LIFE CYCLE CONSUMPTION AND HOMEOWNERSHIP

by

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ABSTRACT

We consider the following "portfolio" problem of life cycle consumption and saving. In order to achieve an optimal rate of consumption over the life cycle, a consumer must hold corresponding ("desired") stocks of consumer durables. To the extent that the timing of the acquisition and disposition of such durables, by their indivisible nature, is sensitive to liquidity constraints and other market imperfections, the shape of the optimal life cycle profile of consumption may be affected. We investigate this possible influence through a study of one particular component of the consumer's wealth, namely, home ownership. We assume that the consumer can secure a mortgage loan, if he or she can afford the required down-payment, but that borrowing for any other purpose is impossible.

The results show that the homebuyer's optimal consumption profile over the life cycle is discontinuous. This result is in sharp contrast to the existing literature, according to which an individual's consumption expenditures over time are "smooth" (even if the individual's income profile is discontinuous). Secondly, we provide a method for estimating the gap between borrowing and lending rates, caused by the homeownership-related indivisibility and market imperfection.

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1. CONSUMPTION PROFILES AND THE LIFE CYCLE THEORY

In Irving Fisher's "ideal loan market", a household's consumable resources are measured by its wealth. At any point in time, the household's wealth is defined as the sum of its non-human wealth, net of liabilities, and the present value of its prospective earnings from work. Within the bound of solvency set by this concept of wealth, the Fisherian household will engage in borrowing and lending (dissaving and saving) in order to make its rate of consumption "more nearly uniform" [5,p.112] over time.

Modigliani and Brumberg [18] elaborated on this idea and formulated what has since become known as the life cycle theory of saving and consumption. A major tenet of their theory holds that an individual's current rate of consumption and saving can be satisfactorily explained, not by the individual's current rate of income, but by the individual's current position -- described by age and present wealth -- in the life cycle. In particular, the role of savings is to serve as a cushion against variations in income during the life cycle, and to provide for retirement and emergencies.

Both greater precision and richness were added to the life cycle theory by Yaari [26]. He provides a precise rule for the allocation of an individual's consumption over his or her lifetime in the form of a utility function, with well-defined properties, to be maximized subject only to a wealth constraint. Of particular interest in the present context, Yaari proves that under his assumptions the optimal plan of consumption over the life cycle is a continuous function of time (and differentiable in the interior of the opportunity set). Indeed, smoothness characterized the consumption profile in the original Modigliani-Brumberg framework -- but as an assumption, not as a result.

Tobin [24] also deals with the issue of the shape of the consumption profile. He uses specific, illustrative, utility functions, and one of his examples is of the following well-known form:

$$\text{Max } U(c) = \int_0^T e^{-\delta t} \log(c(t))s(t)dt \tag{1.1}$$

where c(t) is the time profile of the individual's consumption plan over the interval [0,T]; s(t) is the probability of surviving from birth to age t; T is defined as an age such that $s(t) = 0$ for $t \geq T$; and δ is a subjective rate of discount of future utility. Tobin shows that with a given market interest rate, an individual who behaves according to the rule (1.1), will allocate consumption "evenly" over the life cycle. Evenly allocated consumption here means consumption discounted at the difference between the market and the subjective rates of interest. The "undiscounted" (current-value) stream of consumption will be increasing, constant, or decreasing, respectively, if the market rate of interest is greater than, equal to, or smaller than the subjective rate of discount -- as proven by Yaari [26,p.309] and as illustrated numerically by Tobin [24,p.151]. In all cases, it will be a differentiable stream.

Some authors have questioned the assumed ability of individuals and households to adjust their consumption, over time, to the optimal life cycle profile [3,10,20,23,25]. Among these, Tobin's studies are particularly illuminating, because he compares the wealth-constrained consumption profile, and the corresponding wealth profile, with the profiles which would result if the optimizing individual were subjected to more stringent liquidity constraints. He points out that the human-wealth component of total wealth is often illiquid, especially for the young and the poor. Consequently, he analyzes an alternative to the pure life cycle theory, in which an individual must maintain non-negative assets at all times. Comparing the rates of consumption which result, Tobin finds much smaller values of the short-run marginal propensity to consume in the case where behavior is constrained by wealth, than when it is constrained by a restriction that the household must maintain non-negative assets ("liquidity-constraint"). In turn, he suggests, such differences in behavior imply very different responses to other parametric changes: "Wealth-constrained households will, liquidity-constrained households will not, alter their current consumption in response to marginal changes in their illiquid resources -- such as increases in expected future labor incomes, improvements in prospective retirement benefits, capital gains on houses and other imperfectly liquid assets. Wealth-constrained households will, liquidity-constrained households will not, respond to small changes in interest rates, either for lending or borrowing" [25,p.48].

Some of the macroeconomic policy implications are analyzed and elucidated by Dolde and Tobin [3]. For this purpose, they use what may best be described as a microeconomic simulation model. In an attempt to capture existing differences in access to capital markets, they classify the population into two income classes, and one of their findings is that liquidity constraints are binding on poorer segments of the population, as well as on many of those in younger age groups.

In both the mentioned papers, it is suggested that even with liquidity constraints imposed on the consumer, the optimal consumption profile is still continuous over the life cycle.[1]

A major thrust of the literature oriented towards life cycle saving and consumption has been to provide a more satisfactory microfoundation for macroeconomic theory, especially with respect to the determinants of the aggregate consumption function. From that perspective, it is perhaps not surprising that the literature has generally ignored "portfolio" considerations of life cycle savings. Yet, at any point of an individual's life cycle, the chosen rate of consumption is merely a scalar-valued representation of an implied, optimally composed, basket of goods and services. In turn, to attain and sustain any such basket requires the holding of desired stocks of consumer durables, as part of the individual's non-human wealth. To the extent that the timing of the acquisition and disposition of such durables, by their indivisible nature, is sensitive to liquidity constraints and other market imperfections, the shape of the life cycle profile may be affected.

That is the problem we wish to investigate. As the object of the analysis we shall select one particular component of indivisible wealth, namely residential housing; buying a home is typically the largest investment a household ever makes over its life cycle.

All available empirical studies -- ranging from the pioneering work by Lansing and Morgan in the early 1950's [13], to the panel study

[1] Tobin's suggestion is based on diagrammatic representations of cumulative non-capital earnings, on the one hand, and cumulative consumption streams on the other, and an inspection of the diagrams indicates to us that the profile may be non-differentiable at the points where the constraints become active or cease to be active. In section 3 of this paper, we prove that Tobin's suggestion about the continuity is indeed correct, under his assumptions.

recently conducted by the Survey Research Center at the University of Michigan [19], and to the Housing Assistance Supply Experiment studied by Rand [16][2] -- show a strong relationship between homeownership status and stage in the life cycle, and generally only a weak linkage between homeownership and current income, thus lending support to the pure life cycle theory. On the other hand, current income is much more significant, according to many of these studies, as a predictor of home purchase[3] -- giving support to Tobin's liquidity-constraint version of the life cycle theory.

In the next sections, we present a formal analysis of the effects of tenure (homeownership vs. rental), and its timing, upon the shape of the optimal profile of consumption over the life cycle. Since we wish to isolate and lay bare those effects which result from a particular set of liquidity constraints and (to homeownership related) market imperfections, we allow ourselves to make highly simplified assumptions about other factors influencing the life cycle. We begin by stating our assumptions.

