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OF NONLINEAR DEVICES AND SYSTEMS

by

L. O. Chua and S. M. Kang

Memorandum No. ERL-M603

20 October 1976

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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ABSTRACT

Several canonical structures of lumped dynamical systems having prescribed qualitative behaviors are presented. In particular, two canonical models are presented for simulating the transient and steady state response of nonlinear devices or systems driven by step inputs of arbitrary amplitudes. Another canonical model is presented for simulating the steady-state response of nonlinear devices or systems driven by a dc-superimposed sinusoidal inputs of arbitrary amplitude and frequency. Finally a canonical model is presented for simulating the steady state response of nonlinear devices or systems driven by periodic input signals of fixed frequency but arbitrary waveforms. Explicit methods for identifying the model parameters are given in each case.

†This work was supported by the Office of Naval Research Grant # N00014-76-C-0572.

††L.O. Chua is with the Department of Electrical Engineering and Computer Sciences and Electronics Research Laboratory, University of California, Berkeley, CA 94720.

S. M. Kang is with the Department of Electrical Engineering, Rutgers University, New Brunswick, N.J. 08903.

I. INTRODUCTION

In recent years many papers have considered various aspects of device and system modeling. A recent paper by Åström and Eykhoff [1] summarizes the state of the art in this field. The task of model making generally involves two steps: (1) the determination of the circuit or system structure of the model, and (2) the identification of the associated model parameters. The first step is essentially a synthesis problem where the resulting structure may take the form of a circuit diagram or a black box with an input-output mathematical structure. Circuit models can usually be synthesized for electronic devices having well-understood physical operating principles [2-3]. For more exotic devices, such as TRAPPAT diodes [4], or complex systems, such as biological systems [5], the internal operating mechanism may be so poorly understood that a black-box model may be the only recourse.

A general black-box representation for single-input single-output lumped systems is given by

$$\text{State Equation:} \quad \dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, u; \underset{\sim}{\alpha}) \qquad (1\text{-a})$$

$$\text{Output Equation:} \quad y = g(\underset{\sim}{x}, u; \underset{\sim}{\beta}) \qquad (1\text{-b})$$

where $\underset{\sim}{x}$ is an n-dimensional state vector characterizing the internal dynamics, u is a scalar representing the input and y is a scalar representing the output. The vectors $\underset{\sim}{\alpha} = [\alpha_1, \alpha_2, \ldots \alpha_p]^T$ and $\underset{\sim}{\beta} = [\beta_1, \beta_2, \ldots, \beta_q]^T$ are parameter vectors which are identified through input-output measurements. Equation (1) is often referred to as a dynamical system [6].

The structure of the above black-box model is therefore specified by the functional form of $\underset{\sim}{f}(\underset{\sim}{x}, u; \underset{\sim}{\alpha})$ and $\underset{\sim}{g}(\underset{\sim}{x}, u; \underset{\sim}{\beta})$. Once the model structure is synthesized, we can determine an optimum set of model parameters so that the error relative to certain criteria between the predicted and measured data is minimized. This second task is often referred to as the parameter identification problem. Most papers on modeling are addressed to this problem and many parameter identification techniques are now available. On the other hand, very few results concerning the synthesis of model structures - - i.e., the functional forms of $\underset{\sim}{f}(\underset{\sim}{x}, u, \underset{\sim}{\alpha})$ and $g(\underset{\sim}{x}, u, \underset{\sim}{\beta})$ - - are presently known. This problem is much more difficult and it is unlikely that a completely general and practical solution can ever be found [7].

A more modest and realistic approach to the structural modeling problem is to derive various classes of canonical structures which possess certain general properties of practical interest. For example, a simple canonical structure can be synthesized for simulating the steady state behavior of hysteretic systems driven by sinusoidal inputs [8-9]. Another canonical structure can be synthesized for simulating the class of dissipative systems having memory; namely, memristive systems [10]. Our objective in this paper is to derive even more general canonical structures for simulating the response of nonlinear systems driven by various classes of testing signals. In particular, the following classes will be considered:

Class $\mathcal{U}_1$: Step Signals of Arbitrary Amplitudes

$$\mathcal{U}_1 = \left\{ u(t) = \begin{matrix} 0, & t \leq 0 \\ A, & t > 0 \end{matrix} : \; A \in \mathbb{R} \right\}$$

where $\mathbb{R}$ denotes the set of all real numbers.

Class $\mathcal{U}_2$: DC-Superimposed Sinusoidal Signals of Arbitrary Amplitudes and Frequencies

$$\mathcal{U}_2 = \{u(t) = A_o + A \cos \omega t: (A_o, A, \omega) \in \mathbb{R} \times \mathbb{R}^\circ_+ \times \mathbb{R}^\circ_+\}$$

where $\mathbb{R}_+^\circ \triangleq (0,\infty)$.

Class $\mathcal{U}_3$: Periodic Signals with Arbitrary Series Coefficients

$$\mathcal{U}_3 = \{u(t) = A_o + \sum_{k=1}^{n} A_k \cos k\,\omega t + B_k \sin k\,\omega t: (A_o, A_k, A_k) \in \mathbb{R}^3$$

Our choice of these three signals are motivated by the fact that they represent the most common deterministic signals used in practice. We will assume throughout this paper that the systems being modeled either have unique steady-state response independent of the initial conditions, or are known to be initially relaxed. In the case of periodic inputs, we assume that the steady-state response has the same frequency as that of the input [11].

Several canonical model structures will be presented in the following sections. In particular, two canonical models will be presented in Section II for simulating the complete - - transient and steady state - - zero-state response of nonlinear systems driven by any number "M" of inputs signals belonging to $\mathcal{U}_2$ and $\mathcal{U}_3$, respectively. Since our models in Sections III and IV are designed only for simulating the steady state response, they are more suitable for modeling devices and systems operating under periodic inputs. We emphasize that our models are guaranteed to simulate exactly only the measured set of input-output waveforms. However, it follows from continuity arguments that the larger the number M of such waveforms used to determine the model parameters, the more realistic it will be in simulating other signals.

II. MODELING THE TRANSIENT AND STEADY-STATE RESPONSE OF NONLINEAR SYSTEMS DRIVEN BY STEP INPUTS

Let $y(t; A_1), y(t;A_2),\ldots,y(t;A_M)$ denote a family of "M" measured zero-state responses[1] of a nonlinear device or system to a family of step inputs $u(t) \in \mathcal{U}_1$ with amplitudes $A_1, A_2, \ldots, A_M$. Let each response be decomposed into

$$y(t;\, A_j) = y_o(A_j) + y_{ac}(t;A_j) \tag{2}$$

where $y_o(A_j)$ and $y_{ac}(t;A_j)$ denote respectively the "dc" and the "ac" components. We assume that each "ac" component belongs to the function space $L^2(0,\infty)$ [12] and is therefore square integrable over the time interval $(0,\infty)$. Our problem is to synthesize a canonical model which is capable of simulating the "M" measured input-output pairs to within any desired degree of accuracy. Two canonical models which satisfy this requirement will be presented in this section. The two models differ from each other in that the dynamics is linear in one model but nonlinear in the other.

2.1 Step-Input Canonical Model 1

Consider the following dynamical system:

State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_N \end{bmatrix} = - \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & 0 \\ 1 & 1 & \frac{1}{2} & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ \frac{5}{2} \\ \vdots \\ \frac{2N+1}{2} \end{bmatrix} u \tag{3-a}$$

1 By zero-state response we mean the output waveform for $t > 0$ due to an input applied for $t \geq 0$ when the system is at rest at $t=0$; i.e., assuming zero initial states [6]. The zero-state response would in general consists of a transient and a steady-state component. For step inputs, the steady-state component is just the dc average value. For periodic inputs, the steady-state component is assumed to be the periodic component of the output waveform.

Output Equation:

$$y = g_o(u) + \sum_{k=1}^{N} g_k(u)\, x_k \tag{3-b}$$

where

$$g_o(u) \triangleq h_o(u) + \sum_{k=1}^{N} h_k(u) \tag{4-a}$$

$$g_k(u) \triangleq h_k(u)/u \tag{4-b}$$

The function $h_o(u)$ is obtained by passing a smooth curve through the set of "M" points $y_o(A_j)$ associated with the dc component of the output waveform, i.e., $h_o(A_j) = y_o(A_j)$. The function $h_k(u)$ is similarly determined by the "ac" components $y_{ac}(t;A_j)$ and $h_k(A_j)$ is defined by

$$h_k(A_j) \triangleq \int_0^{\infty} y_{ac}(t;\, A_j)\ \phi_k(t)\, dt, \tag{5-a}$$

where

$$\phi_k(t) \triangleq e^{-1/2t}\left[\frac{1}{(k-1)!}\, e^t \frac{d^{k-1}}{dt^{k-1}}\left(t^{k-1}\, e^{-t}\right)\right] \tag{5-b}$$

denotes the _kth_ Laguerre function [13-14]. Observe that our standing assumption "$y_{ac}(t;A_j)$ belongs to $L^2(0,\infty)$" guarantees that $h_k(A_j)$ in (5-a) is integrable and is therefore well-defined. The following theorem shows that the _canonical model 1_ can simulate the input-putput pairs to within any desired degree of accuracy by increasing the number "N" of state varaibles.

Theorem 1. For each step input of amplitude A_j, the zero-state response y(t) given by (3) converges to the measured output waveform $y(t;A_j)$ in the mean-square sense as $N \longrightarrow \infty$

Proof. Let us define a new state variable

$$z_k(t) \triangleq x_k(t) + u(t),\ k=1,2,\ldots,N \tag{6}$$

and transform (3) into the following equivalent system:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{z}_N \end{bmatrix} = - \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & \frac{1}{2} & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 1 & \frac{1}{2} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \frac{1}{2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \dot{u}(t) \tag{7-a}$$

$$y = h_o(u) + \sum_{k=1}^{N} g_k(u)\, z_k \tag{7-b}$$

Since $u(t)$ is a step input of amplitude A_j, $\dot{u}(t) = A_j \delta(t)$ is a delta function with area equal to A_j. By transforming (7-a) into the s-domain via Laplace transform, it is easily shown that the zero-state response $z_k(t)$ of (7-a) due to a step input of amplitude A_j is given by [15]:

$$z_k(t) = A_j\, \phi_k(t) \tag{8}$$

Substituting (5-a), (8), and (4-b) into (7-b) and assuming $u = A_j$, we obtain

$$\begin{aligned} y(t) &= h_o(A_j) + \sum_{k=1}^{N} \left\{ \frac{1}{A_j} \int_0^\infty y_{ac}(t;A_j)\, \phi_k(t)\, dt \right\} A_j\, \phi_k(t) \\ &= y_o(A_j) + \sum_{k=1}^{N} \left\{ \int_0^\infty y_{ac}(t;A_j)\, \phi_k(t)\, dt \right\} \phi_k(t) \\ &= y_o(A_j) + \sum_{k=1}^{N} \alpha_k(A_j)\, \phi_k(t) \end{aligned} \tag{9}$$

where

$$\alpha_k(A_j) \triangleq \int_0^\infty y_{ac}(t;A_j)\, \phi_k(t)\, dt \tag{10}$$

Now observe that the family $\{\phi_1(t), \phi_2(t), \ldots, \phi_N(t)\}$ of zero-state solutions $z_k(t)$ of (7-a) can be identified as the first N members of the well-known orthonormal set of Laguerre functions [13-14]. Hence $\alpha_k(A_j)$ is just the kth Fourier coefficient of the expansion of $y_{ac}(t;A_j)$ in terms of Laguerre functions. Moreover, since the infinite family of Laguerre functions is complete [12-13] we have

$$\int_0^\infty \left\{ y_{ac}(t;A_j) - \sum_{k=1}^{N} \alpha_k(A_j)\, \phi_k(t) \right\}^2 dt \longrightarrow 0 \text{ as } N \longrightarrow \infty \tag{11}$$

It follows from (9) and (11) that

$$y(t) \longrightarrow y_o(A_j) + y_{ac}(t;A_j) = y(t;A_j) \text{ as } N \longrightarrow \infty \tag{12}$$

Hence the response of the preceding step input model 1 converges to the measured response $y(t;A_j)$ in the mean-square sense as $N \longrightarrow \infty$.

