RECURSIVE CAUSAL LINEAR FILTERING
FOR TWO-DIMENSIONAL RANDOM FIELDS

by

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1. Introduction

The basic problem that we shall consider in this paper is the estimation of a two-parameter Gaussian random field in the presence of an additive independent white Gaussian noise. Specifically, consider an observation equation of the form

(1) $$\xi(t_1,t_2) = x(t_1,t_2)+\eta(t_1,t_2), \qquad (t_1,t_2) \in T$$

where ξ denotes the observation, x a Gaussian random field, η a Gaussian random field independent of x and having a covariance function

(2) $$E\eta(t_1,t_2)\eta(s_1,s_2) = N_0\delta(t_1-s_1)\delta(t_2-s_2)$$

and T is a rectangle in the plane, say $T = (a_1,b_1)\times(a_2,b_2)$. Because x and η are jointly Gaussian, the general estimator

(3) $$E[x(t_1,t_2)|\eta(s_1,s_2) \in S]$$

is linear in η for any set S in T. Thus, the meaning of the word "linear" in the title is clear.

For each t in T, denote by A_t the quadrant below and to the left

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of t, i.e.,

(4) $$A_t = \{(s_1,s_2): \ s_1 \leq t_1, \ s_2 \leq t_2\}$$

We shall call A_t the <u>past</u> at t. Let A_t denote the boundary of A_t, i.e.,

(5) $$\partial A_t = \{(s_1,t_2): s_1 \leq t_1\} \cup \{(t_1,s_2): s_2 \leq t_2\}$$

We shall call ∂A_t the <u>present</u> at t. Denote

(6) $$\hat{x}(t|s) = E(x(t)|\eta(\tau), \ \tau \in A_s)$$

By the "causal" estimator we shall mean $\hat{x}(t|t)$, which is the estimator of x(t) that uses only the past data at t.

By "recursive filtering" we shall refer to a means of suitably embedding $\hat{x}(t|t)$ in a state $X(\cdot)$ so that for $t' \succ t$ (which means $t_1' \geq t_1$ and $t_2' \geq t_2$) $X(t')$ can be computed using $X(t)$ and the observed data in the area between $A_{t'}$ and A_t. The results of [1] show that if x(t) has a Markovian property (in the sense that $t' \succ t$ implies the conditional independence of $x(t')$ from the past of x at t given the present of x at t) then recursive filtering is indeed possible and the state can be taken to be

(7) $$X(t) = \{\hat{x}(s|t), \ \ s \in \partial A_t\}$$

To model the dynamics of x(t), we shall take a class of partial differential equations, which are often used as such in the literature [2,3,4,5]. From these modeling equations, we shall derive the recursive filtering equations and the generalized Riccati equation. The Riccati equation will be solved for one specific example, corresponding to the case where x(t) is a homogeneous random field with a spectral density function given by

$$S_x(\nu_1,\nu_2) = \frac{K}{|(i\nu_1+\alpha_1)(i\nu_2+\alpha_2)|^2}$$

The problem considered here is to be distinguished from the problem of computing the estimation $\hat{x}(t_1,t_2|t_1,T_2)$ for a fixed T_2 as t_1 changes. The latter might be described as the one-sided-recursive half-plane-causal filtering which has been considered in a recent paper by Wong and Tsui [6]. The problem treated in this paper is considerably more complex, owing to the inherently two-dimensional dynamics of information change. Whether the results are more useful is arguable, but we think that aside from its mathematical interest the two-dimensional recursive causal filter is important for a number of reasons which include the following:

(1) It allows data to be added in either or both directions, and reduces to the one-sided filter as a degenerate limiting case.

(2) The state X(t) as given by (7) plays a special role in the computation of the likelihood ratio [7], which in turn is essential in hypothesis testing and parameter estimation.

(3) One expects, and the results vindicate this, that the dynamic of $\hat{x}(t|t)$ follows closely that of x(t).

(4) Most importantly, the state $X(t) = \{\hat{x}(s|t), s \in \partial A_t\}$ plays a generic role in recursive computation for all estimation problems much as that played by the causal estimator in one dimension. Thus, whether or not one is

interested in causal estimators per se, recursive computation forces them to be considered.

2. Wiener Process and Stochastic Integrals

As in the one-dimensional case, the pathology of white noise can be avoided by dealing with its integral. Let W(A) be a Gaussian random function parameterized by sets A in the plane such that EW(A) = 0 and

$$EW(A)W(B) = \text{Area}(A \cap B) \tag{2.1}$$

We shall call W(A) a <u>standard</u> two-parameter Wiener process. Formally, we can view W(A) as the integral over A of a Gaussian white noise with spectral density equal to 1. We can now rewrite (1) in terms of a standard Wiener process W as

$$Z(t) = \int_{A_t} x(s)ds + \sqrt{N_0}\, W(A_t) \tag{2.2}$$

where A_t is the lower-left quadrant of T with tip at t as defined earlier, and Z is now the observed field. The Wiener process W is independent for nonoverlapping areas and this captures the independence property of white noise.

Our experience with one-parameter processes suggests that in dealing with a Wiener process and its transformations a stochastic calculus is necessary and such a calculus must be closely related to a theory of martingales. These considerations motivated Wong and Zakai [8,9] and Cairoli and Walsh [10] to undertake a systematic study of martingales with a two-dimensional parameter and their associated stochastic calculus. These results will be briefly summarized in this section.

We begin with some notations and terminology. For two points t and s on the plane $t \succ s$ will mean $t_1 \geq s_1$ and $t_2 \geq s_2$. Let T be a rectangle. A family of σ-fields $\{\mathscr{F}_t, t \in T\}$ is said to be <u>increasing</u> if $t \succ s$ implies that $\mathscr{F}_t \supset \mathscr{F}_s$. A random field $\{M(t), t \in T\}$ is said to be <u>adapted</u> to $\{\mathscr{F}_t, t \in T\}$ if for each t in T M(t) is $\mathscr{F}_t$-measurable, and a <u>martingale</u> with respect to $\{\mathscr{F}_t, t \in T\}$ if $t \succ s$ implies that

$$(2.3) \qquad E(M(t) \mid \mathscr{F}_s) = M(s) \quad \text{almost surely}$$

To be brief, we shall say M(t) is an $\mathscr{F}_t$-martingale or $\{M(t), \mathscr{F}_t\}$ is a martingale.

