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by

A. R. Bergen and G. Gross

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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A. R. Bergen and G. Gross

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

This paper presents a unified treatment of the problem of choosing appropriate state space descriptions of the swing-equation model for transient stability studies. An application of Lyapunov theory to these representations is given.

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1 Introduction

The study of transient stability of multimachine power systems via Lyapunov's direct method has received considerable attention in the last few years.[2-12] The starting point for these analyses is usually the swing-equation model which describes the dynamics of an n-machine interconnection in terms of n coupled second-order differential equations. While there is general agreement concerning these differential equations, some confusion has arisen with regard to the appropriate state space representation[7,10] for the stability analysis. Indeed, the recent exchanges of views between Ribbens-Pavella and J. L. Willems,[11] and Sastry and Murthy and J. L. Willems,[16] indicate that there are even questions concerning the order of an appropriate state-space description.

For an n-machine system, there are possible system representations of dimension 2n, 2n-1, or even (in special cases) 2n-2. It is not possible to specify which representation is "correct" based solely on the rules of consistency of a state-space representation; rather, the appropriate choice is dictated by the needs of the stability analysis. This point will be clarified in the following.

The object of this paper is to present a unified treatment of the problem of choosing system representations of the swing-equation model suitable for transient stability studies. The paper consists of two sections. In Section 2 the appropriate system representations of the swing-equation model in the general, as well as special cases, are derived. These state space descriptions are convenient in applying

Lyapunov theory. Lyapunov functions associated with these representations are constructed in Section 3. Computation of the transient stability regions of the proposed Lyapunov functions may then be carried out by techniques discussed in Reference 12.

2 System Representations of Swing Equation Model

The swing equation for an n-machine interconnection (and with the usual simplifying assumptions) are taken to be:[2,5,6]

$$M_i \ddot{\delta}_i + D_i \dot{\delta}_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} b_{ij} \sin(\delta_i - \delta_j) = P_i \qquad (1)$$

$$i = 1, 2, \ldots, n$$

where δ_i is the angle between the rotor of the i^{th} machine and a reference axis rotating at synchronous speed, M_i is the inertia coefficient (necessarily positive), D_i is the damping constant (nonnegative), P_i is the "excess" power[12] at the i^{th} machine, and the b_{ij} are coupling coefficients. An important fact is that $b_{ij} = b_{ji}$, for i, j = 1, 2, ..., n, $i \neq j$.

To start with, we present a 2n-state system representation in terms of the state variables

$$\underline{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix}$$

and

$$\underline{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix} \triangleq \dot{\underline{\delta}} .$$

This representation constitutes a valid state space description; however, for the purposes of stability analysis there is an essential defect: the equilibrium condition is met not at isolated equilibrium points but along certain sets of equilibrium lines. The reason may easily be seen from Eq. (1) where the δ_i's are involved only through their differences. Thus if $\underline{\delta}^0$ satisfies the equilibrium condition so does any point of form $\underline{\delta}^0 + \eta\, \underline{1}$ with η an arbitrary real number and $\underline{1}$ the n-column vector with each entry equal to 1. Indeed, this situation conforms to physical reality. A system is considered stable (in an appropriate sense) if it returns to any point on the aforementioned equilibrium line, e.g. after a momentary upset we do not require the δ_i's to return to their original values but rather require this of their differences.

Although there are complications, it may be possible to carry out a stability analysis in 2n-space as long as stability is defined with respect to a return to the equilibrium line.[11] For example, El-Abiad and Nagappan[2] chose this 2n-state system representation and defined their Lyapunov function on $\mathbb{R}^n \times \mathbb{R}^n$ such that it is zero on the equilibrium line. Their Lyapunov function is consequently only positive semi-definite on $\mathbb{R}^n \times \mathbb{R}^n$ and care is needed in applying the results of

Lyapunov theory. Willems[6,9] used a 2n-state representation and defined stability with respect to the equilibrium line.[11] Again his Lyapunov function is positive semi-definite.