Observe that since the family of Laguerre functions is weighted in such a way that only the "immediate past" of the signal $y_{ac}(t;A_j)$ is emphasized, it is clear that the preceding model can be truncated with a finite "N" only for systems having a "fading" memory [14]. For such systems, the Fourier series often converges rapidly so that only a few Laguerre functions are sufficient. A physical realization of the "step-input canonical model 1" is shown in Fig. 1.

Example.

Consider the celebrated Hodgkin-Huxley model of the nerve membrane [16]. The experimental data obtained by Hodgkin and Huxley consists of a family of step inputs u(t) in "membrane potential" of varying amplitudes while the output y(t) of interest here is the "conductance" of the potassium channel of the nerve membrane.[2] The ac component $y_{ac}(t;A_j)$ of the potassium

2 For many biological systems, the step input of varying amplitudes is the most convenient testing signals. Note that since the physical mechanism of the nerve membrane is still unknown, the Hodgkin-Huxley model is, by necessity, a black-box model.

channel conductance is easily verified to be $L^2(0,\infty)$ and hence our preceding model is applicable. The resonse of the Hodgkin-Huxley model corresponding to 5 different amplitudes (A_j = -20, -40, -60, -80, -100 mV) of "membrane potential" step inputs are plotted in Figs. 2(a) and (b) (solid curves) and will be taken as the "measured" response for this example. The response predicted by our model 1 using only 2 Laguerre functions (N = 2) is shown in Fig. 2(a) (dotted curves). Observe that the approximation is already quite acceptable. The predicted response using 4 Laguerre functions (N = 4) as shown in Fig. 2(b) (dotted curves) is virtually identical to the measured response.

2.2 Step-Input Canonical Model 2

Another model for simulating the response of nonlinear systems due to step inputs of arbitrary amplitudes has been given in the form of an integral equation [17]. By introducing appropriate state variables, this model can be shown to be equivalent to the following dynamical system:

State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} - \begin{bmatrix} f_{N1}(u) \\ 2f_{N2}(u) \\ \vdots \\ Nf_{NN}(u) \end{bmatrix} \tag{13-a}$$

Output Equation:

$$y = \hat{h}_o(u) + \sum_{k=1}^{N} x_k$$

$$\triangleq \left[h_o(u) + \sum_{k=1}^{N} f_{Nk}(u)\right] + \sum_{k=1}^{N} x_k \tag{13-b}$$

Where $h_o(u)$ is as defined in Model 1; i.e., $h_o(A_j) = y_o(A_j)$, and $f_{Nk}(u)$ is obtained by passing a smooth curve through the set of points

$$f_{Nk}(A_j) \triangleq \sum_{\ell=k}^{N} \beta_{\ell k} \left\{ \int_0^{\infty} y_{ac}(t;A_j) \left[\sum_{k=1}^{\ell} \beta_{\ell k}\, e^{-kt} \right] dt \right\} \tag{14-a}$$

for each amplitude $u = A_j$, where

$$\beta_{\ell k} \triangleq \frac{(-1)^k (2\ell)^{1/2} \prod_{m=1}^{\ell-1} (k+m)}{\prod_{\substack{m=1 \\ m\neq k}}^{\ell} (k-m)} \tag{14-b}$$

A comparison between the step-input canonical models 1 and 2 shows that the state equations for model 1 is a linear function of both $\underset{\sim}{x}$ and u, whereas that for model 2 is a nonlinear function of u. On the other hand, the output equation for model 1 is a nonlinear function in both $\underset{\sim}{x}$ and u, whereas that for model 2 is a linear function of $\underset{\sim}{x}$. In other words, model 1 has a linear memory whereas model 2 has a nonlinear memory. Both models expand the "ac" component $y_{ac}(t;A_j)$ as a Fourier series of complete orthonormal functions - - Laguerre functions for model 1 and a weighted sum of exponentials for model 2. Consequently the choice of one model over the other will depend mainly on the nature of $y_{ac}(t;A_j)$: Given a family of $y_{ac}(t;A_j)$, the Fourier series which results in a fewer terms would lead to the simpler model. For digital computer simulation, the sensitivity of the model 1 to perturbations in the model parameters is not an important consideration. However, for physical realizations, model 1 is expected to be more sensitive because it is known that networks for realizing Laguerre functions are quite sensitive to parameter variations.

III. MODELING THE STEADY-STATE RESPONSE OF NONLINEAR SYSTEMS DRIVEN BY DC-SUPERIMPOSED SINUSOIDAL INPUTS.

Let $y_s(t;\, A_{o_1},\, A_1,\, \omega_1)$, $y_s(t;\, A_{o_2},\, A_2,\, \omega_2)$,..., $y_s(t;\, A_{o_M},\, A_M,\, \omega_M)$

denote a family of "M" measured steady-state responses of a nonlinear device or system to a family of sinusoidal inputs $u(t) \varepsilon \mathcal{U}_2$ having amplitudes $A_1, A_2, \ldots, A_M$, frequencies $\omega_1, \omega_2, \ldots, \omega_M$, and superimposed on top of a dc bias $A_{o_1}, A_{o_2}, \ldots, A_{o_M}$. By steady-state measurement, we mean that the response $y(t)$ is recorded only after its transient component has settled down. Hence, each of the "M" measured steady-state response is periodic and we assume that it can be approximated by a truncated Fourier series with N frequency components; namely,

$$y_s(t;A_{o_j},A_j,\omega_j)=a_o(A_{o_j},A_j,\omega_j)+\sum_{k=1}^{N}\left\{a_k(A_{o_j},A_j,\omega_j)\cos k\omega_j t + b_k(A_{o_j},A_j,\omega_j)\sin k\,\omega_j t\right\} \tag{15}$$

$$j = 1,2,\ldots,M,$$

where the (2N+1) Fourier coefficients $\left\{a_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N\right\}$ are identified by the parameters A_{o_j}, A_j, and ω_j of the corresponding input signal $u(t) = A_{o_j} + A_j \cos \omega_j t$. Our problem is to synthesize a canonical model which is capable of simulating the "M" measured steady-state responses to within any desired degree of accuracy.

Our first step in synthesizing a model to simulate (15) is to represent the (2N+1) Fourier coefficients $a_o(A_{o_j}, A_j, \omega_j)$, $a_k(A_{o_j}, A_j, \omega_j)$, and $b_k(A_{o_j}, A_j, \omega_j)$ by suitable functions passing through the set of M data points corresponding to the M measurements. This step is strictly a memoryless operation involving the approximation of a scalar function of several variables. One could approximate these functions by polynomials [18], splines [19], or other appropriate functions. The state-of-the-art in the approximation of functions of several variables is unfortunately still quite unsatisfactory because the amount of computations and the number of terms needed in the approximation often turn out to be excessive. However, a recent result using section-wise piecewise-linear functions [20] seems rather promising and an example illustrating this new approximation approach will be given in Section IV. Regardless of the method chosen, we will assume in the following discussion that the (2N+1) "Fourier-coefficinet Functions" $\alpha_o(A_o, A, \omega)$, $\alpha_k(A_o, A, \omega)$, and $\beta_k(A_o, A, \omega)$, where $k=1,2,\ldots,N$, have already been found which accurately approximated the exact Fourier coefficients at the set of M data points; i.e.,

$$\alpha_o(A_{o_j}, A_j, \omega_j) \doteq a_o(A_{o_j}, A_j, \omega_j) \;,\; j = 1,2,\ldots,M \tag{16-a}$$

$$\alpha_k(A_{o_j}, A_j, \omega_j) \doteq a_k(A_{o_j}, A_j, \omega j) \;,\; j = 1,2,\ldots,M \tag{16-b}$$

$$\beta_k(a_{o_j}, A_j, \omega_j) \doteq b_k(A_{o_j}, A_j, \omega j) \;,\; j = 1,2,\ldots,M \tag{16-c}$$

The following canonical model will be formulated in terms of these Fourier-Coefficient Functions.

Sinusoidal-Input Canonical Model

State Equation:

$$\dot{x}_1 = x_2 \tag{17-a}$$

$$\dot{x}_2 = \frac{1}{\varepsilon_1}(1 - x_2) \quad , \; 0 < \varepsilon_1 << 1 \tag{17-b}$$

$$\dot{x}_3 = -\left(\frac{x_2}{x_1}\right) x_3 + \left(\frac{1}{x_1}\right) u \tag{17-c}$$

$$\dot{x}_4 = \mathrm{sgn}\,(u - x_3) \tag{17-d}$$

$$\dot{x}_5 = \frac{1}{2\varepsilon_2}\left\{(1 + \varepsilon_2^2)[r(x_4) - x_5] + (1 - \varepsilon_2^2)\cdot\left|r(x_4) - x_5\right|\right\}, 0 < \varepsilon_2 << 1 \tag{17-e}$$

$$\dot{x}_6 = \frac{1}{2\varepsilon_2}\left\{(1 + \varepsilon_2^2)[r(u - x_3) - x_6] + (1 - \varepsilon_2^2)\cdot\left|r(u - x_3) - x_6\right|\right\} \;,\; 0 < \varepsilon_2 << 1 \tag{17-f}$$

$$\dot{x}_7 = \frac{\pi}{2x_5} x_8 \tag{17-g}$$

$$\dot{x}_8 = -\frac{\pi}{2x_5} x_7 \tag{17-h}$$

<u>Output Equation:</u>

$$y=\alpha_o(x_3,x_6,\frac{\pi}{2x_5})+\sum_{k=1}^{N}\left\{\alpha_k(x_3,x_6,\frac{\pi}{2x_5})T_k(x_8)+\beta_k(x_3,x_6,\frac{\pi}{2x_5})x_7\,U_{k-1}(x_8)\right\} \tag{17-i}$$

where ε_1 and ε_2 are small parameters for controlling the rate of transient decay, sgn(·) denotes the sigmum function,

$$\begin{aligned} \operatorname{sgn}(x) &= 1,\ x > 0 \\ &= 0,\ x = 0 \\ &= -1,\ x < 0 \end{aligned}$$

r(·) denotes the unit ramp function

$$\begin{aligned} r(x) &= x,\ x \geq 0 \\ &= 0,\ x < 0 \end{aligned}$$

and where $T_k(\cdot)$ and $U_k(\cdot)$ denote the <u>kth</u> Chebyshev polynomials of the first and second kind, respectively [21].

<u>Theorem 2</u>

For each sinusoidal input $u(t)=A_{o_j} + A_j \cos \omega_j t$, with dc component A_{o_j}, amplitude A_j, and frequency ω_j, $j=1,2,\ldots,M$, the solution $y(t)$ of the <u>sinusoidal-input canonical model</u> under the initial state[3]

$$[x_1(0),\ x_2(0),\ldots,x_8(0)]^T = [\varepsilon_1,0,0,0,\rho,0,0,1]^T$$

tends to a steady-state response which differs from the <u>measured</u> response $y_s(t;\ A_{o_j},A_j,\ \omega_j)$ by at most a phase shift θ as $t\to\infty$ and $\varepsilon_2 \to 0$[3].

