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OPTIMAL CONTROL WITH RANDOM INPUTS

by

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SUMMARY

The problem considered is that of controlling the input of a given system (not necessarily linear or time-invariant) in such a way that the output of the system follows a random process as closely as possible, with the restriction that the input is to remain within prescribed bounds.

The problem is first developed at a quite general level, and a necessary condition for optimal feedforward and feedback control is formulated. This condition is also shown to be sufficient for linear systems. The condition is subsequently specialized to linear time-varying differential systems and random processes with a finite-dimensional state. It is derived that for a certain class of systems bang-bang type control is optimal.

Finally, the constraint on the input is dropped, which necessitates provisions with regard to the stability of the interconnection of controller and system. The optimal control problem is completely solved for a quadratic error criterion and a very large class of random processes, within the constraint that the optimal controller is to be stable.

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I. INTRODUCTION

A very typical characteristic of present day control theory is the desire not just to make control systems work well, but to optimize them in one way or another. The general approach is first to exhibit the control situation, then to choose as a design criterion some function which has to be maximized or minimized, and to design accordingly.

It seems that in particular due to this attitude control theory often appears to have little to do with control engineering. The reason is that often it is impossible to justify the particular choice of optimization criterion, since only very few of the factors which in any practical case in fact determine the properties of the design have been taken into consideration. Furthermore, only the simplest problems can be solved, so that many realistic problems remain untouched by theoreticians.

The general feeling is, however, that the mathematical and academic interests of these problems amply justify the efforts that are being spent on them, and moreover, that even from the inadequate and oversimplified examples considered much can be learned about the design of good systems.

In addition to being characterized by this craving for optimization, modern control theory distinguishes itself from the "classical" theory of poles and Nyquist plots by a number of concepts and ideas, which in many cases (in particular where nonlinear systems are involved) make more powerful and general approaches to various questions possible. One of the most outstanding of the modern features of control and system theory is the notion of s t a t e, which makes it possible to describe large classes of systems and properties of systems very adequately.

In this work a problem which is not at all new is approached along the lines of modern control and system theory. The question is that of controlling the input of a given system in such a way that the output of the system follows a random process as closely as possible. This is a typical servo-mechanism problem and applications may be found in various fields. Instances where the problem arises occur in the design of radar-tracking devices, but also in the design of many types of measuring instruments. As a specific, elementary example one might consider a servo pen-recorder. The dynamic properties of the combination of servomotor and pen are given; the question is to design a servo amplifier in such a way that the recorder is a faithful measuring instrument for a given class of random signals.

The problem just outlined, which will be referred to as the optimal following problem, has mainly been studied in the context of linear systems, quadratic error functions and linear controllers; it is then known as the Wiener problem. In this study, a much broader approach to the problem is taken, although in the end only little more than the Wiener problem is actually solved.

The general problem has received some attention in the literature, but much work still remains to be done. Fuller[1] has exhibited the problem, almost in the form in which it will be given here, and has brought attention to the concept of feedback control. In the light of the approach of Wonham[2] the problem of this work could be considered as an example of a large class of stochastic optimal control problems, but it has some particular features of its own.

The problem will be developed at a quite general level in Section II, employing an abstract characterization of the system and the random process. Later on the results are specialized almost all the way down to the Wiener problem in Section IV.

Fuller[1] has first pointed out how the concept of the state of a random process is connected with feedback control. It is hoped that by the treatment of Sections II. 4 and III. 2 this concept will be made a little more familiar. In Section III. 1, more or less incidentally, the ideas of instantaneous controllability and observability of a linear time-varying system are introduced. These notions seem to be relevant in the description of such systems. Finally, in Section IV some questions concerning the intrinsic nature of optimal stable controllers are brought up, but only partially answered.

No attempt has been made to achieve complete mathematical rigor, but an effort has been made to formulate the problems and conclusions precisely.

II. GENERAL THEORY

II.1 Problem Statement and Preliminaries

Consider a system Σ, linear or nonlinear, time-invariant or time-varying, with scalar input $u(t)$ and scalar output $y(t)$. Also consider a (measurable) random process $\{r(t), \; t \in (-\infty, \infty)\}$, also scalar.

The topic of this research is the problem of providing an input to the system in such a way that the output $y(t)$ is at each instant t as close as possible to the observed value of the random process, $r(t)$, with the constraint on the input to the system

$$|u(t)| \leq \gamma(t), \text{ all } t \tag{1.1}$$

where $\gamma(t)$, $\infty \geq \gamma(t) \geq 0$, is a prescribed function of time. This statement will be made precise in the following. In this section, various notions and assumptions which are involved in the problem statement will be discussed.

System. It is supposed that the system can be described by a input-output-state relationship of the form[3, 4]

$$y(t) = H\left\{x(t_o); u_{(t_o, t]}\right\} \tag{1.2}$$

where $y(t)$ denotes the output at time t, $x(t_o)$ the initial state at $t_o \leq t$, and $u_{(t_o, t]}$ the input during the interval $(t_o, t]$. No assumptions on the nature of the state $x(t_o)$ are made; typically, it will be a finite-dimensional vector, however.

From now on it will be supposed that interest is taken in the problem from time $t_o=0$ on. The initial state $x(0)$ is presumed to be known. It will figure as a parameter throughout the following and will usually be suppressed in the notation.

Only systems will be considered that are <u>bounded</u> and that are <u>differentiable</u>. These two notions will now be explained. Let U be the space of allowable input functions $u_{[0, \infty)}$.

<u>Definition:</u> A system Σ characterized by an input-output-state relationship of the form (1.2) is called <u>bounded</u> if

$$\sup_{\substack{u \in U \\ 0 \leq t \leq T}} |y(t)| < \infty \tag{1.3}$$

for all $T < \infty$, and for all initial states $x(0)$ that one may wish to take into consideration.

By this restriction, systems with "finite escape time" are excluded.

<u>Definition:</u> Let Σ be a system characterized by a relationship of the form (1.2). Suppose that an input is applied of the following form

$$u(t) = u^o(t) + \varepsilon\, \bar{u}\,(t) \tag{1.4}$$

with $u^o(t)$ and $\bar{u}(t)$ both in U. Then the system Σ is called <u>differentiable,</u> if for ε small enough the output of the system at any time $t \geq 0$

can be expressed as

$$y(t) = H\left\{x(0);\, u^o_{(0,t]}\right\} + \varepsilon H_{u^o}\bar{u}(t) + o(\varepsilon;\, u^o_{(0,t]};\, \bar{u}_{(0,t]}) \tag{1.5}$$

In this expression, H_{u^o} is a linear operator depending upon $x(0)$ and $u^o_{(0,\infty)}$. Its domain consists of time functions $\bar{u}_{(0,\infty)}$ in U, its range consists of time functions $\bar{y}_{(0,\infty)}$. The notation $H_{u^o}\bar{u}(t)$ indicates the value that the function $H_{u^o}\bar{u}$ assumes at time t. Furthermore, $o(\varepsilon;\, u^o_{(0,t]},\, \bar{u}_{(0,t]})$ is a term with the property

$$\lim_{\epsilon\to 0} \frac{o(\varepsilon;\, u^o_{(0,t]},\, \bar{u}_{(0,t]})}{\varepsilon} = 0 \tag{1.6}$$

where the limit is uniform in $u^o_{(0,t]}$, $\bar{u}_{(0,t]}$ and t; i.e., the quantity of which the limit is taken can always be made smaller than any fixed number for all $u^o \in U$, all $\bar{u} \in U$, and all t.

The system with the input-output relationship

$$\bar{y}(t) = H_{u^o}\bar{u}(t), \text{ all } t \geq 0 \tag{1.7}$$

will be called the variational system of Σ about u^o, and will be denoted as Σ_{u^o}. Note that this is a linear system which is always in zero state at time $t = 0$; i.e., the response to zero input from $t = 0$ on is the zero function. Furthermore, if Σ is bounded, Σ_{u^o} is also bounded; since in (1.5) all terms besides $\varepsilon H_{u^o}\bar{u}(t)$ are bounded, $H_{u^o}\underline{u}(t)$ must also be bounded.

The adjoint of the variational system of Σ about u^o, denoted as $\Sigma^*_{u^o}$, is introduced as the system with input-output relationship

$$y^*(t) = H^*_{u^o} u^*(t), \text{ all } t \geq 0 \tag{1.8}$$

where the linear operator $H^*_{u^o}$ is the adjoint of H_{u^o}, defined by the requirement that

$$\langle u^*, H_{u^o} \bar{u} \rangle = \langle H^*_{u^o} u^*, \bar{u} \rangle \tag{1.9}$$

for all functions u^* and $\bar{u}$ with finite norms. The inner product that will be found convenient is

$$\langle u, v \rangle = \frac{1}{T} \int_0^T u(t)\, v(t)\, dt \tag{1.10}$$

The norm of a function u is, as usual, defined as

$$|| u || = \langle u, u \rangle^{1/2} \tag{1.11}$$

To make the notion of variational and adjoint variational system somewhat more tangible, one might represent the operation of the variational system in the form

$$\bar{y}(t) = H_{u^o} \bar{u}(t) = \int_0^t h_{u^o}(t, \tau)\, \bar{u}(\tau)\, d\tau \tag{1.12}$$

By this relation, $h_{u^o}(t, \tau)$ is defined as a generalized integral operator kernel; it is the familiar impulse response. It is easy to show that the adjoint variational system $\Sigma^*_{u^o}$ may be represented by

$$y^*(t) = H^*_{u^o} u^*(t) = \int_t^T h^*_{u^o}(t, \tau)\, u^*(t)\, d\tau \tag{1.13}$$

where

$$h^*_{u^o}(t, \tau) = h_{u^o}(\tau, t) \tag{1.14}$$

Controller. A controller is defined as a device that performs the operation

$$u(t) = F\left\{r_{(-\infty, t]}\right\} \tag{1.15}$$

i.e., it is assumed that at each instant t the entire past of the sample function of the random process up to and including t is known, and that the instantaneous input is chosen on the basis of this knowledge. A controller characterized by (1.15) will be termed admissible if

$$\left| F\left\{r_{(-\infty, t]}\right\} \right| \leq \gamma(t) \tag{1.16}$$

for all sample functions $r_{(-\infty, t]}$ and all t, where $\gamma(t)$, $\infty \geq \gamma(t) \geq 0$, is a prescribed function of time.

