ON THE UNIQUENESS OF THE TIME DOMAIN SOLUTION
OF NONLINEAR DYNAMIC NETWORKS AND SYSTEMS

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ABSTRACT

In this report it will be shown, for a fairly broad class of nonlinear time-varying dynamic networks and systems, that while global passivity does not imply uniqueness, however, local passivity implies the uniqueness of the time domain solution. Furthermore, necessary and sufficient conditions will be presented ensuring the uniqueness of the solution also in case of non-Lipschitz systems. The conditions are given also in terms of element characteristics and network topology. Finally, the results will be generalized for networks containing multiport time-varying and nonlinear elements.

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I. INTRODUCTION

Mathematical models for describing the time domain behaviors of physical systems must have a unique solution for all initial values and excitations. This is one of the physical realizability conditions. This means that if a network (or system) $\mathcal{N}$ has the vector-valued input, state and output functions $\underset{\sim}{u}(t)$, $\underset{\sim}{x}(t)$ and $\underset{\sim}{q}(t)$ respectively (see Fig. 1). Then for any initial value $\underset{\sim}{x}(t_o)$ and for all permissible excitations (inputs $\underset{\sim}{u}(t)$) the state and output functions must be unique for $t \geq t_o$ (t is the scalar time variable).

In the following, we suppose that $\underset{\sim}{q}(t)$ is a single valued continuous function of $\underset{\sim}{u}$ and $\underset{\sim}{x}$, and hence we will investigate the uniqueness of $\underset{\sim}{x}(t)$ only. We set $t_o = 0$ (t_o is finite) and define uniqueness as follows.

Definition 1

The network (system) $\mathcal{N}$ has a unique (time domain) solution if and only if for any bounded $\underset{\sim}{u}_2(t)$ and $\underset{\sim}{u}_1(t)$ defined on $t \in [0,T)$ $T > 0$, the state vector functions $\underset{\sim}{x}_2(t)$ and $\underset{\sim}{x}_1(t)$ are equals on this interval whenever $\underset{\sim}{u}_2(t) = \underset{\sim}{u}_1(t)$ $t \in [0,T)$ and $\underset{\sim}{x}_1(0) = \underset{\sim}{x}_2(0) = \underset{\sim}{x}_0$.

It has been shown that linear passive networks always have a unique solution and that they are causal[1] and finite time stable [1]. The necessary and sufficient conditions for causality and finite time stability of active linear (lumped-distributed) networks [2], some analytic sufficient conditions for the causality of nonlinear operators [3], as well as results concerning active infinite lines [4[and linear active lumped networks [15] are also well known. In case of lumped nonlinear networks sufficient conditions have been determined for checking the unique time domain solvability of a class of networks [5] and conditions were derived for ensuring the unique solution of a class of monotone uncoupled RLC networks [6, Theorem 5]. For some related results we refer to [7,8].

Let us suppose, now, that the network is described by the state equation

$$(1) \quad \frac{d\underset{\sim}{x}}{dt} = \dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}(t)); \ \underset{\sim}{x}_o = \underset{\sim}{x}(0)$$

where $\underset{\sim}{f}$ is continuous in $\underset{\sim}{x}$ and $\underset{\sim}{u}$ (and single-valued).

Equation (1) has a unique solution if $\underset{\sim}{f}$ is either Lipschitz continuous[2] or the Jacobian $J = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}$ is continuous (and bounded) in a domain $\underset{\sim}{x} \in \mathcal{D}$.

In the case of nonlinear lumped networks, however, it is possible for the element characteristics to satisfy a Lipschitz condition in $\mathcal{D}$ but $\underset{\sim}{f}$ is not Lipschitz and vice-versa. On the other hand, many "well behaved" nonlinear networks contain non-Lipschitz elements and/or state equations ($\underset{\sim}{f}$). It is interesting to note that even very simple circuits (containing positive RC elements, operational amplifiers and diodes with simple models) have more than one distinct time domain solutions to a given input (see Example 2. in [5]).

The circuit in Fig. 2 shows a more interesting feature, namely, it is a passive circuit and has two different solutions (responses) to a simple unique input.

Our ultimate goal is to find conditions for ensuring the uniqueness of the solution (possible necessary and sufficient ones) which may be stated in an algorithmic form, or which can be formulated in terms of the element characteristics and the topology of the network.

In what follows it will be shown for a fairly broad class of nonlinear dynamic networks which may contain n-port elements that global passivity does not imply uniqueness, (see Fig. 2) and that local passivity (and the reciprocity of the lossless subnetwork) implies the uniqueness of the solution (Section II). Furthermore, necessary and sufficient conditions for the uniqueness of the solution will be presented (Section III). For an important class of practical networks, these conditions will be given in terms of the element characteristics and network topology (Section IV). Finally, these results will be generalized for networks containing multiport time-varying and nonlinear elements and the conditions under which the Lipschitz property of the elements imply the Lipschitz property of the state equation will be determined (Section V).

The results are based on two mathematical theorems the proof of which can be found in the Appendix.

Throughout this paper we denote the Euclidean k-space by $\mathbb{R}^k$, the set of positive real numbers by $\mathbb{R}_+$, the usual Euclidean norm by $\|\cdot\|$ and the Cartesian product of A and B by A x B. Vector quantities are column vectors denoted by lower case letters, matrices are denoted by capital letters. The elements of a vector $\underset{\sim}{x}$ are denoted by x_i, namely, $x_1, \ldots x_n$. We denote the Euclidean inner product by $\langle\cdot,\cdot\rangle$. A one-to-one onto mapping $\underset{\sim}{f}: \mathbb{R}^n \to \mathbb{R}^n$ is called a C^k-diffeomorphism if $\underset{\sim}{f}(\cdot)$ and $\underset{\sim}{f}^{-1}(\cdot)$ are C^k functions.

II. PASSIVITY AND THE UNIQUENESS OF THE SOLUTION

Consider the network $\mathcal{N}$ of Fig. 3. It is supposed that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are ermissible excitations of the memoryless (n+r) -port M (later only the independence of $u_1, \dots u_r$ will be required). So, M can be described by

$$(2) \quad -\underset{\sim}{z} = \underset{\sim}{g}\,(\underset{\sim}{y},\underset{\sim}{u}(t))$$

and the lossless n-port L has a constitutive relation

$$(3) \quad \begin{array}{l} \underset{\sim}{x} = \underset{\sim}{h}(\underset{\sim}{y}) \\ \underset{\sim}{y} = \underset{\sim}{p}(\underset{\sim}{x}) \end{array} \quad \to \quad \underset{\sim}{p}(\cdot) = \underset{\sim}{h}^{-1}(\cdot)$$

where $\underset{\sim}{g}$, $\underset{\sim}{h}$ and $\underset{\sim}{p}$ are C° functions: $\mathbb{R}^n \to \mathbb{R}^n$, $\underset{\sim}{g}$ and $\underset{\sim}{p}$ are single valued in a domain $\mathcal{D} \subset R^n$. Using the above relations the state equations can be written as follows

$$(4) \quad \begin{array}{l} \dot{\underset{\sim}{x}} = -\underset{\sim}{g}(\underset{\sim}{p}(\underset{\sim}{x}),\ \underset{\sim}{u}(t)) = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}) \\ \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \in \mathcal{D} \end{array}$$

Suppose a solution exists in $\mathcal{D}$.

It will be shown that local passivity plays an important role in ensuring the uniqueness of the solution.

First we will present an important theorem.

<u>Theorem A</u> (a generalization of Theorem 6.2. in [9])

Let us consider the differential equation

$$(5) \quad \begin{array}{l} \dfrac{d}{dt}\,(\underset{\sim}{C}y) = -\underset{\sim}{g}(\underset{\sim}{y},t) = \underset{\sim}{C}\dot{y} \\ \underset{\sim}{y}(t_0) = \underset{\sim}{y}_0 \end{array}$$

where $\underset{\sim}{g}$ is a C° function: $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ in the domain $\mathcal{D} \subset R^n$ for $t \in [t_0, t_0 + a]$ $a > 0$; $\underset{\sim}{y}$, $\underset{\sim}{y}_0 \in \mathcal{D}$ and $\underset{\sim}{C}$ is an n-dimensional bounded real matrix.

If for all $\underset{\sim}{y}_1$, $\underset{\sim}{y}_2$ in $\mathcal{D}$ and for all $t \in [t_0, t_0 + a]$

$$\left\langle \underset{\sim}{g}(\underset{\sim}{y}_2,t)\ -\underset{\sim}{g}(\underset{\sim}{y}_1,t),\ \underset{\sim}{y}_2-\underset{\sim}{y}_1 \right\rangle \geq 0$$

and $\underset{\sim}{C}$ is symmetric positive-definite then equation (5) has at most one solution in $\mathcal{D}$ and on $[t_0, t_0 + \alpha]$, $a > \alpha > 0$.

The proof is given in the Appendix. Using this result the following theorem can be stated.

Theorem 1

In the case of network $\mathcal{N}$ of Figure 3, let us suppose that the lossless n-port L is locally passive, reciprocal and C^1-diffeomorphic in the E neighborhood d_e of $\underset{\sim}{x}_0 = \underset{\sim}{x}(0)$ $(d_e = \{\underset{\sim}{x}: \| \underset{\sim}{x}-\underset{\sim}{x}_0 \| \leq \varepsilon\})$ $\varepsilon > 0$ $\varepsilon \to 0$ for all $\underset{\sim}{x}_0$ in $\mathcal{D} \subset \mathbb{R}^n$.

Or equivalently, we assume the incremental reactance matrix $\underset{\sim}{C}_1$ is bounded symmetric positive-definite and in these d_e neighborhoods $\underset{\sim}{h}(\cdot)$ is a $\underset{\sim}{C}^1$ diffeomorphism and

$$(6) \quad \left.\frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{y}}\right|_{\underset{\sim}{y}=\underset{\sim}{y}_0,\ \underset{\sim}{x}_0=\underset{\sim}{h}(\underset{\sim}{y}_0)} = \underset{\sim}{C}_1 \quad ; \quad \frac{d}{dt}\ \underset{\sim}{h}(\underset{\sim}{y}) \cong \underset{\sim}{C}_1\dot{\underset{\sim}{y}} \quad ;$$

Under these conditions if the memoryless n-port subnetwork M is locally passive in these d_e domains in $\mathcal{D}$ with respect to $\underset{\sim}{y}$; namely,

$$\langle \underset{\sim}{g}(\underset{\sim}{y}_2) - \underset{\sim}{g}(\underset{\sim}{y}_1),\ \underset{\sim}{y}_2 - \underset{\sim}{y}_1 \rangle \geq 0$$

($\underset{\sim}{g}$ is increasing), then the solution $\underset{\sim}{x}(t)$ is unique in $\mathcal{D}$ and $t > 0$.