2. ASSUMPTIONS

1. The study is formulated from the view point of a consumer whose life time T, initial assets a_0 and constant wage rate w, are all known with certainty.
2. There is only one kind of house that the consumer considers for purchase, or rental. It is available at any time during the consumer's

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[2] For other significant studies, see [6,12,14,15]

[3] The difference between these two ways of studying tenure is stressed by Maisel [15].

life cycle at a fixed purchase price P, or fixed rental rate R.

3. The consumer can borrow up to an amount $(1-k)P$ towards the purchase of a home at any time when the assets held by the consumer exceed or equal the downpayment, kP. The amount borrowed must be repaid with interest, r, in the form of a constant annuity over s years, where $r \geqq 0$ and $s > 0$ are constants. When the home is sold, the seller receives the equity. Hence, it is assumed that there is no depreciation of the value of the home.
4. The consumer cannot borrow for any other purpose. Hence, the consumer must at all times maintain non-negative assets. When a home is purchased, it is considered as an illiquid component of wealth and not as an asset.
5. The consumer's assets yield an interest r.
6. If and when a consumer does not own a home, he or she rents an identical dwelling-unit at a fixed rent $R > rP$.[4]
7. The consumer selects the life-cycle profile $c = (c(t))$, consisting of non-housing expenditures, so as to maximize

$$U(c) = \int_0^T e^{-\delta t} \log(c(t))\, dt + \phi(a(T)) \tag{2.1}$$

where $\phi(a(T))$ is the utility derived from the bequest $a(T)$; ϕ is concave and $\frac{\partial \phi}{\partial a} > 0$. This is the same preference function as (1.1), with a bequest motive added and with two other modifications. First, since the consumer's life span is known with certainty (by assumption 1), $s(t)$ becomes

[4] This inequality guarantees that the average cost of owning is less than the cost of renting. Estimates of the relative costs of owning vs. renting a home are given in [21]. Primarily due to tax advantages and to a vacancy allowance, homeownership is found to be less costly, on the average, whenever the length of homeownership exceeds three or four years.

$$s(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T \\ 0 & \text{for } t > T. \end{cases}$$

Second, by our assumption 2, housing consumption is a constant throughout the life cycle (independent of tenure). For convenience, $c(t)$ is therefore defined here as consumption at time t, exclusive of housing consumption. This simplifies the analysis, but it does not influence the conclusions.[5]

3. PRELIMINARY ANALYSIS AND RESULT

The optimal consumption profile $c^* = c(t)$ defined over the interval $[0,T]$, and its corresponding asset profile $a^* = a(t)$, can each have two very different forms which we shall call the owner profiles and the tenant profiles. However, before deriving these, we shall find it useful to pose and solve another problem whose solution will facilitate the derivation and analysis of the profiles specified with respect to tenure.

Consider the problem of choosing a consumption profile $(c(t))$ so as to

$$\text{maximize} \int_{t_1}^{t_2} e^{-\delta t} \log(c(t))\, dt \tag{3.1}$$

subject to

$$\dot{a}(t) = ra(t) + I - c(t), \qquad t_1 \leq t \leq t_2 \tag{3.2}$$

[5] It may be worth noting that if in (2.1) $\log(c(t))$ is replaced by $u(c(t))$, where $u(c)$ is any increasing concave function, the qualitative properties of the optimal profile $(c^*(t), a^*(t))$ to be derived in the following analysis, would continue to hold. However, we would then be unable to specify the functional form of $c^*(t)$, as in Theorem 3.1.

$$a(t_1) = A_1, \quad a(t_2) = A_2 \tag{3.3}$$

$$a(t) \geq 0 \quad , \qquad t_1 \leq t \leq t_2, \tag{3.4}$$

where t_1, t_2, $A_1 \geq 0$, $A_2 \geq 0$, $I > 0$ are constants.

The right-hand side of (3.2) can be interpreted as the difference between current income (including interest) and current consumption which gives the time rate of change of the net assets. The initial and terminal values of the assets are given in (3.3), and (3.4) stipulates that the net assets must never become negative.

THEOREM 3.1. A consumer who behaves according to (3.1) and who is constrained by (3.2), (3.3) and (3.4), selects the optimal profile $(c^*(t))$ given by

$$c^*(t) = \begin{cases} c_1^* e^{(r-\delta)(t-t_1)} & , \quad t_1 \leq t \leq t_1^* \\ c^*(t_1^*) = I & , \quad t_1^* \leq t \leq t_2^* \\ I e^{(r-\delta)(t-t_2^*)} & , \quad t_2^* \leq t \leq t_2 \end{cases} \tag{3.5}$$

This profile is continuous. The optimal level c_1^* and the "switch" times t_1^*, t_2^* are unique and determined by the condition that the asset profile $(a^*(t))$, given by (3.2), satisfies (3.3), (3.4) and

$$a^*(t) = 0 \quad \text{for } t_1^* \leq t \leq t_2^* \tag{3.6}$$

Remark: It is quite possible that the values of t_1^*, t_2^* are such that one or two of the three intervals in (3.5) are empty.

Figure 3.1 displays the optimal profiles (solid lines) for the case where $\delta > r$.

We sketch a proof of the theorem.

LEMMA 3.1. $(c^*(t), a^*(t))$ constitute an optimal solution if and only if there exist non-negative functions $p(t)$, $\lambda(t)$ such that for $t_1 \le t \le t_2$

$$\dot{p}(t) = -rp(t) - \lambda(t) , \tag{3.7}$$

$$e^{-\delta t}[c^*(t)]^{-1} = p(t) , \tag{3.8}$$

$$a^*(t)\lambda(t) = 0 , \tag{3.9}$$

and $(c^*(t), a^*(t))$ satisfy (3.2), (3.3), (3.4).

PROOF. Define the Hamiltonian

$$H(t,c,a,p(t),\lambda(t)) = e^{-\delta t}\log c + p(t)[ra+I-c] + \lambda(t)a . \tag{3.10}$$

By Hestenes [9,p.354] it is necessary for optimality that there exist $p(t) \ge 0$, $\lambda(t) \ge 0$ such that (3.7), (3.9) hold and

$$H(t,c^*(t),a^*(t),p(t),\lambda(t)) = \max_{c \ge 0} H(t,c,a^*(t),p(t),\lambda(t)). \tag{3.11}$$

From (3.10) and (3.11):

$$e^{-\delta t}\log c^*(t) - p(t)\, c^*(t) = \operatorname*{Max}_{c \ge 0} \{e^{-\delta t}\log c - p(t)\, c\},$$

which immediately gives (3.8); we have thus proven the necessity. It remains to prove suffiency. So let $(c(t), a(t))$ be any pair satisfying (3.2), (3.3), (3.4). From (3.11) we conclude

$$e^{-\delta t}\log c^*(t) + p(t)[ra^*(t)+I-c^*(t)]$$

$$\ge e^{-\delta t}\log c(t) + p(t)[ra^*(t)+I-c(t)]$$

$$= e^{-\delta t}\log c(t) + p(t)[ra(t)+I-c(t)] - rp(t)[a(t)-a^*(t)],$$

which, by (3.2), can be rewritten as

$$e^{-\delta t}\log c^*(t) - e^{-\delta t}\log c(t) \geq p(t)[\mathring{a}(t)-\mathring{a}^*(t)]$$

