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NECESSARY CONDITIONS FOR REPRESENTABILITY*

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Abstract

This paper concerns the metatheory of measurement axioms; specifically, of first-order conditions necessary for the existence of representations. As (a subtheory of) the universal theory of the representing structure is satisfied by all representable structures (Łos-Tarski theorem), we distinguish between necessary conditions which are entailed by this (sub-) theory, trivially necessary conditions, and those which are not logical consequences of this subtheory, nontrivially necessary conditions.

Every necessary ∀∃-sentence (in a language without function symbols) is trivially necessary. In many cases, nontrivially necessary ∃∀-sentences exists; i.e. such sentences are satisfied by all representable structures, but not satisfied by some structures representable in elementary equivalents of the representing structure. In such cases, the class of representable structures is not a first-order class. Examples of such nontrivially necessary ∃∀-sentences are given for extensive and conjoint measurement and for other common measurement models.

If the representing structure has many automorphisms, this excludes necessary conditions of certain quantificational form from being nontrivially necessary. Likewise, if the first-order theory of the representing

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structure has only one countable model (up to isomorphism), as is the case with ordinal measurement into the ordered real numbers, there are no non-trivially necessary axioms.

Acknowledgment

The research reported here was sparked by comments of Ernest Adams in a seminar on Measurement Theory given by William Craig at Berkeley, Spring 1975. In successive stages of development, the work was reported to that seminar, to a Berkeley-Stanford summer seminar on Measurement Theory in August 1975, and to a seminar on Measurement Theory given by Zoltan Domotor at Berkeley, Spring 1976. Throughout, Ernest Adams and Zoltan Domotor supported the project with encouragement, suggestions, and criticism. Their enthusiasm has been very important to me.

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Contents

1. Introduction

This paper is a contribution to general axiomatic measurement theory along the lines developed in Scott and Suppes' (1958) classic paper. Scott and Suppes deal in general with first-order conditions which must be satisfied by "empirical structures" or models in order that there should exist numerical assignments, measures or representations over the domains of these structures which satisfy specified _hypotheses of measurement_; namely that objects satisfying certain "empirical relations" are assigned numbers satisfying specified corresponding numerical relations. (For an overview of measurement theory, see Pflanzagl (1968) and Krantz et al. (1971).)

We will here generalize Scott and Suppes' treatment in several ways. First, Scott and Suppes require numerical assignments to be _homomorphisms_ in the sense that a numerical relation holds between numerical measurement values if _and only if_ the corresponding empirical relation holds between the empirical objects. This formulation fails to encompass many theories, such as, for example, Adams' (1965) model of inexact extensive measurement, which requires the existence of a numerical assignment f over the domain which satisfies only the one-way implication

$$\text{if } xRy \text{ then } f(x) > f(y)$$

and not its converse (as required in homomorphic embedding), where R is an empirical ordering relation. We will use a more general model to accommodate the foregoing and analogous hypotheses among the theories we consider.

Second, Scott and Suppes consider primarily purely _universal_ first-order axioms for measurement. Indeed a central theorem of their paper demonstrates the impossibility of a finite universal axiomatization of finite interval (and related) structures homomorphically embeddable in the interval structure of the reals. Although it must be admitted that universal axioms play a central role in displaying the "empirical content" of measurement hypotheses, we wish to argue that other sorts of axioms can be of importance in this regard, and to initiate their investigation. In particular, we wish to consider what sentences (to be called "nontrivially necessary"), not entailed by necessary universal axioms are necessary conditions for the existence of representations satisfying a given hypothesis of measurement. It appears to us that this question has never been studied heretofore; the nonuniversal axioms (e.g., the Luce-Tukey "solutions" axioms) considered in fundamental measurement theory are not necessary conditions for the existence of representations. We will show that there do in fact exist nontrivially necessary conditions for a great variety of theories, including extensive, interval, and conjoint measurement. Furthermore, it is possible to give an analysis of their logical form, and also to say something about what they "mean."

Third, we emphasize general insight into representation by not requiring, as did Scott and Suppes, that the representing structure (to be called measurement structure in the sequel) be a numerical structure. Rather we allow arbitrary infinite structures.

We consider an analysis of axiomatizability in the present abstract, general form valuable in that it may yield

-- general methods or guidelines for axiomatization

-- general insights into the relationship between properties of measurement structures and the corresponding measurement axioms (and, ultimately, measurement procedures)

-- insights into the special role played by numerical measurement structures (i.e. real numbers) in measurement, as contrasted with other mathematical structures which could in principle be used as measurement structures.

This kind of information is often not obtained by the usual methods of measurement theory because these methods only apply to the study of specific measurement models, in which the measurement structure and representation conditions are fixed at the onset of theoretical consideration.

The methods used in our analysis are those of the model theory of first-order logic. A model-theoretic setup is natural at the level of generality at which we are working. We choose a first-order language L; the measure structure M may be any infinite L-structure (i.e. interpretation of L); representations are mappings from other L-structures into M satisfying chosen requirements of structure preservation. Necessary axioms for measurement are all those L-sentences true of all L-structures which can be represented in M. Treatment of unique representability would seem to require further concepts, and no results for this or other aspects of measurement theory are obtained in the present paper.

Many measurement models of importance in measurement theory actually require a more intricate setup than that suggested in the preceding paragraph but one. It has been our policy to study the simple setup, which we call <u>Simple Measurement</u>, systematically, and later indicate strategies to apply the methods and results to more intricate situations. This allows a unified and, we hope, clear presentation of the ideas.

In Section 2 the basic concepts used in our analysis of simple measurement are introduced. Section 3 contains an exposition of the theory of necessary axioms for representability of finite empirical structures: these are given by the logical consequences of finiteness together with an appropriately selected subtheory of the set of universal sentences true in the measure structure M. (Prop. 3.2.2; essentially well-known.) A surprising further result, Theorem 3.2.6, is that any $\forall\exists$-sentence true of all finite representable structures is actually a logical consequence of one of these universal structures.

The heart of the paper is Section 4, containing the theory of necessary axioms for representability in M of arbitrary structures, i.e. of the first-order sentences true in all structures representable in M. By Prop. 2.3.3, necessary and sufficient conditions for the representability of arbitrary structures in some model of the first-order theory of M (rather than in M itself) are given by an appropriately selected subtheory of the set of all universal sentences true in M. Thus our problem reduces to finding those necessary first-order sentences which are consequences of this subtheory of the set of all universal sentences true in M and those that are not. The second kind will be called <u>nontrivially necessary sentences</u>. As any sentence true in all representable (in M) structures is true in all finite representable structures, Theorem 3.2.6, quoted above, shows that no $\forall\exists$-sentence is nontrivially necessary.

Section 4.2 contains a closer analysis of what it means for a sentence to be nontrivially necessary; on this basis, $\exists\forall$ and $\forall\exists\forall$ nontrivially necessary sentences are constructed in Section 4.3 for extensive measurement and two variants of difference measurement, thus proving that such sentences actually exist. In Section 4.4 the analysis is pursued further: It is shown that "if the structure M has many automorphisms, nontrivially necessary axioms must have many quantifiers." Analysis shows that the non-trivially necessary axioms constructed in Section 4.3 actually contain the minimal possible number of quantifiers allowed by the theorems of Section 4.4! The results of Sections 4.2-4.4 are mainly restricted to sentences of quantifier forms no more complicated than $\forall\exists\forall$; Section 4.5 contains a sketch of how the results of 4.2 at least can be generalized for arbitrary first-order sentences.

Section 4.1 contains some rather modeltheoretic considerations on how the choice of a specific measure structure M within the model class of the first-order theory of M may influence whether there are any nontrivially necessary sentences for representability in M. It follows easily from Theorem 4.1.1, a kind of Löwenheim-Skolem theorem, that there are no nontrivially necessary axioms for ordinal measurement. Another simple but important result in this subsection is that any type of measurement with nontrivially necessary sentences cannot have first-order necessary and sufficient conditions for representability in M.

Finally, Section 5 considers generalizations and more intricate applications of the preceding theory. A major application is Section 5.4, to binary additive conjoint measurement (introduced in Luce and Tukey (1964); see also Krantz et al. (1971)). In this section, all minimal $\forall\exists\forall$-quantifier prefixes of nontrivially necessary axioms are determined, a total of eight. Their minimality is proved, and examples of all eight forms are given (Theroem 5.4.11 and proceding arguments). The proof of minimality requires generalizations of the results of Sections 4.2 and 4.4 to two-sorted first-order logic, as well as a new technique called <u>definability analysis</u> which allows us to relate the two-sorted measurement situation of binary conjoint measurement to certain single-sorted measurement situations, among others the ordinal measurement discussed in Section 4.1 and the two types of difference measurement (originally introduced by Hölder (1901); see also Krantz et al. (1971)) for which nontrivially necessary sentences are given in Section 4.3. Examples of nontrivially necessary sentences for binary additive conjoint measurement are then obtained from these examples for difference measurement.

Another application, in Section 5.5, shows how ordinal (utility) measurement with a constant threshold ε (see Luce (1956)) can be reformulated as simple measurement. A nontrivially necessary axiom is then given for this type of measurement.

2. Basic Concepts of Abstract Measurement Theory

2.1. Notation and Conventions

Standard definitions and notation from set theory will be used throughout this paper. Specifically, a sequence of objects $X_0,X_1,X_2,\ldots$ of any kind will be denoted by $\langle X_0,X_1,X_2,\ldots\rangle$; an initial segment of a sequence $\langle X_0,\ldots,X_k\rangle$ is a subsequence $\langle X_0,X_1,\ldots,X_\ell\rangle$, $\ell \leq k$; here the original sequence may be infinite as well. If the terms $x_0,x_1,\ldots$ of a sequence $\langle x_0,x_1,\ldots\rangle$ are elements of a given set X (for example, the individuals of an L-structure, see below), the sequence is also denoted by $\vec{x}$. The set of all n-term sequences of elements of a set X is denoted by $(X)^n$; the set of all subsets of a set X which consist of exactly n distinct elements is denoted by $[X]^n$. The cardinality of the set X is denoted by $\bar{\bar{X}}$.

We adopt the usual definitions pertaining to first-order languages with equality with logical symbols: $\vee$, $\neg$, $\wedge$ (sometimes replaced by '&'), $\forall$, $\exists$; individual variables: $x,y,z,\ldots$; finitely many relation symbols R each specified to be n-ary for some $n \in \omega$; and possibly finitely many individual constant symbols $c_1,\ldots,c_k$ (which will be neglected after Section 3). Repeated conjunctions and disjunctions are denoted by $\bigwedge$, $\bigvee$ as in: $\bigwedge_{i<k} \phi_i$. Atomic formulas are of the forms $\alpha_1 = \alpha_2$, $R\alpha_1\cdots\alpha_n$, where the α's denote constant symbols or variables, and R is an n-ary relation symbol. Quantifier free formulas result from atomic formulas by finite application of $\vee$, $\wedge$, $\neg$. If ϕ is a formula, $\forall x\,\phi$ and $\exists x\,\phi$ are formulas (for any variable x) and in $\forall x\ \phi$ the universal quantifier $\forall x$ (or the existential quantifier $\exists x$ in $\exists x\ \phi$) binds any occurrence of the variable x in ϕ not already bound in ϕ; any unbound occurrence of a variable in a formula is free. A formula without free variables is a sentence. Formulas which can be obtained by the means explicitly described

have the property that all quantifiers precede all occurrences of $\wedge$, $\vee$, $\neg$ and are said to be in prenex form. More generally, conjunctions, disjunctions and negations of formulas are again formulas; but we will usually assume that all formulas are in prenex form, which is possible as any formula can be effectively converted to an equivalent (see below) formula in prenex form.

An existential (universal) formula or sentence is a formula or sentence (in prenex form!) with only existential (universal) quantifiers. More generally, the sequence of quantifiers in a formula ψ will be called the quantifier prefix Q, and the formula may be abbreviated as $Q\phi$, where ϕ is the quantifierfree part or matrix of ψ; then ψ is called a Q-formula, and similarly for sentences. We abbreviate repeated quantification of a single type, i.e. '$\exists x_1 \exists x_2 \ldots \exists x_n\ \phi$' as '$\exists x_1 x_2 \ldots x_n\ \phi$' or '$\exists \vec{x}\ \phi$'.

A formula is n-ary if and only if it contains exactly n distinct free variables; more precisely, if a formula is written so as to indicate which variables it contains, i.e. $\phi(\vec{x},\vec{y},\vec{z})$, then we will call ϕ a $(\ell+m+n)$-ary formula to indicate that $\vec{x} = \langle x_1 \ldots x_\ell \rangle$, $\vec{y} = \langle y_1 \ldots y_m \rangle$, $\vec{z} = \langle z_1 \ldots z_n \rangle$, and that these are all the free variables which may occur in ϕ.

Given a first-order language L an L-structure is an interpretation for L in the standard sense; i.e. a set with relations and constants corresponding to the relation symbols and constants of R. If A is an L-structure, the underlying set or domain is denoted by $|A|$; A is infinite (finite, countable) if and only if $|\bar{\bar{A}}|$ is infinite (finite, countable). The elements of the domain are called individuals. If $S \subseteq |A|$, then the substructure $A \upharpoonright S$ is the L-structure with domain S (which must contain all constants of A), constants as in A, and relations restricted to S, i.e. if R^A is an n-ary relation of A corresponding to the n-ary relation symbol R of L, then the relation on $A|S$ corresponding to R is $R^A \cap S^n$. A_1 is an <u>extension</u> of A_2 if and only if A_2 is a substructure of A_1. If m is a map from $|A_1|$ to $|A_2|$, we allow the notation

$$m: A_1 \rightarrow A_2 ;$$

i.e. m always induces a map from the L-structure A_1 to the L-structure A_2. Much of the paper will study such maps which satisfy extra requirements to be stated in the next subsection.

The notions of <u>satisfaction</u> and truth of L-formulas in L-structures can be defined using any of the accepted formalisms; our notation will be that if A is an L-structure $\phi(x_1 \ldots x_n)$ is an L-formula, and $a_1 \ldots a_n \in |A|$, we write $A \models \phi(a_1 \ldots a_n)$ for: ϕ is satisfied by $\langle a_1 \ldots a_n \rangle$ in A. A sentence is <u>true</u> in A if and only if it is satisfied in A. For any set of L-sentences T, $A \models T$ if and only if $A \models \phi$ for all $\phi \in T$; we then say that A <u>is a model</u> of T; $Md(T)$ denotes the set of all L-structures which are models of T; T is <u>satisfiable</u> if and only if it has a model. T_1 is a logical consequence of T_2, notation $T_2 \models T_1$, if and only if $Md(T_1) \supseteq Md(T_2)$.

$Th(A)$ denotes the set of all L-sentences true in A, and if K is a class of L-structures, $Th(K)$ denotes the set of all L-sentences true in <u>all</u> members of K. Two L-structures A_1 and A_2 are said to be <u>elementarily equivalent</u> if and only if they satisfy exactly the same L-sentences, i.e. if and only if $Th(A_1) = Th(A_2)$. The term 'elementary' is used throughout as a synonym for 'first-order'.

In the paper, basic metatheorems of first-order logic will be assumed, notably the effective reducibility to a logically equivalent prenex form (mentioned above) and <u>compactness</u>: a set of sentences is satisfiable if and only if all its finite subsets are satisfiable. (See Schoenfeld (1967) for statements and proofs.)

2.2. The Basic Model: Simple Measurement

We are now ready for a rigorous description of the basic model to be considered: Simple Measurement. Let L be a fixed first-order language with finitely many relation parameters and constant symbols appropriate to the description of hypothesized empirical relationships. The notion of a structure-preserving mapping m:

$$m: A_1 \rightarrow A_2$$

between L-structures will be characterized by a set Γ

$$\Gamma = \{\phi_1(\vec{x}),\ldots,\phi_n(\vec{x})\}$$

of quantifierfree L-formulas as follows:

m is a <u>Γ-morphism</u> if and only if for each $\vec{a}$ from $|A|$

$$A_1 \models \phi_i(\vec{a}) \Rightarrow A_2 \models \phi_i(m(\vec{a})) \ , \quad i = 1,2,\ldots,n \ .$$

Some examples will illustrate the manner in which we can specify the empirical relationships to be preserved by a representation m by requiring m to be a Γ-morphism for an appropriately chosen Γ. Let L be a language appropriate for the description of concatenation and order relations, i.e. L has a ternary relation parameter R^3 and a binary relation parameter R^2; let A_1, A_2 be L-structures, and m a mapping $A_1 \rightarrow A_2$.

(a) The classical notion of A_1 being a substructure of A_2 is obtained if we require m to be a Γ-morphism for

$$\Gamma = \{x_1 = x_2, x_1 \neq x_2, R^3x_1x_2x_3, \neg R^3x_1x_2x_3, R^2x_1x_2, \neg R^2x_1x_2\} ,$$

i.e. such Γ-morphisms are 1-1 mappings, and

$$\forall a_1a_2 \in |A_1|: A_1 \models R^2a_1a_2 \Leftrightarrow A_2 \models R^2m(a_1)m(a_2)$$
$$\forall a_1a_2a_3 \in |A_1|: A_1 \models R^3a_1a_2a_3 \Leftrightarrow A_2 \models R^3m(a_1)m(a_2)m(a_3)$$

(b) The notion of representation appropriate to extensive measurement (for a complete definition and references see below) is obtained by taking only

$$\Gamma = \{x_1 = x_2, R^3x_1x_2x_3, R^2x_1x_2, \neg R_2x_1x_2\} ,$$

i.e. these mappings are not required to be 1-1, and we may have $a_1, a_2, a_3 \in |A_1|$:

$$A_1 \models \neg R^3a_1a_2a_3 \quad \text{and} \quad A_2 \models R^3m(a_1)m(a_2)m(a_3)$$

Simple Measurement will consist of the representation by a Γ-morphism

$$m: E \to M$$

of ("empirical") L-structures E in a specified L-structure M, to be called the measurement structure. Thus a particular kind of simple measurement will be characterized by a pair $\langle M,\Gamma\rangle$ (which implicitly presupposes some particular first-order language L).

For example, extensive measurement (Hölder (1901); see also Krantz et al. (1971)) will be characterized by the choice of language with R^3, R^2 made in the examples above, and the pair $\langle M,\Gamma\rangle$ where

$$M = \langle \mathbb{R}, R^3, R^2\rangle = \langle \mathbb{R}, +, \leq\rangle \quad , \quad \mathbb{R}: \text{the real numbers}$$

and Γ was given in example (b) above.

The model of measurement just described is basically similar to that given by Scott and Suppes (1958); however, our notion of Γ-morphism is a generalization of the notion of homomorphism used there.

Many kinds of measurement studied in the literature require somewhat more general models, or seem to do so. Thus one may wish to replace the single measurement structure M by a class K of such structures. For simultaneous consideration of distinct attributes, such as in conjoint measurement of such attributes, it is useful to introduce structures consisting of distinct types of objects, with relations among objects of different types as well as among objects of a common type. Often it is appropriate to describe a measurement structure in a different language than the empirical structures.

We will consider such models in section 5.3. Often it will be possible to reduce the study of a more generalized measurement model to the study of simple measurement; or the techniuqes developed in the analysis of simple measurement prove similarly useful in the analysis of generalized measurement models. This will be demonstrated by examples in sections 5.4-5.5.

Thus the phenomena discovered below in the analysis of simple measurement are manifested throughout measurement theory; the choice of simple measurement as the model in the theoretical development was made in order to enhance conceptual clarity. Once the ideas are properly understood, it will be sufficiently clear how they apply in different settings.

We now establish notation for some modeltheoretic concepts which will play a crucial role in the analysis of simple measurement models. Let, for any L-structure A, and any class of L-structures K

$$\mathrm{Eq}(A) \overset{\mathrm{def}}{=} \text{the set of L-structures elementarily equivalent to } A$$

(i.e. which satisfy exactly the same L-sentences as A);

$\Gamma^{-1}(K) \overset{\text{def}}{\equiv}$ the set of L-structures Γ-morphically representable in some member of K

$\Gamma^{-1}(A) \overset{\text{def}}{\equiv} \Gamma^{-1}(\{A\})$, i.e. the set of L-structures Γ-morphically representable in A

(i.e. $\Gamma^{-1}(\cdot)$ is the inverse Γ-morphic image operator)

$\mathbb{F} \overset{\text{def}}{\equiv}$ the set of finite L-structures Γ-morphically representable in M

(Note that $\langle M,\Gamma\rangle$ is assumed in context.)

The classes of L-structures which will be considered in the following are indicated in Diagram 1, which is drawn under the assumption that $|M|$ is infinite (points on the page indicate individual L-structures and the universe of points is the class of all L-structures).

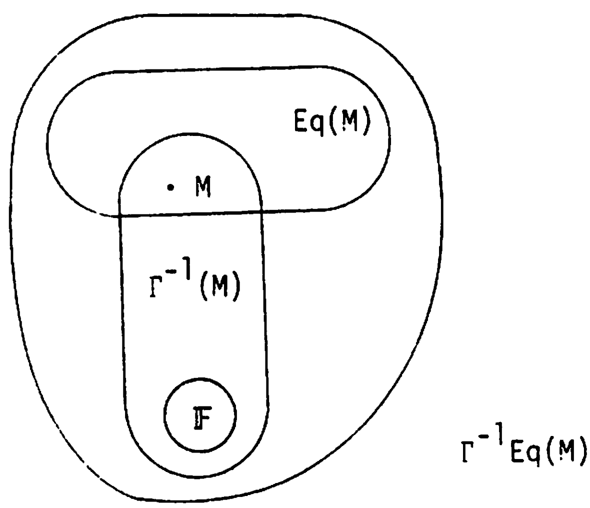


Diagram 1

For measurement theory, the relevant aspect of the basic classes is the first-order theory associated with each class. For example, Th $\Gamma^{-1}(M)$ is the set of first order consequences in L of the hypothesis (which itself can obviously not be formulated in L) that an arbitrary empirical structure A is Γ-embeddable into M. Mathematically, investigation of properties and relationships of the classes of models is more convenient than direct investigation of the associated first-order theories. Also, the relationships between the classes of models are more easily visualized.

The following relationships indicated in Diagram 1 follow trivially from the definitions and the assumption that M has an infinite universe.

1. $M \in \Gamma^{-1}(M)$
2. $M \in Eq(M)$
3. $\Gamma^{-1}(M) \subseteq \Gamma^{-1}Eq(M)$
4. $Eq(M) \subsetneq \Gamma^{-1}Eq(M)$
5. $\mathbb{F} \subseteq \Gamma^{-1}Eq(M)$, $M \notin \mathbb{F}$
6. $\mathbb{F} \cap Eq(M) = \emptyset$

On the other hand, '$\Gamma^{-1}(M) \neq \Gamma^{-1}Eq(M)$' or equivalently, '$Eq(M) \subsetneq \Gamma^{-1}(M)$' is a nontriviality postulate, and not always satisfied. In most cases of interest, M satisfies various non-first-order statements not shared by most other structures in Eq(M) but preserved under inverse Γ-morphic images, for instance cardinality restrictions and archimedian axioms. Such properties are connected in not entirely understood but essential ways with the usefulness of M as a measure structure. For example, archimedian axioms often entail that measurement procedures terminate within a finite number of steps, a property of obvious importance in actual measurement. These issues will be clarified to some extent in section 4.

The only relationship indicated in Diagram 1 which requires further justification is:

7. $\mathbb{F} \subseteq \Gamma^{-1}(M)$

This will be demonstrated as Lemma 3.2.1.

2.3. Axiomatization of $\mathrm{Th}\,\Gamma^{-1}\mathrm{Eq}(M)$

Given a measurmment model $\langle M,\Gamma\rangle$, one can consider various classes of structures which are representable according to that model, and the axioms satisfied by the structures in such classes. We wish to find general properties of such axiomatizations, i.e. properties which are true for all possible $\langle M,\Gamma\rangle$, or for wide varieties of $\langle M,\Gamma\rangle$; and also to gain an insight as to what properties of an $\langle M,\Gamma\rangle$ affect properties of axiomatizations. We are especially interested in necessary axioms for $\langle M,\Gamma\rangle$-representability of arbitrary L-structures, or finite L-structures. Can these axioms be first-order? Can they be universal axioms (i.e. only universal quantifiers occur)? Are finite sets of axioms sufficient? Are there systematic procedures for obtaining such axioms?