Let $\{\mathscr{F}_t, t \in T\}$ be an increasing family of σ-fields. Let W be a standard Wiener process such that W(A) is $\mathscr{F}_t$-measurable if $A \subset A_t$ and $\mathscr{F}_t$-independent if A and A_t are disjoint. Then $\{W(A_t), \mathscr{F}_t, t \in T\}$ is a martingale and we shall investigate stochastic integrals with respect to W. The first stochastic integral to be introduced, and it is an obvious generalization of the Ito integral, is of the form

$$(2.4) \qquad M = \int_T \phi(s)W(ds)$$

where ϕ is an $\mathscr{F}_t$-adapted random field and satisfies

$$(2.5) \qquad \int_T E\phi^2(s)ds < \infty$$

This integral can be defined as the quadratic-mean limit of a sequence of approximating sums, i.e.,

$$(2.6) \qquad M = \lim_{n\to\infty} \text{ in q.m} \sum_{i,j} \phi(t_{ij}^{(n)})W(\Delta_{ij}^{(n)})$$

where for each n $\{t_{ij}^{(n)}\}$ is a rectangular partition of T, $\Delta_{ij}^{(n)} = t_{i+1,j+1}^{(n)} - t_{i+1,j}^{(n)} - t_{i,j+1}^{(n)} + t_{ij}^{(n)}$, and $\{t_{ij}^{(n)}\}$ refers to zero as $n \to \infty$. So defined, M has the martingale property that

$$E(M|\mathscr{F}_t) = \int_{A_t} \phi(s)W(ds) \tag{2.7}$$

In [8] Wong and Zakai raised the question as to whether every square-integrable functional of a Wiener process is expressible in the form of (2.4). While the answer for the one-parameter case is affirmative, the answer for a parameter space of dimension two or more is no. For a two-parameter Wiener process every square-integrable functional is of the form

$$M = \int_T \phi(s)W(ds) + \int_{T\times T} \psi(s,s')W(ds)W(ds') \tag{2.8}$$

The second integral on (2.8), to be defined shortly, will be called a multiple stochastic integral. Cairoli and Walsh [10] introduced a class of mixed integrals, which slightly modified were found to be required for a complete stochastic calculus in the plane [9,11].

Multiple stochastic integrals and mixed integrals can all be expressed as

$$\int_{T\times T} \psi(s,s')X(ds)Y(ds')$$

where X and Y are either the Wiener process W or the Lebesgue measure. Let $s \curlywedge s'$ denote the fact that $s_1 \leqslant s_1'$ and $s_2 \geqslant s_2'$. Then for ψ which satisfy the condition

(a) $\psi(s,s')$ is measurable with respect to $\mathscr{F}_{(s_1',s_2)}$ and

(b) $\int_{s \wedge s'} E\psi^2(s,s')dsds' < \infty$

the integral is defined by

$$(2.9) \qquad \int_{T\times T} \psi(s,s')X(ds)Y(ds') = \lim_{n\to\infty} \text{ in q.m.} \sum_{\substack{s\wedge s' \\ s,s'\in T_n}} \psi(s,s')X(\Delta s)Y(\Delta s')$$

where T_n is a sequence of partitions of T which refine to zero and Δs stands for the forward incremental area

$$\Delta s = (s_1+\delta_1, s_2+\delta_2)-(s_1+\delta_1,s_2)-(s_1,s_2+\delta_2)+(s_1,s_2)$$

In [11] it was shown that if a random field X(t) is defined by

$$(2.10) \qquad X(t) = \int_{A_t} \phi(s)W(ds) + \int_{A_t\times A_t} \alpha(s,s')W(ds)W(ds')$$

$$+\int_{A_t\times A_t} \beta(s,s')W(ds)ds' + \int_{A_t\times A_t} \gamma(s,s')dsW(ds')$$

$$+\int_{A_t} \theta(s)ds$$

then F(X(t)) for a suitably differentiable function F is again expressible as the sum of integrals of the same types.

It is tempting to write the multiple stochastic integral and the mixed integrals as iterated integrals. This can be done as follows: Suppose that Γ is an <u>increasing</u> path connecting the minimal point and the maximal point of the rectangle T. For any point s in T we define $s(\Gamma)$ as the smallest point on Γ which dominates s, i.e.,

$$s(\Gamma) = \min\{t: t \in \Gamma, t \succ s\} \tag{2.11}$$

Let $\phi(s)$ be a random function and let Γ be an increasing path such that $\phi(s)$ is $\mathscr{F}_{s(\Gamma)}$-measurable for each s. We shall say ϕ is Γ-adapted. If ϕ is Γ-adapted for some Γ then the stochastic integral

$$M = \int_T \phi(s)W(ds) \tag{2.12}$$

can again be defined. We note that if $\phi(s)$ is $\mathscr{F}_s$-measurable then it is $\mathscr{F}_{s(\Gamma)}$-measurable for any increasing Γ. Hence, the definition of (2.12) represents a considerable generalization over (2.4). If ϕ is Γ-adapted then the integral

$$M_t = \int_{A_t} \phi(s)W(ds)$$

is a martingale (one-parameter) on the path $t \in \Gamma$. With this generalization we can now express the multiple stochastic integral and the mixed integrals as interated integrals, i.e.,

$$\int_{T\times T} \psi(s,s')X(ds)Y(ds')$$

$$= \int_T [\int_T I(s\curlywedge s')\psi(s,s')X(ds)]Y(ds')$$

$$= \int_T [\int_T I(s\curlywedge s')\psi(s,s')Y(ds')]X(ds)$$

where $I(s\curlywedge s')$ denotes the indicator function of the set in $T\times T$ of (s,s') which satisfy $s\curlywedge s'$ and the inner and outer integrals are either a Lebesgue integral or a stochastic integral in the sense of (2.12). For example, let T be the rectangle $\{s : a_1 \leqslant s_1 \leqslant b_1,\ a_2 \leqslant s_2 \leqslant b_2\}$. Then

$$\int_{T\times T} \psi(s,s')dsW(ds')$$

$$= \int_T [\int_T I(s\curlywedge s')\Psi(s,s')ds]W(ds')$$

where the integrand $\int_T I(s\curlywedge s')\psi(s,s')ds$ of the outer integral is adapted to the path $\Gamma_1 = \{(a_1,s_2),\ a_2 \leqslant s_2 \leqslant b_2\} + \{(s_1,b_2),\ a_1 \leqslant s_1 \leqslant b_1\}$

<u>Figure 1</u>: The path Γ_1

It follows that

$$M_t = \int_{A_t\times A_t} \psi(s,s')dsW(ds')$$

is a one-parameter martingale with respect to $\{\mathscr{F}_{(t_1,b_2)},\ a_1 \leqslant t_1 \leqslant b_1\}$. Similarly,

$$M_t = \int_{A_t\times A_t} \psi(s,s')W(ds)ds'$$

is a martingale with respect to $\{\mathscr{F}_{(b_1,t_2)}, a_2 \leq t_2 \leq b_2\}$. We shall call a random field $\{M_t, t \in T\}$ an adapted $\{\mathscr{F}_t\}$ 1-martingale (2-martingale) if M_t is $\mathscr{F}_t$-measurable for each t and is a one-parameter martingale with respect to $\mathscr{F}_{(t_1,b_2)}$ (resp. $\mathscr{F}_{(b_1,t_2)}$).

3. Filtering Equations

Once again, consider the observation equation

(3.1) $$Z(t) = \int_{A_t} x(s)ds + \sqrt{N_0}W(t)$$

Let $\mathscr{F}_t$ denote the σ-field generated by $\{x(s), W(s), s < t\}$ and let $\mathscr{F}_{zt}$ denote the σ-field generated by $\{Z(s), s < t\}$. Let $\hat{x}(s|t)$ denote the conditional expectation

(3.2) $$\hat{x}(s|t) = E[x(s) | \mathscr{F}_{zt}]$$

A generalization of the innovations representation in one dimension is that if M(t) is an adapted $\mathscr{F}_{zt}$ 1-martingale (2-martingale) then it must be of the form

(3.3-1) $$M(t_1,t_2) = M_0 + \int_{a_1}^{t_1}\int_{a_2}^{t_2} k(t_2,s_1,s_2)[A(ds_1ds_2)-\hat{x}(s_1,s_2|s_1,t_2)ds_1ds_2]$$

respectively

(3.3-2) $$M(t_1,t_2) = M_0 + \int_{a_1}^{t_1}\int_{a_2}^{t_2} k(t_1,s_1,s_2)[Z(ds_1ds_2)-\hat{x}(s_1,s_2\ t_1,s_2)ds_1ds_2]$$

We note that the only difference between these two representations is the term involving $\hat{x}$.