$$- rp(t)[a(t)-a^*(t)].$$

Integration of this gives

$$\int_{t_1}^{t_2} [e^{-\delta t}\log c^*(t) - e^{-\delta t}\log c(t)]dt \geq \int_{t_1}^{t_2} p(t)[\mathring{a}(t)-\mathring{a}^*(t)]dt$$

$$- \int_{t_1}^{t_2} rp(t)[a(t)-a^*(t)]dt. \qquad (3.13)$$

Integrating the first term on the right by parts:

$$\int_{t_1}^{t_2} p(t)[\mathring{a}(t)-\mathring{a}^*(t)]dt = - \int_{t_1}^{t_2} \dot{p}(t)[a(t)-a^*(t)]dt + p(t)[a(t)-a^*(t)]\Big|_{t_1}^{t_2}$$

$$= - \int_{t_1}^{t_2} \dot{p}(t)[a(t)-a^*(t)]dt, \text{ by (3.3).}$$

Substituting into (3.13), we get:

$$\int_{t_1}^{t_2} [e^{-\delta t}\log c^*(t)-e^{-\delta t}\log c(t)]dt \geq - \int_{t_1}^{t_2} [\dot{p}(t)+rp(t)][a(t)-a^*(t)]dt$$

$$= \int_{t_1}^{t_2} \lambda(t)[a(t)-a^*(t)]dt, \text{ by (3.7)}$$

$$\geq 0,$$

since $\lambda(t)\ a^*(t) = 0$ and $\lambda(t)\ \mathring{a}(t) \geq 0$. Hence $(c^*(t), a^*(t))$ is optimum.

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We proceed to complete the proof of Theorem 1. First, from (3.7) it follows that $p(t)$ is continuous. Hence, we conclude from (3.8) that $c^*(t)$ is continuous. Second, we can deduce (3.5) by observing that on any interval $[\tau_1,\tau_2]$ on which $a^*(t) > 0$, $p(t)$ has the form:

$$p(t) = p(\tau_1)\, e^{-r(t-\tau_1)}.$$

Therefore, by (3.8):

$$c^*(t) = c^*(\tau_1)\, e^{(r-\delta)(t-\tau_1)}.$$

On the other hand, if $a^*(t) = 0$ on $[\tau_1,\tau_2]$ then by (3.2):

$$c^*(t) = I.$$

It is now straightforward to show that the optimal profile must have the form (3.5). Third, there can be only one $(c^*(t))$ such that (3.2), (3.3), and (3.4) hold; more directly, the uniqueness of the optimal profile is a consequence of the strict concavity of the utility function (3.1).

As we noted in a previous section, most of the literature on the life cycle theory ignores liquidity constraints and specifies only an over-all wealth constraint. It is thus of interest to investigate how the profile of optimal life-cycle consumption, as specified in Theorem 1, changes, if the liquidity constraint (3.4) is absent. The result is described in the following lemma.

LEMMA 3.2. Let $(\hat{c}(t),\hat{a}(t))$ maximize (3.1) subject to the constraints (3.2) and (3.3). Then $(\hat{c}(t))$ is given by

$$\hat{c}(t) = \hat{c}_1 e^{(r-\delta)(t-t_1)}, \quad t_1 \le t \le t_2. \tag{3.14}$$

where the unique constant $\hat{c}_1$ is determined by the condition that the asset profile $(\hat{a}(t))$, given by (3.2), satisfies (3.3).

PROOF. The proof of Lemma 3.2 is identical to that of Lemma 2.1 upon setting $\lambda(t) = 0$, or one can use the result of Yaari [26]. ◻

The optimal profiles $(\hat{c}(t), \hat{a}(t))$ are displayed in Fig. 1 (dashed lines). We note that the effect of the constraint (3.4) is to induce the consumer to save more in the early part of the life cycle, and to increase consumption later.

If instead of specifying the terminal asset $a(t_2)$ in (3.3), a bequest motive is added in (3.1), Theorem 3.1 needs to be changed slightly as follows.

COROLLARY 3.1. Suppose (3.1) is changed to

$$\text{maximize} \int_{t_1}^{t_2} e^{-\delta t} \log(c(t))dt + \phi(a(t_2)),$$

and (3.3) is changed to

$$a(t_1) = A_1, \quad a(t_2) \geq 0.$$

Then the optimal profile is of the form (3.5); however c_1^*, t_1^*, t_2^* are determined by the condition that the terminal asset $a^*(t_2)$ is given implicitly by the condition

$$\frac{\partial \phi}{\partial a} (a^*(t_2)) = p(t_2),$$

where $p(t)$ is the function introduced in Lemma 3.1.

4. THE OPTIMUM TENANT PROFILE

We are now ready to analyze and describe how the choice of tenure (homeownership vs. rental) influences the optimal profile of life cycle consumption. We begin with the analytically simpler case -- that of the consumer who <u>rents</u> a dwelling unit throughout the life cycle.

Even if the consumer's circumstances are such that, at some point in the life cycle, a sufficient mortgage loan can be obtained, the consumer may still prefer to remain a tenant -- despite the fact that according to our assumptions the cost of owning is less than the cost of renting. The explanation lies in the liquidity constraint imposed. The consumer must save enough money to make the required downpayment. Therefore, the benefit of lower housing cost, in the case of homeownership, must be weighed against the loss of utility resulting from the reduction of consumption early in the life cycle (and possibly not outweighed by the gain of consumption later). Hence, if the consumer's subjective discount factor is sufficiently high, and the consumer's wage and initial assets are sufficiently low, the tenant profile could be the optimal one. The crucial nature of the liquidity constraint is evident in the massive increase in homeownership in the United States during 1953 - 1955 due primarily to the reduction in down payment (see Grebler [27]).

Using the notation and assumptions of section 2, we now state the tenant's optimization problem. Select (c(t)), (a(t)) so as to

$$\text{maximize} \int_0^T e^{-\delta t} \log(c(t))dt + \phi(a(T)) \tag{4.1}$$

subject to

$$\dot{a}(t) = ra(t) + w - R - c(t) \ , \quad 0 \leq t \leq T$$

$$a(0) = a_0, \; a(T) \geq 0, \qquad (4.2)$$

$$a(t) \geq 0 , \qquad 0 \leq t \leq T .$$

As seen, the problem is of the same form as the one formulated and studied in Corollary 3.1. Hence the optimum profile c* is continuous and, by (3.5), it is of the form

$$c^*(t) = \begin{cases} c_1^* e^{(r-\delta)t} & , \; 0 \leq t \leq t_1^* \\ c^*(t_1^*) = w - R & , \; t_1^* \leq t \leq t_2^* \\ (w-R)e^{(r-\delta)(t-t_2^*)} & , \; t_2^* \leq t \leq T. \end{cases} \qquad (4.3)$$

The optimum tenant profiles are sketched in Fig. 4.1 (dashed lines) for $\delta > r$.

5. THE OPTIMUM OWNER PROFILE

If the consumer finds it worthwhile to purchase a home, then the optimal profile of consumption is best described as a sequence over three segments of time: an <u>initial</u> period $[0,T_1]$ during which a home is rented, and during which the consumer is forced to save until at least the downpayment kP has been accumulated; a <u>middle</u> period $[T_1,T_1+T_2]$ during which the consumer is a homeowner; and a <u>final</u> period $[T_1+T_2,T]$ at the beginning of which the home is sold thus liquidating the equity. Depending on the size of the individual's initial assets and on the form of the bequest motive, the initial or final period may be empty.