<u>Proof</u>. Let us first observe that (17-a) and (17-b) are uncoupled from the remaining equations and can therefore be solved separately; namely,

$$x_1(t) = t + \varepsilon_1\, e^{-t/\varepsilon_1} \tag{18-a}$$

3 The initial condition ε_1 for x_1 is assigned equal to the parameter defined in (17-b), whereas the initial condition ρ for x_5 can be assigned any positive constant. Observe that since our objective is to model the steady state behavior of the system, the presence of a phase-shift θ is of no concern to us.

$$x_2(t) = 1 - e^{-t/\varepsilon_1} \tag{18-b}$$

Substituting (18-a) and (18-b) into (17-c), we obtain a linear time-varying equation

$$\dot{x}_3 = -\left\{\frac{1 - e^{-t/\varepsilon_1}}{t + \varepsilon_1 e^{-t/\varepsilon_1}}\right\} x_3 + \left\{\frac{1}{t + \varepsilon_1 e^{-t/\varepsilon_1}} \; u(t)\right\} \tag{19}$$

whose zero-state solution is given by

$$x_3(t) = \left\{\frac{1}{t + \varepsilon_1 e^{-t/\varepsilon_1}}\right\}\left\{\int_0^t u(\tau)d\tau\right\} \tag{20}$$

Now observe that (20) implies

$$x_3(t) \longrightarrow \overline{u(t)} \quad , \text{ as } t \longrightarrow \infty \tag{21}$$

where $\overline{u(t)}$ denotes the average value[4] of $u(t)$. Now for $u(t) = A_{o_j} + A_j \cos \omega_j t$, (21) implies

$$x_3(t) \longrightarrow A_{o_j} \quad , \text{ as } \quad t \longrightarrow \infty \tag{22}$$

and hence

$$u(t) - x_3(t) \longrightarrow A_j \cos \omega_j t, \quad \text{as } t \longrightarrow \infty \tag{23}$$

Substituting (23) into (17-d), we found that the zero-state response $x_4(t)$ tends to a triangular waveform having the same fundamental frequency ω_j and a peak value equal to $\frac{\pi}{2\omega_j}$; i.e.,

$$\max x_4(t) \longrightarrow \frac{\pi}{2\omega_j} \quad , \text{ as } t \longrightarrow \infty \tag{24}$$

Now (17-e) can be recast into the simplified form

$$\dot{x}_5 = f(r(x_4) - x_5) \tag{25}$$

4 Equations (17-a), (17-b), and (17-c) play the role here of a dynamical system which is capable of extracting the average value $\overline{u(t)} = x_3(t)$ of the input signal $u(t)$, as $t \to \infty$. Any other system having a similar capability can therefore be substituted in place of (17-a), (17-b) and (17-c).

where

$$f(z) \triangleq \frac{1}{2\varepsilon_2} \left\{(1 + \varepsilon_2^{\,2})\, z + (1 - \varepsilon_2^{\,2})\, |z|\right\} \tag{26}$$

and $0 < \varepsilon_2 < 1$. Observe that the time function $r(x_4(t))$ represents a rectified waveform having the same peak value $\frac{\pi}{2\omega_j}$ as $x_4(t)$.[5] It follows from Proposition 2 of [17] that

$$x_5(t) \longrightarrow \frac{\pi}{2\omega_j} \quad , \text{ as } t \longrightarrow \infty \text{ and } \varepsilon_2 \longrightarrow 0 \tag{27}$$

Next observe that (17-f) can be recast as

$$\dot{x}_6 = f(r(u - x_3) - x_6) \tag{28}$$

where $f(\cdot)$ is as defined in (26). It follows from (23) and Proposition 2 of [17] that the zero-state response

$$x_6(t) \longrightarrow A_j \quad , \text{ as } t \longrightarrow \infty \text{ and } \varepsilon_2 \longrightarrow 0 \tag{29}$$

Finally, if we substitute the solution $x_5(t)$ from (17-e) into (17-g) and (17-h), we would obtain a pair of linear time-varying equations whose solution under the initial state $x_7(0) = 0$ and $x_8(0) = 1$ is readily seen to be given by

$$x_7(t) = \sin\left\{\int_0^t \frac{\pi}{2x_5(\tau)}\, d\tau\right\} \tag{30a}$$

$$x_8(t) = \cos\left\{\int_0^t \frac{\pi}{2x_5(\tau)}\, d\tau\right\} \tag{30b}$$

Equations (30a) and (30b) can be recast into the following equivalent form:

---

5 Observe that (17-d) and (17-d) play the role here of a dynamical system which is capable of extracting the frequency $\omega_j = \pi/2x_5(t)$ of the input signal $u(t)$, as $t \to \infty$. Any other system having a similar capability can therefore be substituted in place of (17-d) and (17-e). The ramp function $r(\cdot)$ introduced in (25) has the desirable effect of speeding up the transient decay and represents therefore an improvement over the "frequency detector" given in [17]. Observe that since the expression $\pi/2x_5$ occurs as one of the arguments in (17-i), we must assume the initial condition $x_5(o) = \rho > 0$.

$$x_7(t) = \sin\left\{\omega_j t - \theta - \int_{t_1}^{t} [\omega_j - \frac{\pi}{2x_5(\tau)}]\, d\tau\right\} \tag{31-a}$$

$$x_8(t) = \cos\left\{\omega_j t - \theta - \int_{t_1}^{t} [\omega_j - \frac{\pi}{2x_5(\tau)}]\, d\tau\right\} \tag{31-b}$$

where

$$\theta \triangleq \int_0^{t_1} \left\{\omega_j - \frac{\pi}{2x_5(\tau)}\right\} d\tau \tag{32}$$

and where t_1 is chosen sufficiently large so that the integrand in (31-a) and (31-b) are negligible for all $t \geq t_1$. Given any allowable error, such a t_1 can always be found in view of (27). Substituting (27) into (31) and (32), we obtain

$$x_7(t) \longrightarrow \sin\,(\omega_j t - \theta),\text{ as } t \longrightarrow \infty \text{ and } \varepsilon_2 \longrightarrow 0 \tag{33-a}$$

$$x_8(t) \longrightarrow \cos\,(\omega_j t - \theta),\text{ as } t \longrightarrow \infty \text{ and } \varepsilon_2 \longrightarrow 0 \tag{33-b}$$

Substituting (22), (27), (29), (33-a) and (33-b) into (17-i), and making use of the trigonometric identity $T_k(\cos\omega t) = \cos k\,\omega t$ and $\sin \omega t\; U_{k-1}(\cos\omega t) = \sin k\,\omega t$ [21], we obtain

$$y(t)=\alpha_0(A_{o_j},A_j,\omega_j) + \sum_{k=1}^{N}\left\{\alpha_k(A_{o_j},A_j,\omega_j)\cos k(\omega_j t-\theta) + \beta_k(A_{o_j},A_j,\omega_j)\sin k(\omega_j t-\theta)\right\},$$

$$\text{as } t \longrightarrow \infty \text{ and } \varepsilon_2 \longrightarrow 0 \tag{34}$$

With the exception of the phase-shift θ, (34) can now be identified with the measured steady state response $y_s(t;A_{o_j},A_j,\omega_j)$ defined in (15).

IV. MODELING THE STEADY-STATE RESPONSE OF NONLINEAR SYSTEMS DRIVEN BY PERIODIC INPUTS.

Let $y_s(t;\, A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j})$, $j = 1,2,\ldots,M$ denote a family of "M" measured steady-state response of a nonlinear device or system to a family of periodic inputs $u(t) \in \mathcal{U}_3$ characterized by $(2n+1)$

Fourier coefficients $A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}$ and a fixed frequency ω. Let the corresponding steady-state response be represented by a truncated Fourier series with N frequency components; namely,

$$y_s(t; A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}) = a_o(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j})$$
$$+ \sum_{k=1}^{N} \left\{ a_k(a_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}) \cos k\,\omega t + b_k(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}) \sin k\,\omega t \right\}, \quad j = 1,2,\ldots,M. \tag{35}$$

where the (2N + 1) "output" Fourier coefficients $\{A_o, A_1, \ldots, A_N, B_1, B_2, \ldots, B_N\}$ are identified by the corresponding "input" Fourier coefficients $\{A_{o_j}, A_{1_j}, \ldots, A_{N_j}, B_{1_j}, \ldots, B_{N_j}\}$ of the periodic input signal

$$u(t) = A_{o_j} + \sum_{k=1}^{n} \left\{ A_{k_j} \cos k\,\omega t + B_{k_j} \sin k\,\omega t \right\}, \quad j = 1,2,\ldots,M \tag{36}$$

Our problem in this section is to synthesize a canonical model which is capable of simulating the "M" measured steady-state responses to within any desired degree of accuracy. Just as in the sinusoidal input case, our first step is to approximate the (2N+1) "output" Fourier coefficients as (2N+1) functions $\alpha_0(\cdot), \alpha_1(\cdot), \ldots, \alpha_N(\cdot), \beta_1(\cdot), \ldots, \beta_N(\cdot)$ of the (2n+1) "input" Fourier coefficients; namely,

$$\begin{aligned}
\alpha_0(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}) &\simeq a_0(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}), \quad j=1,2,\ldots,M \\
\alpha_1(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}) &\simeq a_1(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}), \quad j=1,2,\ldots,M \\
&\vdots \\
\alpha_N(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}) &\simeq a_N(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}), \quad j=1,2,\ldots,M \\
\beta_1(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}) &\simeq b_1(A_{1_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}), \quad j=1,2,\ldots,M \\
&\vdots
\end{aligned} \tag{37}$$

$$\beta_N(a_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{j_n}) \triangleq b_N(A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}),\ j=1,2,\ldots,M$$

Again, these "functions of several variables" may be represented by various basis functions [18-20] and we will simply assume here that this has been done. The following canonical model will therefore be formulated in terms of these "output" Fourier-coefficient functions:

Periodic Input Canonical Model

State Equation:

$$\dot{x}_1 = x_2 \tag{38-a}$$

$$\dot{x}_2 = \frac{1}{\varepsilon}(1 - x_2), \quad 0 < \varepsilon << 1 \tag{38-b}$$

$$\dot{x}_3 = \omega x_4 \tag{38-c}$$

$$\dot{x}_4 = -\omega x_3 \tag{38-d}$$

$$\dot{p}_0 = -(x_2/x_1)p_0 + (1/x_1)u \tag{38-e}$$

$$\dot{p}_1 = -(x_2/x_1)p_1 + (2x_4/x_1)u \tag{38-f}$$

$$\dot{p}_2 = -(x_2/x_1)p_2 + (2T_2(x_4)/x_1)u \tag{38-g}$$

$$\vdots \qquad\qquad \vdots$$

$$\dot{p}_N = -(x_2/x_1)p_N + (2T_N(x_4)/x_1)j \tag{38-h}$$

$$\dot{q}_1 = 1(x_2/x_1)\ q_1 + (2\ x_3/x_1)u \tag{38-i}$$

$$\dot{q}_2 = -(x_2/x_1)q_2 + (2\ x_3\ \ U_1(x_4)/x_1)u \tag{38-j}$$

$$\vdots \qquad\qquad \vdots$$

$$\dot{q}_N = 1(x_2/x_1)q_N + (2x_3\ \ U_{N-1}(x_4)/x_1)u \tag{38-k}$$

Output Equation:

$$\begin{aligned} y = {} & \alpha_0(p_0, p_1, \ldots, p_N, q_1, \ldots, q_N) \\ & + \sum_{k=1}^{N} \Big\{ \alpha_k(p_0, p_1, \ldots, p_N, q_1, \ldots, q_N)\ T_k(x_4) \\ & + \beta_k(p_0, p_1, \ldots, p_N, q_1, \ldots, q_N)\ x_3\ U_{k-1}(x_4) \Big\} \end{aligned} \tag{38-1}$$

where ω is the fixed frequency of the periodic input signals, ε is a small parameter for controlling the rate of transient decays, and $T_k(\cdot)$ and $U_{k-1}(\cdot)$ denote as before the kth Chebyshev polynomials of the first and second kind, respectively.