As a first step in answering such questions, we will find an important class of necessary axioms for Γ-morphic representability. Consider the set of all L-formulas of the following form:

$$\neg\,[\psi_1 \wedge \cdots \wedge \psi_k] \ ,$$

$\psi_1,\ldots,\psi_k$ variable substitution variants of formulas in Γ; $k \geq 1$. These formulas and conjunctions of such formulas will be called <u>$\bar{\Gamma}$-formulas</u>. Sentences obtained by quantifying over all variables in a $\bar{\Gamma}$-formula will be called <u>$\bar{\Gamma}$-sentences</u>, or, if the quantifier sequence used is Q, <u>$Q\bar{\Gamma}$-sentences</u>.

For example,

$$\forall\vec{y}\ \gamma(\vec{y})\ , \quad \gamma\text{: } \bar{\Gamma}\text{-formula with free variables } \vec{y}$$

is a <u>$\forall\bar{\Gamma}$-sentence</u>.

$\forall\bar{\Gamma}$-sentences have an important property, preservation under inverse images of Γ-morphisms: Let A_1, A_2 be L-structures, and $\forall\vec{y}\ \gamma(\vec{y})$ a $\forall\bar{\Gamma}$-sentence, such that

$$A_2 \models \forall\vec{y}\ \gamma(\vec{y})\ .$$

Let $\phi\colon A_1 \to A_2$ be a Γ-morphism. If $\forall\vec{y}\ \gamma(\vec{y})$ were false in A_1, i.e.

$$A_1 \models \neg\,\gamma(\vec{a})$$

for some $\vec{a}$ from $|A_1|$, then for some variable substitution instances $\psi_1,\ldots,\psi_k$ of formulas in Γ, and such an $\vec{a}$ from $|A_1|$,

$$A_1 \models \neg\neg\,[\psi_1(\vec{a}) \wedge \cdots \wedge \psi_k(\vec{a})]\ ,$$

i.e. $A_1 \models \psi_i(\vec{a})$, $i = 1,\ldots,k$; $\psi_i(\vec{x}) \in \Gamma$. So because ϕ is a Γ-morphism,

$$A_2 \models [\psi_1(\phi(\vec{a})) \wedge \cdots \wedge \psi_k(\phi(\vec{a}))]$$

$$A_2 \models \exists\vec{y}\ \neg\,\gamma(\vec{y})\ , \quad \text{namely } \phi(\vec{a})$$

contradicting the assumption that $A_2 \models \forall\vec{y}\ \gamma(\vec{y})$. Thus we have the following (taking A_2 in the above to be a measurement structure M).

<u>Proposition 2.3.1</u>. Let $\langle M,\Gamma\rangle$ specify a type of simple measurement. Every structure E Γ-morphically representable in M satisfies the set $Th_{\forall\bar{\Gamma}}(M)$ of all $\forall\bar{\Gamma}$-sentences true in M.

Thus the sentences in $Th_{\forall\bar{\Gamma}}(M)$ are necessary axioms for $\langle M,\Gamma\rangle$-measurability. It will be helpful to have a characterization of the $\langle M,\Gamma\rangle$-measurement properties common to all structures E satisfying these axioms.

Let A be an L-structure, $\{d_a: a \in |A|\}$ a set of new constant symbols, and $L^A = L \cup \{d_a: a \in |A|\}$. The <u>Γ-diagram</u> $D_\Gamma(A)$ <u>of</u> A is the set of L^A-sentences (quantifierfree):

$$D_\Gamma(A) = \{\phi(d_{a_1} \cdots d_{a_n}): \phi(x_1,\ldots,x_n) \in \Gamma \text{ and } A \models \phi(a_1,\ldots,a_n)\} .$$

<u>Theorem 2.3.2</u>. Let K be a first-order class of L-structures. For any L-structure A:

$$A \in \Gamma^{-1}(K) \Leftrightarrow \text{All } \forall\bar{\Gamma}\text{-sentences true in } K \text{ are true in } A .$$

<u>Proof</u>. Let $A \in \Gamma^{-1}(K)$, i.e. $A \in \Gamma^{-1}(T)$ for some structure $T \in K$. Let ϕ be a $\forall\bar{\Gamma}$-sentence true in K, hence in T. Then ϕ is true in A because truth of $\forall\bar{\Gamma}$-sentences is preserved in Γ-substructures.

To show the converse, let A be an L-structure, $A \notin \Gamma^{-1}(K)$. Then the set of L^A-sentences:

$$Th(K) \cup D_\Gamma(A) \qquad (*)$$

must be unsatisfiable. For if this set of sentences was satisfiable, one could assign all constants in $\{d_a: a \in |A|\}$ to elements of the universe of some $T \in K$ in such a way that the resulting expansion of T satisfies $D_\Gamma(A)$. This would mean that the map

$$m: m(a) = d_a \in |T| , \quad \text{all } a \in |A|$$

defined by any one such assignment is a Γ-morphism

$$m: A \rightarrow T \in K$$

contradicting the assumption that $A \notin \Gamma^{-1}(K)$.

It follows from the unsatisfiability of (*) by the compactness theorem of first-order logic that for some finite subset $\{\phi_1,\ldots,\phi_n\} \subseteq D_\Gamma(A)$,

$$Th(K) \cup \{\phi_1,\ldots,\phi_n\}$$

must still be unsatisfiable. Hence we cannot choose elements in any structure in K to represent the new constants, say $d_{a_1},\ldots,d_{a_m}$, occurring in $\phi_1,\ldots,\phi_n$, such that all the ϕ_i are satisfied. But this means

$$K \models \forall a_1 \cdots a_m \neg [\phi_1 \wedge \cdots \wedge \phi_m] .$$

Thus we have a $\forall\bar{\Gamma}$-sentence which is true in K and false in A_1 for because $\{\phi_1,\ldots,\phi_n\} \subseteq D_\Gamma(A)$, we must have

$$A \models \exists a_1 \cdots a_m [\phi_1 \wedge \cdots \wedge \phi_n] . \qquad \text{Q.E.D.}$$

We can apply this result in the case $K = Eq(M)$, and thus we find that $Th_{\forall\bar{\Gamma}}(Eq(M)) = Th_{\forall\bar{\Gamma}}(M)$ axiomatizes $\Gamma^{-1}Eq(M)$. As we have assumed that L has only finitely many relation parameters the set of $\forall\bar{\Gamma}$-sentences is decidable. It follows that if $Th(M)$ is decidable, so is $Th_{\forall\bar{\Gamma}}(M)$; and if $Th(M)$ is recursively enumerable (and hence, by a device due to Craig (1953), has a decidable axiomatization), the same is true for $Th_{\forall\bar{\Gamma}}(M)$.

From the point of view of measurement theory, these results can be reformulated as:

Proposition 2.3.3. There is a necessary and sufficient condition for Γ-morphic representability of arbitrary structures in structures elementarily equivalent to M, consisting of the set $Th_{\forall\bar{\Gamma}}(M)$ of universal $\bar{\Gamma}$-sentences. Furthermore:

(a) These sentences contain only those relation parameters of L which occur in Γ; hence all other structural knowledge of M is irrelevant.

(b) If $Th(M)$ is decidable or axiomatizable, so is $Th_{\forall\bar{\Gamma}}(M)$.

Another application of Theorem 2.3.2 is:

<u>Theorem 2.3.4</u>. $\Gamma^{-1}(M)$ is an elementary class $\Leftrightarrow$ $\Gamma^{-1}(M) = \Gamma^{-1}Eq(M)$.

<u>Proof</u>. One direction is trivial; for the other, assume $\Gamma^{-1}(M)$ is a first-order class. Now $\Gamma^{-1}(\Gamma^{-1}(M)) = \Gamma^{-1}(M)$; so taking $K = \Gamma^{-1}(M)$ in Theorem 2.3.2, $\Gamma^{-1}(M)$ is axiomatized by $Th_{\forall\bar{\Gamma}}(\Gamma^{-1}(M))$. As $\Gamma^{-1}(M) \subseteq \Gamma^{-1}Eq(M)$, $Th(\Gamma^{-1}(M)) \supseteq Th(\Gamma^{-1}Eq(M)) = Th_{\forall\bar{\Gamma}}(M)$. So $\Gamma^{-1}(M) \neq \Gamma^{-1}Eq(M)$ only if $Th_{\forall\bar{\Gamma}}(\Gamma^{-1}(M)) \supsetneq Th_{\forall\bar{\Gamma}}(M)$, i.e. there is a $\forall\bar{\Gamma}$-sentence true in $\Gamma^{-1}(M)$ but not in M. But this is impossible, as $M \in \Gamma^{-1}(M)$. So indeed $\Gamma^{-1}(M) = \Gamma^{-1}Eq(M)$.

In measurement-theoretical terms, Theorem 2.3.4 means that there are only necessary and sufficient first-order conditions for $\langle M,\Gamma\rangle$-representability if all models of $Th_{\forall\bar{\Gamma}}(M)$ are $\langle M,\Gamma\rangle$-representable.

2.4. <u>Notes on Section 2</u>

The model of simple measurement, in a somewhat less general form, was formulated in Scott and Suppes (1958); their model is obtained from ours by requiring Γ to be the set of all atomic formulas of L and their negations, with the exception that the language L need not contain equality-formulae (i.e. '$x_1 = x_2$'); otherwise such a Γ would force Γ-morphisms to be 1-1, which was not required by Scott and Suppes.

In model theory, sets of formulae such as Γ have often been used to define classes of mappings between structures and to prove theorems about categories of structures with such mappings which depend solely on syntactical properties of the formulas in Γ. Our Γ-morphisms are special cases of the Γ-morphisms of Schoenfeld (1967), and the F-maps of Fittler (1969).

The result of Theorem 2.3.2 is classical, having been demonstrated in various contexts by Tarski (1959) and Łoś (1955) (extensions of structures), Fittler (1969, 1972) and others; application to measurement theory occurs in Suppes and Scott (1958).

3. Representability of Finite Structures

3.1. Introduction

Because any body of actual discrete observations is finite, finite structures have been important in methodological considerations in measurement theory. Thus, for example, Adams, Fagot and Robinson (1970) evaluate the empirical content of axioms in theories of measurement by considering which finite lists of "observation" statements are consistent with an axiomatic theory (that is, a model of the theory contains a finite substructure whose elements satisfy such a finite list of "observation" statements). From an axiomatic point of view, the consistency consists in a certain universal sentence (asserting that there do not exist elements satisfying a given finite list of "observation" statements) not being entailed by the axiomatic theory in question. For similar reasons, Scott and Suppes (1958) pose the problem of first order axiomatization of measurability of finite structures. They especially stress finite universal axiomatizations because the empirical validity of such axiomatizations can generally be established more convincingly than that of infinite or nonuniversal axiomatizations. Section 3.2 gives an overview of the facts about axiomatization of measurability of finite structures which can be shown in the framework of simple measurement. (Discussion of applications of Vaught's Criterion for finite universal axiomatizability has been omitted, as this subject has received treatment elsewhere, see Scott and Suppes (1958), and Titiev (1969).)

At this point, however, we wish to caution against restricting methodological discussion to consideration of measurability of finite structures. In many situations, the domain of potential observations (i.e. lengths of objects) is considered infinite. We will argue that in such cases infinitary considerations are unavoidably involved in justification of a measurement procedure, even though such justification, as well as initial calibration of the procedure, presumably would be based on a finite body of observations.

A measurement procedure consists of an empirically executable algorithm which yields (arbitrarily good approximations to) a measurement value for any given object. Among other things it is required that subsequent measurement values for various objects be consistent, i.e. it must not be necessary to go back and modify an earlier measurement value in the light of later observations. (This does not exclude later improvement of an approximate measurement value.) This requirement is not a special feature of measurement procedures, but rather follows from the definition of representation as a function from an entire domain of empirical objects, assuming that we intend to use the measurement procedure to construct such a representation.

In a mathematical model, such a sequence of subsequent measurements may be represented by (1) a sequence of finite empirical structures, each a substructure of the next, corresponding to the totality of objects which have been measured at each subsequent instant of time, with (2) a sequence of mappings from these structures to the measurement structure, each mapping extending all the previous ones. This corresponds to the requirement that previous measurement values remain consistent with later obtained measurements. By assumption that the potential domain of observation be infinite, we have that (3) the structures in the sequence become arbitrarily large.

Now assume that a sequence of finite structures satisfying (1) and (3) satisfied a set of necessary and sufficient axioms for measurability of finite structures. That would imply that there would be a sequence of mappings from the structures to the measurement structure. However, and this is the crucial point, it does <u>not</u> follow that (2) is satisfied; in fact in sections 4.2-4.5 examples will be provided of such sequences of structures, each measurable but only by constant revision at finite stages, of previous measurement values. These examples occur in settings of practical importance, such as extensive measurement into $\langle\mathbb{R},+,\leq\rangle$; moreover, such occurrences seem to be unavoidably connected with the existence of empirically executable measurement procedures, of whatever kind. This may seem paradoxical (as we seem to be arguing that measurement procedures can't work in such circumstances) but of course the union of the sequence of finite structures is an infinite structure which simply cannot be measured. The point is that no axioms for measurement of finite structures, and no finite set of observations, can strictly exclude this structure. (Note that, as we shall see below in section 3.2, necessary and sufficient axioms for measurability of finite structures can always be universal, so that the union of our chain of structures does in fact satisfy all these axioms.)

3.2. Axiomatization of Th($\mathbb{F}$)

The initial fact about measurement of finite structures is that we inherit a set of necessary and sufficient axioms from $\Gamma^{-1}\mathrm{Eq}(M)$. This was already argued by Scott and Suppes (1958):

Lemma 3.2.1. $\mathbb{F} \subseteq \Gamma^{-1}(M)$

Proof. Let $A \in \mathbb{F}$, that is, $|A|$ is finite and A is Γ-embeddable into some $T \in \mathrm{Eq}(M)$. Let $\{x_1,\ldots,x_n\}$ be the image of A in T under some Γ-morphism $A \to T$; let $\phi(x_1\cdots x_n)$ be the quantifierfree description of the relations among $x_1\cdots x_n$ in T:

$$\phi(x_1\cdots x_n) = R_1(x_{i_1}\cdots x_{i_k}) \wedge\cdots\wedge \neg R_1(x_{j_1}\cdots x_{j_k}) \wedge\cdots\wedge R_m(\cdots)$$

which is a finite L-formula because by assumption there are only finitely many parameters $R_1\cdots R_m$ in L, and $\{x_1\cdots x_n\}$ is a finite set. Then clearly

$$T \models \exists a_1 \cdots a_n \ \phi(a_1 \cdots a_n)$$

and so, because $T \in Eq(M)$,

$$M \models \exists a_1 \cdots a_n \ \phi(a_1 \cdots a_n) \ .$$

But now any instance $\{y_1 \cdots y_n\}$ in M of this existential assertion satisfies

$$M \restriction \{y_1 \cdots y_n\} \simeq T \restriction \{x_1 \cdots x_n\} \ .$$

[$\simeq$ = isomorphic; $T \restriction S$ = T restricted to subuniverse S] as ϕ describes $T \restriction \{x_1 \cdots x_n\}$ up to isomorphism. As we can by hypothesis Γ-embed A into $T \restriction \{x_1 \cdots x_n\}$, we can Γ-embed A into $M \restriction \{y_1 \cdots y_n\}$ and so into M itself, so that $A \in \Gamma^{-1}(M)$. As A was arbitrary in $\mathbb{F}$, we find that $\mathbb{F} \subseteq \Gamma^{-1}(M)$.

<u>Remark</u>. The argument remains valid under the assumption that only finitely many relation parameters $R_1 \cdots R_m$ occur in formulas in Γ, regardless of the number of relation parameters in L.

Lemma 3.2.1 justifies the following conclusion:

<u>Proposition 3.2.2</u>. A finite structure is Γ-morphically representable in M if and only if it satisfies $Th_{\forall\bar{\Gamma}}(M)$.

This entails that any axiomatization of $Th_{\forall\bar{\Gamma}}(M)$ is an axiomatization for Γ-morphic representability of finite structures in M. Conversely, as will be shown next, axiomatizations for Γ-morphic representability of finite structures in M yield axiomatizations of $Th_{\forall\bar{\Gamma}}(M)$.

<u>Lemma 3.2.3</u>. Let T be a set of universal L-sentences (with possibly one $\exists\forall$-L-sentence). If all finite models of T are in $\Gamma^{-1}(M)$, then all models of T are in $\Gamma^{-1}Eq(M)$.

Proof. Let $A \in Md(T)$. If $A \notin \Gamma^{-1}Eq(M)$, then, as was shown in the proof of Theorem 2.3.2, there are $\phi_1 \cdots \phi_n \in D_\Gamma(A)$ involving $d_{x_1},\ldots,d_{x_m}$, $x_1,\ldots,x_m \in |A|$, such that

$$M \models \neg \exists a_1 \cdots a_m \, [\phi_1 \wedge \cdots \wedge \phi_n]$$

If T contains an $\exists\forall$-sentence, let $y_1 \cdots y_k \in |A|$ form an instance satisfying the existential quantifiers of that sentence; otherwise let $k = 0$. The finite L-structure

$$A \restriction \{x_1 \cdots x_m, y_1 \cdots y_k\} \overset{\text{def}}{=} A'$$

contains $x_1,\ldots,x_m$, and therefore satisfies

$$A' \models \exists a_1 \cdots a_m \, [\phi_1 \wedge \cdots \wedge \phi_n]$$

so that $A' \notin \Gamma^{-1}(M)$. But also, by the preservation of universal formulae (e.g. sentences) under substructures, $A' \in Md(T)$, and hence by hypothesis $A' \in \Gamma^{-1}(M)$. We conclude that $A \in \Gamma^{-1}Eq(M)$.

Proposition 3.2.4. Let T be a set of universal sentences. T axiomatizes $\mathbb{F}$ (i.e. Γ-morphic representability in M) in the sense

$$\forall \text{L-structure } A\text{: } A \text{ finite} \Rightarrow [A \in \Gamma^{-1}(M) \Leftrightarrow A \models T] \qquad (*)$$

if and only if T axiomatizes $\Gamma^{-1}(Eq(M))$ in the sense

$$\forall \text{L-structure } A\text{: } [A \in \Gamma^{-1}Eq(M) \Leftrightarrow A \models T] \,. \qquad (**)$$

Proof. If T satisfies (**), then $T \models Th\, \forall\bar{\Gamma}(M)$ and $\mathbb{F} \models T$, so Proposition 3.2.2. T satisfies (*).

Let T satisfy (*). Then all finite models of T are in $\Gamma^{-1}(M)$, so by Lemma 3.2.3, all models of T are in $\Gamma^{-1}Eq(M)$, so

$$A \models T \Rightarrow A \in \Gamma^{-1}Eq(M) .$$

If $A \in \Gamma^{-1}Eq(M)$, not $A \models T$, then for some $\phi \in T$

$$\text{not } A \models \phi , \quad \phi = \forall x_1 \cdots x_n \ \psi(x_1 \cdots x_n) , \quad \psi \text{ quantifierfree}$$

$$A \models \exists x_1 \cdots x_n \neg\psi(x_1 \cdots x_n)$$

$$A \restriction \{a_1 \cdots a_n, c_1 \cdots c_k\} \models \exists x_1 \cdots x_n \neg\psi(x_1 \cdots x_n)$$

where $a_1,\ldots,a_n \in |A|$, $A \models \psi(a_1 \cdots a_n)$, and $c_1,\ldots,c_k$ are the constants of L (if any). But $A \restriction \{a_1 \cdots a_n, c_1 \cdots c_k\}$ is a substructure of A, hence in $\Gamma^{-1}Eq(M)$, and is finite, hence in $\mathbf{F}$, and

$$\underline{\text{not}}\ A \restriction \{a_1 \cdots a_n, c_1 \cdots c_k\} \models T$$

contradicting the assumption (*). Hence we have $A \in \Gamma^{-1}Eq(M) \Rightarrow A \models T$, completing the proof of (**).

The reverse process, for obtaining axiomatizations for $\Gamma^{-1}Eq(M)$ is of interest because in actual practice one can sometimes obtain efficient axiomatizations of Γ-morphic representability of finite structures in M, in the sense of (*), directly, for example cancellation laws for conjoint measurement. In such cases this is probably more economical than use of a decision procedure for the $\forall\bar{\Gamma}$-theory of M or more general and unnecessarily powerful procedures.

We note that Lemma 3.2.3 is the strongest possible generally true proposition of its type. In fact, if we consider the next possible case (with respect to the classification of L-sentences by their quantifier prefixes in prenex form) we find a counterexample to the assertion of 3.2.3 as extended to that case:

Counterexample 3.2.5. Let L, M, Γ be arbitrary subject to our general assumptions. Let T consist of the set of existential axioms:

$$\{'\exists a_0 \cdots a_n \, [\bigwedge_{i \neq j} a_i \neq a_j]' : n \in w\} = T .$$

Then $M \models T$, all finite models of T are in $\Gamma^{-1}(M)$ (as there are none), and not all models of T will belong to $\Gamma^{-1}Eq(M)$ unless this set contains all infinite L-structures (which it normally would not).

On the other hand, the reasoning employed in proving the implication $(*) \Rightarrow (**)$ is of significantly wider application. We showed then that a universal axiom true in $\mathbb{F}$ is true in $\Gamma^{-1}Eq(M)$; among other things it follows that such an axiom is a logical consequence of some $\forall\bar{\Gamma}$-sentence in $Th_{\forall\bar{\Gamma}}(M)$. All this and more is true for arbitrary $\forall\exists$-sentences (which by cumulativity of our prefix classification, include existential and universal sentences).

Theorem 3.2.6. Let L have constants $\{c_1, \ldots, c_k\}$, if any.

Any $\forall\exists$-axiom necessary for Γ-morphic representation in M of finite structures is necessary for Γ-morphic representation of arbitrary structures in (members of) $Eq(M)$, and hence a consequence of $Th_{\forall\bar{\Gamma}}(M)$.

In fact, if $\phi = \forall x_1 \cdots x_m \, \exists y_1 \cdots y_n \, \psi(x_1 \cdots x_m, y_1 \cdots y_n)$, ψ quantifierfree, is an L-sentence, then

1) either

(i) ϕ is unsatisfied in a structure in $\Gamma^{-1}(M)$ with at most m elements distinct from the constants of L, $m \geq 1$, or

(ii) $Th_{\forall\bar{\Gamma}}(M) \models \phi$

and 2) we can effectively find a universal sentence χ such that

(i) $\vdash \chi \rightarrow \phi$

(ii) $\chi \in \mathrm{Th}\ \Gamma^{-1}\mathrm{Eq}(M) \Leftrightarrow \phi \in \mathrm{Th}(\mathbb{F})$

(iii) $\chi = \forall x_1\cdots x_m \bigvee_{y_1\cdots y_n \in \{x_1\cdots x_m, c_1\cdots c_k\}} \psi(x_1\cdots x_m, y_1\cdots y_n)$

Proof. Assume, contrary to 1(ii), that not $\mathrm{Th}_{\forall\mathbb{F}}(M) \models \phi$. Then there is an L-structure $A \in \Gamma^{-1}\mathrm{Eq}(M)$ such that $A \models \neg\phi$, i.e.

$$A \models \exists x_1\cdots x_m\ \forall y_1\cdots y_n \neg\psi(x_1\cdots x_m, y_1\cdots y_n)\ .$$

Choose, on the strength of this, an $a_1,\ldots,a_m \in |A|$ such that

$$A \models \forall y_1\cdots y_n \neg\psi(a_1\cdots a_m, y_1\cdots y_n)\ .$$

Because substructures are Γ-substructures,

$$A' \overset{\mathrm{def}}{\equiv} A\restriction\{a_1\cdots a_m, c_1\cdots c_k\} \in \Gamma^{-1}\mathrm{Eq}(M)\ .$$

Clearly A' has at most m elements distinct from the constants of L unless L has no constants and we were considering an existential sentence, i.e. $m = 0$. In this case, let $a \in |A|$ be arbitrary, and set $A' \overset{\mathrm{def}}{\equiv} A\restriction\{a\}$.) and by preservation of universal formulas in substructures,

$$A' \models \forall y_1\cdots y_n \neg\psi(a_1\cdots a_m, y_1\cdots y_n)$$
$$A' \models \neg\ \forall x_1\cdots x_m\ \exists y_1\cdots y_n\ \psi(x_1\cdots x_m, y_1\cdots y_n)$$

But by Lemma 3.2.1, $A' \in \mathbb{F} \subseteq \Gamma^{-1}(M)$, so 1(i) is true.

