ARITHMETIC TESTS FOR A-STABILITY

A[α]-STABILITY, AND STIFF-STABILITY

by

T. A. Bickart and E. I. Jury

Memorandum No. UCB/ERL M78/1

4 January 1978

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ARITHMETIC TESTS FOR

A-STABILITY, A[α]-STABILITY, AND STIFF-STABILITY

T. A. Bickart† and E. I. Jury

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

Arithmetic tests for A-stability, A[α]-stability, and stiff-stability are presented as special cases of a general stability test for numerical integration methods. The test evolves from extracted properties of the characteristic polynomial (in two variables) of the numerical method applied to the prototype scalar ordinary differential equation $\dot{x} = qx$, $Re\{q\} < 0$. The several steps of the test impose root clustering conditions — such as being Hurwitz — on a polynomial of one variable.

Research sponsored in part by a special grant of the Syracuse University Senate Research Committee to the first author and by the National Science Foundation under Grant ENG76-21816 to the second author.

†On leave from the Department of Electrical and Computer Engineering, Syracuse University, Syracuse, NY 13210.

INTRODUCTION

A-stability [4] and its weaker associates A[α]-stability [17] and stiff-stability [6], have become generally accepted as appropriate properties of numerical methods suitable for solving a stiff initial value problem, as described by a first order vector ordinary differential equation

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{f}[\underset{\sim}{x}(t),t] \tag{1a}$$

with initial condition

$$\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0. \tag{1b}$$

We shall develop herein a test for a general stability property, which includes those above as special cases. This test is a generalization within a new framework of that reported by Rubin [14] for A-stability. As invoked to test for A[α]-stability or stiff-stability it is simpler than that elsewhere reported by Bickart and Rubin [3].

STABILITY CONDITION

The archtypical initial value problem by which the foregoing stability properties are given definition is that in which (1a) is the scalar, linear equation

$$\dot{x}(t) = qx(t), \tag{2a}$$

subject to the constraint $Re\{q\} < 0$, and, correspondingly, (1b) is

$$x(t_0) = x_0. \tag{2b}$$

Note: The solution to the archtypical initial value problem is asymptotic to the origin.

Our concern is with those methods defining with (2) a linear difference equation for x_n $(n = 0,1,2,\ldots)$ — a unique approximation of $x(t)$ at $t_n = nh + t_0$ $(n = 0,1,2,\ldots)$ — and having a real (with integer coefficients) characteristic polynomial P in two variables, $\lambda = hq$ and ζ, such that $\{x_n\}$ is asymptotic to the origin if and only if $P(\lambda,\zeta) = 0$ implies $|\zeta| < 1$. Such methods include multistep methods [9], wherein P is of degree 1 in λ, and include block one-step methods [15] and Runge-Kutta methods [7], wherein P is of degree 1 in ζ. Methods for which there is no a priori limit on the degree in λ or ζ include multistep-multiderivative methods [7,13], which as a class subsume the multistep methods as special cases, composite multistep methods [14,16], which as a class subsume the multistep methods and the block one-step methods as special cases, and multistep-multiderivative-multistage methods [8], which subsume all the thusfar noted methods.

We are now in a position to present[†]

DEFINITION 1: *Let* $\mathcal{T}$ *denote a simply connected open region of the extended complex plane* $\mathbb{C}^*$, *such that* $\partial\mathcal{T}$ *— the boundary of* $\mathcal{T}$ *— is piecewise regular. Then, a method is said to be stable with respect to* $\mathcal{T}$ *if*

$$\{\lambda \in \mathcal{T}\} \wedge \{P(\lambda,\zeta) = 0\} \Rightarrow \zeta \in \mathcal{D}\,, \qquad (3)$$

where $\mathcal{D}$ *denotes the open unit disk, or, equivalently, if*

$$\{\lambda \in \mathcal{T}\} \wedge \{\zeta \in \mathcal{D}^c\} \Rightarrow P(\lambda,\zeta) \neq 0, \qquad (4)$$

where $\mathcal{D}^c$ *is the complement of* $\mathcal{D}$ *in* $\mathbb{C}^*$.

[†]The closure of a set will be denoted by a "-" overscore of the set symbol; the complement, by a superscript "c"; and the boundary by an anticedent "∂". Though specifically noted in this definition, such will not subsequently be the case.

Our principal result will be a test by which to validate (4). This test will evolve from the intimate relationship between the polynomial P and the transformed polynomial Q defined as

$$Q(\lambda,z) = (z-1)^{m_p} P(\lambda,\frac{z+1}{z-1}), \qquad (5)$$

where m_p denotes the degree of P in ζ. Correspondingly, we let m_q denote the degree of Q in z. The intermediate result we next need is expressed in

THEOREM 1: *The implication (4) is valid if and only if* $m_q = m_p$ *and*

$$\{\lambda \in \mathcal{T}\} \wedge \{z \in \mathcal{L}^c\} \Rightarrow Q(\lambda,z) \neq 0, \qquad (6)$$

where $\mathcal{L}$ *is the open left half plane.*

PROOF: *[only if]* By (4), $P(\cdot,1)$ is not the zero function; therefore, by (5) the degree of Q with respect to z must be m_p. That is, $m_q = m_p$. Furthermore, (5) follows as a consequence of (4). *[if]* The inverse relationship to that in (5) is

$$P(\lambda,\zeta) = (\frac{\zeta-1}{2})^{m_p} Q(\lambda,\frac{\zeta+1}{\zeta-1}) = (\frac{\zeta-1}{2})^{m_q} Q(\lambda,\frac{\zeta+1}{\zeta-1}) . \qquad (7)$$

The second equality follows from the fact that $m_p = m_q$. The implication (4) is now an immediate consequence of the implication (6). ∎

STABILITY TEST

The transition from the stability condition as manifested in (6) to the stability test is by way of the result embodied in

THEOREM 2: *The implication (6) is valid if and only if*

$$\{\lambda \in \mathcal{T}\} \wedge \{z=z_0 \in \mathcal{L}^c\} \Rightarrow Q(\lambda,z) \neq 0, \qquad (8a)$$

$$\{ \lambda \in \partial\mathcal{T} \cap \{\lambda\colon Q(\lambda,\cdot) \not\equiv 0\}\} \wedge \{z \in \bar{\mathcal{L}}^c\} \Rightarrow Q(\lambda,z) \neq 0, \qquad \text{(8b)}$$

and

$$\{ z \in \partial\mathcal{L}\} \Rightarrow Q(\cdot,z) \not\equiv 0. \qquad \text{(8c)}$$

PROOF: [*only if*] Since $z_0 \in \bar{\mathcal{L}}^c$ implies $z_0 \in \mathcal{L}^c$, (8a) is rather immediately a consequence of (6). The logical equivalent of (6)

$$\{ \lambda \in \mathcal{T}\} \wedge \{Q(\lambda,z) = 0\} \Rightarrow z \in \mathcal{L}$$

implies, by continuity properties of the algebraic function $z = \phi(\lambda)$ defined by $Q(\lambda,z) = 0$ on $\mathbb{C}^* - \{\lambda\colon Q(\lambda,\cdot) \equiv 0\}$,

$$\{ \lambda \in \bar{\mathcal{T}} \cap \{\lambda\colon Q(\lambda,\cdot) \not\equiv 0\}\} \wedge \{Q(\lambda,z) = 0 \Rightarrow z \in \bar{\mathcal{L}}. \qquad \text{(9)}$$