Proof

The state equation (4) in d_e is approximated by

$$\underset{\sim}{C}_1\dot{\underset{\sim}{y}} = -\underset{\sim}{g}(\underset{\sim}{y}, \underset{\sim}{u}(t)); \ \underset{\sim}{x}_0 = \underset{\sim}{h}(\underset{\sim}{y}_0)$$

However since $\underset{\sim}{g}$ is locally passive and continuous if follows from Theorem A that the solution $\underset{\sim}{y}(t)$ is unique. Hence $\underset{\sim}{x}(t)$ is also unique in d_e. This $\underset{\sim}{y}(t)$ is a solution of an approximation, namely if $\underset{\sim}{h}(\underset{\sim}{y})$ is a piecewise-linear approximation where the domains are arbitrarily small. Using Theorem 3.2 of [9] we conclude to the result that this approximation approaches the exact solution. But the uniqueness holds in every open neighborhoods of $\mathcal{D}$, so the solution is unique in $\mathcal{D}$. q. e. d.

Corollary 1.1

If the lossless subnetwork L has capacitors forming linear C-E loops (loops containing linear capacitors and independent voltage sources only) and inductors forming linear L-J cut sets (cut sets containing linear inductors and independent current sources only) then the statement of Theorem 1 is still valid.

Proof:

According to the Reference [10] these C-E loops and L-J cut sets can be transformed in such a way that one C(L) element is deleted from each loop (cut-set) provided the remaining elements become mutually coupled. These coupled elements remain lossless, passive and reciprocal.

Remarks 1

1. If M contains only passive linear resistors and locally passive (increasing) nonlinear one-ports and multi-ports, then M will be locally passive [16]. Hence if L is locally passive and reciprocal, the solution is unique.

2. The boundedness of the Jacobian $\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}}$ is not needed.

3. Corollary 1.1 remains true even if the specified loops and cut sets contain nonlinear elements. For in each infinitesimally small neighborhoods, the characteristics can be considered linear and we can use the limit theorem for the convergent series of solutions [9, Theorem 3.2].

4. These results are in good agreement with the result of [6, Theorem 5] for monotone uncoupled RLC networks.

5. Consider the circuit of Fig. 2. Observe that even though $\underset{\sim}{h}$ is

locally strictly passive and reciprocal, the function $\underset{\sim}{g}$ is locally active in the neighborhood of $\underset{\sim}{x}_0$.

III. NECESSARY AND SUFFICIENT CONDITIONS FOR UNIQUENESS OF SOLUTIONS - ANALYTIC RESULTS

Let us consider again the network $\mathcal{N}$ of Fig. 3 and the state equation (4). Suppose there are some points $\underset{\sim}{x}^*(\underset{\sim}{y}^*)$, henceforth referred to as irregular points, where the memoryless element characteristics are "very steep" in the sense that the derivative is infinite (though the curve is continuous and bounded but not Lipschitz continuous). More precisely, in these points the Jacobian $\underset{\sim}{J} = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}$ does not exist, it contains unbounded (infinite) elements. Since we have seen in Section II that the sign of the slope, even if it is unbounded, is very important (local passivity), let us first define the different Jacobians near the irregular points. Throughout this section we suppose that near the irregular points $\underset{\sim}{p}(\underset{\sim}{x})$ is a C^1 diffeomorphism.

Definition 2

Consider the ε neighborhoods of the irregular points $\underset{\sim}{x}^*(\underset{\sim}{y}^*)$ and denote these neighborhoods by d_e^*, $d_e^* = \{x: \| x-x^* \| \leq \varepsilon,\ \varepsilon > 0,\ x_i \neq x_i^*\}$. ε is chosen in such a way that the elements of the finite Jacobians, defined next, have the same sign in d_e^*. (Observe that since $\underset{\sim}{p}(x)$ is a C^1 diffeomorphism, d_e^* is also a given neighborhood of y^*.) We define the finite Jacobian $\underset{\sim}{J}_{\Delta x}$ of $\underset{\sim}{f}$ by the equation

(8) $\Delta \underset{\sim}{f} \triangleq \underset{\sim}{J}_{\Delta x} \ \Delta \underset{\sim}{x}$

near x^*, where $\underset{\sim}{J}_{\Delta x}$ is an n x n matrix whose jk element $\underset{\sim}{J}_{\Delta x_{jk}}$ is

$$\underset{\sim}{J}_{\Delta x_{jk}} = \left.\frac{\Delta f_j}{\Delta x_k}\right|_{\Delta x_i = 0 \quad i \neq k}$$

where $\Delta \underset{\sim}{x} = \underset{\sim}{x}_2 - \underset{\sim}{x}_1$, $\underset{\sim}{x}_2, \underset{\sim}{x}_1 \in d_e^*$.

By the definition of the irregular points, at least for one element of $\underset{\sim}{J}_{\Delta x}$ $(\underset{\sim}{J}_{\Delta x_{ij}})$, it is true that for all $\gamma > 0$ there exists an ε such that $|\underset{\sim}{J}_{\Delta x_{ij}}| > \gamma$. These elements are the unbounded elements (of course for the bounded elements $\underset{\sim}{J}_{\Delta x_{jk}} = \left.\frac{\partial f_j}{\partial x_k}\right|_{\underset{\sim}{x} = \underset{\sim}{x}^*}$).

The finite Jacobian $\underset{\sim}{J}_{\Delta y}$ of $\underset{\sim}{g}$ near $\underset{\sim}{x}^*$ is defined by the equation

(9) $\Delta \underset{\sim}{g} \triangleq \underset{\sim}{J}_{\Delta y} \Delta \underset{\sim}{y}$

in the same way as for $\underset{\sim}{J}_{\Delta x}$ with the additional assumption, the elements of $\frac{\partial \underset{\sim}{p}(\underset{\sim}{x})}{\partial \underset{\sim}{x}}$ have also the same sign in d_e^*.

Before showing the conditions of the uniqueness of the solution of network $\mathcal{N}$ we introduce an important theorem.

Theorem B

Given

(1) $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},t)$, $\underset{\sim}{x}_0 = \underset{\sim}{x}(0)$

where $\underset{\sim}{f}$: $R^n \times R_+ \to R^n$ is single valued and Lipschitz continuous in $\underset{\sim}{x}$ in the domain $\mathcal{D} \subset R^n$, $\underset{\sim}{x}_0 \in \mathcal{D}$, except at the irregular points x^*. $\underset{\sim}{J}_{\Delta x}$ has unbounded elements only in the diagonal positions and these elements have the same sign in a neighborhood of the irregular points d_e^*.

Under these conditions equation (1) has a unique solution in $\mathcal{D}$ if and only if the sign of the unbounded elements of $\underset{\sim}{J}_{\Delta x}$ in these neighborhoods are negative.

The proof of Theorem B is given in the Appendix.

The necessary part means that if the sign is positive then in the neighborhood of the irregular points there will be at least two solutions.

Now we define a class of permissible characteristics for the lossless and memoryless n-ports.

Property A

The lossless n-port L of Fig. 3 is characterized by a constitutive relation $\underset{\sim}{y} = \underset{\sim}{p}(\underset{\sim}{x})$ which is a single-valued Lipschitz continuous bounded diagonal mapping in $\mathcal{D} \subset \mathbb{R}^n$ and in the d_e^* neighborhoods of the irregular points it is a C^1 diffeomorphism (the "diagonal" constraint will be dropped later on).

Property B

The memoryless n-port M of Fig. 3 is characterized by a constitutive relation $\underset{\sim}{g}(\underset{\sim}{y})$ which is single-valued Lipschitz continuous bounded in $\mathcal{D}$ except in the d_e^* neighborhoods of $\underset{\sim}{x}^*(\underset{\sim}{y}^*)$. $\underset{\sim}{J}_{\Delta y} = \underset{\sim}{J}_\delta + \underset{\sim}{P}$ where $\underset{\sim}{J}_\delta$ is a diagonal matrix containing the unbounded and only the unbounded elements of $\underset{\sim}{J}_{\Delta y}$.

Theorem 2

If the lossless and memoryless n-ports of network $\mathcal{N}$ of Fig. 3 have the Property A and B respectively then $\mathcal{N}$ has a unique solution in $\mathcal{D} \subset \mathbb{R}^n$ $\underset{\sim}{x}_0 \in \mathcal{D}$ if and only if the following terms have the same sign and they are positive in the neighborhoods d_e^*:

$$J_{\delta k} \frac{\partial p_k(x_k)}{\partial x_k}$$

where $J_{\delta k} = J_{\delta kk}$ and k are the indices where $J_{\delta k} \neq 0$ (This means that if the specified terms are all negative then there is always at least two solutions near the irregular points).

Proof:

Since

(4) $\quad \dot{\underset{\sim}{x}} = -\underset{\sim}{g}(\underset{\sim}{p}(\underset{\sim}{x}), \underset{\sim}{u}(t))$

and $\underset{\sim}{J}_{\Delta x} = \underset{\sim}{J}_{\Delta y} \dfrac{\partial p(\underset{\sim}{x})}{\partial \underset{\sim}{x}}$

where $\dfrac{\partial p(\underset{\sim}{x})}{\partial \underset{\sim}{x}}$ is diagonal, Theorem 2 follows from Theorem B in the domain d_e^*. Outside d_e^* $\underset{\sim}{g}$ and $\underset{\sim}{f}$ are Lipschitz continuous and uniqueness of the solution is already assured.

Remarks 2

1. If $\underset{\sim}{p}(\underset{\sim}{x})$: $R^n \to R^n$ in the specified k-th equations does not depend on the variables x_i $i \neq k$, i.e., $\dfrac{\partial p(\underset{\sim}{x})}{\partial \underset{\sim}{x}}$ does not have off-diagonal elements in the k-th rows and columns in d_e^*, then couplings are permitted. Namely, then Theorem 2 is valid under arbitrary couplings between the lossless elements. So the constraint of Property B is partially dropped.

2. Theorem 2 assures unique solutions even if the characteristics or the constitutive relations are not Lipschitz.

3. It is possible that the irregular points form an n-m dimensional subspace m>0.

IV. THE CONDITIONS OF UNIQUENESS IN TERMS OF NETWORK ELEMENT CHARACTERISTICS AND TOPOLOGY

Next, we try to determine the conditions of Theorem 2, at least for an important class of practical nonlinear dynamic networks, in terms of the network element characteristics and network topology. First, we consider networks containing memoryless nonlinear resistive one-ports, linear resistive multiports and energy storage elements (Corollaries 2.1, 2.2, 2.3). Then we drop this restriction to obtain Theorem 3 which represents the most general, main result within this context.

