STABILITY THEORY OF INTERCONNECTED SYSTEMS

PART I: ARBITRARY INTERCONNECTIONS

PART II: STRONG CONNECTED SUBSYSTEMS

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Stability Theory of Interconnected Systems

Part I: Arbitrary Interconnections

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ABSTRACT

The object of this paper is a general study of the input-output stability of arbitrary interconnections of nonlinear, time-varying, multivariable subsystems which may be either continuous-time or discrete-time. The paper shows how the overall system can be algorithmically decomposed into a hierarchy of strongly connected subsystems interacting through interconnection subsystems. Theorem I establishes that the overall system is stable once the strongly connected subsystems and the interconnection subsystems are stable. Theorem II shows that, under very reasonable assumptions, these sufficient conditions are actually necessary. Part II of the paper will consider subsystems whose dynamics are restricted in several ways.

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A. Introduction

The object of this paper is a general study of the input-output stability of arbitrary interconnections of nonlinear, time-varying, multivariable subsystems which may be either continuous-time or discrete-time. This problem can be viewed as a generalization of that dealing with the feedback interconnection of multivariable systems: see for example references [1-5]. On the other hand, since an arbitrary interconnection can always, by suitable reformulation, be viewed as a single overall feedback system (as is done in Eqs. (7) and (8) below), the task of this paper is to analyze the details of the interconnections and to bring them to bear on the stability study. Arbitrary interconnections of systems can be treated by Lyapunov techniques [6-10], the cost is that the dynamics are restricted to ordinary and functional differential equations, [23]. Our functional analysis approach is much more general. For other papers using this approach see [11,12].

The thrust of our approach lies in exploiting the structure of the overall system. Thus, in spirit, our approach is very close to that of signal flow graphs, see for example [13]. Following Kevorkian [14-16] we decompose the overall system into a hierarchical structure of strongly connected components. This was also done in [17] however in contrast to [17], we use a much more efficient algorithm due to Tarjan [18]. In Part I of this paper we prove two very <u>general structural</u> theorems which, together with their corollaries, make technically precise and correct the intuitive notion that if all strongly connected subsystems are stable and if all interconnecting subsystems are stable, then the overall system is stable. Theorem I shows that under very mild assumptions these conditions are necessary and Theorem II specifies a number of technical assumptions which guarantee that these conditions are both necessary and sufficient. Part II of the paper will consider special interconnections of subsystems whose dynamics are restricted to special classes.

B. Preliminaries, System Description and Assumptions

Throughout this paper we consider an interconnection of subsystems each one having the following standard descrption [2,Sec III.1]. Let $\mathcal{T}$ be the time set of observation (typically $\mathcal{T} = \mathbb{R}_+$ or $\mathbb{Z}_+$), $\mathcal{V}$ be a normed space with norm $|\cdot|$ (typically $\mathcal{V} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$ or $\mathbb{C}^n$), and $\mathcal{F}$ be the set of all the functions mapping $\mathcal{T}$ into $\mathcal{V}$. The function space $\mathcal{F}$ is a linear space over $\mathbb{R}$ (or $\mathbb{C}$) under pointwise addition and pointwise multiplication by scalars. Introducing a norm $\|\cdot\|$ on $\mathcal{F}$, we obtain a normed linear subspace $\mathcal{L}$ of the linear space $\mathcal{F}$, given by

$$\mathcal{L} \triangleq \{f : \mathcal{T} \to \mathcal{V} | \|f\| < \infty\}$$

For any $T \in \mathcal{T}$, we define $f_T(t) = f(t)$ if $t \leq T$, and zero for $t > T$. We say that f_T is obtained by truncating f at T. Associated with the normed space $\mathcal{L}$ is the extended space $\mathcal{L}_e$ defined by

$$\mathcal{L}_e \triangleq \{f : \mathcal{T} \to \mathcal{V} | \ \forall T \in \mathcal{T}, \|f_T\| < \infty\}$$

We shall often write $\|f\|_T$ instead of $\|f_T\|$. From now on we take $\mathcal{V} = \mathbb{R}$.

The object of our study is the overall system S which consists of m subsystems described by the operator equations (Fig. 1)

$$\eta_i = \underset{\sim}{G}_i e_i + v_i \qquad i = 1,2,\ldots,m \tag{1}$$

where (i) $\underset{\sim}{G}_i : \mathcal{L}_e^{n_{ii}} \to \mathcal{L}_e^{n_{io}}$ † is a causal†† operator sending e_i into the

†By definition, $\mathcal{L}_e^n = \mathcal{L}_e \times \mathcal{L}_e \times \ldots \times \mathcal{L}_e$, n times.

††By definition, $\underset{\sim}{G}_i$ is causal iff for all $e \in \mathcal{L}_e^{n_i}$, for all $T \in \mathcal{T}$, $(\underset{\sim}{G}_i e_T)_T = (\underset{\sim}{G}_i e)_T$, [2].

ith subsystem undisturbed output $y_i = \underset{\sim}{G}_i e_i$; (ii) $v_i \in \mathcal{L}_e^{n_{io}}$ is an output disturbance and (iii) $y_i + v_i = \eta_i \in \mathcal{L}_e^{n_{io}}$ is the ith subsystem available output. Each subsystem input, say e_i, is the output of a summing node fed by the interconnection operators $\underset{\sim}{F}_{ij}$ and the ith subsystem external input u_i; more precisely e_i is given by the operator equations (Fig. 2)

$$e_i = \sum_{j=1}^{m} \underset{\sim}{F}_{ij}\eta_j + u_i \qquad i = 1,2,\ldots,m \tag{2}$$

where (i) for $j = 1,\ldots,m$, $\underset{\sim}{F}_{ij} : \mathcal{L}_e^{n_{jo}} \to \mathcal{L}_e^{n_{ii}}$ is a causal interconnection operator (which may be the zero operator) sending η_j into $\underset{\sim}{F}_{ij}\eta_j$ which is fed into the summing node (Fig. 2) and (ii) $u_i \in \mathcal{L}_e^{n_{ii}}$ is the ith subsystem external input.

The 2m Eqs. (1) and (2) describing the overall system S can be condensed to describe the causal relationship between inputs and outputs of S.

Let $$n_i \triangleq \sum_{i=1}^{m} n_{ii}, \quad n_o \triangleq \sum_{i=1}^{m} n_{io} \tag{3}$$

$$u \triangleq (u_1,u_2,\ldots,u_m) \triangleq (u_i)_{i=1}^{m} \in \mathcal{L}_e^{n_i}\text{†††} \tag{4a}$$

$$v \triangleq (v_1,v_2,\ldots,v_m) \triangleq (v_i)_{i=1}^{m} \in \mathcal{L}_e^{n_o} \tag{4b}$$

$$e \triangleq (e_1,e_2,\ldots,e_m) \triangleq (e_i)_{i=1}^{m} \in \mathcal{L}_e^{n_i} \tag{4c}$$

$$\eta \triangleq (\eta_1,\eta_2,\ldots,\eta_m) \triangleq (\eta_i)_{i=1}^{m} \in \mathcal{L}_e^{n_o} \tag{4d}$$

†††To avoid clumsy notations, we shall often represent column vectors as ordered sets.

By referring to Table I, the reader will perceive the logic of our notation scheme. We choose to view (u,v) as the overall system input and (e,η) as the overall system output.

Let $\underset{\sim}{G} : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ be defined by (5)

$$\underset{\sim}{G}e \triangleq (\underset{\sim}{G}_i e_i)_{i=1}^m$$

and $\underset{\sim}{F} : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ be defined by (6)

$$\underset{\sim}{F}\eta \triangleq \left(\sum_{j=1}^m \underset{\sim}{F}_{ij}\eta_j \right)_{i=1}^m$$

With these definitions, the 2m Eqs. (1) and (2) become two operator equations

$$e - \underset{\sim}{F}\eta = u \tag{7}$$

$$-\underset{\sim}{G}e + \eta = v \tag{8}$$

Thus at this abstract level the overall system S, with its complicated interconnected structure defined by (1) and (2), is just like a standard feedback system (Fig. 3). Our task is to take advantage of the structure.

In order to perceive (7) and (8) in simpler form, let $\underset{\sim}{H} : \mathcal{L}_e^{n_i+n_o} \to \mathcal{L}_e^{n_i+n_o}$ be the causal operator defined by

$$\underset{\sim}{H}(e,\eta) \triangleq (\underset{\sim}{F}\eta, \underset{\sim}{G}e) \tag{9}$$

and let $\underset{\sim}{I}$ denote the identity operator. Then (7) and (8) give rise to the equation

$$(\underset{\sim}{I}-\underset{\sim}{H})(e,\eta) = (u,v) \tag{10}$$

Assumption I:

Throughout this paper, we assume that

$$\left.\begin{array}{l}\text{the operator } \underset{\sim}{H}_e \triangleq (\underset{\sim}{I}-\underset{\sim}{H})^{-1} \text{ is a } \underline{\text{well defined causal}} \\ \text{map from } \mathcal{L}_e^{n_i+n_o} \text{ into } \mathcal{L}_e^{n_i+n_o}\end{array}\right\} \tag{11}$$

Conditions under which this assumption is satisfied can be found in [1, Chap. 2], [2, Sec III.5]. If $(\underset{\sim}{I}-\underset{\sim}{H})$ is not one-to-one, then $\underset{\sim}{H}_e$ is a relation.††††

If we define $\underset{\sim}{H}_y : (u,v) \mapsto (\underset{\sim}{G}e, \underset{\sim}{F}\eta)$ and the linear isometric map $\underset{\sim}{K} : (u,v) \mapsto (v,u)$, then

$$\underset{\sim}{H}_y = \underset{\sim}{K}(\underset{\sim}{H}_e - \underset{\sim}{I}) \text{ and } \underset{\sim}{H}_e = \underset{\sim}{I} + \underset{\sim}{K}\,\underset{\sim}{H}_y \tag{12}$$

Hence $\underset{\sim}{H}_e$ exists (resp. is causal) if and only if $\underset{\sim}{H}_y$ exists (resp. is causal).

The causal operator $\underset{\sim}{G} : \mathcal{L}_e^n \to \mathcal{L}_e^m$ is said to be <u>$\mathcal{L}$-stable</u> iff there exists constants b, γ in $\mathbb{R}_+$ such that $\forall e \in \mathcal{L}_e^n$, $\forall T \in \mathcal{T}$

$$\|\underset{\sim}{G}e\|_T \le b + \gamma\|e\|_T \tag{13}$$

(b and γ stand for bias and gain respectively).

Using the sum norm in the product space $\mathcal{L}^{n_i+n_o}$, i.e. $\|(u,v)\| \triangleq \|u\| + \|v\|$, we say that $\underset{\sim}{H}_e$ is <u>$\mathcal{L}$-stable</u> iff there exists constants b, γ in $\mathbb{R}_+$ such that $\forall (u,v) \in \mathcal{L}_e^{n_i+n_o}$, $\forall T \in \mathcal{T}$

$$\|e\|_T + \|\eta\|_T \le b + \gamma(\|u\|_T + \|v\|_T). \tag{14}$$

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††††If $\underset{\sim}{H}_e$ is a relation, then $\underset{\sim}{H}_e$ is defined to be $\mathcal{L}$-stable iff (14) holds for all (e,η) in the relation. With this definition, all the stability results presented below also hold, with obvious modification, for the case where $\underset{\sim}{H}_e$ is a relation.

Similarly we define $\mathcal{L}$-stability of $\underset{\sim}{H}_y$. A <u>system</u> whose input-output relation is given by a causal operator $\underset{\sim}{G}$ is said to be <u>$\mathcal{L}$-stable</u> iff $\underset{\sim}{G}$ is $\mathcal{L}$-stable. In view of (12), we have $\underset{\sim}{H}_e$ is $\mathcal{L}$-stable if and only if $\underset{\sim}{H}_y$ is $\mathcal{L}$-stable. Hence, as far as the stability is concerned, we can choose either (e,η) or $(\underset{\sim}{G}e,\underset{\sim}{F}\eta)$ as the overall system output of S. We chose (e,η) because $\underset{\sim}{H}_e$ has a simpler expression.

For causal operators, the above definition of $\mathcal{L}$-stability can be shown to be equivalent to the usual one [2, Sec III.7], [1], [19].

C. Graph Theoretic Preliminary System Decomposition

By definition, a <u>digraph</u> $\mathcal{D} \triangleq (V,E)$ consists of a set of vertices V and a set of directed edges $E = \{(\nu_i, \nu_j) | \nu_i, \nu_j \in V\}$. (ν_i,ν_j) is an edge <u>directed</u> from ν_i to ν_j and is said to be <u>incident</u> to both ν_i and ν_j [20,21]. A <u>section graph</u> of $\mathcal{D} = (V,E)$ is defined to be a digraph $\mathcal{D}(U) \triangleq (U \subseteq V, \{(\nu_i,\nu_j) \in E | \nu_i,\nu_j \in U\})$. $\mathcal{D}(U)$ is said to be <u>connected</u> iff disregarding the direction of the edges, every pair of vertices in U are mutually reachable by going through edges in $\mathcal{D}(U)$. $\mathcal{D}(U)$ is said to be <u>strongly connected</u> iff respecting the direction of the edges, every pair of vertices in U are mutually reachable by traversing along edges in $\mathcal{D}(U)$. A maximal strongly connected section graph $\mathcal{D}(U)$ is called a <u>strongly connected component</u> (abbr. SCC) of $\mathcal{D}$. A <u>connected component</u> is similarly defined. The vertex ν_i is said to have a <u>self-loop</u> iff $(\nu_i,\nu_i) \in E$. A <u>circuit</u> of length $\ell > 1$ is defined to be an ordered set of ℓ distinct vertices $(\pi_1,\pi_2,\ldots,\pi_\ell)$ such that $(\pi_\ell,\pi_1) \in E$ and $(\pi_k,\pi_{k+1}) \in E$ for $k = 1,2,\ldots,\ell-1$. A digraph is said to be <u>acyclic</u> iff it does not contain any circuit. The <u>indegree</u> (resp. <u>outdegree</u>) of a vertex ν_i is defined to be the number of edges coming into (resp. out of) ν_i.

The adjacency matrix of a digraph $\mathcal{D} = (V,E)$ is defined to be an $n\times n$ matrix A where n is the number of vertices in $\mathcal{D}$, such that $a_{ij} = 1$ iff $(\nu_j, \nu_i) \in E$[§] and $a_{ij} = 0$ otherwise.

Consider the overall system S described above. The interconnection digraph $\mathcal{D}_{int}$ of S is defined as follows: each subsystem operator $\underset{\sim}{G}_i$ corresponds to a vertex ν_i, and there is a directed edge from ν_j to ν_i iff the interconnection operator $\underset{\sim}{F}_{ij}$ is not the zero operator. Since each connected component of $\mathcal{D}_{int}$ can be analyzed separately, without loss of generality, we assume that $\mathcal{D}_{int}$ is a connected digraph.

We now perform a graph theoretic decomposition on the connected digraph $\mathcal{D}_{int}$.

Step 1: Find all the SCC's $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\mu$ of $\mathcal{D}_{int}$.

Step 2: Make a condensation of $\mathcal{D}_{int}$ with respect to these SCC's. That is, we define a new digraph called the structural digraph $\mathcal{D}_s$ of S as follows: each SCC $\mathcal{C}_\alpha$ of $\mathcal{D}_{int}$ corresponds to a vertex $\bar{\nu}_\alpha$ in $\mathcal{D}_s$ and there is a directed edge from $\bar{\nu}_\alpha$ to $\bar{\nu}_\beta$ iff the set of directed edges in $\mathcal{D}_{int}$ from any vertex in $\mathcal{C}_\alpha$ to any vertex in $\mathcal{C}_\beta$ is not empty. By construction, $\mathcal{D}_s$ is a connected acyclic digraph.

Step 3: Relabel the vertices of $\mathcal{D}_s$ so that its adjacency matrix A_s is a lower triangular matrix. Hence, with respect to the new labeling, a SCC, say $\mathcal{C}_\alpha$, can only feed its output to SCC's, say $\mathcal{C}_\beta, \mathcal{C}_\gamma, \ldots$, with a higher subscript, i.e. $\beta, \gamma > \alpha$.

[§]Most graph theorists define $a_{ij} = 1$ iff $(\nu_i, \nu_j) \in E$. Hence our adjacency matrix is the transpose of theirs.

Step 4: Relabel the vertices of $\mathcal{D}_{int}$ so that (a) those that belong to the same SCC are numbered consecutively and (b) those that belong to the lower numbered SCC are numbered lower than those belong to the higher numbered SCC. ¤

An example illustrating this decomposition algorithm is given in Appendix AII. Step 1, the identification of SCC's, can be done by using Tarjan's efficient algorithm STRONGCONNECT [18]. An English language description of it is in Appendix AII. Step 2 can easily be done by inspection. Step 3, labeling of a connected acyclic digraph is called topological sort [20,pp.462], [22,pp.258]. It is done in μ iterations (μ = the number of SCC's) by deleting a vertex with zero indegree and all its incident edges at each iteration and, then, by relabeling the vertices in the order they were deleted.

A little thought reveals that the adjacency matrix A_{int} of $\mathcal{D}_{int}$ after Step 4 will be in the lower block triangular form:

$$A_{int} = \begin{array}{c} \\ m_1 \\ m_2 \\ \vdots \\ m_\mu \end{array} \begin{array}{c} \begin{array}{cccc} m_1 & m_2 & \cdots & m_\mu \end{array} \\ \begin{bmatrix} A^c_{11} & 0 & \cdots & 0 \\ A^c_{21} & A^c_{22} & \cdots & 0 \\ \vdots & & & \\ A^c_{\mu 1} & A^c_{\mu 2} & \cdots & A^c_{\mu\mu} \end{bmatrix} \end{array} \tag{15}$$

where (i) m_α is the number of vertices in $\mathcal{C}_\alpha$, (ii) each diagonal block $A^c_{\alpha\alpha}$ is the adjacency matrix of $\mathcal{C}_\alpha$ and (iii) each off diagonal block $A^c_{\alpha\beta}$, $\alpha > \beta$ is the adjacency matrix of $\mathcal{C}_{\alpha\beta}$ which is defined to be the bipartite digraph [20, pp. 168] consisting of (a) all the vertices of

$\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$, and (b) all edges of $\mathcal{D}_{int}$ directed from a vertex in $\mathcal{C}_\beta$ to a vertex in $\mathcal{C}_\alpha$.