Theorem 3

For each periodic input $u(t) = a_{o_j} + \sum_{k=1}^{n} \left\{ a_{k_j} \cos k\,\omega t + b_{k_j} \sin k\,\omega t \right\}$, with "input" Fourier coefficients $\left\{ a_{o_j}, a_{1_j}, \ldots, a_{n_j}, b_{1_j}, \ldots, b_{n_j} \right\}$, $j=1,2,\ldots,M$, the solution $y(t)$ of the periodic input canonical model under the initial state[6]

$$\underset{\sim}{x}(0) = [\,\varepsilon\,,\ 0,\ 0,\ 1]^T, \quad \underset{\sim}{p}(0) = \underset{\sim}{0}\ , \quad \underset{\sim}{q}(0) = \underset{\sim}{0}$$

tends to the measured steady-state response

$$y_s(t;\, A_{o_j}, A_{1_j}, \ldots, A_{n_j}, B_{1_j}, \ldots, B_{n_j}), \text{ as } t \longrightarrow \infty\ .$$

Proof. Let us first observe that the solution of (38-a) and (38-b) have already been found earlier in (18-a) and (18-b). The solutions of (38-c) and (38-d) with initial state $x_3(0) = 0$ and $x_4(0) = 1$ are given respectively by

$$x_3(t) = \sin\ \omega\, t \tag{39-a}$$

$$x_4(t) = \cos\ \omega\, t \tag{39-b}$$

Substituting (18-a) and (18-b) for x_1 and x_2 in (38-c), we obtain the same "average-value detection" dynamical system as given in (17-c) and hence

$$p_0(t) \longrightarrow A_{o_j}\ , \quad \text{as } t \longrightarrow \infty \tag{40}$$

Now observe that each of the remaining equations (38-c) and (38-k) are also

6 The initial condition ε for x_1 is assigned equal to the parameter defined in (38-a). Observe that unlike in Theorem 2, no phase-shift in the steady-state response is involved here because the frequency ω is fixed in this model. The phase shift occuring in Theorem 2 comes directly from the "frequency detection" state equations (17-d) and (17-e).

identical in form as (17-c) provided we replace $u(t)$ by a new input

$$u_p(t) \triangleq 2T_i(x_4(t))u(t) \quad , i = 1,2,\ldots,N \qquad (41\text{-a})$$

for (38-e) - (38-h), where $T_1(x_4) \triangleq x_4$,

and another input

$$u_q(t) \triangleq 2x_3(t) \; U_{i-1}(x_4(t))u(t) \quad , i = 1,2,\ldots,N \qquad (41\text{-b})$$

for (38-i) - (38-k), where $U_0(x_4) \triangleq 1$.

Substituting (36), (39-a), and (39-b) for $u(t)$, $x_3(t)$ and $x_4(t)$ in (41-a) and (41-b), respectively, we obtain

$$u_p(t) = 2(\cos i \omega t) \left\{ A_{o_j} + \sum_{k=1}^{n} [A_{k_j} \cos k \omega t + A_{k_j} \sin k \omega t] \right\}$$

$$= 2A_{o_j} \cos i \omega t + \sum_{k=1}^{n} \left\{ A_{k_j} [\cos(k+i)\omega t + \cos(k-i)\omega t] + B_{k_j} [\sin(k+i)\omega t + \sin(k-i)\omega t] \right\} \qquad (42\text{-a})$$

$$u_q(t) = 2(\sin i \omega t) \left\{ A_{o_j} + \sum_{k=1}^{n} [A_{k_j} \cos k \omega t + B_{k_j} \sin k \omega t] \right\}$$

$$= 2A_{o_j} \sin i \omega t + \sum_{k=1}^{n} \left\{ A_{k_j} [\sin(k+i)\omega t - \sin(k-i)\omega t] + B_{k_j} [\cos(k-i)\omega t - \cos(k+i)\omega t] \right\}$$

(42-b)

Taking the time average of $u_p(t)$ and $u_q(t)$, we obtain

$$u_p(t) = A_{i_j} \quad , i = 1,2,\ldots,N \qquad (43\text{-a})$$

$$u_q(t) = B_{i_j} \quad , i = 1,2,\ldots,N \qquad (43\text{-b})$$

It follows that each of the average-value detection dynamical systems given by (38-f) - (38-k) settles to a constant equal to the corresponding Fourier coefficients as $t \longrightarrow \infty$; namely;

$$p_i(t) \longrightarrow A_{i_j} \text{, as } t \longrightarrow \infty, \; i = 1,2,\ldots,N \qquad (44\text{-a})$$

$$q_i(t) \longrightarrow B_{i_j} \text{, as } t \longrightarrow \infty, \; i = 1,2,\ldots,N \qquad (44\text{-b})$$

Substituting (39), (40) and (44) into the output equation (38-1), we obtain

$$y(t) = \alpha_0(A_{o_j}, A_{1_j}, \ldots, A_{N_j}, B_{1_j}, \ldots, B_{N_j})$$

$$+ \sum_{k=1}^{N} \Big\{ \alpha_k(A_{o_j}, A_{1_j}, \ldots, A_{N_j}, B_{1_j}, \ldots, B_{N_j}) \cos k\,\omega t$$

$$+ \beta_K(A_{o_j}, A_{1_j}, \ldots, A_{N_j}, B_{1_j}, \ldots, B_{N_j}) \sin k\,\omega t \Big\} \quad \text{as } t \longrightarrow \infty \tag{45}$$

The <u>periodic-input</u> canonical model can be represented in a block diagram form consisting of a memory and a memoryless subsystem as shown in Fig.3(a). The memory subsystem is characterized by state equations (38-a) - (38-b) and can be realized by the circuit diagram shown in Fig.3(b), where the boxes denoted by $2T_i(\cdot)$ and $2\,U_i(\cdot)$ are single-input single-output memoryless systems characterized by appropriate Chebyshev nonlinear transfer functions. The box in the lower left-hand corner is a sinusoidal oscillator for simulating (38-c) and (38-d).

To illustrate the application of the <u>periodic-input canonical model</u>, we will present next an example using a hypothetical system in order to avoid the time-consuming process of taking actual measurements. In other words, our "measured" data will be generated by simply solving the system equations using a computer.

<u>Example</u>.

Consider a 4<u>th</u> order dynamical system characterized as follows:

<u>State Equation</u>:

$$\dot{x}_1 = -2x_1 + 2x_2\,u \tag{46-a}$$

$$\dot{x}_2 = -x_2 + u \tag{46-b}$$

$$\dot{x}_3 = -4x_3 + 2x_4\,u^2 \tag{46-c}$$

$$\dot{x}_4 = -2x_4 + u^2 \qquad (46\text{-d})$$

Output Equation:

$$y = x_1 + x_2^2 + x_3 + x_4^2 \qquad (46\text{-e})$$

For our input testing signals, we choose

$$u(t) = c_1 \sin t + c_3 \sin 3\, t \qquad (47)$$

where c_1 and c_3 each ranges over the integers $1,2,\ldots,5$. Using our notation in (36), we have $M=25$, $A_{k_j} = 0$ and $B_{k_j} = 0$ for all k except when $k = 1$ or 3; i.e., $(B_{1_j}, B_{3_j}) = (I,J)$, $I,J = 1,2,\ldots,5$. The corresponding 25 steady-state responses as simulated by the computer (with initial state $\underset{\sim}{x}(o) = \underset{\sim}{0}$) are found to be relatively smooth and can be represented in a truncated Fourier series with $N = 12$; i.e.,

$$y_s(t;B_{1_j},B_{3_j}) = a_0(B_{1_j},B_{3_j}) + \sum_{k=1}^{12}\left\{a_{k_j}(B_{1_j},B_{3_j})\cos k\,\omega t + b_{k_j}(B_{1_j},B_{3_j})\sin k\,\omega t\right\}$$

$$= a_0(I,J) + \sum_{k=1}^{12}\left\{a_{k_j}(I,J)\ \mathrm{Cos}\ k\,\omega t + b_{k_j}(I,J)\ \sin k\,\omega t\right\}, \qquad (48)$$

where $I,J = 1,2,\ldots,5$.

The 25 sets of "output" Fourier coefficients are tabulated in Table 1 where each block of data is arranged as follows:

$a_0\ (I,J)$	
$a_1\ (I,J)$	$b_1(I,J)$
$\vdots$	$\vdots$
$a_{12}\ (I,J)$	$b_{12}\ (I,J)$

Each Fourier coefficient is a function of two variables, namely B_{1_j} and B_{3_j}. To approximate these functions, we have chosen the section-wise piecewise-linear representation [20]. This representation is chosen because, unlike other methods of representation - - such as polynomial or splines - -, the coefficients associated with this new representation can be easily computed via explicit formulas requiring no derivative information. Using this representation, the output Fourier coefficient functions $\alpha_0(q_1,q_3)$, $\alpha_k(q_1,q_3)$, and $\beta_k(q_1,q_3)$ assume the following explicit section-wise piecewise-linear form:

$$\alpha_0(q_1,q_3) = \sum_{i=1}^{5} \sum_{j=1}^{5} \gamma_0(i,j) \; \phi_j(q_1) \, \phi_i \, (q_3) \tag{49-a}$$

$$\alpha_k(q_1,q_3) = \sum_{i=1}^{5} \sum_{j=1}^{5} \gamma_k(i,j) \; \phi_j(q_1) \, \phi_i \, (q_3), \; k=1,2,\ldots,12 \tag{49-b}$$

$$\beta_k(q_1,q_3) = \sum_{i=1}^{5} \sum_{j=1}^{5} \delta_k(i,j) \; \phi_j \, (q_1) \, \phi_i \, (q_3), \; k=1,2,\ldots,12 \tag{49-c}$$

where $\phi_1(x) \triangleq 1$, $\phi_2(x) \triangleq x$, $\phi_3(x) \triangleq |x-2|$, $\phi_4(x) \triangleq |x-3|$, and $\phi_5(x) \triangleq |x-4|$. Observe that each Fourier coefficient function in (49) is in turn characterized by 25 coefficients denoted by $\gamma_0(i,j)$ for (49-a), $\gamma_k(i,j)$ for (49-b), and $\delta_k(i,j)$ for (49-c). These coefficients have been computed using the explicit formulas given in [20] and are tabulated in Table 2. The first block $k = 0$ gives the coefficients $\gamma_0(i,j)$ as i and j each ranges over $1,2,\ldots,5$. Hence $\gamma_0(i,j)$ is located at the ith row and jth column in the $k = 0$ block. Each of the remaining 12 blocks is divided into two parts, the left side gives the coefficients $\gamma_k(i,j)$ while the right side gives the coefficients $\delta_k(i,j)$.