The controller represented by (1.15), which for brevity's sake will be denoted as F, produces an input function $u_{(0, \infty)}$ corresponding to each sample function $r_{(-\infty, \infty)}$ of the random process; F is therefore an operator. In its turn, each input function $u_{(0, \infty)}$ produces an output function $y_{(0, \infty)}$. In this manner, one obtains from the random process $\{r(t), t \in (-\infty, \infty)\}$ two new random processes, $\{u(t), t\in(0, \infty)\}$, and $\{y(t), t\in(0, \infty)\}$.

<u>Optimization criterion.</u> Consider the expression

$$\mathcal{E}(F, T) = \frac{1}{T} E\left(\int_0^T W[y(t) - r(t)]\, dt\right) \qquad (1.17)$$

Here $W(e)$ is a positive, convex, twice-differentiable weighting function. The following three restrictions are imposed, which involve at the same time the weighting function W; the random process $\{r(t)\}$, the system Σ, and the controller F. Let $w(e) = \frac{dW(e)}{de}$ and $w'(e) = \frac{dw(e)}{de}$; then the requirements are

$$R_1 : E(W[y(t) - r(t)]) \text{ is bounded for all } t \geq 0$$

$$R_2 : E(|w[y(t) - r(t)]|^2) \text{ is bounded for all } t \geq 0 \qquad (1.18)$$

$$R_3 : E(|w'[y(t) - r(t) + \eta]|) \text{ is bounded uniformly in } \eta \text{ for } \eta \text{ in some interval } (-\eta_o, \eta_o) \text{ and for all } t \geq 0.$$

(Typically, $W(e) = \frac{1}{2} e^2$, hence $w(e) = e$ and $w'(e) = 1$; then both R_1 and R_2 amount to the condition that $E(|y(t) - r(t)|^2)$ be bounded for all $t \geq 0$; R_3 is trivial). The conditions can be interpreted in several ways. When the random process, the system, and $W(e)$ are given, R_1, R_2 and R_3 restrict the class of controllers that is taken into consideration. If the random process, the system, and the class of controllers are given, R_1, R_2 and R_3 restrict the weighting functions $W(e)$ that can be used. Other interpretations are also possible. In the following, the <u>first</u> of these approaches is taken.

By theorem 2. 7, of Doob (p. 62),[4] the expression (1.17) for $\mathcal{E}(F, T)$ makes sense when

$$\frac{1}{T} \int_0^T E\left(W\left[y(t) - r(t) \right] \right) dt < \infty \tag{1.19}$$

This condition is fulfilled by restriction R_1. Because of the validity of (1.19), interchanging the expectation and integration is now also allowed in (1.17).

That $\mathcal{E}(F, T)$ depends upon T is evident; its dependence on F is through the statistical properties of the random process $\{y(t)\}$. A controller F^o is called T-optimal, if it is admissible, and

$$\mathcal{E}(F^o, T) \leq \mathcal{E}(F, T), \text{ for all admissible } F \tag{1.20}$$

Of great interest are T-optimal controllers for very large values of T. A controller F^o will be called optimal if it is admissible, if $\lim_{T \to \infty} \mathcal{E}(F^o, T)$ exists, and if

$$\lim_{T \to \infty} \mathcal{E}(F^o, T) \leq \lim_{T \to \infty} \mathcal{E}(F, T) \tag{1.21}$$

for all admissible F for which the limit on the right exists.

II. 2 A Necessary Condition for a T-optimal Controller

In this section a condition will be derived which a T-optimal controller must satisfy, in the general setup of Section II. 1. It is desired to minimize

$$\mathcal{E}(F, T) = E\left(\frac{1}{T}\int_0^T W[y(t) - r(t)]\,dt\right) \tag{2.1}$$

Consider a controller of the form

$$F\left\{r_{(-\infty, t]}\right\} = F^o\left\{r_{(-\infty, t]}\right\} + \varepsilon\,\bar{F}\left\{r_{(-\infty, t]}\right\} \tag{2.2}$$

where F^o is a supposedly T-optimal controller, $\varepsilon \geq 0$ is small, and $\bar{F}$ is arbitrary within the following conditions:

$$\begin{aligned} \infty > \bar{F}\left\{r_{(-\infty, t]}\right\} \geq 0 \quad &\text{wherever} \quad F^o\left\{r_{(-\infty, t]}\right\} = -\gamma(t) \\ \left|\bar{F}\left\{r_{(-\infty, t]}\right\}\right| < \infty \quad &\text{wherever} \quad \left|F^o\left\{r_{(-\infty, t]}\right\}\right| < \gamma(t) \\ -\infty < \bar{F}\left\{r_{(-\infty, t]}\right\} \leq 0 \quad &\text{wherever} \quad F^o\left\{r_{(-\infty, t]}\right\} = +\gamma(t) \end{aligned} \tag{2.3}$$

for all t. Here $\gamma(t)$ is the function that constrains the input amplitude, and "wherever" is to be read as: "for all sample functions $r_{(-\infty, t]}$ for which." By imposing these conditions on $\bar{F}$ the controller F can always be made admissible by choosing ε small enough.

Let $u(t)$, $u^o(t)$, and $\bar{u}(t)$ be the inputs generated by the controllers F, F^o and $\bar{F}$, respectively, from a sample function $r_{(-\infty, t]}$; then

$$u(t) = u^o(t) + \varepsilon\,\bar{u}(t), \quad t > 0 \tag{2.4}$$

The output which corresponds to the input $u_{(0, t]}$ is

$$y(t) = H\left\{x(0)\,;\,u^o_{(0, t]} + \varepsilon\,\bar{u}_{(0, t]}\right\} \tag{2.5}$$

For this it can be written by the assumption of the differentiability of the system

$$y(t) = H\left\{x(0);\, u^{o}_{(0,t]}\right\} + \varepsilon H_{u^{o}}\, \bar{u}(t) + o(\varepsilon; t) \tag{2.6}$$

where it is abbreviated

$$o(\varepsilon; t) = o(\varepsilon;\, u^{o}_{(0,t]},\, \bar{u}_{(0,t]}) \tag{2.7}$$

If one also abbreviates

$$y^{o}(t) = H\left\{x(0);\, u^{o}_{(0,t]}\right\} \tag{2.8}$$

i.e., $y^{o}(t)$ is the output generated by the T-optimal controller F^{o} from the sample function $r_{(-\infty,t]}$, expression (2.1) can be written in the form

$$\mathcal{E}(F,T) = E\left(\frac{1}{T}\int_{0-}^{T} W[y^{o}(t) - r(t) + \varepsilon H_{u^{o}}\, \bar{u}(t) + o(\varepsilon;t)]\, dt\right) \tag{2.9}$$

It is noted that $o(\varepsilon; t)$ is here a random process, which is bounded (because of the uniform convergence of the limit (1.6)), and such that

$$\lim_{\varepsilon \to 0} \frac{o(\varepsilon;\, t)}{\varepsilon} = 0 \tag{2.10}$$

almost surely. The following property of any differentiable function is now invoked, here applied to the weighting function $W(e)$:

$$W(x+y) = W(x) + y\, w(x) + o'(y) \tag{2.11}$$

where, as before, $w(e) = \frac{dW(e)}{de}$, and $o'(y)$ has the property

$$\lim_{y \to 0} \frac{o'(y)}{y} = 0, \text{ for all } x \tag{2.12}$$

Employing this in (2.9) one obtains

$$\mathcal{E}(F;T) = E\left(\frac{1}{T}\int_o^T W[y^o(t) - r(t)]\, dt\right)$$

$$+ E\left(\frac{1}{T}\int_o^T [\varepsilon H_{u^o} \bar{u}(t) + o(\varepsilon;t)]\, w[y^o(t)-r(t)]\, dt\right)$$

$$+ E\left(\frac{1}{T}\int_o^T o'[\varepsilon H_{u^o} \bar{u}(t) + o(\varepsilon;t)]\, dt\right)$$

$$= \mathcal{E}(F^o, T) + \varepsilon E\left(\frac{1}{T}\int_o^T H_{u^o} \bar{u}(t)\, w[y^o(t) - r(t)]\, dt\right)$$

$$+ E\left(\frac{1}{T}\int_o^T o(\varepsilon;t)\, w[y^o(t) - r(t)]\, dt\right)$$

$$+ E\left(\frac{1}{T}\int_o^T o'[\varepsilon\ H_{u^o} \bar{u}(t) + o(\varepsilon;t)]\, dt\right) \tag{2.13}$$

It is proved in Appendix 1 that the last two terms on the right-hand side of (2.13) can be replaced by $o''(\varepsilon)$, a number depending upon ε with the property

$$\lim_{\varepsilon \to 0} \frac{o''(\varepsilon)}{\varepsilon} = 0 \tag{2.14}$$

Thus one can write for (2.13):

$$\mathcal{E}(F, T) - \mathcal{E}(F^o, T) = \varepsilon E\left(\frac{1}{T}\int_o^T H_{u^o} \bar{u}(t)\ w[y^o(t) - r(t)]\, dt\right) + o''(\varepsilon) \tag{2.15}$$

Now if F^o is to be a T-optimal controller, the change of the controller from F^o to F should not be an improvement. Hence for all admissible F

$$\mathcal{E}(F, T) - \mathcal{E}(F^o, T) \geq 0 \tag{2.16}$$

By choosing ε small enough, the first term of the right-hand side of (2.15) can always be made to dominate the second term. This means that, since $\varepsilon \geq 0$, condition (2.16) cannot be satisfied unless

$$E\left(\frac{1}{T}\int_0^T H_{u^o}\,\bar{u}(t)\;w[y^o(t) - r(t)]\;dt\right) \geq 0 \tag{2.17}$$

for all $\bar{F}$ allowable according to (2.3).