Equation (3.3) can usefully be rewritten in a differential form as

(3.4-1) $$M(dt_1,t_2) = \int_{a_2}^{t_2} k(t_2,t_1,s_2)[A(dt_1ds_2)-\hat{x}(t_1,s_2\ t_1,t_2)dt_1ds_2]$$

respectively

(3.4-2) $$M(t_1,dt_2) = \int_{a_1}^{t_1} k(t_1,s_1,t_2)[Z(ds_1dt_2)-\hat{x}(s_1,t_2\ t_1,t_2)ds_1dt_s]$$

These forms suggest certain limitations on the recursion of the causal estimation $\hat{x}(t|t)$. For example, consider an incremental change

$t_1 \to t_1+dt_1$

$$d_{t_1}\hat{x}(t|t)-E[d_{t_1}x(t)|\mathscr{F}_{zt}] = M(dt_1,t_2) \tag{3.5}$$

where $M(t)$, $\mathscr{F}_{zt}$ is a martingale in t_1 for each t_2. Therefore,

$$d_{t_1}\hat{x}(t|t)-E[d_{t_1}x(t)|\mathscr{F}_{zt}] = \int_{a_2}^{t_2} k(t_2,t_1,s_2)]Z(dt_1ds_2)-\hat{x}(t_1,s_2|t_1,t_2)dt_1ds_2] \tag{3.6}$$

and this means that no matter what the dynamics of x is, the best recursion that can be hoped for in t_1 is for the line $\{\hat{x}(t_1,s_2|t_1,t_2),\ a_2 \leq s_2 \leq t_2\}$. A similar conclusion follows in the t_2 direction.

The foregoing also suggests that for a line-by-line recursive filtering to be possible in both directions; an appropriate model for the dynamics of x is given by

$$\begin{aligned} x(dt_1,t_2) &= \alpha_1(t_1,t_2)x(t_1,t_2)dt_1+dt_1\int_{a_2}^{t_2} f_1(t_1,t_2;s_2)x(t_1,s_2)ds_2 \\ &\quad + \int_{a_2}^{t_2} h_1(t_1,t_2;s_2)V(dt_1ds_2) \\ x(t_1,dt_2) &= \alpha_2(t_1,t_2)x(t_1,t_2)dt_2\_dt_2\int_{a_1}^{t_1} f_2(t_1,t_2;s_1)x(s_1,t_2)ds_1 \\ &\quad + \int_{a_1}^{t_1} h_2(t_1,t_2;s_1)V(ds_1dt_2) \end{aligned} \tag{3.7}$$

where V is a standard Wiener process independent of the observation noise W and α_i, f_i, h_i are deterministic functions. Naturally, since $d_{t_2}x(dt_1,t_2) = d_{t_1}x(t_1,dt_2)$, the function α_i, f_i, h_i must be interrelated. It is easy to verify that subject to some obvious differentiability conditions on the kernels, a sufficient condition for x to satisfy (3.7) is that it should satisfy the symmetric equation

$$\text{(3.8)}\quad x(dt_1dt_2) = \alpha_1(t_1,t_2)dt_1x(t_1,dt_2)+\alpha_2(t_1,t_2)dt_2x(dt_1,t_2) +\beta(t_1,t_2)x(t_1,t_2)dt_1dt_2+\gamma(t_1,t_2)V(dt_1dt_2)$$

from which the functions f_i and h_i can be found to be expressible as

$$\text{(3.9)}\quad \begin{aligned} f_1(t;s_2) &= [\beta(t)+\alpha_1(t)\alpha_2(t)-\frac{\partial}{\partial t_2}\alpha_1(t)]\exp[\int_{s_2}^{t_2}\alpha_2(t_1,\tau_2)d\tau_2 \\ f_2(t_1t_2;s_1) &= [\beta(t)+\alpha_1(t)\alpha_2(t)-\frac{\partial}{\partial t_1}\alpha_2(t)]\exp[\int_{s_1}^{t_1}\alpha_1(\tau_1,t_2)d\tau_1] \\ h_1(t_1,t_2;s_2) &= \gamma(t_1,t_2)\exp[\int_{s_2}^{t_2}\alpha_2(t_1,\tau_2)d\tau_2] \\ h_2(t_1,t_2;s_1) &= \gamma(t_1,t_2)\exp[\int_{s_1}^{t_1}\alpha_1(\tau_1,t_2)d\tau_1] \end{aligned}$$

At any t the state of the filtering equation is $X(t) = \{\hat{x}(s|t), s \in \partial A_t\}$. The first set of filtering equations will exhibit the change in these quantities as $t_1 \to t_1+dt_1$ or as $t_2 \to t_2+dt_2$, one at a time. These equations would be "partial differential" equations similar to (3.7). Since only t_1 or t_2 changes, these are basically one-dimensional equations and the only aspect of their derivation which is two dimensional is the innovations representation (c.f. (3.6)). Generally, let $\theta(t_1,t_2)$ be any random function and let $\hat{\theta}(t|t)$ denote $E[\theta(t)|t]$. Then, the same line of reasoning leading to (3.6) will also yield the representation

$$d_{t_1}\hat{\theta}(t|t) = E[d_{t_1}\theta(t)|\mathscr{F}_{zt}] + \int_{a_2}^{t_2} k_1(t,s_2)\hat{Z}(dt_1 ds_2|t)$$

$$d_{t_2}\hat{\theta}(t|t) = E[d_{t_2}\theta(t)|\mathscr{F}_{zt}] + \int_{a_1}^{t_1} k_2(t,s_1)\hat{Z}(ds_1 dt_2|t)$$

where $\hat{Z}(ds|t)$ denotes the "innovations"

$$\hat{Z}(ds|t) = Z(ds)-\hat{x}(s|t)ds = [x(s)-\hat{x}(s|t)]ds + \sqrt{N_0}\,W(ds) \tag{3.10}$$

Now, consider a point on the boundary ∂A_t, say (τ_1,t_2) with $\tau_1 \leq t_1$. Since $d_{t_1}x(\tau_1,t_2) = \delta_{\tau_1 t_1}x(dt_1,t_2)$ ($\delta_{ab}=1$ if $a=b$, $=0$ otherwise) we have

$$d_{t_1}\hat{x}(\tau_1,t_2|t) = \delta_{\tau_1 t_1}E[x(dt_1,t_2)|\mathscr{F}_{zt}] + \int_{a_2}^{t_2} k(\tau_1,t_2;t_1,s_2)\hat{Z}(dt_1 ds_2|t) \tag{3.11}$$

Using (3.7), we get

$$E[x(dt_1,t_2)|\mathscr{F}_{zt}] = dt_1[\alpha_1(t)\hat{x}(t|t)+\int_{a_2}^{t_2} f_1(t;s_2)\hat{x}(t_1,s_2|t)ds_2] \tag{3.12}$$