Thus either 1(i) or 1(ii) is true. This implies the initial assertion of the theorem; to show 2, first note that for the choice of χ in 2(iii), 2(i) is true, and hence $\chi \in \mathrm{Th}\ \Gamma^{-1}(M) \Rightarrow \phi \in \mathrm{Th}(\mathbb{F})$. If χ were false in some $A \in \Gamma^{-1}(M)$, then "cutting out" a counterexample instance $\{x_1\cdots x_m\}$

as in the proof of 1, we obtain $A' \in \mathbb{F}$ in which ϕ must be false. So $\chi \notin \mathrm{Th}\,\Gamma^{-1}(M) \Rightarrow \phi \notin \mathrm{Th}(\mathbb{F})$. Q.E.D.

Thus any $\forall\exists$-axiom for measurement is either replaceable by a specific necessary $\forall$-axiom (which we can effectively obtain from it), or is not necessary for measurability of finite structures (the cardinalities of which we can effectively obtain from the $\forall\exists$-axiom). This essentially settles the general questions concerning the kinds of first-order consequences of measurability which have been considered of direct empirical significance in the methodological literature: $\forall\exists$ and simpler sentences. It also sheds light on the fact that $\forall\exists$-axioms in representation theorems tend to be unnecessary for measurability (of course these structural axioms play a crucial role in the existence of measurement procedures and unique representations).

Again, Theorem 3.2.6 is as strong as possible for a general assertion of this kind. For we have:

<u>Counterexample 3.2.7</u>. Let M be the real numbers.

(a) For any language for the real numbers containing "+", and any Γ such that $\mathbb{F} \neq \emptyset$, let ϕ be $\exists xy\, \forall z\, [x+y \neq z \vee z+z = z]$ (non-closure of $+$).

(b) For any language for the real numbers containing "$\geq$" or $>$, and any Γ such that $\mathbb{F} \neq \phi$, let ϕ be $\exists x\, \forall y\, [x \not> y]$ (minimal elements).

In both cases $\mathbb{F} \models \phi$, but $M = \mathbb{R} \models \neg\phi$, so $M \in \Gamma^{-1}(M)$, $\phi \notin \mathrm{Th}\,\Gamma^{-1}(M)$, $\phi \notin \mathrm{Th}\,\Gamma^{-1}\mathrm{Eq}(M)$. The example is fairly general. In case (a) we could replace $\mathbb{R}$ by any infinite group without nontrivial finite subgroups; in case (b) by any infinite partial order without minimal elements.

Theorem 3.2.6 yields the following result on finite axiomatizability:

Proposition 3.2.8. The following assertions are pairwise equivalent and the first pair implies the second. (See Diagram 2.)

i(a) There is a $\forall\bar{\Gamma}$-sentence ϕ

$$\mathrm{Th}(\mathbb{F}) = \{\psi: \phi \vdash \psi,\ \psi: \text{L-sentence}\}$$

i(b) There is a $\forall\exists$-sentence ϕ'

$$\mathrm{Th}(\mathbb{F}) = \{\psi: \phi' \vdash \psi,\ \psi: \text{L-sentence}\}$$

$$\begin{tikzcd} \text{i(a)} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{i(b)} \\ \text{ii(a)} \arrow[r, Leftrightarrow] & \text{ii(b)} \end{tikzcd}$$

Diagram 2

ii(a) There is a $\forall\bar{\Gamma}$-sentence ϕ

$$\text{For any L-structure A: A finite} \Rightarrow [A \in \mathbb{F} \Leftrightarrow A \models \phi]$$

ii(b) There is a $\forall\exists$-sentence ϕ'

$$\text{For any L-structure A: A finite} \Rightarrow [A \in \mathbb{F} \Leftrightarrow A \models \phi'] .$$

Proof. In both cases, (a) $\Rightarrow$ (b) is trivial.

i(b) $\Rightarrow$ i(a): If a $\forall\exists$-sentence ϕ' axiomatizes $\mathrm{Th}(\mathbb{F})$, then $\phi' \in \mathrm{Th}(\mathbb{F})$, and so by Theorem 3.2.6, part 2, ϕ' is implied by a $\forall\bar{\Gamma}$-sentence ϕ in $\mathrm{Th}\,\Gamma^{-1}\mathrm{Eq}(M) \subseteq \mathrm{Th}(\mathbb{F})$. But then ϕ axiomatizes $\mathrm{Th}(\mathbb{F})$, as $\vdash \phi \leftrightarrow \phi'$. Note that in this case $\mathrm{Th}(\mathbb{F}) = \mathrm{Th}\,\Gamma^{-1}\mathrm{Eq}(M)$.

ii(b) $\Rightarrow$ ii(a): If a $\forall\exists$-sentence ϕ' axiomatizes $\mathbb{F}$, then $\phi' \in \mathrm{Th}(\mathbb{F})$ and so by Theorem 3.2.6, part 2, ϕ' is implied by a $\forall\bar{\Gamma}$-sentence ϕ in $\mathrm{Th}\,\Gamma^{-1}\mathrm{Eq}(M) \subseteq \mathrm{Th}(\mathbb{F})$. Assume that A is a finite L-structure. Then

$$A \models \phi \Rightarrow A \models \phi' \Rightarrow A \in \mathbb{F} \Rightarrow A \models \mathrm{Th}\,\mathbb{F} \Rightarrow A \models \phi .$$

i(a) $\Rightarrow$ ii(a): Assume i(a). If $A \in \mathbb{F}$, then obviously $A \models \phi$. If A is finite and $A \models \phi$ then $A \in \Gamma^{-1}\mathrm{Eq}(M)$, because $\Gamma^{-1}\mathrm{Eq}(M)$ is an elementary class, by Theorem 2.3.2 and as we noted above $\mathrm{Th}\,\Gamma^{-1}\mathrm{Eq}(M) = \mathrm{Th}(\mathbb{F})$ in this case. But then $A \in \mathbb{F}$ by the definition of $\mathbb{F}$.

4. Representability of Arbitrary Structures

4.1. Nontrivially Necessary Sentences

In section 4, necessary sentences for $\langle M,\Gamma\rangle$-measurability of arbitrary L-structures will be considered; that is, Th $\Gamma^{-1}(M)$. As $\Gamma^{-1}(M) \subseteq \Gamma^{-1}Eq(M)$, Th $\Gamma^{-1}(M)$ contains Th $\Gamma^{-1}Eq(M)$, which was considered in detail in previous sections. Thus it will be sufficient at present to find the sentences of Th $\Gamma^{-1}(M)$ which are not contained in Th $\Gamma^{-1}Eq(M)$. These sentences will be called *nontrivially necessary* whereas the sentences of Th $\Gamma^{-1}Eq(M)$ will be called *trivially necessary*.

From these definitions and Theorem 2.3.4, we directly obtain a basic result:

Proposition 4.1.0. If there are any nontrivially necessary axioms for $\langle M,\Gamma\rangle$-measurement, then $\Gamma^{-1}(M)$ is not a first-order class.

In the present subsection, some basic relationships between the situation of M in $Eq(M)$ and nonexistence of nontrivially necessary sentences will be discussed. In 4.2, "inductive" processes on structures will be introduced in terms of which nontrivially necessary sentences are characterized. Subsection 4.3 provides examples of nontrivially necessary sentences in a familiar setting, viz. extensive measurement and difference measurement. In 4.4 it is shown that certain properties of the structure M exclude sentences of certain syntactical forms from being nontrivially necessary. In fact the examples for extensive measurement in subsection 4.3 are sentences of the simplest possible forms not excluded by the results of 4.4 as applied to $\langle \mathbb{R},+,\leq\rangle$. The theory of 4.2-4.4 applies mainly to $\forall\exists\forall$-sentences. In 4.5 we show how to generalize the ideas of 4.2 to arbitrary sentences. The basic nonexistence theorem for nontriivally necessary sentences is an analogue of the Löwenheim-Skolem theorem of first-order logic. (See Bell and Slomson (1971), p. 30.)

Theorem 4.1.1. If all countable models of Th(M) are in $\Gamma^{-1}(M)$, then $\Gamma^{-1}(M)$ has no nontrivially necessary axioms.

Proof. Let ϕ be a nontrivially necessary sentence. Then there is some L-structure $T \in \Gamma^{-1}Eq(M)$ such that $T \models \neg\phi$. By the Löwenheim-Skolem theorem, there is then a countable $T' \in \Gamma^{-1}Eq(M)$ with this property (i.e. $\{\neg\phi\} \cup Th\ \Gamma^{-1}Eq(M)$ is satisfiable, so countably satisfiable). Again by the Löwenheim-Skolem theorem, $T' \in \Gamma^{-1}(M')$ for some countable $M' \in Eq(M)$ (i.e. $D_\Gamma(T') \cup Th(M)$ is satisfiable, so countably satisfiable; $D_\Gamma(T')$ was defined in preceding Theorem 2.3.2). But by hypothesis, $M' \in \Gamma^{-1}(M)$, so $T' \in \Gamma^{-1}(M)$, $T' \models \neg\phi$; this contradicts the assumption that ϕ was nontrivially necessary. So no such ϕ exists.

In some cases, the hypothesis of Theorem 4.1.1 will be true irrespective of the choice of M within the class K = Eq(M). An important example is the case where all countable models of Th(M) are isomorphic; Th(M) is then called an $\aleph_0$-categorical theory. (Note that $\aleph_0$-categoricity does not imply that all models of a given other cardinality are isomorphic, or vice versa.) For example, $Th\langle\mathbb{R},\leq\rangle$ is $\aleph_0$-categorical, by a famous theorem of Cantor (1895). With Theorem 4.1.1, this entails that there are no nontrivially necessary axioms for ordinal measurement (Krantz et al. (1971)), where $M = \langle\mathbb{R},\leq\rangle$, $\Gamma = \{x \leq y,\ x \nleq y\}$.

The following applications of Theorem 4.1.1 are slightly more technical than the rest of this paper.

<u>Theorem 4.1.2</u>. For any Γ and M: There is a countable $M' \in Eq(M)$ such that $\Gamma^{-1}(M')$ satisfies no nontrivially necessary sentences. A fortiori, such M' of any larger cardinality exist.

<u>Proof</u>. Enumerate all L-sentences ϕ_i such that

$$M \models \phi_i \text{ and } \exists T_i,\ T_i \in Eq(M): \Gamma^{-1}(T_i) \not\models \phi_i \ . \qquad (*)$$

Note that there are at most countably many ϕ_i. We will be assuming that there are infinitely many ϕ_i and T_i; otherwise the following argument can be drastically simplified. By the reasoning of the proof of Theorem 4.1.1, we may take all the T_i to be countable; by renaming elements we can get all the T_i to have pairwise disjoint universes.

We will construct a sequence of L-structures S_i, $i \in \omega$, such that

(i) $S_i \in Eq(M)$ and S_i countable for all i

(ii) S_{i+1} is an elementary extension of S_i for all i

(iii) $T_j \in \Gamma^{-1}(S_i)$ for all $j \leq i$

(iv) T_j and S_i have disjoint universes, for all $j > i$

as follows: Let

(v) $S_0 = T_0$

By a compactness argument, given $S_0,\ldots,S_i$ satisfying (i)-(v), there exists S_{i+1} satisfying (i), (ii), (iv), and

$$T_{i+1} \in \Gamma^{-1}(S_{i+1}) \ .$$

But then clearly $S_0,\ldots,S_{i+1}$ satisfies (i)-(v).

Now consider the union of the elementary chain $S_0,\ldots,S_i$, $i \in \omega$:

$$M' = \bigcup_{i\in\omega} S_i \ .$$

It follows that

(i) $M' \in Eq(M)$, M' countable

(ii) For all $i \in \omega$, $T_i \in \Gamma^{-1}(M')$

and by the choice of the T_i, $\Gamma^{-1}(M')$ has no nontrivially necessary sentences.

An M' of any given larger cardinality with this property can be found by a compactness argument. (See Bell and Slomson (1971), p. 82.)

Thus, given the theory of a measure structure, we are still free to choose a model of any infinite cardinality avoiding all nontrivially necessary sentences, if we so wish.

Given Γ, and M or $Th(M)$, we can define the <u>nontrivially necessary sentences in</u> $Th(M)$ as the L-sentences ϕ satisfying

$$\begin{cases} \exists T_1 : T_1 \models Th(M) \ , \ \Gamma^{-1}(T_1) \models \phi \\ \exists T_2 : T_2 \models Th(M) \ , \ \Gamma^{-1}(T_2) \not\models \phi \end{cases}$$

Hence it can be asked whether a countable measure structure $M' \models Th(M)$ exists such that all these sentences are in fact nontrivially necessary axioms for $\Gamma^{-1}(M')$. An answer to this question requires sharper analysis of nontrivially necessary sentences such as given in the next subsection.

We conclude the present subsection with some similar considerations concerning the hypothesis of Theorem 4.1.1.

<u>Proposition 4.1.3</u>. Let $Th(M)$ have only countably many countable models (up to isomorphism). Then

(i) There is a countable $M' \models Th(M)$ such that all countable models of $Th(M)$ are in $\Gamma^{-1}(M')$.

(ii) There is a countable $M' \models Th(M)$ such that for any countable $T \models Th(M)$: $T \in \Gamma^{-1}(M') \Rightarrow$ for any countable $T' \models Th(M)$, $T \in \Gamma^{-1}(T')$.

The hypothesis of Proposition 4.1.3 can be weakened somewhat but not omitted. (i) can be shown by the argument of 4.1.2; M' in (ii) can be taken to be a prime model of $\mathrm{Th}(M)$; see Chang and Keisler (1973). For example, if $M = \langle \mathbb{R},\leq,+\rangle$, then a prime model exists, namely the rationals $\langle \mathbb{Q},\leq,+\rangle$. Thus $\mathrm{Th}\ \Gamma^{-1}(\langle \mathbb{Q},\leq,+\rangle) \supseteq \mathrm{Th}\ \Gamma^{-1}(\langle \mathbb{R},\leq,+\rangle)$. It would be interesting to know whether these two theories differ for any Γ; e.g. does $\Gamma^{-1}(\langle \mathbb{R},\leq,+\rangle)$ miss any of the nontrivially necessary sentences of $\Gamma^{-1}(\langle \mathbb{Q},\leq,+\rangle)$?

4.2. Nontrivially Necessary (∀)∃∀-sentences

We wish to discover the simplest quantificational forms of nontrivially necessary axioms for Simple Measurement models $\langle M,\Gamma\rangle$. As we shall soon see, this will entail looking for a mathematically informative characterization of nontrivially necessary ∃∀-sentences. In a broad sense, the idea of this characterization applies to nontrivially necessary axioms of any quantificational form. This will be exploited in section 4.5, where some of the basic results of the present section will be generalized for arbitrary quantifier prefixes. For the more detailed investigations of sections 4.3, 4.4, and 5.4, however, more careful attention to the details of the description of nontrivial necessity for ∃∀-sentences will be required. In this degree of detail, our description generalizes easily to the case of ∀∃∀-sentences, and in a less useful way, to ∀∃∀∃-sentences; beyond that, significant new aspects arise. Thus the discussion of the present section focusses on ∀∃∀∃-sentences and eventually only on ∀∃∀-sentences.

Our reason for starting with ∃∀-sentences is given by the following basic result:

Proposition 4.2.1. Any $\forall\exists$-sentence necessary for $\langle M,\Gamma\rangle$-measurability is trivially necessary (i.e. in $\mathrm{Th}\,\Gamma^{-1}\mathrm{Eq}(M)$ as well as $\mathrm{Th}\,\Gamma^{-1}(M)$).

Proof. Assume ϕ is an $\forall\exists$-sentence, $\phi \in \mathrm{Th}\,\Gamma^{-1}(M)$. As $\mathbb{F} \subseteq \Gamma^{-1}(M)$ by Lemma 3.2.1, $\phi \in \mathrm{Th}(\mathbb{F})$. As ϕ is $\forall\exists$, $\phi \in \mathrm{Th}\,\Gamma^{-1}\mathrm{Eq}(M)$ by Theorem 3.2.6.

Together with Proposition 2.3.3, this would seem to solve the problem raised by Adams (1974): Characterize the $\forall\exists$-theory of $\Gamma^{-1}(M)$. For clearly this theory coincides with $\mathrm{Th}_{\forall\bar{\Gamma}}(M)$. This result would seem to contradict Adams' motivation for his question. For in his discussion of extensive measurement (i.e. representation of order by order and concatenation by addition, in $M = \langle\mathbb{R},\leq,+\rangle$; see section 2.2 above), Adams gives the "empirically confirmable condition"

$$(a < b)\ \&\ (\forall x)(x < b \rightarrow x{*}x < b)$$

which is consistent with $\mathrm{Th}_{\forall\bar{\Gamma}}(M)$, but clearly contradicts the $\forall\exists$-sentence (which is in fact nontrivially necessary for Γ-morphic representation into $\langle\mathbb{R}^+,\leq,+\rangle$), $\mathbb{R}^+$: positive real numbers.

$$\forall ab\ \exists x\ \neg[a < b\ \&\ (x < b \rightarrow x+x < b)]$$

i.e. $$\forall b[(\exists a\ a < b) \rightarrow (\exists x)(x < b\ \&\ x+x \nless b)] \qquad (*)$$

We resolve this conflict by noting that Adams' formalization of extensive measurement includes treatment of '+' as an operation symbol. Such symbols are excluded throughout the present paper, and reformulation of the example with a ternary relation symbol for '+' would raise the quantificational complexity of the sentence (*) above $\forall\exists$. (See Example 4.3.0.)

Examination of the proof of Theorem 4.2.6 shows that it is in general incorrect for languages with operation symbols. The example (*) shows further that in fact Theorem 4.2.6 and Proposition 4.2.1 are in general false for such languages. Thus the restriction to languages without such symbols is essential at this point.

We now proceed to the description of trivial necessity of $\exists\forall$-sentences. The first step is a description of necessity of $\exists\forall$-sentences: An $\exists\forall$-sentence ψ is necessary for $\langle M,\Gamma\rangle$-measurement if and only if no model of the $\forall\exists$-sentence $\neg\psi$ can be embedded in a Γ-morphic pre-image of M (i.e. a structure $T \in \Gamma^{-1}(M)$). The embeddability in a structure T of a model of $\neg\psi$ can be described more carefully using an idea due to Löwenheim (1915) and Skolem (1922). We illustrate this by an example before proceeding to a general formal description (which will apply to arbitrary $\neg\forall\exists\forall\exists$-sentences). Let $\neg\psi = \forall x\ \exists y\ \phi$, and let a structure T be given. Consider the construction (possibly noneffective):

S_0: pick $x_0 \in |T|$

S_1: unless $T \models \forall y\ \neg\phi(x_0,y)$, pick $y = x_1 \in |T|$: $T \models \phi(x_0,x_1)$

$\vdots$

S_i: unless $T \models \forall y\ \neg\phi(x_{i-1},y)$, pick $y = x_i \in |T|$: $T \models \phi(x_{i-1},x_i)$

If, for any $i \geq 0$, $T \models \forall y\ \neg\phi(x_i,y)$, we say the <u>process terminates</u> (<u>at stage</u> i). It is easy to see that if the process never terminates, the substructure of T:

$$T' = T\upharpoonright\{x_0,x_1,\ldots\}$$

will satisfy $T' \models \forall x\ \exists y\ \phi(x,y)$. Conversely, if T' is a substructure of T satisfying $\forall x\ \exists y\ \phi(x,y)$, then the construction can be executed on T in such a way as not to terminate at any finite stage. This is Löwenheim's idea, which we proceed to formalize.

<u>Definition 4.2.2(a)</u>. Let ℓ, m, n $\in \omega$, and $\phi(x_1 \cdots x_\ell, y_1 \cdots y_m, z_1 \cdots z_n)$ be a universal L-formula. Let T be an L structure, $\vec{t} \in |T|^\ell$, $X \subseteq |T|$.

(i) A <u>$\phi(\vec{t},m,n)$-induction from X on T</u> is a countable (finite or infinite) sequence

$$<\vec{t},X,X_0,X_1,\ldots>$$

corresponding to a process of the following type or an initial segment of such a sequence, containing at least $<\vec{t},X,X_0>$. (Note that such processes need <u>not</u> be effective in any sense.)

"Given X and $\vec{t}$, set $X_0 = X \cup \{t_1,\ldots,t_\ell\}$, where $\vec{t} = \{t_1,\ldots,t_\ell\}$. If $X_0 \neq \emptyset$, <u>go to stage 0</u>.

<u>Stage k</u> ($k \in \omega$): Given X_k.

If $\forall \vec{y} \in (X_k)\ \exists \vec{z}(\vec{y}) \in |T|^n$: $T \models \phi(\vec{t},\vec{y},\vec{z}(\vec{y}))$,

let $X_{k+1} = X_k \cup \left(\bigcup_{\vec{y} \in (X_k)^m} \{z_1(\vec{y}),\ldots,z_n(\vec{y})\}\right)$

for some such $\vec{z}(\vec{y}) = <z_1(\vec{y}),\ldots,z_n(\vec{y})> \in |T|^n$,

and <u>go to stage k+1</u>

Otherwise, <u>terminate at stage k</u>."

The $\phi(\vec{t},m,n)$-induction from X on T corresponding to such a process is the sequence

$$<\vec{t},X,X_0,\ldots,X_k>$$

if the process terminated at stage k; otherwise the sequence

$$<\vec{t},X,X_0,X_1,\ldots,X_i,\ i \in \omega> .$$

In the first case we say the induction has <u>length k</u>, in the second case the induction has <u>infinite length</u>.

(ii) $\vec{y} \in |T|^m$ is a <u>termination vector</u> if and only if $T \models \forall \vec{z} \neg\phi(\vec{t},\vec{y},\vec{z})$.

It should be noted that each $\phi(\vec{t},m,n)$-induction is a realization of a formal process which can be described formally without reference to any particular objects or any particular structure. In this description, variables are used to denote the objects which play a role in actual inductions, and the sets of objects $X, X_0, \ldots$ are replaced by sets of variables denoting the objects. Thus the fixed vector $\vec{t} = \langle z_1,\ldots,z_n\rangle$ (where before $t_1,\ldots,t_n$ were particular objects) is now a sequence of variables $t_1,\ldots,t_n$; each time a new object is called for, a new variable is introduced (even though in some actual inductions, an old object will sometimes be able to fulfill the new role). Each stage, k, of an induction process is described by a formula ψ_k, which depends only on the original ϕ, ℓ, m, n, and the size of initial set X, which we will denote by c. In defining ψ_k it will be helpful to simultaneously define a function $f\colon \omega \to \omega$ such that for each k, $\{t_1\cdots t_\ell, x_1\cdots x_{f(k)}\}$ is the complete list of variables used in describing the process up to and including stage k. Thus ψ_k is generated by the process:

<u>Definition 4.2.2(b)</u>.

"ψ_0 is the empty formula. $f(0) = c$. <u>Go to stage 0</u>.

<u>Stage k</u>, $k \geq 1$. Given ψ_{k-1}, $f(k-1)$.