At this point the adjoint of the operator H_{u^o} is introduced. The two time-functions that occur in (2.17) are of finite norm: Eq. (2.17) can therefore be written as

$$E\left(\frac{1}{T}\int_0^T \bar{u}(t)\;H^*_{u^o} w[y^o(t) - r(t)]\;dt\right) \geq 0 \tag{2.18}$$

In this expression, $\bar{u}(t)$ can be replaced by $\bar{F}\left\{r_{(-\infty, t]}\right\}$. Introducing conditional expectation, (2.18) can be put into the form

$$\frac{1}{T}\int_0^T E\Big(\bar{F}\left\{r_{(-\infty, t]}\right\}\;E(H^*_{u^o} w[y^o(t) - r(t)]\,|\,r_{(-\infty, t]})\Big)dt \geq 0$$

(2.19)

If now a functional $\mu_T\left\{r_{(-\infty, t]}\right\}$ is defined by

$$\mu_T\left\{r_{(-\infty, t]}\right\} = E(H^*_{u^o}\, w[y^o(t) - r(t)] \mid r_{(-\infty, t]}) \qquad (2.20)$$

the condition for optimality of F^o can finally be put into the form

$$\frac{1}{T} \int_0^T E(\bar{F}\left\{r_{(-\infty, t]}\right\} \mu_T\left\{r_{(-\infty, t]}\right\}\ )\, dt \geq 0 \qquad (2.21)$$

for all $\bar{F}$ allowable according to (2. 3).

From this requirement the following theorem can easily be derived:

Theorem 1 : Let F^o be a controller such that restrictions R_1, R_2, and R_3 are satisfied. Let the functional $\mu_T\left\{r_{(-\infty, t]}\right\}$ be defined by (2. 20). Then a necessary condition for F^o to be T-optimal with respect to all admissible controllers satisfying R_1, R_2, and R_3 is

$$\mu_T\left\{r_{(-\infty, t]}\right\} \begin{cases} \geq 0 & \text{wherever} \quad F^o\left\{r_{(-\infty, t]}\right\} = -\gamma(t) \\ = 0 & \text{wherever} \quad \left|F^o\left\{r_{(-\infty, t]}\right\}\right| < \gamma(t) \\ \leq 0 & \text{wherever} \quad F^o\left\{r_{(-\infty, t]}\right\} = +\gamma(t) \end{cases} \qquad (2.22)$$

almost everywhere with respect to the probability measure induced by the random process $\left\{r(t)\,,\ t \in (-\infty, +\infty)\right\}$.

Proof : The necessity of condition (2. 21) has been established. Suppose that condition (2. 22) is violated in some region of non-zero measure. Then it is easily recognized that an $\bar{F}$ can be chosen, within the requirements imposed by (2. 3), such that (2. 21) is violated. This contradicts the assumption that (2. 22) is not necessary; hence (2. 22) is necessary and the theorem has been proved.

It may be enlightening to adopt the following point of view. With the notation

$$(\mu_T, \overline{F}) = \frac{1}{T} \int_0^T E(\overline{F}\left\{r_{(-\infty, t]}\right\} \mu_T \left\{r_{(-\infty, t]}\right\}) \; dt \qquad (2.23)$$

it is possible to rewrite (2.15) in the form

$$\mathcal{E}(F, T) \approx \mathcal{E}(F^o, T) + (\mu_T, \varepsilon \overline{F}) \qquad (2.24)$$

for ε small. It is seen that in some sense μ_T is the derivative of $\mathcal{E}(F, T)$ with respect to F at F^o; it corresponds to the notion of the Fréchet-derivative.[5]

II.3 Sufficiency of Condition for Optimal Controller for Linear Systems

In general very little can be said about the sufficiency of the conditions of Theorem 1. In the case of a linear system, however, the following theorem can be proved:

Theorem 2 : Suppose that the system Σ is linear. Let F^o be an admissible controller such that the restrictions R_1, R_2, and R_3 are satisfied. Then (2.22) is a necessary and sufficient condition for F^o to be T-optimal with respect to all admissible controllers satisfying R_1, R_2, and R_3.

Proof : The necessity of condition (2.22) has been established in Theorem 1. To prove the sufficiency of (2.22), it must be shown that if a controller F^o satisfies (2.22), it is T-optimal.

In order to do this, consider any admissible controller F satisfying R_1, R_2, and R_3, and write it in the form

$$F\left\{r_{(-\infty, t]}\right\} = F^o\left\{r_{(-\infty, t]}\right\} + \bar{F}\left\{r_{(-\infty, t]}\right\} \tag{3.1}$$

Due to the admissibility of F and F^o, it must hold

$$\bar{F}\left\{r_{(-\infty, t]}\right\} \begin{cases} \geq 0 & \text{wherever} \quad F^o\left\{r_{(-\infty, t]}\right\} = -\gamma(t) \\ \leq 0 & \text{wherever} \quad F^o\left\{r_{(-\infty, t]}\right\} = +\gamma(t) \end{cases} \tag{3.2}$$

Corresponding to (3.1), the input to the system can be written as

$$u(t) = u^o(t) + \bar{u}(t) \tag{3.3}$$

and the output as

$$y(t) = H\left\{x(0)\, ;\, u^o_{(0, t]} + \bar{u}_{(0, t]}\right\} \tag{3.4}$$

Now by the definition of linearity[3, 4] this can be equivalently written as

$$y(t) = H\left\{x(0)\, ;\, u^o_{(0, t]}\right\} + H\left\{0\, ;\, \bar{u}_{(0, t]}\right\} \tag{3.5}$$

where the second term on the right-hand side indicates the zero-state response of the system to $\bar{u}_{(0, t]}$. Thus, in the notation of II.1,

$$H_{u^o}\, \bar{u}(t) = H\left\{0\, ;\, \bar{u}_{(0, t]}\right\} \tag{3.6}$$

and

$$o(\varepsilon\,;\,u^{o}_{(0,t]}\,;\,\overline{u}_{(0,t]}) = 0 \tag{3.7}$$

Furthermore, the following property of any differentiable convex function W is invoked:

$$W(x+y) \geq W(x) + y\,w(y) \tag{3.8}$$

where, as previously, $w(e) = \dfrac{dW(e)}{de}$.

Now, using (3.5) and (3.8), it follows

$$\begin{aligned}
\mathcal{E}(F,T) &= E\left(\frac{1}{T}\int_{o}^{T} W\,[y(t) - r(t)]\,dt\right) \\
&= E\left(\frac{1}{T}\int_{o}^{T} W\,[y^{o}(t) - r(t) + H_{u^{o}}\,\overline{u}(t)]\,dt\right) \\
&\geq E\left(\frac{1}{T}\int_{o}^{T} W\,[y^{o}(t) - r(t)]\,dt\right) \\
&\quad + E\left(\frac{1}{T}\int_{o}^{T} H_{u^{o}}\,\overline{u}(t)\,w\,[y^{o}(t) - r(t)]\,dt\right)
\end{aligned} \tag{3.9}$$

where, as before,

$$y^{o}(t) = H\left\{x(0)\,;\,u^{o}_{(0,t]}\right\} \tag{3.10}$$

Going through the same steps as by which Eqs. (2.18) and (2.19) were obtained, and with the use of μ_T as defined by (2.20), it follows from (3.9):

$$\mathcal{E}(F,T) - \mathcal{E}(F^{o},T) \geq \frac{1}{T}\int_{o}^{T} E\left(\overline{F}\left\{r_{(-\infty,t]}\right\}\;\mu_T\left\{r_{(-\infty,t]}\right\}\right)\,dt \tag{3.11}$$

The right-hand side of this expression is obviously ≥ 0 because of the hypothesis (2.22) on the behavior of the functional μ_T and the inequalities (3.2) that $\overline{F}$ must satisfy. Thus the proof that condition (2.22) is sufficient for F^o to be T-optimal, in the case of a linear system, has been completed, since the inequality

$$\mathcal{E}(F, T) - \mathcal{E}(F^o, T) \geq 0 \tag{3.12}$$

expresses that F^o is T-optimal.

II.4 Optimal Feedback Control

Controllers which are expressed in terms of functionals on the random process in the form of (1.15) have the disadvantage of being practically unfeasible as soon as it is necessary to compute and implement them, except when simple analytical expressions can be found. This difficulty may to some extent be overcome by the use of feedback controllers, as opposed to controllers of the type that have been considered so far, which will be denoted as feedforward controllers. Before defining a feedback controller, the notions of the state of the system Σ and the state of the random process $\{r(t), t \in (-\infty, +\infty)\}$ should be discussed.

The notion of the state of the system Σ was already touched upon in II.1. For the purpose of this investigation it is sufficient to use the following concept of state.[3,4] At all times t the system Σ is described by the state $x(t)$ in the sense that there exist two relations of the form

$$x(t) = \mathcal{X}\{x(t_o) ; u_{(t_o, t]}\} , \qquad 0 \leq t_o \leq t \tag{4.1}$$

$$y(t) = \mathcal{Y}(x(t) , u(t) , t) \tag{4.2}$$

That is, at each instant t the state $x(t)$ is uniquely determined by the state at some previous instant $x(t_o)$ and the input during the interval $(t_o, t]$. The output $y(t)$ is at each instant uniquely determined by the instantaneous state $x(t)$ and input $u(t)$. Upon combining (4.1) and (4.2), the relation (1.2) that was used previously is obtained:

$$y(t) = H\,\{x(t_o) ; u_{(t_o, t]}\} \tag{4.3}$$

The <u>state of the random process $\{r(t), t \in (-\infty, +\infty)\}$</u> is a random process $\{z(t), t \in [0, \infty)\}$, not necessarily scalar or even vector-valued, which is defined on the sample space of $\{r(t)\}$ and has the following properties:

i) For all $t_1, t_2, \ldots, t_n \geq t$, for all $r_1, r_2, \ldots, r_n$, and for all integers n,

$$P\,\{r(t_1) \leq r_1, r(t_2) \leq r_2, \ldots, r(t_n) \leq r_n \mid r_{(-\infty, t]}\}$$

$$= P\,\{r(t_1) \leq r_1, r(t_2) \leq r_2, \ldots\ldots, r(t_n) \leq r_n \mid z(t)\} \tag{4.4}$$

ii) At each instant t, $z(t)$ is completely determined by its value $z(t_o)$ at any previous instant $t_o \leq t$, and $r_{(t_o, t]}$, i.e., there exists a relation

$$z(t) = \mathfrak{Z}\left\{z(t_o)\,;\, r_{(t_o,\, t]}\right\}, \text{ for all } 0 \le t_o \le t \tag{4.5}$$

It is seen that in a sense $z(t)$ "summarizes" all information concerning the future of $r(t)$ which is contained in $r_{(-\infty,\, t]}$. By condition ii) the state can be found by observing the random process. Clearly, the notion of state is related to that of sufficient statistic. In the following it will be assumed that either $z(0)$ is known, or else that as $t_o \to -\infty$, the right-hand side of (4.5) becomes independent of $z(t_o)$ and thus $z(0)$ can be obtained from $r_{(-\infty,\, 0]}$.