It is also important that constructive algorithms are given for determining the irregular points $\underset{\sim}{x}^*(\underset{\sim}{y}^*)$.

Throughout this section it is supposed that the state equation exists (see conditions, e.g. in [10]). We use the same notations and definitions as in the preceding sections.

Property C

This includes Properties A and B and the requirement that all nonlinear memoryless elements of M are one-ports. Moreover each voltage-controlled resistor is connected across a parallel voltage-controlled capacitor while each current-controlled resistor is connected in series with a current-controlled inductor.

Corollary 2.1

If the network $\mathcal{N}$ of Fig. 3 has the Property C, then

(i) $\underset{\sim}{J}_{\Delta y}$ has unbounded elements only in the diagonal positions.

(ii) N has a unique time domain solution in $\mathcal{D} \subset \mathbb{R}^n$, for all $x_0 \in \mathcal{D}$ if and only if in the neighborhoods d_e^* of $\underset{\sim}{x}^*(\underset{\sim}{y}^*)$ for each nonlinear one-ports the terms

$$\frac{\Delta n_i(y_i)}{\Delta y_i} \cdot \frac{\partial p_i(x_i)}{\partial x_i}$$

have the same sign and they are positive. The functions $n_i(y_i)$ are the characteristics of the nonlinear resistors and $y_i = p_i(x_i)$ are the characteristics of the energy storage elements attached to the nonlinear resistors.

(iii) $\underset{\sim}{x}^*(\underset{\sim}{y}^*)$ are determined only by $n_i(y_i)$; namely the irregular points of n_i are the irregular points $\underset{\sim}{y}^*$.

The necessary part of this and the subsequent two corollaries means that if all the specified terms have the same sign and they are negative then in the neighborhoods d_e^* there exist at least two different solutions.

Proof:

In this case M has the following description

(10) $\underset{\sim}{g}(\underset{\sim}{y}) = \underset{\sim}{H}\underset{\sim}{y} + \underset{\sim}{\phi}(\underset{\sim}{y}) + \underset{\sim}{B}\,\underset{\sim}{u}(t)$

where $\underset{\sim}{\phi}(\underset{\sim}{y})$ is a diagonal mapping of $\phi_i = n_i(y_i)$ (in case no resistor is attached to an energy storage element $n_i = 0$), $\underset{\sim}{H}$ and $\underset{\sim}{B}$ are the hybrid matrices of the linear part of M.

Hence in d_e^*

(11) $\underset{\sim}{J}_{\Delta y} = \underset{\sim}{H} + \dfrac{\Delta\underset{\sim}{\phi}(\underset{\sim}{y})}{\Delta\underset{\sim}{y}} = \underset{\sim}{H} + \text{diag}\left\{\dfrac{\Delta n_i(y_i)}{\Delta y_i}\right\}$

where diag $\{\alpha_i\}$ is a diagonal matrix with diagonal elements α_i. Because $\dfrac{\Delta nk(y_k)}{\Delta y_k} = J_{\delta k}$ it follows from Theorem 2 that the sign condition of our corollary is a necessary and sufficient condition of the uniqueness in d_e^*. Outside d_e^* the Lipschitz continuity guarantees the uniqueness q. e. d.

We note that if the network M has one-and two-port elements only then equation (4) is equivalent to equation (10-90) of [10] with the following restrictions

$$\underset{\sim}{C}_L = \underset{\sim}{0};\ \underset{\sim}{L}_T = \underset{\sim}{0};\ \underset{\sim}{H}_{C_T R_L} = \underset{\sim}{0};\ \underset{\sim}{H}_{L_L R_T} = \underset{\sim}{0};\ \underset{\sim}{H}_{C_T R_T} = \underset{\sim}{1};\ \underset{\sim}{H}_{L_L R_L} = \underset{\sim}{1};$$

$\underset{\sim}{v}_{c_T}$, $\underset{\sim}{i}_{L_L}$, $\hat{\underset{\sim}{v}}_{R_T}$, $\hat{\underset{\sim}{i}}_{R_L}$ are diagonal mappings.

Property D

This includes Property C. However, the following couplings are allowed within the lossless reactances and between the nonlinear oneports:

(i) between resistors: any coupling which does not contain the non-Lipschitz resistors; (ii) between reactances: any coupling which does not contain the energy-storage elements attached to the non-Lipschitz resistors. (It is supposed that the couplings are either linear or if it is nonlinear, then the corresponding partial derivatives are continuous.

Corollary 2.2

Suppose the network N of Fig. 3 has the Property D. This means that

$\underset{\sim}{\phi}(\underset{\sim}{y})$ and $\underset{\sim}{p}(\underset{\sim}{x})$ are block diagonal mappings and hence

$$\underset{\sim}{\phi}(\underset{\sim}{y}) = \begin{bmatrix} \underset{\sim}{\phi}_1(\underset{\sim}{y}_1) \\ \underset{\sim}{\phi}_2(\underset{\sim}{y}_2) \end{bmatrix} ; \qquad \underset{\sim}{p}(\underset{\sim}{x}) = \begin{bmatrix} \underset{\sim}{y}_1 = \underset{\sim}{p}_1(\underset{\sim}{x}_1) \\ \underset{\sim}{y}_2 = \underset{\sim}{p}_2(\underset{\sim}{x}_2) \end{bmatrix}$$

$\underset{\sim}{\phi}_1$ contains the non-Lipschitz resistor characteristics, $\underset{\sim}{\phi}_1$ and $\underset{\sim}{p}_1$ are diagonal mappings.

If $\dfrac{\partial \underset{\sim}{\phi}_2}{\partial y_2}$ and $\dfrac{\partial \underset{\sim}{p}_2}{\partial \underset{\sim}{x}_2}$ contain Lipschitz-continuous functions in $\mathcal{D} \subset \mathbb{R}^n$, then

(i) N has a unique solution in D for all $\underset{\sim}{x}_0 \in \mathcal{D}$ if and only if in the neighborhoods d_e^* for the unbounded elements of $\dfrac{\Delta \underset{\sim}{\phi}_1}{\Delta \underset{\sim}{y}_1}$ all the terms $\dfrac{\Delta\phi_{1i}(y_{1i})}{\Delta y_{1i}}$ $\dfrac{\partial p_{1i}(x_{1i})}{\partial x_{1i}}$ of the functions $\underset{\sim}{\phi}_1$ and $\underset{\sim}{p}_1$ have the same sign and they are positive.

(ii) $\underset{\sim}{x}^*(\underset{\sim}{y}^*)$ are determined only by $\underset{\sim}{\phi}_1$ $(\phi_{1i} = \phi_{1i}(y_{1i}))$.

Proof:

In d_e^*, $\varepsilon \to 0$,

$$(12) \quad \underset{\sim}{J}_{\Delta x} = -\underset{\sim}{J}_{\Delta y} \frac{\partial p(\underset{\sim}{x})}{\partial \underset{\sim}{x}}$$

and $\quad \underset{\sim}{J}_{\Delta y} = \underset{\sim}{H} + \dfrac{\Delta \underset{\sim}{\phi}(y)}{\Delta \underset{\sim}{y}}$

(see equations (10) and (11)).

So, in this domain

$$J_{\underset{\sim}{\Delta y}} = \underset{\sim}{H} + \begin{bmatrix} \vdots \\ \dfrac{\Delta\phi_{1i}}{\Delta y_{i}} \\ \vdots \\ \dfrac{\Delta\underset{\sim}{\phi}_2}{\Delta \underset{\sim}{y}_2} \end{bmatrix}$$

Since $\dfrac{\partial\underset{\sim}{\phi}_2}{\partial y_2}$ and $\dfrac{\partial\underset{\sim}{p}_2}{\partial \underset{\sim}{x}_2}$ are Lipschitz continuous and $\dfrac{\partial\underset{\sim}{p}_1}{\partial\underset{\sim}{x}_1}$ is a diagonal matrix $J_{\underset{\sim}{\Delta x}}$ can have unbounded elements only in the diagonal positions. These terms are the terms of the condition in the corollary (consider equation (12)). Hence using Theorem 1 the proof is complete q. e. d. We note that if the network $\mathcal{N}$ contains one-ports and two-ports only, equation (4) is equivalent to equation (10-90) at [10] with the restrictions mentioned after Corollary 2.1, and provided that in the mappings $\underset{\sim}{v}_{c_T}$, $\underset{\sim}{i}_{L_L}$, $\hat{\underset{\sim}{v}}_{R_T}$ and $\hat{\underset{\sim}{i}}_{R_L}$ the couplings in the sense of Property D are allowed.

Corollary 2.3

Corollary 2.2 is true also in the case the network $\mathcal{N}$ contains linear C-E loops and L-J cut sets (specified in Corollary 1.1). However these loops and cut sets may not contain the energy storage elements attached to non-Lipschitz resistors (in the sense of Property C).

Proof:

The problem can be derived back to the network of Corollary 2.2 by the same reasoning as in the case of Corollary 1.1.

Let us realize that the networks containing linear RLC elements, transistors and diodes (modelled by the usual Eber-Moll models) have

Property D (in the generalized sense of Corollary 2.3).

Next we drop the restrictions of Property C and introduce the most important result of this section. In this case the state equation will be an <u>implicit function</u>.

Consider the network of Fig. 4 where it can be seen that memoryless nonlinear elements (multiports) are allowed which are not attached to energy storage elements. These nonlinear resistive elements form an m-port characterized by the constitutive relation

$$\underset{\sim}{\eta} = \underset{\sim}{\phi}^*(\underset{\sim}{\xi})$$

where η_i and ξ_i are the port variables (voltage and current) of the i-th port. The linear resistive multi-port has the hybrid description

$$(13) \quad \begin{bmatrix} -\underset{\sim}{\eta} \\ -\underset{\sim}{z}' \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H}_{11} & \underset{\sim}{H}_{12} \\ \underset{\sim}{H}_{21} & \underset{\sim}{H}_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{y} \end{bmatrix} + \underset{\sim}{B}\,\underset{\sim}{u}(t) \; ; \quad \underset{\sim}{B}\,\underset{\sim}{u}(t) = \begin{bmatrix} \underset{\sim}{b}_\eta(t) \\ \underset{\sim}{b}_z(t) \end{bmatrix}$$

and $\underset{\sim}{g}'(\underset{\sim}{y})$ exists.