Therefore,

$$d_{t_1}\hat{x}(\tau_1,t_2|t) = \delta_{\tau_1 t_1}dt_1[\alpha_1(t)\hat{x}(t|t)+\int_{a_2}^{t_2} f_1(t;s_2)\hat{x}(t_1,s_2|t)ds_2] + \int_{a_2}^{t_2} k(\tau_1,t_2;t_1,s_2)\hat{Z}(dt_1 ds_2|t) \tag{3.13}$$

The gain function k can be computed by noting that for $\tau_2 \leq t_2$

$$E\{d_{t_1}[\hat{x}(\tau_1,t_2|t)\hat{Z}(t_1,\tau_2|t)|\mathscr{F}_{zt}\} = \hat{Z}(t_1,\tau_2|t)E[d_{t_1}\hat{x}(\tau_1,t_2|t)|\mathscr{F}_{zt}]$$
$$+ \hat{x}(\tau_1,t_2|t)E[\hat{Z}(dt_1,\tau_2|t)|\mathscr{F}_{zt}]+ N_0\int_{a_2}^{\tau_2} k(\tau_1,t_2;t_1,s_2)dt_1 ds_2$$

On the other hand, since W and V are independent Wiener processes, we have

$$E\{d_{t_1}[x(\tau_1,t_2)Z(t_1,\tau_2|t)]|\mathscr{F}_{zt}\} = \hat{Z}(t_1,\tau_2|t)E[d_{t_1}x(\tau_1,t_2)|\mathscr{F}_{zt}]$$
$$+ E[x(\tau_1,t_2)\hat{Z}(t_1,\tau_2|t)|\mathscr{F}_{zt}]$$

Because

$$E[d_{t_1}\hat{x}(\tau_1,t_2|t)|\mathscr{F}_{zt}] = E[d_{t_1}x(\tau_1,t_2)|\mathscr{F}_{zt}]$$

we have

$$N_0\int_{a_2}^{\tau_2} k(\tau_1,t_2;t_1,s_2)ds_2dt_1$$
$$= E[\varepsilon(\tau_1,t_2|t)\hat{Z}(dt_1,\tau_2|t)|\mathscr{F}_{zt}]$$
$$= \int_{a_2}^{\tau_2} E[\varepsilon(\tau_1,t_2|t)\varepsilon(t_1,s_2|t)|\mathscr{F}_{zt}]ds_2dt_1$$

where $\varepsilon(s|t)$ denotes the estimation error $x(s)-\hat{x}(s|t)$. Therefore,

$$k(\tau_1,t_2;t_1,\tau_2) = \frac{1}{N_0}\rho(\tau_1,t_2;t_1,\tau_2|t) \tag{3.14}$$

where ρ denotes the conditional covariance function

$$\rho(s,\tau|t) = E[\varepsilon(s|t)\varepsilon(\tau|t)|\mathscr{F}_{zt}] \tag{3.15}$$

The filtering equation (3.13) can now be written as:

$$d_{t_1}\hat{x}(\tau_1,t_2|t) = \delta_{\tau_1t_1}dt_1[\alpha_1(t)\hat{x}(t|t)+\int_{a_2}^{t_2} f_1(t;s_2)\hat{x}(t_1,s_2|t)ds_2]$$
$$+\frac{1}{N_0}\int_{a_2}^{t_2}\rho(\tau_1,t_2;t_1,s_2|t)\hat{Z}(dt_1ds_2|t)$$

Similarly, we can derive the equations for $d_{t_2}\hat{x}(\tau_1,t_2|t)$ and $d_{t_1}\hat{x}(t_1,\tau_2|t)$. These can all be expressed in a generic form as follows: For any point τ on the boundary ∂A_t, we can write

$$
\begin{aligned}
d_{t_1}\hat{x}(\tau|t) &= \delta_{\tau_1 t_1} dt_1[\alpha_1(\tau)\hat{x}(\tau|t)+\int_{a_2}^{\tau_2} f_1(\tau;s_2)\hat{x}(t_1,s_2|t)\,ds_2] \\
&\quad + \frac{1}{N_0}\int_{a_2}^{t_2} \rho(\tau;t_1,s_2|t)Z(dt_1ds_2|t) \qquad (3.16) \\
d_{t_2}\hat{x}(\tau|t) &= \delta_{\tau_2 t_2} dt_2[\alpha_2(\tau)\hat{x}(\tau|t)+\int_{a_1}^{\tau_1} f_2(\tau;s_1)\hat{x}(s_1,t_2|t)\,ds_1] \\
&\quad + \frac{1}{N_0}\int_{a_1}^{t_1} \rho(\tau;s_1,t_2|t)\hat{Z}(ds_1dt_2|t)
\end{aligned}
$$

Equation (3.16) represents the basic filtering equation for the two-dimensional problem, even though other forms derived by it may be more useful. Before deriving the alternative forms, however, we shall first find the generalized Riccati equation which is satisfied by the covariance function ρ.

For τ, $\tau' \in \partial A_t$ we have two distinct situations. Either τ and τ' are on the same segment of ∂A_t or they are on different segments. We shall first consider $\rho(\tau,\tau'|t)$ for τ and τ' on the same segment of ∂A_t, say $\rho(t_1,\tau_2;t_1,\tau_2'\ t)$. Using (3.7) and (3.16) we have

$$
\begin{aligned}
(3.17)\qquad d_{t_1}\varepsilon(t_1,\tau_2|t) &= dt_1[\alpha_1(t_1,z_2)\varepsilon(t_1,\tau_2|t)+\int_{a_2}^{\tau_2} f_1(t_1,\tau_2;s_2)\varepsilon(t_1,s_2|t)\,ds_2] \\
&\quad + \int_{a_2}^{\tau_2} h_1(t_1,\tau_2;s_2)V(dt_1ds_2) \\
&\quad - \frac{1}{N_0}\int_{a_2}^{t_2} \rho(t_1,\tau_2;t_1,s_2|t)\,[\varepsilon(t_1,s_2|t)\,dt_1ds_2+\sqrt{N_0}\,\dot{W}(dt_1ds_2)]
\end{aligned}
$$

It follows that

$$d_{t_1}\rho(t_1,\tau_2;t_1,\tau_2'|t) = E\{d_{t_1}[\varepsilon(t_1,\tau_2|t)\varepsilon(t_1,z_2'|t]|\mathscr{F}_{zt}\}$$

$$= E\{\varepsilon(t_1,\tau_2'\ t)d_{t_1}\varepsilon(t_1,\tau_2|t)+\varepsilon(t_1,\tau_2|t)d_{t_1}\varepsilon(t_1,\tau_2'|t)$$

$$+\int_{a_2}^{\min(\tau_2,\tau_2')} h_1(t_1,z_2;s_2)h_1(t_1,z_2';s_2)ds_2dt_1$$

$$+\frac{1}{N_0}\int_{a_2}^{t_2}\rho(t_1,\tau_2;t_1,s_2|t)\rho(t_1,\tau_2';t_1,s_2|t)ds_2dt_1$$

which yields

$$(3.18)\quad \frac{\partial}{\partial t_1}\rho(t_1,\tau_2;t_1,\tau_2'|t) = [\alpha_1(t_1,\tau_2)+\alpha_1(t_1,\tau_2')\rho(t_1,\tau_2;t_1,\tau_2'|t)$$

$$+\int_{a_2}^{t_2} f_1(t_1,\tau_2;s_2\rho(t_1,\tau_2';t_1,s_2|t)ds_2$$

$$+\int_{a_2}^{\tau_2'} f_1(t_1,\tau_2';s_2)\rho(t_1,\tau_2;t_1,s_2|t)ds_2$$

$$+\int_{a_2}^{\min(\tau_2,\tau_2')} h_1(t_1,\tau_2;s_2)h_1(t_1,\tau_2';s_2)ds_2$$

$$-\frac{1}{N_0}\int_{a_2}^{t_2}\rho(t_1,\tau_2;t_1,s_2|t)\rho(t_1,\tau_2';t_1,s_2|t)ds_2$$