Two examples of the state of a random process are 1) in the least favorable case $z(t) = r_{(-\infty,\, t]}$, and 2) the case of a simple Markov process $z(t) = r(t)$.

Now the meaning of feedback controller can be explained.

<u>Definition</u> : A <u>feedback controller</u> is a controller which generates an input to the system as an <u>instantaneous</u> function of the state of the random process and the state of the system, i.e.,

$$u(t) = F(x(t),\ z(t),\ t) \tag{4.6}$$

with the restriction that the interconnection of the controller F and the system Σ be <u>determinate.</u> The interconnection of the controller F and the system Σ is said to be determinate if for all t the state of the system $x(t)$ is u n i q u e l y determined by the state $x(t_o)$ at some preceding instant t_o <u>and</u> the input to the interconnection of the controller and the system during the interval $(t_o, t]$. The input in this case is $z_{(t_o, t]}$.

The sketch of Fig. 1 gives the interconnection of the controller and the system.

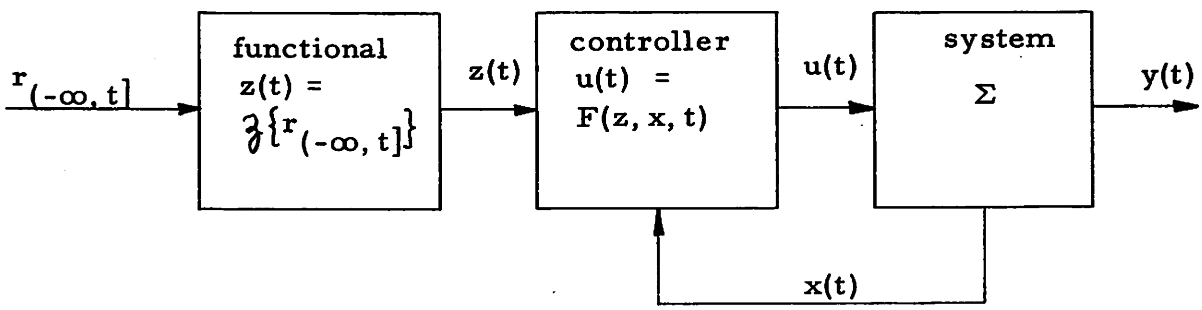

Figure 1. Interconnection of Controller and System.

The condition of determinateness is necessary to ensure that there is no non-uniqueness in the operation of the interconnection of system and controller. It is always satisfied when the value of $u(t)$ has no immediate effect on the value of $x(t)$.

To say that the interconnection of system and feedback controller is determinate is equivalent to saying that to the feedback controller there corresponds a unique feedforward controller of the form

$$u(t) = F\left\{r_{(-\infty, t]}\right\} \tag{4.7}$$

such that the operation of the feedforward controller is the same as that of the feedback controller.[+] The converse of this is not true, however: there does not correspond a feedback controller to every feedforward controller. That nevertheless it is possible to limit the search for optimal controllers to feedback controllers will become clear in the following. First of all the following lemma is proved.

Lemma 1 : Suppose that the system Σ is interconnected with a controller F of the feedback type, i.e.,

$$u(t) = F(x(t), z(t), t) \tag{4.8}$$

Then the functional $\mu_T\left\{r_{(-\infty, t]}\right\}$ as defined in (2.20), but extended to any controller F, can be expressed as an instantaneous function of $x(t)$ and $z(t)$, where $x(t)$ is the state into which the system has been brought by the controller F as a result of the sample function $r_{(-\infty, t]}$; i.e.,

$$\mu_T\left\{r_{(-\infty, t]}\right\} = \mu_T(x(t), z(t), t) \qquad (4.9)^*$$

[+] To simplify the notation the same symbol F is used for both the feedback and the feedforward controller. When confusion is possible the argument of the functional will be written out in full.

[*] A remark similar to footnote[+] applies to μ_T.

Sketch of Proof : No attempt will be made to prove the lemma rigorously, although intuitively there are few difficulties. First of all, write the function $\mu_T\left\{r_{(-\infty, t]}\right\}$ with the aid of the impulse response of the variational system, according to (1.13) and (1.14), in the form

$$\mu_T\left\{r_{(-\infty, t]}\right\} = E\left(\int_t^T h_{u^o}(\tau, t)\, w\left[y(\tau) - r(\tau)\right] d\tau \;\middle|\; r_{(-\infty, t]} \right) \tag{4.10}$$

Consider the right-hand side of this expression. By giving $r_{(-\infty, t]}$, also $z(t)$ and $x(t)$ are known. It will be argued that it is sufficient to know these two quantities to find the right-hand side of (4.10). Consider the various functions of time which occur under the integral and expectation sign. The function $h_{u^o}(\tau, t)$ is defined as the response at time τ due to an impulse applied at time t to the variational system about u^o. Loosely speaking, $h_{u^o}(\tau, t)$ can therefore be considered as the derivative with respect to τ of the change of response of the system itself due to an infinitesimally small step applied at time t (with an appropriate multiplying factor). But the behavior of the system, from time t on, and hence also $h_{u^o}(\tau, t)$, is completely determined by $z(t)$, $x(t)$ and $r_{(t, \infty)}$. But statistically, $r_{(t, \infty)}$ depends entirely on $z(t)$; hence, statistically $h_{u^o}(\tau, t)$ depends completely upon $z(t)$ and $x(t)$.

By the same argument, $y(\tau)$ and $r(\tau)$ also depend statistically only upon $x(t)$ and $z(t)$ for $\tau \geq t$. Thus it follows that all quantities under the integral and expectation sign depend statistically only upon $x(t)$ and $z(t)$; hence $\mu_T\{r_{(-\infty,t]}\}$ can be expressed as a function of $x(t)$ and $z(t)$ alone.

As a consequence of this lemma the following theorem, which brings out the usefulness of considering feedback controllers, can easily be established.

Theorem 3 : Let F^o be an admissible controller of the feedback type, such that conditions R_1, R_2, and R_3 are satisfied, and such that the corresponding μ_T-function satisfies the condition

$$\mu_T(x(t), z(t), t) \begin{cases} \geq 0 & \text{wherever} \quad F^o(x(t), z(t), t) = -\gamma\ (t) \\ = 0 & \text{wherever} \quad |F^o(x(t), z(t), t)| < \gamma\ (t) \\ \leq 0 & \text{wherever} \quad F^o(x(t), z(t), t) = +\gamma\ (t) \end{cases} \tag{4.11}$$

almost everywhere with respect to the probability measure induced on $x(t)$ and $z(t)$ by the random process $\{r(t)\}$ and for almost all t. Then the feedforward version of F^o satisfies the conditions for a T-optimal controller with respect to all admissible controllers satisfying R_1, R_2, and R_3.

Proof : Once Lemma 1 is established, the proof of Theorem 3 is very simple. Let $r_{(-\infty, t]}$ be a sample function for which condition (2. 22) is to be checked. From this sample function, the corresponding $x(t)$ and

$z(t)$ can be found, and from these the values of $F^o\{r_{(-\infty, t]}\} =$ $F^o(x(t), z(t), t)$ and $\mu_T\{r_{(-\infty, t]}\} = \mu_T(x, (t), z(t), t)$. But then it follows immediately from (4.11) that F^o and μ_T stand in the correct relation to each other; hence, F^o satisfies condition (2.22) and therefore qualifies as a candidate for a T-optimal controller.

II. 5 Conditions for a T-optimal Controller with $T = \infty$

In the preceding discussions the question of the requirements for an optimal controller, i.e., a T-optimal controller for $T = \infty$, has been avoided. The difficulty is that not much can be said about the existence of the various limits as T tends to ∞. What can be said is summarized in the following theorem.

Theorem 4 : Suppose that F^o is an admissible controller such that R_1, R_2, and R_3 are satisfied, and such that

$$\lim_{T \to \infty} \mathcal{E}(F^o, T) \tag{5.1}$$

exists and is finite. Suppose furthermore that for this controller the function $\mu_T\{r_{(-\infty, t]}\}$ as defined by Eq. (2.20) has a limit $\mu_\infty\{r_{(-\infty, t]}\}$ as $T \to \infty$ in the sense that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E\left(\left|\mu_T\{r_{(-\infty, t]}\} - \mu_\infty\{r_{(-\infty, t]}\}\right|^2\right) dt = 0 \tag{5.2}$$

Now define a neighborhood of F^o by permitting variations of the type

$$F\{r_{(-\infty, t]}\} = F^o\{r_{(-\infty, t]}\} + \varepsilon \bar{F}\{r_{(-\infty, t]}\} \tag{5.3}$$

with F such that R_1, R_2, and R_3 are satisfied, $0 < \varepsilon < \varepsilon_m$, $\overline{F}$ allowable according to conditions (2.3), and $\overline{F}$ such that

$$\lim_{T\to\infty} \mathcal{E}(F^o + \varepsilon\overline{F},\ T) \tag{5.4}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\left(\left|\overline{F}\left\{r_{(-\infty,t]}\right\}\right|^2\right) dt \tag{5.5}$$

and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\left(\mu_\infty\left\{r_{(-\infty,t]}\right\}\overline{F}\left\{r_{(-\infty,t]}\right\}\right) dt \tag{5.6}$$

exist and are finite. Then there exists an ε_m such that F^o is locally optimal in this neighborhood if

$$\mu_\infty\left\{r_{(-\infty,t]}\right\} \begin{cases} \geq 0 & \text{wherever} \quad F^o\left\{r_{(-\infty,t]}\right\} = -\gamma(t) \\ = 0 & \text{wherever} \quad \left|F^o\left\{r_{(-\infty,t]}\right\}\right| < \gamma(t) \\ \leq 0 & \text{wherever} \quad F^o\left\{r_{(-\infty,t]}\right\} = +\gamma(t) \end{cases} \tag{5.7}$$

almost surely with respect to the probability measure induced by the random process $\left\{r(t),\ t \in (-\infty, +\infty)\right\}$.