For this class of networks Theorem 3 gives the conditions which ensures the uniqueness of the solution.

<u>Theorem 3</u>

Consider the network of Fig. 4 with the constitutive relations just specified. The lossless subnetwork has Property A, however, loops of capacitors and cut sets of inductors are allowed as well as couplings in the sense of Corollary 2.2 and 2.3.

$\underset{\sim}{\phi}^*$ is Lipschitz continuous in $\mathbb{R}^m$. $\underset{\sim}{\phi}(\underset{\sim}{y})$ is Lipschitz-continuous in $\mathcal{D} \subset \mathbb{R}^n$ except near the irregular points $\underset{\sim}{x}^*(\underset{\sim}{y}^*)$. These irregular points are determined solely by $\underset{\sim}{\phi}$ namely these are the irregular points of $\underset{\sim}{\phi}(\underset{\sim}{y})$.

Under these conditions the network $\mathcal{N}$ of Fig. 4 has a unique solution in $\mathcal{D}$ for all $\underset{\sim}{x}_0 \in \mathcal{D}$ $t > 0$, if

$$\det \underset{\sim}{J}_H^* = \det\left(\frac{\partial \underset{\sim}{\phi}^*(\underset{\sim}{\xi})}{\partial \underset{\sim}{\xi}} + \underset{\sim}{H}_{11}\right) \neq 0 \text{ in } \mathcal{D} \text{ and in the neighbor-}$$

hoods d_e^*

$$\frac{\Delta\phi_i\ (y_i)}{\Delta y_i} \quad \frac{\partial p_i(x_i)}{\partial x_i} > 0$$

i: the indices of the ports of L attached by the non-Lipschitz resistors.

Proof:

We will show that the conditions ensure the uniqueness in the domain d_e^*. Outside of these domains the condition ensures the Lipschitz-continuous character of the state equation.

According to Figure 4 using the constitutive relations we have

$$-\underset{\sim}{z} = \underset{\sim}{H}_{21}\xi + \underset{\sim}{H}_{22}\underset{\sim}{y} + \underset{\sim}{b}_z(t)$$

$$-\underset{\sim}{\eta} = -\underset{\sim}{\phi}^*(\underset{\sim}{\xi}) = \underset{\sim}{H}_{11}\underset{\sim}{\xi} + \underset{\sim}{H}_{12}\underset{\sim}{y} + \underset{\sim}{b}_\eta(t)$$

Let us define $\underset{\sim}{e}(\underset{\sim}{\xi})$ by

$$\underset{\sim}{H}_{12}\underset{\sim}{y} = -(\underset{\sim}{\phi}^*(\underset{\sim}{\xi}) + \underset{\sim}{H}_{11}\underset{\sim}{\xi} + \underset{\sim}{b}_\eta(t)) =$$
$$= -\underset{\sim}{e}(\underset{\sim}{\xi})$$

Hence

$$\underset{\sim}{\xi} = \underset{\sim}{e}^{-1}(-\underset{\sim}{H}_{12}\underset{\sim}{y})$$

and then

$$-\underset{\sim}{z}' = \underset{\sim}{H}_{21}\underset{\sim}{e}^{-1}(-\underset{\sim}{H}_{12}\underset{\sim}{y}) + \underset{\sim}{H}_{22}\underset{\sim}{y} + \underset{\sim}{b}_z(t) = \underset{\sim}{g}'(\underset{\sim}{y})$$

From Fig. 4 we have

$$\underset{\sim}{g}(\underset{\sim}{y}) = \underset{\sim}{g}'(\underset{\sim}{y}) + \underset{\sim}{\phi}(\underset{\sim}{y})$$

and hence

$$(14)\quad \underset{\sim}{J}_{\Delta y} = \frac{\partial \underset{\sim}{g}'(\underset{\sim}{y})}{\partial \underset{\sim}{y}} + \frac{\underset{\sim}{\Delta}(\underset{\sim}{y})}{\Delta \underset{\sim}{y}} = \frac{\Delta \underset{\sim}{g}(\underset{\sim}{y})}{\Delta \underset{\sim}{y}}$$

in d_e^*.

Using the expression for $g'(y)$ we have

$$(15) \qquad \frac{\partial \underset{\sim}{g}'(\underset{\sim}{y})}{\partial \underset{\sim}{y}} = \underset{\sim}{H}_{21}\left(\frac{\partial \underset{\sim}{\phi}^*(\underset{\sim}{\xi})}{\partial \underset{\sim}{\xi}}\Bigg|_{\underset{\sim}{\xi} = \underset{\sim}{e}^{-1}(\underset{\sim}{y})} + \underset{\sim}{H}_{11}\right)^{-1} \underset{\sim}{H}_{12} + \underset{\sim}{H}_{22}$$

i.e.

$$\underset{\sim}{J}_H^* = \frac{\partial \underset{\sim}{\phi}^*(\underset{\sim}{\xi})}{\partial \underset{\sim}{\xi}}\Bigg|_{\underset{\sim}{\xi} = \underset{\sim}{e}^{-1}(\underset{\sim}{y})} + \underset{\sim}{H}_{11}$$

(Let us realize that in (15) there are two types of inverses: an inverse function, $\underset{\sim}{e}^{-1}(\cdot)$, and an inverse matrix, $\underset{\sim}{J}_H^{*-1}$).

Because of J_H^* is nonsingular in $\mathcal{D}$ (according to our condition), considering the fact that outside d_e^*:

$$\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} = -\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} \frac{\partial \underset{\sim}{p}(\underset{\sim}{x})}{\partial \underset{\sim}{x}}$$

We see that $\underset{\sim}{f}$ is Lipschitz continuous in $\mathcal{D}$ outside d_e^*. This ensures the uniqueness in $\mathcal{D}$ outside d_e^*.

However in d_e^* $\frac{\partial \underset{\sim}{g}'}{\partial \underset{\sim}{y}}$ is bounded and hence only the terms

$$\frac{\Delta \underset{\sim}{\phi}(\underset{\sim}{y})}{\Delta \underset{\sim}{y}} \quad \text{of } \underset{\sim}{J}_{\Delta y}$$

will introduce the unbounded diagonal terms. So we have reduced the problem to the special case of Corollary 2.2 and Corollary 2.3, where the second condition of Theorem 3 ensures the uniqueness in the domains d_e^*.

q. e. d.

Remarks 3

1. If $\underset{\sim}{\phi}^*(\underset{\sim}{\xi})$ is a diagonal mapping with strictly increasing characteristics and $\underset{\sim}{H}_{11} \in P_0$ (the class of matrices with nonnegative principal minors) then $\det \underset{\sim}{J}_H^* > 0$ (i.e. $\neq 0$).

This is true, because, if $\underset{\sim}{A} \in P_0$ and $\underset{\sim}{D} > 0$ is a positive diagonal matrix then $\det(\underset{\sim}{A}+\underset{\sim}{D}) > 0$ [13,8].

2. Let us realize that the condition $\det J_H^* \neq 0$ in $\mathbb{R}^m$ ensures all the same time also the uniqueness of $\underset{\sim}{e}^{-1}(\cdot)$ because of the global implicit function theorem [12] (the second condition in this theorem, the behavior at the infinity is satisfied too).

3. If $\frac{\partial \phi}{\partial \underset{\sim}{\xi}}^*$ and $\underset{\sim}{H}_{11}$ are the elements of the class of a pair of matrices $\mathcal{W}_o$ then $\det \underset{\sim}{J}_H^* > 0$. This is true because if $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are a pair of matrices of the class $\mathcal{W}_o$ then for every positive diagonal matrix $\underset{\sim}{D}$ $\det(\underset{\sim}{A}\underset{\sim}{D}+\underset{\sim}{B}) \neq 0$. To our special case $\underset{\sim}{D} = \underset{\sim}{1}$.

V. GENERALIZATIONS: Networks Containing Nonlinear and Time-varying Multiports and the Lipschitz Property Invariance.

Next, we consider a fairly broad class of nonlinear and time-varying networks containing multiport elements. After the state equations are formulated we give conditions under which the Lipschitz property (continuity) of the element characteristics imply the Lipschitz continuity of the state equation (Theorem 4). Finally we apply Theorem 2 for these class of networks (Corollary 2.4).

Consider the network $\mathcal{N}$ of Fig. 5. The one-ports are linear or nonlinear, time invariant or time-varying elements (R, L, C-s). The N-port elements are defined by the following type of constitutive relations

$$\underset{\sim}{y}_N = \underset{\sim}{\Gamma}_N(\underset{\sim}{x}_N)\,\dot{\underset{\sim}{x}}_N + \underset{\sim}{T}_N\,\underset{\sim}{\phi}_N(\underset{\sim}{x}_N) + \underset{\sim}{\Gamma}_{NO}(t)\,\underset{\sim}{x}_N \tag{16}$$

where $\underset{\sim}{\Gamma}_N(\underset{\sim}{x}_N)$, $\underset{\sim}{T}_N$ and $\underset{\sim}{\Gamma}_{NO}(t)$ are $N \times N$ matrices. For convenience a few one-port elements are listed on Fig. 6. It is important that the N-port need not have a circuit model.

The state equation of these networks can be written as follows.

The constitutive relations of the extracted N-ports can be collected in the following equation

$$y = \Gamma(x)\,\dot{x} + T\phi(x) + \Gamma_t\, x \tag{17}$$

where: y contains the y_N-type port variables (in case of one-ports e.g. currents of capacitors, voltages of inductors etc.);

x contains the x_N-type variables (e.g. voltages of capacitors etc.);

$\Gamma(x)$ is a block-diagonal matrix of the $\Gamma_N(x_N)$-type matrices (e.g. L or C in case of linear and $C_t(t)$ or $L_t(t)$ for time varying energy storage elements etc.);

Γ_t is a block diagonal matrix of the $\dot{\Gamma}_{NO}(t)$-type time-varying characteristics;

T is a block diagonal matrix describing the linear couplings between the nonlinear memoryless ports (within the N-ports);

$\phi(x)$: The characteristics of the nonlinear memoryless parts of the N-ports, these are supposed to be diagonal mappings;

The elements of Γ are C^1 diffeomorphism in the domain considered, the elements of $\dot{\Gamma}_t$ and $\phi(x)$ are continuous functions.