From now on, without loss of generality, we assume that we start out with the overall system S which has been relabeled. For each SCC $\mathcal{C}_\alpha$, $\alpha = 1,2,\ldots,\mu$, we define

$$n_{\alpha i}^c \triangleq \sum_k n_{ki} \quad , \quad n_{\alpha o}^c \triangleq \sum_k n_{ko} \tag{16}$$

$$u_\alpha^c \triangleq (u_k)_k \quad , \quad v_\alpha^c \triangleq (v_k)_k \tag{17}$$

$$e_\alpha^c \triangleq (e_k)_k \quad , \quad \eta_\alpha^c \triangleq (\eta_k)_k \tag{18}$$

$$\underset{\sim}{G}_\alpha^c : \mathcal{L}_e^{n_{\alpha i}^c} \to \mathcal{L}_e^{n_{\alpha o}^c} \text{ such that } \underset{\sim}{G}_\alpha^c e_\alpha^c \triangleq (\underset{\sim}{G}_k e_k)_k \tag{19}$$

and

$$\underset{\sim}{F}_{\alpha\beta}^c : \mathcal{L}_e^{n_{\beta o}^c} \to \mathcal{L}_e^{n_{\alpha i}^c} \text{ such that } \underset{\sim}{F}_{\alpha\beta}^c \, \eta_\beta^c \triangleq (\sum_j \underset{\sim}{F}_{kj}\eta_j)_k \tag{20}$$

where in every case, k ranges from $\sum_{\lambda=1}^{\alpha-1} m_\lambda + 1$ to $\sum_{\lambda=1}^{\alpha} m_\lambda$ and j ranges from $\sum_{\lambda=1}^{\beta-1} m_\lambda + 1$ to $\sum_{\lambda=1}^{\beta} m_\lambda$. Since $\underset{\sim}{G}_k$, $\underset{\sim}{F}_{kj}$ are assumed to be causal, $\underset{\sim}{G}_\alpha^c$, $\underset{\sim}{F}_{\alpha\beta}^c$ are also causal.

Equations (1) and (2) (or equivalently (7) and (8)) can now be rewritten as 2μ equations

$$e_\alpha^c - \sum_{\beta=1}^{\alpha} \underset{\sim}{F}_{\alpha\beta}^c \eta_\beta^c = u_\alpha^c \qquad \alpha = 1,2,\ldots,\mu \tag{21}$$

$$-\underset{\sim}{G}_\alpha^c e_\alpha^c + \eta_\alpha^c = v_\alpha^c \qquad \alpha = 1,2,\ldots,\mu \tag{22}$$

Observe that due to relabeling, the index β in (21) does not go beyond α.

For $\alpha = 1,2,\ldots,\mu$, we denote by S_α^c the <u>strongly connected subsystem</u> (abbr. SCS) described by Eq. (22) and

$$e_\alpha^c - \underset{\sim}{F}_{\alpha\alpha}^c \eta_\alpha^c = u_\alpha^c \quad . \tag{23}$$

For $\alpha > \beta$, $\alpha,\beta = 1,\ldots,\mu$, we denote by $S_{\alpha\beta}^c$ the <u>interconnection subsystem</u> (abbr. IS) represented by the interconnection operators $\underset{\sim}{F}_{\alpha\beta}^c$.

By Assumption I in Sec. B, the SCS S_α^c is represented by a causal operator mapping from $\mathcal{L}_e^{n_{\alpha i}^c + n_{\alpha o}^c}$ into itself. Thus the $\mathcal{L}$-stability of the SCS S_α^c and IS $S_{\alpha\beta}^c$ are defined unambiguously using the definition given in Sec. B.

D. <u>Structural Theorems</u>

The two structural theorems presented below are based on the form of the Eqs. (1) and (2) as well as the structure of the interconnection exhibited by (21) and (22). We emphasize the fact that these theorems are valid for a very general class of systems: linear or nonlinear, time-invariant or time-varying, continuous-time or discrete-time.

Given the decomposition described in Sec. C, it seems intuitively obvious that the statement "the overall system S is $\mathcal{L}$-stable if and only if each SCS S_α^c and each IS $S_{\alpha\beta}^c$ are $\mathcal{L}$-stable" should be true. Theorems I and II below should be viewed as attempts to delineate technical conditions under which that statement is true. Although many sets of conditions were considered, the conditions below seem to the authors to be the most general and elegant.

Theorem I[§§]

Consider the overall nonlinear system S originally described by (7) and (8) and by (21) and (22) after relabelling:

(a) If every SCS S^c_α and every IS $S^c_{\alpha\beta}$ are $\mathcal{L}$-stable, then the overall system S is $\mathcal{L}$-stable.

(b) For the overall system $\underset{\sim}{S}$, assume that for i = 2,...,μ, there is an input $(\hat{u}^c_\alpha,\hat{v}^c_\alpha) \in \mathcal{L}^{n^c_{\alpha i}+n^c_{\alpha o}}$, $\alpha = 1,\ldots,i-1$ such that for the corresponding output $\hat{\eta}^c_\alpha$, $\alpha = 1,\ldots,i-1$

$$\sum_{\beta=1}^{i-1} \underset{\sim}{F}^c_{i\beta}\hat{\eta}^c_\beta \in \mathcal{L}^{n^c_{ii}}, \tag{24}$$

under these conditions, if the overall system S is $\mathcal{L}$-stable then every every SCS S^c_α is $\mathcal{L}$-stable. ¤

Assumption (24) above is very mild. Indeed if $\underset{\sim}{G}(\theta) = \theta$ and $\underset{\sim}{F}(\theta) = \theta$, then (24) is satisfied by taking the inputs $(\hat{u}^c_\alpha,\hat{v}^c_\alpha) = \theta$ for $\alpha = 1,\ldots,\mu-1$. In particular, (24) is satisfied for all <u>linear</u> $\underset{\sim}{G}$ and $\underset{\sim}{F}$.

Since each IS $S^c_{\alpha\beta}$ does not involve feedback loops, its stability problem is straight forward. The following corollary is an obvious result of Theorem I. It is useful because it gives us an assumption under which the question of stability of the overall system S reduces to the stability of each SCS S^c_α.

Corollary I.1

Consider the overall nonlinear system S described by (21) and (22).

[§§] All the proofs are relegated to Appendix AI.

Suppose $\underset{\sim}{G}(\theta) = \theta$, $\underset{\sim}{F}(\theta) = \theta$ and suppose that every IS $S^c_{\alpha\beta}$ is $\mathcal{L}$-stable. Under these conditions, the overall system S is $\mathcal{L}$-stable if and only if every SCS S^c_α is $\mathcal{L}$-stable. ¤

In order to obtain the desired if and only if statement for the $\mathcal{L}$-stability of S versus the $\mathcal{L}$-stability of each S^c_α and each $S^c_{\alpha\beta}$, it turns out that more delicate assumptions are needed.

Theorem II

Consider the overall nonlinear system S described by (21) and (22). Assume that

$$\text{for } \alpha = 1,\dots,\mu, \quad \underset{\sim}{F}^c_{\alpha\alpha}(\mathcal{L}^{n^c_{\alpha o}}) \subset \mathcal{L}^{n^c_{\alpha i}} \tag{25a}$$

$$\text{for } \alpha = 1,\dots,\mu, \quad \text{there exists an } \bar{e}^c_\alpha \in \mathcal{L}^{n^c_{\alpha i}} \text{ such that } \underset{\sim}{G}^c_\alpha \bar{e}^c_\alpha \in \mathcal{L}^{n^c_{\alpha o}}, \tag{25b}$$

for $\beta = 1,\dots,\mu-1$, there exists an $\bar{\eta}^c_\beta \in \mathcal{L}^{n^c_{\beta o}}$ such that for all

$$\alpha > \beta, \quad \underset{\sim}{F}^c_{\alpha\beta}\bar{\eta}^c_\beta \in \mathcal{L}^{n^c_{\alpha i}}; \tag{25c}$$

for every unstable $\underset{\sim}{F}^c_{\alpha\beta}$, there exists an $\bar{\bar{\eta}}^c_{\alpha\beta} \in \mathcal{L}^{n^c_{\beta o}}$

$$\text{such that } \underset{\sim}{F}^c_{\alpha\beta}\bar{\bar{\eta}}^c_{\alpha\beta} \notin \mathcal{L}^{n^c_{\alpha i}}. \tag{26}$$

Under these conditions, the overall system S is $\mathcal{L}$-stable if and only if every SCS S^c_α and every IS $S^c_{\alpha\beta}$ are $\mathcal{L}$-stable. ¤

Assumption (25a) appears to be more restrictive than necessary. It is required by the form of the system equations and by the possibly devilish behavior of nonlinear maps: suppose that $\underset{\sim}{F}^c_{22}(\theta) = \theta$ but for all $\theta \neq \eta^c_2 \in \mathcal{L}^{n^c_{20}}$, $\underset{\sim}{F}^c_{22}\eta^c_2 = y$, a fixed element of $\mathcal{L}_e^{n^c_{2i}} - \mathcal{L}^{n^c_{2i}}$ and similarly suppose that $\underset{\sim}{F}^c_{21}(\theta) = \theta$ for all $\theta \neq \eta^c_1 \in \mathcal{L}^{n^c_{10}}$, $\underset{\sim}{F}^c_{21}\eta^c_1 = -y$. Then in the equation for SCS S^c_2, $\underset{\sim}{F}^c_{21}\eta^c_1 + \underset{\sim}{F}^c_{22}\eta^c_2$ is always θ, hence in $\mathcal{L}^{n^c_{2i}}$, even though most of the time the two terms are not in $\mathcal{L}^{n_{2i}^c}$. Obviously, in general, in the linear case, such cancellation will not occur for all $\eta_1^c \neq \theta$ and $\eta_2^c \neq \theta$.

Assumption (25b) and (25c) are very mild. They are of a similar nature as assumption (24) of Theorem I(b). If $\underset{\sim}{G}(\theta) = \theta$, $\underset{\sim}{F}(\theta) = \theta$ and every $\underset{\sim}{F}_{\alpha\alpha}^c$ is $\mathcal{L}$-stable, then (25) holds: choose all $\bar{e}_\alpha^c, \bar{\eta}_\beta^c$ in (25) to be θ. In the proof of Theorem II, we use Theorem I(b) after showing that (25) implies (24).

Assumption (26) is required because our definition of $\mathcal{L}$-stability not only requires that any input in $\mathcal{L}$ gives rise to an output in $\mathcal{L}$ but also that the "gain factor" γ in (13) be finite. From a strictly mathematical elegance point of view, deleting the second requirement would have made assumption (26) unnecessary.

E. Conclusion

This paper has treated in a very general setting the stability of an arbitrary interconnection of subsystems. Due to the functional analysis approach the assumptions required on the subsystems are minimal. The paper shows how the overall system can be algorithmically decomposed into a hierarchy of strongly connected subsystems interacting through interconnection subsystems. Theorem I establishes that the overall system is stable once the strongly connected subsystems and the interconnection subsystems are stable. Theorem II shows that, under very reasonable assumptions, these sufficient conditions are actually necessary. Part II of the paper will consider strongly connected subsystems whose dynamics are restricted in several ways.

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APPENDIX AI: PROOFS

PROOF OF THEOREM I

Proof of (a): By induction. (I) observe that because SCS S_1^c is $\mathcal{L}$-stable, the system $(u_1^c, v_1^c) \mapsto (e_1^c, \eta_1^c)$ is $\mathcal{L}$-stable. (II) Using the induction principle, we only need to prove that if the system $(u_1^c, \ldots, u_{i-1}^c; v_1^c, \ldots, v_{i-1}^c)$ $\mapsto (e_1^c, \ldots, e_{i-1}^c; \eta_1^c, \ldots, \eta_{i-1}^c)$ is $\mathcal{L}$-stable, then the system $(u_1^c, \ldots, u_i^c; v_1^c, \ldots, v_i^c)$ $\mapsto (e_1^c, \ldots, e_i^c; \eta_1^c, \ldots, \eta_i^c)$ is also $\mathcal{L}$-stable.

Set $\alpha = i$ in the system Eqs. (21), (22) and obtain

$$e_i^c - \underset{\sim}{F}_{ii}^c \eta_i^c = u_i^c + \sum_{\beta=1}^{i-1} \underset{\sim}{F}_{i\beta}^c \eta_\beta^c \triangleq \tilde{u}_i^c \tag{A1}$$

$$-\underset{\sim}{G}_i^c e_i^c + \eta_i^c = v_i^c \triangleq \tilde{v}_i^c \tag{A2}$$

Because of the $\mathcal{L}$-stability of $S_{\alpha\beta}^c$ and the inductive assumption in (II), there exists constants $\bar{b}$ and $\bar{\gamma}$ in $\mathbb{R}_+$ such that $\forall\ (u_\beta^c, v_\beta^c) \in \mathcal{L}_e^{n_{\beta i}^c + n_{\beta o}^c}$, $\beta = 1, \ldots, i-1$, and $\forall T \in \mathcal{J}$

$$\left\| \sum_{\beta=1}^{i-1} \underset{\sim}{F}_{i\beta}^c \eta_\beta^c \right\|_T \leq \bar{b} + \bar{\gamma} \left\{ \sum_{\beta=1}^{i-1} \left(\|u_\beta^c\|_T + \|v_\beta^c\|_T \right) \right\} \tag{A3}$$

Hence the conclusion in (II) follows from (A3) and the $\mathcal{L}$-stability of SCS S_i^c. ◻

Proof of (b): Consider an arbitrary SCS S_i^c. Set $\alpha = i$ in the system equation (21), (22) and obtain (A1), (A2).

Thus the operator $(\tilde{u}_i^c, \tilde{v}_i^c) \mapsto (e_i^c, \eta_i^c)$ represents the SCS S_i^c. By assumption, the overall system S is $\mathcal{L}$-stable, i.e. there exists b, γ in $\mathbb{R}_+$ such that $\forall (u_\alpha^c, v_\alpha^c)_{\alpha=1}^{\mu} \in \mathcal{L}_e^{n_i + n_o}$, $\forall T \in \mathcal{J}$

$$\sum_{\alpha=1}^{\mu} \left(\|e_\alpha^c\|_T + \|\eta_\alpha^c\|_T \right) \leq b + \gamma \left\{ \sum_{\alpha=1}^{\mu} \left(\|u_\alpha^c\|_T + \|v_\alpha^c\|_T \right) \right\} \tag{A4}$$

Consider first $i = 1$: the term $\sum_{\alpha=1}^{i-1} \underset{\sim}{F}_{i\alpha}^{c} \eta_{\alpha}^{c}$ in (A1) is absent. Hence $(\tilde{u}_1^c, \tilde{v}_1^c) = (u_1^c, v_1^c)$. Setting $(u_\alpha^c, v_\alpha^c)_{\alpha=2}^{\mu} = \theta$ in (A4), we have b, γ in $\mathbb{R}_+$ such that $\forall (\tilde{u}_1^c, \tilde{v}_1^c) \in \mathcal{L}_e^{n_{1i}+n_{1o}}$, $\forall T \in \mathcal{J}$,

$$\|e_1^c\|_T + \|\eta_1^c\|_T \le b + \gamma(\|\tilde{u}_1^c\|_T + \|\tilde{v}_1^c\|_T)$$

i.e. the SCS S_1^c is $\mathcal{L}$-stable.

Now consider $i \ge 2$: setting $(u_\alpha^c, v_\alpha^c)_{\alpha=1}^{i-1} = (\hat{u}_\alpha^c, \hat{v}_\alpha^c)_{\alpha=1}^{i-1}$, the inputs specified by the assumption in part (b) of the theorem; and setting $(u_\alpha^c, v_\alpha^c)_{\alpha=i+1}^{\mu} = \theta$, we obtain from (A4)

$$\forall (u_i^c, v_i^c) \in \mathcal{L}_e^{n_{ii}^c + n_{io}^c}, \ \forall T \in \mathcal{J},$$

$$\|e_i^c\|_T + \|\eta_i^c\|_T \le b + \gamma\{\sum_{\alpha=1}^{i-1} (\|\hat{u}_\alpha^c\|_T + \|\hat{v}_\alpha^c\|_T) + (\|u_i^c\|_T + \|v_i^c\|_T)\} \tag{A5}$$

By (24), $(\hat{u}_\alpha^c, \hat{v}_\alpha^c)_{\alpha=1}^{i-1} \in \mathcal{L}^{\sum_{\alpha=1}^{i-1}(n_{\alpha i}^c + n_{\alpha o}^c)}$, hence there exists b_1 in $\mathbb{R}_+$ such that $\forall T \in \mathcal{J}$,

$$\sum_{\alpha=1}^{i-1} (\|\hat{u}_\alpha^c\|_T + \|\hat{v}_\alpha^c\|_T) \le b_1 \tag{A6}$$

Using (A6), the second equalities in (A1) and (A2), and triangular inequality of the norm, we obtain from (A5)

$$\forall (\tilde{u}_i^c, \tilde{v}_i^c) \in \mathcal{L}_e^{n_{ii}^c + n_{io}^c}, \ \forall T \in \mathcal{J}$$

$$\|e_i^c\|_T + \|\eta_i^c\|_T \le b + \gamma b_1 + \gamma(\|\tilde{u}_i^c\|_T + \|\tilde{v}_i^c\|_T) + \gamma \|\sum_{\alpha=1}^{i-1} \underset{\sim}{F}_{i\alpha}^{c} \hat{\eta}_\alpha^c\|_T \tag{A7}$$

By (24), $\sum_{\alpha=1}^{i-1} \underset{\sim}{F}_{i\alpha}^{c} \hat{\eta}_\alpha^c \in \mathcal{L}^{n_{ii}^c}$, hence there exists b_2 in $\mathbb{R}_+$ such that $\forall T \in \mathcal{J}$,

$$\|\sum_{\alpha=1}^{i-1} \underset{\sim}{F}_{i\alpha}^{c} \hat{\eta}_\alpha^c\|_T \le b_2 \tag{A8}$$

Substituting (A8) into (A7), we have $\tilde{b} \triangleq b + \gamma b_1 + \gamma b_2$ and γ in $\mathbb{R}_+$ such that $\forall (\tilde{u}_i^c, \tilde{v}_i^c) \in \mathcal{L}_e^{n_{ii}^c + n_{io}^c}$, $\forall T \in \mathcal{J}$

$$\|e_i^c\|_T + \|\eta_i^c\|_T \leq \tilde{b} + \gamma(\|\tilde{u}_i^c\|_T + \|\tilde{v}_i^c\|_T)$$

i.e. the SCS S_i^c is $\mathcal{L}$-stable.