But, as $\{\lambda\colon\{\lambda \in \partial\mathcal{T}\} \wedge \{Q(\lambda,\cdot) \not\equiv 0\} \subset \{\lambda\colon\{\lambda \in \bar{\mathcal{T}}\} \wedge \{Q(\lambda,\cdot) \not\equiv 0\}$, it follows that

$$\{ \lambda \in \partial\mathcal{T} \cap \{\lambda\colon Q(\lambda,\cdot) \not\equiv 0\}\} \wedge \{Q(\lambda,z) = 0\} \Rightarrow z \in \bar{\mathcal{L}}, \qquad \text{(10)}$$

The implication (8b) is the logical equivalent of this in (10). As $\partial\mathcal{L} \subset \mathcal{L}^c$ and as $Q(\lambda,z) \neq 0 \Rightarrow Q(\lambda,\cdot) \not\equiv 0$, (8c) follows directly from (6). [*if*] Let the definition of the algebraic function be extended to the (finite, discrete) set of points $\{\lambda\colon Q(\lambda,\cdot) \equiv 0\}$ such that it is an algebraic function on $\mathbb{C}^*$. Then it maps $\partial\mathcal{T}$ into a finite number of points — the set of points $\{z\colon Q(\cdot,z) \equiv 0\}$ — and a finite number of piecewise regular, simple closed curves. The complement of the union of these points and curves is a finite set of connected, disjoint open regions — called components — of $\mathbb{C}^*$. By (8b) and continuity of the extended algebraic function $z = \phi(\lambda)$ we know that $\bar{\mathcal{L}}^c$ is contained in one of the components. By (8a) we know that the preimage of $z_0 \in \bar{\mathcal{L}}^c$ cannot be a subset of $\mathcal{T}$; it must be a subset of $\mathcal{T}^c$.

It then follows, because the components are connected and because the preimages of their boundaries, exclusive of the set of points $\{z:Q(\cdot,z) \equiv 0\}$, are subsets of $\partial\mathcal{T}$, that the preimage of every point of $\mathcal{L}^c$ is a subset of $\mathcal{T}^c$. Therefore,

$$\{ z \in \bar{\mathcal{L}}^c\} \wedge \{Q(\lambda, z) = 0\} \Rightarrow \lambda \in \bar{\mathcal{T}}^c, \tag{11}$$

which by continuity of the extended algebraic function $z = \phi(\lambda)$, implies

$$\{ z \in \bar{\mathcal{L}}^c \cup \{z:\{z \in \partial\mathcal{L}\} \wedge \{Q(\cdot,z) \not\equiv 0\}\}\} \wedge \{Q(\lambda,z) = 0\} \Rightarrow \lambda \in \mathcal{T}^c. \tag{12}$$

But, by (8c), the set of points $\{z:\{z \in \partial \mathcal{L}\} \wedge \{Q(\lambda,z) \equiv 0\}$ is the empty set; hence, we have that

$$\{ z \in \mathcal{L}^c\} \wedge \{Q(\lambda,z) = 0\} \Rightarrow \lambda \in \mathcal{T}^c. \tag{13}$$

This implication is the logical equivalent of (6). ∎

When z_0 is real, $Q(\lambda,z_0)$ of (8a) is a real polynomial in λ. This means that the implication (8a) can be validated by a root clustering test — roots all in $\mathcal{T}^c$, in this case — for a real polynomial.† Fortunately, such tests — with integer arithmetic being sufficient — are known for such regions $\mathcal{T}$ as are associated with A-stability, A[α]-stability, and stiff-stability. Note: When, in particular, $z_0 = 1$ (and $m_q = m_p$ as required by Theorem 1) that the root clustering test can be applied, equivalently, to the easily

---

†Note: When the set of points $\{\lambda:\{\lambda \in \partial\mathcal{T}\} \wedge \{Q(\lambda,\cdot) \equiv 0\}\}$ is empty, (8a) together with (8b), in which z_0 is taken to be the value of z, imply the roots must all be in $\bar{\mathcal{T}}^c$.

evaluated real polynomial $P(\cdot,\infty)$.[†] This is also noteworthy because clustering of the roots in $\mathcal{T}^c$ is essentially the condition needed to guarantee the existence of a unique $\{x_n\}$.

Now let us examine the implication (8b). Let $\partial\mathcal{T}$ be parametrized by the continuous one-to-one complex valued function β mapping a, not necessarily finite, interval $\mathcal{I}$ of the extended real line into the extended complex plane; that is $\beta:\mathcal{I} \rightarrow \mathbb{C}^*$. With $\lambda = \beta(\mu)$ we find $Q[\beta(\mu),z]$ is a complex polynomial in z which, if (8b) is valid, must be Hurwitz for all $\mu \in \mathcal{I}$ except the preimages by of the points in $\{\lambda:\{\lambda \in \partial\mathcal{T}\} \wedge \{Q(\lambda,\cdot) \equiv 0\}$. Let $\mathcal{A}$ denote this set of preimages. Let $\Delta_i(\mu)$ $(i = 1,2,\ldots,m_q)$ be the set of inners determinants — real functions of $\mu \in \mathcal{I}$ — associated with this polynomial. [10,p.20][††] Assume $\Delta_{m_q}(\cdot) \not\equiv 0$.[†††] It then follows that $\Delta_j(\cdot) \not\equiv 0$ $(i = 1,\ldots,m_q-1)$. Let $\mathcal{Z}$ denote the finite set of (discrete) values of μ such that $\Delta_i(\mu) = 0$ for some $i \in \{1,\ldots,m_q\}$. Note $\mathcal{A} \subset \mathcal{Z}$. Then, we can state [10,p.20]: $Q[\beta(\mu),z]$ is Hurwitz — in fact, strictly Hurwitz — for

---

[†] The characteristic equation $P(\lambda,\zeta) = 0$ defines the algebraic function $\zeta = \phi(\lambda)$ which exhibits a pole for those λ for which $P(\lambda,\infty) = 0$. That is, for those λ which are the zeros of the polynomial coefficient of the highest degree term in ζ. We shall mean that polynomial (coefficient) in λ when writing $P(\cdot,\infty)$.

[††] These inners determinants are the Hurwitz determinants of $Q[\beta(\mu),z]$ [5,pp.248-250]; only their anticedent arrays differ — by an even row-permutation. Furthermore, if the imaginary part of $Q[\beta(\mu),\infty]$ is identically zero, then the inners determinants are to be those of $jQ[\beta(\mu),z]$.