This is seen to be a quadratic differential-integer equation for the quantity

$$\rho_{t_1}(\cdot) = \{\rho(t_1,\tau_2;t_1,\tau_2'|t);\quad \tau_2,\tau_2'\varepsilon[a_2,t_2]\}$$

Both (3.17) and (3.18) can be expressed more simply if we agree to the convention

$$f_1(t;s_2) = 0, \quad h_1(t;s_2) = 0 \quad \text{for } s_2 > t_2$$
$$f_2(t;s_1) = 0, \quad h_2(t;s_1) = 0 \quad \text{for } s_1 > t_1$$

Adopting this convention, we have the following equations for $\rho(\tau,\tau'|t)$ when τ and τ' are on the same segment of ∂A_t.

$$\begin{aligned}(3.19) \quad \frac{\partial}{\partial t_1}\rho(t_1,\tau_2;t_1,\tau_2'|t) &= [\alpha_1(t_1,\tau_2)+\alpha_1(t_1,\tau_2')]\rho(t_1,\tau_2;t_1,\tau_2'|t) \\ &+ \int_{a_2}^{t_2}[f_1(t_1,\tau_2;s_2)\rho(t_1,\tau_2';t_1,s_2|t) \\ &\qquad + f_1(t_1,\tau_2';s_2)\rho(t_1,\tau_2;t_1,s_2|t)]ds_2 \\ &+ \int_{a_2}^{t_2} h_1(t_1,\tau_2;s_2)h_1(t_1,\tau_2';s_2)ds_2 \\ &- \frac{1}{N_0}\int_{a_2}^{t_2}\rho(t_1,\tau_2;t_1,s_2|t)\rho(t_1,\tau_2';t_1,s_2|t)ds_2\end{aligned}$$

$$\begin{aligned}(3.20) \quad \frac{\partial}{\partial t_2}\rho(\tau_1,t_2;\tau_1',t_2|t) &= [\alpha_2(\tau_1,t_2)+\alpha_2(\tau_1',t_2)]\rho(\tau_1,t_2;\tau_1',t_2|t) \\ &+ \int_{a_1}^{t_1}[f_2(\tau_1,t_2;s_1)\rho(\tau_1',t_2;s_1,t_2|t) \\ &\qquad + f_2(\tau_1',t_2;s_1)\rho(\tau_1,t_2;s_1,t_2|t)]ds_1 \\ &+ \int_{a_1}^{t_1} h_2(\tau_1,t_2;s_1)h_2(\tau_1',t_2;s_1)ds_1 \\ &+ \frac{1}{N_0}\int_{a_1}^{t_1}\rho(\tau_1,t_2;s_1,t_2|t)\rho(\tau_1',t_2;s_1,t_2|t)ds_1\end{aligned}$$

Now for τ and τ' on opposite legs of ∂A_t, $\rho(\tau,\tau'|t)$ has the form $\rho(\tau_1,t_2;t_1,\tau_2|t)$. To find $\frac{\partial}{\partial t_1}\rho(\tau_1,t_2;t_1,\tau_2|t)$ we first write for $\tau_1 < t_1$

$$(3.21)\quad d_{t_1}\varepsilon(\tau_1,t_2|t) = -d_{t_1}\hat{x}(\tau_1,t_2|t)$$

$$= -\frac{1}{N_0}\int_{a_2}^{t_2}\rho(\tau_1,t_2;t_1,s_2|t)[\varepsilon(t_1,s_2|t)dt_1ds_2+\sqrt{N_0}\,W(dt_1ds_2)$$

We can now use (3.17) and (3.21) to get

$$(3.22)\quad d_{t_1}\rho(t_1,\tau_2;\tau_1,t_2|t) = E\{d_{t_1}[\varepsilon(t_1,\tau_2|t)\varepsilon(\tau_1,t_2|t)]|\mathscr{F}_{zt}\}$$

$$= E\{\varepsilon(t_1,\tau_2|t)d_{t_1}\varepsilon(\tau_1,t_2|t)+\varepsilon(\tau_1,t_2|t)d_{t_1}\varepsilon(t_1,\tau_2|t)$$

$$+\frac{1}{N_0}\int_{a_2}^{t_2}\rho(\tau_1,t_2;t_1,s_2|t)\rho(t_1,\tau_2;t_1,s_2|t)ds_2dt_1\}$$

$$= dt_1\{\alpha_1(t_1,\tau_2)\rho(t_1,\tau_2;\tau_1,t_2|t)$$

$$+\int_{a_2}^{t_2}f_1(t_2,\tau_2;s_2)\rho(t_1s_2;\tau_1,t_2|t)ds_2$$

$$-\frac{1}{N_0}\int_{a_2}^{t_2}\rho(t_1,\tau_2;t_1,s_2|t)\rho(t_1,s_2;\tau_1,t_2|t)ds_2\}$$

We observe that if $\rho(t_1,\tau_2;t_1,s_2|t)$ is determined by solving (3.19) then (3.22) is a linear equation for the function $\rho(t_1,\tau_2;\tau_1,t_2|t)$. By symmetry, we also have

$$(3.23)\quad \frac{\partial}{\partial t_2}\rho(t_1,\tau_2;\tau_1,t_2|t) = \alpha_2(\tau_1,t_2)\rho(t_1,\tau_2;\tau_1,t_2|t)$$

$$+\int_{a_1}^{t_1}f_2(\tau_1,t_2;s_1)\rho(t_1,\tau_1;s_1,t_2|t)ds_1$$

$$-\frac{1}{N_0}\int_{a_1}^{t_1}\rho(\tau_1,t_2;s_2,t_2|t)\rho(t_1,\tau_2;s_1,t_2|t)ds_1$$

Equations (3.19), (3.20), (3.22) and (3.23) provide us with a complete set of equations which determines $\rho(\tau,\tau'|t)$, $\tau,\tau' \in \partial A_t$.