Generally the port variables of x are not independent. A part of these, x_2, can be expressed by the others x_1, and by the inputs $u(t)$:

$$x_2 = L_1 x_1 + L_u\, u(t); \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

The linear memoryless n-port $N\ell$ can be described by the hybrid equation

(19) $$\underset{\sim}{y}_1 - \underset{\sim}{P}\underset{\sim}{y}_2 = \underset{\sim}{R}\underset{\sim}{x}_1 + \underset{\sim}{Q}\underset{\sim}{u}(t)$$

It is derived from the form: $\underset{\sim}{P}'\underset{\sim}{y} = \underset{\sim}{R}'\underset{\sim}{x} + \underset{\sim}{Q}'\underset{\sim}{u}$.

Now, partitioning equation (17) according to $\underset{\sim}{x}$ and introducing the notations

$$\underset{\sim}{T} = \begin{bmatrix} \underset{\sim}{T}_1 & \underset{\sim}{T}_{12} \\ \underset{\sim}{T}_{21} & \underset{\sim}{T}_2 \end{bmatrix}; \quad \underset{\sim}{\Gamma} = \begin{bmatrix} \underset{\sim}{\Gamma}_1(\underset{\sim}{x}_1) & 0 \\ 0 & \underset{\sim}{\Gamma}_2(\underset{\sim}{x}_2) \end{bmatrix}; \dot{\underset{\sim}{\Gamma}}_t = \begin{bmatrix} \dot{\underset{\sim}{\Gamma}}_{t1}(t) & \dot{\underset{\sim}{\Gamma}}_{t12}(t) \\ \dot{\underset{\sim}{\Gamma}}_{t21}(t) & \dot{\underset{\sim}{\Gamma}}_{t_2}(t) \end{bmatrix}$$

$$\text{and } \underset{\sim}{\phi} = \begin{bmatrix} \underset{\sim}{\phi}_1(\underset{\sim}{x}_1) \\ \underset{\sim}{\phi}_2(\underset{\sim}{x}_2) \end{bmatrix}$$

let us realize that in a lot of practically important cases the off-diagonal submatrices of $\underset{\sim}{T}$ and $\dot{\underset{\sim}{\Gamma}}_t$ are zero. Under this assumption the state equation can be determined easily using the above equations:

$$\begin{aligned} \dot{\underset{\sim}{x}}_1 = -\underset{\sim}{P}_e^{-1}[&\underset{\sim}{T}_1\underset{\sim}{\phi}_1(\underset{\sim}{x}_1) + [\underset{\sim}{R} + \dot{\underset{\sim}{\Gamma}}_{t1} + \underset{\sim}{P}\dot{\underset{\sim}{\Gamma}}_{t2}\,\underset{\sim}{L}_1]\,\underset{\sim}{x}_1 + \\ &+ [\underset{\sim}{Q} + \underset{\sim}{P}\dot{\underset{\sim}{\Gamma}}_{t2}\,\underset{\sim}{L}_u]\,\underset{\sim}{u}(t) + \\ &+ \underset{\sim}{P}\underset{\sim}{T}_2\,\underset{\sim}{\phi}_2(\underset{\sim}{L}_1\underset{\sim}{x}_1(t) + \underset{\sim}{L}_u\,\underset{\sim}{u}(t)) + \\ &+ [\underset{\sim}{P}\underset{\sim}{\Gamma}_2(\underset{\sim}{L}_1\underset{\sim}{x}_1 + \underset{\sim}{L}_u\,\underset{\sim}{u}(t))]\,\underset{\sim}{L}_u\,\dot{\underset{\sim}{u}}(t)]; \end{aligned} \tag{20}$$

$$\underset{\sim}{P}_e = \underset{\sim}{\Gamma}_1(\underset{\sim}{x}_1) + [\underset{\sim}{P}\underset{\sim}{\Gamma}_2(\underset{\sim}{L}_1\underset{\sim}{x}_1 + \underset{\sim}{L}_u\,\underset{\sim}{u}(t))]\,\underset{\sim}{L}_1;$$

* If $\underset{\sim}{x} = \underset{\sim}{x}_1$ (all the variables are independent), we have

(21) $$\dot{\underset{\sim}{x}} = -\underset{\sim}{\Gamma}^{-1}(\underset{\sim}{x})\,\{\underset{\sim}{H}\underset{\sim}{x} + \dot{\underset{\sim}{\Gamma}}_t\,\underset{\sim}{x} + \underset{\sim}{T}\underset{\sim}{\phi}(\underset{\sim}{x}) + \underset{\sim}{B}\,\underset{\sim}{u}(t)]$$

where $\underset{\sim}{\Gamma}^{-1}$ means an inverse matrix and $\underset{\sim}{H},\underset{\sim}{B}$ can be read from (20).

---

*In this equation the bracket [ ] has been used for emphasizing the matrix vector product against the functional relationship (e.g. $\underset{\sim}{\Gamma}_2(\cdot)$).

We note that N-ports described by their own state equation and input-output equation can be incorporated in the very similar way [14].

Property E

The nonlinear and time-varying elements have independent $\underset{\sim}{x}$-type variables (e.g. they do not form loops if they are voltages or do not form cut sets if they are currents, and also the controlled sources of $N\ell$ do not destroy this independence).

If N has Property E then it can be shown [14] that if we denote these independent variables by $\underset{\sim}{x}_{1a}$ and hence

$$\underset{\sim}{x}_1 = \begin{bmatrix} \underset{\sim}{x}_{1a} \\ \underset{\sim}{x}_{1b} \end{bmatrix},$$

then $\underset{\sim}{P}_e^{-1}$ will be

$$\underset{\sim}{P}_e^{-1} = \begin{bmatrix} \underset{\sim}{\Gamma}_a^{-1}(\underset{\sim}{x}_{1a}) & 0 \\ 0 & 1 \end{bmatrix} \underset{\sim}{M} \tag{22}$$

where $\underset{\sim}{M}$ is a matrix of real constants. This can be shown by using the following identity:

$$\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{A}_1 & \underset{\sim}{A}_2 \\ 0 & \underset{\sim}{A}_3 \end{bmatrix}; \quad \underset{\sim}{A}^{-1} = \begin{bmatrix} \underset{\sim}{A}_1^{-1} & \underset{\sim}{A}_1^{-1}\underset{\sim}{A}_2\underset{\sim}{A}_3^{-1} \\ 0 & \underset{\sim}{A}_3^{-1} \end{bmatrix}$$

and realizing the specific structure of $\underset{\sim}{L}_1$ as a consequence of Property E.

Now, considering (20) and(22) we can state the following theorem.

Theorem 4

If the network $\mathcal{N}$ of Fig. 5 has the Property E, $\underset{\sim}{\Gamma}_{1a}(\underset{\sim}{x}_{1a})$ and $\underset{\sim}{\phi}(\underset{\sim}{x})$ are Lipschitz continuous in $\mathcal{D} \subset \mathbb{R}^{n_1}$ (n_1 is the order of $\underset{\sim}{x}_1$), then the state equation (20) is also Lipschitz continuous in $\mathcal{D}$. So $\mathcal{N}$ has a unique solution in $\mathcal{D}$ for all $\underset{\sim}{x}_o \in \mathcal{D}$.

Proof: For a C^1 function $\mu(\xi)$ for which $\frac{\partial \mu(\xi)}{\partial \xi} \neq 0$ the function $\frac{1}{\mu}$ is also Lipschitz continuous. The sum and product of the Lipschitz continuous function is also Lipschitz continuous. Now, substituting equation (22) into equation (20) we see that equation (20) is also Lipschitz continuous Lipschitz continuous q.e.d.

Next we apply Theorem 2 for these generalized networks in the case when $\underset{\sim}{x} = \underset{\sim}{x}_1$.

Corollary 2.4.

Consider the network $\mathcal{N}$ of Fig. 5 and suppose that $\underset{\sim}{x} = \underset{\sim}{x}_1$ (the variables of $\underset{\sim}{x}$ are independent). In a domain $\mathcal{D} \subset \mathbb{R}^u$ $\underset{\sim}{\Gamma}(\underset{\sim}{x})$ and $\underset{\sim}{\Gamma}^{-1}(\underset{\sim}{x})$ (the inverse matrix) are continuous, single-valued and bounded. $\underset{\sim}{\phi}(\underset{\sim}{x})$ is a diagonal mapping containing non-Lipschitz functions $\phi_i(x_i)$ at the indices $i=i_1,\ldots i_k$ only, those functions have the irregular points $\underset{\sim}{x}^*$.

Under these conditions (i) the irregular points of equation (21) will be $\underset{\sim}{x}^*$ and (ii) $\mathcal{N}$ has a unique solution in $\mathcal{D}$ for all $\underset{\sim}{x}_o \in \mathcal{D}$ if and only if the terms

$$\frac{\Delta\ \phi_i(x_i)}{\Delta x_i} \quad \cdot \quad \frac{\partial \Gamma_i(x_i)}{\partial x_i}$$

have the same sign in d_e^* and they are positive. (The necessity is understood in the same sense as in the case of Corollaries 2.1 - 2.3).

Proof:

After realizing that equation (21) has the same structure as it was for Corollary 2.2 the proof goes in the same way. Because the time-varying terms occur in the state equation as an explicit function of t, these terms are vanishing at the application of Theorem B or Theorem 2.

Conclusions

As a consequence of the results presented here it is obvious that the Lipschitz continuity of the state equation is sometimes a too strong, superfluous requirement. On the other hand the Lipschitz continuity of the element characteristics does not imply the uniqueness of the solution of the network.

It turned out that local passivity and continuity is a sufficient condition of the uniqueness of the solution.

It is important that necessary and sufficient conditions of the uniqueness have been determined in terms of element characteristics and topology. Finally the results are applicable for a broad class of nonlinear time-varying networks and systems.

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Appendix

A: The proof of Theorem A

Suppose, there are two solutions $\underset{\sim}{y}_1(t)$ and $\underset{\sim}{y}_2(t)$ on $[t_o, t_o+\alpha]$. Let us consider the scalar function.

$$\text{(A1)} \qquad \delta(t) = \langle \underset{\sim}{C}(\underset{\sim}{y}_2-\underset{\sim}{y}_1), \underset{\sim}{y}_2 - \underset{\sim}{y}_1 \rangle \geq 0$$

where $\underset{\sim}{y}_2$, $\underset{\sim}{y}_1$, $\underset{\sim}{y}_o$ are in Ω and $\underset{\sim}{C}$ is positive-definite. For, $\underset{\sim}{y}_2(t_o) = \underset{\sim}{y}_1(t_o) = \underset{\sim}{y}_o$, $\delta(t_o) = 0$. Furthermore,

$$\delta(t) = 0 \quad t > t_o \quad \text{if and only if } \underset{\sim}{y}_2 = \underset{\sim}{y}_1 = \underset{\sim}{0}.$$

Let us realize that

$$\dot{\delta}(t) = 2\langle \underset{\sim}{C}(\underset{\sim}{y}_2-\underset{\sim}{y}_1), \underset{\sim}{y}_2 - \underset{\sim}{y}_1 \rangle$$

(because $\underset{\sim}{C}$ is symmetric) and substituting Eq. 5 into (A2) we get

$$\dot{\delta}(t) = -2 \langle \underset{\sim}{g}(\underset{\sim}{y}_2,t) - \underset{\sim}{g}(\underset{\sim}{y}_1,t), \underset{\sim}{y}_2 - \underset{\sim}{y}_1 \rangle.$$

However the condition of Theorem A, namely the monotone increasing property of $\underset{\sim}{g}$ assures that $\dot{\delta}(t) \leq 0$.