The periodic-input canonical model for this hypothetical example can now be expressed in terms of the output Fourier coefficient functions given in (49) as follow:

<u>State Equation</u>:

$$\dot{x}_1 = x_2 \tag{50-a}$$

$$\dot{x}_2 = \frac{1}{\varepsilon}(1-x_2) \tag{50-b}$$

$$\dot{x}_3 = x_4 \tag{50-c}$$

$$\dot{x}_4 = -x_3 \tag{50-d}$$

$$\dot{q}_1 = -(x_2/x_1)\, q_1 + (2x_3/x_1)u \tag{50-e}$$

$$\dot{q}_3 = -(x_2/x_1)q_3 + (2x_3\; U_2(x_4)/x_1)u \tag{50-f}$$

<u>Output Equation</u>:

$$y = \alpha_0(q_1,q_3) + \sum_{k=1}^{12}\left\{\alpha_k(q_1,q_3)\; T_k(x_4) + \beta_k(q_1,q_3)\; x_3\, U_{k-1}(x_4)\right\} \tag{50-g}$$

The preceding model guarantees that the steady-state response y(t) will tend to the "computer simulated" <u>measured</u> response so long as the input u(t) coincides with one of the 25 testing signals. By continuity argument, however, we can expect the steady state response to other inputs u(t) should also be close to the measured response so long as the Fourier coefficients of u(t) do not differ significantly from those characterizing the 25 testing signals. To verify this prediction, we simulated (50) on a computer with the initial state

$$[x_1(0),\ x_2(0),\ x_3(0),\ x_4(0),\ q_1(0),\ q_3(0)]^T = [\varepsilon,0,0,1,0,0]^T$$

for the two input signals $u_1(t) = 3 \sin t + 2 \sin 3t$ and $u_2(t) = 1.5 \sin t + 3.5 \sin 3t$, respectively. The solutions in both cases settled quickly into the steady state and the corresponding waveforms recorded during the 10<u>th</u> cycle are shown as <u>dotted</u> curves in Figs.4(a) and (b), respectively. The corresponding steady-state response simulated from the actual system given by (4b) is also shown (solid curves) in Fig.4 for comparison purposes. Observe that the agreement is excellent in Fig.4(a) since the input is a member of the original testing signal $(B_1,B_3) = (3,2)$. On the other hand, the input signal $(B_1,B_3) = (1.5, 3.5)$ used in Fig.4(b) is not one of the original testing signals but the predicted response remains close to the actual response since the Fourier coefficients (1.5, 3.5) of the input is still relatively close to the four coefficients (1.0, 3.0), (1.0, 4.0), (2.0, 3.0), and (2.0, 4.0) belonging to the original set of testing signals.

V. CONCLUDING REMARKS

In contrast to the sinusoidal input in Section III where we vary both the amplitude and frequency, only the Fourier coefficients of the periodic input waveform are varied in Section IV; i.e., only the shape of the input waveform is changed while the frequency ω remains fixed. Our main reason for considering non-sinusoidal periodic testing signals is that there exist many low-impedance devices wherein the input voltage source cannot be connected directly across the device without destroying or damaging it. For example, the operation of most arc-discharge devices, such as fluorescent lamps, requires that a ballast be connected in series with any input voltage source [22]. On such occasions, even if the input voltage waveform is sinusoidal, the associated voltage waveform across the device being modeled is still far from being sinusoidal, and in fact will change its shape as we vary either the amplitude or the frequency of the input sinusoidal voltage source. Hence even though the input waveform is sinusoidal, the actual input across the device is not.

To simplify our problem, we have assumed apriori in Section IV that the frequency ω of the input waveform is fixed. If we also vary the frequency, then ω must be added as an extra parameter as in Section III.

In this case, it would be necessary to conceive a dynamical system which is capable of detecting the frequency ω of <u>any</u> periodic input $u(t)$ as $t \longrightarrow \infty$. We have not been able to devise such a system without at least restricting $u(t)$ to some more tractable albeit still fairly general sub-classes of periodic signals.

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FIGURE CAPTIONS

Fig. 1. A physical realization of the step-input canonical model 1.

Fig. 2. The predicted potassium conductance (in dotted curves) due to 5 "membrane-potential" step inputs of amplitudes -20, -40, -60, -80, and -100 mV is shown in (a) for n = 2 and in (b) for n = 4. The solid curves in each case give the corresponding "measured" response obtained by simulating the Hodgin-Huxley model.

Fig. 3. The Periodic-Input Canonical Model: (a) block diagram (b) circuit realization of the memory subsystem. The circle $\otimes$ denotes multiplication and the box labelled $\mathcal{A}$ denotes a time-averaging system.

Fig. 4. The steady-state response predicted by the model (dotted curve) and the corresponding response (solid curve) simulated from the actual system due to two periodic inputs: (a) $u(t) = 3 \sin t + 2 \sin 3t$ and (b) $u(t) = 1.5 \sin t + 3.5 \sin 3t$.

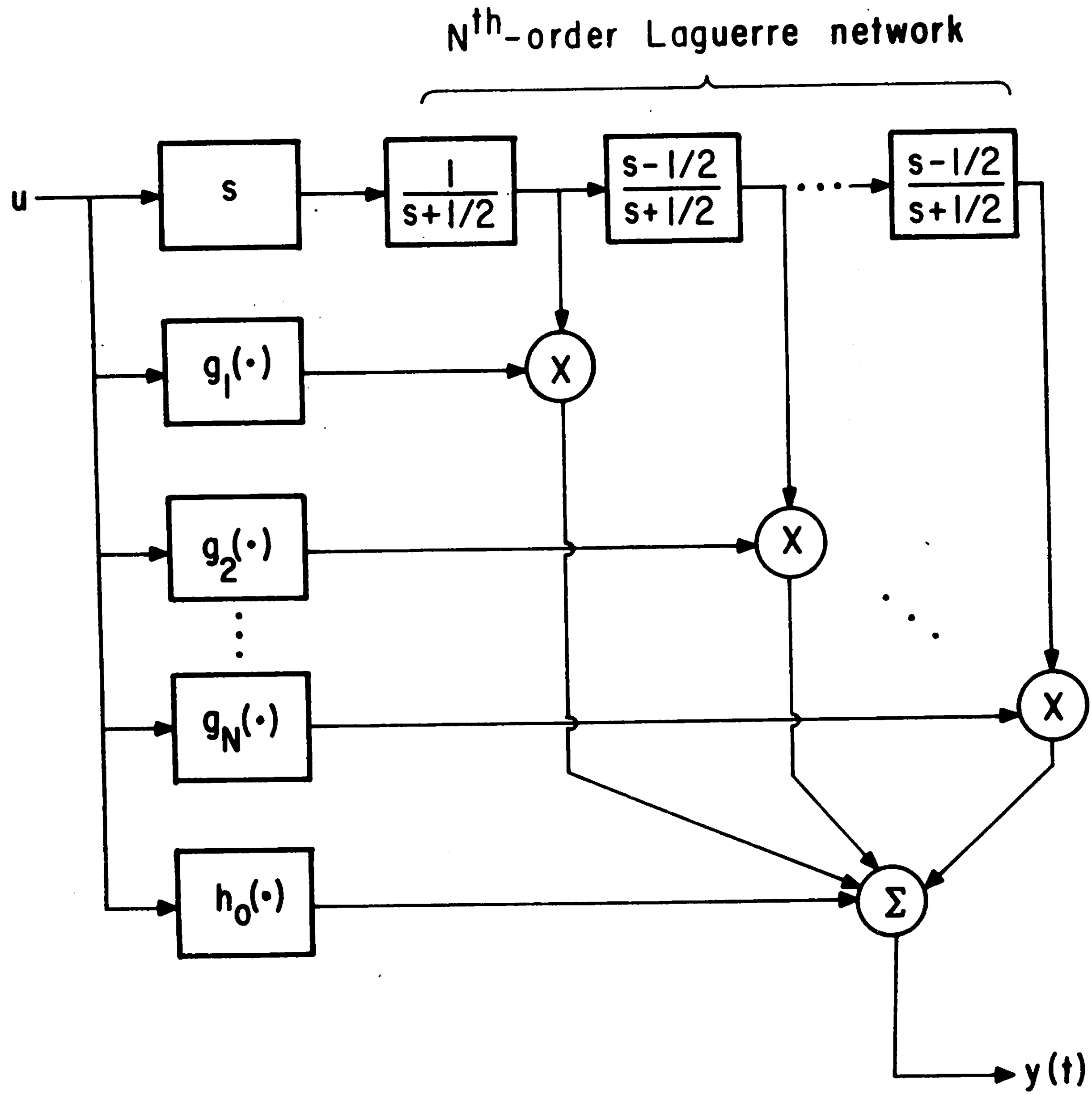
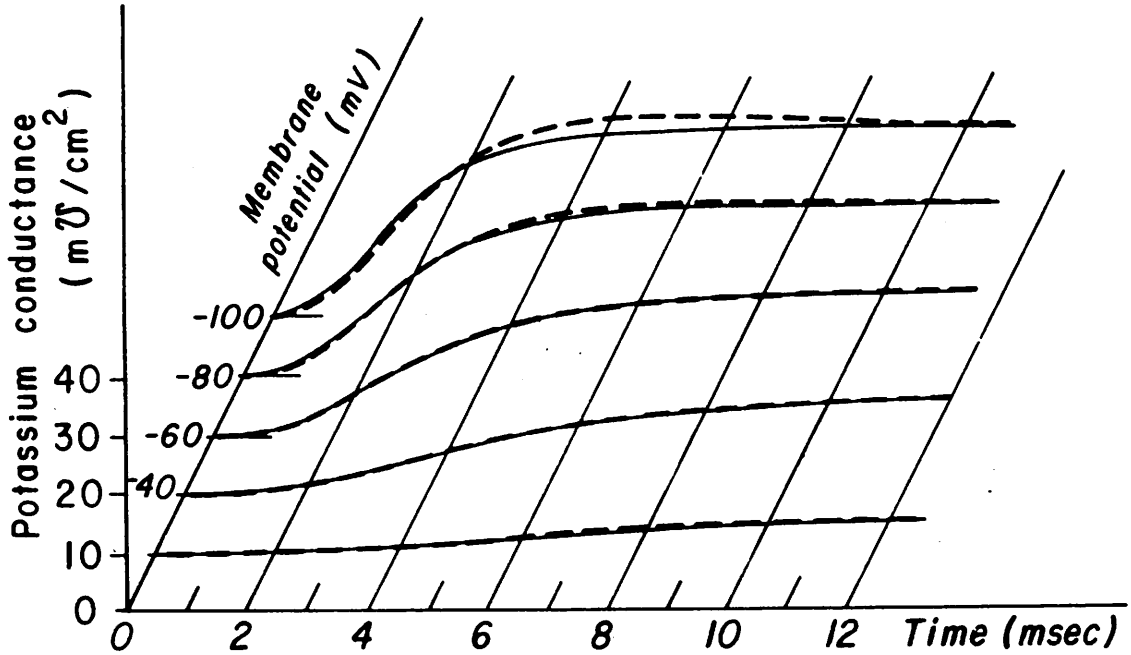

N th -order Laguerre network
u
s
1 / s+1/2
s-1/2 / s+1/2
s-1/2 / s+1/2
g1(·)
g2(·)
gN(·)
h0(·)
X
X
X
Σ
y(t)

Fig. 1.