Since this conclusion holds for all $i = 1,\ldots,\mu$, the proof is complete. ¤

PROOF OF COROLLARY I.1.

Since $\underset{\sim}{G}(\theta) = \theta$, $\underset{\sim}{F}(\theta) = \theta$, assumption (24) is satisfied with $(\hat{u}_\alpha^c, \hat{v}_\alpha^c) = \theta$ for $\alpha = 1,\ldots,\mu$. The result follows immediately from Theorem I. ¤

PROOF OF THEOREM II

$\Leftarrow$ Follows immediately from Theorem I(a)

$\Rightarrow$ (i) We prove first the $\mathcal{L}$-stability of SCS S_α^c. By Theorem I(b), we only need to show (25) implies (24). We want to choose inputs, say $(\hat{u}_\alpha^c, \hat{v}_\alpha^c)_{\alpha=1}^\mu$ such that the corresponding outputs are the $(\bar{e}_\alpha^c, \bar{\eta}_\alpha^c)_{\alpha=1}^\mu$ specified in (25b) and (25c). By (21) and (22) these inputs are given by

$$\hat{u}_\alpha^c \triangleq \bar{e}_\alpha^c - \sum_{\beta=1}^{\alpha} \underset{\sim}{F}_{\alpha\beta}^c \bar{\eta}_\beta^c \qquad \alpha = 1,\ldots,\mu \tag{A9}$$

$$\hat{v}_\alpha^c \triangleq \bar{\eta}_\alpha^c - \underset{\sim}{G}_\alpha^c \bar{e}_\alpha^c \qquad \alpha = 1,\ldots,\mu \tag{A10}$$

Note that $(\hat{u}_\alpha^c, \hat{v}_\alpha^c)_{\alpha=1}^\mu \in \mathcal{L}^{n_i + n_o}$ since each term in the right hand sides of (A9) and (A10) are in $\mathcal{L}^{n_{\alpha i}^c}$ and $\mathcal{L}^{n_{\alpha o}^c}$ respectively. Thus, by the uniqueness assumption (11), assumptions (25) imply the existence of the inputs $(\hat{u}_\alpha^c, \hat{v}_\alpha^c)_{\alpha=1}^\mu$ which produce $\hat{\eta}_\alpha^c$ (denoted here by $\bar{\eta}_\alpha^c$) which satisfy (24).

(ii) We prove the $\mathcal{L}$-stability of IS $S^c_{\alpha\beta}$ by contradiction. Suppose there is at least one $\mathcal{L}$-unstable IS $S^c_{\alpha\beta}$. Among all the $\mathcal{L}$-unstable $S^c_{\alpha\beta}$'s let S^c_{ij} be the one, first, with the smallest value of α and, second, with the largest value of β among $\mathcal{L}$-unstable $S^c_{i\beta}$'s. Hence for $\alpha,\beta = 1,\dots,i-1$, $\alpha > \beta$, $S^c_{\alpha\beta}$ is $\mathcal{L}$-stable and for $\beta = j+1,\dots,i-1$, $S^c_{i\beta}$ is $\mathcal{L}$-stable. Rewrite the system Eq. (21) for S^c_i as follows

$$e^c_i - \underset{\sim}{F}^c_{ii}\eta^c_i = u^c_i + \sum_{\beta=1}^{j-1}\underset{\sim}{F}^c_{i\beta}\eta^c_\beta + \underset{\sim}{F}^c_{ij}\eta^c_j + \sum_{\beta=j+1}^{i-1}\underset{\sim}{F}^c_{i\beta}\eta^c_\beta \tag{A11}$$

where the last term is absent when $j = i-1$.

We shall reach a contradiction by showing that for some input, say $(\hat{u},\hat{v})$, the left hand side of (A11) is in $\mathcal{L}^{n^c_{ii}}$ while the right hand side is not.

By $\mathcal{L}$-stability of S and assumption (25a), for all $(u,v) \in \mathcal{L}^{n_i+n_o}$, the left hand side of (A11) is in $\mathcal{L}^{n^c_{ii}}$. (A12)

Using arguments similar to those in (i), we first pick $(\hat{u}^c_\alpha,\hat{v}^c_\alpha) \in \mathcal{L}^{n^c_{\alpha i}+n^c_{\alpha o}}$, $\alpha = 1,\dots,j$ so as to obtain

$$\hat{\eta}^c_\alpha = \begin{cases} \bar{\eta}^c_\alpha, \text{ as given by (25c)} & \text{for } \alpha = 1,\dots,j-1 \\ \bar{\bar{\eta}}^c_{ij} \text{ as given by (26)} & \text{for } \alpha = j. \end{cases}$$

Second we pick any $(\hat{u}^c_\alpha,\hat{v}^c_\alpha) \in \mathcal{L}^{n^c_{\alpha i}+n^c_{\alpha o}}$ for $\alpha = j+1,\dots,\mu$ so that the overall input $(\hat{u},\hat{v}) \in \mathcal{L}^{n_i+n_o}$. Now substituting $\hat{u}^c_i$ and $(\hat{\eta}^c_\alpha)^i_{\alpha=1}$ into the right hand side of (A11) we have,

$$\hat{u}^c_i + \sum_{\beta=1}^{j-1} \underset{\sim}{F}^c_{i\beta}\bar{\eta}^c_\beta + \underset{\sim}{F}^c_{ij}\bar{\bar{\eta}}^c_{ij} + \sum_{\beta=j+1}^{i-1} \underset{\sim}{F}^c_{i\beta}\hat{\eta}^c_\beta$$

where the first term is in $\mathcal{L}^{n^c_{ii}}$ by choice, the second term is in $\mathcal{L}^{n^c_{ii}}$

by (25c), the third term is not in $\mathcal{L}^{n^c_{ii}}$ by (26); the last term is in $\mathcal{L}^{n^c_{ii}}$: for β = j+1,...,i-1, by the $\mathcal{L}$-stability of S, $\hat{\eta}^c_\beta \in \mathcal{L}^{n^c_{\beta o}}$ and by the $\mathcal{L}$-stability of $S^c_{i\beta}$, $\underset{\sim}{F}^c_{i\beta}\hat{\eta}^c_\beta \in \mathcal{L}^{n^c_{\beta i}}$.

In conclusion, there exists a $(\hat{u},\hat{v}) \in \mathcal{L}^{n_i+n_o}$ such that the right hand side of (A11) is not in $\mathcal{L}^{n^c_{ii}}$. This is a contradiction to (A12) and hence every IS $S^c_{\alpha\beta}$ must be $\mathcal{L}$-stable. ¤

APPENDIX AII: IDENTIFICATION OF STRONGLY CONNECTED COMPONENTS OF A DIGRAPH

By definition, a search of a digraph $\mathcal{D}$ is a process in which one traverses† all the directed edges of $\mathcal{D}$, each exactly once; and a *depth-first search* is a search in which the edge to be traversed at each step is always an unexplored edge emanating from the vertex most recently reached. Tarjan's algorithm is based on using depth-first search: it will *neglect* edges which are irrelevant in identification of SCC's and it will classify the remaining edges of $\mathcal{D}$ as either a *tree arc* or a *nontree arc.* The algorithm uses a *stack* which is a linear list for which all the insertions and deletions are made at the same end of the list, hence it is characterized by last-in-first-out. We *number* the vertices consecutively in the order they are reached in depth-first search; these numbers will be used in the algorithm. The algorithm STRONGCONNECT can now be described in the English language as follows:

Algorithm STRONGCONNECT [18]

Step 1: If there is a vertex not yet numbered, take it as the active vertex, number it and insert it into the stack; *otherwise stop*.

Step 2: Choose an unexplored edge emanating from the active vertex ν.

Step 3: Case (a) If the chosen edge leads to a vertex π not yet numbered (Fig. 4(a)), then classify (ν,π) as a *tree arc*, number π and insert π into the stack. Take π as the active vertex and go to Step 2.

Case(b) If the chosen edge leads to a vertex π already numbered and π is not reachable from ν by traversing along any number (≥ 0) of tree arcs (Fig. 4(b)), then classify (ν,π) as a *nontree arc* and

†Throughout this paper, *traversing* means moving along the directed edges respecting their direction.

go to Step 2.

Case (c) If the chosen edge leads to a vertex π already numbered and π is reachable from ν by traversing along some number (≥ 0) of tree arcs (Fig. 4(c)), then neglect (ν,π) and go to Step 2.

Case (d) If there is no more unexplored edge emanating from ν, then (i) evaluate LOWLINK (ν) $\triangleq$ min{number of vertex ν, number of vertex $\pi | \pi$ is in the stack and is reachable from ν by traversing along some number (≥ 0) of tree arcs followed by exactly one nontree arc}; (ii) If LOWLINK (ν) = number of vertex ν, then, from the stack, delete ν and all the vertices that come after ν in the stack because they all belong to the same SCC; if the stack is now empty, go to Step 1, else go to Step 4; if LOWLINK (ν) $\neq$ number of vertex ν, then go to Step 4.

Step 4: Take the preceding active vertex as the active vertex and go to Step 2. ¤

Tarjan proves the correctness of the algorithm and shows that the run time and memory storage required is bounded by a linear function of the number of vertices and number of edges in $\mathcal{D}$.

Example 1:

Consider the interconnection digraph $\mathcal{D}_{int}$ given in Fig. 5(a) and associated adjacency matrix in Fig. 5(b). When algorithm STRONGCONNECT is applied on $\mathcal{D}_{int}$, it generates the digraph as shown in Fig. 5(c) where the solid lines represent the tree-arcs, the dashed lines represent the nontree-arcs and the LOWLINK values are given in square brackets. The SCC $\mathcal{C}_1,\ldots,\mathcal{C}_5$, labelled in the order they are detected in the Algorithm STRONGCONNECT, are the section graphs $\mathcal{D}(\{7\})$, $\mathcal{D}(\{3,4,5,6,8\})$,

$\mathcal{D}(\{10,11,12\})$, $\mathcal{D}(\{1,2,9\})$ and $\mathcal{D}(\{13,14,15\})$ respectively. The corresponding structural digraph $\mathcal{D}_s$ is given in Fig. 5(d). Figure 5(e) is the table relating the old labeling to the new labeling after steps 3 and 4 of the decomposition algorithm described in Sec. C. Figure 5(f) gives the adjacency matrix of $\mathcal{D}_{int}$ with respect to the new labeling and observe that it is indeed in block lower-triangular form. Although such relabeling is not unique, the corresponding adjacency matrices will always be in block lower-triangular form with the same number of diagonal blocks of the same sizes.

Table I: Summary of Notations

	subsystem level	strongly connected subsystem level	overall system level
number of subsystems	m	μ	1
system	(see Fig. 1,2)	S^c_α, $S^c_{\alpha\beta}$	S (see Fig. 3)
number of inputs	n_{ii}	$n^c_{\alpha i} \triangleq \sum_{i\in\alpha} n_{ii}$	$n_i \triangleq \sum_{i=1}^{m} n_{ii}$
number of outputs	n_{io}	$n^c_{\alpha o} \triangleq \sum_{i\in\alpha} n_{io}$	$n_o \triangleq \sum_{i=1}^{m} n_{io}$
subsystem operator	$\underset{\sim}{G}_i$	$\underset{\sim}{G}^c_\alpha \triangleq \mathrm{diag}(\underset{\sim}{G}_i)_{i\in\alpha}$	$\underset{\sim}{G} \triangleq \mathrm{diag}(\underset{\sim}{G}_i)_{i=1,\ldots,m}$
interconnection operator	$\underset{\sim}{F}_{ij}$	$\underset{\sim}{F}^c_{\alpha\beta} \triangleq (\underset{\sim}{F}_{ij})_{i\in\alpha,j\in\beta}$	$\underset{\sim}{F} \triangleq (\underset{\sim}{F}_{ij})_{i,j=1,\ldots,m}$
external input	u_i	$u^c_\alpha \triangleq (u_i)_{i\in\alpha}$	$u \triangleq (u_i)^m_{i=1}$
output disturbance	v_i	$v^c_\alpha \triangleq (v_i)_{i\in\alpha}$	$v \triangleq (v_i)^m_{i=1}$
local input	e_i	$e^c_\alpha \triangleq (e_i)_{i\in\alpha}$	$e \triangleq (e_i)^m_{i=1}$
disturbed output	η_i	$\eta^c_\alpha \triangleq (\eta_i)_{i\in\alpha}$	$\eta \triangleq (\eta_i)^m_{i=1}$
undisturbed output	y_i	$y^c_\alpha \triangleq (y_i)_{i\in\alpha}$	$y \triangleq (y_i)^m_{i=1}$
digraph	$\mathcal{D}_{int}$	$\mathcal{D}_s$	
adjacency matrix	A_{int}	A_s	
vertices	$\underset{\sim}{G}_i \leftrightarrow \nu_i$	S^c_α or $\underset{\sim}{G}^c_\alpha \leftrightarrow \bar{\nu}_\alpha$	
directed edges	$\underset{\sim}{F}_{ij} \leftrightarrow (\nu_j,\nu_i)$	$S^c_{\alpha\beta}$ or $\underset{\sim}{F}^c_{\alpha\beta} \leftrightarrow (\bar{\nu}_\beta,\bar{\nu}_\alpha)$	
Strongly connected component (SCC)		$\mathcal{C}_\alpha$	
interconnection between SCC's		$\mathcal{C}_{\alpha\beta}$	
adjacency matrix of $\mathcal{C}_\alpha$		$A^c_{\alpha\alpha} \triangleq ([A_{int}]_{ij})_{i,j\in\alpha}$	
adjacency matrix of $\mathcal{C}_{\alpha\beta}$		$A^c_{\alpha\beta} \triangleq ([A_{int}]_{ij})_{i\in\alpha,j\in\beta}$	
number of vertices in $\mathcal{C}_\alpha$		m_α	

Figure Caption

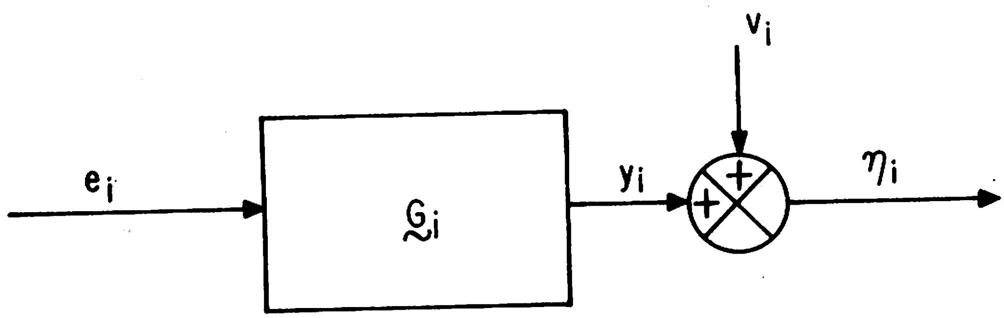
Fig. 1. The ith subsystem.

Fig. 2. Summing node associated with the ith subsystem.

Fig. 3. The overall system S.

Fig. 4. The solid lines represent tree arcs. The dashed line represents the edge (ν, π) under consideration in Step 3 of the algorithm STRONGCONNECT.