These complications caused by dealing with the stability of sets[15] can be easily avoided by choosing a (2n-1) state system representation when there is damping (i.e. $D_i > 0$) and a (2n-2) state description when there is no damping (i.e. $D_i = 0$). In this way we study the stability of isolated equilibrium points whose theory is well established.

2.1 A (2n-1) state system representation

To derive a (2n-1) state description of (1) we introduce the angle differences as state variables. More specifically, we shall choose (arbitrarily) the n^{th} machine as the reference machine[†] and define the (n-1) intermachine angles

$$\alpha_i \triangleq \delta_i - \delta_n \qquad i = 1, \ldots, n-1$$

constituting the (n-1) vector

$$\underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} .$$

We choose the (2n-1) variables $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \omega_1, \omega_2, \ldots, \omega_n$ as the state variables of the system (1). Let

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† If an infinite bus is included in the system, it is chosen as the reference machine.

$$f_i(\underline{\alpha}) \triangleq \sum_{\substack{j=1 \\ j\neq i}}^{n-1} b_{ij} \sin(\alpha_i-\alpha_j) + b_{in} \sin \alpha_i \tag{2}$$

$$i = 1, \ldots, n-1$$

and

$$f_n(\underline{\alpha}) \triangleq -\sum_{j=1}^{n-1} b_{nj} \sin \alpha_j. \tag{3}$$

Note that for any $\underline{\alpha} \in \mathbb{R}^{n-1}$

$$\sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ j\neq i}}^{n-1} b_{ij} \sin(\alpha_i-\alpha_j) = 0,$$

so that

$$f_n(\underline{\alpha}) = -\sum_{i=1}^{n-1} f_i(\underline{\alpha}). \tag{4}$$

If we define the two (n-1) vectors

$$\underline{e} = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$$

and

$$\underline{f}(\underline{\alpha}) = \begin{bmatrix} f_1(\underline{\alpha}) \\ f_2(\underline{\alpha}) \\ \vdots \\ f_{n-1}(\underline{\alpha}) \end{bmatrix},^{\dagger}$$

† See bottom of page 6.

then (4) may be written as

$$f_n(\underline{\alpha}) = \langle \underline{e}, \underline{f}(\underline{\alpha}) \rangle ,$$

where $\langle \cdot,\cdot \rangle : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ denotes the inner product defined for two vectors $\underline{x} = (x_1, x_2, \ldots, x_{n-1})^t$ and $\underline{y} = (y_1, y_2, \ldots, y_{n-1})^t$ by $\langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{n-1} x_i y_i$. Similarly, by defining the excess power vector $\underline{P} \in \mathbb{R}^{n-1}$ by

$$\underline{P} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \end{bmatrix} ,$$

and using the assumption $\sum_{i=1}^{n} P_i = 0$, which involves no loss of generality,[11] we may write:

$$P_n = \langle \underline{e}, \underline{P} \rangle .$$

We define next the n-order diagonal matrices

$$\underline{M} = \text{diag } (M_1, M_2, \ldots, M_n),$$

$$\underline{D} = \text{diag } (D_1, D_2, \ldots, D_n),$$

and, the (n-1) x n transformation matrix

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†As f(.) is a periodic function (with period 2π) we restrict its domain to the periodic frame of reference $-\pi < \alpha_i \leq \pi$, $i = 1, 2, \ldots, n-1$.

$$\underline{T} = [\underline{I}_{n-1} \vdots \underline{e}]$$

where, $\underline{I}_{n-1}$ is the (n-1) identity matrix. Note that the matrix $\underline{T}$ has rank (n-1) and consequently represents a singular transformation.