Proof : As in the proof of Theorem 1, it holds for any finite T

$$\mathcal{E}(F^o + \varepsilon\overline{F}, T) - \mathcal{E}(F^o, T) = \varepsilon\frac{1}{T}\int_0^T E\left(\overline{F}\left\{r_{(-\infty,t]}\right\}\mu_T\left\{r_{(-\infty,t]}\right\}\right)dt + o(\varepsilon) \tag{5.8}$$

Now as T tends to ∞, the two quantities on the left have a limit by hypothesis. Furthermore

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T E(\bar{F}\{r_{(-\infty,t]}\} \mu_T\{r_{(-\infty,t]}\})\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T E(\bar{F}\{r_{(-\infty,t]}\} \mu_\infty\{r_{(-\infty,t]}\})\, dt \tag{5.9}$$

since

$$\Big|\lim_{T\to\infty} \frac{1}{T} \int_0^T E(\bar{F}\{r_{(-\infty,t]}\} \mu_T\{r_{(-\infty,t]}\})\, dt$$

$$- \lim_{T\to\infty} \frac{1}{T} \int_0^T E(\bar{F}\{r_{(-\infty,t]}\} \mu_\infty\{r_{(-\infty,t]}\})\, dt \Big|^2$$

$$= \lim_{T\to\infty} \Big| \frac{1}{T} \int_0^T E(\bar{F}\,(\mu_T - \mu_\infty))\, dt \Big|^2$$

$$\leq \lim_{T\to\infty} \frac{1}{T} \int_0^T E(\bar{F})^2\, dt \cdot \lim_{T\to\infty} \frac{1}{T} \int_0^T E(\mu_T - \mu_\infty)^2\, dt$$

$$= 0 \tag{5.10}$$

because of relations (5.2) and (5.5) and with the use of Schwartz's inequality. Now since the limits as T goes to ∞ of all terms in (5.8) besides $o(\varepsilon)$ exist, also $\lim_{T\to\infty} o(\varepsilon)$ must exist and remain a term $o'(\varepsilon)$ of order ε. Thus one can write for (5.8) as $T\to\infty$

$$\lim_{T\to\infty} \mathcal{E}(F^o + \varepsilon \overline{F}, T) - \lim_{T\to\infty} \mathcal{E}(F^o, T) =$$

$$\varepsilon \lim_{T\to\infty} \frac{1}{T} \int_o^T E(\overline{F}\left\{r_{(-\infty, t]}\right\} \mu_\infty \left\{r_{(-\infty, t]}\right\}) \, dt + o'(\varepsilon) \tag{5.11}$$

By the same arguments as in the proof of Theorem 1 it can be reasoned that the function μ_∞ has to satisfy condition (5.7) if the controller F^o is to be locally optimal. This completes the proof.

By means of this theorem one now has a tool in μ_∞ to investigate the optimality for $T = \infty$ of any given controller; it is not necessary, for example, to investigate sequences of T-optimal controllers with $T \to \infty$. It is clear that Theorem 4 expresses necessary conditions for a ∞-optimal controller, since optimality implies local optimality in any neighborhood.

Of course, corresponding to Theorem 4 there exists a stronger statement for linear systems in the following form.

Theorem 5 : Let the system Σ be linear. Suppose that F^o is an admissible controller such that R_1, R_2, and R_3 are satisfied, and such that

$$\lim_{T\to\infty} \mathcal{E}(F^o, T) \tag{5.12}$$

exists and is finite. Also suppose that for this controller $\mu_T \longrightarrow \mu_\infty$ as $T \longrightarrow \infty$ in the sense required by (5. 2). Let F be any other admissible controller such that R_1, R_2, and R_3 are satisfied and

$$\lim_{T \to \infty} \mathcal{E}(F, T) \tag{5.13}$$

$$\lim_{T \to \infty} \int_0^T E\left(\left| F\left\{ r_{(-\infty, t]} \right\} - F^o \left\{ r_{(-\infty, t]} \right\} \right|^2 \right) dt \tag{5.14}$$

and

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E\left(\mu_\infty \left\{ r_{(-\infty, t]} \right\} \left(F\left\{ r_{(-\infty, t]} \right\} - F^o \left\{ r_{(-\infty, t]} \right\} \right)\right) dt \tag{5.15}$$

exist and are finite. Then F^o is optimal with respect to all such controllers F, i.e.,

$$\lim_{T \to \infty} \mathcal{E}(F^o, T) \leq \lim_{T \to \infty} \mathcal{E}(F, T), \quad \text{all such } F, \tag{5.16}$$

if and only if

$$\mu_\infty \left\{ r_{(-\infty, t]} \right\} \begin{cases} \geq 0, & \text{whenever} \quad F^o \left\{ r_{(-\infty, t]} \right\} = -\gamma(t) \\ = 0 & \text{whenever} \quad \left| F^o \left\{ r_{(-\infty, t]} \right\} \right| < \gamma(t) \\ \leq 0 & \text{whenever} \quad F^o \left\{ r_{(-\infty, t]} \right\} = +\gamma(t) \end{cases} \tag{5.17}$$

almost surely with respect to the probability measure induced by the random process $\left\{ r(t), \ t \in (-\infty, +\infty) \right\}$.

Proof : The proof can be very short. The necessity follows from Theorem 4. To prove the sufficiency of condition (5.17), it is recalled that for a linear system it can be written for all finite T (see proof of Theorem 2)

$$\mathcal{E}(F, T) - \mathcal{E}(F^o, T) \geq \frac{1}{T} \int_o^T E(\bar{F}\{r_{(-\infty, t]}\} \; \mu_T\{r_{(-\infty, t]}\}) \, dt \tag{5.18}$$

where

$$\bar{F} = F - F^o \tag{5.19}$$

By the same argument as in the proof of Theorem 4, and by the use of (5.13), (5.14) and (5.15), it follows that as $T \rightarrow \infty$, the right-hand side of (5.18) approaches

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_o^T E(\bar{F} \{r_{(-\infty, t]}\} \; \mu_\infty\{r_{(-\infty, t]}\}) \, dt \tag{5.20}$$

which exists by hypothesis (5.15). Now since both F and F^o are admissible,

$$\bar{F}\{r_{(-\infty, t]}\} \begin{cases} \geq 0 & \text{if} \quad F^o\{r_{(-\infty, t]}\} = -\gamma(t) \\ \leq 0 & \text{if} \quad F^o\{r_{(-\infty, t]}\} = +\gamma(t) \end{cases} \tag{5.21}$$

From this it follows immediately that

$$\begin{aligned} &\lim_{T \rightarrow \infty} \mathcal{E}(F, T) - \lim_{T \rightarrow \infty} \mathcal{E}(F^o, T) \\ &\quad \geq \lim_{T \rightarrow \infty} \frac{1}{T} \int_o^T E(\bar{F} \{r_{(-\infty, t]}\} \mu_\infty \{ r_{(-\infty, t]}\}) \, dt \\ &\quad \geq 0 \end{aligned} \tag{5.22}$$

which implies that F^o is optimal. This completes the proof.

III. APPLICATION TO LINEAR, TIME-VARYING DIFFERENTIAL SYSTEMS AND A CLASS OF MARKOV-TYPE RANDOM PROCESSES

In this section, the results of Section II will be specialized to

- linear, time-varying differential systems
- non-stationary random processes of which the state is a

finite-dimensional diffusion process. The main results will consist of a number of manipulatory results which will be useful for finding actual solutions, and the fact that for a subclass of the cases considered the optimal controller is of the bang-bang type.

However, first considerable attention should be given to the characterization of the systems and the random processes that will be considered.

III.1 The Characterization of Linear, Time-Varying Differential Systems

The type of system to be considered is that which can be described by the relations

$$\Sigma: \begin{cases} \dot{\underline{x}}(t) = \underline{A}(t)\, \underline{x}(t) + \underline{b}(t)\, u(t) \\ y(t) = \underline{c}(t)'\underline{x}(t) + k(t)\, u(t) \end{cases} \tag{1.1}$$

where, as before, u(t) and y(t) (scalars) are the input and output, respectively, and $\underline{x}(t) = \mathrm{col}(x_1(t), x_2(t), \ldots, x_n(t))$ is the state of the system. The dot indicates differentiation with respect to time; $\underline{A}(t)$ is a time-varying n x n matrix; $\underline{b}(t)$ and $\underline{c}(t)$ are time-varying column vectors of dimension n; k(t) is a time-varying scalar function; the prime denotes the transpose. All quantities are supposed to be real.