Potassium conductance (m℧/cm^2)
Membrane potential (mV)
-100
-80
-60
-40
0
10
20
30
40
0
2
4
6
8
10
12
Time (msec)
(a) N = 2

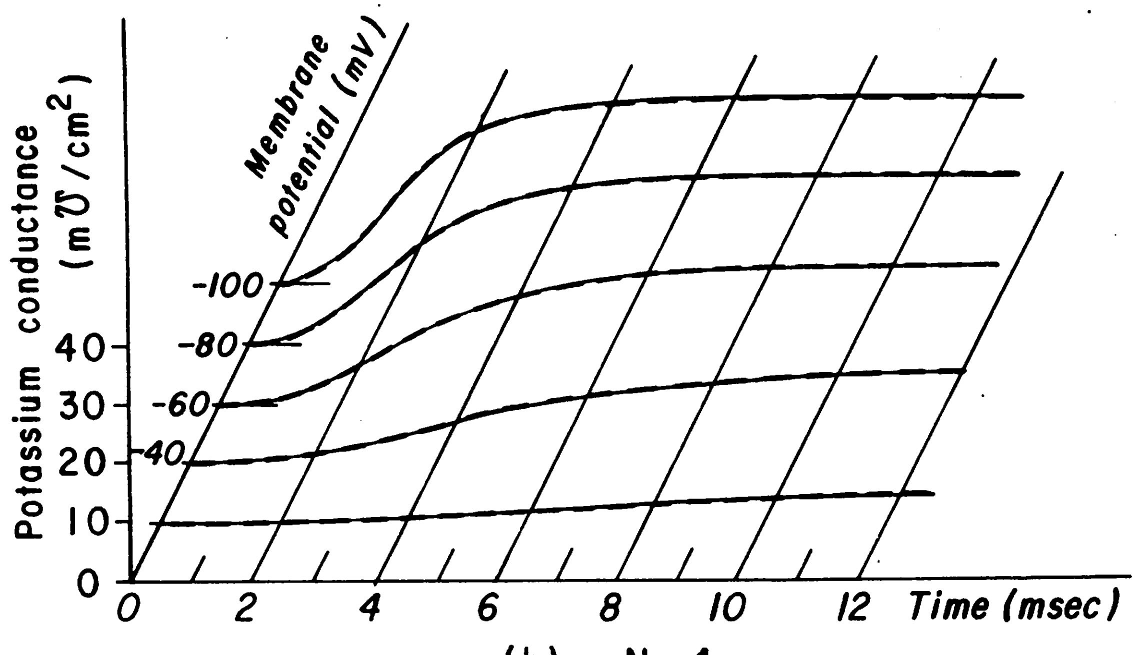
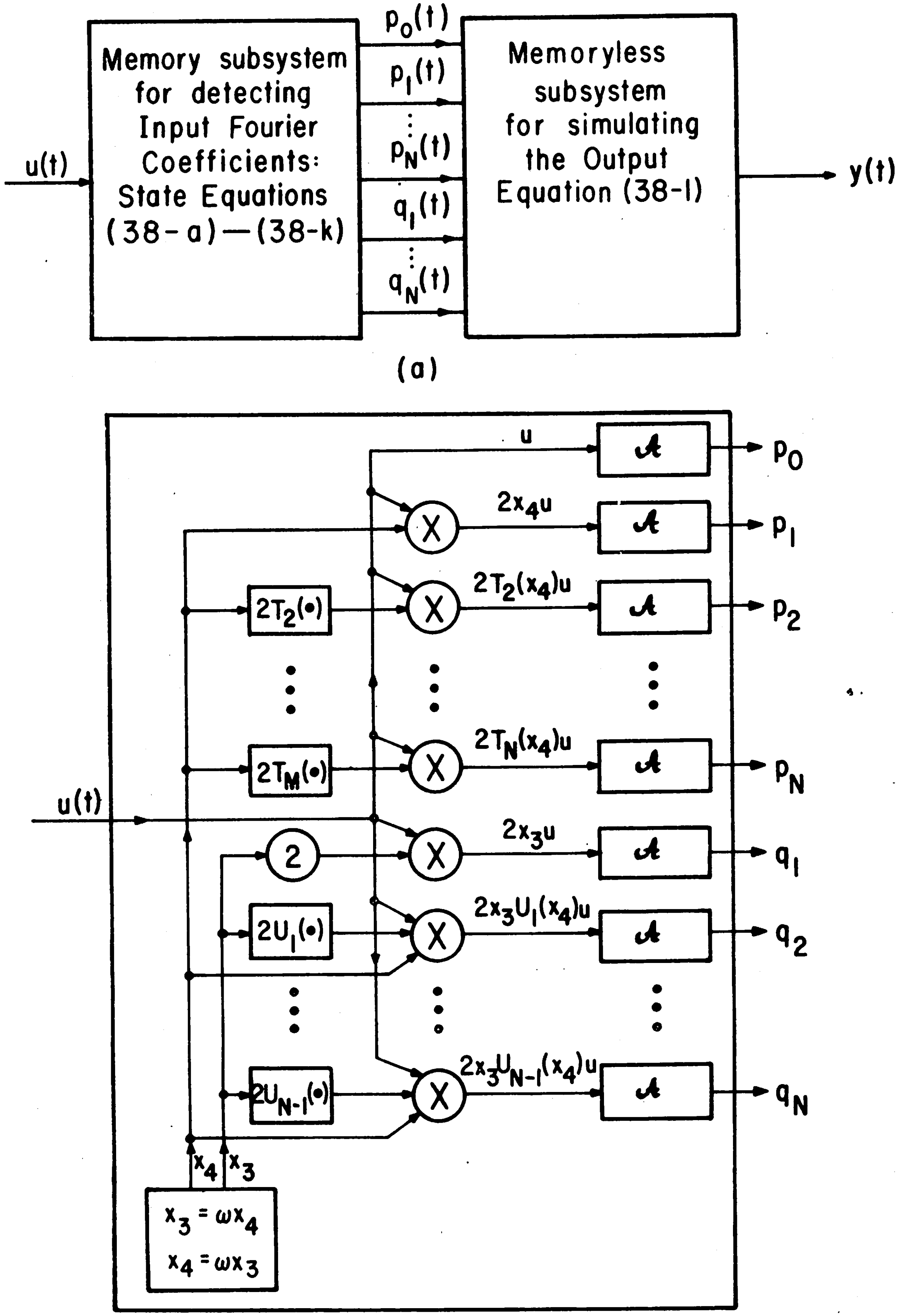

Potassium conductance (m℧/cm^2)
Membrane potential (mV)
-100
-80
-60
-40
0
10
20
30
40
0
2
4
6
8
10
12
Time (msec)
(b) N = 4

Fig. 2.


u(t)
Memory subsystem for detecting Input Fourier Coefficients: State Equations (38-a)—(38-k)
p0(t)
p1(t)
pN(t)
q1(t)
qN(t)
Memoryless subsystem for simulating the Output Equation (38-l)
y(t)
u
2x4u
2T2(x4)u
2TN(x4)u
2x3u
2x3U1(x4)u
2x3UN-1(x4)u
2T2(•)
2TM(•)
2
2U1(•)
2UN-1(•)
X
A
p0
p1
p2
pN
q1
q2
qN
x4
x3
x3 = ωx4
x4 = ωx3

(a)

(b)

Fig. 3.

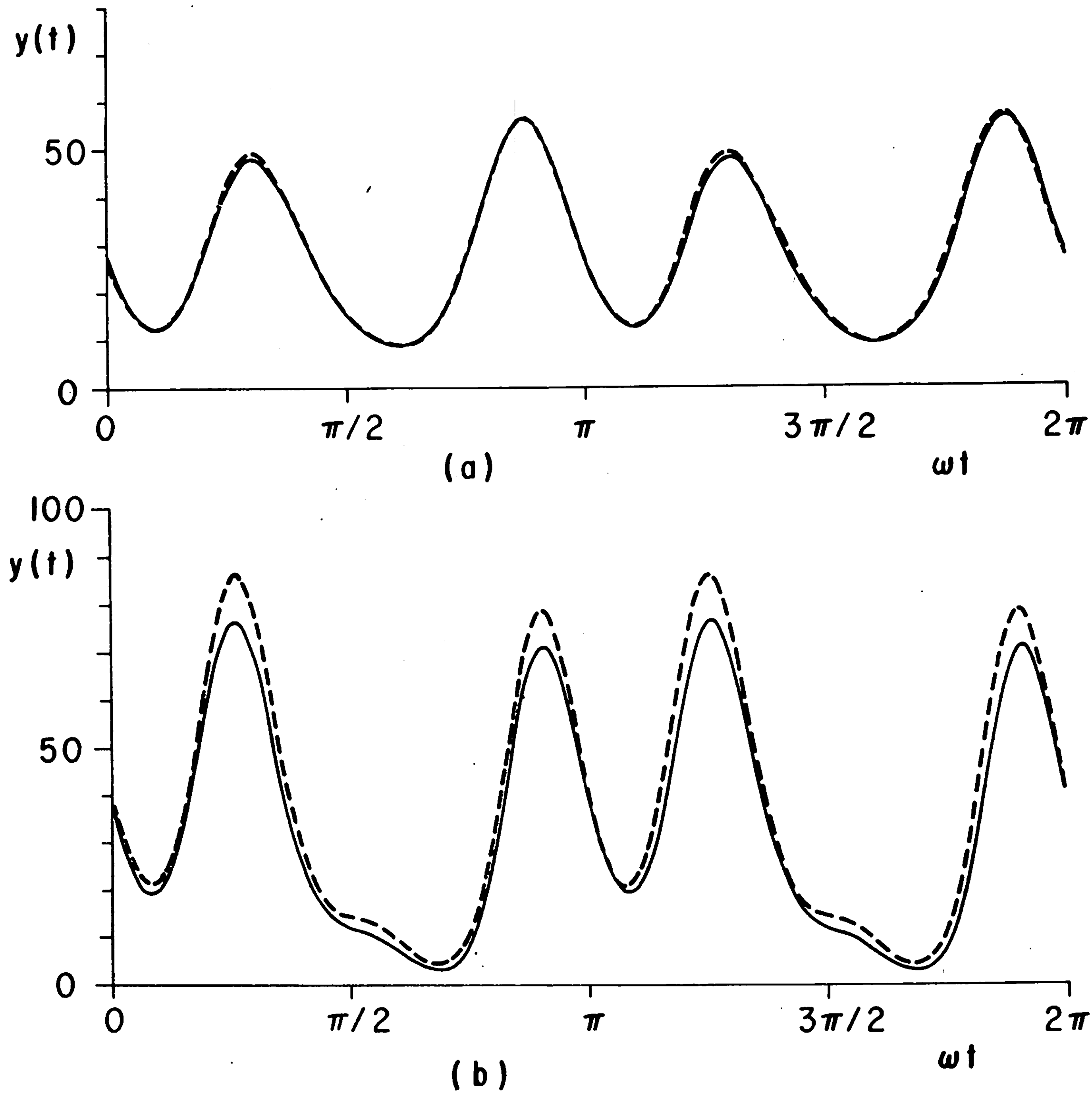


Fig. 4.

TABLE 1. Output Fourier Coefficients for the Hypothetical Example.