Fig. 5. Illustrations for Example 1. (a) Interconnection digraph $\mathcal{D}_{int}$ with vertex numbers, the corresponding new vertex numbers after the relabeling are given in parentheses. (b) Original adjacency matrix A_{int} of $\mathcal{D}_{int}$. (c) Digraph generated by the algorithm STRONGCONNECT with LOWLINK values given in square brackets. (d) Structural digraph $\mathcal{D}_s$. (e) Table relating the old label to the new label. (f) Adjacency matrix of $\mathcal{D}_{int}$ after relabeling.


v_i
e_i
G_i
y_i
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+
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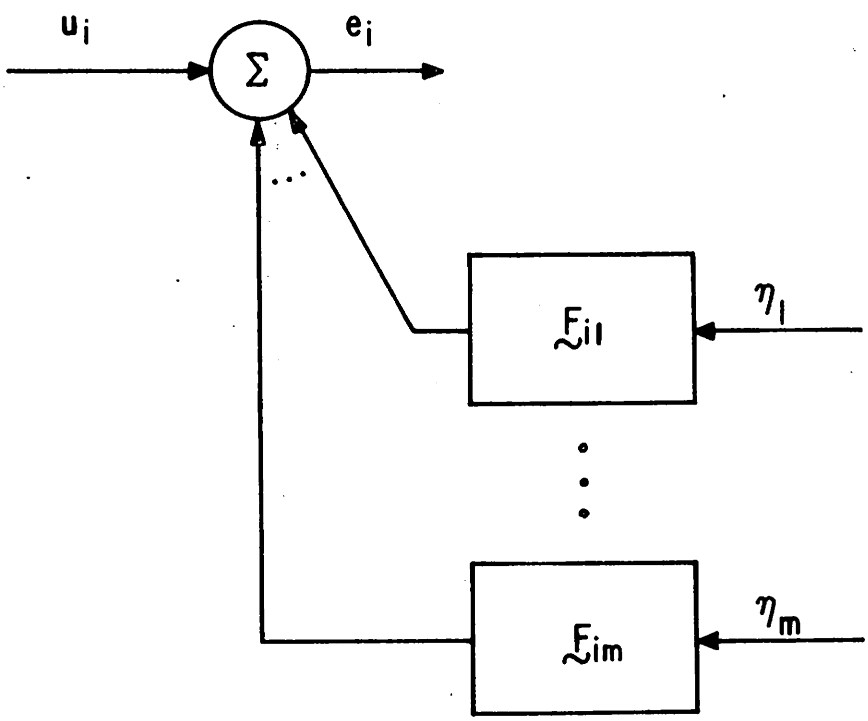
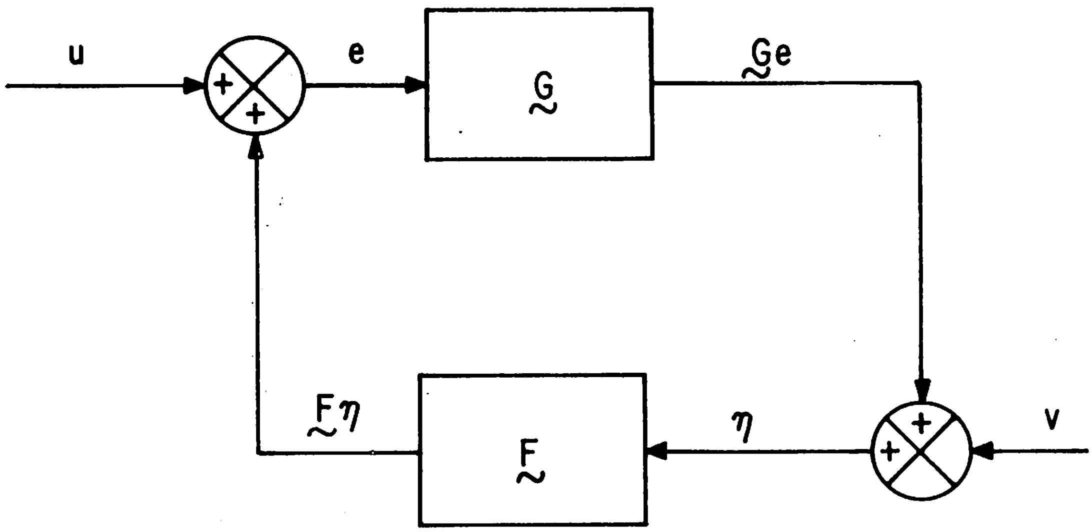

u_i
Σ
e_i
$\underset{\sim}{F}_{i1}$
η_1
$\underset{\sim}{F}_{im}$
η_m



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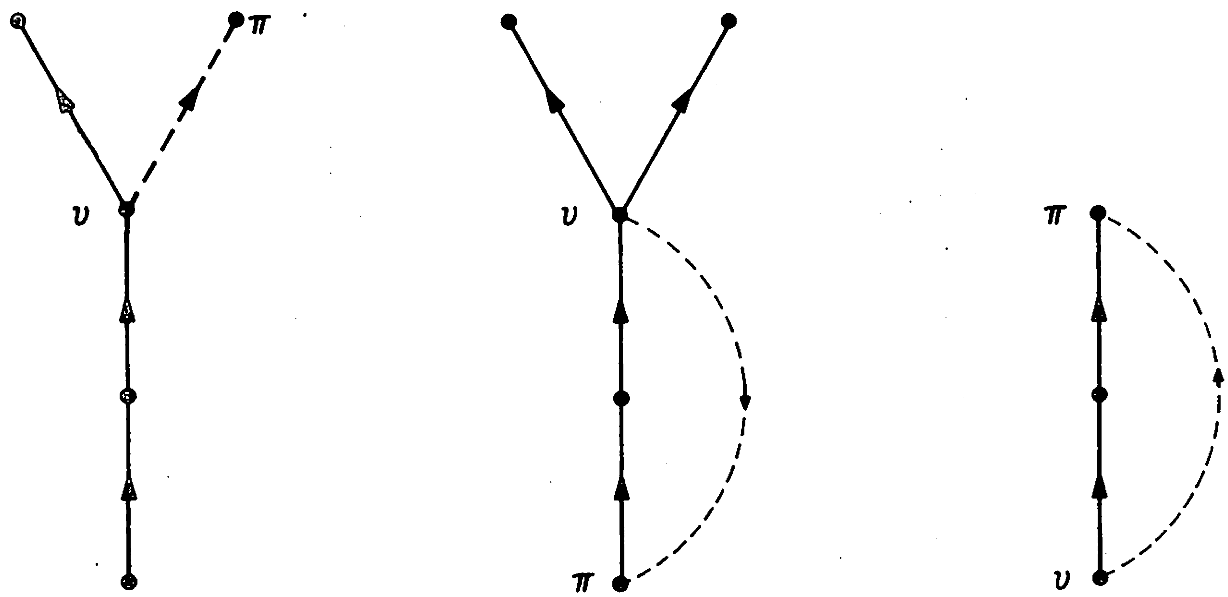
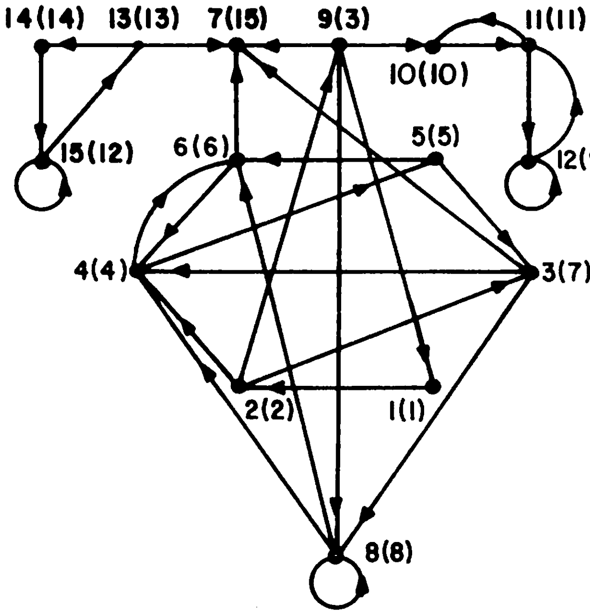


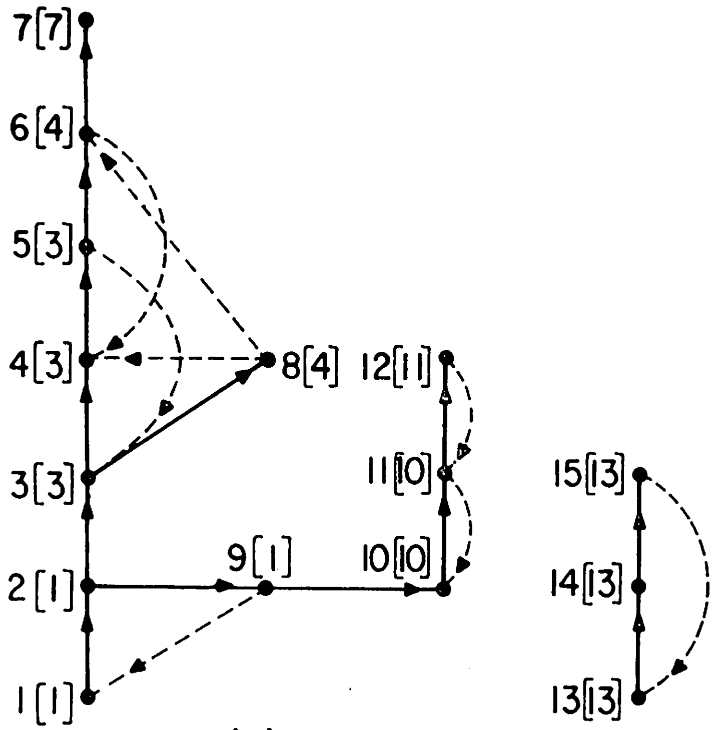
(a) (b) (c)



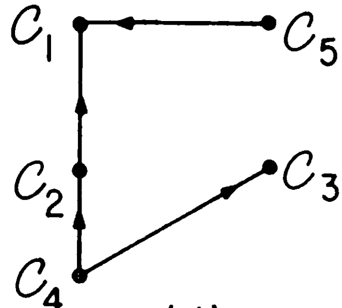
(a)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1									1						
2	1														
3		1			1										
4		1	1			1		1							
5				1											
6				1	1			1							
7			1			1			1				1		
8			1					1	1						
9		1													
10									1		1				
11										1		1			
12											1	1			
13															1
14													1		
15														1	1

(b)



(c)



(d)

OLD LABEL		NEW LABEL	
C_1		C_5	
	7		15
C_2		C_2	
	3		7
	4		4
	5		5
	6		6
	8		8
C_3		C_3	
	10		10
	11		11
	12		9
C_4		C_1	
	1		1
	2		2
	9		3
C_5		C_4	
	13		13
	14		14
	15		12

(e)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1			1												
2	1														
3		1													
4		1				1	1	1							
5				1											
6				1	1			1							
7		1			1										
8			1				1	1							
9									1		1				
10			1								1				
11									1	1					
12												1		1	
13												1			
14													1		
15			1			1	1						1		

(f)

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Stability Theory of Interconnected Systems

Part II: Strongly Connected Subsystems

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ABSTRACT

Part II studies the stability of a strongly connected subsystem (SCS) exclusively. Using the concept of minimum essential set, we partition each SCS into two parts: one part corresponds to the "forward subsystem" whose only feedbacks are self-loops; the other part corresponds to the vertices of the minimum essential set; together they form the overall feedback system. Simplified stability conditions for the SCS are obtained by exploiting this structural decomposition. Theorem III gives sufficient condition for stability of nonlinear time-varying SCSs. The other three theorems consider linear time-invariant continuous-time, lumped as well as distributed SCSs. We provide translation rules for reformulating these three theorems for the discrete-time case. We also show how to reduce the amount of calculation involved in finding the SCS characteristic polynomial. The paper ends by an example illustrating the structural decomposition and the computational advantages obtained from it.

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F. Introduction

Even though Part II of this paper can be read independently from Part I, Part II obtains its justification from Part I. Indeed Part I considered an arbitrary interconnection of subsystems and by studying the structure of the overall system, it was shown that, roughly speaking, the overall system is input-output stable if and only if the strongly connected subsystems (SCS) and the interconnection subsystems are stable. Part II studies exclusively the stability of strongly connected subsystems. In order to save space we do not replicate references and equations from Part I: so for Part II, references start with item [24], equations start with (27) and the first section is labelled "F. Introduction."

Throughout Part II we study the stability of a single strongly connected subsystem, namely, S_α^c. For convenience and to alleviate the already burdensome notation, we will drop the subscript α throughout Part II. Thus Eqs. (22) and (23) which in Part I describe the SCS S_α^c, now labelled S^c, are now written as

$$e^c - \underset{\sim}{F}^c \eta^c = u^c \tag{27}$$

$$-\underset{\sim}{G}^c e^c + \eta^c = v^c \tag{28}$$

The thrust of Part II lies in obtaining a graph theoretic decomposition of the SCS under study by using the concept of minimum essential set. Once a minimum essential set is obtained, the SCS can be viewed as partitioned into a "forward subsystem" whose only feedbacks are strictly local (self-loops) and the subsystem corresponding to the vertices of the chosen minimum essential set. Together they form an overall multiloop feedback system. The task is to obtain stability conditions for the SCS by making use of this structural decomposition. One theorem considers nonlinear time-varying subsystems. The remaining six occur in pairs, one for the lumped case and one for the distributed case, and they consider only linear time-invariant subsystems. The techniques involved are those developed recently in the study of feedback systems [2,4,5,32]. In section J we use the structure of the SCS to calculate the characteristic polynomial. In section K we give translation rules for continuous-time to discrete-time. Finally a simple example shows how the techniques of the paper are applied. The reader may find it helpful to use the example as a vehicle to illustrate the theoretical developments of the paper.

G. Graph Theoretic Decomposition on SCS

In addition to the graph theoretic terms defined in Part I Sec. C we will need the following terms. By definition, $U \subset V$ is called an essential set of a digraph $\mathcal{D} \triangleq (V,E)$ iff the section graph $\mathcal{D}(V-U)$ is acyclic. Given a digraph, an essential set with minimum number of vertices is called a minimum essential set of the digraph. It should be noted that our definitions allow an acyclic digraph to have self-loops; this follows from our requirement that a circuit be of length >1.

Consider the strongly connected subsystem S^c and its interconnection digraph $\mathcal{G} \triangleq (V,E)$ which by construction is strongly connected. We now perform a graph theoretic decomposition on $\mathcal{G}$.

Step 1: Find an essential set V^2 of $\mathcal{G}$ and define $V^1 \triangleq V-V^2$. By construction, the section graph $\mathcal{G}(V^1)$ is acyclic.

Step 2: Relabel the vertices of $\mathcal{G}$ so that every vertex in V^1 is numbered lower than all the vertices in V^2.

Step 3: Relabel the vertices of $\mathcal{G}(V^1)$ so that its adjacency matrix A^{11} is a lower triangular matrix.† ⌑

A little thought reveals that the adjacency matrix A^c of $\mathcal{G}$ after Step 3 will be in the bordered lower triangular form:

$$A^c = \begin{array}{c} \\ m^1 \\ \\ m^2 \end{array} \begin{array}{c} \begin{array}{cc} m^1 & m^2 \end{array} \\ \left[\begin{array}{c|c} A^{11} & A^{12} \\ \hline A^{21} & A^{22} \end{array}\right] \end{array} \tag{29}$$

where (i) for $i = 1,2, m^i$ is the number of vertices in V^i and (ii) A^{11} is a lower triangular matrix.

To exploit the structure of $\mathcal{G}$ as much as possible, it is obvious that one should use a <u>minimum</u> essential set in the decomposition. The problem of finding a minimum essential set has been studied by many researchers [24-29,39]. Theoretically speaking, the problem can be considered solved since it requires a finite amount of work; however the amount of work required can become potentially excessive for some large digraphs. To perform Step 1, we must first compensate for the fact that we allow self-loops, so we first remove all the self-loops in $\mathcal{G}$, then apply the algorithm given in [28] to find a minimum essential set and then put back the self-loops. Step 2 of the decomposition can be done easily. Step 3 is carried out by using the <u>topological sort</u> described in Part I, Sec. C.

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†Take note of the definition of adjacency matrix in Part I, Sec. C.

Remark. Consider the operator $\underset{\sim}{F}^c : \mathcal{L}_e^{n_o^c} \to \mathcal{L}_e^{n_i^c}$. The algorithm above can be viewed as a reordering of the scalar equations representing $\underset{\sim}{F}^c$ and of its variables; in fact since only a relabelling of the vertices of $\mathcal{G}$ is involved, the same permutation is applied to the equations as well as the variables. Clearly we are perfectly free to permute the equations independently from the permutation of the variables, this increased flexibility will, in general, decrease the size of A^{22} in A^c of (29). There is as yet no practical algorithm for doing so: however, as soon as such an algorithm is available it can be used and the theory below is applicable, except that the description of the variables associated with A^{22} as "associated with the minimum essential set" is no longer appropriate.

From now on, without loss of generality, we assume that we start out with the SCS S^c which has been relabelled after decomposition with respect to a minimum essential set. We define, for i = 1,2,

$$n_i^i \triangleq \sum_k n_{ki}, \qquad n_o^i \triangleq \sum_k n_{ko}$$

$$u^i \triangleq (u_k)_k, \qquad v^i \triangleq (v_k)_k$$

$$e^i \triangleq (e_k)_k, \qquad \eta^i \triangleq (\eta_k)_k$$

$$\underset{\sim}{G}^i : \mathcal{L}_e^{n_i^i} \to \mathcal{L}_e^{n_o^i} \text{ such that } \underset{\sim}{G}^i e^i \triangleq (\underset{\sim}{G}_k e_k)_k$$

and for j = 1,2,

$$\underset{\sim}{F}^{ij} : \mathcal{L}_e^{n_o^j} \to \mathcal{L}_e^{n_i^i} \text{ such that } \underset{\sim}{F}^{ij}\eta^j \triangleq \Big(\sum_\ell \underset{\sim}{F}_{k\ell}\eta_\ell\Big)_k$$

where in every case, k ranges over V^i and ℓ ranges over V^j. Since the $\underset{\sim}{G}_k$'s and $\underset{\sim}{F}_{k\ell}$'s in (1) and (2) are assumed to be causal nonlinear operators, the $\underset{\sim}{G}^i$'s and $\underset{\sim}{F}^{ij}$'s are also causal nonlinear operators.