We can now replace (1) by

$$\underline{M}\,\dot{\underline{\omega}} + \underline{D}\,\underline{\omega} + \underline{T}^t[\underline{f}(\underline{\alpha}) - \underline{P}] = \underline{0}, \tag{5}$$

or, equivalently by a state representation

$$\dot{\underline{\alpha}} = \underline{T}\,\underline{\omega} \tag{6}$$

$$\dot{\underline{\omega}} = -\,\underline{M}^{-1}\,[\underline{T}^t\underline{f}(\underline{\alpha}) + \underline{D}\,\underline{\omega}] + \underline{M}^{-1}\,\underline{T}^t\underline{P}\ . \tag{7}$$

2.2 Equilibrium points

An equilibrium point of (5) is specified by

$$\underline{T}\,\underline{\omega}^0 = \underline{0} \tag{8}$$

$$\underline{T}^t\ [\underline{f}(\underline{\alpha}^0) - \underline{P}] + \underline{D}\,\underline{\omega}^0 = \underline{0} \tag{9}$$

For a system with damping $\underline{D}$ is a positive definite matrix since each damping coefficient D_i is positive. Then, from (9)

$$\underline{\omega}^0 = -\,\underline{D}^{-1}\,\underline{T}^t[\underline{f}(\underline{\alpha}^0) - \underline{P}]$$

so that

$$\underline{T}\,\underline{\omega}^0 = -\,\underline{T}\,\underline{D}^{-1}\,\underline{T}^t\ [\underline{f}(\underline{\alpha}^0) - \underline{P}] = \underline{0} \tag{10}$$

by (8). Now, it follows by repeated application of Sylvester's inequality[13] that $\underline{T}\,\underline{D}^{-1}\,\underline{T}^t$ is an (n-1) rank matrix and hence nonsingular

Therefore, (10) is satisfied if and only if

$$\underline{f}(\underline{\alpha}^0) = \underline{P} \; . \tag{11}$$

This relation then determines the $\underline{\alpha}$-component of an equilibrium point of (5). To specify the $\underline{\omega}$-component, we note that when (11) holds, (9) can be satisfied if and only if

$$\underline{\omega}^0 = \underline{0}. \tag{12}$$

Then the 2n-1 equations in (11) and (12) specify isolated equilibrium points in $\mathbb{R}^{n-1} \times \mathbb{R}^n$ of a damped system.

Consider next an undamped system - the case $\underline{D} \equiv \underline{0}$. An equilibrium point is determined by solving

$$\underline{T}\,\underline{\omega}^0 = 0 \tag{13}$$

$$\underline{T}^t\,[\underline{f}(\underline{\alpha}^0) - P] = \underline{0}. \tag{14}$$

Since $\underline{T}^t$ is full rank, (14) can hold if and only if (11) is satisfied. On the other hand, (13) holds for any $\underline{\omega}$ of the form

$$\underline{\omega} = \eta \underline{e} \qquad \eta \in \mathbb{R} \; . \tag{15}$$

Thus, a solution of (13), (14) specifies not a point in $\mathbb{R}^{n-1} \times \mathbb{R}^n$ but rather a continuum of points: a line in $\mathbb{R}^{n-1} \times \mathbb{R}^n$. Consequently, we should reduce the order of the system representation by one and consider a (2n-2) representation in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$.

2.3 A (2n-2) state system representation

We replace the n frequency variables $\underline{\omega}$ by the (n-1) intermachine frequencies

$$\underline{\gamma} = \begin{bmatrix} \gamma_1 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{bmatrix} \triangleq \underline{T}\,\underline{\omega} = \begin{bmatrix} \omega_1 - \omega_n \\ \cdot \\ \cdot \\ \cdot \\ \omega_{n-1} - \omega_n \end{bmatrix} . \tag{16}$$

Using (6) and (7) we then obtain the following state representation

$$\dot{\underline{\alpha}} = \underline{\gamma} \tag{6'}$$

$$\dot{\underline{\gamma}} = - \underline{S}\,[\underline{f}(\underline{\alpha}) - P], \tag{7'}$$

where

$$\underline{S} \triangleq \underline{T}\,\underline{M}^{-1}\,\underline{T}^t . \tag{17}$$

Many authors have treated the case of zero damping.[1,5,8]

The equilibrium points are given by

$$\underline{\gamma}^0 = \underline{0} ,$$

and (because $\underline{S}$ is nonsingular)

$$\underline{f}(\underline{\alpha}^0) = \underline{P} .$$

Thus isolated equilibrium points are specified in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ space. Physically, stability would imply that the differences between angular velocities return to zero following a disturbance even though the individual angular velocities change. Thus synchronism is maintained although at a new synchronous speed. For a lossless system without damping this is a reasonable objective. This is in contrast to a stable system with damping for which each machine must settle to the synchronous frequency since in this case the system incurs losses due to damping.