But, $\delta(t) = 0$, $\delta(t) \geq 0$ $t \geq t_o$, $\dot{\delta}(t) \leq 0$ means that $\delta(t) = 0$, $t \geq t_o$, i.e. $\underset{\sim}{y}_2(t) = \underset{\sim}{y}_1(t)$ on $[t_o, t_o + \alpha]$. q.e.d.

B: The Proof of Theorem B

Consider the initial value problem of equation (1).

$$\text{(1)} \qquad \dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},t) \quad , \quad \underset{\sim}{x}(t_o) = \underset{\sim}{x}(o) = \underset{\sim}{x}_o$$

First we prove the <u>sufficiency</u> of the condition. The following

Lemma will be used (see [9] pp. 34-35).

Lemma: If for the initial value problem (1)

$$\text{(B1)} \qquad \gamma = \langle \underset{\sim}{x}_2 - \underset{\sim}{x}_1, \underset{\sim}{f}(\underset{\sim}{x}_2, t) - f(x_1,t) - \frac{1}{t-t_0}(x_2-x_1) \rangle \leq 0$$

in a domain $G \subset \mathbb{R}^n$, $\underset{\sim}{x}_2$, $\underset{\sim}{x}_1$, $\underset{\sim}{x}_o \in G$ and $t_o \leq t \leq t_o + a$, then equation (1) has at most one solution in G and $t \in [t_o, t_o+a)$ if $\underset{\sim}{f}$ is continuous and bounded in G.

In what follows we prove the Theorem B in the ε neighbourhoods of $\underset{\sim}{x}^*$ d_e^*, $\underset{\sim}{x}_o \in d_e^*$ because elsewhere the Lipschitz continuity assures the uniqueness (the notations are according to Section III). ε is greater than zero and can be arbitrary small.

Let $\underset{\sim}{J}_{\Delta x}$ be decomposed as

$$\text{(B2)} \qquad \underset{\sim}{J}_{\Delta x} = \underset{\sim}{J}_d + \underset{\sim}{H}$$

where $\underset{\sim}{J}_d$ is a diagonal matrix containing the unbounded elements and only these elements of $\underset{\sim}{J}_{\Delta x}$. Hence $\underset{\sim}{H}$ is a bounded real n×n matrix.

Using (8), (B1) and (B2) the condition (B1) can be written as follows (in d_e^*)

$$\text{(B3)} \qquad \gamma = \langle \Delta \underset{\sim}{x}, (\underset{\sim}{J}_d + \underset{\sim}{H} - \frac{1}{t-t_o} \underset{\sim}{1}) \Delta \underset{\sim}{x} \rangle =$$

$$= \langle \Delta \underset{\sim}{x}, \underset{\sim}{J}_d \Delta \underset{\sim}{x} \rangle - \langle \Delta \underset{\sim}{x}, \underset{\sim}{A} \Delta \underset{\sim}{x} \rangle;$$

$$\text{(B4)} \qquad \underset{\sim}{A} = (\frac{1}{t-t_o} \underset{\sim}{1} - \underset{\sim}{H})$$

However for any given $\underset{\sim}{H}$ a Δt can be given for which $\underset{\sim}{A}$ is positive-definite if $t-t_o \leq \Delta t$. But because of $\underset{\sim}{f}$ is bounded and continuous it follows that ε_o can be determined such that $\|\Delta x\| < \varepsilon_o$ if $t-t_o < \Delta t$. Now let us choose $0 < \varepsilon < \varepsilon_o$, then in d_e^* A is positive definite and in view of the condition of the Theorem B, $\langle \Delta \underset{\sim}{x}, \ \underset{\sim}{J}_d \Delta \underset{\sim}{x} \rangle < 0$ (the unbounded elements are negative). Hence $\gamma < 0$ and the Lemma assures the uniqueness of the solution in d_e^*. (if $\underset{\sim}{J}_d$ is negative definite the statement remains true) q.e.d.

Next we prove that the condition of Theorem B is also necessary. It means that if all unbounded terms are positive then in the d_e^* neighbourhoods there are at least two solutions passing through $\underset{\sim}{x}^*$.

First we prove it for the <u>case n = 1</u>, $x^* = 0 = f(x^*)$ and $\frac{df}{dx} = +\infty$ at $x = 0$. (See Fig. B.1.)

Let us expand $x = f^{-1}(z)$ i.e. the inverse function in Taylor series:

$$x = \alpha_i z^i + \alpha_{i+1} z^{i+1} + \ldots. = \alpha_i \, z^i (1 + \frac{\alpha_{i+1}}{\alpha_i} z + \ldots); \text{ i is odd}$$

$i > 2$ because f is single-valued and $\left.\frac{df}{dx}\right|_0 = +\infty$. Now in d_e^* if $\varepsilon \to 0$ we have

$$x^{\frac{1}{i}} = \alpha_i^{\frac{1}{i}} z \; (1 + \theta_1(z))^{\frac{1}{i}} = \alpha_i^{\frac{1}{i}} z (1 + \theta_2(z))$$

$$\|\theta_k\| \to 0 \qquad k=1,2,. \quad , \varepsilon \to 0$$

Hence

$$z = \alpha_i^{-\frac{1}{i}} x^{\frac{1}{i}} - \theta_3 \; (f(x)) = \frac{1}{i\sqrt{\alpha_i}} \; x^{\frac{1}{i}} + \theta(x); \; \Theta_3 = f(x) \; \Theta_2;$$

$$\left\| \frac{\theta(x)}{f(x)} \right\| \to 0, \; \varepsilon \to 0$$

However, then using Theorem 3.2 of [9] we conclude that the solution of $z = f(x) = x$ near $x = 0$ will be the solution of

(B5) $$\dot{x} = \frac{1}{i\sqrt{\alpha_i}} \; x^{\frac{1}{i}} = c\, x^{\frac{1}{\beta}};\; x(0) = 0;\; \beta > 2.$$

But (B5) has at least two solutions, e.g.

$$x_1(t) = 0 \qquad t > 0$$

$$x_2(t) = kt^{\alpha} \qquad t > 0$$

where $\alpha = \frac{\beta}{\beta-1} > 0;\; k\alpha = ck^{\frac{1}{\beta}}$

Now, let us turn to the n-dimensional case. We will prove that if the solution passes through the irregular point $\underset{\sim}{x}^*$ then the solution will not be unique in d_e^*. (The terms of Definition 2 are used.)

Let us suppose that (1) has two solutions, namely $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$; $\underset{\sim}{x}_1(0) = \underset{\sim}{x}_2(0)$. Let us consider the function

(B6) $$\delta_1(t) = \langle \dot{\underset{\sim}{x}}_2(t) - \dot{\underset{\sim}{x}}_1(t),\; \dot{x}_2(t) - \dot{\underset{\sim}{x}}_1(t) \rangle =$$

$$= \langle \Delta\dot{\underset{\sim}{x}}(t),\; \Delta\dot{\underset{\sim}{x}}(t) \rangle = \|\Delta\dot{\underset{\sim}{x}}(t)\|^2 \geq 0$$

If $\underset{\sim}{x}_2(t) = \underset{\sim}{x}_1(t)$ i.e. if the solution is unique, then

$$\delta_1(t) = 0 \quad \dot{\delta}_1(t) = 0 \qquad \text{for all } t > 0$$

Now, we will show that at least in a small neighborhood of $\underset{\sim}{x}^*$ $\dot{\delta}_1(t) > 0$ and this proves the necessary part of Theorem B.

Using (B6)[3]

$$\dot{\delta}_1(t) = 2\langle \Delta\ddot{\underset{\sim}{x}},\; \Delta\dot{\underset{\sim}{x}} \rangle$$
$$\cong 2\langle \underset{\sim}{J}_{\Delta x}\, \Delta\dot{\underset{\sim}{x}},\; \Delta\dot{\underset{\sim}{x}} \rangle^3$$
$$= 2\langle \underset{\sim}{J}_{\Delta x}\, \Delta\underset{\sim}{f},\; \Delta\underset{\sim}{f} \rangle$$

(B7) $$= 2\langle \underset{\sim}{J}_{\Delta x}\, \underset{\sim}{J}_{\Delta x}\, \Delta\underset{\sim}{x},\; \underset{\sim}{J}_{\Delta \underset{\sim}{x}}\, \Delta\underset{\sim}{x} \rangle$$

Using (B2) equation (B7) will be

$$\dot{\delta}_1(t) = 2[\langle \underset{\sim}{J}_d^2 \Delta\underset{\sim}{x}, \underset{\sim}{J}_d \Delta\underset{\sim}{x}\rangle +$$

$$+ \langle \underset{\sim}{J}_d \Delta\underset{\sim}{x}, \underset{\sim}{J}_d \underset{\sim}{H} \Delta\underset{\sim}{x}\rangle + \langle \underset{\sim}{J}_d \underset{\sim}{H}\Delta\underset{\sim}{x}, \underset{\sim}{J}_d \Delta x\rangle + \langle \underset{\sim}{J}_d \underset{\sim}{H}\Delta\underset{\sim}{x}, \underset{\sim}{H}\Delta\underset{\sim}{x}\rangle$$

$$+ \langle \underset{\sim}{H} \underset{\sim}{J}_d \Delta\underset{\sim}{x}, \underset{\sim}{J}_d \Delta\underset{\sim}{x}\rangle + \langle \underset{\sim}{H} \underset{\sim}{J}_d \Delta\underset{\sim}{x}, \underset{\sim}{H}\Delta\underset{\sim}{x}\rangle +$$

$$+ \langle \underset{\sim}{H}^2 \Delta\underset{\sim}{x}, \underset{\sim}{J}_d \Delta\underset{\sim}{x}\rangle + \langle \underset{\sim}{H}^2 \Delta\underset{\sim}{x}, \underset{\sim}{H}\Delta\underset{\sim}{x}\rangle$$

where in the second term we realized that

$$\langle \underset{\sim}{J}_d^2 \Delta\underset{\sim}{x}, \underset{\sim}{H}\Delta\underset{\sim}{x}\rangle = \langle \underset{\sim}{J}_d \Delta\underset{\sim}{x}, \underset{\sim}{J}_d \underset{\sim}{H}\Delta\underset{\sim}{x}\rangle$$

(because $\underset{\sim}{J}_d$ is symmetric).