Equations (27) and (28) which describe the SCS S^c can now be rewritten as four equations

$$e^1 - \underset{\sim}{F}^{11}\eta^1 - \underset{\sim}{F}^{12}\eta^2 = u^1 \tag{30}$$

$$-\underset{\sim}{G}^1 e^1 + \eta^1 = v^1 \tag{31}$$

$$-\underset{\sim}{F}^{21}\eta^1 + e^2 - \underset{\sim}{F}^{22}\eta^2 = u^2 \tag{32}$$

$$-\underset{\sim}{G}^2 e^2 + \eta^2 = v^2 \tag{33}$$

Observe that since A^{11} is lower triangular, $\underset{\sim}{F}^{11}$ as defined above is a <u>block lower triangular</u> matrix which is partitioned columnwise according to $(n_{ko})_{k\in V^1}$ and row-wise according to $(n_{ki})_{k\in V^1}$. Diagonal blocks of $\underset{\sim}{F}^{11}$ are $(\underset{\sim}{F}_{ii})_{i\in V^1}$. We define

$$\tilde{\underset{\sim}{F}}^{22} \triangleq \underset{\sim}{F}^{22} + \underset{\sim}{F}^{21}\underset{\sim}{G}^1(\underset{\sim}{I}-\underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1}\underset{\sim}{F}^{12} \tag{34}$$

The <u>matrix signal flowgraph</u> [13] associated with the nonlinear Eqs.(30)-(33) is given in Fig. 6. The flowgraph interpretation of $\tilde{\underset{\sim}{F}}^{22}$ defined by (34) is given in Fig. 7.

H. Nonlinear Time-varying Case

We first consider nonlinear time-varying subsystems. We give below a theorem which makes use of the structual decomposition of the SCS S^c.

The <u>incremental gain</u>, $\tilde{\gamma}(\underset{\sim}{G})$, <u>of a causal operator</u> $\underset{\sim}{G} : \mathcal{L}_e^n \to \mathcal{L}_e^m$ is defined as

$$\tilde{\gamma}(\underset{\sim}{G}) \triangleq \inf\{\gamma \in \mathbb{R}_+ | \forall x_1, x_2 \in \mathcal{L}_e^n,\ \forall T \in \mathcal{J},$$

$$\|\underset{\sim}{G}x_1 - \underset{\sim}{G}x_2\|_T \le \gamma \|x_1 - x_2\|_T\} \tag{40}$$

From (40), it follows that when $\underset{\sim}{G}$ is a linear operator, $\underset{\sim}{G}$ is $\mathcal{L}$-stable if and only if $\tilde{\gamma}(\underset{\sim}{G}) < \infty$.

Theorem III

Consider the SCS S^c described by (30)-(33). We assume that for all $(u^1, u^2, v^1, v^2) \in \mathcal{L}_e^{(n_i^c + n_o^c)}$, these equations have at least one solution $(e^1, e^2, \eta^1, \eta^2) \in \mathcal{L}_e^{(n_i^c + n_o^c)}$.

If (a) for $i,j = 1,2$, each $\underset{\sim}{F}^{ij}$ is $\mathcal{L}$-stable and its incremental gain $\tilde{\gamma}(\underset{\sim}{F}^{ij}) < \infty$;

(b) $\underset{\sim}{G}^1(\underset{\sim}{I} - \underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1}$ and $\underset{\sim}{G}^2(\underset{\sim}{I} - \tilde{\underset{\sim}{F}}^{22}\underset{\sim}{G}^2)^{-1}$ are $\mathcal{L}$-stable

(c) $\tilde{\gamma}[\underset{\sim}{G}^1(\underset{\sim}{I} - \underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1}] < \infty$

then the SCS S^c is $\mathcal{L}$-stable. ⌑

Comments: (i) Neither $\underset{\sim}{G}^1$ nor $\underset{\sim}{G}^2$ are required to be $\mathcal{L}$-stable. (ii) If all the $\underset{\sim}{F}^{ij}$'s are linear, then assumption (a) need only require them to be $\mathcal{L}$-stable. (iii) By referring to Fig. 6, the physical meaning of assumption (b) becomes clear: first, set u^1, v^1, v^2 identically zero in (30)-(33), a careful computation with nonlinear operators show that $\eta^2 = \underset{\sim}{G}^2(\underset{\sim}{I} - \tilde{\underset{\sim}{F}}^{22}\underset{\sim}{G}^2)^{-1} u^2$ and hence $\underset{\sim}{G}^2(\underset{\sim}{I} - \tilde{\underset{\sim}{F}}^{22}\underset{\sim}{G}^2)^{-1}$ can be viewed as the closed loop operator taking u^2 into η^2 with $u^1 \equiv 0$, $v^1 \equiv 0$, $v^2 \equiv 0$; second, set v^1 identically zero and set $\underset{\sim}{F}^{12}$ as the zero operator in (31), (32), again by computation we have $\eta^1 = \underset{\sim}{G}^1(\underset{\sim}{I} - \underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1} u^1$ and hence $\underset{\sim}{G}^1(\underset{\sim}{I} - \underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1}$ can be viewed as the <u>open loop</u> operator (since $\underset{\sim}{F}^{12}$ is the zero operator) taking u^1 into η^1, with $v^1 \equiv 0$. (iv) Another benefit of the decomposition of SCS S^c is that $\underset{\sim}{G}^1$ has a block diagonal structure which conforms with the block lower triangular structure of $\underset{\sim}{F}^{11}$; consequently $(\underset{\sim}{I}-\underset{\sim}{F}^{11}\underset{\sim}{G}^1)$ is block lower triangular with square diagonal blocks. Hence its inversion is greatly simplified: it requires only the inversion of the $(\underset{\sim}{I}-\underset{\sim}{F}_{ii}\underset{\sim}{G}_i)$'s for each $i \in V^1$.

I. <u>Linear Time-Invariant Continuous-Time Case</u>

It is well known that a very large class of linear time-invariant operators can be represented as convolution operators [30]. We shall be concerned with two classes of convolution kernels. First, we define the convolution algebra $\mathcal{A}$ [2]: f belongs to $\mathcal{A}$ iff, for $t < 0$, $f(t) = 0$, and for $t \geq 0$, $f(t) = f_a(t) + \sum_{i=0}^{\infty} f_i\, \delta(t-t_i)$ where $f_a \in L_1(\mathbb{R}_+)$, $f_i \in \mathbb{R}$ for all i, $\sum_{i=0}^{\infty} |f_i| < \infty$, $0 = t_o$, $0 < t_i$ for $i \geq 1$, and $\delta(\cdot)$ is the Dirac delta "function." An n-vector v, (n×n matrix A), is said to be in $\mathcal{A}^n$, ($\mathcal{A}^{n\times n}$, resp.), iff all its elements are in $\mathcal{A}$. Let $\hat{}$ denote Laplace transforms; $\hat{\mathcal{A}}$ denotes the commutative algebra (with pointwise product) of the $\hat{f}$'s where $f \in \mathcal{A}$. We note that (i) f belongs to the convolution algebra $\mathcal{A}$ iff $\hat{f}$ belongs to the algebra $\hat{\mathcal{A}}$ (with pointwise product); (ii) $f \in \mathcal{A}$ is invertible in $\mathcal{A}$ iff $\inf_{\mathrm{Re}\, s \geq 0} |\hat{f}(s)| > 0$; (iii) $A \in \mathcal{A}^{n\times n}$ is invertible in $\mathcal{A}^{n\times n}$ iff $\inf_{\mathrm{Re}\, s \geq 0} |\det[\hat{A}(s)]| > 0$ [37,2]. $\hat{\mathcal{A}}$,($\hat{\mathcal{A}}^{n\times n}$), is a commutative (noncommutative, resp.) algebra over the field $\mathbb{R}$ [31]. A linear time-invariant <u>distributed</u> system with input u and output y is said to be <u>$\mathcal{A}$-stable</u> iff its transfer function $\hat{H}(s) : \hat{u} \mapsto \hat{y}$ is a matrix with all its elements in $\hat{\mathcal{A}}$. It is well known that if a system is

$\mathcal{A}$-stable then (i) for any $p \in [1,\infty]$, it takes an L_p-input into an L_p-output and (ii) it takes continuous and bounded inputs (periodic inputs, almost-periodic inputs, resp.) into outputs belonging to the same classes, resp. [2,38]. For linear time-invariant lumped systems, we introduce the algebra $\mathbb{R}(s)$ of rational functions with real coefficients. An $n \times r$ rational function $\hat{H}(s)$ is said to be exponentially stable iff (i) all its elements are in $\mathbb{R}(s)$ and are proper (i.e. bounded at infinity) and (ii) $\hat{H}(s)$ has all its poles in the open left-half plane. It is easy to see that $\mathbb{R}_e(s)$, the class of all scalar exponentially stable transfer functions is an algebra over $\mathbb{R}$, in fact a subalgebra of $\hat{\mathcal{A}}$.

The following well-established identities will be used repeatedly throughout this paper. Let M, N be matrices of appropriate sizes with elements in a commutative ring, say $\mathcal{A}$, $\mathbb{R}(s)$, we have

$$\det(I-NM) = \det(I-MN) \tag{50}$$

$$M(I-NM)^{-1} = (I-MN)^{-1}M \tag{51}$$

$$I + M(I-M)^{-1} = (I-M)^{-1} \tag{52}$$

Let M be a square matrix partitioned into four submatrices M^{ij} $i,j = 1,2$ where M^{11} is square and nonsingular, then

$$\det\left(\begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}\right) = \det(M^{11}) \times \det(\tilde{M}^{22}) \tag{53}$$

and

$$\begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}^{-1} = \begin{bmatrix} (M^{11})^{-1} + (M^{11})^{-1}M^{12}(\tilde{M}^{22})^{-1}M^{21}(M^{11})^{-1} & -(M^{11})^{-1}M^{12}(\tilde{M}^{22})^{-1} \\ -(\tilde{M}^{22})^{-1}M^{21}(M^{11})^{-1} & (\tilde{M}^{22})^{-1} \end{bmatrix} \tag{54}$$

$$\text{where } \tilde{M}^{22} \triangleq M^{22} - M^{21}(M^{11})^{-1}M^{12} \tag{55}$$

We denote by $(\hat{G},\hat{F})$ the linear time-invariant feedback system described by

$$\begin{bmatrix} I & -\hat{F}(s) \\ -\hat{G}(s) & I \end{bmatrix} \begin{bmatrix} \hat{e}(s) \\ \hat{\eta}(s) \end{bmatrix} = \begin{bmatrix} \hat{u}(s) \\ \hat{v}(s) \end{bmatrix} \tag{56}$$

where $(\hat{u},\hat{v})$, $(\hat{e},\hat{\eta})$ are the Laplace-transformed input and output, respectively. It is the same system shown in Fig. 3 except that the operators $\underset{\sim}{G}$, $\underset{\sim}{F}$ are now replaced by the transfer function $\hat{G}(s)$, $\hat{F}(s)$ respectively.

Direct calculation [4] shows that the transfer function $\hat{H}_e$ of the linear time-invariant feedback system $(\hat{G},\hat{F})$ is given by

$$\hat{H}_e(s) \triangleq \begin{bmatrix} I & -\hat{F}(s) \\ -\hat{G}(s) & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (I-\hat{F}\hat{G})^{-1}(s) & \hat{F}(I-\hat{G}\hat{F})^{-1}(s) \\ \hat{G}(I-\hat{F}\hat{G})^{-1}(s) & (I-\hat{G}\hat{F})^{-1}(s) \end{bmatrix} \tag{57}$$

By definition, <u>the linear time-invariant feedback system</u> $(\hat{G},\hat{F})$ is said to be <u>exp. stable</u> (resp. <u>$\mathcal{A}$-stable</u>) iff every one of the four submatrices in (57) is exp. stable (resp. $\mathcal{A}$-stable).

In the following, we list a set of known stability results for the linear time-invariant feedback system $(\hat{G},\hat{F})$. Throughout this section, we use L and D to indicate Facts and Theorems associated with lumped and distributed systems, respectively. The similarity between these two cases is noteworthy. We first list three facts whose proofs except for Fact ID, are available in the literature [5,32]. For completeness, we include the proof of Fact ID in the Appendix AIII.

Fact IL

If $\hat{G}(s)$, $\hat{F}(s)$ are exp. stable, then the feedback system $(\hat{G},\hat{F})$ is exp. stable if and only if $\det(I-\hat{F}\hat{G})(s) \neq 0 \quad \forall s \in \bar{\mathbb{C}}_+$ ¤

Fact ID

If $\hat{G}(s)$, $\hat{F}(s)$ are $\mathcal{A}$-stable, then the feedback system $(\hat{G},\hat{F})$ is $\mathcal{A}$-stable if and only if $\inf_{s\in\mathbb{C}_+} |\det(I-\hat{F}\hat{G})(s)| > 0$. ¤

Fact IIL (resp. Fact IID)

If $\hat{F}(s)$ is exp. stable (resp. $\mathcal{A}$-stable), then the feedback system $(\hat{G},\hat{F})$ is exp. stable (resp. $\mathcal{A}$-stable) if and only if $\hat{G}(I-\hat{F}\hat{G})^{-1}(s)$ is exp. stable (resp. $\mathcal{A}$-stable). ¤

Fact IIIL

If $\hat{G}(s)$ and $\hat{F}(s)$ have no common $\bar{\mathbb{C}}_+$-pole, then the feedback system $(\hat{G},\hat{F})$ is exp. stable if and only if $\hat{G}(I-\hat{F}\hat{G})^{-1}(s)$ and $\hat{F}(I-\hat{G}\hat{F})^{-1}(s)$ are exp. stable. ¤

Fact IIID below requires the concept of pseudo-right-coprime factorization (abbreviated p.r.c.f.). Given an $m\times r$ transfer function $G(s)$.

the ordered pair $(\hat{\mathcal{N}},\hat{\mathcal{D}})$ is said to be a p.r.c.f. of $\hat{G}$ iff (i) $\hat{G} = \hat{\mathcal{N}}\hat{\mathcal{D}}^{-1}$, (ii) $\hat{\mathcal{N}} \in \hat{\mathcal{A}}^{m\times r}$ and $\hat{\mathcal{D}} \in \hat{\mathcal{A}}^{r\times r}$, (iii) there is some $\hat{\mathcal{U}} \in \hat{\mathcal{A}}^{r\times m}$, $\hat{\mathcal{V}} \in \hat{\mathcal{A}}^{r\times r}$ and $\hat{\mathcal{W}} \in \hat{\mathcal{A}}^{r\times r}$ such that

$$\hat{\mathcal{U}}\hat{\mathcal{N}} + \hat{\mathcal{V}}\hat{\mathcal{D}} = \hat{\mathcal{W}}$$

and $\det \hat{\mathcal{W}}(s) \neq 0$ for all $s \in \mathbb{C}_+$, finally (iv) for all sequences $(s_i)_1^\infty \subset \mathbb{C}_+$ with $|s_i| \to \infty$, $\liminf |\det \hat{\mathcal{D}}(s_i)| > 0$. The important fact is that if $\hat{G}(s) = R(s) + \hat{G}_b(s)$ where $R(s)$ is a proper rational matrix with poles in $\mathbb{C}_+$ and $\hat{G}_b(s) \in \hat{\mathcal{A}}^{m\times r}$ then there is an algorithm which gives a p.r.c.f. of $\hat{G}$ [2]. The concept of <u>pseudo-left-coprime factorization</u> is defined similarly except for interchange of factors.

<u>FACT IIID</u>

Suppose that $\hat{G}(s)$, $\hat{F}(s)$ have no common $\mathbb{C}_+$-pole. Suppose that $\hat{G}$ has p.ℓ.c.f. and $\hat{F}$ has p.r.c.f. or $\hat{G}$ has p.r.c.f. and $\hat{F}$ has p.ℓ.c.f. Suppose that $\forall$ sequences $(s_i)_{i=1}^\infty \subset \mathbb{C}_+$ and $|s_i| \to \infty$, $\liminf_{i\to\infty} |\det[I-\hat{F}(s_i)\hat{G}(s_i)]| > 0$. Under these conditions, the feedback system $(\hat{G},\hat{F})$ is $\mathcal{A}$-stable if and only if $\hat{G}(I-\hat{F}\hat{G})^{-1}(s)$ and $\hat{F}(I-\hat{G}\hat{F})^{-1}(s)$ are $\mathcal{A}$-stable. ⌑

For the linear time-invariant case, Eqs. (27), (28) translated into the frequency domain become

$$\begin{bmatrix} I & -\hat{F}^c(s) \\ -\hat{G}^c(s) & I \end{bmatrix} \begin{bmatrix} \hat{e}^c(s) \\ \hat{\eta}^c(s) \end{bmatrix} = \begin{bmatrix} \hat{u}^c(s) \\ \hat{v}^c(s) \end{bmatrix} \tag{60}$$

In other words, SCS S^c is simply the linear time-invariant feedback system $(\hat{G}^c,\hat{F}^c)$. In terms of decomposition quantities, we have frequency domain version of Eqs. (30)-(33),

$$\begin{bmatrix} I & 0 & -\hat{F}^{11}(s) & -\hat{F}^{12}(s) \\ 0 & I & -\hat{F}^{21}(s) & -\hat{F}^{22}(s) \\ -\hat{G}^{1}(s) & 0 & I & 0 \\ 0 & -\hat{G}^{2}(s) & 0 & I \end{bmatrix} \begin{bmatrix} \hat{e}^{1}(s) \\ \hat{e}^{2}(s) \\ \hat{\eta}^{1}(s) \\ \hat{\eta}^{2}(s) \end{bmatrix} = \begin{bmatrix} \hat{u}^{1}(s) \\ \hat{u}^{2}(s) \\ \hat{v}^{1}(s) \\ \hat{v}^{2}(s) \end{bmatrix} \tag{61}$$