The solution of (1.1) for some input and some initial state can most conveniently be given in terms of the transition matrix $\underline{\phi}(t, \tau)$, which is defined as the solution of

$$\begin{cases} \dfrac{d}{dt}\,\underline{\phi}(t, \tau) = \underline{A}(t)\,\underline{\phi}(t, \tau) \\ \\ \underline{\phi}(\tau, \tau) = \underline{I} \end{cases} \tag{1. 2}$$

where $\underline{I}$ is the identity matrix. With the aid of the transition matrix the solution of (1.1) takes on the form

$$y(t) = \underline{c}(t)' \left(\underline{\phi}(t, t_o)\underline{x}(t_o) + \int_{t_o}^{t} \underline{\phi}(t, \tau)\, \underline{b}(\tau)\, u(\tau)\, d\tau \right)$$

$$+ \; k(t)\, u(t) \;, \quad t \geq t_o \tag{1. 3}$$

where $\underline{x}(t_o)$ is the initial state at $t = t_o$. This relation is in fact the explicit form of the representation which has been used to characterize the system up till this point:

$$y(t) = H \left\{ x(t_o) \; ; \; u_{(t_o, t]} \right\} \tag{1.4}$$

It was seen in Section II. 3 that in the case of a linear system the variational system is exactly the same as the system itself, with the restriction that the initial state at time 0 is always 0; hence the variational system is characterized by

$$\bar{y}(t) = H \left\{0 \, ; \bar{u}_{(0, t]}\right\}$$

$$= \underline{c}(t)' \int_0^t \underline{\phi}(t, \tau) \, \underline{b}(\tau) \, \bar{u}(\tau) \, d\tau + k(t)\bar{u}(t) \tag{1.5}$$

From this expression it is seen that the impulse response of the variational system is given by

$$h(t, \tau) = \underline{c}(t)' \, \underline{\phi}(t, \tau) \, \underline{b}(\tau) \; + \; k(t) \, \delta(t-\tau) \tag{1.6}$$

where the subscript u^o has been dropped from $h_{u^o}(t, \tau)$, since the impulse response does in this case not depend upon the trajectory.

It follows that the impulse response of the adjoint variational system is given by

$$h^*(t, \tau) = h(\tau, t) = \underline{b}(t)' \, \underline{\phi} \, (\tau, t)'\underline{c}(\tau) \; + \; k(t) \, \delta(t-\tau) \tag{1.7}$$

Hence the adjoint variational system is characterized by

$$y^*(t) = \underline{b}(t)' \int_t^T \underline{\phi} \, (\tau, t)'\underline{c}(\tau) \;\; u^*(\tau) \, d\tau \; + \; k(t) \, u^*(t) \tag{1.8}$$

From the transitivity property of the matrix $\underline{\phi}$, i.e., the fact that

$$\underline{\phi}(t, t_o) = \underline{\phi} \, (t, t_1) \, \underline{\phi}(t_1, t_o) \tag{1.9}$$

for all t, t_1, and t_o, it is easily derived by differentiation with respect to t_1 that the matrix $\underline{\phi}(\tau, t)'$ satisfies

$$\begin{cases} \dfrac{d}{dt}\ \underline{\phi}(\tau, t)' = -\underline{A}(t)'\underline{\phi}(\tau, t)' \\ \\ \underline{\phi}(\tau, \tau)' = \underline{I} \end{cases} \tag{1.10}$$

If therefore the state of the adjoint variational system is defined by

$$\underline{x}^*(t) = -\int_t^T \underline{\phi}(\tau, t)'\underline{c}(\tau)\, u^*(\tau)\, d\tau \tag{1.11}$$

it follows easily that in state form the adjoint variational system is described by

$$\begin{cases} \dot{\underline{x}}^*(t) = -\underline{A}(t)'\underline{x}^*(t) + \underline{c}(t)u^*(t) \\ y^*(t) = -\underline{b}(t)'\underline{x}^*(t) + k(t)u^*(t) \\ \underline{x}^*(T) = \underline{0} \end{cases} \tag{1.12}$$

Because of the close connection of the system Σ itself and the variational system, the system described by Eqs. (1.12), but without the restriction $\underline{x}^*(T) = \underline{0}$, will be called the adjoint system of Σ; it will be denoted as Σ^*.

In the sequel some facts concerning the equivalent representation of the systems Σ and Σ^* by scalar differential equations will be useful. In order to do this, first the following notions are introduced.

Definition : A linear differential system Σ described by Eqs. (1.1) is called instantaneously controllable at time t if the system can be brought into any state $\underline{x}(t)$, from zero state, during an arbritrarily short interval of time preceding t.

<u>Definition</u>: A linear differential system Σ described by Eqs. (1.1) is called <u>instantaneously observable at time t</u> if it is possible to determine any initial state $\underline{x}(t)$ by observing the zero-input response from this state during an arbitrarily short interval of time following t.

Define two chains of vectors as follows

$$\underline{b}_{(i)}(t) = (-1)^{i+1} \left(\frac{d}{dt} + \underline{A}(t)\right)^{i} \underline{b}(t) \ , \quad i = 0, 1, \ldots, n$$

$$\underline{c}_{(i)}(t) = \left(\frac{d}{dt} + \underline{A}(t)\right)^{i} \underline{c}(t), \quad i = 0, 1, \ldots, n \tag{1.13}$$

under the assumption that all vectors $\underline{b}_{(i)}(t)$, $i = 0, 1, \ldots, n$, and $\underline{c}_{(i)}(t)$, $i = 0, 1, \ldots, n$ exist and are continuous. The relevance of the vectors $\underline{b}_{(i)}(t)$ and $\underline{c}_{(i)}(t)$ will become evident from the following lemma:

<u>Lemma 2</u> : Suppose that a linear differential system Σ is described by Eqs. (1.1), and that the vectors $\underline{b}_{(i)}(t)$ and $\underline{c}_{(i)}(t)$, as defined in (1.13), exist and are continuous. Then if the vectors $\underline{b}_{(i)}(t)$, $i = 0, 1, \ldots, (n-1)$, are linearly independent during an arbitrarily short interval of time preceding t_o, the system Σ is instantaneously controllable at time t_o. Furthermore, if the vectors $\underline{c}_{(i)}(t)$, $i = 0, 1, \ldots, (n-1)$, are linearly independent during an arbitrarily short interval following t_o, the system Σ is instantaneously observable at time t_o.

Proof: To prove the first part of the lemma, consider the following expression for the state at time t_o into which the system has been brought from zero state at some time $t < t_o$ by some input $u_{(t, t_o]}$

$$\underline{x}(t_o) = \int_t^{t_o} \underline{\phi}(t_o, \tau)\, \underline{b}(\tau)\, u(\tau)\, d\tau \tag{1.14}$$

Now apply an input which consists of a linear combination of delta functions just prior to t_o:

$$u(t) = \sum_{i=0}^{n-1} u_i \; \delta^{(i)}(t - t_o^-) \tag{1.15}$$

where t_o^- indicates an "instant just prior to t_o", and the u_i are arbitrary constants. With the use of (1.10) it is not difficult to verify that the response to this input at time t_o, according to (1.14), is precisely

$$\underline{x}(t_o) = \sum_{i=0}^{n-1} (-1)^{i+1} \; u_i \underline{b}_{(i)}(t_o^-) \tag{1.16}$$

Since by hypothesis the vectors $\underline{b}_{(i)}(t_o^-)$, $i = 0, 1, \ldots, (n-1)$, are linearly independent, they span the space, and every state $\underline{x}(t_o)$ can be reached, which implies that Σ is instantaneously controllable at t_o.

To prove that the linear independence of the vectors $\underline{c}_{(i)}(t)$, $i = 0, 1, \ldots, (n-1)$, during an arbitrarily short interval of time following t_o means that the system Σ is instantaneously observable at time t_o, consider the zero-input response from any state $\underline{x}(t_o)$

$$y(t) = \underline{c}(t)' \underline{\phi}(t, t_o)\, \underline{x}(t_o) \quad , \; t \geq t_o \tag{1.17}$$

By differentiating repeatedly with respect to t and setting $t = t_o^+$, where t_o^+ is an "instant just after t_o", it follows with the aid of (1.2)

$$y(t_o^+) = \underline{c}_{(0)}(t_o^+)' \, \underline{x}(t_o)$$

$$y^{(1)}(t_o^+) = \underline{c}_{(1)}(t_o^+)' \, \underline{x}(t_o)$$

$$\vdots \tag{1.18}$$

$$y^{(n-1)}(t_o^+) = \underline{c}_{(n-1)}(t_o^+)' \, \underline{x}(t_o)$$

Since by hypothesis the vectors $\underline{c}_{(i)}(t_o^+)$, $i = 0, 1, \ldots, (n-1)$ are linearly independent, the Eqs. (1.18) can always be solved for $\underline{x}(t_o)$ if the $y^{(i)}(t_o^+)$ are observed, which completes the proof that Σ is instantaneously observable.

Now for a system Σ that is instantaneously controllable or observable during some interval of time (t_o, t_1), the following result can be shown to hold:

Lemma 3 : Let Σ be a system described by Eqs. (1.1) such that the vectors $\underline{c}_{(i)}(t)$ exist, are continuous, and are linearly independent during some interval (t_o, t_1). Then during this interval the input $u(t)$ and the output $y(t)$ obey a differential equation of the form

$$\sum_{k=0}^{n} \beta_k(t) \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{n} \alpha_k(t) \frac{d^k u(t)}{dt^k} \tag{1.19}$$

which will be abbreviated as

$$D(\frac{d}{dt})y = N(\frac{d}{dt})u \tag{1.20}$$

such that $\beta_n(t) \neq 0$ during this interval.

Furthermore, if the vectors $\underline{b}_{(i)}(t)$, $i = 0, 1, \ldots, (n-1)$ exist, are continuous, and are linearly independent during (t_o, t_1), the input and output ($u^*(t)$ and $y^*(t)$, respectively) of the adjoint system Σ^* obey a differential equation of the form

$$\sum_{k=0}^{n} \beta^*_k(t) \frac{d^k y^*(t)}{dt^k} = \sum_{k=0}^{n} \alpha^*_k(t) \frac{d^k u^*(t)}{dt^k} \tag{1.21}$$

which will be abbreviated to

$$D^*\left(\frac{d}{dt}\right) y^* = N^*\left(\frac{d}{dt}\right) u^* \tag{1.22}$$

such that $\beta^*_n(t) \neq 0$ during this interval.

