equilibrium point $\underset{\sim}{z}^* \triangleq \underset{\sim}{g}^{-1}(\underset{\sim}{x}^*)$. Moreover, $\underset{\sim}{z}^*$ is *globally asymptotically stable.*

2. If the constitutive relation $\underset{\sim}{h}(\cdot)$ of the n-port resistor N is a *strictly increasing homeomorphism* mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$, then there exists a *unique* $\underset{\sim}{x} \in \mathbb{R}^n$ such that $\underset{\sim}{h}(\underset{\sim}{x}^*) = \underset{\sim}{0}$ and $\underset{\sim}{z}^* \triangleq \underset{\sim}{g}^{-1}(\underset{\sim}{x}^*)$ is a *globally asymptotically stable* equilibrium point.

Our next theorem provides some more explicit criteria.

Theorem 7b. Explicit Global Asymptotic Stability Criterion [71].

Let an RLC network $\mathcal{N}$ be described by the autonomous state equation (64), and assume that the capacitor-inductor constitutive relation $\underset{\sim}{g}(\cdot)$ is a C^1 *strictly increasing state function* mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$. Assume that $\mathcal{N}$ contains *no CE cut sets, no LJ loops,* and *no loops and cut sets made up of both capacitors and inductors and/or dc sources.*[11] Then we have the following properties:

1. If $\mathcal{N}$ contains *no dc sources* and if each multi-terminal or multiport resistor in $\mathcal{N}$ is *strictly passive,* then $\mathcal{N}$ has a *unique* equilibrium point $\underset{\sim}{z}^* = \underset{\sim}{g}^{-1}(\underset{\sim}{0})$. Moreover, $\underset{\sim}{z}^*$ is *globally asymptotically stable.*

2. If each multi-terminal or multi-port resistor R_α in $\mathcal{N}$ is *strictly passive,* and if *every loop* of $\mathcal{N}$ containing a dc voltage source also contains a capacitor, and *every cut set* of $\mathcal{N}$ containing a dc current source also contains an inductor, then $\mathcal{N}$ has a *unique* equilibrium point $\underset{\sim}{z}^* \in \mathbb{R}^n$. Moreover, $\underset{\sim}{z}^*$ is *globally asymptotically stable.*

---

[11] $\mathcal{N}$ may contain CE loops and LJ cut sets.

3. If each multi-terminal or multi-port resistor R_α in $\mathcal{N}$ is characterized by a *strictly increasing homeomorphic* function $h_\alpha(\cdot)$ mapping $\mathbb{R}^{n_\alpha}$ *onto* $\mathbb{R}^{n_\alpha}$ then $\mathcal{N}$ has a *unique* equilibrium point $\underset{\sim}{z}^* \in \mathbb{R}^{\tilde{n}}$. Moreover, $\underset{\sim}{z}^*$ is *globally asymptotically stable.*[12]

The hypotheses in Theorems 6 and 7 are sufficient but *not necessary* for global asymptotic stability. For example, the linear circuit shown in Fig. 29(a) has a globally asymptotically stable equilibrium point; namely, the origin. Yet, it is easy to verify that the hypotheses of Theorem 6 are violated because the constitutive relation $\underset{\sim}{h}(\cdot)$ of the associated 2-port resistor N is passive, but *not strictly* passive with respect to the origin. Moreover, N is increasing, but *not strictly* increasing. Similarly, the hypotheses of Theorem 7 are violated because $\mathcal{N}$ contains a loop made up of a capacitor and an inductor. Hence neither theorem can be used to show that the origin is globally asymptotically stable in this case. This example clearly demonstrates that loops and cut sets made up of both capacitors and inductors may be allowed in certain cases. On the other hand, the linear circuit shown in Fig. 29(b) can

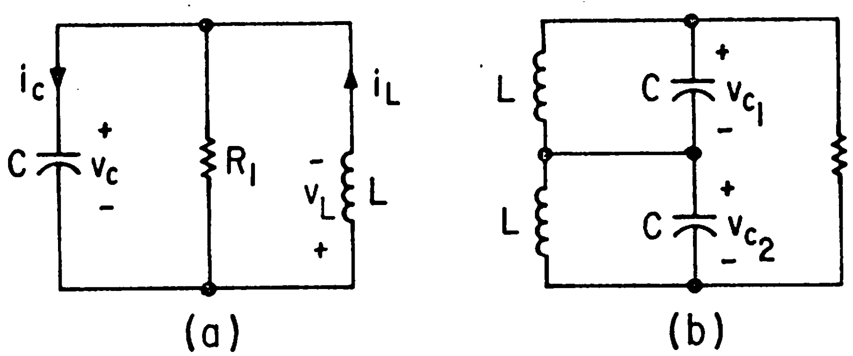

Fig. 29. (a) The origin is a globally asymptotically stable equilibrium point for this network. (b) The origin is an unstable equilibrium point for this oscillatory network.

support a non-trivial periodic solution in view of the presence of the capacitor-inductor loops and cut set. To distinguish these two "qualitatively" distinct networks requires additional topological conditions of a rather subtle nature:

Theorem 8. Explicit Global Asymptotic Stability Criterion [71]

Let an RLC network $\mathcal{N}$ be described by the autonomous state equation (64), and assume that the capacitor-inductor constitutive relation $g(\cdot)$ is a C^1 *strictly increasing diffeomorphic state function* mapping $\mathbb{R}^{\tilde{n}}$ *onto* $\mathbb{R}^{\tilde{n}}$. Assume that $\mathcal{N}$ contains *no CE loops* and *no LJ cut sets*. Assume further that the following *inductor-capacitor loop-cut set hypothesis (LC Hypothesis)* is satisfied:

Let $\mathcal{S}$ be *any* subset of capacitors and inductors in $\mathcal{N}$ such that any capacitor or inductor in $\mathcal{S}$ forms a loop and/or cut set exclusively with any combination of dc voltage and current sources, and other capacitors and

[12] $\mathcal{N}$ may contain dc sources in this case. The internal resistors may have any number of terminals and ports. We let R_α denote an n_α-terminal or an n_α-port resistor.

inductors of $\mathcal{S}$. For *each and every* such set $\mathcal{S}$, assume that at least one of the following conditions is satisfied:

a) There is a capacitor C_j in $\mathcal{S}$ which is not in a cut set formed exclusively with any combination of current sources and elements of $\mathcal{S}$. Moreover, this capacitor is not coupled to any other capacitor of $\mathcal{S}$.

b) There is an inductor L_j in $\mathcal{S}$ which is not in a loop formed exclusively with any combination of voltage sources and elements of $\mathcal{S}$. Moreover, this inductor is not coupled to any other inductor of $\mathcal{S}$.

Under the above condition, we have the following properties:

1. If $\mathcal{N}$ contains *no sources* and if each multi-terminal or multi-port resistor of $\mathcal{N}$ is *strictly passive*, then $\mathcal{N}$ has *unique* equilibrium point $\underset{\sim}{z}^* = g^{-1}(0)$ which is *globally asymptotically stable*, provided all voltage and current solution waveforms of $\mathcal{N}$ are C^1 functions of t.

2. If the capacitor-inductor constitutive relation $g(\cdot)$ is a C^3-function, and if each multi-terminal or multiport resistor in $\mathcal{N}$ is characterized by a C^3 strictly increasing diffeomorphism mapping $\mathbb{R}^{n_\alpha}$, then $\mathcal{N}$ has a *unique* equilibrium point $\underset{\sim}{z}^* \in \mathbb{R}^n$. Moreover, $\underset{\sim}{z}^*$ is *globally asymptotically stable*.

Observe that the preceding *LC Hypothesis* requires that *all possible* subsets of capacitors and inductors which qualify as $\mathcal{S}$ must be tested. The circuit shown in Fig. 29(a) is easily seen to satisfy this hypothesis since $\mathcal{S}$ in this case consists of the single capacitor C and inductor L, and since the capacitor C does not form a cut set with L. It follows from Theorem 8 that the origin is globally asymptotically stable. On the other hand, the "LC Hypothesis" is clearly violated by the circuit in Fig. 29(b). Indeed, if we choose $\mathcal{S}$ to consist of the two capacitors *and* the two inductors, then both a) and b) of this hypothesis are violated.

C. Exponential Decay of Transients to the Globally Asymptotically Stable Equilibrium Point.

The class of globally asymptotically stable networks studied in the preceding section behaves qualitatively like a *linear* network in many respects. In particular, it has a *unique* equilibrium point and all trajectories approach it regardless of the initial condition. Our next theorem shows that it is even possible to bound each trajectory by two *exponential* waveforms with a time constant τ_{max} and τ_{min}, respectively.

Theorem 9. Exponential Transient Decay Property [71]

Let an RLC network $\mathcal{N}$ be described by the autonomous state equation (64) and assume that the capacitor-inductor constitutive relation $g(\cdot)$ is a *C^1 strictly increasing diffeomorphic state function* mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$. Let the constitutive relation $\underset{\sim}{h}(\cdot)$ of the n-port resistor N be C^1 and *strictly passive with respect to some* $\underset{\sim}{x}^* \in \mathbb{R}^n$ in the sense of (68), and assume that N is *strictly locally passive* at $\underset{\sim}{x}^*$.[13] Then we have the following properties.