[†††] The restriction of this assumption can be removed at the expense of further complicating the now evolving stability test. The (numerical integration) methods excluded by this assumption are few and, to the authors, appear to be of little consequence. Therefore, we have relegated the treatment of the special case $\Delta_{m_q}(\cdot) \equiv 0$ to Appendix I.

all $\mu \in \mathcal{G} - \mathcal{Z}$ if and only if

$$\{ i \in \{1,\ldots,m_q\}\} \wedge \{\mu \in \mathcal{G} - \mathcal{Z}\} \Rightarrow \Delta_i(\mu) > 0. \tag{14}$$

That is, (14) implies and is implied by

$$\{\mu \in \mathcal{G} - \mathcal{Z}\} \wedge \{Q[\beta(\mu),z] = 0\} \Rightarrow z \in \mathcal{L}. \tag{15}$$

By continuity of $\Delta_i(\mu)$ $(i = 1,\ldots,m_q)$ and of $z = \phi[\beta(\mu)]$ with respect to μ we find that

$$\{ i \in \{1,\ldots,m_q\}\} \wedge \{\mu \in \mathcal{G}\} \Rightarrow \Delta_j(\mu) \geq 0 \tag{16}$$

implies and is implied by

$$\{\mu \in \mathcal{G} - \mathcal{N}\} \wedge \{Q[\beta(\mu),z] = 0\} \Rightarrow z \in \bar{\mathcal{L}}. \tag{17}$$

This last expression is the statement that $Q[\beta(\mu),z]$ is Hurwitz — not necessarily, strictly Hurwitz — for all $\mu \in \mathcal{G} - \mathcal{N}$. This, however, is equivalent to (8b). Note: For such regions $\mathcal{T}$ as are associated with A-stability, A[α]-stability, and stiff-stability there exist tests — with integer arithmetic being sufficient — by which to validate (16).

Because Q is a real polynomial, if $\{z \in \partial\mathcal{T}\} \wedge \{z \neq 0\}$ implies $Q(\cdot,z) \equiv 0$, then such is also true of z^*, the conjugate of z. Hence, any z for which $Q(\cdot,z) \equiv 0$ is true must be a zero of an even or an odd factor of Q in the single variable z. The assumption $\Delta_{m_q}(\cdot) \not\equiv 0$ precludes such a factor.† Therefore, (8c) is trivially true under this assumption.

---

†This fact is brought out in the treatment of the special case $\Delta_{m_q}(\cdot) \equiv 0$ in Appendix I.

The foregoing discussion, together with Theorems 1 and 2, validates the

GENERAL STABILITY TEST: *A method for which* $\Delta_{m_q}(\cdot) \not\equiv 0$ *is stable with respect to* $\mathcal{T}$ *if and only if*

(a) *the zeros of the real polynomial* $P(\cdot,\infty)$ *are (clustered) in* $\mathcal{T}^c$.

(b) $m_q = m_p$, *and*

(c) $\Delta_i(\mu) \geq 0$ *for all* $\mu \in \mathcal{J}$ *and* $i = 1,\ldots,m_q$.

A-STABILITY

Let us now specialize this result. Suppose $\mathcal{T} = \mathcal{L}$, which corresponds to the method being A-stable. Then, in (a), $\mathcal{T}^c$ becomes the closed right-half plane. Furthermore, $\partial\mathcal{T}$, being the imaginary axis, can be parametrized by the function $\lambda = j\mu$, defined on the entire extended real axis. Thus, $\Delta_i(\mu)$ $(i = 1,\ldots,m_q)$ is a real — and, as is easily shown — even polynomial in μ. As a consequence, in (c), $\Delta_i(\mu) \geq 0$ for all $\mu \in \mathcal{J}$ becomes $\Delta_i(\infty) > 0$ and $\Delta_i(\cdot)$ has no positive real zeros of odd multiplicity. Taken together we have the†

A-STABILITY TEST: *A method for which* $\Delta_{m_q}(\cdot) \not\equiv 0$ *is A-stable if and only if*

---

†Inclusion of the precondition *P has no factors in one variable alone* would validate item (b) and would, because $\{\lambda:\{\lambda \in \partial\mathcal{T}\} \wedge \{Q(\lambda,\cdot) \equiv 0\}\}$ would then be the empty set, allow "*closed*" in item (a) to be replaced by "*open*". (See the footnote on page 6.) The resulting test would then be equivalent to that given by Genin [7,pp.7-12].

(a) *the zeros of the real polynomial* $P(\cdot,\infty)$ *are in the closed right-half plane,*

(b) $m_q = m_p$,

(c_1) $\Delta_i(\infty) > 0$ *for all* $i = 1,\ldots,m_q$, *and*

(c_2) $\Delta_i(\cdot)$ *for all* $i = 1,\ldots,m_q$ *has no positive real zeros of odd multiplicity.*

Validation of items (b) and (c_1) is a trivial task, so we turn to items (a) and (c_2). The zeros of $P(\lambda,\infty)$ are in the closed right-half plane, if and only if those of $P(-\lambda,\infty)$ are in the close left-half plane. Thus, $P(-\lambda,\infty)$ must be a Hurwitz polynomial. This property — hence, item (a) also — can be verified by known methods,† using only integer arithmetic. Of course, if $P(-\lambda,\infty)$ is strictly Hurwitz — an easier property to verify [10,pp.22-23] — it is Hurwitz as well. There is also a method [10,pp.156-158] extending the classical results of Sturm, for which integer arithmetic is sufficient and by which to verify that a real polynomial has no positive real zeros of odd multiplicity. Of course, if a polynomial has no positive real zeros — an easier condition to verify — it has none of odd multiplicity. In this way (c_2) can be verified. Do keep in mind the fact that these noted stronger properties are not necessary for items (a) and (c_2) and that their violation does not imply a method is not A-stable. Only violation of the weaker properties, requiring rather sophisticated verification methods, can do that.

---

†A procedure to follow in using these methods to verify that a real polynomial is Hurwitz is given in Appendix II.

Let us examine an illustration of the results thusfar by applying the A-stability test to the (A-stable) composite multistep method, elsewhere reported by Sloate and Bickart [16], having

$$P(\lambda,\zeta) = [1 \;\; \lambda \;\; \lambda^2] \begin{bmatrix} 0 & 48 & -48 \\ 5 & 8 & 35 \\ 3 & 0 & -9 \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \end{bmatrix}$$

as its characteristic polynomial. We first observe that

$$P(\lambda,\infty) = -48+35\lambda-9\lambda^2 = -(48-35\lambda+9\lambda^2)$$

and, hence, that

$$P(-\lambda,\infty) = -(48+35\lambda+9\lambda^2).$$

By inspection $P(-\lambda,\infty)$ is seen to have both its roots in the open left-half plane. Thus, it is strictly Hurwitz and, perforce, item (a) is verified. By (5) we determine that

$$Q(\lambda,z) = [1 \;\; \lambda \;\; \lambda^2] \begin{bmatrix} -96 & -96 & 0 \\ 32 & 60 & 48 \\ -6 & -24 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}.$$

Clearly $m_q = 2$ equals $m_p = 2$; hence, item (b) is verified. To determine the inners determinants associated with $Q(\lambda,z)$ on the boundary of the left-half plane — parametrized by $\lambda = j\mu$ — we must first write $Q(j\mu,jz)$ as a complex polynomial in z. After some trivial complex algebra it is found that

$$Q(j\mu,jz) = [1 \;\; j] \begin{bmatrix} -96+6\mu^2 & -60\mu & -6\mu^2 \\ 32\mu & -96+24\mu^2 & -48\mu \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}.$$

It now follows that the sought inners determinants $\Delta_1(\cdot)$ and $\Delta_2(\cdot)$ are the determinants of the following two bordered arrays — one nested within the other [10,p.20]:

$$\Delta_2 \quad \begin{array}{|cccc|}\hline -48\mu & -96+24\mu^2 & 32\mu & 0 \\ \cline{2-3} 0 \quad \Delta_1 & \multicolumn{1}{|c}{-48\mu} & \multicolumn{1}{c|}{-96+24\mu^2} & 32\mu \\ 0 & \multicolumn{1}{|c}{-6\mu^2} & \multicolumn{1}{c|}{-60\mu} & -96+6\mu^2 \\ \cline{2-3} -6\mu^2 & -60\mu & -96+6\mu^2 & 0 \\ \hline \end{array}\ .$$