A (2n-2) state description also arises in another way. If the system is uniformly damped, i.e.,

$$\frac{D_i}{M_i} = k, \quad i = 1, 2, \ldots, n \tag{18}$$

then it is possible to reduce the order ofthe (2n-1) system although there are no compelling reasons to do so, particularly since the system has isolated equilibrium points. The case of uniform damping has received wide attention in the literature.[5-11]

We define the relative angular velocities γ_i as in (16), noting that with uniform damping $\underline{M}^{-1}\,\underline{D} = k\,\underline{I}_n$, and using (6), (7) we obtain

$$\dot{\underline{\alpha}} = \underline{\gamma} \tag{6''}$$

$$\dot{\underline{\gamma}} = -\,\underline{S}\,[\underline{f}(\underline{\alpha}) - \underline{P}] - k\underline{\gamma} \tag{7''}$$

which is a valid state representation. Note that with zero damping (k = 0) the equations reduce to (6') and (7'). It is worth noting that this representation only yields information regarding the differences between the angular velocities.

3 Application of Lyapunov Theory

The system representations (5)-(7) and (6")-(7") are convenient forms to which Lyapunov theory may be applied to determine transient stability regions in $\mathbb{R}^{n-1} \times \mathbb{R}^{n}$ and $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ state spaces, respectively. Energy-like Lyapunov functions are presented for these system representations.

To start with, let us define the "potential energy" of our system

$$W(\underline{\alpha}) \triangleq \int_{\underline{\alpha}^0}^{\underline{\alpha}} \langle [\underline{f}(\underline{\xi}) - \underline{f}(\underline{\alpha}^0)], d\underline{\xi} \rangle$$

where the integration is performed over an arbitrary path between $\underline{\alpha}^0$ and $\underline{\alpha}$. The integral is well defined and path independent since $\frac{\partial f_i}{\partial \alpha_j} = \frac{\partial f_j}{\partial \alpha_i}$, i,j = 1, ..., n-1, i.e., $\frac{\partial \underline{f}}{\partial \underline{\alpha}}$ is symmetric.

The "kinetic energy" of the system (5) is given by $\frac{1}{2} \langle \underline{\omega}, \underline{M}\, \underline{\omega} \rangle$. Consequently a proposed "total energy" Lyapunov function $V(\cdot,\cdot) : \mathbb{R}^{n-1} \times \mathbb{R}^{n} \mapsto \mathbb{R}$ is given by

$$V(\underline{\alpha},\underline{\omega}) = \frac{1}{2} \langle \underline{\omega}, \underline{M}\, \underline{\omega} \rangle + W(\underline{\alpha}).$$

This corresponds to the Lyapunov function in References 2 and 6. For the uniform damping case, the "kinetic energy" is given by $\frac{1}{2} \langle \underline{\gamma}, \underline{S}^{-1} \underline{\gamma} \rangle$ and the corresponding Lyapunov function $\tilde{V}(\cdot,\cdot) : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \mapsto \mathbb{R}$ is proposed

$$\tilde{V}(\underline{\alpha},\underline{\gamma}) = \frac{1}{2}\langle \underline{\gamma},\underline{S}^{-1}\,\underline{\gamma}\rangle + W(\underline{\alpha}),$$

which corresponds to the Lyapunov function in References 1, 5, 8 and 9. An explicit expression for $\underline{S}^{-1}$ is given in the Appendix.