[13] That is, $\partial\underset{\sim}{h}(\underset{\sim}{x}^*)/\partial\underset{\sim}{x}$ is positive definite.

1. All solutions $z(t)$ tend to the *unique* equilibrium point $z^* = g^{-1}(x^*)$.

2. Let $D \subset \mathbb{R}^n$ be any convex and compact set such that $z(t) \in D\ \forall t \geq 0$. Then one can find constants $\bar{\gamma}_g \geq \underline{\gamma}_g > 0$ and $\bar{\gamma}_h \geq \underline{\gamma}_h > 0$ such that for all z' and z'' belonging to D, the following basic inequalities hold:[14]

$$\underline{\gamma}_g \|z'-z''\|^2 \leq (z'-z'')^T\left[g(z')-g(z'')\right] \leq \bar{\gamma}_g \|z'-z''\|^2 \tag{69}$$

$$\underline{\gamma}_h \|g(z')-x^*\|^2 \leq \left[g(z')-x^*\right]^T\left[h\left(g(z')\right)\right] \leq \bar{\gamma}_h \|g(z')-x^*\|^2 \tag{70}$$

3. If we define the two *time constants*

$$\tau_{min} \triangleq \frac{\underline{\gamma}_g}{\bar{\gamma}_g^2 \bar{\gamma}_h} \quad , \quad \tau_{max} \triangleq \frac{\bar{\gamma}_g}{\underline{\gamma}_g^2 \underline{\gamma}_h} \tag{71}$$

where $\bar{\gamma}_g$, $\underline{\gamma}_g$, $\bar{\gamma}_h$ and $\underline{\gamma}_h$ are the constants obtained from (69) and (70), then each trajectory $z(t)$ with initial condition $z(0)$ tends to z^* *exponentially* in the sense that

$$K_1 e^{-(t/\tau_{min})} \leq \|z(t)-z^*\| \leq K_2 e^{-(t/\tau_{max})} \tag{72}$$

where

$$K_1 \triangleq (\underline{\gamma}_g/\bar{\gamma}_g)^{1/2} \|z(0)-z^*\| \ , \ K_2 \triangleq (\bar{\gamma}_g/\underline{\gamma}_g)^{1/2} \|z(0)-z^*\| \tag{73}$$

D. Complete Stability

The results in Sections C and D are valid only for circuits having a unique equilibrium state. For circuits having *multiple* equilibrium states, such as flip flops, one must settle for a weaker form of stability which we define next.

Definition 7. Completely stable networks

An autonomous RLC network $\mathcal{N}$ is said to be *completely stable* if each trajectory tends to an equilibrium point of the associated state equation.

Theorem 10. Completely stable RC networks [71]

An autonomous RC network characterized by the state equation

$$\dot{v}_C = C^{-1}(v_C)h(v_C) \tag{74}$$

is *completely stable* if:

1. The incremental capacitance matrix $C(v_C)$ is positive definite.

2. The associated n-port resistor N is *reciprocal, voltage-controlled,*

[14] The inequality (69) is equivalent to requiring that $g(\cdot)$ be a *strongly uniformly increasing* function.

and its *co-content*

$$\hat{\mathcal{G}}(\underset{\sim}{v}) \triangleq \int_0^{v} \underset{\sim}{h}(\underset{\sim}{v}) \cdot d\underset{\sim}{v} \to \infty \quad \text{as } \|\underset{\sim}{v}\| \to \infty \tag{75}$$

<u>Theorem 11</u>. <u>Completely Stable RL network</u> [71].

An autonomous RL network characterized by the state equation

$$\dot{\underset{\sim}{i}}_L = \underset{\sim}{L}^{-1}(\underset{\sim}{i}_L)\underset{\sim}{h}(\underset{\sim}{i}_L) \tag{76}$$

is *completely stable* if:

1. The incremental inductance matrix $\underset{\sim}{L}(\underset{\sim}{i}_L)$ is positive definite.

2. The associated n-port resistor N is *reciprocal, current-controlled,* and its *content*

$$\mathcal{G}(\underset{\sim}{i}) \triangleq \int_0^{i} \underset{\sim}{h}(\underset{\sim}{i}) \cdot d\underset{\sim}{i} \to \infty \quad \text{as } \|\underset{\sim}{i}\| \to \infty \tag{77}$$

<u>Theorem 12</u>. <u>Complete Stability Criterion 1 for Complete RLC Networks</u> [54].

A *complete* RLC network described by the state equations (34)-(35) is *completely stable* if the following conditions are satisfied:[15]

1. The incremental capacitance matrix $\underset{\sim}{C}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)$ and the incremental inductance matrix $\underset{\sim}{L}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ are symmetric and positive definite.

2. All resistors in $\mathcal{J}_1$ and $\mathcal{L}_1$ are voltage-controlled.

3. All elements in $\mathcal{J}_2$ are *linear positive* resistors, i.e., $\underset{\sim}{v}_{\mathcal{J}_2} = \underset{\sim}{R}_{\mathcal{J}_2}\underset{\sim}{i}_{\mathcal{J}_2}$, where $R_{\mathcal{J}_2}$ is a positive definite diagonal matrix.

4. Any element in series with an inductor in each "inductive" composite branch in $\mathcal{L}_2$ is a dc voltage source.

5. The topological submatrix $B_{\mathcal{L}_2\mathcal{J}_2}$ has maximal *row rank*.

6. $$\left\|\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}\right\| + \hat{\mathcal{G}}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) + \hat{\mathcal{G}}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}\right) \to \infty, \text{ as } \|\underset{\sim}{v}_{\mathcal{J}_1}\| \to \infty \tag{78}$$

7. $$\|\underset{\sim}{K}\|^2 \triangleq \left\|\underset{\sim}{C}^{-1/2}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{R}^{-1}\underset{\sim}{L}^{1/2}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right)\right\|^2 < 1-\delta,\ \delta > 0 \tag{79}$$

where $\underset{\sim}{R} \triangleq \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{R}_{\mathcal{J}_2}\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}$, $\underset{\sim}{C}^{-1/2}(\cdot)\underset{\sim}{C}^{-1/2}(\cdot) = \underset{\sim}{C}^{-1}(\cdot)$ and $\underset{\sim}{L}^{1/2}(\cdot)\underset{\sim}{L}^{1/2}(\cdot) = \underset{\sim}{L}(\cdot)$, and where $\|\underset{\sim}{K}\|$ denotes any convenient induced *norm* of the matrix $\underset{\sim}{K}$.

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[15] Definitions 3 and 7 are unrelated even though the word "complete" appears in both. All symbols in this theorem have been defined earlier in Sec. IV-1-A.

The matrix $\underset{\sim}{R}$ in (79) may be *singular* for some complete network. Moreover (78) may not be easy to verify if the voltage-controlled resistors in $\mathcal{T}_1$ and $\mathcal{L}_1$ are coupled to each other. Our next theorem overcomes these two objections:

Theorem 13. Complete Stability Criterion 2 for Complete RLC Networks [77].

A *complete* RLC network described by the state equations (34)-(35) is *completely stable* if the following conditions are satisfied:

1. The incremental capacitance matrix $\underset{\sim}{C}\left(\underset{\sim}{v}_{\mathcal{T}_1}\right)$ and the incremental inductance matrix $\underset{\sim}{L}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ are symmetric and positive definite.

2. All resistors in $\mathcal{T}_1$ and $\mathcal{L}_1$ are voltage-controlled which may be coupled to each other so long as the couplings are reciprocal.

3. All elements in $\mathcal{T}_2$ are *linear, reciprocal* and *strictly passive* resistors; i.e., $\underset{\sim}{v}_{\mathcal{T}_2} = \underset{\sim}{R}_{\mathcal{T}_2}\,\underset{\sim}{i}_{\mathcal{T}_2}$ where $R_{\mathcal{T}_2}$ is a positive definite symmetric matrix.

4. Any element in series with an inductor in each "inductive" composite branch in $\mathcal{L}_2$ is a dc voltage source.

5. $\mathcal{R}\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}\right) \subset \mathcal{R}\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2}\right)$ (80)

when $\mathcal{R}(A)$ denotes the *range space* of $\underset{\sim}{A}$.

6. Let $\underset{\sim}{R} \triangleq \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2}\,\underset{\sim}{R}_{\mathcal{T}_2}\,\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{T}_2}$ and let $\underset{\sim}{R}^I$ denote the *generalized inverse* of R [77], then

$$\|\underset{\sim}{K}\|^2 \triangleq \left\| \underset{\sim}{L}^{\frac{1}{2}}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right)\underset{\sim}{R}^I\,\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}\,\underset{\sim}{C}^{-\frac{1}{2}}\left(\underset{\sim}{v}_{\mathcal{T}_1}\right)\right\|^2 < 1-\delta\ , \quad \delta > 0 \qquad (81)$$

where $\|\underset{\sim}{K}\|$ denotes any convenient induced *norm* of the matrix $\underset{\sim}{K}$.