Now, we will prove that $\underset{\sim}{m} = \underset{\sim}{J}_d \underset{\sim}{H}\Delta\underset{\sim}{x} \to \underset{\sim}{0}$ if $\|\Delta x\| \leq \varepsilon \to 0$, $\varepsilon > 0$. We prove that all the components of $\underset{\sim}{m} \to 0$. Let the <u>k</u>th nonzero term of $\underset{\sim}{J}_d$ be denoted by J_{dk} hence the <u>k</u>th component of $\underset{\sim}{m}$ will be

(B9) $$m_k = J_{dk} \, (H_{k1}\Delta x_1 \cdots H_{kn}\Delta x_n)$$

Using the same arguments for the approximation by a Taylor series as in the case n = 1 we have, upon differentiating (B5) with respect to x,

(B10) $$J_{dk} = K_k(\Delta x_k)^{\frac{1}{\beta} - 1} \qquad \beta > 2$$

(in case $n = 1$ $f = c(\Delta x)^{\frac{1}{\beta}} \to f' = \frac{c}{\beta}(\Delta x)^{\frac{1}{\beta} - 1}$, $\frac{c}{\beta} = K$)

Because $\|\Delta\underset{\sim}{x}\| \leq \varepsilon \to 0$ we can write

(B11) $$\Delta x_i = \kappa_i \, \varepsilon \, ; \quad |\kappa_i| < \infty \qquad \kappa_k > 0$$

Now, using (B9), (B10) and (B11) we get

(B12) $$m_k = K_k\, \varepsilon^{\frac{1}{\beta}} [\kappa_k^{\frac{1}{\beta}-1} (H_{k1}\kappa_1 + \dots H_{ku}\,\kappa_n)]$$

and because the term in the bracket [ ] is finite $m_k \to 0$ as $\varepsilon \to 0$. And because this is true for all components of $\underset{\sim}{m}$ we conclude to

$$\underset{\sim}{J}_d\, \underset{\sim}{H}\, \Delta \underset{\sim}{x} \to 0 \quad \text{if } \|\Delta x\| \leq \varepsilon \to 0.$$

As a special case $\underset{\sim}{J}_d\, \Delta\underset{\sim}{x} \to \underset{\sim}{0}$ too.

But hence, except the first term, all the terms of equation (B8) go to zero as $\varepsilon \to 0$. It is because the two elements of all these scalar products go to zero as $\varepsilon \to 0$. We have just shown that $\underset{\sim}{J}_d\, \underset{\sim}{H}\Delta\underset{\sim}{x}$ and $\underset{\sim}{J}_d\, \Delta\underset{\sim}{x} \to \underset{\sim}{0}$. But, because H and H^2 are finite so the terms $\underset{\sim}{H}\Delta\underset{\sim}{x}$, $\underset{\sim}{H}^2\Delta\underset{\sim}{x}$ and $\underset{\sim}{H}\underset{\sim}{J}_d\, \Delta\underset{\sim}{x}$ go also to $\underset{\sim}{0}$.

Let us now investigate the remaining first term of (B8). Using (B10) we have:

(B13) $$\dot{\delta}_{11}(t) = 2\, \langle \underset{\sim}{J}_d^2\, \Delta\underset{\sim}{x},\ \underset{\sim}{J}_d\, \Delta\underset{\sim}{x} \rangle =$$

$$= \sum_{k(J_{dk} \neq 0)} K_k^3\, (\Delta x_k)^{\frac{3}{\beta}-3}\, \Delta x_k^2 =$$

$$= \sum_{k(J_{dk} \neq 0)} K_k^3\, (\Delta x_k)^{\frac{3}{\beta}-1}\ ; \qquad \beta > 2$$

This means, however, that

$$\dot{\delta}_1(t) \Rightarrow \sum_k K_k^3\, (\Delta x_k)^{\frac{3}{\beta}-1}\ ; \text{ if } \varepsilon \to 0 \quad \varepsilon > 0$$

but hence if K_k are greater than zero (this is the case if the condition of Theorem B does not fulfill) then $\dot{\delta}_1(t) > 0$ i.e. the solution is not unique. q.e.d.

Footnotes

[1] If in Definition 1 we make the additional assumption $\underset{\sim}{u}_1(t) = \underset{\sim}{u}_2(t) = \underset{\sim}{0}$ and t_o can be $-\infty$, then the network is called causal. For a more precise definition of causality see [1-4].

[2] A function $\underset{\sim}{f}(\underset{\sim}{x})$ is Lipschitz continuous in a domain $\mathcal{D}$ if $\|\underset{\sim}{f}(\underset{\sim}{x}_2) - \underset{\sim}{f}(\underset{\sim}{x}_1)\| \leq k\|\underset{\sim}{x}_2 - \underset{\sim}{x}_1\|$ for all $\underset{\sim}{x}_2, \underset{\sim}{x}_1 \in \mathcal{D}$, where k is a constant and $\|\cdot\|$ is the usual Euclidean norm.

[3] Here we have used the fact that in d_e^* if $\varepsilon \to 0$

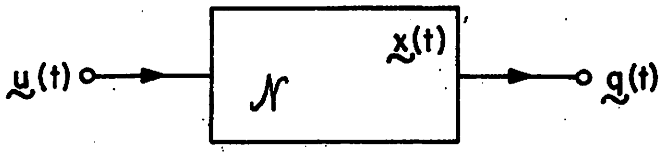
$$\ddot{\underset{\sim}{x}} \cong \underset{\sim}{J}\Delta x \, \dot{\underset{\sim}{x}} + \left.\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}}\right|_{\underset{\sim}{x}_2} \dot{\underset{\sim}{u}}(t) + \left.\frac{\partial \underset{\sim}{f}(t)}{\partial t}\right|_{\underset{\sim}{x}_2} .$$

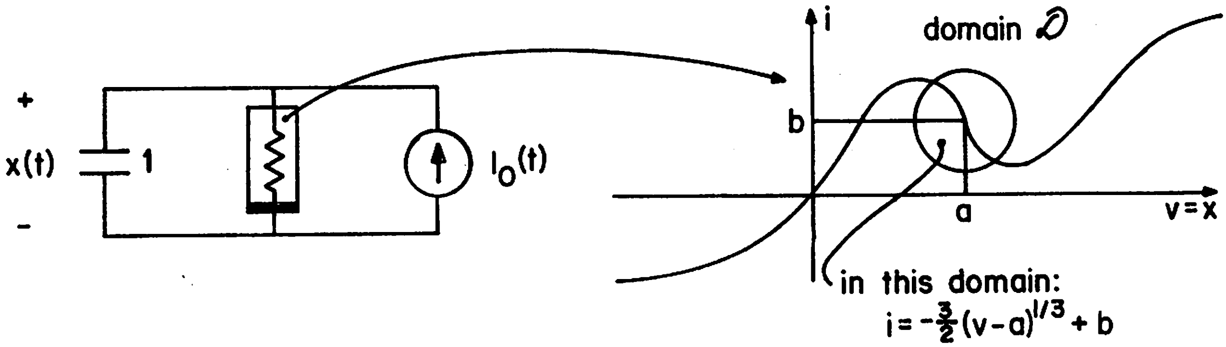
The last two terms are vanishing as we express the $\Delta\ddot{\underset{\sim}{x}}$ at a given t.

We point out, using [16], that we are <u>near</u> $\underset{\sim}{x}^*$ i.e. $\Delta\ddot{\underset{\sim}{x}}$, $\ddot{\underset{\sim}{x}}$ exist. Similarly the last line of (B7) is as follows.

$$2 \left\langle (\underset{\sim}{J}_{\Delta x} + \underset{\sim}{\varepsilon}_1)(\underset{\sim}{J}_{\Delta x} + \underset{\sim}{\varepsilon}_1)\Delta \underset{\sim}{x} + \underset{\sim}{\varepsilon}_2, (\underset{\sim}{J}_{\Delta x} + \underset{\sim}{\varepsilon}_1)\Delta \underset{\sim}{x} + \underset{\sim}{\varepsilon}_2 \right\rangle \|\underset{\sim}{\varepsilon}_i\| \to 0, \ \varepsilon \to 0.$$

The terms containing the $\underset{\sim}{\varepsilon}_i$ - s in (B7) go to zero as $\varepsilon \to 0$ as it can be shown in the same way as was used for proving $\underset{\sim}{m} \to 0$. In fact we will prove that generally if $\varepsilon \to 0$ $\dot{\delta}_1 \to \infty$.