	J=1		J=2		J=3		J=4		J=5	
I = 1	0.12E 01	0.0	0.46E 01	0.0	0.16E 02	0.0	0.42E 02	0.0	0.95E 02	0.0
	0.95E-02	-0.20E-02	0.26E-01	0.54E-03	0.56E-01	0.65E-02	0.14E 00	0.19E-01	0.16E 00	0.42E-02
	0.61E 00	-0.71E-01	0.25E 01	0.17E 01	0.72E 01	0.61E 01	0.16E 02	0.15E 02	0.32E 02	0.29E 02
	-0.36E-02	-0.13E-01	-0.35E-01	-0.72E-01	-0.13E 00	-0.23E 00	-0.34E 00	-0.54E 00	-0.71E 00	-0.11E 01
	-0.16E-01	-0.78E 00	-0.87E 00	-0.27E 01	-0.35E 01	-0.71E 01	-0.87E 01	-0.16E 02	-0.18E 02	-0.29E 02
	0.12E-02	0.98E-02	0.11E-01	-0.49E-02	0.19E-01	-0.88E-01	-0.30E-02	-0.33E 00	-0.10E 00	-0.85E 00
	0.60E-01	-0.27E 00	0.17E-01	-0.22E 01	-0.97E 00	-0.84E 01	-0.43E 01	-0.23E 02	-0.12E 02	-0.54E 02
	-0.24E-02	0.16E-01	0.15E-02	0.80E-01	0.40E-01	0.28E 00	0.16E 00	0.74E 00	0.41E 00	0.17E 01
	-0.19E-01	0.24E-01	0.32E-01	-0.88E-01	0.32E 00	-0.58E 00	0.10E 01	-0.17E 01	0.22E 01	-0.36E 01
	-0.55E-03	0.77E-02	-0.83E-02	0.48E-01	-0.29E-01	0.18E 00	-0.65E-01	0.47E 00	-0.12E 00	0.10E 01
	-0.25E-01	0.30E-01	-0.21E 00	0.22E 00	-0.70E 00	0.76E 00	-0.16E 01	0.18E 01	-0.32E 01	0.38E 01
	0.15E-02	0.39E-01	0.83E-02	0.18E-01	0.23E-01	0.68E-01	0.46E-01	0.20E 00	0.66E-01	0.48E 00
	-0.40E-02	0.72E-02	-0.75E-01	0.73E-01	-0.39E 00	0.34E 00	-0.12E 01	0.11E 01	-0.31E 01	0.26E 01
I = 2	0.54E 01	0.0	0.12E 02	0.0	0.28E 02	0.0	0.63E 02	0.0	0.13E 03	0.0
	0.97E-02	-0.15E-01	0.40E-01	-0.86E-02	0.13E 00	0.85E-02	0.21E 00	0.45E-01	0.33E 00	0.80E-01
	0.88E 00	-0.16E 01	0.49E 01	0.25E 01	0.14E 02	0.11E 02	0.30E 02	0.28E 02	0.58E 02	0.55E 02
	-0.18E-01	-0.27E-01	-0.90E-01	-0.17E 00	-0.28E 00	-0.46E 00	-0.65E 00	-0.10E 01	-0.13E 01	-0.19E 01
	-0.57E 00	-0.28E 01	-0.27E 01	-0.78E 01	-0.83E 01	-0.17E 02	-0.19E 02	-0.34E 02	-0.37E 02	-0.60E 02
	0.70E-02	0.56E-01	0.33E-01	0.60E-01	0.86E-01	-0.58E-02	0.15E 00	-0.21E 00	0.18E 00	-0.68E 00
	-0.34E-01	-0.40E 00	0.39E-02	-0.38E 01	-0.72E 00	-0.13E 02	-0.36E 01	-0.32E 02	-0.11E 02	-0.68E 02
	-0.71E-04	0.45E-01	-0.33E-02	0.18E 00	0.28E-01	0.50E 00	0.15E 00	0.11E 01	0.42E 00	0.23E 01
	-0.12E 00	0.19E 00	-0.29E 00	0.34E 00	-0.17E 00	-0.54E-01	0.56E 00	-0.15E 01	0.23E 01	-0.47E 01
	0.58E-02	0.18E-01	0.95E-03	0.83E-01	-0.30E-01	0.28E 00	-0.10E 00	0.69E 00	-0.22E 00	0.15E 01
	-0.46E-01	0.64E-01	-0.40E 00	0.44E 00	-0.14E 01	0.15E 01	-0.33E 01	0.35E 01	-0.65E 01	0.72E 01
	0.67E-02	0.12E-01	0.27E-01	0.33E-01	0.72E-01	0.91E-01	0.15E 00	0.23E 00	0.25E 00	0.50E 00
	0.61E-04	0.15E-01	-0.62E-01	0.90E-01	-0.36E 00	0.38E 00	-0.12E 01	0.11E 01	-0.32E 01	0.27E 01
I = 3	0.18E 02	0.0	0.28E 02	0.0	0.53E 02	0.0	0.10E 03	0.0	0.18E 03	0.0
	-0.12E-01	-0.49E-01	0.63E-01	-0.33E-01	0.14E 00	-0.41E-02	0.25E 00	0.44E-01	0.39E 00	0.86E-01
	-0.18E 01	-0.74E 01	0.67E 01	0.12E 01	0.20E 02	0.16E 02	0.44E 02	0.41E 02	0.84E 02	0.81E 02
	-0.34E-01	-0.22E-01	-0.19E 00	-0.32E 00	-0.50E 00	-0.81E 00	-0.10E 01	-0.16E 01	-0.20E 01	-0.30E 01
	-0.22E 01	-0.69E 01	-0.67E 01	-0.18E 02	-0.16E 02	-0.35E 02	-0.33E 02	-0.62E 02	-0.62E 02	-0.10E 03
	0.30E-01	0.18E 00	0.80E-01	0.24E 00	0.18E 00	0.21E 00	0.34E 00	0.38E-01	0.52E 00	-0.43E 00
	-0.37E 00	-0.33E-01	-0.40E 00	-0.53E 01	-0.88E 00	-0.18E 02	-0.34E 01	-0.44E 02	-0.11E 02	-0.89E 02
	0.15E-01	0.94E-01	0.30E-02	0.35E 00	0.16E-01	0.86E 00	0.11E 00	0.18E 01	0.40E 00	0.33E 01
	-0.30E 00	0.49E 00	-0.94E 00	0.13E 01	-0.14E 01	0.17E 01	-0.13E 01	0.82E 00	0.12E 00	-0.23E 01
	0.23E-01	0.38E-01	0.35E-01	0.13E 00	0.14E-01	0.39E 00	-0.73E-01	0.94E 00	-0.22E 00	0.19E 01
	-0.59E-01	0.11E 00	-0.58E 00	0.68E 00	-0.20E 01	0.22E 01	-0.49E 01	0.54E 01	-0.98E 01	0.11E 02
	0.19E-01	0.30E-01	0.58E-01	0.63E-01	0.14E 00	0.14E 00	0.28E 00	0.30E 00	0.49E 00	0.58E 00
	0.10E-01	0.33E-01	-0.40E-01	0.12E 00	-0.31E 00	0.43E 00	-0.12E 01	0.12E 01	-0.32E 01	0.28E 01
I = 4	0.47E 02	0.0	0.61E 02	0.0	0.96E 02	0.0	0.16E 03	0.0	0.26E 03	0.0
	-0.15E 00	-0.15E 00	-0.39E-01	-0.12E 00	0.78E-01	-0.73E-01	0.24E 00	-0.32E-01	0.40E 00	0.37E-01
	-0.13E 02	-0.24E 02	0.42E 01	-0.64E 01	0.26E 02	0.17E 02	0.59E 02	0.54E 02	0.11E 03	0.11E 03
	-0.26E-01	0.66E-01	-0.34E 00	-0.50E 00	-0.81E 00	-0.13E 01	-0.16E 01	-0.25E 01	-0.28E 01	-0.44E 01
	-0.56E 01	-0.13E 02	-0.14E 02	-0.35E 02	-0.29E 02	-0.64E 02	-0.54E 02	-0.11E 03	-0.95E 02	-0.17E 03
	0.87E-01	0.41E 00	0.17E 00	0.63E 00	0.33E 00	0.70E 00	0.59E 00	0.59E 00	0.96E 00	0.28E 00
	-0.11E 01	0.13E 01	-0.15E 01	-0.58E 01	-0.20E 01	-0.24E 02	-0.43E 01	-0.58E 02	-0.11E 02	-0.12E 03
	0.57E-01	0.17E 00	0.39E-01	0.57E 00	0.29E-01	0.14E 01	0.85E-01	0.27E 01	0.29E 00	0.47E 01
	-0.55E 00	0.95E 00	-0.19E 01	0.29E 01	-0.35E 01	0.47E 01	-0.45E 01	0.53E 01	-0.43E 01	0.40E 01
	0.59E-01	0.76E-01	0.10E 00	0.19E 00	0.11E 00	0.51E 00	0.48E-01	0.12E 01	-0.80E-01	0.24E 01
	-0.58E-01	0.17E 00	-0.74E 00	0.94E 00	-0.26E 01	0.30E 01	-0.67E 01	0.73E 01	-0.13E 02	0.15E 02
	0.42E-01	0.66E-01	0.11E 00	0.12E 00	0.24E 00	0.22E 00	0.47E 00	0.41E 00	0.82E 00	0.75E 00
	0.30E-01	0.67E-01	-0.17E-02	0.18E 00	-0.24E 00	0.54E 00	-0.11E 01	0.14E 01	-0.28E 01	0.29E 01
I = 5	0.10E 03	0.0	0.12E 03	0.0	0.17E 03	0.0	0.25E 03	0.0	0.39E 03	0.0
	-0.52E 00	-0.37E 00	-0.29E 00	-0.30E 00	-0.83E-01	-0.22E 00	-0.28E-01	-0.11E 00	0.10E 01	0.38E-01
	-0.41E 02	-0.59E 02	-0.91E 01	-0.28E 02	0.25E 02	0.10E 02	0.73E 02	0.62E 02	0.14E 03	0.13E 03
	0.63E-01	0.35E 00	-0.48E 00	-0.66E 00	-0.12E 01	-0.19E 01	-0.24E 01	-0.36E 01	-0.42E 01	-0.62E 01
	-0.11E 02	-0.20E 02	-0.26E 02	-0.61E 02	-0.49E 02	-0.11E 03	-0.85E 02	-0.17E 03	-0.14E 03	-0.26E 03
	0.20E 00	0.79E 00	0.33E 00	0.13E 01	0.56E 00	0.16E 01	0.98E 00	0.16E 01	0.15E 01	0.13E 01
	-0.24E 01	0.40E 01	-0.36E 01	-0.44E 01	-0.46E 01	-0.28E 02	-0.72E 01	-0.72E 02	-0.14E 02	-0.14E 03
	0.14E 00	0.27E 00	0.13E 00	0.86E 00	0.87E-01	0.20E 01	0.15E 00	0.39E 01	0.28E 00	0.66E 01
	-0.86E 00	0.16E 01	-0.32E 01	0.50E 01	-0.63E 01	0.91E 01	-0.90E 01	0.13E 02	-0.11E 02	0.14E 02
	0.12E 00	0.14E 00	0.21E 00	0.28E 00	0.27E 00	0.68E 00	0.20E 00	0.15E 01	0.16E 00	0.29E 01
	-0.35E-01	0.26E 00	-0.86E 00	0.12E 01	-0.33E 01	0.40E 01	-0.84E 01	0.93E 01	-0.16E 02	0.18E 02
	0.83E-01	0.13E 00	0.18E 00	0.20E 00	0.38E 00	0.34E 00	0.64E 00	0.58E 00	0.12E 01	0.11E 01
	0.66E-01	0.13E 00	0.61E-01	0.27E 00	-0.59E-01	0.77E 00	-0.81E 00	0.17E 01	-0.25E 01	0.34E 01

TABLE 2. Coefficients Characterizing the Output Fourier Coefficient Functions.