7. The network has at least one equilibrium point.

8. All solutions of (34)-(35) are bounded.

It is easy to see that Theorem 13 is a generalization of Theorem 12. In particular, if $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2}$ has maximal row rank, then (80) is satisfied so that $\underset{\sim}{R}$ is non-singular and $\underset{\sim}{R}^I = \underset{\sim}{R}^{-1}$.

Interchanging the roles of capacitors and inductors, the following "dual" of Theorem 13 is easily obtained.

Theorem 14. Complete Stability Criterion 3 for Complete RLC Networks [77].

A *complete* RLC network described by the state equations (34)-(35) is *completely stable* if the following conditions are satisfied:

1. The incremental capacitance matrix $\underset{\sim}{C}\left(\underset{\sim}{v}_{\mathcal{T}_1}\right)$ and the incremental

inductance matrix $\underset{\sim}{L}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ are symmetric and positive definite.

2. All resistors in $\mathcal{T}_2$ and $\mathcal{L}_2$ are current-controlled which may be coupled to each other so long as the couplings are reciprocal.

3. All elements in $\mathcal{L}_1$ are *linear, reciprocal* and *strictly passive* resistors, i.e., $\underset{\sim}{i}_{\mathcal{L}_1} = \underset{\sim}{G}_{\mathcal{L}_1} \underset{\sim}{v}_{\mathcal{L}_1}$ where $\underset{\sim}{G}_{\mathcal{L}_1}$ is a positive definite symmetric matrix.

4. Any element in parallel with a capacitor in each "capacitive" composite branch in $\mathcal{T}_1$ is a dc current source.

5. $$\mathcal{R}\left(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{T}_1}\right) \subset \mathcal{R}\left(\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{T}_1}\right) \tag{82}$$

6. Let $\underset{\sim}{G} \triangleq \underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{T}_1} \underset{\sim}{G}_{\mathcal{L}_1} \underset{\sim}{B}_{\mathcal{L}_1\mathcal{T}_1}$ and let $\underset{\sim}{G}^I$ denote the generalized inverse of $\underset{\sim}{G}$, then

$$\|\underset{\sim}{S}\|^2 \triangleq \left\| \underset{\sim}{C}^{\frac{1}{2}}\left(\underset{\sim}{v}_{\mathcal{T}_1}\right) \underset{\sim}{G}^I \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1} \underset{\sim}{L}^{-\frac{1}{2}}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right) \right\|^2 < 1-\delta, \quad \delta > 0 \tag{83}$$

7. The network has at least one equilibrium point.

8. All solutions of (34)-(35) are bounded.

Most of the conditions in theorems 12, 13, and 14 are either graph or circuit-theoretic in nature and are therefore easily checked. For example, condition 1 in these theorems is equivalent to the condition that all multi-terminal and multiport capacitors and inductors are *reciprocal* and *strictly locally passive*. Conditions 2, 3, and 4 in these theorems can be checked by inspection. Condition 5 of Theorems 13 and 14 can be tested either by numerical methods, or by a simple explicit *topological test* given in [77]. Only condition 7 in Theorem 12 and condition 6 in Theorems 13 and 14 require numerical computation. This condition can be used to derive an upper bound on $\|\underset{\sim}{L}(\cdot)\|$ in terms of $\|\underset{\sim}{C}(\cdot)\|$, or vice-versa. To show that this upper bound is rather sharp, an example is given in [77] which shows that a completely stable circuit becomes oscillatory when this bound is violated at its boundary!

The most serious drawback of Theorems 12, 13, and 14 is the *completeness* hypothesis since many RLC networks are not complete. However, if one is willing to introduce "parasitic" inductances and capacitances at appropriate locations, any network may be transformed into a complete network. Such parasitic elements would tend to make the circuit model more realistic anyway. On the other hand, the completeness hypothesis is needed only if we insist on writing the state equation explicitly via the *hybrid content* as in (34)-(35). There is no fundamental reason why this hypothesis is needed. In fact, by defining a *pseudo hybrid content*, theorems 12, 13, and 14 have been generalized for non-complete networks in [77].

Finally, note that condition 6 in Theorem 12 has been replaced by the condition that "all solutions of (34)-(35) are bounded." We will show in the next section that this boundedness hypothesis (condition 8) as well as the existence of an equilibrium point (condition 7) are satisfied by almost all practical networks. Since the basic hypotheses which guarantee boundedness are identical for both autonomous and non-autonomous networks, let us now turn to the latter.

3. Qualitative Properties of Non-Autonomous Networks

Consider now the case where $\mathcal{N}$ contains ac sources so that $\underset{\sim}{u}_S = \underset{\sim}{u}_S(t)$ in the state equations (15) and (17). For the results to be presented in this section, it is more convenient to work with (17) which we reproduce as follow:

$$\boxed{\dot{\underset{\sim}{z}} = -\underset{\sim}{h}\left(\underset{\sim}{g}(\underset{\sim}{z});\underset{\sim}{u}_S(t)\right)} \tag{84}$$

A. Boundedness and Eventual Uniform Boundedness

Definition 8. Boundedness

The solutions $\underset{\sim}{z}(\cdot)$ of the non-autonomous system (84) are said to be *bounded* if given any bounded $\underset{\sim}{u}_S(\cdot)$, the solution $\underset{\sim}{z}(\cdot)$ is bounded in the sense that there exists a constant K such that

$$\|\underset{\sim}{z}(t)\| < K, \quad \text{for all } t \geq t_0 \tag{85}$$

where K may depend on both the initial state $\underset{\sim}{z}(t_0)$ and the initial time t_0.

Definition 9. Eventual Uniform Boundedness.

The solutions $\underset{\sim}{z}(\cdot)$ of the non-autonomous system (84) are said to be *eventually uniformly bounded* if given any bounded $\underset{\sim}{u}_S(\cdot)$, there exists a compact set $\mathcal{Z} \subset \mathbb{R}^n$ such that for any solution $\underset{\sim}{z}(\cdot)$ of (84), there is a time $T \in \mathbb{R}^1$ such that

$$\|\underset{\sim}{z}(t)\| \in \mathcal{Z}, \quad \text{for all } t \geq T \tag{86}$$

where T may depend on both the initial state and on the initial time.

Observe that there is a subtle difference between Definitions 8 and 9. For example, the periodic solutions of a linear passive autonomous Lossless LC network are all bounded but not eventually uniformly bounded.

Theorem 15. Boundedness and Eventual Uniform Boundedness Criterion [72].

Let an RLC non-autonomous network $\mathcal{N}$ be described by the state equation (84). Assume the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(\cdot)$ is a C^1 *state function* and let $\mathcal{E}: \mathbb{R}^n \to \mathbb{R}^1$ denote its associated scalar "energy" function, i.e., $\nabla\mathcal{E}(\underset{\sim}{z}) = \underset{\sim}{g}(\underset{\sim}{z})$. Assume that $\underset{\sim}{g}(\cdot)$ and $\mathcal{E}(\cdot)$ satisfy

$$\lim_{\|\underset{\sim}{z}\|\to\infty} \|\underset{\sim}{g}(\underset{\sim}{z})\| \to +\infty \quad (87) \qquad \lim_{\|\underset{\sim}{z}\|\to\infty} \mathcal{E}(\underset{\sim}{z}) \to +\infty \quad (88)$$

Assume further that the *source vector* $\underset{\sim}{u}_S(\cdot)$ is bounded; i.e.,

$$\|\underset{\sim}{u}_S(t)\| \leq k_1, \quad \text{for all } t \in \mathbb{R}^1 \tag{89}$$

Then we have the following:

1. The solutions $\underset{\sim}{z}(\cdot)$ of (84) are *bounded* if the associated *n-port resistor N* is *eventually passive* in the sense that there exists a finite constant $k_0 > 0$ such that

$$\underset{\sim}{x}^T\underset{\sim}{h}(\underset{\sim}{x};\underset{\sim}{u}_S) \geq 0, \quad \text{for all } \|\underset{\sim}{x}\| > \underset{\sim}{k}_0 \tag{90}$$

and for all $\underset{\sim}{u}_S(\cdot)$ satisfying (89)

2. The solutions $\underset{\sim}{z}(\cdot)$ of (84) are *eventually uniformly bounded* if the associated n-port resistor N is *eventually strictly passive*; i.e., (90) is satisfied with a *strict* inequality. Moreover, if $\underset{\sim}{u}_S(\cdot)$ is *periodic* with period T, then (84) has a periodic solution with the same period T. In particular, in the autonomous case where $\underset{\sim}{u}_S(\cdot)$ is a constant vector, the associated autonomous equation has at least one equilibrium point.

To derive an explicit version of Theorem 15, we note first that it is easy to find examples showing that *eventual passivity* does not obey the closure property [71]. Consequently, an additional condition must be imposed in our next theorem.

Theorem 16. Explicit Eventual Uniform Boundedness Criterion [72].