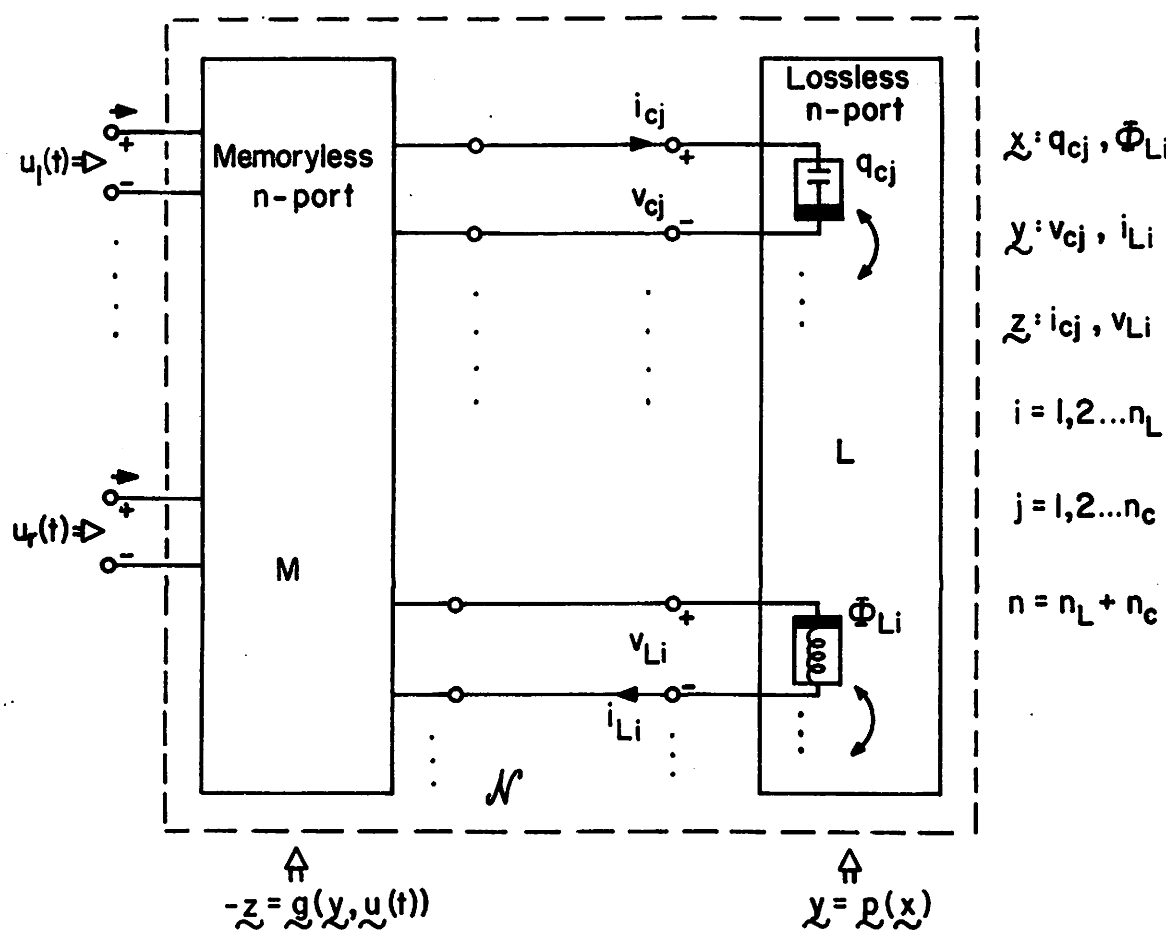

u(t)
x(t)
N
q(t)

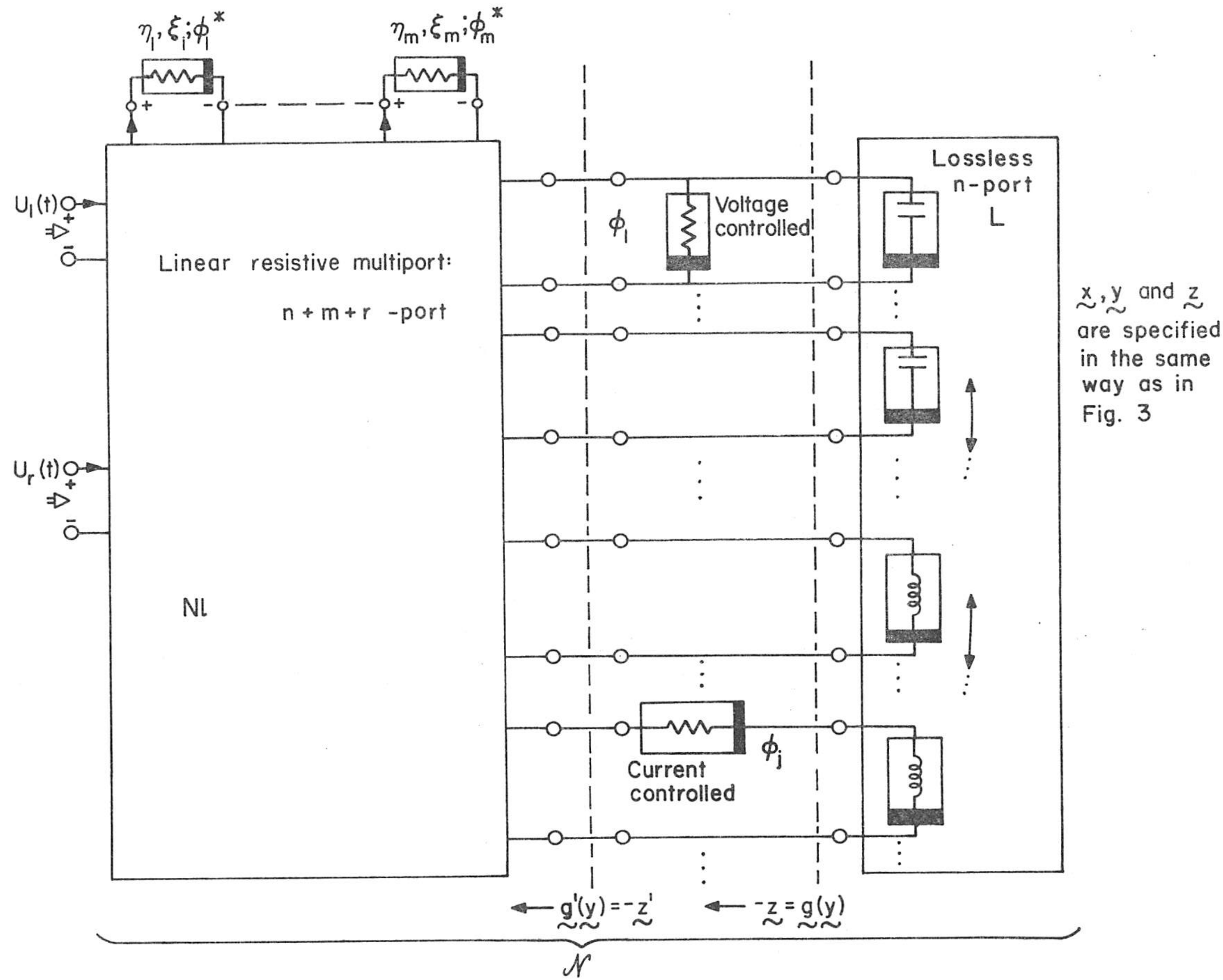


$$\dot{x} = \frac{3}{2}(x-a)^{1/3} - b + I_0(t)\ ; \qquad x(0) = a$$
$$I_0(t) = b = \text{constant}$$

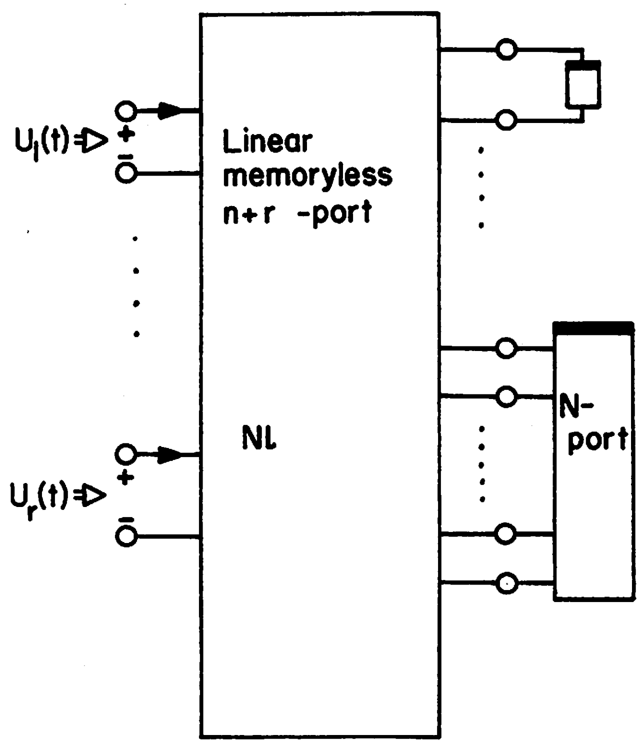
$$x_1(t) = a \quad ; \quad t > 0 \qquad x_1 \in \mathcal{D}$$

$$x_2(t) = a + t^{3/2}; \quad t > 0 \qquad x_2 \in \mathcal{D}$$

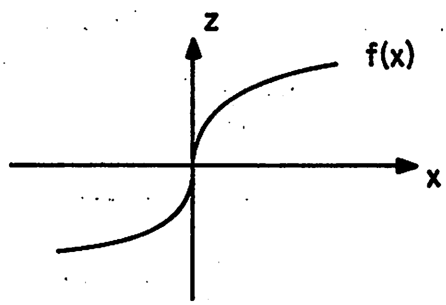

Memoryless
n-port
M
Lossless
n-port
L
$u_1(t)$
$u_r(t)$
i_{cj}
v_{cj}
q_{cj}
Φ_{Li}
v_{Li}
i_{Li}
$\mathcal{N}$
$\underset{\sim}{x}: q_{cj}, \Phi_{Li}$
$\underset{\sim}{y}: v_{cj}, i_{Li}$
$\underset{\sim}{z}: i_{cj}, v_{Li}$
$i = 1,2 \ldots n_L$
$j = 1,2 \ldots n_c$
$n = n_L + n_c$
$-\underset{\sim}{z} = \underset{\sim}{g}(\underset{\sim}{y}, \underset{\sim}{u}(t))$
$\underset{\sim}{y} = \underset{\sim}{p}(\underset{\sim}{x})$



$\eta_1, \xi_1; \phi_1^*$
$\eta_m, \xi_m; \phi_m^*$
$U_1(t)$
$U_r(t)$
Linear resistive multiport:
n + m + r -port
NL
Voltage controlled
ϕ_1
Current controlled
ϕ_j
Lossless n-port L
$\underset{\sim}{x}, \underset{\sim}{y}$ and $\underset{\sim}{z}$ are specified in the same way as in Fig. 3
$\underset{\sim}{g}'(\underset{\sim}{y}) = -\underset{\sim}{z}'$
$-\underset{\sim}{z} = \underset{\sim}{g}(\underset{\sim}{y})$
$\mathcal{N}$


$U_1(t)$
+
Linear
memoryless
n+r -port
NL
$U_r(t)$
+
N-
port

Name	Constitutive relations
Linear time-varying capacitor	[v,i]: $i = C_t(t)\frac{dv}{dt} + \dot{C}_t(t)\,v$; $q = C_t(t)\,v$
Linear time varying inductor	[v,i]: $v = L_t(t)\frac{di}{dt} + \dot{L}_t(t)\,i$; $\phi = L_t(t)\,i$
Nonlinear time-varying one port	[v,i]: $y = \mathcal{C}(x)\,\dot{x} + F(x) + \dot{\mathcal{C}}_0(t)\,x$ $\mathcal{C}$ F and $\mathcal{C}_0$ are real functions $y = i$, $x = O$ or $y = O$, $x = i$



$$\left.\frac{\partial f}{\partial x}\right|_{o} = +\infty$$

$\alpha_i > 0!$