During each of these three periods, of lengths T_1, T_2, and T_3, respectively, the consumer must satisfy constraints similar to (3.2), (3.3), and (3.4). The optimization problem can then be formulated as follows. Select (c(t), a(t)) so as to

$$\text{maximize} \int_0^T e^{-\delta t}\log(c(t))dt + \phi(a(T)) \tag{5.1}$$

subject to

$$\left.\begin{aligned} &\dot{a}(t) = ra(t) + w - c(t) - R &&, \ 0 \leq t < T_1 \\ &a(0) = a_0 \ , \quad a(T_1-) \triangleq a_1 \geq kP && \\ &a(t) \geq 0 &&, \ 0 \leq t < T_1 \end{aligned}\right\} \tag{5.2}$$

$$\left.\begin{aligned} &\dot{a}(t) = ra(t) + w - c(t) - \nu &&, \ T_1 \leq t < T_1 + T_2 \\ &a(T_1) = a_1 - kP, \quad a(T_1+T_2-) \triangleq a_2 \geq 0 && \\ &a(t) \geq 0 &&, \ T_1 \leq t < T_1 + T_2 \end{aligned}\right\} \tag{5.3}$$

$$\left.\begin{aligned} &\dot{a}(t) = ra(t) + w - c(t) - R &&, \ T_1 + T_2 \leq t \leq T_1 + T_2 + T_3 = T \\ &a(T_1+T_2) = a_2 + \alpha(T_1+T_2), \ a(T) \triangleq a_3 \geq 0 && \\ &a(t) \geq 0 &&, \ T_1 + T_2 \leq t < T \end{aligned}\right\} \tag{5.4}$$

The differential equation in (5.2) describes the accumulation of assets in the initial period during which a home is rented. Correspondingly, the differential equation in (5.3) describes the time rate of change of the assets during the period of homeownership. Here, the symbol ν denotes the constant annuity through which the mortgage loan plus interest must be repaid over s years.[6] The annuity can easily be evaluated from the relation

$$(1-k)P = \int_{T_1}^{T_1+s} \nu e^{-r(t-T_1)} dt$$

to be

[6]We assume that $s \geq T_2$. We have also investigated the case where $T_2 > s$ i.e. where the home loan is paid off completely before the sale, and we have not discovered anything significantly different to justify the additional notational burden.

$$\nu = r(1-k)(1-e^{-rs})^{-1} P. \tag{5.5}$$

Finally, the differential equation in (5.4) shows the decumulation of assets in the last period. The boundary conditions are self-explanatory; $\alpha(T_1+T_2)$ denotes the money received by the consumer upon sale of the home and it is determined by

$$\dot{\alpha}(t) = \nu - r[P-\alpha(t)], \ \alpha(T_1) = kP.$$

Upon integration, we obtain:

$$\alpha(T_1+T_2) = [(1-k)e^{r(T_2-s)} - e^{-rs}+k](1-e^{-rs})^{-1} P. \tag{5.6}$$

If we ignore the intra-period constraint $a(t) \geq 0$ and concentrate on the inter-period constraints, $a_1 \geq kP$, $a_2 \geq 0$, $a_3 \geq 0$, then we see that the individual faces a sequence of constraints. (If the capital market imperfection were absent, there would remain only the one budget constraint $a_3 \geq 0$.[7]) The situation is similar to the one discussed in an abstract setting by Hahn [8] where it is shown that the imperfection can lead to an inefficient solution.[8] We verify that this is indeed the case here.

The optimum owner profile is obtained by selecting $a_1 \geq kP$; $a_2 \geq 0$; $T_1 \geq 0$; $T_2 \geq 0$; and $(c(t))$, $(a(t))$ so as to maximize (5.1). Now suppose that the variables a_1, a_2, T_1, T_2 have been chosen optimally.

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[7] This is the standard formulation as studied for example by Yaari [26].

[8] There are two differences from Hahn's analysis. First, the length of the periods in the sequence economy considered by him are fixed in advance whereas here the three periods T_1, T_2, T_3 can be chosen by the individual. Secondly, his is a general equilibrium model whereas ours is partial.

Then the problem of selecting $(c(t))$, $(a(t))$ in each of the three periods is of the same form as the problem considered in Theorem 3.1. Hence, in each of these periods the optimal c^* is of the form (3.5). It follows that <u>within</u> each period c^* is continuous. We next study whether c^* is continuous at the boundaries of the periods.

LEMMA 5.1. $c^*(T_1-) \leq c^*(T_1)$ and $c^*(T_1+T_2-) \leq c^*(T_1+T_2)$

PROOF. The asset constraints $a(t) \geq 0$, $a_1 \geq kP$ are such that it is always feasible to reduce consumption in the initial period and increase consumption in the middle period. Formally, for $\sigma > 0$, $\varepsilon > 0$ sufficiently small, the consumption profile c^σ given by

$$c^\sigma(t) = \begin{cases} c^*(t) & , t \leq T_1 - \varepsilon \text{ , } t \geq T_1 + \varepsilon \\ c^*(t) - \sigma & , T_1 - \varepsilon \leq t < T_1 \\ c^*(t) + \sigma & , T_1 \leq t < T_1 + \varepsilon \end{cases}$$

is feasible. We have

$$\int_0^T e^{-\delta t}[\log c^\sigma(t) - \log c^*(t)]dt = \varepsilon e^{-\delta T_1}[\log(c^*(T_1)-\sigma)$$

$$- \log c^*(T_1-) + \log(c^*(T_1) + \sigma) - \log c^*(T_1)]$$

$$+ o(\varepsilon) \leq 0, \tag{5.7}$$

Since c^* is optimal. It is immediate that this can hold for $\sigma > 0$ only if $c^*(T_1-) \leq c^*(T_1)$. In the same way we can prove the second assertion. ¤

LEMMA 5.2. If $a_1 > kP$, then $c^*(T_1-) = c^*(T_1)$. If $a_2 > 0$, then $c^*(T_1+T_2-) = c^*(T_1+T_2)$.

PROOF. Suppose $a_1 > kP$. Then it is feasible to <u>increase</u> consumption in the initial period and reduce consumption correspondingly in the middle period. Hence, the expression in (5.7) must be non-positive for $\sigma < 0$, which implies $c^*(T_1-) \geq c^*(T_1)$. Hence $c^*(T_1-) = c^*(T_1)$. The second assertion follows in a similar way. ¤

LEMMA 5.3. If $T_1 + T_2 < T$, then $c^*(T_1+T_2-) < c^*(T_1+T_2)$.

PROOF. We prove this by contradiction. So suppose

$$c^*(T_1+T_2-) = c^*(T_1+T_2) \tag{5.8}$$

We shall compare c^* with an alternative profile c^ε in which the middle period is extended from T_2 to $T_2 + \varepsilon$. In the interval $[0,T_1+T_2+\varepsilon)$ c^ε is specified to be

$$c^\varepsilon(t) = \begin{cases} c^*(t) & , \quad 0 \leq t \leq T_1 + T_2 \\ c^*(T_1+T_2) & , \quad T_1 + T_2 \leq t < T_1 + T_2 + \varepsilon \end{cases}$$

We must show that this is indeed feasible for $\varepsilon > 0$ small, i.e. the corresponding asset profile $a^\varepsilon(t)$ is non-negative. This is certainly true if $a_2 = a^*(T_1+T_2-) > 0$. On the other hand if $a_2 = 0$, then by (3.5), (5.3) we must have $c^*(T_1+T_2) = w - \nu$, and then $c^\varepsilon(t) = w - \nu$ and $a^\varepsilon(t) = 0$ for $T_1 + T_2 \leq T_1 + T_2 + \varepsilon$. Hence c^ε is feasible for $t < T_1 + T_2 + \varepsilon$.