k = 0 $\gamma_0(i,j)$	0.12E 01	0.46E 01	0.16E 02	0.42E 02	0.95E 02					
	0.54E 01	0.12E 02	0.28E 02	0.63E 02	0.13E 03					
	0.18E 02	0.28E 02	0.53E 02	0.10E 03	0.18E 03					
	0.47E 02	0.61E 02	0.96E 02	0.16E 03	0.26E 03					
	0.10E 03	0.12E 03	0.17E 03	0.25E 03	0.39E 03					
k = 1 $\gamma_1(i,j),\delta_1(i,j)$	0.43E 01	-0.12E 01	0.54E-01	-0.26E-01	-0.10E 01	0.60E 00	-0.21E 00	0.28E-01	-0.55E-01	-0.90E-01
	-0.98E 00	0.29E 00	-0.11E-01	-0.45E-02	0.25E 00	-0.11E 00	0.43E-01	-0.90E-02	0.64E-02	0.19E-01
	0.17E-01	0.49E-02	-0.26E-01	0.16E-01	-0.70E-02	0.82E-02	-0.46E-04	-0.12E-02	-0.15E-02	-0.71E-03
	0.24E 00	-0.65E-01	0.26E-01	-0.94E-02	-0.52E-01	-0.45E-01	0.14E-01	-0.37E-02	-0.48E-02	0.16E-01
	-0.11E 01	0.28E 00	-0.26E-01	-0.92E-02	0.27E 00	-0.12E-01	0.78E-02	-0.68E-02	0.89E-02	-0.66E-02
k = 2 $\gamma_2(i,j),\delta_2(i,j)$	0.45E 02	-0.33E 02	-0.69E 00	-0.43E 01	-0.88E 01	0.60E 02	-0.39E 02	-0.17E 01	-0.45E 01	-0.11E 02
	-0.12E 02	0.11E 02	0.44E 00	0.14E 01	0.22E 01	-0.14E 02	0.12E 02	0.60E 00	0.12E 01	0.23E 01
	0.33E 01	-0.22E-01	-0.28E 00	-0.30E 00	-0.32E 00	0.19E 01	0.48E 00	-0.17E 00	-0.19E 00	-0.16E 00
	0.21E 01	0.11E 01	-0.28E 00	-0.24E 00	-0.78E-01	0.20E 01	0.12E 01	-0.19E 00	-0.13E 00	-0.22E 00
	0.31E 01	0.18E 01	-0.12E 00	-0.17E 00	-0.46E 00	0.23E 01	0.19E 01	-0.16E 00	-0.46E 00	-0.92E-01
k = 3 $\gamma_3(i,j),\delta_3(i,j)$	-0.32E 00	0.54E 00	0.40E-01	0.30E-01	0.86E-01	-0.66E 00	0.82E 00	0.88E-01	0.63E-01	0.16E 00
	0.30E 00	-0.26E 00	-0.24E-01	-0.25E-01	-0.55E-01	0.45E 00	-0.39E 00	-0.45E-01	-0.47E-01	-0.78E-01
	-0.21E-01	-0.18E-01	0.42E-02	0.39E-02	-0.17E-02	-0.43E-01	-0.21E-01	0.41E-02	0.41E-02	0.17E-02
	0.15E 00	-0.74E-01	0.17E-02	0.10E-01	-0.51E-01	-0.47E-01	-0.40E-01	0.48E-02	0.14E-02	-0.40E-03
	-0.14E 00	-0.27E-01	-0.80E-02	-0.59E-02	0.37E-01	-0.62E-01	-0.66E-01	-0.16E-01	0.63E-02	0.39E-02
k = 4 $\gamma_4(i,j),\delta_4(i,j)$	0.26E 01	0.13E 02	0.36E 00	-0.22E 00	-0.58E 00	-0.31E 01	0.19E 02	-0.23E 00	0.12E 01	0.74E 00
	0.80E 01	-0.77E 01	-0.60E 00	-0.77E 00	-0.10E 01	0.22E 02	-0.15E 02	-0.13E 01	-0.24E 01	-0.29E 01
	0.13E 00	-0.85E 00	0.26E-01	0.13E-01	-0.70E-01	-0.16E 01	-0.53E 00	0.14E 00	0.14E 00	0.15E 00
	0.77E 00	-0.15E 01	-0.60E-02	-0.69E-01	-0.16E 00	-0.52E 00	-0.15E 01	0.15E 00	0.11E 00	-0.11E 00
	-0.12E 01	-0.16E 01	-0.11E 00	0.87E-01	0.28E 00	-0.22E 01	-0.19E 01	0.80E-02	-0.77E-01	0.53E 00
k = 5 $\gamma_5(i,j),\delta_5(i,j)$	0.74E-01	-0.12E 00	-0.11E-01	-0.60E-02	0.88E-01	-0.33E 00	0.31E 00	0.92E-01	-0.75E-01	0.32E 00
	-0.13E 00	0.83E-01	0.69E-02	0.94E-02	-0.73E-02	-0.41E 00	0.77E-01	-0.11E-02	0.62E-01	-0.19E-01
	-0.81E-02	0.13E-01	0.26E-03	0.54E-03	-0.22E-02	0.35E-01	-0.23E-01	-0.67E-02	-0.70E-02	-0.84E-02
	-0.93E-01	0.35E-01	-0.21E-02	0.23E-02	0.14E-01	-0.67E-01	-0.13E-02	-0.47E-02	-0.61E-02	0.24E-02
	0.22E-01	0.50E-02	0.55E-02	0.28E-02	-0.26E-01	0.88E-01	-0.49E-01	-0.13E-01	0.29E-01	-0.75E-01
k = 6 $\gamma_6(i,j),\delta_6(i,j)$	0.17E 02	-0.13E 01	0.14E 00	-0.22E 00	-0.58E 00	0.19E 02	0.16E 02	0.29E 01	0.23E 01	0.24E 01
	-0.33E 01	-0.37E-01	-0.10E 00	-0.94E-01	-0.10E 00	-0.69E 01	-0.57E 01	-0.10E 01	-0.63E 00	-0.42E 00
	-0.60E 00	0.12E 00	0.28E-01	0.19E-01	-0.18E-01	0.56E 00	-0.13E 01	-0.22E 00	-0.21E 00	-0.24E 00
	-0.13E 01	0.93E-01	-0.99E-03	0.12E-01	0.16E-02	-0.12E 01	-0.15E 01	-0.25E 00	-0.35E 00	-0.26E 00
	-0.28E 01	0.12E 00	-0.61E-01	0.24E-01	0.14E 00	-0.52E 01	-0.14E 01	-0.32E 00	-0.88E-01	-0.17E 00
k = 7 $\gamma_7(i,j),\delta_7(i,j)$	-0.76E 00	0.79E-01	0.37E-02	0.12E 00	0.61E-02	-0.74E 00	-0.46E 00	-0.11E 00	-0.15E 00	0.25E-01
	0.17E 00	-0.12E-01	-0.21E-02	-0.27E-01	0.80E-02	-0.83E-02	0.27E 00	0.53E-01	0.62E-01	0.21E-01
	0.44E-01	-0.82E-02	-0.23E-02	-0.20E-02	-0.39E-02	-0.42E-01	0.51E-01	0.85E-02	0.89E-02	0.10E-01
	-0.10E-02	0.10E-01	-0.38E-02	-0.73E-03	0.12E-01	-0.39E-01	0.68E-01	0.10E-01	0.77E-02	0.26E-01
	0.14E 00	-0.12E-01	0.50E-02	-0.26E-01	-0.42E-02	0.25E 00	0.22E-01	0.13E-01	0.26E-01	-0.31E-01
k = 8 $\gamma_8(i,j),\delta_8(i,j)$	-0.47E 01	0.22E 00	0.50E 00	0.68E 00	0.22E 00	0.64E 01	0.34E 00	-0.76E 00	-0.13E 01	-0.63E-01
	0.27E 01	-0.62E 00	-0.27E 00	-0.30E 00	-0.19E 00	-0.41E 01	0.75E 00	0.41E 00	0.55E 00	0.27E 00
	0.58E 00	-0.13E 00	-0.44E-01	-0.51E-01	-0.62E-01	-0.85E 00	0.15E 00	0.67E-01	0.77E-01	0.96E-01
	0.29E 00	0.24E-01	-0.48E-01	-0.19E-01	-0.71E-02	-0.88E 00	0.19E-01	0.73E-01	0.46E-01	0.13E 00
	0.42E 00	0.92E-01	-0.45E-01	-0.11E 00	0.51E-02	-0.17E 00	-0.30E 00	0.70E-01	0.22E 00	-0.16E 00
k = 9 $\gamma_9(i,j),\delta_9(i,j)$	0.31E 00	0.14E-01	-0.26E-01	-0.21E-01	-0.57E-01	-0.49E 00	-0.34E 00	0.45E-02	0.42E-01	-0.60E-01
	-0.15E 00	0.17E-01	0.14E-01	0.15E-01	0.21E-01	0.14E 00	0.13E 00	0.76E-03	-0.34E-02	0.18E-01
	-0.13E-01	-0.18E-02	0.17E-02	0.21E-02	0.83E-03	0.14E-01	0.22E-01	-0.52E-03	-0.26E-03	0.29E-02
	0.43E-01	-0.21E-01	0.20E-04	0.31E-02	-0.12E-01	0.67E-01	0.27E-01	0.31E-02	-0.26E-02	-0.37E-02
	-0.11E 00	0.16E-01	0.53E-02	0.22E-02	0.25E-01	0.37E-01	0.56E-01	-0.21E-02	-0.64E-02	0.20E-01
k = 10 $\gamma_{10}(i,j),\delta_{10}(i,j)$	0.13E 01	0.21E 01	-0.11E 00	-0.14E 00	-0.22E 00	-0.14E 01	-0.24E 01	-0.10E-01	0.60E-01	0.23E 00
	-0.36E 00	-0.80E 00	0.26E-01	0.31E-01	0.66E-01	0.29E 00	0.92E 00	0.12E-01	-0.46E-02	-0.51E-01
	0.32E-01	-0.17E 00	0.69E-03	-0.67E-03	-0.21E-01	-0.12E 00	0.19E 00	0.49E-03	0.86E-02	0.27E-01
	-0.72E-01	-0.23E 00	-0.21E-01	-0.34E-01	0.54E-01	0.22E 00	0.21E 00	0.24E-01	0.52E-02	-0.62E-01
	-0.32E 00	-0.23E 00	0.42E-01	0.62E-01	0.26E-01	0.22E 00	0.32E 00	-0.15E-01	-0.22E-01	-0.21E-01
k = 11 $\gamma_{11}(i,j),\delta_{11}(i,j)$	0.30E 00	-0.18E 00	-0.17E-01	-0.13E-01	-0.31E-01	0.71E-01	-0.14E 00	0.20E-01	-0.41E-01	-0.10E 00
	-0.11E 00	0.74E-01	0.71E-02	0.94E-02	0.80E-02	-0.51E-01	0.61E-01	-0.51E-02	0.15E-01	0.31E-01
	-0.26E-01	0.14E-01	0.55E-03	0.11E-02	0.44E-02	0.21E-01	-0.40E-02	0.91E-02	0.28E-03	0.30E-02
	0.65E-01	-0.45E-02	-0.61E-03	0.70E-02	-0.23E-01	0.43E-01	-0.22E-03	0.27E-02	-0.64E-03	-0.17E-02
	-0.12E 00	0.42E-01	0.45E-02	-0.20E-02	0.25E-01	-0.62E-01	0.33E-01	-0.24E-02	0.12E-01	0.28E-01
k = 12 $\gamma_{12}(i,j),\delta_{12}(i,j)$	0.29E 01	0.14E 00	-0.18E 00	-0.21E 00	0.25E 00	-0.23E 01	-0.62E-01	0.67E-01	0.43E-02	-0.14E 00
	-0.10E 01	-0.47E-02	0.43E-01	0.57E-01	-0.46E-01	0.64E 00	0.50E-01	-0.12E-01	0.42E-02	0.47E-01
	-0.20E 00	0.24E-01	0.41E-03	0.26E-02	0.16E-01	0.73E-02	0.28E-01	0.51E-03	0.72E-02	0.17E-01
	-0.31E 00	0.46E-02	-0.14E-01	-0.13E-03	0.17E-01	[illegible]	[illegible]	0.47E-02	-0.12E-01	-0.13E-01
	-0.39E 00	-0.51E-01	0.55E-01	0.52E-01	[illegible]	[illegible]	[illegible]	-0.25E-01	0.53E-02	0.38E-01