Upon evaluation of these two determinants, we obtain

$$\Delta_1(\mu) = 2304\mu^2 + 144\mu^4 = 144\mu^2(16+\mu^2)$$

and

$$\Delta_2(\mu) = 184320\mu^6 + 20736\mu^8 = 2304\mu^6(80+9\mu^2).$$

As $\Delta_{m_q}(\cdot) = \Delta_2(\cdot) \not\equiv 0$, the precondition of the A-stability test is satisfied. Obviously $\Delta_1(\infty) > 0$ and $\Delta_2(\infty) > 0$; therefore, item (c_1) is verified. Also, by inspection $\Delta_1(\cdot)$ and $\Delta_2(\cdot)$ have no positive real zeros — thence, none of odd multiplicity; so, item (c_2) is verified. It now follows — all items of the test being verified — that, as claimed, the method is A-stable.

A[α]-STABILITY

For $\alpha \in (0,\pi/2]$, let $\mathcal{W}_\alpha = \{\lambda : |\arg\{-\lambda\}| < \alpha\}$. Set $\mathcal{T} = \mathcal{W}_\alpha$, which corresponds to the method being A[α]-stable. Note: $\mathcal{W}_\alpha$ is the open wedge, within the left-half plane and symmetric with respect to the real axis, depicted in Fig. 1. Of course, $\mathcal{T}^c = \mathcal{W}_\alpha^c$, the complement of this wedge.

To the authors' knowledge there is no reported method by which to verify that a polynomial has all its zeros in such a region.

So we digress to develop one possible method. Observe: $\mathcal{W}_\alpha$ is the union, over all $\gamma \in (0,\infty)$, of the open disks $\mathcal{D}_{\alpha,\gamma} = \{\lambda: |\lambda+\gamma| < \rho,$ $\rho = \gamma \sin \alpha,\ \gamma > 0\}$ as depicted in Fig. 2. Therefore, item (a) is equivalent to: For all $\gamma \in (0,\infty)$ the zeros of the real polynomial $P(\cdot,\infty)$ are in the complement of the open disk $\mathcal{D}_{\alpha,\gamma}$. This condition can be converted to the requirement that a polynomial be Hurwitz. The bilinear function

$$\lambda = -\frac{(\gamma-\rho)\eta+(\gamma+\rho)}{\eta+1} \tag{18}$$

maps the closed left-half plane onto the complement of the open disk $\mathcal{D}_{\alpha,\gamma}$. Set

$$D_{\alpha,\gamma}(\eta) = (\eta+1)^n\, P[-\frac{(\gamma-\rho)\eta+(\gamma+\rho)}{\eta+1},\ \infty]\ , \tag{19}$$

where n is the degree of $P(\cdot,\infty)$. Clearly, item (a) is now equivalent to: For all $\gamma \in (0,\infty)$ the zeros of the real polynomial $D_{\alpha,\gamma}(\cdot)$ are in the closed left-half plane — $D_{\alpha,\gamma}(\cdot)$ is Hurwitz. As previously noted, there exist methods by which to verify that a polynomial is Hurwitz. To do so for all γ adds some complications to be considered shortly.

The boundary $\partial\mathcal{T}$, which consists of two rays from the origin, can be parametrized by the continuous function

$$\begin{aligned} \lambda &= a\mu + jb\mu \quad (\mu\le 0) \\ &= -a\mu + jb\mu \quad (\mu\ge 0), \end{aligned} \tag{20}$$

defined on the entire extended real axis. The relationship between the parameter α and the parameters a and b is $\sin \alpha = b/(a^2+b^2)^{1/2}$. Thus, $\Delta_i(\mu)$ $(i = 1,\ldots,m_q)$ is a real — and, as is easily shown — even function in μ, which is a polynomial on each of the two semi-infinite intervals $\mu \leq 0$ and $\mu \geq 0$. As a consequence, item (c) separates into the same two parts as for A-stability. Taken together we have the

A[α]-STABILITY TEST: *A method for which* $\Delta_{m_q}(\cdot) \not\equiv 0$ *is* $A[\alpha]$*-stable if and only if*

(a) *for all* $\gamma \in (0,\infty)$ *the real polynomial* $D_{\alpha,\gamma}(\cdot)$ *is a Hurwitz polynomial,*

(b) $m_q = m_p$,

(c_1) $\Delta_i(\infty) > 0$ *for all* $i = 1,\ldots,m_q$, *and*

(c_2) $\Delta_i(\cdot)$ *for all* $i = 1,\ldots,m_q$ *has no positive real zeros of odd multiplicity.*

As before, validation of items (b) and (c_1) is a trivial task. Verification of (c_2) can be accomplished by following the guidelines given in the discussion of the A-stability test. We turn our attention, therefore, to item (a). Let $\delta_i(\gamma)$ $(i = 1,\ldots,n)$ be the set of inners determinants — real polynomials in $\gamma \in (0,\infty)$ — associated with the real polynomial $D_{\alpha,\gamma}(\cdot)$. Assume $\delta_n(\cdot) \not\equiv 0$. Then, $\delta_i(\cdot) \not\equiv 0$ for $i = 1,\ldots,n-1$. By arguments similar to those advanced when (c) of the general stability test was established and, in the last section, separated into two parts it can be established (when $\delta_n(\cdot) \not\equiv 0)^{\dagger}$ that item (a) is equivalent to (a_1) $\delta_i(\infty) > 0$ for all $i = 1,\ldots,n$ and (a_2) $\delta_i(\cdot)$ for all $i = 1,\ldots,n$ has no positive real zero of odd multiplicity. Verification of (a_1) is a trivial task. Verification of (a_2), though not trivial can be accomplished, as previously noted, by a known

---

† This restriction can be removed by the methods presented in Appendix I.

method. [10,pp.156-158]. Also, as previously noted, it is sufficient in verifying (a_2) to show that $\delta_i(\cdot)$ for all $i = 1,\ldots,n$ has no positive real zero. Since the underlying condition (a) — see the general stability test — is that the zeros of $P(\lambda,\infty)$ be all in $\mathcal{W}_\alpha^c$ and since $\mathcal{W}_\alpha^c$ contains the open right-half plane, it is evident that another property, sufficient for item (a), is: $P(-\lambda,\infty)$ is a strictly Hurwitz polynomial.

If the validating methods are to be invoked with just integer — possibly, quadratic integer — arithmetic, we assume that the value of α is determined subsequent to specification of a and b, as integers.