Next, in the profile c^ε the home is owned during $[T_1+T_2, T_1+T_2+\varepsilon)$ at a cost of rP per unit time, whereas in c^* it is rented at a cost of $R > rP$ per unit time. It follows that

$$a^\varepsilon(T_1+T_2+\varepsilon) - a^*(T_1+T_2+\varepsilon) = \varepsilon(R-rP) + o(\varepsilon)$$

Thus the profile c^ε, specified for $t \geq T_1 + T_2 + \varepsilon$ by

$$c(t) = \begin{cases} c^*(t) + R - rP + o(\varepsilon) & T_1 + T_2 + \varepsilon \leq t < T_1 + T_2 + 2\varepsilon \\ c^*(t) & t \geq T_1 + T_2 + 2\varepsilon, \end{cases}$$

is feasible where $o(\varepsilon)$ is such that $a^\varepsilon(T_1+T_2+2\varepsilon) = a^*(T_1+T_2+2\varepsilon)$. The difference in utility is given by

$$\int_0^T e^{-\delta t}[\log c^\varepsilon(t) - \log c^*(t)]dt = \{e^{-\delta(T_1+T_2)} \log[c^*(T_1+T_2) + R - rP]$$

$$- \log c^*(T_1+T_2)\} + o(\varepsilon) > 0$$

for ε sufficiently small. Hence c^* cannot be optimal. ◻

Thus the market imperfection which prevents the owner from borrowing against the equity in the home, introduces a discontinuity in the consumption profile at the time of sale, $T_1 + T_2$. This is a result of "forced" saving induced during the homeownership period. Now we show that if R is sufficiently large, then there is a discontinuity also at the time of purchase, T_1.

Let $R = (r+\pi)P$ where $\pi > 0$ denotes the landlord's "entrepreneurial" profit rate. Rewrite (5.5) as $\nu = r(1-k)P + \xi P$ with

$$\xi = r(1-k)(e^{rs}-1)^{-1} .$$

Then

$$R - \nu - rkP = (\pi-\xi)P.$$

Table 5.1 lists values of ξ for some sample values of r, k, s. It is evident that π may easily exceed ξ.

As before let c^*, a^* be the optimal profiles and T_1, T_2 the switch times.

LEMMA 5.4. Assume that $\pi > \xi$ and suppose that $T_1 > 0$. Then $a^*(T_1-) = kP$ so that $a^*(T_1) = 0$.

PROOF. We proceed by contradiction. Suppose $a^*(T_1-) > kP$. Then for $\varepsilon > 0$ sufficiently small it is possible to purchase the home at an earlier time $T_1-\varepsilon$. Suppose this is done but that the consumption $c^*(t)$ is maintained unchanged over $0 \leq t \leq T_1$. The asset profile will of course be different. Call it $a^\varepsilon(t)$. We wish to compare $a^\varepsilon(T_1)$ with $a^*(T_1)$. Note that $a^\varepsilon(t) = a^*(t)$ for $t < T_1-\varepsilon$. From (5.2) we see that

$$a^*(T_1) = a^*(T_1-\varepsilon) + ra^*(T_1-\varepsilon)\varepsilon + o(\varepsilon) + (w-R)\varepsilon - \int_{T_1-\varepsilon}^{T_1} c^*(t)dt - kP,$$

and from (5.3) we see that

$$a^\varepsilon(T_1) = a^*(T_1-\varepsilon) - kP + r[a^*(T_1-\varepsilon)-kP]\varepsilon + o(\varepsilon) + (w-\nu)\varepsilon - \int_{T_1-\varepsilon}^{T_1} c^*(t)dt.$$

Hence,

$$a^\varepsilon(T_1) - a^*(T_1) = (R-\nu)\varepsilon - rkP\varepsilon + o(\varepsilon) \tag{5.9}$$

$$= (\pi-\xi)P\varepsilon + o(\varepsilon).$$

By assumption then $a^\varepsilon(T_1) > a^*(T_1)$ for $\varepsilon > 0$ small. It is therefore possible to increase consumption beyond c^* after T_1 and so c^* cannot be optimal. □

From (5.9) we see that the assumption $\pi > \xi$ means that the rent exceeds the annuity plus the interest on the downpayment and so it is not unexpected that the individual will purchase the home as soon as he has accumulated enough savings to cover the downpayment. The next result shows that this incentive is another source of "forced" savings.

LEMMA 5.5. Assume that $\pi > \xi$ and suppose that $T_1 > 0$. Then $c^*(T_1-) < c^*(T_1)$.

PROOF. We again proceed by contradiction. Suppose that $c^*(T_1-) \geq c^*(T_1)$. By Lemma 5.1 then we must have $c^*(T_1-) = c^*(T_1)$, so c^* is continuous at T_1. We shall compare c^* with another profile c^ε in which the home is purchased before T_1, at $T_1 - \varepsilon$. To do this savings must be increased and hence consumption reduced, prior to $T_1 - \varepsilon$. The amount of excess assets relative to a^* which must be accumulated at time $T_1 - \varepsilon$ is, by (5.2), equal to

$$a = a^*(T_1-) - a^*(T_1-\varepsilon) = \int_{T_1-\varepsilon}^{T_1} ra^*(t)dt + \int_{T_1-\varepsilon}^{T_1} [w-c^*(t)-R]dt$$

$$= ra^*(T_1-)\varepsilon + (w-c^*-R)\varepsilon + o(\varepsilon),$$

where $c^* = c^*(T_1)$. By Lemma 5.4, $a^*(T_1-) = kP$ and so

$$a = [rkP + (w-c^*-R)]\varepsilon + o(\varepsilon). \tag{5.10}$$

Next suppose that these assets are accumulated by reducing consumption by level α over the interval $[T_1-\varepsilon-\beta\varepsilon, T_1-\varepsilon]$ i.e. consider the profile

$$c^\varepsilon(t) = \begin{cases} c^*(t) & , \ 0 \leq t < T_1 - \varepsilon - \beta\varepsilon \\ c^*(t) - \alpha & , \ T_1 - \varepsilon - \beta\varepsilon \leq t \leq T_1 - \varepsilon \end{cases} \tag{5.11}$$

(see Fig. 5.1). Then the additional assets accumulated by time $T_1 - \varepsilon$ equal $\alpha\beta\varepsilon + o(\varepsilon)$ and for this to equal a we must have

$$\alpha\beta\varepsilon = a + o(\varepsilon) \ . \tag{5.12}$$

The reduction in consumption given by (5.10) leads to a loss of utility of

$$u = \int_{T_1-\varepsilon-\beta\varepsilon}^{T_1-\varepsilon} e^{-\delta t}[\log c^*(t) - \log c^{\varepsilon}(t)]dt$$

$$= e^{-\delta T_1}[\log c^*(c^*-\alpha)^{-1}]\beta\varepsilon + o(\varepsilon).$$

Now suppose that after the home is purchased at $T_1 - \varepsilon$, the consumption level $c^*(t)$ is maintained up to time T_1 i.e.,

$$c^{\varepsilon}(t) = c^*(t), \quad T_1 - \varepsilon < t \leq T_1.$$

Then the corresponding assets at time T_1, $a^{\varepsilon}(T_1)$, will exceed $a^*(T_1)$ by

$$a^{\varepsilon}(T_1) - a^*(T_1) = [(w-c^*-R) + (R-\nu)]\varepsilon + o(\varepsilon).$$