Let g be a 1-1 enumeration of the m-tuples to be considered at stage k:

$$g\colon \{t_1\cdots t_\ell, x_1\cdots x_{f(k-1)}\}^m \to \{1,2,\ldots,(\ell+f(k-1))^m-1\}$$

and $\vec{z}(\cdot)$ be the assignment of an n-tuple of new variables to each n-tuple from $\{t_1\cdots t_\ell, x_1\cdots x_{f(k-1)}\}^m$:

$$\vec{z}(\vec{y}) = \langle x_{f(k-1)+ng(\vec{y})+1},\ldots,x_{f(k-1)+ng(\vec{y})+n}\rangle$$

$$\psi_k = \bigwedge\!\!\bigwedge_{\vec{y} \in \{t_1 \cdots t_\ell, x_1 \cdots x_{f(k-1)}\}^m} \phi(\vec{t},\vec{y},\vec{z}(\vec{y}))$$

$$f(k) = f(k-1) + n(\ell + f(k-1))^m \qquad ''$$

The correspondence between the formal description of a $\phi(\vec{t},n,m)$-induction and its realizations can be stated explicitly as:

Lemma 4.2.3. Let T be an L-structure, $\vec{t} \in |T|^\ell$, $X \subseteq |T|$, $\bar{\bar{X}} = c \geq 1$. Let $<\vec{t},X,X_0,X_1,\ldots,X_k>$ be given, with $X_0 = \vec{t} \cup X$.

(i) A sequence $<\vec{t},X,X_0,\ldots,X_k>$, $X_i \subseteq |T|$, $i = 0,\ldots,k$ is (an initial segment of) a $\phi(\vec{t},m,n)$-induction of length $\geq k$ on T from X if and only if there is some mapping of variables to elements of X_k, $\mu\colon \{t_1 \cdots t_\ell, x_1 \cdots x_{f(k)}\} \to X_k$ such that

$$\mu(<t_1 \cdots t_\ell>) = \vec{t}$$

$$\mu(\{t_1 \cdots t_\ell, x_1 \cdots x_{f(i)}\}) = X_i\ , \quad i \leq k$$

$$T \models \bigwedge\!\!\bigwedge_{i \leq k} \psi_i(\vec{t},\mu<x_1 \cdots x_{f(i)}>)$$

(ii) Similarly $<\vec{t},X,X_i,\ i \in \omega>$ is an infinite $\phi(\vec{t},m,n)$-induction if and only if the above conditions hold with $i \leq k$ replaced by '$i \in \omega$' and with

$$\mu\colon \{t_1 \cdots t_\ell, x_i\colon i \in \omega,\ i > 1\} \to \bigcup_{i \in \omega} X_i\ .$$

Lemma 4.2.3 follows directly from Definitions 4.2.2(a) and (b). For later reference we give here a property of $\phi(\vec{t},m,n)$-inductions:

Lemma 4.2.4 (Antitonicity of $\phi(\vec{t},m,n)$-inductions). Let $T_1 = \langle\vec{t},X,X_0,\ldots,X_{k_1}\rangle$ and $T_2 = \langle\vec{t},X',X_0',\ldots,X_{k_2}'\rangle$ be two $\phi(\vec{t},m,n)$-inductions on a structure T, and $X_{k_1} \supseteq X_{k_2}'$. Then T_2 can be extended beyond stage k_2 for at least as many steps as T_1 can be extended beyond stage k_1.

Proof. If T_1 has an extension $\langle\vec{t},X,X_0,\ldots,X_{k_1},\ldots,X_{k_1+\ell}\rangle$, then in fact T_2 has an extension $\langle\vec{t},X',X_0',\ldots,X_{k_2}',\ldots,X_{k_2+\ell}'\rangle$, where

$$X_{k_2+i}' \subseteq X_{k_1+i}\ , \qquad i = 0,1,\ldots,\ell$$

as is obvious from Definition 4.2.2(a); and clearly X_{k_2+i}' cannot contain a terminal vector if X_{k_1+i} does not. Q.E.D.

We note at this point that the theory of inductive definition on L-structures as presented in Moschovakis (1974) does not apply to the inductive processes defined above. Our processes are not monotone. Given sets $X \subseteq X'$, $\phi(\vec{t},m,n)$-inductions from X' will terminate at least as soon as those from X; hence the points of the structure accumulable by $\phi(\vec{t},m,n)$-inductions from X may not form a subset of the set of points so accumulable from X'. This is especially clear when C' is obtained from X by addition of just enough points so that $(X')^m$ contains a terminal vector.

This difference is due to the difference between our notion of termination and Moschovakis' notion of closure. A further difference is that inductive definitions adjoin all solutions $\vec{z}$ in the structure of $\phi(\vec{t},\vec{y},\vec{z})$, and we adjoin only one.

We now proceed to a characterization of nontrivially necessary axioms, which is essentially accomplished by Theorems 4.2.5 and 4.2.6 in conjunction. The theorems have been formulated to deal with $\forall\exists\forall\exists$-sentences which are necessary except for possibly in small finite structures of size less than c for some $c \in \omega$. However, the added generality occasionally requires a slight reformulation; see Remark 4.2.9.

Theorem 4.2.5. Let c be an L-structure, $c \geq 1$; $\phi(\vec{x},\vec{y},\vec{z})$ an $(\ell+m+n)$-ary existential formula; $\psi = \forall\vec{x}\ \exists\vec{y}\ \forall\vec{z}\ \phi(\vec{x},\vec{y},\vec{z})$, a $\forall\exists\forall\exists$-sentence. (i)-(iii) below are equivalent:

(i) $(\forall T \in \Gamma^{-1}(M))[|\bar{\bar{T}}| \geq c \Rightarrow T \models \psi]$

(ii) $(\forall T \in \Gamma^{-1}(M))(\forall\vec{t} \in |T|^{\ell})(\forall X \subseteq |T|)$

$\overline{\overline{\vec{t} \cup X}} = c \Rightarrow$ All $\neg\phi(\vec{t},\vec{m},\vec{n})$-inductions from X on T terminate in a finite number of steps

(iii) $(\forall T \in \Gamma^{-1}(M))$: $c = 1 \Rightarrow T \models \forall\vec{t}\ \forall\vec{x}_w[\bigvee\!\!\bigvee_{1\leq k\leq w} \psi_k(\vec{t},\vec{x}_w)]$

$c > 1 \Rightarrow T \models \forall\vec{t}\ \forall\vec{x}_w[[\bigvee\!\!\bigvee_{1\leq i<j\leq c} x_i = x_j] \vee [\bigvee\!\!\bigvee_{1\leq k\leq w} \psi_k(\vec{t},\vec{x}_w)]]$

where $\vec{x}_w = \langle x_1x_2\cdots\rangle$, and '$\psi_k$' is obtained from '$\neg\phi$' as in Definition 4.2.2(b).

Proof. (ii) $\Leftrightarrow$ (iii) follows from Lemma 4.2.3.

$\neg$(ii) $\Rightarrow \neg$(i): Assume there exist $T \in \Gamma^{-1}(M)$, $\vec{t}$, X, $\bar{\bar{X}} = c$, such that $\langle\vec{t},X,X_0,\ldots\rangle$ is an infinite $\neg\phi(\vec{t},m,n)$-induction from X on T. Set $A = \bigcup_{i\in\omega} X_i$. Then $\vec{t} \in A^{\ell}$, $\bar{\bar{A}} \geq c$, and setting $A = T{\restriction}A$, $A \models \forall\vec{y}\ \exists\vec{z}\neg\phi(\vec{t},\vec{y},\vec{z})$ by construction and the fact that '$\neg\phi$' is a universal formula. So

$$A \in \Gamma^{-1}(M)\ ,\quad |\bar{\bar{A}}| \geq c\ ,\quad A \models \neg\psi\ .$$

(ii) $\Rightarrow$ (i): Let $T \in \Gamma^{-1}(M)$, $|\bar{\bar{T}}| \geq c$. Then

$$(\exists X \subseteq |T|)[\bar{\bar{X}} \geq c\ \&\ (\forall\vec{t} \in |T|^{\ell})\text{: All } \neg\phi(\vec{t},m,n)\text{-inductions from X on T terminate}]\ .$$

Hence there must be a terminal vector for each $\vec{t} \in |T|^{\ell}$, i.e.

$$T \models \forall\vec{t}[\exists\vec{y}\ \forall\vec{z}\neg(\neg\phi(\vec{t},\vec{y},\vec{z})]] \ ,$$

i.e. $$T \models \psi \ .$$

Theorem 4.2.6. Let c, ϕ, and ψ be as in the preceding theorem; let the conditions (i)-(iii) of that theorem hold.

Then (i)-(iv) below are equivalent:

(i) $(\forall T \in \Gamma^{-1}\mathrm{Eq}(M))[|\bar{\bar{T}}| \geq c \Rightarrow T \models \psi]$

(ii) $(\exists N \in \omega)(\forall T \in \Gamma^{-1}(M))(\forall\vec{t} \in |T|^{\ell})(\forall X \subseteq |T|)$:

$\overline{\overline{\vec{t} \cup X}} = c \Rightarrow$ All $\neg\phi(\vec{t},m,n)$-inductions from X on T terminate within N steps.

(iii) $(\exists N \in \omega)(\forall T \in \Gamma^{-1}(M))$:

$$c = 1 \Rightarrow T \models \forall\vec{t}\ \forall\vec{x}[\bigvee\!\!\bigvee_{1 \leq k \leq N} \psi_k(\vec{t},\vec{x})]$$

$$c > 1 \Rightarrow T \models \forall\vec{t}\ \forall\vec{x}[[\bigvee\!\!\bigvee_{1 \leq i < j \leq c} x_i = x_j] \vee [\bigvee\!\!\bigvee_{1 \leq k \leq N} \psi_k(\vec{t},\vec{x})]]$$

(iv) $(\exists N \in \omega)(\exists\theta \in \mathrm{Th}_{\forall\bar{\Gamma}}(M))$: $\models \theta \rightarrow \forall\vec{t}\ \forall\vec{x}[\text{——}]$.

Proof. (ii) $\Leftrightarrow$ (iii): By Lemma 4.2.3.

(iii) $\Rightarrow$ (iv): Note that the sentences in (iii) are $\forall\exists$ first-order L-sentences. Hence by our result $\mathrm{Th}_{\forall\exists}(\Gamma^{-1}(M)) = \mathrm{Th}_{\forall\exists}(\Gamma^{-1}\mathrm{Eq}(M))$, and the axiomatization of $\mathrm{Th}(\Gamma^{-1}\mathrm{Eq}(M))$ by $\mathrm{Th}_{\forall\bar{\Gamma}}(M)$, (iv) follows.

(iv) $\Rightarrow$ (i): From (iv), $\Gamma^{-1}\mathrm{Eq}(M) \models \forall\vec{t}\ \forall\vec{x}[\text{——}]$. Now by the same argument (terminal vectors exist) as used for (iii) $\Rightarrow$ (i) in the previous theorem, (i) follows.

(i) $\Rightarrow$ (ii): Assume (ii) is false; i.e. somewhere in $\Gamma^{-1}(M)$ there are arbitrarily long $\neg\phi(\vec{t},m,n)$-inductions. Noting that $\Gamma^{-1}(M) \subseteq \Gamma^{-1}\mathrm{Eq}(M)$ which is an elementary class, we see such arbitrarily long finite inductions can be found within this elementary class. By a compactness argument, the class contains an infinite such induction (involving $\geq c$ points). Taking the domain $\bigcup_{i\in\omega} X_i$ of this induction as a substructure, still in $\Gamma^{-1}\mathrm{Eq}(M)$, we have a counterexample to (i).

Regrettably, Theorems 4.2.5 and 4.2.6 do not give a strictly intrinsic characterization of nontrivially necessary axioms for $\Gamma^{-1}(M)$; that is, the discussion remains in terms of $\Gamma^{-1}(M)$ rather than in terms of internal properties of M itself. In an important special case, this can be improved as follows.

<u>Theorem 4.2.7</u>. Let c, Γ be as before; let $\phi(\vec{x},\vec{y},\vec{z})$ be an $(\ell+m+n)$-ary quantifierfree $\bar{\Gamma}$-formula; $\psi = \forall\vec{x}\ \exists\vec{y}\ \forall\vec{z}\ \phi(\vec{x},\vec{y},\vec{z})$.

1) (i) and (ii) below are equivalent:

(i) $\forall T \in \Gamma^{-1}(M)\colon\ |\bar{\bar{T}}| \geq c \ \Rightarrow\ T \models \psi$

(ii) $\forall\vec{t} \in |M|^{\ell}\ \forall X \subseteq |M|$

$\overline{\overline{\vec{t}\cup X}} = c \ \Rightarrow$ All $\neg\phi(\vec{t},m,n)$-inductions from X on M terminate in a finite number of steps

2) If $T \in \Gamma^{-1}(M)$, $\vec{t} \in |T|^{\ell}$, $X \subseteq |T|$, and $\mathcal{T} = \langle\vec{t},X,X_0,\ldots\rangle$ is a $\neg\phi(\vec{t},m,n)$-induction on T from X, and $\alpha\colon T \to M$ is a Γ-morphism, then

$$\alpha(\mathcal{T}) = \langle\alpha(\vec{t}),\alpha(X),\alpha(X_0),\ldots\rangle$$

is a $\neg\phi(\alpha(\vec{t}),m,n)$-induction on M from $\alpha(X)$, and can be extended on M at least as fas as $\mathcal{T}$ can be extended on T.

<u>Proof</u>. Note that $\neg\phi(\vec{t},\vec{y},\vec{z})$ is a Γ-formula, i.e. a conjunction of disjunctions of variable substitution instances of members of Γ. Then (2) is direct from Definition 4.2.2(a); (1) follows directly from (2).

We remark that (1) could be extended by a clause (iii) as in Theorem 4.2.5. Theorem 4.2.6 can be modified similarly. So we have

Proposition 4.2.8. If $\phi(\vec{x},\vec{y},\vec{z})$ is an ℓ+m+n-ary quantifierfree $\bar{\Gamma}$-formula, then $\psi = \forall\vec{x}\ \exists\vec{y}\ \forall\vec{z}\ \phi(\vec{x},\vec{y},\vec{z})$ is a nontrivially necessary axiom for $\Gamma^{-1}(M)$ if and only if

(i) All $\neg\phi(\vec{m},m,n)$ - inductions on M terminate in a finite number of steps, for all $\vec{m} \in |M|^{\ell}$, and

(ii) There are arbitrarily long $\neg\phi(\vec{m},m,n)$-inductions on M, not necessarily with the same $\vec{m} \in |M|^{\ell}$.

Remark 4.2.9. If Γ-morphisms are not necessarily 1-1, the additional generality of the case $c > 1$ is illusory, for we have

$$[(\forall T \in \Gamma^{-1}(M))[|\bar{\bar{T}}| \geq c \Rightarrow T \models \psi]] \Rightarrow \Gamma^{-1}(M) \models \psi \ .$$

For assume $c > 1$, there exists $T \in \Gamma^{-1}(M)$, $1 \leq |\bar{\bar{T}}| < c$, $T \models \neg\psi$. Let T' be as follows: fix $y_0 \in |T|$.

$$\begin{cases} |T'| = |T| \cup \{x_0,\dots,x_c\}, \text{ where } \{x_0,\dots,x_c\} \cap |T| = \phi \\ \alpha\colon |T'| \to |T| \\ \alpha(t) = \begin{cases} t & \text{if } t \in |T| \\ y_0 & \text{if } t \in \{x_0,\dots,x_c\} \end{cases} \\ \text{for each relation symbol } R\vec{a} \text{ of } L: \\ \quad T' \models R\vec{t} \Leftrightarrow T \models R\alpha(\vec{t}) \quad [\text{so } T' \in \Gamma^{-1}(M)] \end{cases}$$

It is easy to verify that $|\bar{\bar{T}}'| > c$ and $T' \models \neg\psi$, contradicting $|\bar{\bar{T}}| \geq c \Rightarrow T \models \psi$.

If the language L contains a binary relation parameter for indifference, as is often the case in applications, some simple modifications can be made to obtain genuine theorems for the case $c > 1$: Introduce the notation, for any set X with indifference relation

$$\overline{\langle X,\sim\rangle}\text{: the number of indifference classes of } \langle X,\sim\rangle \ .$$

The clauses of Theorem 4.2.5 become:

(i) $(\forall T \in \Gamma^{-1}(M))[\overline{\langle |T|,\sim\rangle} \geq c \Rightarrow T \models \psi]$

(ii) $(\forall T \in \Gamma^{-1}(M))(\forall \vec{t} \in |T|^{\ell})(\forall X \subseteq |T|)$

$\overline{\langle X,\sim\rangle} = c \Rightarrow$ All $\neg\phi(\vec{t},m,n)$-inductions from X on T terminals are in a finite number of steps.

(iii) $(c > 1)$: $(\forall T \in \Gamma^{-1}(M))$: $T \models \forall \vec{t}\ \forall \vec{x}_w [[\bigvee\!\!\bigvee_{1 \leq i < j \leq c} x_i \sim x_j] \vee [\bigvee\!\!\bigvee_{1 \leq k \leq w} \psi_k(\vec{t},\vec{x}_w)]]$

Thus modified, the theorem is again true for pairs $\langle M,\Gamma\rangle$ for which

$$\begin{cases} M \models x \sim y \leftrightarrow x = y \\ \Gamma \supseteq \forall\!\!\exists[x \sim y]\} \end{cases}$$

This is usually the case in applications; in cases where the language does not contain an indifference relation parameter, it is reasonable to expect that addition of such a parameter, treated as above, should not greatly affect the axiomatization of measurement.

Theorems 4.2.6 and 4.2.7 can be modified similarly.

Remark 4.2.10. As a $\forall\exists$-sentence is a special case of a $\forall\exists\forall\exists$-sentence, i.e. for which the second universal quantifier block is void, it may be asked how the arguments for $\forall\exists$-sentences in Theorem 3.2.6 and Proposition 4.2.1 can be obtained as special cases of the analysis of Theorems 4.2.5 and 4.2.6. This is not only of interest for better understanding of the results; the ideas will find application in the more subtle two-sorted case in the section 5.4 on conjoint measurement. For simplification we restrict the discussion to the setting of Proposition 4.2.8.

So assume $\psi = \forall\vec{x}\ \exists\vec{y}\ \phi(\vec{x},\vec{y})$, ϕ: $(\ell+m)$-ary $\bar{\Gamma}$-sentence, $\vec{s} \in |M|^{\ell}$, and consider

$$\neg\phi(\vec{s},m,0)\text{-inductions on } M .$$

It is clear that such an induction either terminates at stage 0, i.e.

$$\text{for some } \vec{y} \in (X_0)^m\text{: } M \models \neg[\neg\phi(\vec{s},y)]$$

or,

$$\text{for all } \vec{y} \in (X_0)^m\text{: } M \models \neg\phi(\vec{s},y) \tag{*}$$

in which case the induction can be extended infinitely to

$$\langle\vec{s},X,X_0,X_1,X_2,\ldots\rangle$$
$$X_i = X_0 \text{ for all } i \in \omega ,$$

i.e. $M\restriction X_0 \models \neg\psi$; the reasoning is as employed in Theorem 3.2.6(1).

An alternative viewpoint which will be useful in section 5.4 is that one can convert any $\neg\phi(\vec{s},m,0)$-induction to a $\neg\phi'(\vec{s}',0,0)$-induction; this corresponds to the universal sentence X of Theorem 3.2.6(2).

For given $\vec{s} \in |M|^{\ell}$, and a finite $X \subseteq |M|$, say $X = \{x_1\cdots x_k\}$, consider the vector $\vec{s}' = \langle s_1\cdots s_\ell,x_1\cdots x_k\rangle$, and

$$\phi'(\vec{s}') = \bigvee\!\!\bigvee_{\vec{y}\in\{s_1\cdots s_\ell,x_1\cdots x_k\}^m} \phi(\vec{s},\vec{y}) . \tag{**}$$

Then the condition for some $\neg\phi(\vec{s},m,0)$-induction on M from X not to terminate at stage 0 can be written as

$$M \models \neg\phi'(\vec{s}')$$

as can be seen by comparing lines (*) and (**) above; thus the necessity of $\psi = \exists\vec{x}\ \forall\vec{y}\ \phi(\vec{x},\vec{y})$ can be expressed as

$$M \models \forall\vec{s}'\ \phi'(\vec{s}')$$

which for $X = \emptyset$ is exactly the result of Theorem 3.2.6(2) for the case of $\bar{\Gamma}$-sentences; note that '$\forall\vec{s}'\ \phi'(\vec{s}')$' is exactly the sentence 'χ' in Theorem 3.2.6(2). This fits together if we realize that Theorems 4.2.5 and 4.2.6 allow $X = \emptyset$, as long as $\vec{s}$ is nontrivial, i.e. $\ell > 0$; so we could have made this restriction in the above.

4.3. Examples of Nontrivially Necessary Sentences

We are now in a position to discuss examples of nontrivially necessary sentences. Examples will be given for variants of extensive measurement and for several kinds of difference measurement by an interval scale. (References to the literature on these measurement models will be given with the examples.) The examples we give are all of the simplest possible quantificational forms for the measurement models involved. Proof that this is actually the case requires techniques to be developed in subsection 4.4; thus such proof is deferred until there.

Example 4.3.0. The first example to be given has already occurred in the literature: semi-explicitly in Adams (1974) and explicitly in Adams (1975). We have already referred to a related form in section 4.2. It is not known whether the sentence has a minimal quantificational form: The sentence is a disjunction of a $\forall\exists$-sentence and an $\exists\forall$-sentence. This is quantificationally one of the two simplest kinds of sentences which is both $\exists\forall\exists$ and $\forall\exists\forall$. What is not known is whether there could be less quantifiers in a nontrivially necessary axioms for extensive measurement.

$\langle M,\Gamma\rangle$: extensive measurement (Krantz et. al. (1971))

$M = \langle \mathbf{R}^+, \leq, + \rangle$

$\Gamma = \{\leq, \nleq, +\}$

$\psi = (\exists x)(\forall y)[\neg x+x = y] \vee \exists x\ \forall y[y \nleq x] \vee (\forall x)(\exists y_1 y_2)[y_1 < x\ \&\ y_1+y_1 = y_2\ \&\ y_2 \nleq x]$

That the sentence in this example is nontrivially necessary is intuitively obvious once one considers nonstandard models of $\langle \mathbf{R}^+, \leq, + \rangle$ with infinitesimals; we omit a rigorous proof. (Adams' (1975) example was slightly different from ours.)

The construction of nontrivially necessary axioms seems to follow a pattern: Let us call a monotonic sequence

1) $y^{(0)} < y^{(1)} < y^{(2)} < \cdots$

2) $\lim\limits_{i\to\infty} y^{(i)} = \infty$

a <u>basic sequence</u>. (Sometimes it is convenient to let the sequence descend monotonically to $-\infty$.) In constructing an axiom, one tries to find a system of basic sequences in the structure with the following properties:

3) The possibilities for $y^{(i+1)}$ are determined by a quantifierfree L-formula containing $y^{(i)}$ and possibly their parameters.

4) The formula of (3) can be strengthened to include checking $y^{(i+1)}$ for some termination condition, i.e. by comparison with a fixed parameter so as to force finite termination of basic sequences.

5) The number of steps basic sequences can be followed depends on the position of $y^{(0)}$ (possibly with respect to certain parameters) in $\mathbf{R}$, and can be arbitrarily large.

The simplest case obtains when the relations on the structure are already sufficiently strong to determine the steps $y^{(i)} \to y^{(i+1)}$ and the termination conditions without further parameters. An example of this is

<u>Example 4.3.1.</u>

1) $M = \langle \mathbb{R}, Nx, Sxy \rangle$ where

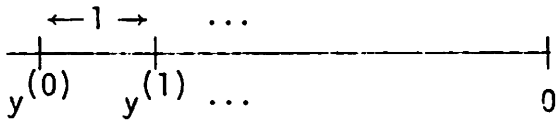
$$\forall x \in \mathbb{R} \qquad M \models Nx \quad \Leftrightarrow \quad \mathbb{R} \models x \leq 0$$

$$\forall x,y \in \mathbb{R} \qquad M \models Sxy \quad \Leftrightarrow \quad \mathbb{R} \models x+1 = y$$

2) $\Gamma = \{Nx_1, Sx_1x_2\}$

3) $\psi = \exists y \; \forall z \; \neg\phi(y,z)$

$\phi(y,z) = Syz \;\&\; Nz$

4) 

basic sequences: $y^{(i+1)} = y^{(i)} + 1$

5) Analysis

(a) $y^{(0)} \in (-1,\infty)$: All $\phi(1,1)$-inductions from $\{y^{(0)}\}$ terminate at stage 0 in 0 steps

(b) $y^{(0)} \in (-(n+1),-n]$: All $\phi(1,1)$-inductions from $\{y^{(0)}\}$ terminate at stage n-1, $n \geq 1$.