To find the transfer function $\hat{H}_e^c(s)$ of SCS S^c, which is the inverse of the matrix in (60) and (61), we first perform the block Gaussian elimination as follows: (i) add $\hat{G}^1$ times row 1 to row 3, (ii) add $\hat{G}^2$ times row 2 to row 4 and (iii) add $\hat{G}^2\hat{F}^{21}(I-\hat{G}^1\hat{F}^{11})^{-1}$ times row 3 to row 4. We now have

$$\begin{bmatrix} I & 0 & -\hat{F}^{11} & -\hat{F}^{12} \\ 0 & I & -\hat{F}^{21} & -\hat{F}^{22} \\ 0 & 0 & (I-\hat{G}^1\hat{F}^{11}) & -\hat{G}^1\hat{F}^{12} \\ 0 & 0 & 0 & (I-\hat{G}^2\hat{\hat{F}}^{22}) \end{bmatrix} \begin{bmatrix} \hat{e}^{1} \\ \hat{e}^{2} \\ \hat{\eta}^{1} \\ \hat{\eta}^{2} \end{bmatrix} = \begin{bmatrix} \hat{u}^{1} \\ \hat{u}^{2} \\ \hat{G}^1\hat{u}^1+\hat{v}^1 \\ \hat{G}^2\hat{F}^{21}(I-\hat{G}^1\hat{F}^{11})^{-1}(\hat{G}^1\hat{u}^1+\hat{v}^1)+\hat{G}^2\hat{u}^2+\hat{v}^2 \end{bmatrix} \tag{62}$$

where $$\hat{\hat{F}}^{22} \triangleq \hat{F}^{22} + \hat{F}^{21}\hat{G}^1(I-\hat{F}^{11}\hat{G}^1)^{-1}\,\hat{F}^{12} \tag{63}$$

By back substitution in (62), we obtain the transfer function $\hat{H}_e^c(s)$. Using (57), four submatrices of $\hat{H}_e^c(s)$ can be described as follows:

$$(I-\hat{F}^c\hat{G}^c)^{-1} = \begin{bmatrix} (I-\hat{F}^{11}\hat{G}^1)^{-1}(I+\hat{F}^{12}\hat{G}^2\hat{M}^{21}\hat{G}^1) & \hat{M}^{12}\hat{G}^2 \\ \hat{M}^{21}\hat{G}^1 & (I-\hat{\hat{F}}^{22}\hat{G}^2)^{-1} \end{bmatrix} \tag{64}$$

$$(I-\hat{G}^c\hat{F}^c)^{-1} = \begin{bmatrix} (I-\hat{G}^1\hat{F}^{11})^{-1}(I+\hat{G}^1\hat{F}^{12}\hat{G}^2\hat{M}^{21}) & \hat{G}^1\hat{M}^{12} \\ \hat{G}^2\hat{M}^{21} & (I-\hat{G}^2\hat{\hat{F}}^{22})^{-1} \end{bmatrix} \tag{65}$$

$$\hat{G}^c(I-\hat{F}^c\hat{G}^c)^{-1} = \begin{bmatrix} \hat{G}_1(I-\hat{F}^{11}\hat{G}^1)^{-1}(I+\hat{F}^{12}\hat{G}^2\hat{M}^{21}\hat{G}^1) & \hat{G}^1\hat{M}^{12}\hat{G}^2 \\ \hat{G}^2\hat{M}^{21}\hat{G}^1 & \hat{G}^2(I-\hat{\hat{F}}^{22}\hat{G}^2)^{-1} \end{bmatrix} \tag{66}$$

$$\hat{F}^c(I-\hat{G}^c\hat{F}^c)^{-1} = \begin{bmatrix} (I-\hat{F}^{11}\hat{G}^1)^{-1}(\hat{F}^{11}+\hat{F}^{12}\hat{G}^2\hat{M}^{21}) & \hat{M}^{12} \\ \hat{M}^{21} & \hat{\hat{F}}^{22}(I-\hat{G}^2\hat{\hat{F}}^{22})^{-1} \end{bmatrix} \tag{67}$$

where $\hat{M}^{12} \triangleq (I-\hat{F}^{11}\hat{G}^1)^{-1}\hat{F}^{12}(I-\hat{G}^2\hat{\hat{F}}^{22})^{-1}$ (68)

and $\hat{M}^{21} \triangleq (I-\hat{\hat{F}}^{22}\hat{G}^2)^{-1}\hat{F}^{21}(I-\hat{G}^1\hat{F}^{11})^{-1}$ (69)

Obviously, Eqs. (64)-(67) can also be obtained by direct calculation using (54).

We shall now present a number of theorems on the stability of the linear time-invariant SCS S^c which make use of the structure obtained in decomposition. The first two theorems give necessary and sufficient conditions for stability.

<u>Theorem IV L</u>

Consider a linear time-invariant <u>lumped</u> SCS S^c described by (60). If $\hat{G}^c(s)$, $\hat{F}^c(s)$ are exp. stable, then S^c is exp. stable if and only if $\forall s \in \bar{\mathbb{C}}_+$, $\forall i \in V^1$, $\det(I-\hat{F}_{ii}\hat{G}_i)(s) \neq 0$ and $\det(I-\hat{\hat{F}}^{22}\hat{G}^2)(s) \neq 0$. ◻

<u>Theorem IV D</u>

Consider a linear time-invariant <u>distributed</u> SCS S^c described by (60). If $\hat{G}^c(s)$, $\hat{F}^c(s)$ are $\mathcal{A}$-stable, then S^c is $\mathcal{A}$-stable if and only if $\forall i \in V^1$, $\inf_{s\in\mathbb{C}_+} |\det(I-\hat{F}_{ii}\hat{G}_i)(s)| > 0$ and $\inf_{s\in\mathbb{C}_+} |\det(I-\hat{\hat{F}}^{22}\hat{G}^2)(s)| > 0$. ◻

Comments (i) Theorems IV L and IV D have the following meaning: In case S^c is formed by an interconnection of stable subsystems, the SCS S^c is stable if and only if every (local) feedback system $(\hat{G}_i, \hat{F}_{ii})$ (where $i \notin V^2$ the minimum essential set) is stable and also the overall feedback system $(\hat{G}^2, \hat{\hat{F}}^{22})$ formed on the one hand by the subsystems associated with the minimum essential set and on the other hand, by the subsystems not associated with the minimum essential set. Note that the feedback systems $(\hat{G}_i, \hat{F}_{ii})_{i \in V^1}$ correspond to self-loops associated with the vertices of $\mathcal{G}(V^1)$.

(ii) The two theorems above, namely, IV L and IV D, are a good illustration of the benefits that follows from the exploitation of the structure of the SCS S^c. If we applied Fact IL and ID, we would have to consider $\det(I-\hat{F}^c\hat{G}^c)$, the determinant of a matrix of dimension $(n_i^1+n_i^2)$. Thanks to the decomposition we need only check $\det(I-\hat{F}_{ii}\hat{G}_i)$, $\forall i \in V^1$ and $\det(I-\hat{\hat{F}}^{22}\hat{G}^2)$. Furthermore in case of instability these new conditions will pinpoint the location of the instability and, hence, help in the stabilization.

We now consider sufficient stability conditions:

Theorem V L (resp. Theorem V D)

Consider a linear time-invariant lumped (resp. distributed) SCS S^c described by (60). If (a) the feedback subsystem $(\hat{G}^1, \hat{F}^{11})$ is exp. stable (resp. $\mathcal{A}$-stable); (b) the feedback subsystem $(\hat{G}^2, \hat{\hat{F}}^{22})$ is exp. stable (resp. $\mathcal{A}$-stable) and (c) $\hat{F}^{12}(s)$ and $\hat{F}^{21}(s)$ are exp. stable (resp. $\mathcal{A}$-stable) then the SCS S^c is exp. stable (resp. $\mathcal{A}$-stable). ¤

Comments (i) Facts IL, IIL and IIIL give three sets of conditions under which conditions (a), (or (b)) of Theorem V L will hold. Hence using these together with Theorem V L we would derive $3\times 3 = 9$ corollaries. A similar comment holds for Theorem V D. (ii) In view of the four (2,2) position submatrices in Eq. (64)-(67), it is obvious that assumption (b) of Theorem V is also a necessary condition. (iii) Theorem V has a form similar to that of Theorem II.

Theorem VI L (resp. Theorem VI D)

Consider a linear time-invariant lumped (resp. distributed) SCS S^c described by (60). If (a) $\hat{F}^c(s)$ is exp. stable (resp. $\mathcal{A}$-stable) and (b) $\hat{G}^1(I-\hat{F}^{11}\hat{G}^1)^{-1}(s)$ and $\hat{G}^2(I-\hat{F}^{22}\hat{G}^2)^{-1}(s)$ are exp. stable (resp. $\mathcal{A}$-stable), then the SCS S^c is exp. stable (resp. $\mathcal{A}$-stable). ¤

Comments: (i) Theorem VI L can be viewed as a specialization of Theorem III by using the linearity of all the subsystems and taking the space $\mathcal{L}$ to be $L^n_\infty(\mathbb{R}_+)$. This is, however, not the case for Theorem VI D because a convolution operator is L^n_∞-stable if and only if its kernel is a bounded measure and $\mathcal{A}$ is a subalgebra of the convolution algebra of bounded measures. The formulation of Theorem VI emphasizes the analogy between the lumped and the distributed case as well as the essentially algebraic nature (i.e. closure) of the result. (ii) The comments (i), (iii) and (iv) of Theorem III also apply to Theorem VI. (iii) Condition (a) and (b) of Theorem VI imply by (63) that $\hat{F}^{22}$ is exp. stable ($\mathcal{A}$-stable)

J. Characteristic Polynomial

In this section we first show that the structure of the matrices $\hat{F}^c$ and $\hat{G}^c$ leads to simplification in the computation of the characteristic polynomial provided the computations are carried out selectively. For simplicity of exposition, we detail the procedure for the lumped case. It is clear that for the distributed case a similar computation can be carried out using pseudo-coprime factorizations [4]. Next we develop some interpretations of the Theorems IV L, V L and VI L which illuminate the effect of the structural decomposition on the statements of these theorems.

J.1. Obtaining the Characteristic Polynomial

Let $\hat{H}(s) \in \mathbb{R}(s)^{n\times m}$ be proper (bounded at infinity), let (A,B,C,E) be any minimal state space realization of $\hat{H}(s)$, then det(sI-A) is called the characteristic polynomial of the transfer function $\hat{H}(s)$. The transfer function $\hat{H}(s) \in \mathbb{R}(s)^{n\times m}$ can be written as a matrix fraction

$$\hat{H}(s) = D_{H\ell}(s)^{-1}N_{H\ell}(s) = N_{Hr}(s)D_{Hr}(s)^{-1}$$

where the four matrices $D_{H\ell}$, $N_{H\ell}$, N_{Hr}, D_{Hr} are polynomial matrices; furthermore $D_{H\ell}$ and $N_{H\ell}$ are left coprime i.e. their only left common factor which is a polynomial matrix is a unimodular matrix (nonsingular with constant determinant); similarly D_{Hr} and N_{Hr} are right-coprime. For algorithms for calculating such factorizations see [33,34,2]. It is well known [4] that the characteristic polynomial $\chi(s)$ of the transfer function of the feedback system $(\hat{G},\hat{F})$ is given by

$$\chi(s) = \det[D_{F\ell}D_{Gr} - N_{F\ell}N_{Gr}](s) \tag{80}$$

$$= \det[D_{G\ell}D_{Fr} - N_{G\ell}N_{Fr}](s) \tag{81}$$

where $\hat{G} = D_{G\ell}^{-1}N_{G\ell} = N_{Gr}D_{Gr}^{-1}$, $\hat{F} = D_{F\ell}^{-1}N_{F\ell} = N_{Fr}D_{Fr}^{-1}$

are left-coprime and right-coprime factorizations of $\hat{G}$ and $\hat{F}$ respectively.

All the "numerators" N and "denominators" D in the formulas above are assumed to be multiplied by a nonzero constant factor so that the polynomials (80) and (81) are monic. We now use (80) to obtain the characteristic polynomial of the feedback system $(\hat{G}^c,\hat{F}^c)$, equivalently the SCS S^c, by taking advantage of its structural decomposition.

First, we find a left-coprime factorization $(D_{F\ell},N_{F\ell})$ of $\hat{F}^c$ as follows:

$$\hat{F}^c(s) = \begin{bmatrix} \hat{F}^{11}(s) & \hat{F}^{12}(s) \\ \hat{F}^{21}(s) & \hat{F}^{22}(s) \end{bmatrix} \tag{82}$$

$$= \begin{bmatrix} D_1(s) & 0 \\ 0 & D_2(s) \end{bmatrix}^{-1} \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \tag{83}$$

where for i = 1,2 $D_i(s)$ is a diagonal matrix and its jth diagonal element is the least common multiple of the denominators of the elements in jth row of $\hat{F}^{i1}(s)$ and $\hat{F}^{i2}(s)$. Since $\hat{F}^{11}(s)$ is a block lower triangular matrix due to the structural decomposition, so is $N_{11}(s)$. To find the greatest common left divisor of the two polynomial matrices in (83), we perform elementary column operation on the polynomial matrix [33,34,2]

$$\begin{bmatrix} N_{11}(s) & N_{12}(s) & D_1(s) & 0 \\ N_{21}(s) & N_{22}(s) & 0 & D_2(s) \end{bmatrix}$$

and obtain a lower triangular polynomial matrix

$$\begin{bmatrix} R_{11}(s) & 0 & 0 & 0 \\ R_{21}(s) & R_{22}(s) & 0 & 0 \end{bmatrix}$$

The square matrix $\begin{bmatrix} R_{11}(s) & 0 \\ R_{21}(s) & R_{22}(s) \end{bmatrix}$ is a greatest common left divisor of the two matrices in (83), hence a left coprime factorization of $\hat{F}^c$ is given by

$$\hat{F}^c(s) = D_{F\ell}(s)^{-1} N_{F\ell}(s) \tag{84}$$

where (dropping the dependence on s)

$$D_{F\ell} = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix}^{-1} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} = \begin{bmatrix} R_{11}^{-1}D_1 & 0 \\ X & X \end{bmatrix}$$

and

$$N_{F\ell} = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} R_{11}^{-1}N_{11} & X \\ X & X \end{bmatrix}$$

where X denotes the appropriate submatrix. Note that by construction, $N_{F\ell}$ and $D_{F\ell}$ are <u>polynomial</u> matrices. Observe that $R_{11}^{-1}N_{11}$ is block lower triangular with the same block partition as $\hat{F}^{11}$ and that $R_{11}^{-1}D_1$ is lower triangular.

Second, we find a right coprime factorization (D_{Gr}, N_{Gr}) of $\hat{G}^c$. For each subsystem i of S^c, i.e. for all $i \in V^1 \cup V^2$, let $\hat{G}_i = N_{G_ir}D_{G_ir}^{-1}$ be a right coprime factorization of $\hat{G}_i$. For j = 1,2 let $N_{Gr}^j \triangleq \text{diag.}(N_{G_ir})_{i\in V^j}$ and let $D_{Gr}^j \triangleq \text{diag.}(D_{G_ir})_{i\in V^j}$. Since $\hat{G}^c = \text{diag.}(\hat{G}_i)_{i\in V^1\cup V^2}$, obviously $(\text{diag.}(D_{G_ir})_{i\in V^1\cup V^2}, \text{diag.}(N_{G_ir})_{i\in V^1\cup V^2})$ is a right coprime factorization of $\hat{G}^c$. In other words,

$$D_{Gr} = \begin{bmatrix} D^1_{Gr} & 0 \\ 0 & D^2_{Gr} \end{bmatrix} \quad \text{and} \quad N_{Gr} = \begin{bmatrix} N^1_{Gr} & 0 \\ 0 & N^2_{Gr} \end{bmatrix} \tag{85}$$

and $$\hat{G}^C = N_{Gr} D^{-1}_{Gr} \tag{86}$$

By (80), we have the characteristic polynomial of SCS S^c,

$$\chi\,(s) = \det\left(\begin{bmatrix} R^{-1}_{11} D_1 D^1_{Gr} - R^{-1}_{11} N_{11} N^1_{Gr} & X \\ X & X \end{bmatrix}\right)(s) \tag{87}$$

Since $R^{-1}_{11}D_1$ is lower triangular and D^1_{Gr} is block diagonal with size of $\underline{i}$th diagonal block being $n_{ii} \times n_{ii}$, $R^{-1}_{11}D_1 D^1_{Gr}$ is block lower triangular with size of $\underline{i}$th diagonal block being $n_{ii} \times n_{ii}$. Similarly, $R^{-1}_{11}N_{11}$ is block lower triangular with size of $\underline{i}$th diagonal block being $n_{ii} \times n_{io}$ and N^1_{Gr} is block diagonal with size of $\underline{i}$th diagonal block being $n_{io} \times n_{ii}$, thus $R^{-1}_{11}N_{11}N^1_{Gr}$ is block lower triangular with size of $\underline{i}$th diagonal block being $n_{ii} \times n_{ii}$. Therefore $(R^{-1}_{11}D_1D^1_{Gr} - R^{-1}_{11}N_{11}N^1_{Gr})$ is block lower triangular with size of $\underline{i}$th diagonal block being $n_{ii} \times n_{ii}$. Hence using (53) to evaluate the determinant in (87), we find that $\det(R^{-1}_{11}D_1D^1_{Gr} - R^{-1}_{11}N_{11}N^1_{Gr})$ is the product of determinants of $n_{ii} \times n_{ii}$ matrices for all $i \in V^1$.

Remark: At first sight one might think that one could save computations by using (81). However this will not work because when we post-multiply $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$ by either $\begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix}^{-1}$ or by $\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}^{-1}$, the block lower triangular form of N_{11} will be destroyed.

J.2. Interpretation of Theorem IV L, V L and VI L

In theorems IV, V, and VI we saw that the stability of the feedback system $(\hat{G}^2(s), \hat{\hat{F}}^{22}(s))$ played a key role in the stability of the SCS S^c.