Let an RLC non-autonomous network $\mathcal{N}$ be described by the state equation (84). Assume the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(.)$ is a C^1 *state function* and let $\mathcal{E}: \mathbb{R}^n \to \mathbb{R}^1$ denote its associated scalar "energy" function; i.e., $\nabla\mathcal{E}(\underset{\sim}{g}) = \underset{\sim}{g}(\underset{\sim}{z})$. Assume that $\underset{\sim}{g}(\cdot)$ and $\mathcal{E}(\cdot)$ satisfy

$$\lim_{\|\underset{\sim}{z}\|\to\infty} \|\underset{\sim}{g}(\underset{\sim}{z})\| \to +\infty \qquad (91) \qquad\qquad \lim_{\|\underset{\sim}{z}\|\to\infty} \mathcal{E}(z) \to +\infty \qquad (92)$$

Assume that $\mathcal{N}$ satisfies the following: *Fundamental Topological Hypothesis:* there is no loop (resp., no cut set) formed exclusively, by capacitors, inductors, and/or dc voltage sources (resp., current sources).

Moreover, let the constitutive relation $\underset{\sim}{h}_\alpha(\cdot)$ of each internal *multi-terminal or multi-port resistor* R_α satisfy the following conditions:

Eventual strict passivity condition:

$$\underset{\sim}{x}_\alpha^T \underset{\sim}{h}_\alpha(\underset{\sim}{x}_\alpha) > 0, \quad \text{for all } \|\underset{\sim}{x}_\alpha\| > k_0 \tag{93}$$

Growth condition:

$$\lim_{\|\underset{\sim}{x}_\alpha\|\to\infty} \frac{1}{\|\underset{\sim}{x}_\alpha\|} \underset{\sim}{x}_\alpha^T \underset{\sim}{h}_\alpha(\underset{\sim}{x}_\alpha) = +\infty \tag{94}$$

Then all voltage and current waveforms of $\mathcal{N}$ are *eventually uniformly bounded*. Moreover, if the *source vector* $\underset{\sim}{u}_S(\cdot)$ is *periodic* with period T, then (84) has a periodic solution with the same period T. In particular, in the autonomous case where $\underset{\sim}{u}_S(\cdot)$ is a constant vector, the associated autonomous equation has at least one equilibrium point.

Remark.

If $\mathcal{N}$ contains only *voltage-controlled resistors* (resp., *current-constrolled resistors*) and only *dc voltage sources* (resp., *dc current sources*), then (94) in Theorem 16 can be replaced by the following condition:

$$\lim_{\|\underset{\sim}{x}_\alpha\|\to\infty} \underset{\sim}{x}_\alpha^T \underset{\sim}{h}_\alpha(\underset{\sim}{x}_\alpha) = +\infty \tag{95}$$

Since most physical multi-terminal and multi-port resistors satisfy (93) and (94), it follows from Theorem 16 that hypotheses 7 and 8 of Theorem 13 and 14 are satisfied by most practical networks.

B. No Finite Escape-Time Solutions

To motivate the materials in this section, consider first the circuit shown in Fig. 30, whose state equation is given by:

$$\dot{v}_C = -\frac{I_S}{C}\left[e^{v_C/V_T}-1\right] \qquad (96)$$

This equation has the following explicit solution:

$$v_C(t) = v_C(0)\,\ell n\left\{\frac{e^{f(t)}}{e^{f(t)}-\text{sgn } v_C(0)}\right\}^{V_T/v_C(0)} \qquad (97)$$

where

$$f(t) = \left(\frac{I_S}{CV_T}\right)t + \ell n\left(\frac{\text{sgn } v_C(0)}{1-e^{-v_C(0)/V_T}}\right) \qquad (98)$$

and $\text{sgn } v_C(0) = v_C(0)/|v_C(0)|$,

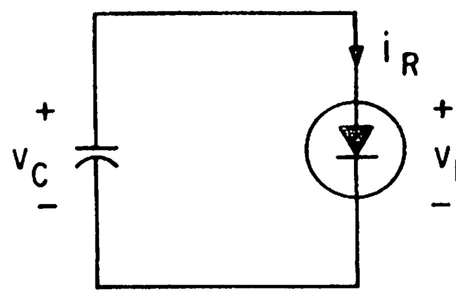

Fig. 30. A simple diode circuit having a finite escape-time solution at $t = t_1 < 0$. The pn junction diode is characterized by $i_R = I_S[\exp(v_R/V_T)-1]$.

$v_C(0) \neq 0$. Observe that for any initial condition $v_C(0) > 0$, the solution tends to *infinity* at a *finite* time

$$t_1 \triangleq -\left(\frac{CV_T}{I_S}\right)\ell n\left(\frac{1}{1-e^{-v_C(0)/V_T}}\right) < 0 \qquad (99)$$

This "non-physical" phenomenon is called a *finite escape-time solution* and is clearly an undesirable property. One could impose conditions on the state equation so that no finite escape-time solutions are possible [85]. Unfortunately, these conditions are much too restrictive that they would exclude most practical networks of interest. However, it is possible to relax these conditions considerably if we wish to exclude only "forward" finite escape-time solutions, i.e., no solution tends to infinity in a *finite* forward time $t_1 > t_0$, where t_0 is the initial time.[16] After all, engineers are usually only interested in solutions *after* the "switch" is thrown. The following theorems have been derived for this purpose.

Theorem 16. No Finite Forward Escape-Time Solution Criterion 1 [71]

Let an RLC non-autonomous network $\mathcal{N}$ be described by the state equation (84). Let the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(\cdot)$ be a C^1 *state function* and assume that $\underset{\sim}{g}(\cdot)$ is *eventually strongly uniformly increasing* in the sense that there exist constants $k_D \geq 0$ and $\bar{\gamma} \geq \underline{\gamma} > 0$ such that

$$\underline{\gamma}\|\underset{\sim}{z}'-\underset{\sim}{z}''\|^2 \leq (\underset{\sim}{z}'-\underset{\sim}{z}'')^T\left(\underset{\sim}{g}(\underset{\sim}{z}')-\underset{\sim}{g}(\underset{\sim}{z}'')\right) \leq \bar{\gamma}\|\underset{\sim}{z}'-\underset{\sim}{z}''\|^2 \qquad (100)$$

[16] Observe that (96) has this property even though it has a "backward" finite escape-time solution.

for all $\|z'\| > k_D$ and $\|z''\| > k_D$.

Moreover, for every compact set $D_u \subset \mathbb{R}^{n_s}$, assume that there exist constants $\gamma > 0$ and $k > 0$ such that for all $u_S \in D_u$, the constitutive relation of the *n-port resistor* N satisfies the condition

$$x^T h(x;u_S) \geq -\gamma\|x\|^2 , \quad \text{for all } \|x\| > k \tag{101}$$

Then the state equation (84) has *no finite forward* escape-time solutions. That is, for any bounded and continuous source function $u_S(t)$ and for any initial time t_0, each solution $z(t)$ of (84) exists for all $t \geq t_0$.

Observe that (101) is an extremely weak condition satisfied by all practical networks. Intuitively speaking, (101) can be interpreted as requiring that the n-port resistor N be *eventually no more active* than some *active linear* n-port resistor. In the next theorem, we relax the "eventually strongly uniform increasing" condition (100) and in turn place a stronger condition than (101) on $h(\cdot)$

Theorem 17. No Finite Forward Escape-Time Solution Criterion 2 [71]

Let an RLC non-autonomous network $\mathcal{N}$ be described by the state equation (84). Let the *capacitor-inductor* constitutive relation $g(\cdot)$ be a C^1 *state function* and let $\mathcal{E}: \mathbb{R}^n \to \mathbb{R}^1$ denote its associated scalar "energy" function, i.e., $\nabla\mathcal{E}(z) = g(z)$. Assume that

$$\lim_{z \to \infty} \mathcal{E}(z) = +\infty \tag{102}$$

Let the constitutive relation $h(\cdot)$ of the associated *n-port resistor* N satisfy the inequality

$$x^T h(x;u_S) \geq -k \tag{103}$$

for all *source vector* $u_S \in \mathbb{R}^{n_s}$ and $x \in \mathbb{R}^n$, where $k \geq 0$. Then (84) has no finite forward escape-time solutions.

Our next theorem gives explicit conditions in terms of the internal elements directly. We assume, without loss of generality, that each voltage sources is in series with a resistor and each current sources is in parallel with a resistor. These resistors with sources attached will be combined as *composite* resistors where each constitutive relation $\hat{h}_\alpha(\cdot)$ is given by

$$y_\alpha = \hat{h}_\alpha(x_\alpha) \triangleq h_\alpha(x_\alpha+b_\alpha) + c_\alpha \tag{104}$$

where the components of the vectors b_α and c_α represent the original dc sources in $\mathcal{N}$.