Hence we have:

$$\begin{cases} \text{All } \phi(1,1)\text{-inductions on finite in } \langle \mathbb{R},N,S \rangle \\ \text{There are arbitrarily long finite } \phi(1,1)\text{-inductions in } \langle \mathbb{R},N,S \rangle \end{cases}$$

By Proposition 4.2.8, ψ is a nontrivially necessary axioms for $\Gamma^{-1}(M)$.

It is of interest to note that the above example still works if we replace Sxy by Rxy: $M = \langle \mathbb{R}, Rxy, Nx \rangle$ where:

$$\forall xy \in \mathbb{R}: \; M \models Rxy \Leftrightarrow \mathbb{R} \models x+1 \leq y$$

More realistic examples are obtained if we consider the structure for classical extensive measurement: $M = \langle \mathbb{R}, +, \leq \rangle$, '+' ternary relation. In the next example, both the step $y^{(i)} \to y^{(i+1)}$ of basic sequences and the termination condition are obtained from one fixed parameter $x \in \mathbb{R}$; that is, we will consider $\phi(x,1,1)$-inductions and get a nontrivially necessary axioms $\forall x\ \exists y\ \forall z\ \neg\phi(x,y,z)$. The example is essentially the same as the previous one; however, two new phenomena occur in $\phi(x,1,1)$-inductions which cannot occur in $\phi(1,1)$-inductions: (1) The parameter x also occurs in X_0, and so for some $x \in \mathbb{R}$ we must find a z satisfying $\phi(x,x,z)$. (Clearly, x may not be a terminal point (vector), as then all $\phi(x,1,1)$-inductions would terminate at stage 0, and $\forall x\ \exists y\ \forall z$ would be trivially necessary. (2) We may have $X = \{x\} = X_0$, so we must in fact find $z \neq x$ satisfying $\phi(x,x,z)$, and z must be such that $\phi(x,1,1)$-inductions from z run for arbitrarily many stages; otherwise we get a trivially necessary axiom.

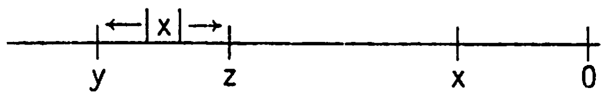
Example 4.3.2 (Extensive measurement, or weaker models).

1. $M = \langle\mathbb{R},+,\leq\rangle$

   $\Gamma \supseteq \{+,<\}$

   $\psi_2 = \forall x\ \exists y\ \forall z\ \neg\phi(x,y,z)$

   $\phi(x,y,z) = [[y = x\ \&\ z < x] \vee [y = z+x\ \&\ y < z < x]]$

2. 

   $y^{(i+1)} = y^{(i)} - x$ basic sequences

3. Analysis

   (a) $x \geq 0 \Rightarrow$ all $\phi(x,1,1)$-inductions on $\langle\mathbb{R},+,\leq\rangle$ terminate at stage 0 or stage 1, namely inductions from $X = \{x\}$, $x = y^{(0)}$ we get $y^{(1)} < x$, and are at stage 1. And now the induction must terminate.

   (b) $x < 0\ \&\ y^{(0)} < x \Rightarrow$ all $\phi(x,1,1)$-inductions on M from $\{y^{(0)}\}$ terminate within $[(x-y^{(0)})/|x|]$ steps, and some such inductions take at least $(x - y^{(0)})/x$ steps.

(c) $x < 0$ & $y^{(0)} > x \Rightarrow$ all $\phi(x,1,1)$-inductions from $y^{(0)}$ terminate at stage 0.

(d) $x < 0$ & $y^{(0)} = x \Rightarrow$ choice of $y^{(1)}$ is free, but smaller than x. Hence these inductions take up to $(x - y^{(1)})/|x|$ steps, which is arbitrarily large.

Theorem 4.3.3. For any L-structure $T \in Eq(<\mathbb{R},+,\leq>)$, for any $\Gamma \supseteq \{+,<\}$, i.e. $\Gamma = \{+,\leq,\neq\}$:

either (i) $T \in \Gamma^{-1}<\mathbb{R},+,\leq>$

or (ii) for some $E \in \Gamma^{-1}(T)$: $E \models \neg\psi_2$.

Proof. By the analysis of Example 4.2.2, ψ_2 is a $\bar{\Gamma}$-sentence for the Γ's considered, and a (nontrivially) necessary axiom in Th $\Gamma^{-1}(<\mathbb{R},+,\leq>)$. Assume (i) is false. Then T is a nonarchimedian ordered group; for T is always an ordered group and isomorphically embeddable into $<\mathbb{R},+,\leq>$ if and only if T is archimedian. (Hölder's (1901) theorem; see Krantz et alii (1971), section 2.2.6.) Let $i > 0$ be infinitesimal in $|T|$. The following is an infinite $\phi(-i,1,1)$-induction from $\{-1\}$ on T:

$$X_0 = \{-1,-i\}$$
$$X_k = \{-1+ni:\ n \leq k\} \cup \{-i\}\ ,\quad k \in \omega$$

and so if we take

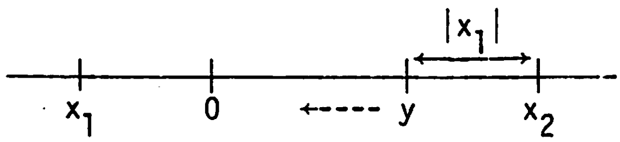
$$E = T\upharpoonright\{-1+ni:\ n \in \omega\} \cup \{-i\}\ ,$$

then

$$E \models \neg\psi_2\ .$$

Thus $\Gamma^{-1}(T) \models \psi_2 \Leftrightarrow T$ is archimedian (for $T \in Eq(<\mathbb{R},+,\leq>)$). So in this (weak) sense, first-order observation could in principle show whether there are infinitesimals in "the real world."

Example 4.3.4 (Weak variants of extensive measurement). We now proceed to an example of the quantificational form $\exists x_1 x_2 \forall y \neg\phi$, again for $\langle \mathbb{R},+,\leq \rangle$. This example is considerably more complicated because no parameters are available to regulate the basic process, which is as follows:



with termination at stage k as soon as X_k contains two distinct positive numbers. The main difficulties are

(1) to make sure that we always get some x_1, x_2 with $x_1 < 0 < x_2$;

(2) not to terminate prematurely, i.e. on pairs $x_1 = x_2 < 0$;

(3) to assure that not all $x > 0$ in X_k are multiples of $-x_1$ for all $k \in \omega$, as this must prevent termination, given (2).

$M = \langle \mathbb{R},+,\leq \rangle$

$\Gamma \supseteq \{+,\leq,\neq\}$, or $\{+,\leq,\nleq\}$, or $\{+,<\}$

$\psi_3 = \exists x_1 x_2 \forall y \neg\phi(x_1,x_2,y)$

$\phi(x_1,x_2,y) =$

(1.1)	$x_2 < x_1 \rightarrow [[y+y=y \;\&\; y \leq x_1 \;\&\; y \neq x_2]$	0 ∈ X_2 (at the latest) no two distinct negative numbers in X_k, $\forall k \in \omega$
	$\vee\; [x_2+x_2=x_2 \;\&\; x_2 < y < x_1]]$	no number is ultimately the minimal number > 0 in X_k, $k \to \infty$
	&	
(2)	$x_1 = x_2 \rightarrow y \neq x_1$	at least two distinct numbers (i.e. in X_1)
	&	
(3.1)	$x_1 < x_2 \rightarrow [[x_1+x_1=x_1 \;\&\; y < x_1]$	X_3 (at latest) contains $y < 0$
(3.2)	$\vee\; [x_2+x_2=x_2 \;\&\; x_2 < y]$	X_3 (at latest) contains $y > 0$
(3.3)	$\vee\; [x_1+x_2=y \;\&\; y \neq x_1 \;\&\; y \neq x_2]]$	If x_1, $x_2 \neq 0$, then "shift x_2 left by $-x_1$"

Lemma 4.3.5. Let ϕ be as in ψ_3, $M = \langle \mathbb{R},\leq,+\rangle$. Every $\phi(2,1)$-induction [abbreviated as ϕ-induction] on M terminates in a finite number of steps.

Proof. Assume not. So $\langle X = X_0, X_1, \ldots\rangle$ is an infinite ϕ-induction on M. Then X_0 is nonempty. Hence, clause (2) of ϕ will be applicable for at least one pair $x_1 = x_2 \in X_0$ in the transition to stage 1. So X_1 contains at least two distinct elements. Clause (1.1) will be applicable to at least one pair $x_2 < x_1$ in the transition to stage 2, so $0 \in X_2$. Then at least one of clauses (3.1) and (3.2) will be applicable to at least one pair $x_2 > x_1$ in the transition to stage 3, so that X_3 contains a positive and a negative element. Moreover, for all i, X_i contains at most one negative element; otherwise clause (1.1) will be applicable to two negative elements, causing termination. Let $i \geq 3$; set

$$m_i = \min\{x > 0,\ x \in X_i\}$$
$$x_0 = \text{the unique negative element of } X_i\ .$$

Now

$$m_i > -x_0 \Rightarrow m_{i+1} = m_i - |x_0| < m_i \quad \text{by clause (3.3)}$$
$$m_i = -x_0 \Rightarrow m_{i+1} < -x_0 \quad \text{by clause (1.2)}$$

Let $N = m_3/|x_0| + 3 + 1$. Then $0 < m_N < -x_0$. By clause (3.3), X_{N+1} will contain an element y: $x_0 < y < 0$. By clause (1.1), the induction terminates at stage N+1.

Lemma 4.3.6. Let ϕ be as in ψ_3, $M = \langle \mathbb{R},\leq,+\rangle$. For any $n \in \omega$, there is a $\phi(2,1)$-induction on M which terminates at stage n.

Proof. The lemma is obvious for $n < 4$. For $n \geq 4$, $\langle X, X_0, X_1, \ldots, X_n\rangle$ is a $\phi(2,1)$-induction on M as required (references in parentheses are to clauses of definition of ϕ).

$$
\begin{aligned}
X = X_0 &= \{0\} \\
X_1 &= \{0\} \cup \{-1\} && (2) \\
X_2 &= \{0,-1\} \cup \{n-3\} && (3.2) \\
X_3 &= \{0,-1\} \cup \{(n-3),(n-4)\} && (3.3) \\
X_4 &= \{0,-1\} \cup \{(n-3),(n-4),(n-5)\} \cup \{2n-7\} && (3.3) \\
&\vdots \\
X_{n-2} &= \{0,-1\} \cup \{1,2,\ldots,n-3\} \cup \{\text{iterated sums of } 2,3,\ldots,(n-3)\} \\
X_{n-1} &= \{0,-1\} \cup \{0,1,\ldots,(n-3)\} \cup \{\text{iterated sums}\} \cup \{\tfrac{1}{2}\} && (3.3) \text{ and } (1.2) \\
X_n &= \{0,-1\} \cup \{0,\ldots,(n-3)\} \cup \{\text{sums}\} \cup \{-\tfrac{1}{2},\tfrac{1}{2}\} && (3.3) \\
&\text{and } \{-1,-\tfrac{1}{2}\} \subseteq X_n \text{ is terminal by (1.1).}
\end{aligned}
$$

From Lemmas 4.3.5 and 4.3.6, and Proposition 4.2.8, it follows that ψ_3 is a nontrivially necessary axiom for $\Gamma^{-1}(M)$, for any Γ such that ψ_3 is equivalent on M to a $\bar{\Gamma}$-sentence. This is clearly the case for the Γ's mentioned in Example 4.3.4.

We now give some examples of nontrivially necessary axioms for the $\langle M,\Gamma\rangle$ appropriate to <u>difference measurement by an interval scale</u>. These examples are given not merely for their own interest, but also because they will yield examples of nontrivially necessary axioms for conjoint measurement, in a later section. This should not be too surprising, as representation theorems for conjoint measurement are sometimes proved by reduction to such theorems for positive difference measurement; see for example Krantz et alii (1971), p. 260.

We fix

$$M = \langle \mathbb{R}, ..\underline{<}..\rangle$$

where $\forall abcd \in \mathbb{R}$: $M \models ab \leq cd \Leftrightarrow \langle \mathbb{R}, \leq, +\rangle \models a-b \leq c-d$

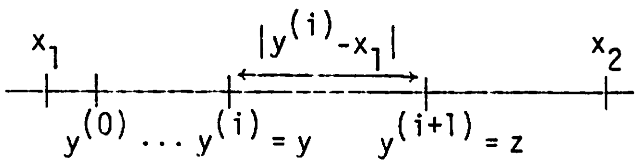
$$\Gamma = \langle ab \leq cd,\ ab \not\leq cd\rangle$$

Example 4.3.7.

$\psi = \forall x_1 x_2 \exists y \forall z \neg\phi$ where

$\phi = [[x_1 < x_2]$ &	proper setup
$[[[y=x_1 \vee y=x_2]$ & $x_1 < z < x_2] \vee$	get a $y^{(0)}$
$[z = 2y - x_1$ & $z < x_2$ & $z \neq x_1]]]$	walk

where 'a $\leq$ b' is an abbreviation of $ax_1 \leq bx_1$, 'a < b', 'a = b', 'a $\neq$ b' are defined in terms of 'a $\leq$ b' in the obvious manner, 'z = 2y - x' abbreviates 'zy $\leq$ yx'. The basic process in this example is



terminate if $y^{(k)} \geq x_2$

and the upper bound on the length of inductions is given by the integer part of

$$\log_2\left(\frac{x_2-y^0}{y^0-x_1}\right) .$$

This is easily shown once one observes that the step size of the basic process doubles at each step.

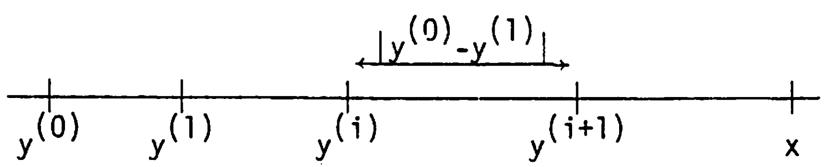
Example 4.3.8. $\psi = \forall x \exists y_1 y_2 \forall z \neg\phi$

$\phi = y_1 = y_2 \rightarrow [z \neq y_1$ & $z \neq x]$

&

$y_1 \neq x \rightarrow [z < x$ & $z \geq 2y_1 - y_2]$

where the abbreviations are as before, and the basic process is



terminate when $y^{(k)} \geq x$

Upper bound (achieved): integer part of

$$\frac{y^{(0)} - x}{y^{(0)} - y^{(1)}} .$$

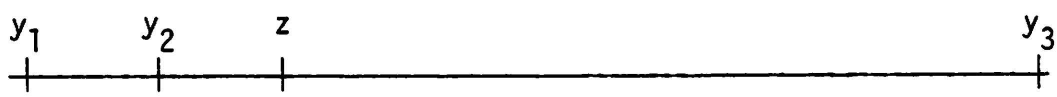
<u>Example 4.3.9</u>. $\psi = \exists y_1 y_2 y_3 \forall z \neg \phi$

$\phi = y_1 = y_2 \vee y_2 = y_3 \vee y_3 = y_1 \rightarrow [z \neq y_1 \;\&\; z \neq y_2 \;\&\; z \neq y_3]$ ($\geq$ 3 points)

&

$y_1 < y_2 < y_3 \rightarrow [y_3 - z \geq y_2 - y_1 \;\&\; y_2 < z]$ (step and not too large)

where the abbreviations are as before, and the basic process is:



$z - y_2 = y_2 - y_1 + \varepsilon$, $\varepsilon > 0$; terminate if $y_2 - y_1 \geq y_3 - y_2$,

i.e. at each stage add a new point between the second and third points of <u>each</u> ordered triple $y_1 < y_2 < y_3$ available at that stage. This process differs significantly from all the preceding ones: In the present case a new point is eventually added between <u>each</u> pair of points (except the leftmost), and such that after all these additions the points are still all well-spaced to the right. Thus an infinite $\phi(3,1)$-process yields a densely ordered discrete subset of the structure.

It is clear that every $\phi(3,1)$-induction terminates in finitely many steps. To show that there exist arbitrarily long $\phi(3,1)$-inductions, first note that at any stage it suffices, in order to get to the next stage, to add a point $z(y)$ between any y (except the leftmost) and the next point to the right, $\bar{y}$, such that

$$\left.\begin{array}{c} y < z(y) < \bar{y} \\ z(y) - y < \bar{y} - z(y) \end{array}\right\}$$

Assume, without loss of generality, that the initial configuration is

$$\underset{0}{+}\!\!-\!\!-\!\!\underset{d}{+}\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\underset{y_3}{+} \quad \text{(stage 0)}$$

Executing the induction for k steps (and reaching stage k, $k > 0$) by the addition of the points $z(y)$ as just described, we would get the configuration

$$\begin{array}{ccccccccccc} 0 & P_0 = d & \cdots & P_{2^{k-1}} & \cdots & P_{2^k} & \cdots & P_{3\cdot 2^{k-1}} & \cdots & P_{2^{k+1}} \\ (0) & (0) & \cdots & (2) & \cdots & (1) & \cdots & (2) & \cdots & (0) \end{array}$$

with a total of $2^{k+1}+1$ points: $0, P_0, P_1, \ldots, P_{2^{k+1}}$; the numbers in parentheses indicate the stage reached by adding the points with a given number.

The following selection of points will give a process of this type which reaches stage k:

$$P_i = 2^i(1+d) - 1 \ , \quad i = 0,1,\ldots,2^{k+1} \ .$$

This example has a special property: In an infinite $\phi(3,1)$-process no three elements y_1, y_2, y_3 are commensurable, i.e. if $y_1 < y_2 < y_3$ then

$$\forall n \in \omega:\ y_1 + n(y_2 - y_1) < y_3$$

Finally, we give two examples of nontrivially necessary axioms for another variant of difference measurement. Again, this is of interest in connection with the discussion of conjoint measurement in section 5.4.

For an arbitrary fixed $d > 0$, we set

$$M(d) = \langle \mathbb{R},\ xy \leq d,\ xy \geq d,\ xy \leq 0 \rangle \quad ,$$

i.e. we are envisaging empirical operations of

-- comparing a difference xy with d (i.e. $y-x \geq d$ or $y-x \leq d$)

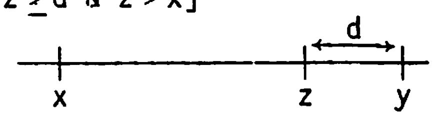
-- checking the orientation of a difference xy (i.e. $y > x$)

$\Gamma(d) = \{xy \leq d,\ xy \nleq d,\ xy \ngeq d,\ xy \geq d,\ xy \leq 0,\ xy \nleq 0\}$

For convenience later, we write the axioms in ordinary mathematical notation (rather than in a language for difference structures); the translation is obvious.

Example 4.3.10. $\psi = \forall x \exists y \forall z \neg\phi(x,y,z)$

$\phi = [y > x \rightarrow y-z \geq d \ \&\ z > x]$

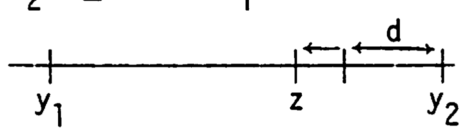
basic process: 

Example 4.3.11. $\psi = \exists y_1 y_2 \forall z \neg\phi(y_1, y_2, z)$

$\phi = [y_1 = y_2 \rightarrow z \neq y_1$

&

$y_1 < y_2 \rightarrow [y_2 - z \geq d \ \&\ z > y_1]]$

basic process: 

4.4. Restrictions on the Form of Nontrivially Necessary Sentences

In this section, methods are considered for showing that $\Gamma^{-1}(M)$ lacks nontrivially necessary sentences of a given form; the results connect properties of M with the nonexistence of such sentences. We will concentrate on properties of the automorphisms of M, and more generally, the invertible Γ-morphisms $M \to M$. Clearly these mappings must be 1-1, onto, Γ^*-morphisms, where Γ^* is obtained from Γ by adding the negations of all formulas in Γ. In all practical cases, these are exactly the automorphisms of M. The basic insight underlying the arguments in this section is that invertible Γ-morphisms of M map ϕ-inductions on M to other ϕ-inductions with exactly the same lengths and configurations.

Let G_Γ be the group of invertible Γ-morphisms of M, and for $\ell \geq 1$, $\vec{s} \in |M|^\ell$, $G_\Gamma(\vec{s})$ the group of invertible Γ-morphisms of M which fix $\vec{s}$, i.e. take each element of $\vec{s}$ to itself. If S is an arbitrary set, and G any group of maps $S \to S$, G induces an equivalence relation on S, namely

$$\forall s_1, s_2 \in S: \; s_1 \equiv s_2 \Leftrightarrow \exists g \in G: \; g(s_1) = s_2 \; .$$

The equivalence classes of S under this relation will be called the <u>orbits of S under G</u>. The theorems in this section will deal with cases in which the sets of such orbits are finite. Certain combinatorial counting arguments allowing optimal use of the hypotheses of these theorems will be stated as separate technical lemmas in order to separate the "logical" content of the theorems from these combinatorial details.

In the following we will deal with strictly necessary axioms (i.e. c = 1 in the notation of section 4.4.2). The results can be generalized to cover the more general case of "almost necessary" axioms. The first theorem (4.4.1) deals with $\exists\vec{x}\forall\vec{y}$-$\bar{\Gamma}$-sentences, the second (4.4.3) with $\forall\vec{x}\exists y\forall\vec{z}$-$\bar{\Gamma}$-sentences. (Note the single variable 'y'.)

<u>Theorem 4.4.1</u>. Let $\psi = \exists\vec{x}\forall\vec{y}\; \phi(\vec{x},\vec{y})$, ϕ a quantifierfree (m+n)-ary $\bar{\Gamma}$-formula, where m is the length of $\vec{x}$, and $m, n \geq 1$. If $[|M|]^m$ consits of a single orbit under G_Γ, then

$$\Gamma^{-1}(M) \models \psi \;\Rightarrow\; \Gamma^{-1}Eq(M) \models \psi \; ,$$

i.e. ψ cannot be nontrivially necessary.

Proof. Assume $\Gamma^{-1}(M) \models \psi$. By Proposition 4.2.8, we only need show that for some $N \in \omega$, all $\neg\phi(m,n)$-inductions on M terminate with N steps. Let k be the smallest number of distinct elements occurring in a terminal vector for $\neg\phi(m,n)$-inductions on M. So $1 \leq k \leq m$. In fact, $N \leq k-1$. For let $T = \langle X, X_0, \ldots, X_k, \ldots, X_p \rangle$ be an arbitrary (of course finite) $\neg\phi(m,n)$-induction on M. Then $X = X_0$, $\bar{\bar{X}} \geq 1$, and therefore $\bar{\bar{X}}_i \geq i+1$, for all $i < p$. Otherwise there exists j such that $X_j = X_{j+1}$, and $M \upharpoonright X_j \models \neg\psi$, but $M \upharpoonright X_j \in \Gamma^{-1}(M)$. Applying this for $i = k-1$, $\bar{\bar{X}}_{k-1} \geq k$, and so X_{k-1} contains a k-element subset α. But now α is in the same orbit of $[|M|]^k$ under G_Γ as the elements of some terminal vector $\vec{x}_0$ (with exactly k distinct elements) by combinatorial Lemma 4.4.2, stated below. Let $\lambda \in G_\Gamma$ map the set of elements occurring in $\vec{x}_0$ to α. Now:

$$M \models \forall\vec{y}\ \phi(\vec{x}_0,\vec{y}) \ , \quad \phi\text{: } \bar{\Gamma}\text{-sentence}, \quad \lambda\text{: } \bar{\Gamma}\text{-morphism} \in G_\Gamma$$

so

$$M \models \forall\vec{y}\ \phi(\lambda(\vec{x}_0),\vec{y}) \ ,$$

i.e. $\lambda(\vec{x}_0)$ is a terminal vector. But the elements of $\lambda(\vec{x}_0)$ are in $\alpha \subseteq X_{k-1}$, so in the fact the induction T terminates at stage $k-1$.

Combinatorial Lemma 4.4.2. Let a group G act on an infinite set S; $m \geq \ell \geq 1$. If $[S]^m$ consists of a single orbit under G, so does $[S]^\ell$.