We now exhibit the relationship between $\chi(s)$, the characteristic polynomial of S^c, and $\chi_2(s)$, the characteristic polynomial of $(\hat{G}^2(s),\hat{\hat{F}}^{22}(s))$.

Start with Eq. (61) rewritten in a different order so that the variables associated with $(\hat{G}^2,\hat{\hat{F}}^{22})$ are written last; furthermore we use factorizations (84) and (86); inserting superscripts (i,j) for the partitioned matrices of the factorization, we obtain successively:

$$\hat{\hat{H}} = \begin{bmatrix} 0 & \hat{F}^{11} & 0 & \hat{F}^{12} \\ \hat{G}^1 & 0 & 0 & 0 \\ 0 & \hat{F}^{21} & 0 & \hat{F}^{22} \\ 0 & 0 & \hat{G}^2 & 0 \end{bmatrix} = \begin{bmatrix} \hat{D}^{11}_{F\ell} & & & \\ & I & & \\ D^{21}_{F\ell} & & D^{22}_{F\ell} & \\ & & & I \end{bmatrix}^{-1} \begin{bmatrix} & N^{11}_{F\ell} & & N^{12}_{F\ell} \\ N^1_{Gr} & & & \\ & N^{21}_{F\ell} & & N^{22}_{F\ell} \\ & & N^2_{Gr} & \end{bmatrix}$$

$$\cdot \begin{bmatrix} D^1_{Gr} & & & \\ & I & & \\ & & D^2_{Gr} & \\ & & & I \end{bmatrix}^{-1} \tag{91}$$

which is of the form

$$\hat{\hat{H}} = D_1^{-1} N D_2^{-1} \tag{92}$$

with (D_1,N) left coprime and (N,D_2) right coprime. From (91), we can calculate the transfer function of S^c

$$(I-\hat{\hat{H}})^{-1} : (\hat{u}^1,\hat{v}^1,\hat{u}^2,\hat{v}^2) \mapsto (\hat{e}^1,\hat{\eta}^1,\hat{e}^2,\hat{\eta}^2) \tag{93}$$

where

$$(I-\hat{\hat{H}})^{-1} = D_2(D_1D_2-N)^{-1}\, D_1. \tag{94}$$

In view of the coprime properties of the polynomial matrices the characteristic polynomial of S^c is given by [35]

$$\chi(s) = \det[D_1D_2-N]\ (s) \tag{95}$$

Let D_2'' denote the last two block-rows of D_2 and D_1'' denote the last two block-columns of D_1,

then (91)-(94) it follows that

$$D_2''(D_1D_2-N)^{-1}D_1'' \tag{96}$$

is a representation of the input-output transfer function of $(\hat{G}^2,\hat{\tilde{F}}^{22})$, namely, that which maps $(\hat{u}^2,\hat{v}^2) \mapsto (\hat{e}^2,\hat{\eta}^2)$. The three polynomial matrices appearing in (96) no longer have the pairwise coprimeness. So let

$$R_\ell(s) \triangleq \text{g.}\ell\text{.c.d. of } (D_1D_2-N) \text{ and } D_1''$$

$$R_r(s) \triangleq \text{g.r.c.d. of } R_\ell^{-1}(D_1D_2-N) \text{ and } D_2''$$

then, it is well known that [35]

$$\det[R_\ell^{-1}(D_1D_2-N)R_r^{-1}]\ (s) = \chi_2(s). \tag{97}$$

Hence using the product of determinants rule, we conclude that

$$\chi(s) = \chi_2(s) \cdot \det R_\ell(s) \cdot \det R_r(s)\ . \tag{98}$$

Now the zeros of the polynomial $\det R_\ell(s)$ are the input decoupling zeros [36] of S^c when only the inputs u^2 and v^2 are operative, i.e. they correspond to all modes of S^c that are <u>uncontrollable</u> by (u^2,v^2); because of the order of our factorizations, the zeros of the polynomial $\det R_r(s)$ are the output decoupling zeros of S^c (when only the outputs e^2 and η^2 are observed) and which are not input decoupling zeros; i.e.

the zeros of det $R_{\ell}(s)$ correspond to modes of S^c that are <u>controllable</u> (from (u^2,v^2)) but <u>unobservable</u> at (e^2,η^2). In the light of this interpretation it becomes clear why the stability condition specified by theorems IV, V and VI require two sets of conditions: one that guarantees the stability of the feedback system $(\hat{G}^2,\hat{\tilde{F}}^{22})$ and the other guarantees the stability of the other modes.

An alternate way of looking at this question is to start by considering a <u>minimal</u> realization of S^c, then restrict the inputs to (u^2,v^2) and restrict the outputs to (e^2,η^2). This gives a realization of the "minimum essential subsystem" $\tilde{S}^{22}$ but the realization thus obtained is not necessarily minimal. If it were minimal, then S^c is stable if and only if $\tilde{S}^{22}$ is stable and $\chi(s) = \chi_2(s)$. If it is not minimal then the stability of S^c cannot be decided by only studying $\tilde{S}^{22}$ because, roughly speaking, S^c contains modes that are not in $\tilde{S}^{22}$. The factors det $R_{\ell}(s)$ and det $R_r(s)$ in (98) serve to check on the stability of these modes.

K. <u>Linear Time-invariant Discrete-time Case</u>

The results in Sections I and J above are stated for the <u>continuous-time case</u>. A study of the proofs would easily show that they extend easily to the <u>discrete-time case</u>. The required changes are listed in the Table I: $B(0,1)$ and $B(0,1)^C$ denote the <u>open</u> unit ball centered on 0 in $\mathbb{C}$ and its complement, resp.; ℓ_1 denotes the convolution algebra of absolutely convergent sequences: $\ell_1 = \{(z_i)_o^\infty \subset \mathbb{C} \mid \sum_o^\infty |z_i| < \infty\}$, (for details see [2]).

<u>Table I</u>

Continuous-time		Discrete-time
Laplace transform	$\rightarrow$	Z-transform
$\mathcal{A}$	$\rightarrow$	ℓ_1
$\mathcal{A}^{n\times n}$	$\rightarrow$	$\ell_1^{n\times n}$
$\mathring{\mathbb{C}}_-$	$\rightarrow$	$B(0,1)$
$\mathbb{C}_+$	$\rightarrow$	$B(0,1)^C$
$s \rightarrow \infty$	$\rightarrow$	$z \rightarrow \infty$
$\mathbb{R}(s)^{n\times n}$	$\rightarrow$	$\mathbb{R}(z)^{n\times n}$

L. Example

Consider the interconnection S^c of linear time-invariant lumped multivariable subsystems shown in Fig. 8(a). Using the labelling of Fig. 8(a), we obtain the interconnection digraph $\mathcal{G}$ of S^c and its associated adjacency matrix which are shown in Fig. 8(b) and (c) respectively. Due to the circuit (3,1,4,5,2), every pair of vertices in the interconnection digraph are mutually reachable and hence the system S^c is a strongly connected subsystem (SCS).

Recalling that we disregard the self-loops in finding a minimum essential set of $\mathcal{G}$, we can easily check that the vertex 2 is contained in every circuit of $\mathcal{G}$. Hence {2} is a minimum essential set of $\mathcal{G}$ and for this particular digraph, it is the <u>only</u> minimum essential set. The acyclic section digraph $\mathcal{G}(\{1,3,4,5\})$ is shown in Fig. 8(d). Applying the <u>topological sort</u> on this section digraph, we obtain the new labeling given in Fig. 8(e). The adjacency matrix with respect to the new labeling is given in Fig. 8(f). Note that it is a <u>bordered lower triangular</u> matrix.

From now on, we will use the new labeling throughout.

Suppose that the transfer function matrices $\hat{G}^c$, $\hat{F}^c$ of the subsystems and the interconnection subsystems are specified by Eqs. (101) and (102) below.

Note that both $\hat{G}^c$, $\hat{F}^c$ are proper but not exp. stable. Hence Theorems IV L and VI L cannot be applied to this system. To show that SCS S^c is exp. stable, we use Theorem V L: first we note that $\hat{F}^{12}$, $\hat{F}^{21}$ are exp. stable, next we have to show that the feedback subsystems $(\hat{G}^1,\hat{F}^{11})$ and $(\hat{G}^2,\hat{F}^{22})$ are both exp. stable.

Since neither $\hat{G}^1$ nor $\hat{F}^{11}$ are exp. stable, no simplifying theorems on stability condition applies to the feedback system $(\hat{G}^1,\hat{F}^{11})$. So the most convenient procedure is to compute its characteristic polynomial χ_1 and to check that it has no $\mathbb{C}_+$-zero. First by finding a right coprime factorization $(D_{G_i r},N_{G_i r})$ of $\hat{G}_i$ for each i = 1,2,3,4, we obtain a right-coprime factorization $(D_{G^1 r},N_{G^1 r})$ of $\hat{G}^1$; next we obtain a left-coprime factorization $(D_{F^{11}\ell},N_{F^{11}\ell})$ of $\hat{F}^{11}$; these are shown in (103)-(106).

It can be checked that for this particular example, the $\underline{i}$th diagonal blocks of $D_{F^{11}\ell}$ and $N_{F^{11}\ell}$ also form a left-coprime factorization of $\hat{F}_{ii}$. In general this is not always true. For convenience, we denote the $\underline{i}$th diagonal blocks of $D_{F^{11}\ell}$, $N_{F^{11}\ell}$ as $D_{F_{ii}\ell}$, $N_{F_{ii}\ell}$ respectively. Since $D_{G^1 r}$, $N_{G^1 r}$, $D_{F^{11}\ell}$, $N_{F^{11}\ell}$ are block lower triangular, so is $(D_{F^{11}\ell}D_{G^1 r} - N_{F^{11}\ell}N_{G^1 r})$. Hence the characteristic polynomial of the feedback system $(\hat{G}^1,\hat{F}^{11})$,

$$\chi_1(s) = \det(D_{F^{11}\ell}D_{G^1 r} - N_{F^{11}\ell}N_{G^1 r})(s)$$

$$= \prod_{i=1}^{4} \det(D_{F_{ii}\ell} D_{G_i r} - N_{F_{ii}\ell} N_{G_i r})\ (s)$$

$$= -\ (s^2+2s+2)(2s^2+9s+1)(s^2+s+2)\ 6(s+1)(s+2)(s+3)(s+1)(s^2+s+8) \tag{107}$$

Since χ_1 has no $\mathbb{C}_+$-zero, the feedback system $(\hat{G}^1, \hat{F}^{11})$ is exp. stable.

To determine the stability of the feedback system $(\hat{G}^2, \hat{\hat{F}}^{22})$ we first note that $\hat{G}^2$ is exp. stable. We compute $\hat{\hat{F}}^{22} \triangleq \hat{F}^{22} + \hat{F}^{21}\hat{G}^1(I-\hat{F}^{11}\hat{G}^1)^{-1}\hat{F}^{12}$ and obtain

$$\hat{\hat{F}}^{22}(s) = \frac{-1}{(s+1)^2(s+2)^2} \begin{bmatrix} 25s^2+67s+25 \\ 29s^2+68s+56 \end{bmatrix} \tag{108}$$

Since $\hat{\hat{F}}^{22}$ is also exp. stable, by Fact IL, the feedback system $(\hat{G}^2, \hat{\hat{F}}^{22})$ is exp. stable if and only if $\det(I-\hat{\hat{F}}^{22}\hat{G}^2)$ has no $\bar{\mathbb{C}}_+$-zero.

$$\det(I-\hat{\hat{F}}^{22}\hat{G}^2)\ (s) = \det(1-\hat{G}^2\hat{\hat{F}}^{22})\ (s)$$

$$= (1-\hat{G}^2\hat{\hat{F}}^{22})\ (s)$$

$$= (4s^4+24s^3+152s^2+187s+38)/4(s+1)^2(s+2)^2 \tag{109}$$

It can be checked by the Liénard-Chipart Test that the numerator polynomial in (109) has no $\mathbb{C}_+$-zero and since the rational function in (109) is bounded away from zero as $|s| \to \infty$, $\det(I-\hat{\hat{F}}^{22}\hat{G}^2)$ has no $\bar{\mathbb{C}}_+$-zero. Therefore $(\hat{G}^2, \hat{\hat{F}}^{22})$ is exp stable.

Hence by Theorem V L, the SCS S^c is exp stable.

$$\hat{G}^{c}(s) = \left[\begin{array}{cc|c|cc|c|cc}
\frac{-3s+2}{s^2-1} & \frac{s}{s-1} & & & & & & \\
\frac{-(2s+1)}{s^2-1} & \frac{s+2}{s-1} & & & & & 0 & \\
\hline
 & & -\frac{2}{s} & & & & & \\
\hline
 & & & -1 & \frac{1}{s+2} & & & \\
 & & & \frac{1}{s+1} & \frac{1}{s+1} & & & \\
 & & & \frac{s}{s+3} & 0 & & & \\
\hline
 & & & & & \frac{-s}{s-2} & & \\
 & 0 & & & & \frac{3(s+1)}{s-2} & & \\
\hline
 & & & & & & \frac{s+1}{s+2} & \frac{-1}{4(s+2)}
\end{array}\right] \tag{101}$$

$$
\hat{F}^c(s) = \left[\begin{array}{cc|c|ccc|cc|c}
\frac{s+1}{(s-1)} & \frac{1}{(s-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2s^2+7s+3}{(s-1)(s-2)} & \frac{s^2+5}{(s-1)(s-2)} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(2s^2+9s+1)(s^2+s+2)}{(s+1)^2(s+2)^2} \\
\hline
\frac{-3s}{(s-1)} & \frac{-(s+\frac{1}{2})}{(s-1)} & \frac{s+1}{(s-1)} & 0 & 0 & 0 & 0 & 0 & \frac{s^2+s+2}{(s+1)(s+2)} \\
\hline
\frac{-18s^2+22s+3}{(s-1)(s-2)} & \frac{-s^2-\frac{19}{2}s+14}{(s-1)(s-2)} & \frac{-s+7}{(s-1)} & 0 & 0 & 0 & 0 & 0 & \frac{-1}{s+1} \\
\frac{-s^2-39s+32}{(s-1)(s-2)} & \frac{-3s^2-5s+4}{(s-1)(s-2)} & -2 & 0 & 0 & 0 & 0 & 0 & \frac{-12}{s+12} \\
\hline
0 & 0 & \frac{2s^2-14s+16}{(s-1)^2(s+1)} & \frac{s}{(s^2-1)} & \frac{1}{(s-1)} & \frac{s+3}{(s^2-1)} & \frac{1}{(s-1)} & \frac{-(s+2)}{(s^2-1)} & 0 \\
\hline
0 & 0 & 0 & \frac{1}{s+2} & \frac{s+1}{s+2} & \frac{s+3}{s+1} & \frac{3}{s+1} & \frac{s}{(s+1)^2} & a(s) \\
0 & 0 & 0 & \frac{1}{s+1} & 1 & \frac{s+3}{s+2} & \frac{s+1}{s+2} & \frac{s}{3(s+2)} & b(s)
\end{array}\right]
$$

$$
a(s) = -\frac{(s+1)}{(s+2)^2} - \frac{s(19s^4-1\frac{1}{2}s^3-55\frac{1}{2}s^2+72s+2)}{(s+1)^3(s+2)^2} - \frac{(s+3)(4s^4+69s^3-53s^2+76s-124)}{(s+1)^2(s+2)^4}
$$

$$
b(s) = -\frac{(s+2)}{(s+1)^2} - \frac{s(19s^4-1\frac{1}{2}s^3-55\frac{1}{2}s^2+72s+2)}{(s+1)^2(s+2)^3} - \frac{(s+3)(4s^4+69s^3-53s^2+76s-124)}{(s+1)^3(s+2)^3} \quad (102)
$$

$$N_{G^1 r}(s) = \left[\begin{array}{cccccc}
s-2 & s & & & \bigcirc & \\
s+1 & s+2 & & & & \\
 & & -2 & & & \\
 & & & s-3 & 6(s+1) & \\
 & & & 2s+3 & 6(s+2) & \\
 & & & s & 0 & \\
 & \bigcirc & & & & -s \\
 & & & & & 3(s+1)
\end{array}\right] \tag{103}$$

$$D_{G^1 r}(s) = \left[\begin{array}{cccccc}
s+1 & 0 & & & \bigcirc & \\
s & s-1 & & & & \\
 & & s & & & \\
 & & & s+3 & 0 & \\
 & & & 2s(s+2) & 6(s+1)(s+2) & \\
 & \bigcirc & & & & s-2
\end{array}\right] \tag{104}$$

$$N_{F^{11}\ell}(s) = \left[\begin{array}{cc|c|ccc|cc}
s+1 & 1 & & & & \bigcirc & & \\
s+2 & s & & & & & & \\
\hline
4 & 3/2 & s+1 & & & & & \\
\hline
s & s+3 & -4 & 0 & 0 & 0 & & \\
15 & 1/2 & s-1 & 0 & 0 & 0 & & \\
\hline
-(19s+33) & -(4s+18) & 2 & s & s+1 & s+3 & s+1 & -(s+2)
\end{array}\right] \tag{105}$$

$$D_{F^{11}\ell}(s) = \left[\begin{array}{cc|c|cc|c} s-1 & 0 & & & & \\ -(s+5) & s-2 & & & 0 & \\ \hline s-7 & s-2 & s-1 & & & \\ \hline -s & s-1 & -3 & 1 & 0 & \\ s+1 & s & s-1 & 0 & 1 & \\ \hline s-5 & s-2 & s & s-2 & s-2 & s^2-1 \end{array}\right] \tag{106}$$

M. Conclusion

Part II has treated the stability of a strongly connected subsystem for various types of dynamics. The paper shows how the SCS can be partitioned using the concept of minimum essential set. Simplified stability conditions and computational advantage obtained from this structural decomposition are presented.