Theorem 18. Explicit No Finite Forward Escape-Time Solution Criterion [71]

Let an RLC non-autonomous network $\mathcal{N}$ be described by the state equation (84). Let the *capacitor-inductor* constitutive relation $g(\cdot)$ be a C^1 *state function* and let $\mathcal{E}: \mathbb{R}^n \to \mathbb{R}^1$ denote its associated scalar "energy" function; i.e., $\nabla\mathcal{E}(z) = g(z)$. Assume that

$$\lim_{\|z\| \to \infty} \mathcal{E}(z) = +\infty \tag{105}$$

Assume further that there is no voltage source (resp., current source) forming a loop (resp., cut set) exclusively with capacitors, inductors and other voltage sources (resp., current sources) of $\mathcal{N}$.

Moreover, if the constitutive relation $\underset{\sim}{h}_\alpha(\cdot)$ of each "composite" internal *multi-terminal or multi-port resistor* satisfies the inequality

$$\underset{\sim}{x}_\alpha^T \underset{\sim}{h}_\alpha(\underset{\sim}{x}_\alpha) \geq -k_\alpha \tag{106}$$

for all $\underset{\sim}{x}_\alpha \in \mathbb{R}^{n_\alpha}$ and $k_\alpha > 0$. Then (84) has no finite forward escape-time solutions.

C. Small-Input Gives Small-Output Property.

All "small-signal" analysis of electronic circuits are based on the implicit assumption that a small "ac" signal applied about a dc operating point will give rise to a small "ac" output signal. It is easy to find example where this assumption is not satisfied. Our objective in this section is to present sufficient conditions which guarantee this property. For complete generality, we will allow any ac signal. Since such signals need not be periodic, or even almost periodic, the following "characteristic" of ac signals will be used.

Definition 10. Eventual Range of a signal $\underset{\sim}{s}(\cdot)$ [72]

Let $\underset{\sim}{s}(\cdot) : \mathbb{R}^1 \rightarrow \mathbb{R}^m$ be a continuous function of time. We define the *eventual range* $\mathcal{R}_S$ of $\underset{\sim}{s}(\cdot)$ to be:

$$\mathcal{R}_S \triangleq \bigcap_{T \in \mathbb{R}^1} \overline{\{\hat{\underset{\sim}{s}} \in \mathbb{R}^m : \text{there exists } t \geq T \text{ such that } \underset{\sim}{s}(t) = \hat{\underset{\sim}{s}}\}} \tag{107}$$

It can be shown that when $\underset{\sim}{s}(\cdot)$ is bounded, then the eventual range $\mathcal{R}_S$ is compact and connected in $\mathbb{R}^m$.

Theorem 19. Small-Input Small-Output Criterion [72].

Let an RLC non-autonomous network $\mathcal{N}$ be described by the state equation (84). Let the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(\cdot)$ be a *C^1 strictly increasing diffeomorphic state* function mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$. Let the constitutive relation $\underset{\sim}{h}(\cdot;\underset{\sim}{u}_S)$ of the associated n-port resistor N be a *strictly increasing eventually strictly passive C^1 diffeomorphism* mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$ for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_s}$. Then $\mathcal{N}$ has the following properties:

1. For any "dc bias" source vector $\underset{\sim}{u}_S^* \in \mathbb{R}^{n_s}$, there exists a unique $\underset{\sim}{z}^* \in \mathbb{R}^n$ such that

$$\underset{\sim}{h}\left(\underset{\sim}{g}(\underset{\sim}{z}^*);\underset{\sim}{u}_S^*\right) = \underset{\sim}{0} \tag{108}$$

2. Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any *continuous* and *bounded* "ac" source vector function $\underset{\sim}{u}_S(\cdot)$ satisfying[17]

$$\left\|\mathcal{R}_{u_S} - \underset{\sim}{u}_S^*\right\| \tag{109}$$

the corresponding solution $\underset{\sim}{z}(\cdot)$ satisfies

$$\left\|\mathcal{R}_z - \underset{\sim}{z}^*\right\| < \varepsilon \tag{110}$$

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[17] See bottom of next page.

regardless of the initial conditions.

3. Given any *continuous* "ac" source vector $\underset{\sim}{u}_S(\cdot)$ satisfying

$$\lim_{t\to\infty} \underset{\sim}{u}_S(t) = \underset{\sim}{u}_S^* \tag{111}$$

the corresponding solution $\underset{\sim}{z}(\cdot)$ satisfies

$$\lim_{t\to\infty} \underset{\sim}{z}(t) = \underset{\sim}{z}^* \tag{112}$$

regardless of the initial conditions.

Remarks:

1. It can be shown that a C^1 strictly increasing diffeomorphism $\underset{\sim}{h}(\cdot;\underset{\sim}{u}_S)$ mapping $\mathbb{R}^n$ onto $\mathbb{R}^n$ is *eventually strictly passive* if it is a state function, or if it is a *uniformly increasing* function.

2. The preceding property 3 is an extension of Theorem 6 where the same conclusion is found assuming $\underset{\sim}{u}_S(t) \equiv \underset{\sim}{u}_S^*$, and without assuming that $\underset{\sim}{h}(\cdot;\underset{\sim}{u}_S)$ is eventually strictly passive *for all* $\underset{\sim}{u}_S \in \mathbb{R}^n$. There is a *subtle* difference here in that $\underset{\sim}{z}^*$ is *not* an equilibrium point of (84) unless $\underset{\sim}{u}_S(t) \equiv \underset{\sim}{u}_S^*$. That is, $\underset{\sim}{z}(t) \triangleq \underset{\sim}{z}^*$ is not a solution of (84) if $\mathcal{N}$ is driven by a *time-varying* input $\underset{\sim}{u}_S(t)$.

Theorem 20. Explicit Small-Input Small-Output Criterion [72]

Let an RLC non-autonomous network $\mathcal{N}$ be described by the state equation (84). Let the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(\cdot)$ be a *C^1 strictly increasing diffeomorphic state function mapping* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$.

Assume that $\mathcal{N}$ satisfies the following:

Fundamental Topological Hypothesis: There is no loop (resp., no cut set) formed exclusively by capacitors, inductors, and/or dc voltage sources (resp., current sources).

Moreover, let the constitutive relation $\underset{\sim}{h}_\alpha(\cdot)$ of each internal *multi-terminal* or *multi-port* resistor $\mathcal{R}_\alpha$ be a *C^1 strictly increasing homeomorphism* mapping $\mathbb{R}^{n_s}$ *onto* $\mathbb{R}^{n_s}$ and let $\underset{\sim}{h}_\alpha(\cdot)$ satisfy the growth condition

$$\lim_{\|\underset{\sim}{x}_\alpha\|\to\infty} \frac{1}{\|\underset{\sim}{x}_\alpha\|}\left\{\underset{\sim}{x}_\alpha^T\, \underset{\sim}{h}_\alpha(\underset{\sim}{x}_\alpha)\right\} = +\infty \tag{113}$$

Then $\mathcal{N}$ has the three properties listed in Theorem 19.

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[17] We have abused our notation slightly in (109) and (110) by using the "eventual range" symbol $\mathcal{R}_{u_S}$ defined earlier in (107) to mean $\|\bar{\underset{\sim}{u}}_S-\underset{\sim}{u}_S^*\| < \delta$ for all $\bar{\underset{\sim}{u}}_S \in \mathcal{R}_{u_S}$, and $\|\bar{\underset{\sim}{z}}-\underset{\sim}{z}^*\| < \varepsilon$ for all $\bar{\underset{\sim}{z}} \in \mathcal{R}_z$, respectively. Here $\mathcal{R}_{u_S}$ and $\mathcal{R}_z$ denote the eventual range of the "ac" source vector $\underset{\sim}{u}_S(\cdot)$ and the solution $\underset{\sim}{z}(\cdot)$ respectively. Roughly speaking, (109) defines a class of source waveforms $\underset{\sim}{u}_S(\cdot)$ with the property that $\|\underset{\sim}{u}_S(t)-\underset{\sim}{u}_S^*\| < \delta$ as $t \to \infty$. Similarly, (110) defines a class of solution waveforms $\underset{\sim}{z}(\cdot)$ such that $\|\underset{\sim}{z}(t)-\underset{\sim}{z}^*\| < \varepsilon$ as $t \to \infty$. Here, the constant vector $\underset{\sim}{u}_S^*$ can be interpreted as the "dc" bias" used in Electronic Circuits to establish a suitable "operating point" $\underset{\sim}{z}^*$.