Proof (sketch). Let $\theta \in [S]^m$, and assume $\{0_\alpha\}$ is the set of orbits of $[S]^\ell$, $\{\bar{\bar{0}}_\alpha\} > 1$. A certain finite set of orbits $\{0_i\}$ are represented by $[\theta]^\ell$, but because $[S]^m$ forms one orbit under G, exactly the same configuration of orbits is represented by any $\theta' \in [S]^m$, hence $\{0_\alpha\}$ is finite. By Ramsey's theorem, get $S' \subseteq S$, $\bar{\bar{S}}' \geq m$, S' homogeneous for one of the 0_α. So this must have been the configuration of orbits of $[S]^\ell$

represented by all $\theta \in [S]^m$, i.e. 0_α is the unique orbit of $[S]^\ell$. (See Ramsey (1928).)

Theorem 4.4.3. Let $\psi = \forall\vec{x}\exists y\forall\vec{z}\ \phi(\vec{x},y,\vec{z})$, ϕ a $(\ell+1+n)$-ary quantifier-free $\bar{\Gamma}$-formula with $\ell \geq 0$, $n > 0$, and ℓ is the length of $\vec{x}$. If $(\forall\vec{s} \in |M|^\ell)$ $G_\Gamma(\vec{s})$ partitions $|M|$ into finitely many orbits, then ψ cannot be a nontrivially necessary axiom.

Proof. Assume $\Gamma^{-1}(M) \models \psi$. By Proposition 4.2.8, we must show that:

$(\exists N \in \omega)(\forall\vec{s} \in |M|^\ell)$: All $\neg\phi(\vec{s},1,n)$-inductions on M terminate within N steps .

Part 1: Initially, we show

$(\forall\vec{s} \in |M|^\ell)(\exists N \in \omega)$: All $\neg\phi(\vec{s},1,n)$-inductions on M terminate within N steps .

Fix $\vec{s} \in |M|^\ell$; set $G = G_\Gamma(\vec{s})$, and let k be the number of orbits of $|M|$ under G. Then in fact, $N < k$. For let $T = \langle\vec{s},X,X_0,\ldots,X_k,\ldots,X_p\rangle$ be an arbitrary $\neg\phi(\vec{s},1,n)$-induction [induction, hereafter], of course finite. Let, for $i = 0,1,\ldots,p$, $n(i)$ be the number of orbits of $|M|$ under G represented among the elements of X_i; so for all i, $n(i) \leq k$. On the other hand,

$$n(i+1) \geq n(i)+1 \ , \quad \text{for all } i; \ n(0) > 0$$

as we will argue directly. Therefore $p < k$; as T was an arbitrary induction, this implies that indeed $N < k$.

So assume there exists i, $n(i+1) = n(i)$, i.e. no representatives of new orbits are added to X_{i+1}. Then there exists an extension of $\langle\vec{s},X,X_0,\ldots,X_i,X_{i+1}\rangle$ to an infinite induction on M contradicting our initial assumption. For, given that no representatives of new orbits are added to X_{i+1}, each $y \in X_{i+1}$ is in the same orbit under G as some $y' \in X_i$; hence

$$\exists \vec{z}' \in X_{i+1}: M \models \neg\phi(\vec{s},y',\vec{z}')$$

and so if $y = \lambda(y')$, $\lambda \in G$, then (as $\lambda(\vec{s}) = \vec{s}$)

$$M \models \neg\phi(\vec{s},y,\lambda(\vec{z}'))$$

hence we set $\lambda(\vec{z}') = \vec{z}(y)$ and add the elements occurring in $\lambda(\vec{z}')$ to X_{i+1} to get our new X'_{i+2}; in this manner we get to stage $i+2$, i.e. an induction

$$<\vec{s},X,X_0,\ldots,X_i,X_{i+1},X'_{i+2}> \quad ,$$

and we note that in this process no representatives of new orbits are added to X'_{i+2} (which were not already represented in X_i): all the elements added to X'_{i+2} were obtained from elements in X_{i+1} by mappings $\lambda \in G$, hence are in "old" orbits. Therefore the process can be continued, obviously indefinitely, giving an infinite induction on M.

This concludes part (1); if $\ell = 0$ the proof is now complete. Otherwise we must use:

Combinatorial Lemma 4.4.4. Let the group G act on the infinite set S; for $\ell \geq 1$ let for $\vec{s} \in S^\ell$: $G(\vec{s}) = \{\lambda \in G: \lambda(\vec{s}) = \vec{s}\}$. Then

$$(\forall \vec{s} \in S^\ell)\ G(\vec{s}) \text{ partitions } S \text{ into finitely many orbits}$$
$$\Rightarrow G \text{ partitions } S^\ell \text{ into finitely many orbits .}$$

Proof. To assign an orbit representative in S^ℓ to an arbitrary $\vec{s} \in S^\ell$: Let $\sigma_0 = \vec{s}$. At stage n, $1 \leq n \leq \ell$, given

$$\sigma_{n-1} = <r_1,\ldots,r_{n-1},\sigma_n^{(n-1)},\ldots,\sigma_\ell^{(n-1)}> \quad ,$$

let $r_n = \lambda_n(s_n^{(n-1)})$ be a representative of the orbit of $s_n^{(n-1)}$ under the action of $G(\vec{r})$, $\vec{r} \in S^\ell$ formed from exactly $\{r_1,\ldots,r_{n-1}\}$. Let

$$\sigma_n \overset{\text{def}}{=} \lambda_n(\sigma_{k-1}) \overset{\text{def}}{=} \langle r_1,\ldots,r_n,\sigma_{n+1}^{(n)},\ldots,\sigma_\ell^{(n)}\rangle \quad .$$

Then $\sigma_\ell = \langle r_1,\ldots,r_\ell\rangle$ is the representative of the orbit of $\vec{s}$ under G.

At each stage n, r_n was chosen from among finitely many candidates (dependent on $\{r_1,\ldots,r_{n-1}\}$, however), so only finitely many different σ_ℓ are possible. (Note that the argument will actually show that G partitions $S^{\ell+1}$ into finitely many orbits!)

Proof of Theorem 4.4.3, Part 2. Let $\vec{s}, \vec{s}' \in |M|^\ell$ be in the same orbit of $|M|^\ell$ under G . Then the maximal length of $\neg\phi(\vec{s},1,n)$-inductions equals the maximal length of $\neg\phi(\vec{s}',1,n)$-inductions, as these are mapped to each other by invertible Γ-morphisms. Thus there is a common maximum length per orbit of $|M|^\ell$, finite by part 1. But by the combinatorial lemma, there are only a finite number of such orbits in $|M|^\ell$ under G; the maximum of the associated maximal length of inductions therefore exists, and is finite; i.e. a uniform finite bound on the length of all $\neg\phi(\cdot,1,n)$-inductions on M, as desired. Q.E.D.

The reader will have noted strong similarities between the (main) arguments for Theorems 4.4.1 and 4.4.3 (part 1). Although these arguments are quite simple, neither the arguments nor the theorems allow generalization in any of the obvious directions. (For instance, one is tempted to try to get results at the level of generality of Theorems 4.2.5 and 4.2.6.) This becomes clear in examining applications; i.e. comparison with the positive results of section 4.3 and section 5.4 on conjoint measurement.

Consider, in comparison with section 4.3:

1. The structure $\langle \mathbb{R}, Nx, Sxy \rangle$ of Example 4.3.1 has no nonidentity invertible Γ-morhpisms ($\Gamma = \{Nx, Sxy\}$.) Thus neither of Theorems 4.4.1 nor 4.4.3 applies. And indeed, there is a nontrivially necessary axiom of the form

$$\exists x \, \forall y \, (\text{——}) \ ,$$

the very simplest quantifier configuration not ruled out by Proposition 4.2.1.

2. $\langle \mathbb{R}, \leq, + \rangle$; Examples 4.3.2 and 4.3.4. This structure, for any reasonable Γ, has the following group of invertible Γ-morphisms (in fact, exactly the automorphisms)

$$\{x \longmapsto \alpha x : \alpha > 0, \ \alpha \in \mathbb{R}\} \ .$$

These partition $\mathbb{R} = [\mathbb{R}]'$ into three distinct orbits; so by Lemma 4.4.2 we see that Theorem 4.4.1 cannot apply. On the other hand, the hypothesis of Theorem 4.4.3, for $\ell = 0$, is satisfied; thus there are no nontrivially necessary axioms of the form

$$\exists x \, \forall \vec{y} \, (\text{——}) \ , \quad (\text{——}) \text{ quantifierfree } \bar{\Gamma} \ .$$

Once we take $\ell > 0$, however, $G_\Gamma(\vec{s})$, $\vec{s} \in \mathbb{R}^\ell$, is trivial, and partitions $\mathbb{R}$ into uncountably many orbits. Thus again Theorem 4.4.3 does not apply. Now Examples 4.3.2 and 4.3.4 give nontrivially necessary axioms of the forms

$$\left.\begin{array}{l} \forall x \exists y \forall z (\text{——}) \\ \exists x_1 x_2 \forall y (\text{——}) \end{array}\right\} \ (\text{——})\text{: quantifierfree } \bar{\Gamma}$$

i.e. again the very simplest quantifier configurations not ruled out by the present analysis.

4.5. Nontrivially Necessary Sentences with Arbitrary Quantifier Prefixes

In this section we outline very briefly how the preceding theory for $\forall\exists\forall$-sentences can be generalized to arbitrary sentences.

The treatment of nontrivially necessary Q-sentences (where 'Q' is a quantifier prefix) is based on a notion of induction processes appropriate to Q; e.g. so far we have discussed $\forall\exists\forall$-inductions. In general, Q-inductions are to have the property that if carried out without termination for countably many steps, the set of all points accumulated forms a model of the negation of the Q-sentence.

Definition 4.5.1. Let $\psi = Q\neg\phi(\text{---})$, ϕ quantifierfree.

(i) The Q,ϕ-induction process (from a given initial set, on a given structure) is defined by:

Stage k, $k \geq 0$.

Extend the list (available at this point) of instances of ϕ over the set S of points accumulated by the beginning of this stage to form a complete verification of "$M \restriction S \models \neg\psi$" except that new points from M may be accumulated into S to satisfy existential quantifiers in $\neg Q$.

If this succeeds, go to stage k+1. Otherwise, terminate at stage k.

Remark. 'Extension' of the list is intended in the following strong sense: If $s \in |M|$ is chosen at stage k to satisfy $\exists x\ \theta(x_1\text{---})$ (i.e. $\exists x\ \forall\vec{y}\ \theta'(x,\vec{y},\text{---})$), then at all stages $\geq k$ we must continue to use s to form the instances of the quantifierfree formula ϕ making up the verification of $\exists x\ \theta(x_1\text{---})$. For example, consider the $\forall\exists\forall$-inductions defined above, where the initial existential instance $\vec{t}$ is chosen at stage 0 and maintained from then on, yielding the concept of a $\phi(\underline{\underline{t}},m,n)$-induction.

(ii) A Q,ϕ-induction is <u>of length</u> <u>n</u> $\in \omega$ if and only if it reaches stage n and then terminates; it is <u>infinite</u> if it reaches stage n for all $n \in \omega$.

This definition yields:

<u>Theorem 4.5.2</u>. Let $\psi = Q\neg\phi$, ϕ quantifierfree.

(i) $\Gamma^{-1}(M) \models \psi \Leftrightarrow$ all Q,ϕ-inductions in $\Gamma^{-1}(M)$ terminate at a finite stage

(ii) $\Gamma^{-1}(Th(M)) \models \psi \Leftrightarrow \exists n \in \omega$: all Q,ϕ-inductions in $\Gamma^{-1}(M)$ terminate by stage n

The proof of Theorem 4.5.2 is entirely analogous to that of Theorems 4.2.5-4.2.6. If ψ is a $\bar{\Gamma}$-sentence, then we obtain a characterization of whether ψ is nontrivially necessary which involves only Q,ϕ-inductions in M, as above in Theorems 4.2.7-4.2.8:

<u>Theorem 4.5.3</u>. Let $\psi = Q\neg\phi$, ϕ quantifierfree, ψ: $\bar{\Gamma}$-sentence.

(i) The image of a Q,ϕ-induction by a Γ-morphism is a Q,ϕ-induction.

(ii) $\Gamma^{-1}(M) \models \psi \Leftrightarrow$ all Q,ϕ-inductions on M terminate at a finite stage.

(iii) $\Gamma^{-1}(Th(M)) \models \psi \Leftrightarrow \exists n \in \omega$: all Q,ϕ-inductions on M terminate by stage n.

4.6. <u>Meaning and Testing of Nontrivially Necessary ∀∃∀-sentences</u>

In usual first-order logic, a ∃∀-sentence $\exists\vec{y}\forall\vec{z}\neg\phi(\vec{y},\vec{z})$ asserts the existence of a (not necessarily definable) constant or constant vector with a certain universal property. On the basis of this, it has been felt that such axioms for measurement would be of "little practical value": "...if an axiomatization in first-order logic,..., involves a combination of several universal and existential quantifiers, then the confirmation of this axiom may be highly contingent on the relatively arbitrary selection of the particular domain of objects. From the empirical standpoint, aside from the possible requirement of a fixed minimal number of objects, results ought to be independent of an exact specification of the extent of the domain" (Scott and Suppes, 1958). In the case of $\exists\forall$-sentence, the outcome of a test would depend, among other things, on whether we happened to look in the right place for the constant vector. Because of this, it has been felt that nonuniversal axioms for measurement would be of little interest.

Let us consider this situation a bit more closely for a nontrivially necessary $\exists\forall$-axiom $\psi = \exists\vec{y}\forall\vec{z}\neg\phi(\vec{y},\vec{z})$. This sentence does indeed assert that there is a terminal vector $\vec{y}_0$ for $\phi(m,n)$-inductions in M: $M \models \forall\vec{z}\neg\phi(\vec{y}_0,\vec{z})$. It would thus seem that testing this axiom in an empirical structure E which we hope is in $\Gamma^{-1}(M)$ would in fact depend upon $|E|$ in the way described above. But if ψ is in fact nontrivially necessary, then $E \in \Gamma^{-1}(M) \Rightarrow \Gamma^{-1}(E) \models \psi$. Thus in this case all $\phi(m,n)$-inductions on E encounter a terminal vector; in this sense such terminal vectors are dense in E and its substructures. Therefore, we would expect to find a terminal vector wherever we look in E, or for that matter, in $\Gamma^{-1}(M)$. For nontrivially necessary axioms, results are independent of an exact specification of the extent of the domain. Certainly for $\exists\forall$-axioms, and in fact similarly for $\forall\exists\forall$-axioms we can see clearly that Scott and Suppes' criticisms do not quite apply to nontrivially necessary axioms. In the case of $\forall\exists\forall$-axioms the induction processes differ only by the fixing of an initial vector, which does not materially affect the arguments. Once the initial vector is fixed, we have essentially an $\exists\forall$-sentence back again.

Specifically, a test of a nontrivially necessary $\exists\forall$-axiom ψ as above would proceed as follows: Execute a $\phi(m,n)$-induction. Regardless of the exact choice of points within an experimental domain E in executing the induction, it must terminate within a finite number of steps if $E = \Gamma^{-1}(M)$. Confirmation of termination amounts to confirmation of $E \models \forall\vec{z}\neg\phi(\vec{y}_0,\vec{z})$ for a $\vec{y}_0$ obtained at the previous stage; this confirmation problem is logically the same as confirmation of a universal sentence. Once a $\vec{y}_0$ has been found for which $\forall\vec{z}\neg\phi(\vec{y}_0,\vec{z})$ has been confirmed, ψ has been confirmed; clearly this is independent of the exact choice of points made in executing the induction, as well as the choice of experimental domain E subject to the restriction that $E \in \Gamma^{-1}(M)$.

On the other hand, refutation of ψ does seem to involve essentially verification of the $\forall\exists$-sentence $\neg\psi$. If ψ is nontrivially necessary for $\langle M,\Gamma\rangle$-measurement, and we are really out to show $E \notin \Gamma^{-1}(M)$ rather than $E \models \neg\psi$, it will in fact be sufficient to show that some substructure of E satisfies $\neg\psi$, i.e. that there is a non-terminating $\phi(m,n)$-induction on E (strictly speaking, in $\Gamma^{-1}(E)$ would be sufficient). Intuitively, this would seem easier than verifying $\neg\psi$ on E, and somewhat less dependent on the exact choice of experimental domain E, but still has the logical strength of verification of an $\forall\exists$-sentence.

A somewhat stronger version of the confirmation problem might be considered: confirmation of "All $\phi(m,n)$-inductions on E terminate in finitely many steps" instead of '$E \models \exists\vec{y}\forall\vec{z}\neg\phi$'. The assertion to be confirmed is analogous to an archimedian axiom, and the methodological analysis of this confirmation problem would seem to be analogous to that for the archimedian axiom.

The analogy between the infinitary principle associated with the nontrivial necessity of any axiom $Q\neg\phi$: 'all Q,ϕ-inductions terminate within finitely many steps', and archimedian axioms is suggestive in another respect. We might hope to use these infinitary principles in proving representation theorems much as archimedian axioms are used. In fact, our "induction processes" are suggestive of the processes underlying measurement procedures. Thus, proper generalization of these "inductions" may yield a formalism appropriate for a metatheory of measurement procedures. As special value is placed on proofs of representation theorems in measurement theory which give measurement procedures, this could be quite useful.

Certain new questions are also raised by these principles. We can ask whether there are such principles of <u>varying strength</u> for a given type $\langle M,\Gamma\rangle$ of measurement. An example of this can be found in Examples 4.3.7-4.3.9. If we consider counterexamples in $\Gamma^{-1}Eq(M)$ to the nontrivially necessary axioms given, we find in Examples 4.3.7 and 4.3.8 that there exist such counterexamples which can be embedded into structures in $Eq(M)$ based on a nonstandard model of $\langle\mathbb{R},\leq,+\rangle$ with only <u>two</u> incommensurable orders of quantities, i.e. finite reals and one order of infinitesimal reals and their sums. On the other hand, the counterexample structures in Examples 4.3.9 can only be embedded into structures in $Eq(M)$ based on a nonstandard model of $\langle\mathbb{R},\leq,+\rangle$ with infinitesimals of at least a countable densely ordered set of orders of infinitesimality. (This follows from the fact that in a counterexample to these axioms, no three individuals can be commensurable.) Thus the sentences of Examples 4.3.7 and 4.3.8 assert essentially "all numbers are commensurable," and Example 4.3.9 makes a much weaker assertion, more like "there is no countable densely ordered set of orders of infinitesimality."

5. More General Measurement Models

5.1. Introduction

The model of Simple Measurement applies to many specific kinds of measurement studied in measurement theory, especially to "classical" forms such as extensive measurement.

In the following sections, we consider extensions of the model of Simple Measurement to more general models applicable to certain important kinds of measurement not directly covered by the model of Simple Measurement, notably additive binary conjoint measurement. We attempt to show how certain kinds of measurement initially not formalizable as Simple Measurement can, after reformulation, nevertheless be treated in this way. (Alternatively, one could attempt to adapt the preceding theory of Simple Measurement to more complicated models. To a slight extent, this is done below for the conjoint measurement case.) The material in this section is intended to suggest ways of dealing with various situations, rather than to provide a completely general theory.

5.2. Representability in a Class of Structures

Class measurement is the model of measurement obtained from simple measurement by allowing a class K of measurement structures instead of a single measurement structure M; so a type of class measurement is characterized by a pair $\langle K,\Gamma\rangle$. A structure E is said to be $\langle K,\Gamma\rangle$-representable if and only if there is a Γ-morphism $E \to M$ for some $M \in K$. Thus if $\langle M,\Gamma\rangle$ is a type of simple measurement, the $\langle Eq(M),\Gamma\rangle$-representable structures are exactly the members of $\Gamma^{-1}Eq(M)$. In general, $\Gamma^{-1}(K)$ denotes the class of all $\langle K,\Gamma\rangle$-representable structures. Class measurement is not a model of independent importance in the sense that important problems in measurement theory are formulated directly in terms of class measurement models. Rather, this is an important model because, on the one hand, many important questions and theories which initially seem to require more general models for their formulization can be shown to be formulable in terms of class measurement, and, on the other hand, much of the theory of simple measurement has a natural generalization to class measurement.

There is a conceptual difficulty in discussing necessary axioms for class measurement. For (given $\langle K,\Gamma\rangle$),

$$\Gamma^{-1}(K) = \cup\{\Gamma^{-1}(M): M \in K\} \tag{1}$$

and so
$$\mathrm{Th}(\Gamma^{-1}(K)) = \cap\{\mathrm{Th}\ \Gamma^{-1}(M): M \in K\} \tag{2}$$

but this theory may be excessively weak, and certainly need not reflect the requirements for Γ-morphic representability of even finite structures in some $M \in K$. This will especially be the case when the structures in K have little to do with each other. In such cases, it seems more appropriate to specify the "necessary axioms for $\langle K,\Gamma\rangle$-representability" as

$$\{\langle M,\ \mathrm{Th}\ \Gamma^{-1}(M)\rangle: M \in K\}\quad . \tag{3}$$

This type of specification is of course unwieldy, though accurate; if the class K is infinite, it seems unreasonable to expect such a disjunctive agglommeration of theories to be tested. On the other hand, if the members of K are sufficiently closely related, as is usually the case in applications, there will be strong agreement between $\mathrm{Th}\ \Gamma^{-1}(M)$, for different $M \in K$; then most of the information about necessary conditions for representability will be given the shared axioms $\mathrm{Th}\ \Gamma^{-1}(K)$.

Subject to these considerations, formulas (1), (2) and (3) show that the theory of necessary axioms for $\langle K,\Gamma\rangle$-representability is reduced to the theory of necessary axioms for simple measurement. Sometimes, $<K,\Gamma>$-representability itself can be reduced to $<M,\Gamma>$-representability. In fact we have:

Theorem 5.2.1. $\exists M$: $\Gamma^{-1}(M) = \Gamma^{-1}(K)$ if and only if $\exists M \in K$: all structures in K can be Γ-morphically represented in M.

The proof is trivial. The condition for full reduction of the theory of necessary axioms is slightly weaker than the condition of Theorem 5.2.1, namely

$$\exists M\text{: Th } \Gamma^{-1}(M) = \text{Th } \Gamma^{-1}(K) \text{ .}$$

It is not clear what value either of these reducibilities might have in cases where the structure M involved was not explicitly known and accessible to study. In the examples in later sections we will not have this problem.

5.3. The Two-Language Model

By far the most important step in generalizing simple measurement to obtain a model of measurement naturally embracing most concepts of measurement studied in measurement theory (i.e. as judged from Krantz et al. (1971)) is to consider a two-language model of measurement. This model reflects a situation which arises naturally in the current state of measurement theory: One is confronted with certain empirical relations on a domain; that is, an empirical structure. One would like to find a representation of this empirical structure in some well known structure, preserving the empirical relations. To find a measure structure, one looks for a numerical structure with continuity and dimensionality properties which appear appropriate to the empirical structure under consideration; then one searches for relations in this numerical structure which seem appropriate for representing the empirical relations. The role of measurement theory in this process is in making the notion of appropriateness more concrete.

If we consider the concept of measurement obtained from this process from the viewpoint of first-order logic, we find 1) a language L_E appropriate for discussing the empirical structure, namely the first-order language with symbols for the empirical relations as its only nonlogical symbols; 2) the numerical structure, which is described in some standard first-order language L_M with symbols for the relations and operations on this structure; and 3) a representation requirement, namely that representations map empirical objects which stand in any empirical relation to numbers which are in a corresponding designated relation.

More precisely, and slightly more generally, we consider <u>$\langle M,\Gamma\rangle$-measurement</u>, where

(i) L_M is an arbitrary first-order language, and M is an L_M-structure (i.e. L_M may contain operation symbols!)