Equation (3.16) represents the filtering counterpart of the state equation (3.7). The question arises as to what is the filtering counterpart of (3.8). After some tedious but routine manipulations, we find that for any $\tau \in \partial A_t$, $\hat{x}(\tau|t)$ satisfies

$$(3.24) \quad d_{t_1}d_{t_2}\hat{x}(\tau|t)-\delta_{\tau,t}\{\alpha_1(t)dt_1d_{t_2}\hat{x}(t|t)+\alpha_2(t)dt_2d_{t_1}\hat{x}(t|t)+\beta(t)\hat{x}(t|t)dt_1dt_2\}$$
$$= \frac{1}{N_0}\int_{a_2}^{t_2}[d_{t_2}\rho(\tau;t_1,s_2|t)]\hat{Z}(dt_1ds_2|t)$$
$$+ \frac{1}{N_0}\int_{a_1}^{t_1}[d_{t_1}\rho(\tau;s_1,t_2|t)]\hat{Z}(ds_1dt_2|t)$$
$$+ \frac{1}{N_0}\rho(\tau;t|t)\hat{Z}(dt|t)$$

If we denote the left hand side by $M(dt;\tau)$ then for each $\tau \in \partial A_t$ M is a "weak martingale" in the sense that

$$E[M(dt;\tau)\,|\,\mathscr{F}_{zt}] = 0$$

Thus, in this sense $M(dt;\tau)$ might be considered to be "white". We should compare the filtering equation:

$$(3.25) \quad d_{t_1}d_{t_2}\hat{x}(t|t) = \alpha_1(t)d_{t_1}d_{t_2}\hat{x}(t|t)+\alpha_2(t)d_{t_2}d_{t_1}\hat{x}(t|t)+\beta(t)\hat{x}(t|t)dt+M(dt;t)$$

where M is a weak martingale with the state equation (3.8)

$$d_{t_1}d_{t_2}x(t) = \alpha_1(t)d_{t_1}d_{t_2}x(t)+\alpha_2(t)d_{t_2}d_{t_1}x(t)+\beta(t)x(t)dt+\gamma(t)V(dt)$$

where V is a Wiener process. This is a direct generalization of the filtering equation in one dimension and as simple a generalization as could have been hoped for.

4. An Example

Possibly the simplest nontrivial example is the so-called "separable-covariance" model which has been used by a number of authors [4,5]. This corresponds to the special case of (3.8) with $\alpha_1(t) = -c_1$, $\alpha_2(t) = -c_2$, $\beta(t) = -c_1c_2$ and $\gamma(t) = 1$. It also corresponds to the case where the state x(t) is a homogeneous Gaussian random field with a spectral density function given by

$$S(\nu_1,\nu_2) = \frac{1}{|(i\nu_1+c_1)(i\nu_2+c_2)|^2} \tag{4.1}$$

We shall attempt to solve for ρ in the infinite-past case, i.e. $a_1 = a_2 = -\infty$. Consider (3.19). For this example $f_1 \equiv 0$, and

$$h_1(t_1,\tau_2;s_2) = e^{-c_2(\tau_2-s_2)}1(\tau_2-s_2) \qquad (1 = \text{unit step}) \tag{4.2}$$

It follows that

$$\int_{a_2}^{t_2} h_1(t_1,\tau_2;s_2)h_1(t_1,\tau_2';s_2)ds_2 = \frac{1}{2c_2}e^{-c_2|\tau_2-\tau_2'|} \tag{4.3}$$

Furthermore, it is clear that with an infinite past in the t_1 direction $\rho(t_1,\tau_2;t_1,\tau_2'|t)$ cannot depend on t_1 and its dependence on the remaining variables t_2, τ_1, τ_2' is only as a function of $t_2-\tau_2$ and $t_2-\tau_2'$. We shall define the function

$$r_1(u,v) = \rho(t_1,t_2-u;t_1,t_2-v|t) \tag{4.4}$$

and use it to rewrite (3.19) in the form

$$0 = -2c_1 r_1(u,v) + \frac{1}{2c_2} e^{-c_2|u-v|}$$

$$- \frac{1}{N_0} \int_0^\infty r_1(u,y) r_1(v,y)\,dy, \quad 0 \leqslant u,\ v < \infty$$

or

$$\int_0^\infty r_1(n,y) r_1(r,y)\,dy + 2c_1 N_0 r_1(u,v) = \frac{N_0}{2c_2} e^{-c_2|u-v|}, \quad 0 \leqslant u,\ v < \infty \tag{4.5}$$

Equation (4.5) can be solved by using the "Karhunen-Loève expansion" for $\frac{N_0}{2c_2} e^{-c_2|u-v|}$ on $0 \leqslant u,\ v < \infty$. Actually, it is a degenerate expansion since because the interval is infinite in length the spectrum will no longer be discrete. In any event, if we represent

$$\frac{N_0}{2c_2} e^{-c_2|u-v|} = \int_0^\infty \lambda(\nu)\phi(\nu,u)\phi(\nu,v)\,d\nu\ , \quad 0 \leqslant u,\ v < \infty,$$

where ϕ satisfies the orthonormality condition

$$\int_0^\infty \phi(\nu,y)\phi(\nu',y)\,dy = \delta(\nu-\nu')$$

then the solution of (4.5) can be expressed as

$$r_1(u,v) = \int_0^\infty \left[\sqrt{c_1^2 N_0^2 + \lambda(\nu)} - c_1 N_0\right] \phi(\nu,u)\phi(\nu,v)\,d\nu$$

The function $\phi(\nu,u)$ and $\lambda(\nu)$ can be obtained by solving the integral equation

$$\int_0^\infty \frac{N_0}{2c_2} e^{-c_2|u-v|} \phi(v)\,dv = \lambda\phi(u), \quad 0 \leqslant u < \infty.$$

which in turn can be converted into a differential equation (see e.g. [13,14])

$$\lambda\left(c_2^2 - \frac{d^2}{du^2}\right)\phi(u) = N_0\phi(u), \quad 0 < u < \infty$$

with the initial condition

$$\dot{\phi}(0) = c_2\phi(0)$$

and a boundedness condition on $\phi(u)$ as $u \to \infty$. Properly normalized, the solutions are

$$\phi(\nu,u) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{c_2^2+\nu^2}}[c_2 \sin \nu u + \nu \cos \nu u]$$

$$\lambda(\nu) = \frac{N_0}{c_2^2+\nu^2}$$

Therefore, the solution for $r_1(u,v)$ is given by

$$r_1(u,v) = \frac{2}{\pi} c_1c_2N_0 \int_0^\infty \left[\sqrt{1+\frac{1}{N_0c_1^2c_2^2(1+\nu^2)}} -1\right]\frac{1}{(1+\nu^2)}$$

$$[\sin c_2\nu u+\nu \cos c_2\nu u][\sin c_2\nu v+\nu \cos c_2\nu v]d\nu$$

It is clear that by symmetry we can define

$$r_2(u,v) = \rho(t_1-u,t_2;\ t_1-v,t_2|t)$$

and r_2 must be given by

$$r_2(u,v) = \frac{2}{\pi} c_1c_2N_0 \int_0^\infty \left[\sqrt{1+\frac{1}{N_0c_1^2c_2^2(1+\nu^2)}} -1\right]\frac{1}{1+\nu^2}$$

$$[\sin c_1\nu u+\nu \cos c_1\nu u][\sin c_1\nu v+\nu \cos c_1\nu v]d\nu$$

Since $r_1(0,0)$ and $r_2(0,0)$ are both equal to $\rho(t|t)$, they must be the same. It is easy to verify that the expressions we have obtained for r_1 and r_2 are consistent with this requirement.