As an illustration of these results on A[α]-stability, consider the characteristic polynomial

$$P(\lambda,\zeta) = [1 \quad \lambda \quad \lambda^2 \quad \lambda^3] \begin{bmatrix} 6 & -6 \\ 6 & 12 \\ 2 & -11 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \end{bmatrix}$$

of an A[88.8°]-stable block one-step method elsewhere reported by Bickart and Picel. [2] Note: The method is not A[89.0°]-stable. Observe that

$$P(\lambda,\infty) = -6 + 12\lambda - 11\lambda^2 + 6\lambda^3 = -(6-12\lambda+11\lambda^2-6\lambda^3)$$

and, hence, that

$$P(-\lambda,\infty) = -(6+12\lambda+11\lambda^2+6\lambda^3).$$

By the Lienard-Chipart criterion [10,pp.22-23] $P(-\lambda,\infty)$ is strictly Hurwitz. Hence, item (a) is verified. In the interest of illustrating the general method of verifying item (a), let us evaluate $D_{\alpha,\gamma}(\cdot)$ as in (19) for a = 2 and b = 95, corresponding to which, $\alpha \approx 88.8°$. The result is

$$D_{88.8°,\gamma}(\eta) = \{[(216624\nu-20583840)\gamma^3 + (198594\nu-18870610)\gamma^2$$
$$+ (108348\nu-10293060)\gamma + 54174\nu]\eta^3$$
$$+ [(72\nu-6840)\gamma^3 + (198682\nu-18870610)\gamma^2$$
$$+ (325044\nu-10293060)\gamma + 162522\nu]\eta^2$$
$$+ [(72\nu+6840)\gamma^3 + (198682\nu+18870610)\gamma^2$$
$$+ (325044\nu+10293060) + 162522\nu]\eta$$
$$+ [(216624\nu+20583840)\gamma^3 + (198594\nu+18870610)\gamma^2$$
$$+ (108348\nu+10293060)\gamma + 54174\nu]\}/9029\nu,$$

where $\nu = \sqrt{9029}$.* For convenience, set

$$D_{88.8°,\gamma}(\eta) = d_3(\gamma)\eta^3 + d_2(\gamma)\eta^2 + d_1(\gamma)\eta + d_0(\gamma).$$

The sought inners determinants $\delta_1(\gamma)$, $\delta_1(\gamma)$, and $\delta_3(\gamma)$ are the determinants of the following three bordered arrays:

$$\delta_3 \quad \delta_2 \quad \delta_1 \quad \left|\begin{array}{cccccc} -d_3(\gamma) & 0 & d_1(\gamma) & 0 & 0 & 0 \\ 0 & -d_3(\gamma) & 0 & d_1(\gamma) & 0 & 0 \\ 0 & 0 & -d_3(\gamma) & 0 & d_1(\gamma) & 0 \\ 0 & 0 & 0 & -d_2(\gamma) & 0 & d_0(\gamma) \\ 0 & 0 & -d_2(\gamma) & 0 & d_0(\gamma) & 0 \\ 0 & -d_2(\gamma) & 0 & d_0(\gamma) & 0 & 0 \end{array}\right|$$

Evaluation of these determinants yields

$$\delta_1(\gamma) = d_2(\gamma)d_3(\gamma),$$

$$\delta_2(\gamma) = d_2(\gamma)d_3(\gamma)[d_1(\gamma)d_2(\gamma)-d_0(\gamma)d_3(\gamma)],$$

*As a matter of convenience, the polynomial $D_{88.8°,\gamma}(\eta)$ is derived from the polynomial $-P(-\lambda,\infty)$ — not $P(-\lambda,\infty)$.

and

$$\delta_3(\gamma) = d_0(\gamma)d_3(\gamma)[d_1(\gamma)d_2(\gamma)-d_0(\gamma)d_3(\gamma)]^2.$$

Examination of the expressions for $d_i(\gamma)$ $(i = 1,2,3,4)$ discloses that each is a real polynomial in γ with positive coefficients. Furthermore, evaluation of $d_1(\gamma)d_2(\gamma)-d_0(\gamma)d_3(\gamma)$ discloses that the same is true of it. Therefore, each of the inners is a real polynomial in γ with positive coefficients. It follows that, for each $i = 1,2,3$, $\delta_i(\infty) > 0$ and $\delta_i(\cdot)$ has no positive real zeros. This validates item (a).

By (5) we now find that

$$Q(\lambda,z) = [1 \ \ \lambda \ \ \lambda^2 \ \ \lambda^3] \begin{bmatrix} -12 & 0 \\ 6 & 18 \\ -13 & -9 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} .$$

Obviously $m_q = 1$ equals $m_p = 1$; so, item (b) is verified. To determine the inners determinants associated with $Q(\lambda,z)$ on the boundary of $\mathcal{W}_{88.8^\circ}$, — parametrized by (20) with $a = 2$ and $b = 95$ — we must first write $Q(-2\mu+j95\mu,jz)$ as a complex polynomial in z. After some trivial, but tedious complex algebra it is found that

$$Q(-2\mu+j95\mu,jz) = [1 \ \ j] \begin{bmatrix} -12-12\mu+117273\mu^2+324852\mu^3 & -1710\mu-3420\mu^2-5137410\mu^3 \\ 570\mu+4940\mu^2+5137410\mu^3 & -36\mu+81189\mu^2+324852\mu^3 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} .$$

The sought inner determinant $\Delta_1(\cdot)$ is the determinant of the following bordered array:

$$\begin{vmatrix} -36\mu+81189\mu^2+324852\mu^3 & 570\mu+4940\mu^2+5137410\mu^3 \\ -1710\mu-3420\mu^2-5137410\mu^3 & -12-12\mu+117273\mu^2+324852\mu^3 \end{vmatrix} .$$

The evaluation of that determinant yields

$$\Delta_1(\mu) = 432\mu + 864\mu^2 + 13\,02480\mu^3 + 2\,12358\,74301\mu^4 + 10\,74195\,25224\mu^5 + 2649\,85103\,30004\mu^6.$$

As $\Delta_{m_q}(\cdot) = \Delta_1(\cdot) \not\equiv 0$, the precondition of the A[α]-stability test is satisfied. Clearly $\Delta_1(\infty) > 0$; therefore, item (c_1) is verified. Since $\Delta_1(\mu)/\mu$ posses only positive coefficients, $\Delta_1(\cdot)$ can have no positive real zeros — item (c_2) is verified. It now follows — all items of the test being verified — that the method is A[88.8°]-stable.

STIFF-STABILITY

A method exhibits the stiff-stability property if the region $\mathcal{T}$ is such that it contains the open half-plane $\{\lambda: \mathrm{Re}\{\lambda\} < -\delta\}$ for some $\delta \geq 0$ and has the origin as a boundary point.† Little can be done to specialize the general stability result in this case because $\mathcal{T}$ is not completely defined. However, for specifically selected regions conforming to the above conditions, more can often be said. This we will illustrate be selecting a particular region.

Let $\mathcal{H}_\delta$ denote the above noted open half-plane. Then, suppose $\mathcal{T} = \mathcal{H}_\delta \cup \mathcal{W}_\alpha$, as illustrated in Fig. 3. The wedge truncated to the left at Re $\lambda = -\delta$ is covered by $\cup_{\gamma\in(0,\hat{\gamma})}\mathcal{D}_{\alpha,\gamma}$, where $\hat{\gamma} = \delta/\cos^2\alpha$ [$= \delta(1+b^2/a^2)$], which is in $\mathcal{T}$. Thus, $\mathcal{T} = \mathcal{H}_\delta \cup \{\cup_{\gamma\in(0,\hat{\gamma})}\mathcal{D}_{\alpha,\gamma}\}$.

†Stiff-stability characterized in this manner conforms in its essential attributes to Gear's definition [6;p.213].