Finally, if both the sets of vectors $\underline{b}_{(i)}(t)$ and $\underline{c}_{(i)}(t)$, $i = 0, 1, \ldots, (n-1)$, are linearly independent during the interval (t_o, t_1), then $\alpha_{n-j}(t) = 0$, $j = 0, 1, \ldots, p$, during (t_o, t_1) implies

$$k(t) = 0, \quad \underline{b}(t)'\underline{c}_{(i)}(t) = 0, \quad i = 0, 1, \ldots, (p-1) \tag{1.23}$$

$$\underline{c}(t)'\underline{b}_{(i)}(t) = 0, \quad i = 0, 1, \ldots, (p-1), \tag{1.24}$$

and

$$\alpha^*_{n-j}(t) = 0, \quad j = 0, 1, \ldots, p \tag{1.25}$$

all during the interval (t_o, t_1).

Proof : From (1.1) it follows by differentiating y repeatedly with respect to t, that

$$y(t) = \underline{c}_{(0)}(t)'\underline{x}(t) + k(t)\, u(t) \tag{1.26}$$

$$\frac{dy(t)}{dt} = \underline{c}_{(1)}(t)'\underline{x}(t) + \underline{c}_{(0)}(t)'\underline{b}(t)u(t) + \frac{d}{dt}k(t)\,u(t) \tag{1.27}$$

$$\frac{d^2y(t)}{dt^2} = \underline{c}_{(2)}(t)'\underline{x}(t) + \underline{c}_{(1)}(t)'\underline{b}(t)\,u(t) + \frac{d}{dt}\,\underline{c}_{(0)}(t)'\underline{b}(t)u(t) + \frac{d^2}{dt^2}k(t)u(t) \tag{1.28}$$

$$\vdots$$

$$\frac{d^ny(t)}{dt^n} = \underline{c}_{(n)}(t)'\underline{x}(t) + \underline{c}_{(n-1)}(t)'\underline{b}(t)u(t) + \frac{d}{dt}\,\underline{c}_{(n-2)}(t)'\underline{b}(t)u(t) + \dots + \frac{d^{n-1}}{dt^{n-1}}\underline{c}_{(0)}(t)'\underline{b}(t)u(t) + \frac{d^n}{dt^n}k(t)u(t) \tag{1.29}$$

Now since the vectors $\underline{c}_{(i)}(t)$, $i = 0, 1, \dots, (n-1)$, span the n-dimensional space by hypothesis, there must be $(n+1)$ time functions $\beta_i(t)$ such that

$$\beta_0(t)\,\underline{c}_{(0)}(t) + \beta_1(t)\,\underline{c}_{(1)}(t) + \dots + \beta_n(t)\,\underline{c}_{(n)}(t) = 0 \tag{1.30}$$

with $\beta_n(t) \neq 0$ during the interval (t_o, t_1). Then it is possible to eliminate $\underline{x}(t)$ from the Eqs. (1.26)-(1.29) with the aid of (1.30); it follows easily that $y(t)$ satisfies the differential equation

$$\beta_o(t)\,y(t) + \beta_1(t)\frac{dy(t)}{dt} + \dots\dots\dots + \beta_n(t)\frac{d^ny(t)}{dt^n} =$$

$$= \beta_n(t) \frac{d^n}{dt^n} k(t)\, u(t)$$

$$+ \beta_n(t) \frac{d^{n-1}}{dt^{n-1}} \underline{c}_{(0)}(t)'\underline{b}(t)\, u(t) + \beta_{n-1}(t) \frac{d^{n-1}}{dt^{n-1}} k(t)\, u(t)$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ \beta_n(t)\, \underline{c}_{(n-1)}(t)'\, \underline{b}(t)\, u(t) + \ldots\ldots + \beta_1(t)\, \underline{c}_{(0)}(t)'\underline{b}(t)\, u(t)$$

$$+ \beta_0(t)\, k(t)\, u(t) \qquad (1.31)$$

The right-hand side of this equation can easily be put into the required form of (1.19).

To prove that the adjoint system leads to a similar differential equation, one proceeds in the same way; the only difference is that in the final result (1.31) the $\underline{c}_{(i)}$ are replaced by $\underline{b}_{(i)}$, $\underline{b}$ by $\underline{c}$, and the $\beta_i(t)$ by $\beta_i^*(t)$.

To show the last part of the lemma, suppose first that $\alpha_n(t) = 0$ during (t_o, t_1). This means that in Eq. (1.31) in the right-hand side no terms in $\frac{d^n u(t)}{dt^n}$ should appear. But since $\beta_n(t) \neq 0$, one must have $k(t) = 0$ during the interval (t_o, t_1). If in addition to $\alpha_n(t) = 0$, also $\alpha_{n-1}(t) = 0$, then neither should terms in $\frac{d^{n-1}u(t)}{dt^{n-1}}$ appear in the right-hand side of (1.31). It follows by inspection of the second line of this right-hand side, that then $\underline{c}_{(0)}(t)'\underline{b}(t) = 0$. Continuing in this fashion, it follows that $\alpha_{n-j}(t) = 0$, $j = 0, 1, .., p$, implies

$$k(t) = 0, \quad \underline{c}_{(i)}(t)'\underline{b}(t) = 0, \quad i = 0, 1, \ldots, (p-1)$$

The proof that $\underline{c}_{(i)}(t)'\underline{b}(t)$, $i = 0,1,\ldots,(p-1)$, implies $\underline{b}_{(i)}(t)'\underline{c}(t)=0$, $i = 0,1,\ldots,(p-1)$, is given in Appendix 2. That this fact together with $k(t) = 0$ implies $\alpha^*_{n-j}(t) = 0$, $j = 0,1,\ldots,p$, is not difficult to recognize by inspection of the differential equation for the adjoint system corresponding to (1. 31).

Observation : The following facts are also true, but will not be proved since they will not be needed.

- The converse of Lemma 2: If the system Σ is instantaneously controllable (observable) at time t_o, then the vectors $\underline{b}_{(i)}(t_o^-)$, $i = 0,1,\ldots,(n-1)$ (respectively $\underline{c}_{(i)}(t_o^+)$, $i = 0,1,\ldots,(n-1)$) are linearly independent.

- A stronger version of Lemma 3: If the vectors $\underline{c}_{(i)}(t)$, $i = 0,1,\ldots,(n-1)$, are linearly independent during an interval (t_o, t_1), the system Σ is equivalent during (t_o, t_1) to the system described by the differential equation (1. 31). Two systems are called equivalent if every input-output pair $(u_{(t_o,t_1)}, y_{(t_o,t_1)})$ of one system is also an input-output pair for the other system, and vice-versa. Similarly, if the vectors $\underline{b}_{(i)}(t)$, $i = 0,1,\ldots,(n-1)$, are linearly independent during (t_o, t_1), the system described by the differential equation derived from the adjoint system is equivalent to the adjoint system Σ^*.

- It is finally remarked that instantaneous controllability and observability, as defined here, differ from the notions of controllability and observability (without adjective) as currently used in the literature (Kalman),[6] in the definitions of which the words "during an arbitrarily short interval of time" are replaced by "during some interval of time." In the case of time-invariant systems, however, instantaneous controllability of course implies controllability, but also is implied by controllability. The same is true for observability.

Results which are related to those obtained in this section can be found in a note by A. Chang.[7]

III. 2 Description of a Class of Random Processes

The random processes that will be considered are those which are in some sense the projection of a finite-dimensional diffusion process. Before defining this class of random processes more precisely, the notions of Markov process and diffusion process will be discussed.

A vector process $\underline{z}(t) = \mathrm{col}(z_1(t), z_2(t), \ldots, z_k(t))$ is a Markov process if

$$P\left\{ \underline{z}(t_1) \leq \underline{z}_1, \underline{z}(t_2) \leq z_2, \ldots, \underline{z}(t_n) \leq \underline{z}_n \mid \underline{z}_{(-\infty, t]} \right\}$$

$$= P\left\{ \underline{z}(t_1) \leq \underline{z}_1, \underline{z}(t_2) \leq z_2, \ldots, \underline{z}(t_n) \leq \underline{z}_n \mid \underline{z}(t) \right\} \qquad (2.1)$$

for all $\underline{z}_1, \ldots, \underline{z}_n$, all $t_1, \ldots, t_n \geq t$, all integers n. Note that a vector inequality has to be taken component by component. A Markov process can be very adequately described in terms of its transition distribution function

$$F_z(t, \underline{z}; \tau, \underline{\zeta}) = P\left\{ \underline{z}(\tau) \underline{\leq} \zeta \mid \underline{z}(t) = \underline{z} \right\} \tag{2.2}$$

usually only defined for $\tau \underline{>} t$. The transition density function f_z of the Markov process, if it exists, is defined as

$$f_z(t, \underline{z}; \tau, \underline{\zeta}) = \frac{\partial^k F_z(t, \underline{z}; \tau, \underline{\zeta})}{\partial \zeta_1 \, \partial \zeta_2 \ldots \partial \zeta_k} \tag{2.3}$$

In the following, the notions that are developed in Doob[8] (VI. 3), are used, except that they have been extended to the vector case.

To define a diffusion process, one can take two different points of view. The first possible approach is to define a diffusion process as a Markov process such that the following set of hypotheses is fulfilled:

H_1 : The transition density f_z exists and has appropriate regularity properties (differentiability and so on).

H_2 : The following limits exist

$$\lim_{\Delta t \to 0} \frac{P\left\{ \| \underline{z}(t, \Delta t) - \underline{z}(t) \| > \varepsilon \right\}}{\Delta t} = 0 \tag{2.4}$$

for any $\varepsilon > 0$;

$$\lim_{\Delta t \downarrow 0} E\left(\frac{\underline{z}(t+\Delta t) - \underline{z}(t)}{\Delta t} \mid \underline{z}(t) = \underline{z} \right) = \underline{p}(\underline{z}, t) \tag{2.5}$$

$$\lim_{\Delta t \downarrow 0} E\left(\frac{[\underline{z}(t+\Delta t) - \underline{z}(t)] \; [\underline{z}(t+\Delta t) - \underline{z}(t)]'}{\Delta t} \mid \underline{z}(t) = \underline{z} \right) = \underline{Q}(\underline{z}, t) \tag{2.6}$$

where $\| \cdot \|$ denotes the Euclidean norm of a vector, $\underline{p}(\underline{z}, t)$ is a vector-valued function of $\underline{z}$ and t, and $\underline{Q}(\underline{z}, t)$ is a symmetric positive semi-definite matrix for each $\underline{z}$ and t. For the limits (2. 5) and (2. 6) to exist it may be necessary to truncate the random variable $\underline{z}(t+\Delta t)$.