The solutions r_1 and r_2 give us $\rho(\tau,\tau'|t)$ for τ,τ' on the same leg of ∂A_t. To complete the solution of the filtering problem, we still need $\rho(t_1,\tau_2;\tau_1,t_2|t)$, which must satisfy (3.22) and (3.23). We note that for the case at hand $\rho(t_1,\tau_2;\tau_1,t_2|t)$ depends only on $t_1-\tau_1$ and $t_2-\tau_2$. Therefore, it is convenient to define $q(\tau_1,\tau_2) = \rho(t_1,t_2-\tau_2;t_1-\tau_1,t_2|t)$ and rewrite (3.22) and (3.23) as

$$\frac{\partial}{\partial\tau_1} q(\tau_1,\tau_2) = -c_1 q(\tau_1,\tau_2) - \frac{1}{N_0}\int_0^\infty r_1(\tau_2,s_2)q(\tau_1,s_2)ds_2$$

$$\frac{\partial}{\partial\tau_2} q(\tau_1,\tau_2) = -c_2 q(\tau_1,\tau_2) - \frac{1}{N_0}\int_0^\infty r_2(\tau_1,s_1)q(s_1,\tau_2)ds_1$$

Consistency requires that

$$\left(\frac{\partial}{\partial\tau_1}+c_1\right)\int_0^\infty r_1(\tau_2,s_2)q(\tau_1,s_2)ds_2 = \left(\frac{\partial}{\partial\tau_2}+c_2\right)\int_0^\infty r_2(\tau_1,s_1)q(s_1,\tau_2)ds_1$$

which is assured provided that $q(\tau_1,s_2) = f(c_1\tau_1,c_2s_2)$ and f is a symmetric function.

We can solve for $q(\tau_1,s_2)$ by assuming it to be of the form

$$q(\tau_1,s_2) = \int_0^\infty A(\nu_1 c_1\tau_1)\theta(\nu_1 c_2 s_2)d\nu$$

where

$$\theta(\nu,\tau) = \sqrt{\frac{2}{\pi}}\ \frac{1}{\sqrt{1+\nu^2}}\ (\sin\nu\tau + \nu\cos\nu\tau)$$

is orthonormalized so that

$$\int_0^\infty \theta(\nu,\tau)\theta(\nu',\tau)d\tau = \delta(\nu-\nu')$$

Since $r_1(\tau_2,s_2)$ is given by

$$r_1(\tau_2,s_2) = c_1c_2N_0\int_0^\infty K(\nu)\theta(\nu,c_2t_2)\theta(\nu,c_2s_2)d\nu$$

with

$$K(\nu) = \sqrt{1 + \frac{1}{N_0 c_1^2 c_2^2 (1+\nu^2)}} \; -1$$

we have

$$\frac{\partial}{\partial \tau_1} A(\nu, c_1\tau_1) + c_1 [1+K(\nu)] A(\nu, c_1\tau_1) = 0$$

or

$$A(\nu, c_1\tau_1) = A(\nu, 0) e^{-[1+K(\nu)]c_1\tau_1}$$

$A(\nu,0)$ can be found by noting that $q(0,\tau_2) = r_1(0,\tau_2)$ so that

$$\int_0^\infty A(\nu,0)\theta(\nu,c_2\tau_2)d\nu = c_1 c_2 N_0 \int_0^\infty K(\nu)\theta(\nu,0)\theta(\nu,c_2\tau_2)d\nu$$

or

$$A(\nu,0) = c_1 c_2 N_0 K(\nu)\theta(\nu,0)$$

Therefore,

$$q(\tau_1,\tau_2) = c_1 c_2 N_0 \int_0^\infty K(\nu) e^{-[1+K(\nu)]c_1\tau_1} \theta(\nu,0)\theta(\nu,c_2\tau_2)d\nu$$

Symmetry dictates that q must also be expressible as

$$q(\tau_1,\tau_2) = c_1 c_2 N_0 \int_0^\infty K(\nu) e^{-[1+K(\nu)]c_2\tau_2} \phi(\nu,0)\phi(\nu,c_1\tau_1)d\nu$$

This is indeed true and we can write q in a symmetric form as

$$q(\tau_1,\tau_2) = c_1 c_2 N_0 \int_0^\infty K(\nu) e^{-[1+K(\nu)]c_2\tau_2} \phi(\nu,0)\phi(\nu,c_1\tau_1)d\nu$$

This is indeed true and we can write q in a symmetric form as

$$q(\tau_1,\tau_2) = \int_0^\infty \int_0^\infty \frac{c_1 c_2 N_0}{[c_1^2 c_2^2 N_0 (1+\nu^2)(1+\mu^2)+1]} \theta(\nu,0)\theta(\mu,0)\theta(\nu,c_1\tau_1)\theta(\mu,c_2\tau_2)d\nu\, d\mu$$

Finally, we can use the solution ρ to obtain the transfer function of the one-quadrant causal Wiener filter. Take the first equation of (3.16) and write it formally for the present example as

$$\frac{\partial}{\partial t_1}\hat{x}(t_1,\tau_2|t_1,t_2) = -c_1\hat{x}(t_1,\tau_2|t_1,t_2) + \frac{1}{N_0}\int_{-\infty}^{t_2} r_1(t_2-\tau_2,t_2-s_2)[\xi(t_1,s_2)-\hat{x}(t_1,s_2|t_1,t_2)]ds_2$$

where ξ represents the observed field in the original white noise form (c.f. (1.1)). The stationarity inherent in the present example suggests that we can write

$$\hat{x}(t_1,t_2-\tau_2|t_1,t_2) = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} h_1(\tau_2;t_1-s_1,t_2-s_2)\xi(s_1,s_2)ds_1ds_2$$

If we define $H_1(\tau_2;\nu_1,\nu_2)$ as the Fourier transform of h_1, i.e.,

$$H_1(\tau_2;\nu_1,\nu_2) = \int_0^\infty\int_0^\infty h_1(\tau_2;s_1,s_2)e^{-i(\nu_1s_1+\nu_2s_2)}ds_1ds_2$$

then H_1 must satisfy the equation

$$(i\nu_1+c_1)H_1(\tau_2;\nu_1,\nu_2) + \frac{1}{N_0}\int_0^\infty r_1(\tau_2,s_2)H_1(s_2;\nu_1,\nu_2)ds_2 = \frac{1}{N_0}\tilde{r}_1(\tau_2,\nu_2)$$

where

$$\tilde{r}_1(\tau_2,\nu_2) = \int_0^\infty e^{-i\nu_2s_2}r(\tau_2,s_2)ds_2$$

H_1 can be obtained by assuming it to be of the form

$$H_1(\tau_2;\nu_1,\nu_2) = \int_0^\infty A_1(\nu;\nu_1,\nu_2)\theta(\nu,c_2\tau_2)d\nu$$

The coefficient A_1 is easily shown to be given by

$$A_1(\nu;\nu_1,\nu_2) = \frac{K(\nu)}{i(\frac{\nu_1}{c_1})+[1+K(\nu)]} \quad \hat{\theta}(\nu, \frac{\nu_2}{c_2})$$

where $K(\nu) = [(1+[N_0 c_1^2 c_2^2 (1+\nu^2)]^{-1})^{1/2} - 1]$ and

$$\hat{\theta}(\nu,\mu) = \int_0^\infty \theta(\nu,\tau)e^{-i\mu\tau}d\tau$$

A more explicit expression for $H_1(\tau_2;\nu_1,\nu_2)$ can be obtained by carrying out the necessary integration for $\hat{\theta}$ to yield

$$H_1(\tau_2;\nu_1,\nu_2) = \frac{-iK(\frac{\nu_2}{c_2})}{[1+K(\frac{\nu_2}{c_2})+i(\frac{\nu_1}{c_1})]} \left[\frac{\sin \nu_2\tau + (\frac{\nu_2}{c_2})\cos \nu_2\tau}{1-i(\frac{\nu_2}{c_2})}\right]$$

$$+ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{K(\nu)[1+i(\frac{\nu_2}{c_2})](i\nu)e^{i\nu c_2\tau_2}}{[1+K(\nu)+i(\frac{\nu_1}{c_1})](1-i\nu)[\nu^2-(\frac{\nu_2}{c_2})^2]}\, d\nu$$

when the integral is to be interpreted as a Cauchy principal-value integral.