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APPENDIX AIII: Proofs of the Theorems in Part II

Proof of Theorem III. From (31),

$$\underset{\sim}{F}^{11}\eta^{1} = \underset{\sim}{F}^{11}(v^{1}+\underset{\sim}{G}^{1}e^{1}) \ . \tag{A13}$$

Define

$$\bar{v}^{1} \triangleq \underset{\sim}{F}^{11}(v^{1}+\underset{\sim}{G}^{1}e^{1}) - \underset{\sim}{F}^{11}\underset{\sim}{G}^{1}e^{1} \ . \tag{A14}$$

By assumption (a), $\forall\ v^{1} \in \mathcal{L}_{e}^{n_0^1}$, $\forall\ T \in \mathcal{J}$

$$\|\bar{v}^{1}\|_{T} \leq \tilde{\gamma}(\underset{\sim}{F}^{11}) \times \|v^{1}\|_{T} \ . \tag{A15}$$

From (A13) and (A14),

$$\underset{\sim}{F}^{11}\eta^{1} = \underset{\sim}{F}^{11}\underset{\sim}{G}^{1}e^{1} + \bar{v}^{1} \ . \tag{A16}$$

Substituting (A16) into (30), we obtain

$$(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^{1})e^{1} = u^{1} + \bar{v}^{1} + \underset{\sim}{F}^{12}\eta^{2} \ .$$

Solving for e^{1} and substituting the result into (31), we have

$$\eta^{1} = v^{1} + \underset{\sim}{G}^{1}(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^{1})^{-1}(u^{1}+\bar{v}^{1}+\underset{\sim}{F}^{12}\eta^{2}) \ . \tag{A17}$$

Define

$$\bar{\bar{u}}^{1}+\bar{\bar{v}}^{1} \triangleq \underset{\sim}{G}^{1}(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^{1})^{-1}(u^{1}+\bar{v}^{1}+\underset{\sim}{F}^{12}\eta^{2}) - \underset{\sim}{G}^{1}(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^{1})^{-1}\underset{\sim}{F}^{12}\eta^{2} \ . \tag{A18}$$

By assumption (c), $\forall\ u^{1},\ \bar{v}^{1} \in \mathcal{L}_{e}^{n_i^1}$, $\forall\ T \in \mathcal{J}$

$$\|\bar{\bar{u}}^{1}+\bar{\bar{v}}^{1}\|_{T} \leq \tilde{\gamma}(\underset{\sim}{G}^{1}(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^{1})^{-1})\|u^{1}+\bar{v}^{1}\|_{T} \ .$$

By (A15), $\forall\ u^{1} \in \mathcal{L}_{e}^{n_i^1}$, $\forall\ v^{1} \in \mathcal{L}_{e}^{n_0^1}$, $\forall\ T \in \mathcal{J}$

$$\|\bar{\bar{u}}^1+\bar{\bar{v}}^1+v^1\|_T = \|\bar{\bar{u}}^1+\bar{\bar{v}}^1\|_T + \|v^1\|_T$$
$$\leq \tilde{\gamma}(\underset{\sim}{G}^1(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1})(\|u^1\|_T+\tilde{\gamma}(\underset{\sim}{F}^{11})\|v^1\|_T) + \|v^1\|_T$$
$$\leq [\tilde{\gamma}(\underset{\sim}{G}^1(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1}) \times \max(1,\tilde{\gamma}(\underset{\sim}{F}^{11}))+1](\|u^1\|_T+\|v^1\|_T)$$
$$\triangleq \gamma_1(\|u^1\|_T+\|v^1\|_T) \ . \tag{A19}$$

From (A17) and (A18),

$$\eta^1 = \underset{\sim}{G}^1(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1}\underset{\sim}{F}^{12}\eta^2 + \bar{\bar{u}}^1 + \bar{\bar{v}}^1 + v^1 \ . \tag{A20}$$

Therefore

$$\underset{\sim}{F}^{21}\eta^1 = \underset{\sim}{F}^{21}\underset{\sim}{G}^1(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1}\underset{\sim}{F}^{12}\eta^2 + \tilde{u}^1 + \tilde{v}^1 \tag{A21}$$

where

$$\|\tilde{u}^1+\tilde{v}^1\| \leq \tilde{\gamma}(\underset{\sim}{F}^{21}) \times \gamma_1 \times (\|u^1\|_T+\|v^1\|_T) \ . \tag{A22}$$

Substituting (A21) into (32), we obtain

$$e^2 - [\underset{\sim}{F}^{22}+\underset{\sim}{F}^{21}\underset{\sim}{G}^1(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1}\underset{\sim}{F}^{12}]\eta^2 = u^2 + \tilde{u}^1 + \tilde{v}^1 \ . \tag{A23}$$

By (34)

$$\tilde{\underset{\sim}{F}}^{22} \triangleq \underset{\sim}{F}^{22} + \underset{\sim}{F}^{21}\underset{\sim}{G}^1(I-\underset{\sim}{F}^{11}\underset{\sim}{G}^1)^{-1}\underset{\sim}{F}^{12} \ .$$

By assumptions (a) and (c), $\tilde{\gamma}(\tilde{\underset{\sim}{F}}^{22}) < \infty$. Define

$$\tilde{v}^2 \triangleq \tilde{\underset{\sim}{F}}^{22}(\underset{\sim}{G}^2e^2+v^2) - \underset{\sim}{F}^{22}\underset{\sim}{G}^2e^2 \ . \tag{A24}$$

Thus, $\forall\ v^2 \in \mathcal{L}_e^{n_0^2}$, $\forall\ T \in \mathcal{J}$

$$\|\tilde{v}^2\|_T \leq \tilde{\gamma}(\tilde{\underset{\sim}{F}}^{22})\|v^2\|_T \ . \tag{A25}$$

From (33),

$$\tilde{F}^{22}\eta^2 = \tilde{F}^{22}(G^2 e^2 + v^2)$$
$$= \tilde{F}^{22}G^2 e^2 + \tilde{v}^2 \, . \tag{A26}$$

Substituting (A26) into (A23), we obtain

$$(I-\tilde{F}^{22}G^2)e^2 = u^2 + \tilde{v}^2 + \tilde{u}^1 + \tilde{v}^1 \, .$$

Solve for e^2 and substitute the result into (33), we have

$$\eta^2 = v^2 + G^2(I-\tilde{F}^{22}G^2)^{-1}(u^2+\tilde{v}^2+\tilde{u}^1+\tilde{v}^1) \, . \tag{A27}$$

In view of (A27), assumption (b), (A25) and (A22), we conclude that the map $(u^1,v^1,u^2,v^2) \longmapsto \eta^2$ is $\mathcal{L}$-stable. In view of (A20), assumptions (b), (a) and (A19), we conclude that the map $(u^1,v^1,u^2,v^2) \longmapsto \eta^1$ is $\mathcal{L}$-stable. Furthermore, from (30) and (32), it follows that the maps $(u^1,v^1,u^2,v^2) \longmapsto e^1$ and $(u^1,v^1,u^2,v^2) \longmapsto e^2$ are also $\mathcal{L}$-stable. Hence S^c is $\mathcal{L}$-stable. //

<u>Proof of Fact ID.</u>

$\hat{G}$, $\hat{F}$ are $\mathcal{A}$-stable implies $(I-\hat{G}\hat{F})$, $(I-\hat{F}\hat{G})$ are $\mathcal{A}$-stable. It is well known that if $\hat{H}_e$ is $\mathcal{A}$-stable, then $\hat{H}_e^{-1}$ is $\mathcal{A}$-stable if and only if $\inf_{s\in\mathbb{C}_+} |\det \hat{H}_e(s)| > 0$ [37,2]. Hence $(I-\hat{G}\hat{F})^{-1}$, $(I-\hat{F}\hat{G})^{-1}$ are $\mathcal{A}$-stable if and only if $\inf_{s\in\mathbb{C}_+} |\det(I-\hat{G}\hat{F})(s)| = \inf_{s\in\mathbb{C}_+} |\det(I-\hat{F}\hat{G})(s)| > 0$. By the closure property of $\hat{\mathcal{A}}^{n\times n}$, the $\mathcal{A}$-stability of $\hat{G}$, $(I-\hat{F}\hat{G})^{-1}$, $\hat{F}$ and $(I-\hat{G}\hat{F})^{-1}$ implies that of $\hat{G}(I-\hat{F}\hat{G})^{-1}$ and $\hat{F}(I-\hat{G}\hat{F})^{-1}$. //

<u>Proof of Theorem IV L.</u>

By Fact IL applied to $(\hat{G}^c,\hat{F}^c)$, S^c is exp. stable if and only if $\forall\, s \in \bar{\mathbb{C}}_+$, $\det(I-\hat{F}^c\hat{G}^c)(s) \neq 0$. Now

$$\det(I-\hat{F}^c\hat{G}^c)(s) = \det\left(\begin{bmatrix} I-\hat{F}^{11}\hat{G}^1 & -\hat{F}^{12}\hat{G}^2 \\ -\hat{F}^{21}\hat{G}^1 & I-\hat{F}^{22}\hat{G}^2 \end{bmatrix}\right)(s)$$
$$= \det(I-\hat{F}^{11}\hat{G}^1)(s) \times \det(I-\hat{F}^{22}\hat{G}^2)(s) \qquad \text{(by 53).}$$

By construction, $\hat{F}^{11}$ is a block lower triangular matrix and $\hat{G}^{1}$ is a conforming block diagonal matrix, hence $(I-\hat{F}^{11}\hat{G}^{1})$ is also a block lower triangular matrix with $(I-\hat{F}_{ii}\hat{G}_{i})$, $i \in V$ on the diagonal. Thus

$$\det(I-\hat{F}^{11}\hat{G}^{1})(s) = \prod_{i \in V^1} \det(I-\hat{F}_{ii}\hat{G}_{i})(s) .$$

Hence the result follows. //

Proof of Theorem IV D.

Follows in the same manner as proof of Theorem IV L by using Fact ID. //

Proof of Theorems V L and V D.

By the closure properties of the algebras $\mathbb{R}_e(s)$ and $\hat{\mathcal{Q}}$, it is easily checked that all sixteen submatrices in Eq. (64)-(67) are stable under the assumptions of the theorem. //

Proof of Theorem VI L and VI D.

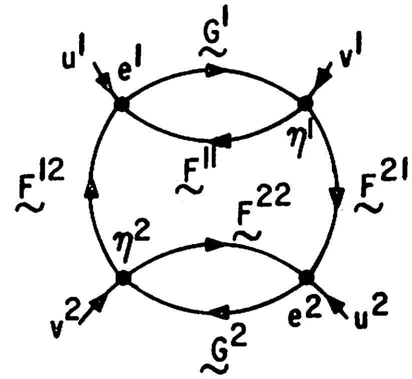
Using Fact 2 on $(\hat{G}^c, \hat{F}^c)$, we have S^c is stable if and only if $\hat{G}^c(I-\hat{F}^c\hat{G}^c)^{-1}$ is stable. By the closure property of the algebras $\mathbb{R}_e(s)$ and $\hat{\mathcal{Q}}$, it can easily be checked that all four submatrices in Eq. (66) are stable under the assumptions of the theorem. //

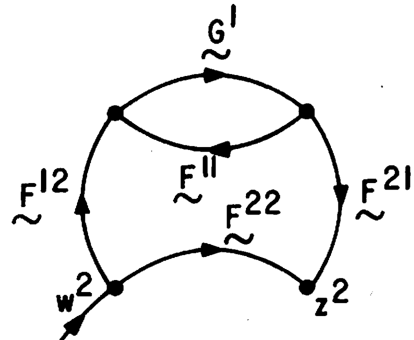
FIGURE CAPTIONS

Fig. 6. Flow graph associated with Eq. (30)-(33).

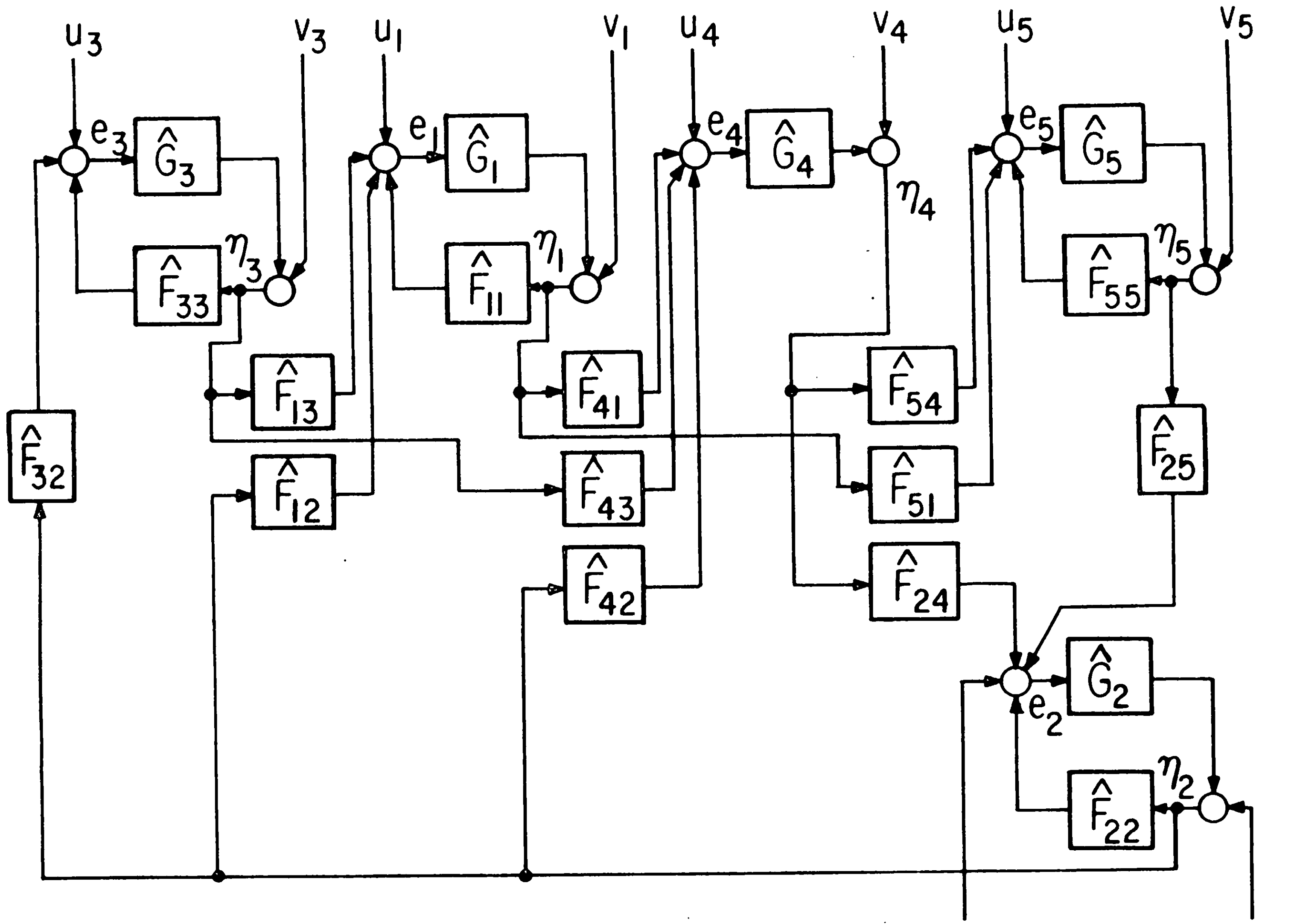
Fig. 7. Flow graph interpretation of $\tilde{\underset{\sim}{F}}^{22}$

Fig. 8. (a) shows the interconnection S^c of linear time-invariant lumped multivariable subsystems used in the example. (b) Interconnection digraph $\mathcal{C}$ of S^c. (c) Adjacency matrix of $\mathcal{C}$ using the original labeling shown in (a) and (b). (d) The acyclic section digraph $\mathcal{C}(\{1,3,4,5\})$. (e) Table relating the old labeling to the new labeling. (f) Adjacency matrix of $\mathcal{C}$ after relabeling.

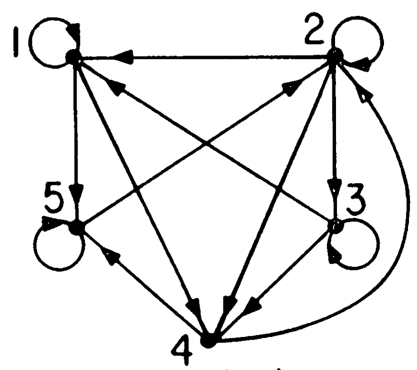

u^1
e^1
$\tilde{G}^1$
v^1
η^1
$\tilde{F}^{11}$
$\tilde{F}^{12}$
$\tilde{F}^{21}$
$\tilde{F}^{22}$
η^2
v^2
e^2
u^2
$\tilde{G}^2$



$$z^{2} = \underset{\sim}{\tilde{F}}^{22} w^{2}$$



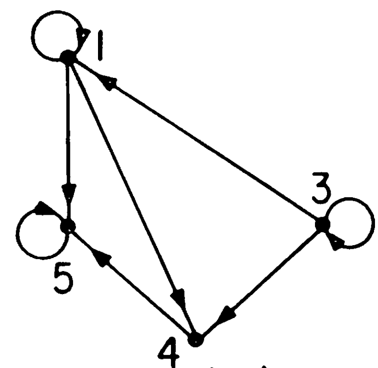
(a)



(b)

	1	2	3	4	5
1	1	1	1		
2		1		1	1
3		1	1		
4	1	1	1		
5	1			1	1

(c)



(d)

OLD LABEL	NEW LABEL
1	2
2	5
3	1
4	3
5	4

(e)

	1	2	3	4	5
1	1				1
2	1	1			1
3	1	1			1
4		1	1	1	
5			1	1	1

(f)