(ii) L_E is a first-order language with as its nonlogical symbols a finite set of relation symbols, and the same variables as L_M

(iii) $\Gamma = \langle\Gamma_E,\Gamma_M\rangle$, where

(iv) Γ_E is a sequence $\langle\phi_1(\vec{x}),\ldots,\phi_n(\vec{x})\rangle$ of <u>existential</u> or quantifier-free L_E-formulas

(v) Γ_M is a sequence $\langle\psi_1(\vec{x}),\ldots,\psi_n(\vec{x})\rangle$ of arbitrary L_M-formulas containing the same number of formulas as Γ_E, and

(vi) exactly the same variables occur in ψ_i as in ϕ_i, for $i = 1,\ldots,n$.

An L_E-structure E is <u>$\langle M,\Gamma\rangle$-representable</u> if and only if there exists a map $f\colon |E| \to |M|$, satisfying

$$\forall\vec{x} \text{ from } |E|:\ E \models \phi_i(\vec{x}) \Rightarrow M \models \psi_i(f(\vec{x}))\ , \quad i = 1,\ldots,n\ .$$

We will denote the set of $\langle M,\Gamma\rangle$-representable (L_E)-structures by $\Gamma^{-1}(M)$.

It is easily seen that extensive measurement and most forms of difference measurement and probability representations discussed in Krantz et al. (1971) have natural formulations in the two-language model. In the next subsection this will be shown for (binary) additive conjoint measurement; polynomial conjoint measurement can be treated entirely analogously.

In these cases, and most cases of practical interest, the two-language model can be replaced by an equivalent simple measurement model.

Theorem 5.3.1. Let $\langle M,\Gamma\rangle$ be a two-language model, $\Gamma = \langle\Gamma_E,\Gamma_M\rangle$. If Γ_E consists of atomic L_E-formulas and negations of atomic L_E-formulas, and if for each such $\phi_{n_1}(\vec{x})$, $\phi_{n_2}(\vec{x}) \in \Gamma_E$ with

$$\phi_{n_1}(\vec{x}) = R\vec{x}\ , \qquad \phi_{n_2}(\vec{x}) = \neg R\vec{x}$$

we have

$$M \models \psi_{n_1}(\vec{x}) \leftrightarrow \neg\, \psi_{n_2}(\vec{x})$$

then there is an L_E-structure M' such that

$$\Gamma^{-1}(M) = \Gamma_E^{-1}(M')$$

and in fact, if $\phi_1,\ldots,\phi_k \in \Gamma_E$ are the atomic formulas occurring in Γ_E, and $\phi_{k+1},\ldots,\phi_n \in \Gamma_E$ are the negations of atomic formulae which do not occur in Γ_E, then

$$M' = \langle|M|, \phi_1^M(\vec{x}),\ldots,\phi_k^M(\vec{x}), \neg\phi_{k+1}^M(\vec{x}),\ldots, \neg\phi_n^M(\vec{x})\rangle$$

where

$$\phi_i^M(\vec{x}) = \{\vec{x} \text{ from } |M| : M \models \psi_i(\vec{x})\}$$

and atomic L_E-formulas which occur neither negated nor unnegated in Γ_E are assigned arbitrary interpretations in M'; such formulas are irrelevant to the measurement problem.

The proof of the theorem is obvious from the statement.

5.4. Binary Additive Conjoint Measurement

The model of additive conjoint measurement was introduced in a pioneering paper by Luce and Tukey (1964). This model allows the measurement of a quantity q (such as momentum in classical physics) which is hypothesized to be related to two other attributes of objects (such as mass and velocity) as $q = a_1 + a_2$ (or $q = a_1 \cdot a_2$, in our example, which is obtained from the additive relation by exponentiation) using empirical operations of adjustment and ordinal comparison of attribute pairs $\langle a_1, a_2 \rangle$. The only operations involving q itself, rather than a_1 or a_2, which are needed are ordinal comparisons. An overview of the representation theory of this model and generalizations is given in Krantz et al. (1971).

In the present subsection, we apply the preceding theory of necessary sentences for measurement to this model. Specifically, we determine the complete list of minimal $\forall\exists\forall$ quantifier prefixes for nontrivially necessary axioms for binary additive conjoint measurement. This allows full use of the exclusion principles developed in subsection 4.4, as well as the development of further techniques. We assume that there are further minimal quantifier prefixes of nontrivially necessary sentences, of $\exists\forall\exists$-quantifier form, but these have as yet not been determined.

As has been remarked by Adams (1975), it is natural to study axiomatization of binary conjoint measurement in a two-sorted logical language; namely, allowing a distinct sort of variable to range over each attribute (much as we speak of masses and velocities, in physics). This is a first-order language which has two sorts of variables

$$x_0, x_1, x_2, \ldots$$
$$x'_0, x'_1, x'_2, \ldots$$

which we shall call the unprimed and primed variables, respectively; further relation parameters, which now come equipped with a specification not only of the number of variable positions, but also of the type of variable appropriate in each position. The atomic formulas are the formulas of the form

$$Rx_1 x'_2 x_3 x'_4$$

where R is a relation parameter followed by an appropriate sequence of primed and unprimed variables. Further formulas and sentences of the language are built up from this atomic formula as in regular, single-sorted first-order languages.

If L is a two-sorted language, then an L-structure

$$\langle D \uplus D'; R, \ldots \rangle$$

has two disjoint sets D and D' as "domain"; the unprimed variables refer to the objects in the unprimed domain and the primed variables refer to the objects in the primed domain. The relations are appropriate subsets of the cartesian product of the domains; thus in our example

$$R \subseteq D \times D' \times D \times D' \ .$$

Quantifiers range over the domain corresponding to the type of variable they bind; thus '$\forall x' \exists y$(——)' means: 'for each primed object there is an unprimed object...".

Now a two-language model for conjoint measurement can be formulated:

L_E: the two-sorted first-order language (without equality) with as sole relation symbol R, to be used as in $Rx_1x_2'x_3x_4'$ which we will abbrevate as $x_1x_2' \leq x_3x_4'$

L_M: the ordinary language in mathematics for $<\mathbb{R},+,\leq>$, with '+' as a binary operation symbol, and '$\leq$' as a binary relation symbol, $M = <\mathbb{R},+,\leq>$

Γ_E: $\{x_1x_2' \leq x_3x_4',\ x_1x_2' \nleq x_3x_4'\}$

Γ_M: $\{x_1+x_2 \leq x_3+x_4,\ x_1+x_2 \nleq x_3+x_4\}$

(Alternatively, we could have taken L_M with '+' as a ternary relation, as in extensive measurement, and

$$\psi_1 = \exists z_1z_2[z_1 = x_1+x_2 \ \&\ z_2 = x_3+x_4 \ \&\ z_1 \leq z_2] \ .$$

This description is clearly less natural, but allowed by the definition of the two-language model.)

In order to analyze the necessary axioms for conjoint measurement, the two-language model will be converted to a Simple Measurement model. This can be done essentially according to the procedure of Theorem 5.3.1, except that we must deal with the fact that L_E is two-sorted. As in the theorem, we wish to find an L_E-structure M', such that

$$\Gamma_E^{-1}(M') = \Gamma^{-1}(M) \ , \qquad (\Gamma = <\Gamma_E,\Gamma_M>) \ .$$

Examination of the situation, especially of Γ_E, shows that a $<M',\Gamma_E>$-representation f of an L_E-structure $E = <D,D';R>$ must

(i) map the unprimed domain to an unprimed domain of M', which "looks just like" $<\mathbb{R},+,\leq>$,

(ii) map the primed domain D' to a primed domain of M', which again looks just like $\langle \mathbb{R},+,\leq\rangle$,

(iii) all this in such a way that R on E is transformed into

$$x_1 + x_2' \leq x_3 + x_4'$$

on M', i.e. M' must be $M' = \langle \mathbb{R} \cup \mathbb{R}'; R\rangle$, where R is the quaternary relation defined by

$$(\forall x_1, x_3 \in \mathbb{R})(\forall x_2', x_4' \in \mathbb{R}'): M' = Rx_1x_2'x_3x_4' \Leftrightarrow \langle \mathbb{R},\leq,+\rangle \models x_1+x_2' \leq x_3+x_4' .$$

The right-hand side of this definition makes sense, once we realize that all of x_1, x_2', x_3, x_4' represent real numbers.

Comparison with Theorem 5.3.1 will show that the above is in fact a natural generalization of the procedure suggested there. It is clear that we have

Proposition 5.4.1. Let M and M' be as defined above.

(a) For any L_E-structure E, there is a 1-1 correspondence between

(i) the $\langle M,\Gamma\rangle$-representations $f: E \to M$

(ii) the $\langle M',\Gamma_E\rangle$-representations $g: E \to M'$

(b) $\Gamma_E^{-1}(M') = \Gamma^{-1}(M)$.

Thus we have found a Simple Measurement model $\langle M',\Gamma_E\rangle$ for binary additive conjoint measurement. In the following, we will denote this model by $\langle M,\Gamma\rangle$, dropping the prime and the subscript 'E'. We procede to analyze the necessary axioms for binary additive conjoint measurement. The trivially necessary axioms are well-known in the literature, see Krantz et al. (1971). They can be given as a countable sequence of so-called cancellation axioms (Adams (1975); Krantz et al. (1971), §6.2.1). In order to determine the minimal quantifier prefixes of nontrivially necessary axioms, we formulate two-sorted analogs of the exclusion principles Theorems 4.4.1 and 4.4.3.

Theorem 5.4.2. Let $\psi = \exists\vec{x}\exists\vec{x}'\forall\vec{y}\forall\vec{y}'\ \phi(\vec{x},\vec{x}',\vec{y},\vec{y}')$, ϕ a quantifierfree $(m+m'+n+n')$-ary $\bar{\Gamma}$-formula, and $|M| = D \cup D'$. If $[D]^m \times [D']^{m'}$ consists of a single orbit under G_Γ, then

$$\Gamma^{-1}(M) \models \psi \Rightarrow \Gamma^{-1}Eq(M) \models \psi \ ,$$

i.e. ψ cannot be nontrivially necessary.

Besides Theorem 5.4.2, there is another two-sorted analogue of Theorem 4.4.1:

Theorem 5.4.3. Let $\psi = \forall\vec{x}\exists\vec{y}'\forall\vec{z}\forall\vec{z}'\ \phi(\vec{x},\vec{y}',\vec{z},\vec{z}')$, ϕ a quantifierfree $(\ell+m'+n+n')$-ary $\bar{\Gamma}$-formula, and $M = D \cup D'$. If, for all $\vec{s} \in D^\ell$, $G_\Gamma(\vec{s})$ partitions $[D']^{m'}$ into a single orbit, then ψ cannot be nontrivially necessary.

An analogous assertion holds if $\psi = \forall\vec{x}'\exists\vec{y}\forall\vec{z}\forall\vec{z}'\ \phi(\vec{x}',\vec{y},\vec{z},\vec{z}')$.

Theorem 5.4.4. Let $\psi = \forall\vec{x}\forall\vec{x}'\exists y\exists y'\forall\vec{z}\forall\vec{z}'\ \phi(\vec{x},\vec{x}',y,y',\vec{z},\vec{z}')$, where ϕ is a $(\ell+\ell'+1+1+n+n')$-ary quantifierfree $\bar{\Gamma}$-formula, and $|M| = D \cup D'$. If, for all $\vec{s} \in D^\ell \times (D')^{\ell'}$, $G_\Gamma(\vec{s})$ partitions $D \times D'$ into finitely many orbits, and

either (i) one of $\exists y$ or $\exists y'$ is missing

or (ii) $\forall\rho \in G_\Gamma(\vec{s})\colon \forall x \in D\ \exists\pi \in G_\Gamma(\vec{s})\colon \pi(x) = x\ \&\ \forall x' \in D'\ \pi\rho(x') = x'$ (*)

then ψ cannot be nontrivially necessary.

Remark. The condition (*) is a weak independence condition for the actions of $G_\Gamma(\vec{s})$ restricted to D respectively D'. It is equivalent to either of the following conditions: for any orbit o of $D \times D'$:

$$\begin{cases} \forall\rho \in G\ \forall x \in D\ \forall x' \in D'\colon \langle x,x'\rangle \in o \Rightarrow \langle x,\rho(x')\rangle \in o \\ \forall\rho \in G\ \forall x \in D\ \forall x' \in D'\colon \langle x,x'\rangle \in o \Rightarrow \langle \rho(x),x'\rangle \in o\ . \end{cases}$$

The proofs of these theorems are analogous to those of Theorems 4.4.1 and 4.4.2. The condition (*) in Theorem 5.4.4, which is shown by the remark to be symmetrical in primed and unprimed variables, is used to show that no new orbits of $D \times D'$ occur in the construction in part (1) of the proof.

We now apply these theorems, using the fact that the invertible Γ-morphisms of M are exactly the automorphisms of M and are given by

$$G = \{\lambda\colon \lambda(x) = \alpha_1 x + \alpha_2,\ \lambda(x') = \alpha_1 x' + \alpha_3;\ \alpha_1,\alpha_2,\alpha_3 \in \mathbf{R},\ \alpha_1 > 0\}\ .$$

The following table indicates the conclusions to be drawn from Theorem 5.4.2-5.4.4 and an analysis of the action of G on $M = \langle \mathbf{R} \,\dot\cup\, \mathbf{R}', R\rangle$; note that for the present $\Gamma = \Gamma_E$, any formula ϕ is a $\bar{\Gamma}$-formula.

Table 5.4.5

Group	Acting on	Orbits (#)	Quantifier Prefixes excluded by — $\forall\vec{z}\forall\vec{z}'\ \phi(—,\vec{z},\vec{z}')$	Theorem (#)
1. a. G	$[\mathbb{R}]^2 \times \mathbb{R}'$, $\mathbb{R} \times [\mathbb{R}']^2$	1	$\exists y_1 y_2 \exists y'$, $\exists y \exists y_1' y_2'$	5.4.2
b.	$[\mathbb{R}]^3 \times \mathbb{R}'$, + symm	∞	---	---
c.	$[\mathbb{R}]^2 \times [\mathbb{R}]^2$	∞	---	---
2. a. G(x);	$[\mathbb{R}']^2$	1	$\forall x \exists y_1' y_2'$	5.4.3
b. $x \in \mathbb{R}$	$\mathbb{R} \times \mathbb{R}'$	$\leq 3(*)$	$\forall x \exists y \exists y'$	5.4.4(ii)
c.	$[\mathbb{R}']^3$	∞	---	---
d.	$[\mathbb{R}]^2 \times \mathbb{R}'$, + symm	∞	---	---
3. a. G(x');	$[\mathbb{R}]^2$	1	$\forall x' \exists y_1 \exists y_2$	5.4.3
b. $x' \in \mathbb{R}'$	$\mathbb{R} \times \mathbb{R}'$	$\leq 3(*)$	$\forall x' \exists y \exists y'$	5.4.4(ii)
c,d	as 2c,d		---	---
4. a. G(x,x');	$\mathbb{R}$, $\mathbb{R}'$	≤ 3	$\forall x \forall x' \exists y$, $\forall x \forall x' \exists y'$	5.4.4(i)
b. $x \in \mathbb{R}$, $x' \in \mathbb{R}'$	$[\mathbb{R}]^2$, $\mathbb{R} \times \mathbb{R}'$, $[\mathbb{R}']^2$	∞	---	---
5. $G(\mathbb{R})$	$\mathbb{R}'$	1	$\forall\vec{x}\exists y'$	5.4.4(i) or 5.4.3
	$\mathbb{R} \times \mathbb{R}'$, $[\mathbb{R}']^2$	∞	---	---
6. $G(\mathbb{R}')$	$\mathbb{R}$	1	$\forall\vec{x}'\exists y$	5.4.4(i) or 5.4.3
	$\mathbb{R} \times \mathbb{R}'$, $[\mathbb{R}]^2$	∞	---	---
7. $G(x_1,x_2)$ $x_1,x_2 \in \mathbb{R}$	$[\mathbb{R}']^2$	∞	---	---
8. $G(x_1',x_2')$	as 7		---	---

These results (excepting 5 and 6, which are stronger) can be summarized as:

<u>Proposition 5.4.6</u>. No nontrivially necessary axiom for conjoint measurement is of the form $\forall\vec{x}\forall\vec{x}'\exists\vec{y}\exists\vec{y}'\forall\vec{z}\forall\vec{z}'\ \phi(\cdots)$, $\phi(\ell+\ell'+m+m'+n+n')$-ary quantifierfree, with

$$\begin{cases} \ell+\ell'+m+m' \geq 3 \\ \ell+m \geq 1 \\ \ell'+m' \geq 1 \end{cases}$$

Thus the simplest quantifier prefixes not excluded by the results of Table 5.4.5 are the following: (We specify only what is to be filled in in — $\forall\vec{z}\forall\vec{z}'\ \phi(—,\vec{z},\vec{z}')$; we give only one of each two forms obtainable from each other by interchanging primed and unprimed variables.)

A. three variables of a kind:

1. $\forall x_1 x_2 \exists y$
2. $\forall x \exists y_1 y_2$
3. $\exists y_1 y_2 y_3$

B. two variables of each kind:

4. $\forall x_1 x_2 \forall x' \exists y'$
5. $\forall x_1 x_2 \exists y_1' y_2'$
6. $\forall x \forall x' \exists y \exists y'$
7. $\forall x \exists y \exists y_1 y_2'$
8. $\exists y_1 y_2 \exists y_1' y_2'$

So far our analysis has paid no attention to the extent of final universal quantifiers, indicated by $n + n'$ in Proposition 5.4.6. Using the considerations of Remark 4.2.10, one obtains the following two-sorted analogue of Theorem 3.2.6 (formulated for general $\langle M,\Gamma\rangle$-measurement, M two-sorted).

Theorem 5.4.7. Let $\psi = \forall\vec{x}\forall\vec{x}'\exists\vec{y}\exists\vec{y}'\forall\vec{z}\forall\vec{z}'\ \phi(\vec{x},\vec{x}',\vec{y},\vec{y}',\vec{z},\vec{z}')$, ϕ a quantifierfree $(\ell+\ell'+m+m'+n+n')$-ary $\bar{\Gamma}$-formula, $m > 0$.

(i) If $n = 0$, $\ell \neq 0$, then ψ is nontrivially necessary if and only if

$$\psi' = \forall\vec{x}\forall\vec{x}'\exists\vec{y}'\forall\vec{z}'\ \phi'(\vec{x},\vec{x}',\vec{y}',\vec{z}')$$

is also, where ϕ' is obtained from ϕ by the procedure of Remark 4.2.10 $(X = \emptyset)$ (or equivalently, as in Theorem 3.2.6(2)).

(ii) If $n = 0$ and $\ell = 0$, then

$$\psi' = \forall x\forall\vec{x}'\exists\vec{y}'\forall\vec{z}'\ \phi''(x,\vec{x}',\vec{y}',\vec{z}')$$

where ϕ'' is obtained from ϕ by the procedure of Remark 4.2.10, for the case $\ell = 0$, $\bar{\bar{X}} = 1$.

(iii) Analogously for the primed variables.

Thus we obtain a drastic simplification of ψ if the final universal quantifiers are omitted. The difference with the single-sorted case of Theorem 3.2.6 is that there the simplification always yields a trivially necessary axiom, whereas in the two-sorted case the simplification may apply only to one of the sorts and the axiom may remain nontrivially necessary.

Now applying Theorem 5.4.7 to our list of minimal nonexcluded quantifier prefixes for nontrivially necessary axioms for conjoint measurement, we find:

(a) In cases (1)-(3), the final universal quantifiers $\forall\vec{z}$ over unprimed variables may not be omitted (on pain of trivial necessity).

(b) In cases (4)-(5), the final universal quantifiers $\forall\vec{z}'$ over primed variables may not be omitted (on pain of trivial necessity).

(c) In cases (6)-(8), neither type of final universal quantifier may be omitted (on pain of trivial necessity).

In all cases the conclusion follows because omission of the quantifiers would make the sentence equivalent (as far as the possibility of being nontrivially necessary) to a sentence with a quantifier prefix which is excluded by the arguments summarized in Table 5.4.5. Moreover, we find that if we allow more variables preceding the final universal quantifiers, the omissions of types (a)-(c) still result in either a quantifier prefix extending (4) or (5) without the omissions, or in a quantifier prefix to be excluded by case (2) of the definability analysis below: $\forall x'\forall\vec{x}\exists\vec{y}\forall\vec{z}$.

This leaves undecided the minimum extent of the final universal quantification in

-- primed variables in (1)-(3)

-- unprimed variables in (4)-(5)

These situations have a common feature: the variables of the sort in question are not essentially involved in the induction processes associated with the quantifier prefixes: If we omit the final universal quantification in these cases, the induction process is carried out entirely on the domain of the other sort. As there are no atomic relations on M among elements of a single sort, it follows directly that ϕ may not omit variables of one sort altogether, so that in cases (1)-(3), the final universal quantification over primed variables may not be omitted.

In order to settle the question in cases (4)-(5) and to get more precise information about cases (1)-(3), we will now study induction processes which are carried out entirely on the domain of a single sort, with only occasional reference to elements of the other sort. This is of independent interest because it will also shed light on the possibilities of conjoint measurement using mainly manipulations of a single kind; for this involves similar processes.

We study inductions which are essentially on the domain of one sort (say the unprimed sort) by a method called <u>definability analysis</u>. Definability analysis is a method of relating the induction processes (on a two-sorted structure M with domain $D \cup D'$) relevant to $\forall\vec{x}'\forall\vec{x}\exists\vec{y}\forall\vec{z}\forall\vec{z}'$-sentences to the induction processes relevant to certain $\forall\vec{x}\exists\vec{y}\forall\vec{z}$-sentences, translations of the original sentences, on a reduced single-sorted structure M_r on the unprimed domain D of M. The definability analysis is used to determine M_r and the language into which the original sentence is to be translated. There is also an inverse application, in which a sentence in a single-sorted language L_r of which we know the relevant induction processes on an L_r-structure M_r, is shown to be translatable into L, such that the translation of the sentence has corresponding induction processes on M.

To see the basic idea, let $\psi = \forall\vec{x}'\forall\vec{x}\exists\vec{y}\forall\vec{z}\forall\vec{z}' \neg\phi$, ϕ an $(\ell'+\ell+m+n+n')$-ary quantifierfree L-formula. To determine whether ψ is nontrivally necessary, we must consider $\phi(\vec{x}',\vec{x},m,n+n')$-inductions on M for all $\vec{x}' \in (D')^{\ell'}$, $\vec{x} \in D^{\ell}$. If we take a particular $\vec{s}' \in (D')^{\ell'}$, and instantiate ψ to $\vec{s}'$, we get $\psi(\vec{s}') = \forall\vec{x}\exists\vec{y}\forall\vec{z}\forall\vec{z}' \neg\phi(\vec{s}',—) = \forall\vec{x}\exists\vec{y}\forall\vec{z}\neg\exists\vec{z}'\ \phi(\vec{s}',—)$. If we define $\phi_r(\vec{s}',—)$ by $\phi_r(\vec{s}',\vec{x},\vec{y},\vec{z}) = \exists\vec{z}'\ \phi(\vec{s}',\vec{x},\vec{y},\vec{z},\vec{z}')$ we see that ϕ_r defines an $(\ell+m+n)$-ary relation on D, and that the nontrivial-necessity of ψ can be determined by considering $\phi_r(\vec{s}',\vec{x},m,n)$-inductions on D, endowed with a relation $\rho(\vec{s}')$, with a definition of the form

$$\left.\begin{array}{l}\rho(\vec{s}\,') = \{\vec{u}\colon M \models \exists\vec{z}\,'\ \theta(\vec{s}\,',\vec{u},\vec{z}\,')\} \\ \theta(\vec{s}\,',u,\vec{z}\,')\ \text{quantifierfree},\ \ \vec{z}\,' = \langle z_1',z_2',\ldots,z_{n'}'\rangle\end{array}\right\}\quad (*)$$

Definability analysis answers the question: Given ℓ', n' and a two-sorted measurement structure, what relations ρ can be defined on the unprimed domain of M by formulas of the form as in (*)?

The information so obtained can be used in two ways:

(i) Let $\{\rho_i\colon i \in I\}$ be the set of all such definable relations. Let $M_r = \langle D,\rho_i\colon i \in I\rangle$. Then any induction on M agrees with one on M_r, so if we can show that, on $M_r = \langle D,\rho_i,\ i \in I\rangle$ all inductions are bounded (independently of $\vec{s}\,'$) or some must be infinite, we may infer that the same is true for any $\phi(\vec{s}\,',\vec{x},m,n+n')$-induction on M. (For examples, see cases 1 and 2 below.)