D. Unique Steady State Property.

It is well-known that autonomous networks containing *locally active* resistors could have several distinct "dc" steady state solutions corresponding to locally asymptotically stable equilibrium points. For *non-autonomous* networks, multiple "ac" steady-state solutions could occur even if all elements are strictly locally passive. For example, consider the network $\mathcal{N}$ shown in Fig. 31. Observe that all elements (except the source) are passive and strictly locally passive. It follows from

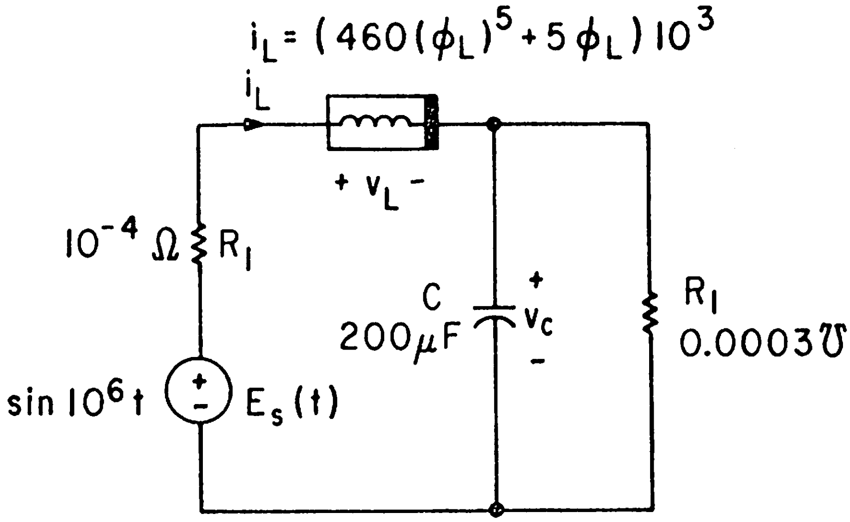

Fig. 31. A non-autonomous RLC network $\mathcal{N}$ containing strictly locally passive elements having more than one steady-state solutions.

Theorem 16 that all solutions of this network are *eventually uniformly bounded*. Moreover, if we replace the voltage source in Fig. 31 by one having a terminal voltage $E(t) = \delta \sin 10^6 t$, where $\delta \ll 1$, then it follows from Theorem 19 that $\mathcal{N}$ has a *unique* steady-state solution $\underset{\sim}{z}(\cdot)$ in the sense that all solutions tend toward $\underset{\sim}{z}(\cdot)$ as $t \to \infty$, regardless of initial conditions. However, for $\delta = 1$ as in Fig. 31, we have found by *computer simulation* that $\mathcal{N}$ *has at least two distinct steady-state solutions* [72]. Our objective in this section is to present some sufficient conditions which guarantee a unique steady-state solution. Since the steady-state solution need *not* be periodic, we will define first the following more general classes of signals.

Definition 11. Almost Periodic Signals [84]

A continuous function $\underset{\sim}{s} : \mathbb{R}^1 \to \mathbb{R}^m$ is said to be *almost periodic* if for any $\varepsilon > 0$, there exists an $\ell(\varepsilon) > 0$ such that every time interval of length $\ell(\varepsilon)$ in $\mathbb{R}^1$ contains a time τ such that

$$\|\underset{\sim}{s}(t+\tau) - \underset{\sim}{s}(t)\| < \varepsilon \text{ , for all } t \in \mathbb{R}^1 \tag{114}$$

Definition 12. Asymptotically Almost Periodic Signals [52,86].

A continuous function $\underset{\sim}{s} : \mathbb{R}^1 \to \mathbb{R}^m$ is said to be *asymptotically periodic* if $\underset{\sim}{s}(\cdot)$ can be uniquely decomposed into

$$\underset{\sim}{s}(t) = \underset{\sim}{s}_0(t) + \bar{\underset{\sim}{s}}_T(s), \text{ for all } t \in \mathbb{R}^1 \tag{115}$$

where $\underset{\sim}{s}_0(t)$ is continuous and almost periodic, and $\underset{\sim}{s}_T(\cdot)$ is continuous and $\lim_{t\to\infty} \underset{\sim}{s}_T(t) = \underset{\sim}{0}$.

It is well-known that a *continuous* function $\underset{\sim}{s}(\cdot) : \mathbb{R}^1 \to \mathbb{R}^m$ is *almost periodic* if, and only if, s(.) can be uniformly approximated by a *multiple Fourier Series* [86]

$$\underset{\sim}{s}(t) \sim \sum_h \underset{\sim}{s}_k e^{j\omega_k t} \tag{116}$$

where $\{\omega_k\}$ is a *countable* set of real numbers called *Fourier exponents*, or *base frequencies*, and the corresponding *countable* set of vectors $\{s_k\}$ is called the *associated Fourier coefficients*.

<u>Definition 13.</u> <u>Spectrum Combinations (σ-module)</u> [52,84]

Let an almost periodic function be represented by (116). Let $\mathcal{S}_S$ denote the *countable* set of *real* numbers which are *integer combinations* of the ω_k. That is, $\mathcal{S}_S = \sum_k n_k\omega_k$ for all possible integers n_k. The set $\mathcal{S}_S$ is called the *spectrum combinations* (or σ-module) of $\underset{\sim}{s}(\cdot)$.

It follows from (116) that an almost periodic function $\underset{\sim}{s}(\cdot)$ is *periodic* if, and only if, for any integers k and ℓ, ω_k/ω_ℓ is a rational number. In this case, the spectrum combinations $\mathcal{S}_S$ is just the set of *harmonics* generated by the base frequencies ω_k. For almost periodic signals, the spectrum combinations would contain all harmonics and modulation products of the base frequencies. The following theorems provide sufficient conditions which guarantee that a network has a *unique* steady-state solution and that the spectrum combinations $\mathcal{S}_z$ of the solution $\underset{\sim}{z}(\cdot)$ is a subset of the spectrum combinations of the input signal $\mathcal{S}_{u_S}$ of $\underset{\sim}{u}_S(\cdot)$. Hence, no unusual frequency components, such as *subharmonics*, will be generated. Since the simple RLC network shown earlier in Fig. 31 fails to have a unique steady-state solution, it is clear that rather *strong hypotheses* will be required in the following theorems.

<u>Theorem 20.</u> <u>Unique Steady-State Criterion for Linear LC Networks</u> [72]

Let an RLC non-autonomous network $\mathcal{N}$ be described by the state equation (84), where the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(\cdot)$ is *linear*; i.e.,

$$\underset{\sim}{g}(\underset{\sim}{z}) = \underset{\sim}{\Gamma}\, \underset{\sim}{z} \tag{117}$$

where Γ is an nxn *symmetric* and *positive definite* matrix. Assume further that the constitutive relation $\underset{\sim}{h}(\cdot;\underset{\sim}{u}_S)$ of the *n-port resistor* N is a C^1 *strictly increasing eventually strictly passive* funtion for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_s}$. Then $\mathcal{N}$ has the following properties:

1. There is a *unique steady-state solution* for any *continuous* and *bounded* source vector $\underset{\sim}{u}_S(\cdot)$.

2. If $\underset{\sim}{u}_S(\cdot)$ is *asymptotically almost periodic* and satisfies a *local Lipschitz* condition, then each solution $\underset{\sim}{z}(\cdot)$ of (84) is *asymptotically almost periodic*, and in the steady state, the *spectrum combinations* of $\underset{\sim}{z}(.)$ is a subset of the spectrum combination of $\underset{\sim}{u}_S(\cdot)$; i.e.,

$$\mathcal{S}_z \subset \mathcal{S}_{u_S} \tag{118}$$

3. If in addition, the constitutive relation $\underset{\sim}{h}(\cdot;\underset{\sim}{u}_S)$ of the *n-port resistor* N is a C^1 *diffeomorphism* mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_s}$, then for any *continuous* and *bounded* $\underset{\sim}{u}_S(\cdot)$ and for any pair of solution $\underset{\sim}{z}'(\cdot)$ and $\underset{\sim}{z}''(\cdot)$ of (84), there exist a constant γ satisfying $1 \geq \gamma > 0$ and time constants τ_{max} and τ_{min} satisfying $\tau_{max} > \tau_{min} > 0$ such that

$$k_1 e^{-t/\tau_{min}} \leq \|\underset{\sim}{z}'(t)-\underset{\sim}{z}''(t)\| \leq k_2 e^{-t/\tau_{max}} \tag{119}$$

for all $t \geq 0$, where

$$k_1 \triangleq \gamma^{\frac{1}{2}} \|\underset{\sim}{z}'(0)-\underset{\sim}{z}''(0)\| \ , \ k_2 \triangleq \left(\frac{1}{\gamma}\right)^{\frac{1}{2}} \|\underset{\sim}{z}'(0)-\underset{\sim}{z}''(0)\| \tag{120}$$

Moreover,

$$\gamma = \underline{\lambda}/\bar{\lambda} \tag{121}$$

where $\bar{\lambda}$ and $\underline{\lambda}$ denote the *maximum* and *minimum eigenvalue* of $\underset{\sim}{\Gamma}$, respectively.

<u>Theorem 21.</u> <u>Explicit Unique Steady-State Criterion for Linear LC Networks</u> [72]

Let an RLC non-autonomous network $\mathcal{N}$ be described by the state equation (84), where the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(\cdot)$ is linear; i.e., it satisfies (117). Assume $\mathcal{N}$ satisfies the following *Fundamental Topological Hypothesis:* There is no loop (resp., no cut set) formed exclusively by capacitors, inductors, and/or dc voltage sources (resp., current sources). Moreover, assume the constitutive relation $\underset{\sim}{h}_\alpha(\cdot)$ of each internal multi-terminal or multi-port resistor R_α is a *strictly increasing homeomorphism* mapping $\mathbb{R}^{n_\alpha}$ *onto* $\mathbb{R}^{n_\alpha}$, and

$$\lim_{\|\underset{\sim}{x}_\alpha\|} \frac{1}{\|\underset{\sim}{x}_\alpha\|} \underset{\sim}{x}_\alpha^T \underset{\sim}{h}_\alpha(\underset{\sim}{x}_\alpha) = +\infty \tag{122}$$

Then $\mathcal{N}$ has the three properties listed in Theorem 20.