It is therefore feasible to increase consumption over the interval $[T_1, T_1+\beta\varepsilon]$ and beyond that time to maintain the original profile $c^*(t)$ i.e. the profile

$$c^{\varepsilon}(t) = \begin{cases} c^*(t) + \beta^{-1}[(w-c^*-R) + (R-\nu)], & T_1 \leq t \leq T_1 + \beta\varepsilon \\ c^*(t) & , \quad T_1 + \beta\varepsilon < t \leq T, \end{cases}$$

is feasible. This increased consumption leads to a utility gain of

$$v = \int_{T_1}^{T_1+\beta\varepsilon} c^{-\delta t}[\log c^{\varepsilon}(t) - \log c^*(t)]dt$$

$$= c^{-\delta T_1}\{\log(c^*+\beta^{-1}[(w-c^*-R)+(R-\nu)])(c^*)^{-1}\}\beta\varepsilon + o(\varepsilon).$$

The net gain in utility resulting from adopting c^{ε} instead of c^* is v-u, and this can be estimated by

$$(v-u)e^{\delta T_1} = \beta\varepsilon \log\{\frac{c^*+\beta^{-1}[(w-c^*-R)+(R-\nu)]}{c^*}\frac{c^*-\alpha}{c^*}\} + o(\varepsilon)$$

Substituting for α from (5.10), (5.12) gives the argument of the logarithm as equal to

$$\frac{c^*+\beta^{-1}[(w-c^*-R)+(R-\nu)]}{c^*}\frac{c^*-[(w-c^*-R)+rkP]}{c^*} = \frac{c^*+\beta^{-1}\Gamma}{c^*}\frac{c^*+\beta^{-1}\gamma}{c^*}$$

say, where

$$\Gamma - \gamma = R - \nu - rkP = (\pi-\xi)P > 0$$

by assumption. Choosing

$$\beta^{-1} = \frac{\Gamma-\gamma}{2\Gamma\gamma}c^* > 0$$

then gives

$$\frac{c^*+\beta^{-1}\Gamma}{c^*}\frac{c^*-\beta^{-1}\gamma}{c^*} = \frac{(\Gamma+\gamma)^2}{4\Gamma\gamma} > 1,$$

and so $v-u > 0$ for $\varepsilon > 0$ small. Hence c^* cannot be optimal. ¤

We will show at the end of the following section that even if $\pi < \xi$, there may be a discontinuity in c^* at T_1. For the moment we combine the results obtained above in the form of a theorem.

THEOREM 5.1. Assume that $\pi > \xi$, and suppose $T_1 > 0$, $T_1 + T_2 < T$. Then the optimum owner profiles satisfy the following properties:

$$a^*(T_1-) = kP, \quad c^*(T_1-) < c^*(T_1),$$
$$a^*(T_1+T_2) = 0\ , \quad c^*(T_1+T_2-) < c^*(T_1+T_2).$$

The optimum owner profiles are sketched in Fig. 4.1 for the case $\delta > r$ (solid lines).

6. MARGINAL MEASURE OF DISTORTION IN OWNER PROFILE

As we have seen, the distortion in the homeowner's consumption profile caused by the inability to borrow against the accumulating equity is severe: it induces a discontinuity in the profile. We study first the discontinuity at $T_1 + T_2$ in greater detail.

In order to eliminate the effect of the intra-period liquidity constraint $a(t) \geq 0$, we consider the special case $\delta = r$. From (3.5) we see that this implies that within each period $c^*(t)$ is <u>constant</u>, and by Theorem 5.1 these constant levels relate to each other as follows:

$$c^*(t) = \begin{cases} c_1^* & , \quad 0 \leq t < T_1 \\ c_2^* > c_1^* & , \quad T_1 \leq t < T_1 + T_2 \\ c_3^* > c_2^* & , \quad T_1 + T_2 \leq t \leq T \quad . \end{cases} \tag{6.1}$$

From the constraints (5.2), (5.3), (5.4), these consumption levels are easily evaluated:

$$c_1^* = w - R - r(ke^{-rT_1}P + a(T_1)e^{-rT_1} - a_0)(1-e^{-rT_1})^{-1} , \tag{6.2}$$

$$c_2^* = w - \nu + r(a_1 - a_2 e^{-rT_2})(1-e^{-rT_2})^{-1} , \tag{6.3}$$

$$c_3^* = w - R + r(a_2 + \alpha(T_1+T_2) - a_3 e^{-rT_3})(1-e^{-rT_3})^{-1} . \tag{6.4}$$

Now, if there were no liquidity constraint then only the constraint $a_3 \geq 0$ would be present which, together with (6.2), (6.3), (6.4) would define the set of feasible consumption levels. The presence of the liquidity constraints $a(T_1) \geq 0$, $a_2 \geq 0$ reduces this feasible set. A marginal measure of the resulting distortion in the optimum consumption profile can be obtained in the following manner.

Suppose an individual, who is a homeowner, were able to obtain a small loan, L, during the interval $[T_1, T_1+T_2]$. How much would he be willing to pay in interest? Let us call that rate ρ.

From (6.2), (6.3), (6.4), we see that the consumer would use the borrowed amount L in such a way that the optimal consumption levels would change to $\hat{c}_1$, $\hat{c}_2$, $\hat{c}_3$, where

$$\begin{aligned} \hat{c}_1 &= c_1^* \\ \hat{c}_2 &= c_2^* + rL(1-e^{-rT_2})^{-1} \\ \hat{c}_3 &= c_3^* - rLe^{\rho T_2}(1-e^{-rT_3})^{-1} \end{aligned} \tag{6.5}$$

Substituting into:

$$U = \int_0^T e^{-rt}\log(c(t))dt,$$

we can then estimate

$$\begin{aligned} dU &= U(\hat{c}_1,\hat{c}_2,\hat{c}_3) - U(c_1^*,c_2^*,c_3^*) \\ &= e^{-rT_1}[(c_2^*)^{-1} - e^{(\rho-r)T_2}(c_3^*)^{-1}]L + o(L) \end{aligned}$$

Hence, at the margin, the consumer would be indifferent between borrowing at rate ρ, and not borrowing at all, if

$$(c_2^*)^{-1} - e^{(\rho-r)T_2}(c_3^*)^{-1} = 0$$

that is, if

$$\rho = r + T_2^{-1}(\log c_3^* - \log c_2^*) \tag{6.6}$$

Since, by Theorem 5.1, $c_3^* > c_2^*$ at the optimum, we can interpret

$$m = \rho - r = T_2^{-1}(\log c_3^* - \log c_2^*)$$

as the <u>premium above the lending rate</u> at which the individual would be willing to borrow against the equity in the home.

Recall Lemma 5.5 where it was shown that if $\pi > \xi$ (see (5.10)) then there is a discontinuity in the optimum owner profile at the time of purchase of the home, T_1. We shall use the result obtained above to show that this discontinuity may occur even if $\pi < \xi$.