By setting $\tau_2 = 0$ in $H_1(\tau_2;\nu_1,\nu_2)$ we get the transfer function for the Wiener filter for $\hat{x}(t|t)$. Symmetry dictates that the transfer function

$$H(\nu_1,\nu_2) = H_1(0;\nu_1,\nu_2)$$

must be a symmetric function of $(\frac{\nu_1}{c_1})$ and $(\frac{\nu_2}{c_2})$. This symmetry is far from obvious in the integral representation for $H_1(\tau_2;\nu_1,\nu_2)$. To obtain a symmetric expression for $H_1(0;\nu_1,\nu_2)$, write

$$H_1(0;\nu_1,\nu_2) = \int_0^\infty A_1(\nu;\nu_1,\nu_2)\theta(\nu,0)d\nu$$

$$= \int_0^\infty \frac{K(\nu)\theta(\nu,0)}{[i(\frac{\nu_2}{c_2})+1+K(\nu)]} \hat{\theta}(\nu, \frac{\nu_2}{c_2})d\nu$$

A comparison with the expression previously obtained for $q(\tau_1,\tau_2)$, viz.,

$$q(\tau_1,\tau_1) = c_1c_2N_0 \int_0^\infty K(\nu)e^{-[1+K(\nu)]c_1\tau_1}\theta(\nu,0)\theta(\nu,c_2\tau_2)d\nu$$

yields a symmetric expression for $H_1(0;\nu_1,\nu_2)$

$$H_1(0;\nu_1,\nu_2) = \frac{1}{N_0}\int_0^\infty\int_0^\infty e^{-i(\nu_1\tau_1+\nu_2\tau_2)} q(\tau_1,\tau_2)d\tau_1 d\tau_2$$

$$= \int_0^\infty\int_0^\infty [1+c_1^2c_2^2N_0(1+\nu^2)(1+\mu^2)]^{-1}\theta(\mu,0)\hat{\theta}(\mu,0)\hat{\theta}(\nu,\frac{\nu_1}{c_1})\hat{\theta}(\nu,\frac{\nu_2}{c_2})d\nu d\mu$$

The relationship between q and H implies that we can write

$$\hat{x}(t_1,t_2|t_1,t_2) = \frac{1}{N_0}\int_{-\infty}^{t_1}\int_{-\infty}^{t_2} q(t_1-s_1,t_2-s_2)\xi(s_1,s_2)ds_1ds_2$$

$$= \frac{1}{N_0}\int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \rho(t_1,s_2;s_1,t_2|t)\xi(s_1,s_2)ds_1ds_2$$

Thus far, we have not been able to determine the degree of generality of this relationship.

5. Discussion

We have chosen to derive the main filtering equation (3.24) by considering the partial differentials in each direction. In effect, we took the path of successive differentiation. A more elegant and a more easily generalized alternative derivation is the following:

(a) If we define $M(\tau, dt)$ to be the left hand side of (3.24) then we can show that M must be a weak martingale.

(b) An innovations representation for $\mathcal{F}_{zt}$-weak martingales can be derived which shows that M must be of the form

$$M(\tau,t) = \int_{A_t} \theta(\tau,s)\hat{Z}(ds|s)+\int_{A_t\times A_t} \psi(\tau;s,s')[\hat{Z}(ds|s\vee s')\hat{Z}(ds'|s\vee s') - \rho(s,s'|s\vee s')dsds']$$

$$+ \int_{A_t\times A_t} \phi_1(\tau;s,s')ds\hat{Z}(ds'|s\vee s')$$

$$+ \int_{A_t\times A_t} \phi_2(\tau;s,s')\hat{Z}(ds|s\vee s')ds'$$

(c) Since M must be linear in $\hat{Z}$ the second integral must vanish, and what remains yields the general form of (3.24).

This derivation suffers from the disadvantage of not readily yielding the gain functions as by-products. Hence we chose a more direct approach.

Let $t > \tau$. The problem of determining $\hat{x}(\tau|t)$ might be termed a smoothing problem. We note that the difference between $\hat{x}(\tau|t)$ and $\hat{x}(\tau|\tau)$ lies entirely in the observation. Using (3.24), we can recursively compute $\hat{x}(\tau|t)$ as t changes by again using $X(t) = \{\hat{x}(s|t), s \in \partial A_t\}$, justifying in part our remark in the introduction that X(t) plays a generic role in all estimation problems.

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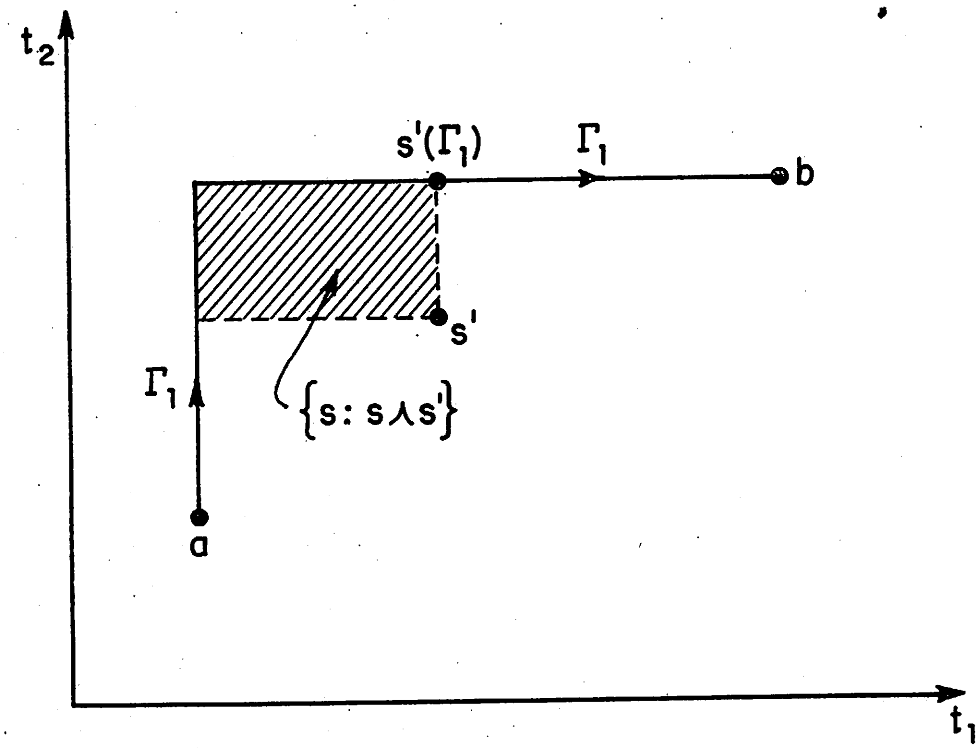

Figure 1 The Path Γ_1