<u>Theorem 22.</u> <u>Unique Steady-State Criterion for RC and RL Networks with Linear Resistors</u> [72].

Let an RLC non-autonomous network $\mathcal{N}$ be described by state equation (84). Let the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(\cdot)$ be a C^1 *strictly increasing diffeomorphic state* function mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$. Let the constitutive relation $\underset{\sim}{h}(\cdot;\underset{\sim}{u}_S)$ of the n-port resistor N be a *linear* function of $\underset{\sim}{x}$ and $\underset{\sim}{u}_S$; i.e.,

$$\underset{\sim}{h}(\underset{\sim}{x};\underset{\sim}{u}_S) = \underset{\sim}{H}_x \underset{\sim}{x} + \underset{\sim}{H}_u \underset{\sim}{u}_S \tag{123}$$

where $\underset{\sim}{H}_x$ is an nxn *symmetric* and *positive definite constant* matrix, and $\underset{\sim}{H}_u$ is an nxn_s *constant* matrix. Then $\mathcal{N}$ has the following properties:

1. Thee exists a *unique* steady-state solution for any continuous and bounded input $\underset{\sim}{u}_S(\cdot)$.
2. Given any pair of solutions $\underset{\sim}{z}'(0)$ and $\underset{\sim}{z}''(\cdot)$, there exist a constant

γ satisfying $1 \geq \gamma > 0$ and time constants τ_{max} and τ_{min} satisfying $\tau_{max} \geq \tau_{min} > 0$ such that

$$k_1 e^{-t/\tau_{min}} \leq \|\underset{\sim}{z}'(t)-\underset{\sim}{z}''(t)\| \leq k_2 e^{-t/\tau_{max}} \tag{124}$$

for all $t \geq 0$, where

$$k_1 \triangleq \gamma^{\frac{1}{2}} \|\underset{\sim}{z}'(0)-\underset{\sim}{z}''(0)\| \;,\; k_2 \triangleq \left(\frac{1}{\gamma}\right)^{\frac{1}{2}} \|\underset{\sim}{z}'(0)-\underset{\sim}{z}''(0)\| \tag{125}$$

3. If $\underset{\sim}{u}_S(\cdot)$ is *asymptotically almost periodic*, and satisfies a *local Lipschitz* condition, then each solution $\underset{\sim}{z}(\cdot)$ of (84) is asymptotically almost periodic, and in the steady state, the spectrum combinations $\mathcal{S}_z$ of $\underset{\sim}{z}(.)$ is a subset of the spectrum combinations $\mathcal{S}_u$ of $\underset{\sim}{u}_S(\cdot)$.

Observe that the *symmetry* condition on the matrix $\underset{\sim}{H}_x$ in (123) cannot be satisfied *in general* if the n-port resistor is terminated by both capacitors and inductors in view of the corollary to Theorem 10 (Sec. II-2). Consequently, the following version of Theorem 22 is stated for either RC or RL networks.

Theorem 23. Explicit Steady-State Criterion for RC and RL Network with Linear Resistors [72]

Let an RC (resp., RL) non-autonomous network $\mathcal{N}$ be described by state equation (84). Let the constitutive relation g(.) of the *capacitors* (resp., *inductors*) be a *C^1 strictly increasing diffeomorphic state* function mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$. Assume that $\mathcal{N}$ satisfies the following *Fundamental Topological Hypothesis:* There is no loop (resp., cut set) formed exclusively by capacitors (resp., inductors) and/or dc voltage sources (resp., dc current sources). Moreover, assume the constitutive relation $\underset{\sim}{h}_\alpha(\cdot)$ of each *internal* multi-terminal or multi-port *resistor* R_α is *linear, passive,* and *reciprocal*; i.e.

$$\underset{\sim}{i}_\alpha = H_\alpha \underset{\sim}{v}_\alpha \qquad (\text{resp., } \underset{\sim}{v}_\alpha = \underset{\sim}{H}_\alpha \underset{\sim}{i}_\alpha) \tag{125}$$

where H_α is an $n_\alpha x n_\alpha$ symmetric and positive definite constant matrix. Then $\mathcal{N}^\alpha$ has the three properties listed in Theorem 22.

Theorem 24. Unique Steady-State Criterion under Small-Signal Inputs [72]

Let an RLC non-autonomous network $\mathcal{N}$ be described by state equation (84). Let the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(.)$ be a *C^1 strictly increasing diffeomorphmic state* function mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$, and let the Jacobian matrix of $\underset{\sim}{g}(\cdot)$ satisfy the following *local Lipschitz* condition:

$$\left\| \frac{\partial \underset{\sim}{g}(\underset{\sim}{z}')}{\partial \underset{\sim}{z}} - \frac{\partial \underset{\sim}{g}(\underset{\sim}{z}'')}{\partial \underset{\sim}{z}} \right\| < \ell_z \|\underset{\sim}{z}'-\underset{\sim}{z}''\| \tag{126}$$

for all $\underset{\sim}{z}', \underset{\sim}{z}'' \in \mathbb{R}^n$, $\|\underset{\sim}{z}'-\underset{\sim}{z}''\| < \delta$, and $\ell_z > 0$ where both δ and ℓ_z may depend on $\underset{\sim}{z}'$.

Let the constitutive relation $\underset{\sim}{h}(\cdot;\underset{\sim}{u}_s)$ of the n-port resistor N be a C^1 *strictly increasing, eventually strictly passive diffeomorphism* mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$ for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_s}$. Then $\mathcal{N}$ has the following properties:

1. For every "dc bias" $\underset{\sim}{u}_S^* \in \mathbb{R}^{n_s}$, there exists $\delta > 0$ such that (84) has a *unique steady-state* solution due to any *continuous* and *bounded* "ac" signal $\underset{\sim}{u}_S(\cdot)$ satisfying

$$\left\| \mathcal{R}_{u_s} - u_S^* \right\| < \delta \tag{127}$$

2. If in addition $\underset{\sim}{u}_S(\cdot)$ is *asymptotically almost periodic* and satisfies a *local Lipschitz* condition, then every solution $\underset{\sim}{z}(\cdot)$ is *asymptotically almost periodic*, and, in the steady state, the spectrum combinations $\mathcal{S}_z$ of $\underset{\sim}{z}(\cdot)$ is a subset of the spectrum combinations $\mathcal{S}_u$ of $\underset{\sim}{u}_S(\cdot)$.

Theorem 25. Explicit Unique Steady-State Criterion under Small-Signal Inputs [72]

Let an RLC non-autonomous network $\mathcal{N}$ be described by state equation (84). Let the *capacitor-inductor* constitutive relation $\underset{\sim}{g}(\cdot)$ be a C^1 *strictly increasing diffeomorphic state* function mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$, and let the Jacobian matrix of $\underset{\sim}{g}(\cdot)$ satisfy a *local Lipschitz* condition in the sense of (126). Assume that $\mathcal{N}$ satisfies the following *Fundamental Topological Hypothesis:* There is no loop (resp., cut set) formed exclusively by capacitors, inductors, and/or dc voltage sources (resp., current sources). Moreover, let the constitutive relation $\underset{\sim}{h}_\alpha(\cdot)$ of each *internal* multi-terminal or multi-port *resistor* R_α be a C^1 *strictly increasing diffeomorphism* mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$, and satisfying the growth condition

$$\lim_{\|x_\alpha\| \to \infty} \frac{1}{\|x_\alpha\|} \left\{ \underset{\sim}{x}_\alpha^T \, \underset{\sim}{h}_\alpha(\underset{\sim}{x}_\alpha) \right\} = +\infty \tag{128}$$

Then $\mathcal{N}$ has the two properties listed in Theorem 24.