It is easily seen that

$$V(\underline{\alpha}^0,\underline{\omega}^0) = \tilde{V}(\underline{\alpha}^0,\underline{\gamma}^0) = 0.$$

Furthermore, the total derivative of V along the trajectories of (5) is

$$\dot{V}_{(5)}\ (\underline{\alpha},\underline{\omega}) = -\langle \underline{\omega},\underline{D}\ \underline{\omega}\rangle\ .$$

Similarly, the total time derivative of $\tilde{V}$ along the trajectories of (6"), (7") is

$$\dot{\tilde{V}}_{(6''),(7'')}\ (\underline{\alpha},\underline{\gamma}) = -\,k\,\langle \underline{\gamma},\underline{S}^{-1}\,\underline{\gamma}\rangle \leq 0$$

since $\underline{S}^{-1}$ is positive definite.[†]

The Lyapunov functions proposed for the two system representations possess derivatives which are negative semidefinite in the entire state space. To obtain a transient stability region we must find some region in the respective state spaces where the Lyapunov functions are positive definite. Now, the kinetic energy term is always positive definite in the velocity subspace since $\underline{M}$ and $\underline{S}^{-1}$ are positive definite matrices in their respective velocity subspaces. It remains to locate a region where $W(\underline{\alpha})$ is positive definite in the $\mathbb{R}^{n-1}$ space. A detailed discussion

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[†] $\underline{S}^{-1}$ is positive definite follows from the fact that $\underline{S}$ is positive definite (see Appendix).

of this matter is presented in Reference 12.

4 Concluding Remarks

The state space representations of the equations of motion of an n-machine interconnection with and without damping were presented in a unified manner. Since more than one representation exists for these equations of motion, the one appropriate for stability studies must be chosen. By considering the two state representations of Eqs. (5)-(7) and (6"), (7") the following points may be observed:

1 In both cases the coordinates of the equilibrium point in the $\underline{\alpha}$-subspace of the state space are the solution of the same nonlinear equation.

2 When an infinite bus is included in the system we choose a (2n-2) state representation since $\omega_n = 0$ and δ_n = constant. The state variables are $\omega_1, \omega_2, \ldots, \omega_{n-1}, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, and the first (n-1) rows of (5) constitute the state representation.

5 Acknowledgment

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7 Appendix

The (n-1) matrix $\underline{S}$ is defined in (17). By inspection $\underline{S}$ is symmetric, also, it has rank (n-1) [see discussion following (10)]. Now, since $\underline{M}^{-1}$ is a diagonal positive definite matrix, $\underline{M}^{-1} = \underline{M}^{-\frac{1}{2}} \underline{M}^{-\frac{1}{2}}$ so that $\underline{S} = \underline{T}\,\underline{M}^{-1}\,\underline{T}^t = (\underline{M}^{-\frac{1}{2}}\,\underline{T}^t)^t\,(\underline{M}^{-\frac{1}{2}}\,\underline{T}^t)$. Now $\underline{M}^{-\frac{1}{2}}\,\underline{T}^t$ has rank (n-1) which implies that $\underline{S}$ is positive definite.[14]

Next, we obtain a more direct expression for $\underline{S}$. To do so, let us first partition the $\underline{M}$ matrix into

$$\underline{M} = \left[\begin{array}{c|c} \underline{M}_r & \\ \hline & M_n \end{array}\right]$$

where,

$$\underline{M}_r = \mathrm{diag}(M_1, M_2, \ldots, M_{n-1}).$$

Then,

$$\underline{S} = \underline{T}\,\underline{M}^{-1}\underline{T}^t = [\underline{I}_{n-1} \,\vdots\, \underline{e}] \left[\begin{array}{c|c} \underline{M}_r^{-1} & \\ \hline & \frac{1}{M_n} \end{array}\right] \left[\begin{array}{c} \underline{I}_{n-1} \\ \hline \underline{e}^t \end{array}\right]$$

$$= \underline{M}_r^{-1}\left[\underline{I}_{n-1} + \frac{1}{M_n}\,\underline{M}_r\,\underline{e}\,\rangle\,\langle\,\underline{e}\right], \tag{A-1}$$

where $\cdot\,\rangle\,\langle\,\cdot : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ denotes the diad defined for two vectors

$\underline{x}, \underline{y} \in \mathbb{R}^{n-1}$ by $\underline{x} \rangle \langle \underline{y} = \underline{x}\, \underline{y}^t$.