Let us first consider item (a) of the test. The zeros of $P(\cdot,\infty)$ will be clustered in $\mathcal{T}^c$ if and only if (i) for all $\gamma \in (0,\hat{\gamma})$ they are in the complement of $\mathcal{D}_{\alpha,\gamma}$ and (ii) they are in the complement of $\mathcal{H}_\delta$. The former can be handled as was done for A[α]-stability, with γ here confined to a finite interval. The latter is easily disposed of. The function

$$\lambda = -\eta-\delta, \tag{21}$$

maps the closed left-half plane onto the complement of the open-half-plane $\mathcal{H}_\delta$. Set

$$H_\delta(\eta) = P(-\eta-\delta,\infty). \tag{22}$$

Clearly, the zeros of $P(\cdot,\infty)$ are contained in $\mathcal{H}_\delta^c$ if and only if $H_\delta(\cdot)$ is a Hurwitz polynomial. Let us next consider item (c) of the test. The boundary $\partial\mathcal{T}$, consisting of four line segments, can be parametrized by the continuous function

$$\begin{aligned} \lambda &= -\delta + j\mu \quad (\mu\leq-\hat{\mu}) \\ &= a\mu+jb\mu \quad (-\hat{\mu}\leq\mu\leq0) \\ &= -a\mu+jb\mu \quad (0\leq\mu\leq\hat{\mu}) \\ &= -\delta+j\mu \quad (\hat{\mu}\leq\mu), \end{aligned} \tag{23}$$

where $\hat{\mu} = \delta\tan\alpha$ $[= \delta(b/a)]$. Thus, $\Delta_i(\mu)$ $(i = 1,\ldots,m_q)$ is a real — and, as is easily shown — even function in μ which is a polynomial on each of the four segments. And, item (c) can be separated into two parts, as before. Taken together we obtain a

STIFF-STABILITY TEST: *A method for which* $\Delta_{m_q}(\cdot) \not\equiv 0$ *is stiffly stable if (or is stable with respect to* $\mathcal{H}_\delta \cup \mathcal{W}_\alpha$ *if and only if)*

(a_1) *the real polynomial* $H_\delta(\cdot)$ *is a Hurwitz polynomial,*

(a_2) *for all* $\gamma \in (0,\hat{\gamma})$ *the real polynomial* $D_{\alpha,\gamma}(\cdot)$ *is a Hurwitz polynomial,*

(b) $m_q = m_p$,

(c_1) $\Delta_i(\infty) > 0$ *for all* $i = 1,\ldots,m_q$, *and*

(c_2) $\Delta_i(\cdot)$ *for all* $i = 1,\ldots,m_q$ *has no positive real zeros of odd multiplicity.*

Note: If α is determined by specifying a and b as integers and if δ is given as a rational number, then verification of each of the items of this test can be accomplished with just integer — possibly quadratic integer — arithmetic.

An an illustration of this result on stiff-stability we will consider the method of the previous illustration. It is known [12] to be stiffly stable for $\delta = 1/50$; on the other hand it is not stiffly stable for $\delta = 1/100$.

By the results established in the previous illustration we know that $D_{\alpha,\gamma}(\cdot)$ with $\alpha = \arcsin[95/(9029)^{1/2}]$ is a Hurwitz polynomial for all $\gamma \in (0,\infty)$; hence, for all $\gamma \in (0,\hat{\gamma})$ where $\hat{\gamma} = \delta(9029/4)$. Thus, to complete the validation of item (a_2), it remains only to consider (a_1). Evaluated as in (22) with $\delta = 1/50$

$$H_{1/50}(\eta) = -\frac{1}{4\text{x}50^3}(195\ 139+388\ 975\eta+355\ 000\eta^2+187\ 500\eta^3).$$

By the Lienard-Chipart criterion, $H_{1/50}(\cdot)$ is strictly Hurwitz. Hence, item (a_1) is validated. The verification of item (a) is now complete. (We of course knew it was valid, because as we had previously shown, $P(-\lambda,\infty)$ is a Hurwitz polynomial — this being sufficient to verify item (a).)

By the previous evaluation of $Q(\lambda,z)$ we know that $m_q = m_p$, so item (b) is verified. Thus, only item (c) remains. On the segment $(0,\hat{\mu}]$ of the positive real-axis, with $\hat{\mu} = 95/100$ we know by results established in the previous illustration that item (c_2) is valid, thus we must consider just item (c_2) on the remaining part of the positive real-axis and item (c_1). To establish the inners determinants on this remaining segment of the boundary of $\mathcal{T}$ — parametrized by the last entry in (23) — we must first write $Q(-1/50+j\mu,jz)$ as a complex polynomial in z. This we find after some complex algebra is

$$Q(-1/50+j\mu,jz) = \frac{1}{50^3}\,[1 \quad j] \cdot \begin{bmatrix} -15\ 15656+16\ 70000\mu^2 & -22\ 95900\mu+7\ 50000\mu^3 \\ 8\ \ 15900\mu-7\ 50000\mu^3 & -45456+11\ 7000\mu^2 \end{bmatrix}\begin{bmatrix} 1 \\ z \end{bmatrix}$$

The sought inner determinant $\Delta_1(\cdot)$ is the determinant of the following bordered array, diminished by the multiplicative factor $1/50^6$

$$\boxed{\begin{array}{ll} -45456+11\ 7000\mu^2 & 8\ 15900\mu-7\ 50000\mu^3 \\ -22\ 95900\mu+7\ 50000\mu^3 & -15\ 15656+16\ 70000\mu^2 \end{array}}$$

Evaluation of that determinant yields

$$\Delta_1(\mu) = \frac{1}{50^6}(688956\ 59136+2\ 39957\ 70000\mu^2 -37\ 99500\ 00000\mu^4+56\ 25000\ 00000\mu^6)$$

By the classical procedure due to Sturm [5,pp.173-176] this polynomial has no (positive real) zeros for $\mu > \hat{\mu}$.† This completes the validation

† An alternate procedure invokes evaluation of a set of inners determinants associated with the polynomial $\Delta_1(\mu-\hat{\mu})$. [10,pp.48-49]

of item (c_2). As $\Delta_1(\infty) > 0$, item (c_1) is also valid. With item (c) verified it follows that the method is stiffly stable with $\delta = 1/50$.

CONCLUDING DISCUSSION

We have established a test for stability of a numerical integration method of the class considered relative to an arbitrary region $\mathcal{T} \subset \mathcal{C}^*$. We then particularized that test to three cases: A-stability, A[α]-stability, and stiff-stability. In each case we gave an illustrative application of the test. It is of particular importance to note that only integer — possibly quadratic integer — arithmetic is required. However, to secure a definitive — no ambiguity — verification of the items of any of these tests, extended precision (sometimes called infinite precision) arithmetic must be used. This can be costly in arithmetic processing time and in data storage space on a computer.

ACKNOWLEDGEMENT

The authors wish to acknowledge their appreciation for the research of Dr. W. B. Rubin [14], without which the results of this paper would not have been conceived, much less carried to actuality.

APPENDIX I

In this appendix we consider the procedure to follow in the event $\Delta_{m_q}(\cdot) \equiv 0$. To continue the discussion, suppose $\bar{m}_q$ is the greatest value of i such that $\Delta_i(\cdot) \not\equiv 0$. Then there exists a real polynomial F in λ and z of degree $m_q - \bar{m}_q$ in z such that $F(\lambda, jz)$ or $jF(\lambda, jz)$ is also a real polynomial and

$$Q(\lambda, z) = F(\lambda, z)\bar{Q}(\lambda, z) \qquad \text{(AI-1)}$$

and such that $\bar{Q}$ possesses no similar real factor.