Our final theorem is addressed to the class $\mathcal{N}$ of *diode-transistor networks* shown in Fig. 32. If we model the pn junction diodes and the pnp and npn transistors by the standard "ac" *Ebers-Moll Model* (with junction capacitances) [11], and if we define the following port variables

$$\underset{\sim}{x} \triangleq \begin{bmatrix} \underset{\sim}{v}_T \\ \underset{\sim}{v}_D \\ \underset{\sim}{v}_C \\ \underset{\sim}{i}_L \end{bmatrix}, \quad \underset{\sim}{y} \triangleq \begin{bmatrix} \underset{\sim}{i}_T \\ \underset{\sim}{i}_D \\ \underset{\sim}{i}_C \\ \underset{\sim}{i}_L \end{bmatrix}, \quad \underset{\sim}{z} \triangleq \begin{bmatrix} \underset{\sim}{q}_T \\ \underset{\sim}{q}_D \\ \underset{\sim}{q}_C \\ \underset{\sim}{\phi}_L \end{bmatrix} \tag{129}$$

where the subscripts T, D, C, L correspond to transistors, diodes, capacitors, and inductors, respectively, then the *state equations* of $\mathcal{N}$ assumes the following form:

$$\boxed{\dot{\underset{\sim}{z}} = -\underset{\sim}{T}\,\underset{\sim}{h} \circ \left(\underset{\sim}{g}(\underset{\sim}{z})\right) - \underset{\sim}{H}\,\underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{u}_S(t)} \tag{130}$$

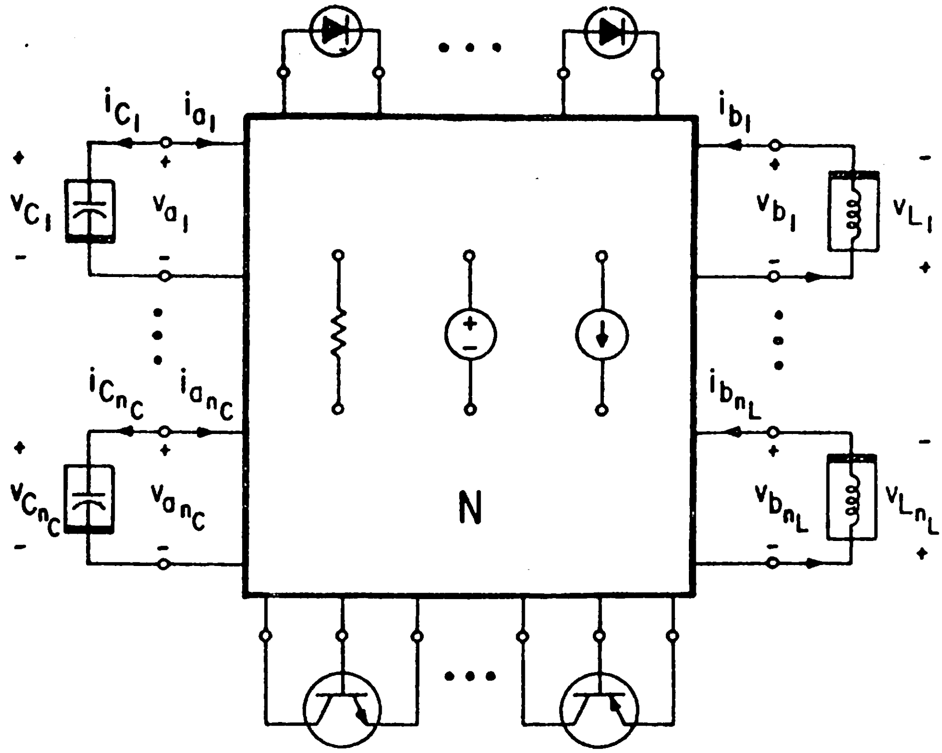

Fig. 32. A diode-transistor dynamic network. The m-port N contains only *linear* 2-terminal resistors and independent sources. The only nonlinear resistors are diodes and transistors.

where $\underset{\sim}{H}$ denotes the *hybrid matrix* associated with N; i.e., $\underset{\sim}{y} = \underset{\sim}{H}\underset{\sim}{x}-\underset{\sim}{u}_S(t)$ and

$$\underset{\sim}{T} \triangleq \left[\begin{array}{cc|c|cc|ccc|ccc|ccc}
1 & -\alpha_r^{(1)} & & & & & & & & & & & & \\
-\alpha_f^{(1)} & 1 & & & & & & & & & & & & \\
\hline
 & & \ddots & & & & & & & & & & & \\
\hline
 & & & 1 & -\alpha_r^{(p)} & & & & & & & & & \\
 & & & -\alpha_f^{(p)} & 1 & & & & & & & & & \\
\hline
 & & & & & 1 & & & & & & & & \\
 & & & & & & \ddots & & & & & & & \\
 & & & & & & & 1 & & & & & & \\
\hline
 & & & & & & & & 0 & & & & & \\
 & & & & & & & & & \ddots & & & & \\
 & & & & & & & & & & 0 & & & \\
\hline
 & & & & & & & & & & & 0 & & \\
 & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & 0
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} \text{Transistors} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{Diodes} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{Capacitors} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{Inductors}
\end{array}$$

is an mxm block diagonal matrix. The vectors $\underset{\sim}{h}(\cdot)$ and $\underset{\sim}{g}(\cdot)$ are defined as follow:

$$\underset{\sim}{h}(\underset{\sim}{x}) \triangleq \begin{bmatrix} \underset{\sim}{f}_T(\underset{\sim}{v}_T) \\ \underset{\sim}{f}_D(\underset{\sim}{v}_D) \\ \underset{\sim}{v}_C \\ \underset{\sim}{i}_L \end{bmatrix}, \quad \underset{\sim}{g}(\underset{\sim}{z}) \triangleq \begin{bmatrix} \underset{\sim}{g}_T(\underset{\sim}{q}_T) \\ \underset{\sim}{g}_D(\underset{\sim}{q}_D) \\ \underset{\sim}{g}_C(\underset{\sim}{q}_C) \\ \underset{\sim}{g}_L(\underset{\sim}{\phi}_L) \end{bmatrix} \tag{132}$$

Each component of $\underset{\sim}{f}_T(\cdot)$ and $\underset{\sim}{f}_D(\cdot)$ of the transistor-diode constitutive relation is defined by $\underset{\sim}{i}_j = \underset{\sim}{f}_j(v_j)$, where $f_j(.)\colon \mathbb{R}^1 \to \mathbb{R}^1$ is a C^1 *strictly increasing* function and $f_j(0) = 0$. Each component of the *capacitor-inductor* constitutive relation $g(\cdot)$ is defined by $q_j = g_j(v_j)$ or $\phi_j = g_j(i_j)$, where $g_j(\cdot)\colon \mathbb{R}^1 \to \mathbb{R}^1$ is a C^1 *uniformly increasing* function. We are now ready to state a slightly generalized version of a theorem due to Sandberg [62].

Theorem 26. Unique Steady-State Criterion for Diode-Transistor Networks

Let $\mathcal{N}$ be a diode-transistor network described by the state equation (130). Assume that there exists a *diagonal* matrix $\underset{\sim}{D}$ with *positive* diagonal elements such that $\underset{\sim}{D}\underset{\sim}{T}$ is *weakly column-sum dominant* and $\underset{\sim}{D}\underset{\sim}{H}$ is *strongly column-sum dominant.*[18] Assume further that $f_j(\cdot)\colon \mathbb{R}^1 \to \mathbb{R}^1$ is C^1 *strictly increasing,* $f_j(0) = 0$, and $g_j(\cdot)\colon \mathbb{R}^1 \to \mathbb{R}^1$ is C^1 *uniformly increasing.* Then $\mathcal{N}$ has the following properties:

1. For any two *bounded* source vectors $\underset{\sim}{u}'_S(t)$ and $\underset{\sim}{u}''_S(t)$, $t \geq 0$, which tend toward each other in the sense that

$$\lim_{t\to\infty} \left[\underset{\sim}{u}'_S(t) - \underset{\sim}{u}''_S(t) \right] = \underset{\sim}{0} \tag{133}$$

the corresponding solutions $\underset{\sim}{z}'(t)$ and $\underset{\sim}{z}''(t)$ also tend toward each other; i.e.,

$$\lim_{t\to\infty} \left[\underset{\sim}{z}'(t) - \underset{\sim}{z}''(t) \right] = \underset{\sim}{0} \tag{134}$$

2. $\mathcal{N}$ has a *unique* steady state response $\underset{\sim}{z}(\cdot)$ corresponding to each bounded source vector $\underset{\sim}{u}_S(\cdot)$.
3. If $\mathcal{N}$ contains only dc independent sources (i.e., $\underset{\sim}{u}'_S(t) = \underset{\sim}{u}''_S(t) = \underset{\sim}{u}^*_S$), then all solutions $\underset{\sim}{z}(\cdot)$ tend to a *unique* and *globally asymptotically stable* equilibrium point.

---

[18] An nxn matrix M is said to be *weakly* (resp., *strongly*) *column-sum dominant* if

$$\min_{k=1,2,\ldots n} \left\{ M_{kk} - \sum_{\substack{j=1 \\ j\neq k}}^{n} |M_{jk}| \right\} \geq \varepsilon$$

for some $\varepsilon \geq 0$ (resp., $\varepsilon > 0$).

V. Recent Tools for Nonlinear Network Analysis

Several powerful and potentially useful tools have found important applications in the *qualitative theory of nonlinear networks*. Due to limitation of space, we will only list them here along with some references on where applications have been found:

1. The colored branch theorem [28,70,91].
2. Degree theory [25,68,75].
3. The Hopf Bifurcation theorem [87-88].
4. Transversality theory [89-91].

These basic tools are likely to find many new applications in future research on nonlinear network theory.

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