Finally, we obtain an explicit expression for $\underline{S}^{-1}$ by means of the following lemma.

Lemma

Let $\underline{R}$ be an arbitrary q x q matrix, $\underline{a}$, $\underline{b}$ be vectors in $\mathbb{R}^q$ and $\underline{I}$ denote the q x q identity matrix. Define the matrix $\underline{Q}$ by

$$\underline{Q} = \underline{I} + \underline{R}\, \underline{a} \rangle \langle \underline{b} \tag{A-2}$$

Given $\underline{Q}$ is nonsingular, then

$$\underline{Q}^{-1} = \underline{I} - \frac{1}{1 + \langle b, \underline{R}_{\underline{a}} \rangle}\, \underline{R}\, \underline{a} \rangle \langle \underline{b} \tag{A-3}$$

Proof. Let $\underline{x}$, $\underline{y}$ be any vectors in $\mathbb{R}^q$ related by $\underline{y} = \underline{Q}\, \underline{x}$. Thus

$$\underline{y} = \underline{x} + \underline{R}\, \underline{a} \rangle \langle \underline{b}\, \underline{x} = \underline{x} + \underline{R}\, \underline{a} \langle \underline{b}, \underline{x} \rangle \tag{A-4}$$

Now

$$\langle \underline{b}, \underline{y} \rangle = \langle \underline{b}, \underline{x} \rangle + \langle \underline{b}, \underline{R}\, \underline{a} \rangle \langle \underline{b}, \underline{x} \rangle ,$$

So that

$$\langle \underline{b}, \underline{x} \rangle = \frac{\langle \underline{b}, \underline{y} \rangle}{1 + \langle \underline{b}, \underline{R}\, \underline{a} \rangle} .$$

Rewriting (A-4) as

$$\underline{x} = \underline{y} - \underline{R}\, \underline{a} \langle \underline{b}, \underline{x} \rangle = \underline{y} - \underline{R}\, \underline{a} \frac{\langle \underline{b}, \underline{y} \rangle}{1 + \langle \underline{b}, \underline{R}\, \underline{a} \rangle}$$

we have

$$\underline{x} = \left[\underline{I} - \frac{1}{1 + \langle \underline{b}, \underline{R}\, \underline{a} \rangle} \; \underline{R}\, \underline{a} \rangle \langle \underline{b} \right] \underline{y}$$

where the bracketed expression is identified to be $\underline{Q}^{-1}$. Q.E.D.

We apply this lemma to the (n-1) square matrix

$$\underline{Q} = \underline{I}_{n-1} + \frac{1}{M_n} \underline{M}_r \, \underline{e} \rangle \langle \underline{e}$$

which is nonsingular since $\underline{M}_r^{-1}$ and $\underline{S}$ are nonsingular. We define $M_T = \sum_{i=1}^{n} M_i$ so that $M_T = \langle \underline{e}, \underline{M}_r \, \underline{e} \rangle + M_n$. Then

$$\underline{Q}^{-1} = \underline{I}_{n-1} - \frac{1}{1 + \frac{1}{M_n} \langle \underline{e}, \underline{M}_r \, \underline{e} \rangle} \; \frac{1}{M_r} \underline{M}_r \, \underline{e} \rangle \langle \underline{e}$$

$$= \underline{I}_{n-1} - \frac{1}{M_T} \underline{M}_r \, \underline{e} \rangle \langle \underline{e} \, ,$$

so that

$$\underline{S}^{-1} = \left[\underline{I}_{n-1} - \frac{1}{M_T} \underline{M}_r \, \underline{e} \rangle \langle \underline{e} \right] \underline{M}_r .$$