Clearly any factor of Q in the single variable λ must be a factor of F. Therefore, the implication of (8b) is equivalent to

$$\{\lambda \in \partial\mathcal{T} \cap \{\lambda: F(\lambda, \cdot) \not\equiv 0\}\} \wedge \{z \in \bar{\mathcal{L}}^c\} \Rightarrow \bar{Q}(\lambda, z) \neq 0 \qquad \text{(AI-2a)}$$

and

$$\{\lambda \in \partial\mathcal{T} \cap \{\lambda: F(\lambda, \cdot) \not\equiv 0\} \wedge \{z \in \bar{\mathcal{L}}^c\} \Rightarrow F(\lambda, z) \neq 0. \qquad \text{(AI-2b)}$$

Now, (AI-2a) can be validated by treating $\bar{Q}$ as was Q in arriving at item (c) of the general stability test. The one, altered developmental fact is: $\{\lambda: Q(\lambda, \cdot)\} = \{\lambda: F(\lambda, \cdot)\}$ may not be the empty set. However continuity of the $\Delta_i(\cdot)$ can be invoked to leave item (c) intact. Thus, we need yet consider only (AI-2b).

Because $F(\lambda, z)\big|_{\lambda \in \partial\mathcal{T}}$ is real and $F(\lambda, jz)\big|_{\lambda \in \partial\mathcal{T}}$ or $jF(\lambda, jz)\big|_{\lambda \in \partial\mathcal{T}}$ is real, the zeros of $F(\lambda, z)\big|_{\lambda \in \partial\mathcal{T}}$ have quadrantial symmetry in $\mathcal{C}^*$. Therefore, (AI-2b) is valid if and only if

$$\{\lambda \in \partial\mathcal{T} \cap \{\lambda: F(\lambda, \cdot) \not\equiv 0\} \wedge \{z \in \mathcal{L} \cup \bar{\mathcal{L}}^c\} \Rightarrow F(\lambda, z) \neq 0 \qquad \text{(AI-3)}$$

Let us combine this implication with that of (8c) which, because $\Delta_{m_q}(\cdot) \equiv 0$, is no longer trivially true. As observed in the development of the general stability test, any z for which $Q(\cdot,z) \equiv 0$ must be a zero of an even or odd factor of Q in the single variable z. That factor must clearly be a factor of F. Thus, (8c) can be replaced by

$$\{ z \in \partial \mathcal{L}\} \Rightarrow F(\cdot,z) \not\equiv 0. \qquad \text{(AI-4)}$$

By (AI-3), F can have only imaginary zeros in z, but by (AI-4) they must not be such that $F(\cdot,z) \equiv 0$. This implies that F must have no factor in the variable z alone. Thus we have

$$\{\nexists z \in \partial\mathcal{L} \ni F(\cdot,z) \equiv 0\} \wedge \{\{ \lambda \in \partial\mathcal{T} \cap \{\lambda: F(\lambda,\cdot) \not\equiv 0\}\} \wedge \{F(\lambda,z) = 0\} \Rightarrow z \in \partial \mathcal{L}\} . \qquad \text{(AI-5)}$$

Validation of this implication is carried out as follows: Trudi's procedure [1,p.33] is invoked to determine the greatest common polynomial factor of the polynomial coefficients of F as a polynomial in z.†

The result is a factorization of F as the product of polynomials

$$F(\lambda,z) = \hat{g}(\lambda)\ \bar{F}(\lambda,z) \qquad \text{(AI-6)}$$

Note: $\{\lambda:F(\lambda,\cdot) \equiv 0\} = \{\lambda:\hat{g}(\lambda) = 0\}$. Trudi's procedure is again invoked, this time to determine the greatest common polynomial factor of the polynomial coefficients of $\bar{F}$ as a polynomial in λ. The result is a factorization of $\bar{F}$ and further factorization of F. Thus, F can be expressed as

---

†As an alternative to Trudi's procedure, a double triangularization on the Sylvester matrix associated with pairs of polynomials can be employed. [11]

$$F(\lambda,z) = \hat{g}(\lambda)\tilde{g}(z)G(\lambda,z) \tag{AI-7}$$

Note: If $\tilde{g}(z)$ is not a constant, then the implication (AI-5) cannot possibly be true. Next, Trudi's procedure is invoked to determine any greatest common polynomial factor of G and G_z, the derivative of G with respect to z, as polynomials in z. Let $\bar{G}$ denote the polynomial remaining when this factor is extracted from G. Then for almost all $\lambda \in \partial\mathcal{T}$ the zeros of $\bar{G}(\lambda,\cdot)$ have unit multiplicity. For such a polynomial we have necessary and sufficient conditions for all its zeros to be imaginary. Let

$$\bar{G}(\lambda,z) = \gamma_\ell(\lambda)z^{2\ell} + \gamma_{\ell-1}(\lambda)z^{2\ell-2}+\ldots+\gamma_0(\lambda) \tag{AI-8a}$$

or

$$\bar{G}(\lambda,z) = \gamma_\ell(\lambda)\, z^{2\ell+1} + \gamma_{\ell-1}(\lambda)z^{2\ell-1}+\ldots+\gamma_0(\lambda)z, \tag{AI-8b}$$

and let $\nu_i(\nu)$ $(i = 1,\ldots,\ell)$ denote the determinants of the inners — nested bordered arrays —

$$\begin{array}{c|cccccccccccc|}
\hline
 & a_\ell & & a_{\ell-1} & & a_{\ell-2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & 0 & & a_\ell & & a_{\ell-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & 0 & & 0 & & a_\ell & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & & \cdot & & \cdot & & & & a_1 & a_0 & 0 \\
 & \cdot & & \cdot & & \cdot & & & & a_2 & a_1 & a_0 \\
 & \cdot & & \cdot & & \cdot & & & & 3a_3 & 2a_2 & a_1 \\
\nu_\ell & \cdot & \nu_{\ell-1} & \cdot & \nu_{\ell-2} & \cdot & & & & 2a_2 & a_1 & 0 \\
 & \cdot & & \cdot & & \cdot & & & & a_1 & 0 & 0 \\
 & 0 & & 0 & & \ell a_\ell & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & 0 & & \ell a_\ell & & (\ell-1)a_{\ell-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \ell a_\ell & & (\ell-1)a_{\ell-1} & & (\ell-2)a_{\ell-2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array} \; .$$

Then [10,p.36], for almost all $\lambda \in \partial\mathcal{T}$ the zeros of $G(\lambda,\cdot)$ are imaginary if and only if $\gamma_i(\lambda) > 0$ [alternatively, $\gamma_i(\lambda) < 0$] $(i = 0,\ldots,\ell)$ and $\nu_i(\lambda) > 0$ $(i = 1,\ldots,\ell)$. By continuity we extend this result to: for all $\lambda \in \partial\mathcal{T}$ the zeros of $G(\lambda,\cdot)$ are imaginary if and only if $\gamma_i(\lambda) \geq 0$ [alternatively, $\gamma_i(\lambda) \leq 0$] $(i = 0,\ldots,\ell)$ and $\nu_i(\lambda) \geq 0$ $(i = 1,\ldots,\ell)$. Collecting these several results we have

SUPPLEMENTAL TEST I: *The implication (AI-2b) is valid if and only if for all* $\lambda \in \partial\mathcal{T}$

(a) degree $\{\tilde{g}(z)\} = 0$,

(b_1) $\gamma_i(\lambda) \geq 0$ [*alternatively,* $\gamma_i(\lambda) \leq 0$] *for* $i = 0,\ldots,\ell$ *and*

(b_2) $\nu(\lambda) \geq 0$ *for* $i = 1,\ldots,\ell$.