Suppose $T_1 + T_2$ is optimal, and the individual considers a marginal change in the time of purchase to $T_1 + \varepsilon$. Then at time $T_1 + \varepsilon$ the individual's money assets will increase by $(\nu-R)\varepsilon + rk\varepsilon P = (\xi-\pi)\varepsilon$, and at time $T_1 + T_2$ the equity in the home will decrease by $\xi\varepsilon$. By the result above this decrease is subjectively evaluated at time T_1 as $e^{-\rho T_2}\xi\varepsilon$. It follows that the individual will choose

$$\varepsilon \gtrless 0 \text{ according as } \xi - \pi \gtrless e^{-\rho T_2}\xi.$$

Substituting from (6.6) this is equivalent to:

$$\varepsilon \gtrless 0 \text{ according as } \xi - \pi \gtrless e^{-rT_2}\left(\frac{c_2^*}{c_3^*}\right)\xi.$$

We can see that even if $\xi - \pi > 0$ it still may be the case that

$$\xi - \pi < e^{-rT_2}\left(\frac{c_2^*}{c_3^*}\right)\xi$$

so that $\varepsilon < 0$, and the individual will purchase the home earlier. From the proof of Lemma 5.5 we can see that this will induce a discontinuity at T_1.

k = 0.3	s = 20	s = 30	k = 0.1	s = 20	s = 30
r = 0.06	0.018	0.008	r = 0.06	0.023	0.011
r = 0.09	0.013	0.004	r = 0.09	0.016	0.005

Table 5.1. Values of ξ.

REFERENCES

[1] Bowden, R., "Risk Premiums and the Life Cycle Hypothesis," American Economic Review, LXIV (March 1974), 211-216.

[2] Diewert, W. E., "Intertemporal Consumer Theory and the Demand for Durables," Econometrica, XVII (May 1974), 497-516.

[3] Dolde, W. and Tobin, J., "Monetary and Fiscal Effects on Consumption," Consumer Spending and Monetary Policy: the Linkages; Boston, The Federal Reserve Bank Conference Series No. 5 (1971).

[4] Farrell, M. J., "The New Theories of the Consumption Function," The Economic Journal, LXIX (December 1959).

[5] Fisher, I., The Theory of Interest, Sentry Press, New York (1930).

[6] Fredland, D., Residential Mobility and Home Purchase, Lexington Books, Lexington, Massachusetts (1974).

[7] Friedman, M., A Theory of the Consumption Function, National Bureau of Economic Research, Princeton (1957).

[8] Hahn, F., "Equilibrium with Transaction Costs," Econometrica, XXXIX (May 1971), 417-439.

[9] Hestenes, M. R., Calculus of Variations and Optimal Control Theory, John Wiley and Sons, New York (1966).

[10] Houthakker, H., "The Permanent Income Hypothesis: Reply," American Economic Review, 48 (December 1958) 991-993.

[11] Hagerstrand, T. "The Impact of Transport on the Quality of Life," European Conference of Ministers of Transport, Fifth Symposium, Athens, Greece (October 1973).

[12] Kain, J. F. and Quigley, J. M., "Housing Market Discrimination, Homeownership, and Savings Behavior," American Economic Review, LXII (June 1972), 263-277.

[13] Lansing, J. B. and Morgan, J., "Consumer Finances over the Life Cycle," Clark, L. (ed.), The Life Cycle and Consumer Behavior, N. Y. University Press, New York (1955).

[14] Maisel, S. and Winnick, L., "Family Housing Expenditures: Elusive Laws and Intrusive Variances," Proceedings of the Conference on Consumption and Saving, Vol. I, University of Pennsylvania, Philadelphia (1960).

[15] Maisel, S., "Rates of Ownership, Mobility, and Purchase," Essays in Urban Land Economics, in Honor of the Sixty-fifth Birthday of Leo Grebler, UCLA Real Estate Research Program, Los Angeles (1966).

[16] McCarthy, K. F., "The Household Life Cycle and Housing Choices," The Rand Corporation Paper Series (January 1976).

[17] Miller, M. H. and Upton, C. W., Macroeconomics, Richard D. Irwin (ed.), Homewood, Illinois (1974).

[18] Modigliani, F. and Brumberg, R., "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," Kurihara, K. (ed.), Post Keynesian Economics, Rutgers University Press, New Brunswick (1954).

[19] Morgan, J. (ed.), Five Thousand American Families -- Patterns of Progress, Vols. I and II, Survey Research Center, Ann Arbor, Michigan (1974).

[20] Nagatani, K., "Life Cycle Saving: Theory and Fact," American Economic Review, LXII (June 1972), 344-353.

[21] Shelton, J. P., "The Cost of Renting vs. Owning a Home," Land Economics, XLIV (January 1968).

[22] Shoup, D. C., "The Optimal Timing of Urban Land Development," Papers of the Regional Science Association, XXV (November 1969), 33-44.

[23] Thurow, L. C., "The Optimum Lifetime Distribution of Consumption Expenditures," American Economic Review, LIX (June 1969), 324-330.

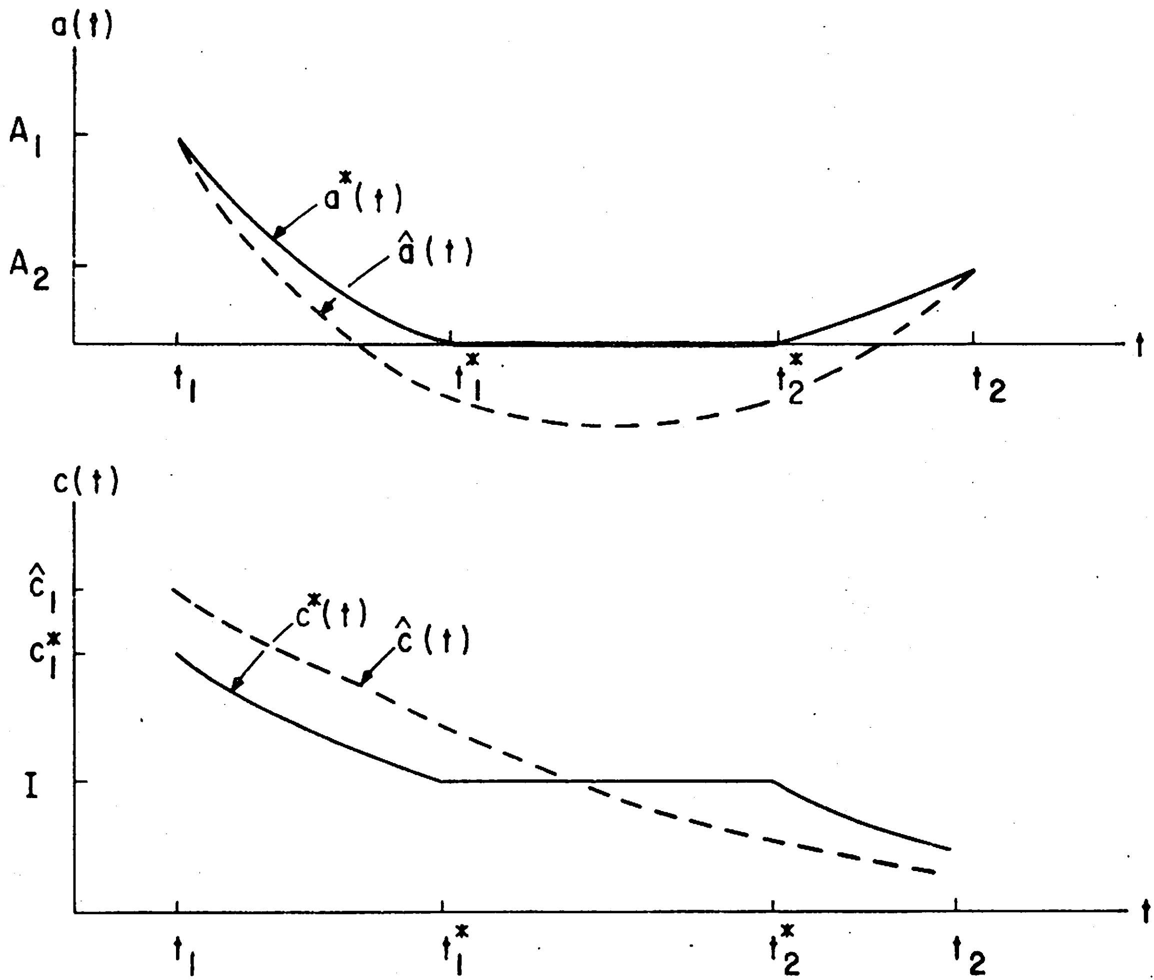
[24] Tobin, J., "Life Cycle Saving and Balanced Growth," Fellner, W. (ed.), Ten Economic Studies in the Tradition of Irving Fisher, John Wiley and Sons, New York (1967).