(ii) If there is a ϕ_r in terms of some of the $\{\rho_i\colon i \in I\}$ such that there are arbitrarily long ϕ_r-inductions on M_r but no infinite ones, then by substituting the defining L-formulas for the ρ_i into ϕ_r we obtain an L-formula ϕ such that there are arbitrarily long finite ϕ-inductions on M but no infinite ones. This may yield nontrivially necessary axioms in L. (For examples, see cases 3 and 4 below.)

In application (i), where it is crucial that we know the full strength of the entire set of definable relations $\{\rho_i\colon i \in I\}$, it will generally occur that most of the ρ_i are quantifierfree definable in terms of a small subset $\{\rho_i\colon i \in I_0\}$, $I_0 \subseteq I$. In that case it is clearly sufficient to analyze inductions in $\langle D,\rho_i\colon i \in I_0\rangle$. Specifically we note that disjunctions of (variable substitution instances of) ρ_i's are again ρ_i's, by the theorem of logic $\vee\!\vee\,\exists\vec{z}\,'\ \theta(\vec{s}\,',\vec{u},\vec{z}\,') \leftrightarrow \exists\vec{z}\,'\vee\!\vee\,\theta(\vec{s}\,',\vec{u},\vec{z}\,')$. Hence $\{\rho_i\colon i \in I_0\}$ need only contain ρ_i whose defining L-formula θ in (*) is disjunctionfree.

Returning to conjoint measurement, we execute the definability analysis for several values of ℓ', n':

Case 1. $\ell' = 0$, $n' = 1$ (will exclude $\forall\vec{x}\exists\vec{y}\forall\vec{z}\forall\vec{z}'$-axioms)

Recalling that in this case $D = \mathbb{R}$, abbreviate $\rho_0(x,y)$ by $x \leq y$. This is a definable relation on $\mathbb{R}$ (in the form of (*)):

$$\mathbb{R} \models x \leq y \Leftrightarrow \langle \mathbb{R} \cup \mathbb{R}', Rxx'yy' \rangle \models \exists z' \, Rxz'yz' \quad .$$

Any other (*)-definable relation is definable in terms of '$\leq$': Let the disjunctionfree L-formula $\theta(\vec{x},z')$ be $[\wedge\wedge Rx_{i_1}z'x_{j_1}z'] \wedge [\wedge\wedge \neg Rx_{i_2}z'x_{j_2}z']$. Then:

$$\langle \mathbb{R} \cup \mathbb{R}, Rxx'yy' \rangle \models \exists z' \, \theta(\vec{x},z') \Leftrightarrow \mathbb{R} \models [\wedge\wedge x_{i_1} \leq x_{j_1}] \wedge [\wedge\wedge x_{i_2} \nleq x_{j_2}] \; .$$

Hence any L-formula (with $\ell' = 0$, $n' = 1$) ϕ can be replaced by a formula ϕ_r of the language L_r appropriate to $\langle \mathbb{R}, \leq \rangle$ such that ϕ-inductions on $\langle \mathbb{R} \cup \mathbb{R}', Rxx'yy' \rangle$ correspond exactly to ϕ_r-inductions on $\langle \mathbb{R}, \leq \rangle$.

However, $Th(\langle \mathbb{R}, \leq \rangle)$ has exactly one countable model, up to isomorphism, so Theorem 4.1.1 applies to this situation, as was noted in section 4.1. There are no arbitrarily long ϕ_r-inductions on $\langle \mathbb{R}, \leq \rangle$ unless some ϕ_r-induction on $\langle \mathbb{R}, \leq \rangle$ is infinite. As the same must then be true of ϕ-inductions on M, we find there are <u>no</u> nontrivially necessary axioms for conjoint measurement of the form $\forall\vec{x}\exists\vec{y}\forall\vec{z}\forall z'$.

Case 2. $\ell' = 1$, $n' = 0$ (excludes $\forall x'\forall\vec{x}\exists\vec{y}\forall\vec{z}$-axioms)

As in Case 1, fixing $s' \in \mathbb{R}'$:

$$\mathbb{R} \models x \leq y \Leftrightarrow \langle \mathbb{R} \cup \mathbb{R}', Rxx'yy' \rangle \models Rxs'ys'$$

$$\langle \mathbb{R} \cup \mathbb{R}, Rxx'yy' \rangle \models \theta(s',\vec{x}) = [\wedge\wedge Rx_{i_1}s'x_{j_1}s'] \wedge [\wedge\wedge \neg Rx_{i_2}s'x_{j_2}s'] \Leftrightarrow$$

$$\mathbb{R} \models [\wedge\wedge x_{i_1} \leq x_{j_1}] \wedge [\wedge\wedge x_{i_2} \nleq x_{j_2}] \; .$$

So again we get $M_r = \langle \mathbf{R}, \leq \rangle$, independent of s'. As in Case 1 it follows that there are no nontrivially necessary axioms for conjoint measurement of the form $\forall x' \forall \vec{x} \exists \vec{y} \forall \vec{z}$. Note that this result agrees in part with the result of the exclusion argument 4a of Table 5.4.5.

Case 3: $\ell' = 0$, $n' = 2$. Transfer of nontrivially necessary axioms for difference measurement by an interval scale to minimal $\forall \vec{x} \exists \vec{y} \forall \vec{z} \forall \vec{z}'$-nontrivially necessary axioms, covering prefixes (1)-(3).

Consider the single-sorted structure $\mathbf{R}_r = \langle \mathbf{R}, x \leq y, x_1 - x_2 \leq y_1 - y_2 \rangle$, for the language L_r with one binary and one quaternary relation symbol. We note that relations on $\mathbf{R}_r$ of the form ϕ_r

$$\vee\vee[[\wedge\wedge x_{i_1} \leq x_{j_1}] \wedge [\wedge\wedge x_{i_2} \nleq x_{j_2}] \wedge [\wedge\wedge x_{i_3} < x_{j_3}] \wedge [\wedge\wedge x_{i_4} \nless x_{j_4}] \wedge [x_1 - x_2 \leq y_1 - y_2]]$$

can be defined in $M = \langle \mathbf{R} \cup \mathbf{R}', Rxx'yy' \rangle$ by a formula of the form:

$$\exists z_1' z_2' [\vee\vee[[\wedge\wedge Rx_{i_1} z_1' x_{j_1} z_1'] \wedge [\wedge\wedge\neg Rx_{i_2} z_1' x_{j_2} z_1'] \wedge [\wedge\wedge\neg Rx_{j_3} z_1' x_{i_3} z_1'] \wedge [\wedge\wedge Rx_{j_4} z_1' x_{i_4} z_1'] \\ \wedge Rx_1 z_1' x_2 z_2' \wedge Ry_2 z_2' y_1 z_1']] .$$

It follows that if an axiom of the form $\forall \vec{x} \exists \vec{y} \forall \vec{z}\, \phi_r$ where ϕ_r is of the special form given above is nontrivially necessary for measurement $\langle \mathbf{R}_r, \Gamma_r \rangle$, $\Gamma_r = \{x \leq y,\ x \nleq y,\ x_1 - x_2 \leq y_1 - y_2,\ x_1 - x_2 \nleq y_1 - y_2\}$, then its translation is a nontrivially necessary axioms for conjoint measurement. Thus, from Examples 4.3.7-4.3.9, we find such axioms of the forms

(1) $\forall x_1 x_2 \exists y \forall z \forall z_1' z_2'$; (2) $\forall x \exists y_1 y_2 \forall z \forall z_1' z_2'$; and (3) $\exists y_1 y_2 y_3 \forall z \forall z_1' z_2'$.

It follows from earlier considerations that these quantifier forms are minimal.

Case 4: $\ell' = 2$, $n' = 0$. Transfer of the axioms Examples 4.3.10-4.3.11 to minimal nontrivially necessary axioms covering prefixes (4)-(5).

Consider the single-sorted structure $\langle \mathbb{R},\ y-x \le d,\ y-x \ge d,\ x \le y\rangle$, $d > 0$, for the language with three binary relation parameters. We note that relations on this structure of the form ϕ_r (see Case 3 above) can be defined on $M = \langle \mathbb{R} \cup \mathbb{R}', Rxx'yy'\rangle$ by a formula of the form

$$Rx_0s_1'x_0s_2' \wedge \neg Rx_0s_2'x_0s_1' \wedge \vee\vee[[\wedge\wedge Rx_{i_1}s_1'x_{j_1}s_2'] \wedge \cdots \wedge Rx_1s_1'x_2s_2' \wedge Ry_2s_2'y_1s_1']$$

where the first two clauses insure that $d = s_2'-s_1' > 0$. Because of this, we can translate the sentences of Examples 4.3.10-4.3.11 into L, and find nontrivially necessary axioms for conjoint measurement of the prefix forms (4) $\forall x_1'x_2'\forall x\exists y\forall z$; (5) $\forall x_1'x_2'\exists y_1y_2\forall z$. Again it follows from earlier considerations that these forms are minimal.

We conclude the analysis of minimal $\forall\exists\forall$-nontrivially necessary axioms by giving three minimal examples.

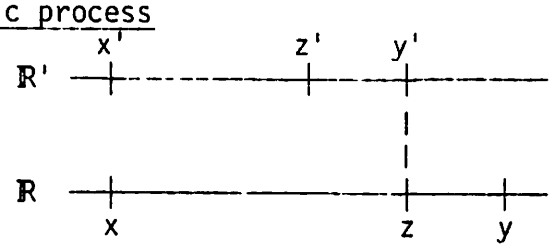
Example 5.4.8. $\psi = \forall x\forall x'\exists y\exists y'\forall z\forall z'\ \neg\phi(x,x',y,y',z,z')$

$$\left.\begin{aligned}\phi = [&x = y \rightarrow z > y\\ &\&\\ &x' = y' \rightarrow z' > y'\\ &\&\\ &[x < y \ \&\ y-x = y'-x'] \rightarrow x' < z' < y'\end{aligned}\right\}\text{(proper initial relations)}$$

$$\&$$

$$[y'-x' < y-x \ \&\ y' \ne x'] \rightarrow z-x = y'-x' \ \&\ y'-z' = y-z \ \&\ z' > x'] \quad \text{(step)}$$

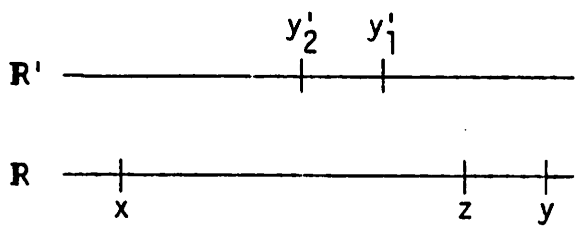
basic process



Example 5.4.9. $\psi = \forall x \exists y \exists y_1' y_2' \forall z \forall z' \neg\phi(x,y,y_1',y_2',z,z')$

$$\begin{aligned}
\phi = [&y_1' = y_2' \rightarrow z' \neq y && \text{(two distinct } y'\text{ s)}\\
&\&\\
&[y_2' < y_1' \ \&\ y_1'-y_2' \geq y-x] \rightarrow z-x > y_1'-y_2' && \text{(step sufficiently small)}\\
&\&\\
&[y_2' < y_1' \ \&\ y_1'-y_2' < y-x] \rightarrow [y-z = y_1'-y_2' \ \&\ y_1'-y_2' < z-x]] && \text{(step)}
\end{aligned}$$

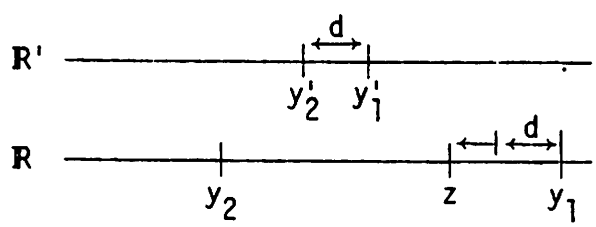
basic process



Example 5.4.10. $\psi = \exists y_1 y_2 \exists y_1' y_2' \forall z \forall z' \neg\phi(y_1,y_2,y_1',y_2',z,z')$

$$\begin{aligned}
\phi = [&\left.\begin{aligned}&y_1' = y_2' \rightarrow z' \neq y_2'\\ &\&\\ &y_1 = y_2 \rightarrow z \neq y_2\end{aligned}\right\} && \text{(four distinct points)}\\
&\&\\
&[y_2' < y_1' \ \&\ y_2 < y_1] \rightarrow [y_1'-y_2' < y_1-z \ \&\ y_2 < z]] && \text{(fill holes on } \mathbb{R})
\end{aligned}$$

basic process



Remark. In this process, points are inserted between any pair on $\mathbb{R}$ until some pair has distance less than d. In the limit this generates a dense subset of $\mathbb{R}$. This example is a slightly simpler version of Example 4.3.9; the phenomenon is discussed in more detail there.

The proof that the sentences ψ in Examples 5.4.8-5.4.10 are nontrivially necessary is left to the reader. The exclusion arguments proceding the definability analysis together with the case $\ell' = 0$, $n' = 1$ of the definability analysis, on the one hand, and on the other hand the arguments in the cases $\ell' = 0$, $n' = 2$ and $\ell' = 2$, $n' = 0$ of the definability analysis, together with Examples 4.3.7-4.3.11 and Examples 5.4.8-5.4.10 finally show:

Theorem 5.4.11. The minimal $\forall\exists\forall$-quantifier forms of nontrivially necessary axioms for conjoint measurement are (up to exchange of primed and unprimed variables):

(1) $\forall x_1 x_2 \exists y \forall z \forall z'$

(2) $\forall x \exists y_1 y_2 \forall z \forall z'$

(3) $\exists y_1 y_2 y_3 \forall z \forall z'$

(4) $\forall x_1' x_2' \forall x \exists y \forall z$

(5) $\forall x_1' x_2' \exists y_1 y_2 \forall z$

(6) $\forall x \forall x' \exists y \exists y' \forall z \forall z'$

(7) $\forall x \exists y \exists y_1' y_2' \forall z \forall z'$

(8) $\exists y_1 y_2 \exists y_1' y_2' \forall z \forall z'$

Of course this may not be a complete list of minimal quantifier forms: The theory does not give any information about $\exists\forall\exists$-quantifier forms; also no example of an $\exists\forall\exists$-nontrivially necessary axiom which is not $\forall\exists\forall$ seems to be known at present. It seems that Theorems 4.4.3 and hence 5.4.2 and 5.4.3, as well as the definability analysis, can be applied to $\exists\forall\exists$-sentences; thus we would expect to obtain exclusions analogous to those summarized in Table 5.4.5, as well as by definability analysis. (Theorem 4.4.1 and hence 5.4.4 seem not to generalize appropriately.) All this would still seem insufficient to settle the complete list of $\exists\forall\exists$-minimal quantifier forms, especially as long as examples are lacking. We will therefore not pursue the matter further.

5.5. Representation in Expanded Structures. Ordinal Measurement with a Constant Threshold

A wide range of problems in measurement theory have the following structure:

"Given a specific L; L-structure S

L_E (empirical language)

To find: axioms A_E on L_E-structures, such that (or which are necessary if...)

-- there exists an expansion of S to an L_T-structure $\bar{S}$, by adding relations $R_1 \cdots R_n$ ($L_T = L \cup \{R_1 \cdots R_n\}$), satisfying L_T-axioms A , such that

-- there exists a mapping α from L_E structures T satisfying A_E to an L_T-structure $\bar{S}$ satisfying A_T which is an expansion of the original given S, such that α "preserves specified empirical structural characteristics," i.e.

-- there exists a pair $\Gamma = \langle\Gamma_E, \Gamma_T\rangle$

$\Gamma_E = \{\phi_1(X_1 \cdots X_k), \ldots, \phi_n(X_1 \cdots X_k)\}$: quantifierfree L_E-formulas

$\Gamma_T = \{\psi_1(X_1 \cdots X_k), \ldots, \psi_n(X_1 \cdots X_k)\}$ " L_T-formulas

such that, for i = 1,2,...,n:

$$(\forall t_1 \cdots t_k \in |T|)\ T \models \phi_i(t_1 \cdots t_k) \Rightarrow \bar{S} \models \psi_i(\alpha(t_1), \ldots, \alpha(t_k))"$$

Thus our model involves:

1) two languages L_E, L_T (E = empirical, T = theoretical)

2) a class of measurement structures K (rather than a single one, as before)

3) a pair of Γ-sets: $\langle\Gamma_E, \Gamma_T\rangle$, otherwise used as before.

Occasionally, a problem initially formulated in this framework will allow simplification to the previous case of "Simple Measurement."

Example 5.5.1. Utility Measurement with Constant Threshold $\varepsilon > 0$. (See Luce (1956), Scott and Suppes (1958).)

(a) L_E: $\{<\}$; $L = \{\leq,+\}$; $L_T = \{\leq,+,\varepsilon\}$

$A_T = Th_L(<\mathbb{R},\leq,+>) \cup \{\varepsilon > 0\}$

K: models of A_T which are expansions (by ε) of $<\mathbb{R},\leq,+>$

$\Gamma_E = \{<, \neg<\}$, i.e. $\{x_1 < x_2; \neg x_1 < x_2\}$

$\Gamma_T =$ $\{x_1 \leq (x_2+\varepsilon); \neg x \leq (x_2+\varepsilon)\}$

i.e. A_E is to concern whether

$$\exists\alpha,\varepsilon: \begin{cases} \varepsilon > 0 \ \& \ a < b \Leftrightarrow \alpha(a)+\varepsilon \leq \alpha(b) \\ \alpha: <D,<> \to <\mathbb{R},+,\leq> \end{cases}$$

(b) First simplification: $(\forall S \in K)$ $(\exists! T$: L_E structure$)$

(i) $|S| = |T|$, and the identity map $|T| \to |S|$ is a Γ-morphism (in the $\Gamma = <\Gamma_E,\Gamma_T>$ sense of our more general model)

(ii) ($\forall E$: L_E-structure): $E \in \Gamma^{-1}(S) \Leftrightarrow E \in \Gamma_E^{-1}(T)$

namely: T is the reduct of $<\mathbb{R},+,\leq,\varepsilon>$ to $<\mathbb{R},R>$ where

$$\forall x,y \in \mathbb{R}: Rxy \leftrightarrow x+\varepsilon \leq y$$

So we are effectively back to a 1-language model, with a class K' of target structures.

(c) Second simplification: Any two structures in K' are isomorphic; the isomorphism $T_1 \to T_2$ is induced by the automorphism of $<\mathbb{R},+,\leq>$ taking $\varepsilon_1 > 0$ to $\varepsilon_2 > 0$. Hence: $\forall E$: L_E-structure: $\forall T_1,T_2 \in K'$:

$$E \in \Gamma_E^{-1}(T_1) \Leftrightarrow E \in \Gamma_E^{-1}(T_2),$$

i.e. $\Gamma_E^{-1}(K') = \Gamma_E^{-1}(T)$ for any $T \in K'$. So we are back to "Simple Measurement," taking a fixed $T \in K'$.

(d) To axiomatize $\Gamma_E^{-1}(T)$, first get trivially necessary axioms:

(i) $Th_\forall(T)$ in L_E

(ii) For nontrivially necessary axioms: Note that

invertible Γ-morphisms of T are automorphisms of T

all quantifierfree formulas of L_E are $\bar{\Gamma}_E$-formulas

The automorphisms of T: all translations: $\{x \mapsto x+\theta,\ \theta \in \mathbf{R}\} = G_\Gamma$.

So by our Theorem 4.4.3, there are no nontrivially necessary axioms of the form

$$\exists x\, \forall y_1 \cdots y_n\, \psi\ , \quad \psi \text{ quantifierfree } L_E\text{-formula}$$

(because G_Γ partitions $|T|$ into one orbit) and this is the best we get from Theorem 4.4.3: $G_\Gamma(\vec{s})$ partitions $|T|$ into ∞ orbits, for any nontrivial $\vec{s}$.

(iii) Example of a nontrivially necessary axiom:

$$\forall x \exists y \forall z \neg (y < z < x)$$

(e) Corollary to d(iii): $\Gamma_E^{-1}(T)$ is not a first-order class, for any Γ_E containing at least '$x_1 < x_2$'. But $\Gamma_E^{-1}(T) = \Gamma^{-1}(K)$, for the original formulation of the problem; hence $\Gamma^{-1}(K)$ is not first-order axiomatizable (using Proposition 4.1.0).

6. Open Problems

1. For which simple measurement models with nontrivially necessary sentences is the set of nontrivially necessary sentences recursively enumerable? Or the set of necessary sentences? Does decidability of $Th(M)$ help?

2. Consider extensive measurement $\langle M_0, \Gamma \rangle$, $M_0 = \langle \mathbb{R}, \leq, + \rangle$. Let M be elementarily equivalent to M_0. Which sets of sentences of L are

$$Th\ \Gamma^{-1}(M)$$

for such M? Clearly we have $Th\ \Gamma^{-1}(M_0)$ and by Theorem 4.1.2, also $Th_{\forall\bar{\Gamma}}(M_0)$, which are distinct by Examples 4.3.2 and 4.3.4. Are there any other possibilities? (Example 4.3.9 is in a different language, but might be converted.)

3. In Problem 2, let M_0 and Γ be arbitrary. Consider the class of sets

$$\{Th\ \Gamma^{-1}(M) : M \text{ elementarily equivalent to } M_0\} ,$$

ordered under set inclusion. Which partially ordered sets can be so realized?

4. Decide the following conjecture: For any $\langle M, \Gamma \rangle$ (Simple measurement model): If there is a nontrivially necessary sentence in $Th\ \Gamma^{-1}(M)$, there is an $\exists\forall$ nontrivially necessary sentence in $Th\ \Gamma^{-1}(M)$.

5. Let $\langle \Gamma, M \rangle$ be a Simple Measurement model, and let $\Gamma^{-1}(M)$ be a nonelementary class (for example, assume there are nontrivially necessary sentences). Can there be subclasses of $\Gamma^{-1}(M)$ which are elementary, and contain an infinite structure? (By Theorem 2.3.2, one may assume that any such class is $\forall\bar{\Gamma}$ and satisfies at least one $\forall\bar{\Gamma}$-sentence false in M.) Can this occur in interesting situations? (Note that if Γ-morphisms are 1-1, the answer is <u>no</u>, as the cardinality

of M is bounded and an elementary class with an infinite structure has infinite structures of every cardinality.)

6. Is there any extension of Theorem 4.4.1 to $\exists\forall\exists$-sentences which is of use in practical cases (e.g. conjoint measurement; see the final comments of section 5.4)?

7. Theorems 4.2.8 and 4.5.3 give a characterization in terms of intrinsic properties of M, of nontrivial necessity of $\bar{\Gamma}$-sentences; we can thus obtain results such as Theorems 4.4.1 and 4.4.3 for $\bar{\Gamma}$-sentences. Is there any way of extending these results to sentences which are not $\bar{\Gamma}$-sentences?

8. For, say, extensive measurement (see section 2.2), not every sentence is a $\bar{\Gamma}$-sentence. By Theorem 4.4.3, there is no nontrivially necessary $\bar{\Gamma}$-sentence for extensive measurement of the form

$$\exists x \forall \vec{y} \neg \phi \ , \quad \phi \text{ quantifierfree} .$$

Is there any nontrivially necessary sentence of this form (non-$\bar{\Gamma}$) for extensive measurement? If not, why not?

9. Consider the two-language model of section 5.3. When all elements of Γ_E are either atomic formulas or their negations, Theorem 5.3.1 sometimes gives reduction to a simple measurement (i.e. single language) model. Find conditions for this result to generalize to more general sets Γ_E of quantifierfree formulas. If that is not possible, can the preceding theory of nontrivially necessary axioms be generalized to apply to the two-language model?

10. Assume that (in a Simple Measurement model $\langle M,\Gamma\rangle$) not all L-structures are considered as candidates for "empirical structures," but only those satisfying a first-order theory T. Thus we now enquire about the

axiomatization of

$$\mathrm{Th}(\Gamma^{-1}(M) \cap \mathrm{Md}(T)) \quad .$$

How much of the preceding theory relativizes to this situation?

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