It remains in validating (8b) only to consider the determination of F of (AI-1). This factor is a serendipitous result of evaluation of the inners determinants $\Delta_i(\cdot)$ by a somewhat modified double triangularization [10,pp.217-224] of the outermost array — associated with $\Delta_{m_q}(\cdot)$.

APPENDIX II

Let $E(\lambda)$ denote a real polynomial of degree k in λ. Furthermore, let κ_i($i = 1\ldots,k$) denote the inners determinants associated with E. Suppose $\bar{k}$ is the largest i such that $\kappa_i \neq 0$. Then there exists a factorization of E as

$$E(\lambda) = B(\lambda)\,C(\lambda), \tag{AII-1}$$

where B is an even or odd real polynomial (of degree $k-\bar{k}$).

Now, C is a Hurwitz — in fact, strictly Hurwitz — polynomial if and only if $\kappa_i > 0$ $(i = 1,\ldots,\bar{k})$. Thus, we must yet just consider B. Being an even or odd real polynomial, its zeros must have quadrantial symmetry. Thus, B is a Hurwitz polynomial if and only if all its zeros are imaginary. We procede as we did with G of (AI-5). By Trudi's procedure we determine the greatest common factor of B and B_λ — the derivative of B — and extract it from B to obtain $\bar{B}$. It is now necessary to show that the distinct zeros of $\bar{B}$ are, all of them, imaginary. This is the case if and only if all its coefficients are of the same sign — positive or negative — and all the inners determinants associated with $\bar{B}$, evaluated as were those associated with $\bar{G}$ of (AI-8), are positive.

REFERENCES

[1] S. Barnett, *Matrices in control theory*, Van Nostrand Reinhold, London, 1971.

[2] T. A. Bickart and Z. Picel, "High order stiffly stable composite multistep methods for numerical integration of stiff differential equations," *BIT* **13** (1973), 272-286.

[3] T. A. Bickart and W. B. Rubin, "Composite multistep methods and stiff stability," in *Stiff differential Systems*, R. A. Willoughby, ed., Plenum, New York, 1974.

[4] G. Dahlquist, "A special stability problem for linear multistep methods," *BIT* **8** (1963), 27-43.

[5] F. R. Gantmacher, *Theory of matrices, vol. 2*, Chelsea, New York, 1959.

[6] C. W. Gear, *Numerical initial value problems in ordinary differential equations*, Prentice-Hall, Englewood Cliffs, N.J., 1971.

[7] Y. Genin, "An algebraic approach to A-stable linear multistep-multiderivative integration formulas," MBLE Research Laboratory Report R244, Brussels, Belgium, 1974.

[8] E. Hairer and G. Wanner, "Multistep-multiderivative-multistage methods for ordinary differential equations," *Computing*, **11** (1973), 287-303.

[9] P. Henrici, *Discrete variable methods in ordinary differential equations*, Wiley, New York, 1962.

[10] E. I. Jury, *Inners and stability of dynamic systems*, Wiley, New York, 1974.

[11] T. Millett and I. Tang, "Using inners to determine multiple roots of polynomials," to be published.

[12] Z. Picel, *Block one-step methods with free parameters for numerical integration of stiff ordinary differential equations*, Ph.D. dissertation, Syracuse University, Syracuse, N.Y., 1975.

[13] M. Reimer, "Finite difference forms containing derivatives of higher order," SIAM J. Num. Anal., 5 (1968), 725-738.

[14] W. B. Rubin, *A-stability and composite multistep methods*, Ph.D. dissertation, Syracuse University, Syracuse, N.Y., 1973.

[15] L. F. Shampine and H. A. Watts, "A-stable block implicit one-step methods," *BIT* 12 (1972), 252-266.

[16] H. M. Sloate and T. A. Bickart, "A-stable composite multistep methods," *J. ACM* 20 (1973), 7-26.

[17] O. B. Widlund, "A note on unconditionally stable linear multistep methods," *BIT* 7 (1967), 65-70.

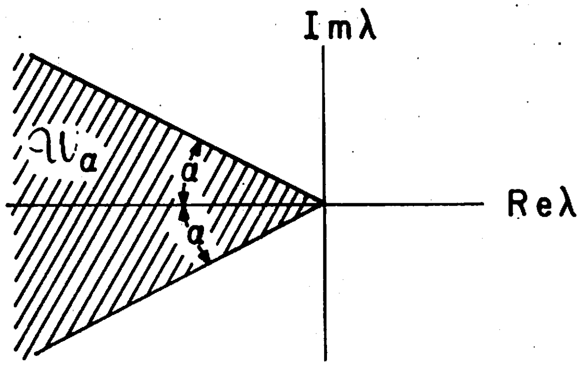

Fig. 1. Open region $\mathcal{W}_\alpha \subset \mathcal{C}^*$.

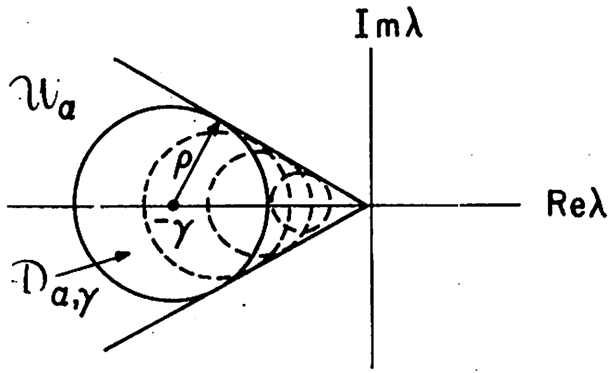

Fig. 2. $\mathcal{W}_\alpha$ — the union $\cup_{\gamma\in(0,\infty)}\mathcal{D}_{\alpha,\gamma}$

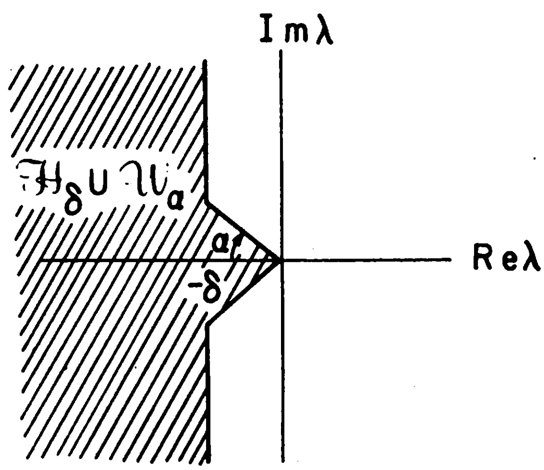

Fig. 3. Open region $\mathcal{H}_\delta \cup \mathcal{W}_\alpha \subset \mathbb{C}^*$