[25] Tobin, J., "Wealth, Liquidity, and the Propensity to Consume," Strumpel, B., Morgan, J. and Zahn, E. (eds.), Human Behavior in Economic Affairs, Essays in Honor of George S. Katona, Elsevier Scientific Publishing Co. (1972).

[26] Yaari, M. E., "On the Consumer's Lifetime Allocation Process," International Economic Review, V (September 1964), 304-317.

[27] Grebler, L., "Housing Issues in Economic Stabilization Policy," Occasional Paper 72, National Bureau of Economic Research, Inc. (1960).

FIGURE CAPTIONS


a(t)
A_1
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t_1^*
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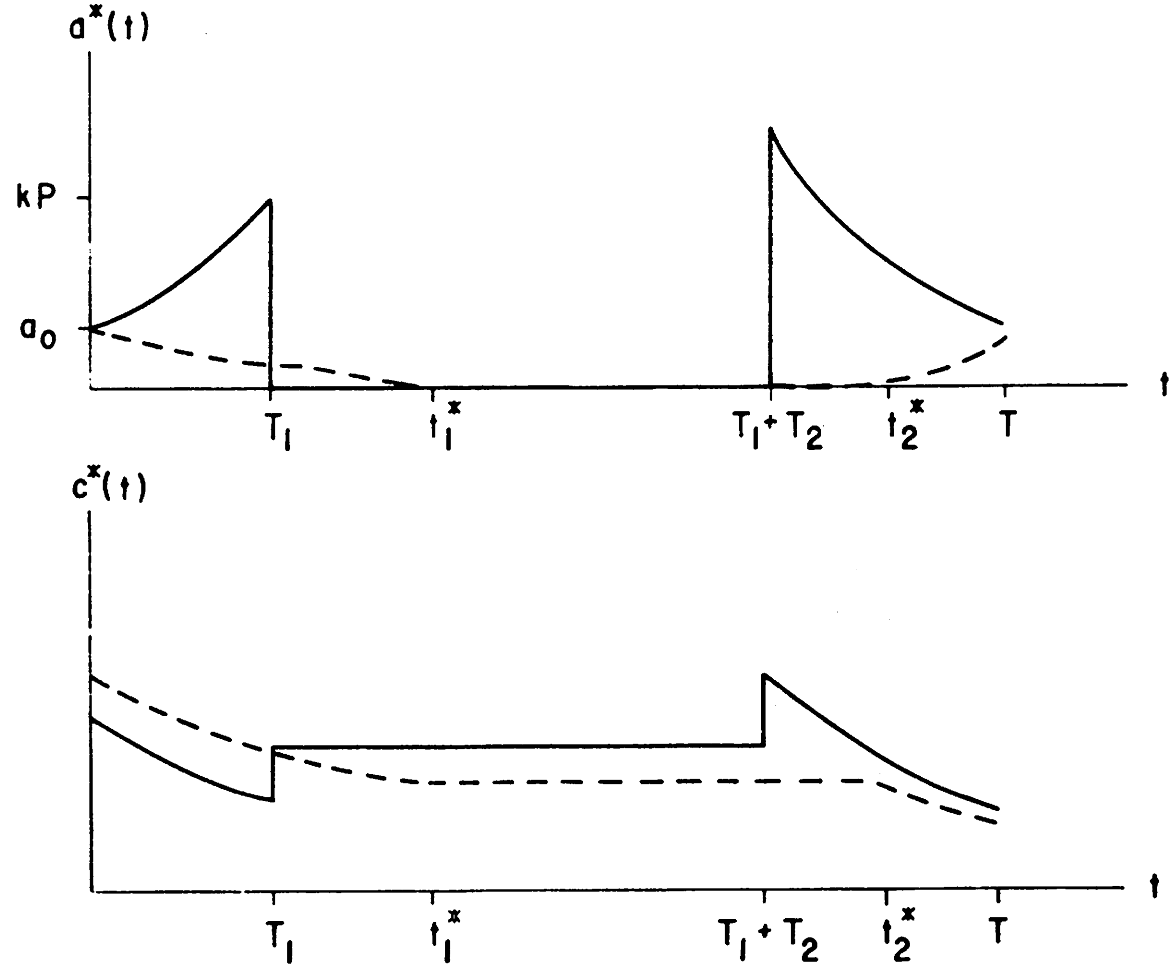

$a^*(t)$
kP
a_0
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T_1
t_1^*
$T_1 + T_2$
t_2^*
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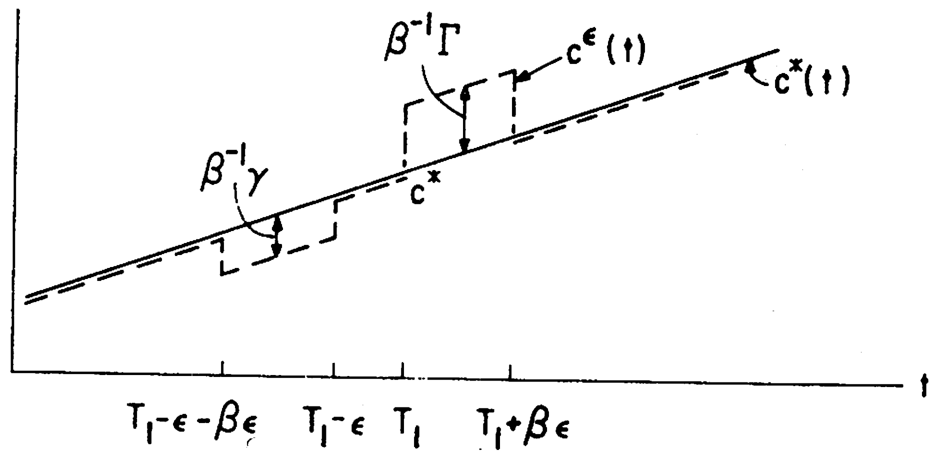
$c^*(t)$
t
T_1
t_1^*
$T_1 + T_2$
t_2^*
T



$\beta^{-1}\Gamma$
$c^{\epsilon}(t)$
$c^{*}(t)$
$\beta^{-1}\gamma$
c^{*}
t
$T_1-\epsilon-\beta\epsilon$
$T_1-\epsilon$
T_1
$T_1+\beta\epsilon$