Condition (2. 4) means (by definition of continuity of a random process) that $\underline{z}(t)$ is a continuous random process. The vector $\underline{p}(\underline{z}, t)$ might be called the infinitesimal mean and the matrix $\underline{Q}(\underline{z}, t)$ the infinitesimal covariance matrix of the process.

H_3 : Both $\underline{p}(\underline{z}, t)$ and $\underline{Q}(\underline{z}, t)$ satisfy appropriate regularity conditions.

The second approach is to define a diffusion process in such a way that each sample function of the process is the solution of a stochastic differential equation of the form

$$d\underline{z}(t) = \underline{p}(\underline{z}(t), t)\, dt + \underline{\Lambda}(\underline{z}(t), t)\, d\underline{v}(t) \tag{2. 7}$$

Here $\underline{p}(\underline{z}, t)$ is the same function as before, and the matrix $\underline{\Lambda}(\underline{z}, t)$ is related to the matrix $\underline{Q}(\underline{z},t)$ as follows

$$\underline{Q}(\underline{z}, t) = \underline{\Lambda}(\underline{z}, t)\, \underline{\Lambda}(\underline{z}, t)' \tag{2. 8}$$

Both $\underline{\Lambda}$ and $\underline{Q}$ are k x k matrices. Finally, $\underline{v}(t)$ is a sample function of a k-dimensional Wiener process with independent components each of which has unit variance parameter. In electrical engineering language, the process $\frac{d\underline{v}(t)}{dt}$ is a k-dimensional Gaussian white noise process with independent components each of which has power density unity.

If again appropriate regularity conditions are imposed upon $\underline{p}(\underline{z}, t)$ and $\underline{\Lambda}(\underline{z}, t)$, it can be shown that the random process $\{\underline{z}(t)\}$ which is defined with the help of the stochastic differential equation (2. 7) has exactly the same transition density function as the process defined under the hypotheses H_1, H_2 and H_3.

Therefore the two approaches can be considered equivalent. In the following the second approach, namely that of considering each sample function of $\{\underline{z}(t)\}$ as a solution of (2. 7), will be taken. One reason for doing this is that presumably in practice many diffusion processes are generated according to a mechanism described by (2. 7); moreover, it will develop into a convenient approach.

It can be shown that $f_z(t, \underline{z}\,; \tau, \underline{\zeta})$ satisfies two partial differential equations, called the Kolmogorov equations, each of which, along with the appropriate boundary and initial conditions, determines f_z completely. The so-called forward equation (also called Fokker-Planck equation) is

$$\frac{\partial f_z(t, \underline{z}; \tau, \underline{\zeta})}{\partial \tau} = \mathcal{L}^* f_z(t, \underline{z}\,; \tau, \underline{\zeta}) \tag{2. 9}$$

where $\mathcal{L}^*$ is the partial differential operator

$$\mathcal{L}^* = -\frac{\partial}{\partial \underline{\zeta}}' \underline{\pi}(\zeta, \tau) + \frac{1}{2}\frac{\partial}{\partial \underline{\zeta}}' \underline{Q}(\zeta, \tau)\frac{\partial}{\partial \underline{\zeta}} \tag{2.10}$$

with

$$\underline{\pi}(\underline{z}, t) = \underline{p}(\underline{z}, t) - \frac{1}{2}\left(\frac{\partial}{\partial \underline{z}}' \underline{Q}(\underline{z}, t)\right)' \tag{2.11}$$

Here $\frac{\partial}{\partial \underline{z}}$ is the vector-"valued" operator

$$\frac{\partial}{\partial \underline{z}} = \mathrm{col}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_k}\right) \tag{2.12}$$

In expressions such as (2.10) and (2.11) it is manipulated as a column vector, but care has to be taken of the order, since $\frac{\partial}{\partial \underline{z}}$ operates on everything that follows it.

The so-called backward equation, which in studies of this kind seems to play the most important role, is derived in Appendix 3; it is given by

$$-\frac{\partial f_z(\underline{z}, t; \underline{\zeta}, \tau)}{\partial t} = \mathcal{L} f_z(\underline{z}, t; \underline{\zeta}, \tau) \tag{2.13}$$

where $\mathcal{L}$ is the partial differential operator

$$\mathcal{L} = \underline{\pi}(\underline{z}, t)' \frac{\partial}{\partial \underline{z}} + \frac{1}{2}\frac{\partial}{\partial \underline{z}}' \underline{Q}(\underline{z}, t)\frac{\partial}{\partial \underline{z}} \tag{2.14}$$

Note that $\mathcal{L}^*$ and $\mathcal{L}$ are adjoint partial differential operators, but that $\mathcal{L}^*$ operates on $\underline{\zeta}$ only, and $\mathcal{L}$ on $\underline{z}$ only. For both (2.9) and (2.13) the initial condition (respectively terminal condition) is

$$f_{\underline{z}}(t, \underline{z}; t, \underline{\zeta}) = \delta(\underline{z} - \underline{\zeta}) \tag{2.15}$$

and the boundary conditions follow from the requirement

$$\int f_{\underline{z}}(t, \underline{z}; t, \underline{\zeta})\, d\underline{\zeta} = 1, \quad \text{all} \quad \tau \geq t \tag{2.16}$$

Two more comments on the diffusion process are the following: If the functions $\underline{p}(\underline{z}, t)$ and $\underline{\Lambda}(\underline{z}, t)$ do not depend upon time, the statistical properties of $\{\underline{z}(t)\}$ are invariant with respect to translations in time; then the process is stationary.

If the differential equation (2. 7), is linear, i.e., it takes on the form

$$d\underline{z}(t) = \underline{P}(t)\, \underline{z}(t)\, dt + \underline{\Lambda}(t)\, d\underline{v}(t) \tag{2.17}$$

where $\underline{P}(t)$ and $\underline{\Lambda}(t)$ are time-varying matrices, $\underline{z}(t)$ is a k-dimensional non-stationary Gaussian process. In this case

$$\underline{\pi}(\underline{z}, t) = \underline{P}(t)\, \underline{z} \tag{2.18}$$

and

$$\underline{Q}(\underline{z}, t) = \underline{Q}(t) = \underline{\Lambda}(t)\, \underline{\Lambda}(t)' \tag{2.19}$$

with an apology for the loose notation,

Now that the properties of a diffusion process are established, one can proceed to outline the requirements for the class of random processes $\{r(t)\}$ that will be considered in this study.

It is supposed that each sample function of $\{r(t)\}$ can be considered an instantaneous scalar function of a k-dimensional diffusion process as described in the preceding pages; i.e., there exists a relation

$$r(t) = \rho(\underline{z}(t), t) \tag{2.20}$$

It is furthermore supposed that it is possible to recover the present value of $\underline{z}(t)$ from the past of $r(t)$; i.e., there exists a relation

$$\underline{z}(t) = \underline{\mathfrak{z}}\{\underline{z}(t_o) ; r_{(t_o, t]}\} \qquad , t \geq t_o \tag{2.21}$$

where it is assumed that either $z(0)$ is known or else the right-hand side of (2.21) becomes independent of $\underline{z}(t_0)$ as $t_o \longrightarrow -\infty$, so that $z(0)$ can be obtained from $r_{(-\infty, 0]}$. It is clear that if $\{r(t)\}$ is of the type described here, the random process $\{\underline{z}(t)\}$ qualifies as the state of the random process $\{r(t)\}$, as defined in Section II.4.

It is not so clear, how large this class of random processes is. At least all stationary Gaussian random processes with rational power densities are included.* To illustrate this, and to clarify the notion of the state of a random process in general, the following example is given.

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* Also many Gaussian non-stationary processes are included. Compare with Kalman and Bucy's prediction theory.[9]

Example : Let $\{r(t)\}$ be a stationary Gaussian random process with zero mean and power density function

$$S(f) = \left|\frac{(j2\pi f + 1)}{(j2\pi f + 2)(j2\pi f + 3)}\right|^2 \tag{2.22}$$

This random process can be thought of as having been obtained by passing Gaussian white noise $\frac{dv(t)}{dt}$ with unity power density through a filter with transfer function

$$H(s) = \frac{(s+1)}{(s+2)(s+3)} \tag{2.23}$$

A possible representation for this system in state form is

$$\dot{\underline{z}}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \underline{z}(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{dv(t)}{dt} \tag{2.24}$$

$$r(t) = (-1 \quad 2)\ \underline{z}(t) \tag{2.25}$$

It is observed that $\{\underline{z}(t)\}$ is a diffusion process since (2.24) can be put in the form

$$d\underline{z}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \underline{z}(t)\, dt + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} d\underline{v}(t) \tag{2.26}$$

where $\underline{v}(t)$ is a two-dimensional Wiener process of which the first component is $v(t)$. It now becomes clear that $\{\underline{z}(t)\}$ is the state of the random process $\{r(t)\}$. If Eqs. (2.24) and (2.25) are in general written as

$$\dot{\underline{z}}(t) = \underline{P}\ \underline{z}(t) + \underline{\Lambda}\ \dot{\underline{v}}(t) \tag{2.27}$$

$$r(t) = \underline{\rho}'\ \underline{z}(t) \tag{2.28}$$

and the value $\underline{z}(t_o)$ of $\underline{z}(t)$ at some instant $t_o \leq t_1$ is known, the value $r(t_1)$ of $r(t)$ at some instant t_1 in the future can be expressed as

$$r(t_1) = \underline{\rho}' \left(e^{\underline{P}(t_1 - t_o)} \underline{z}(t_o) + \int_{t_o}^{t_1} e^{\underline{P}(t_1 - \tau)} \underline{\Lambda}(t)\, d\underline{v}(\tau) \right) \quad (2.29)$$

Of the right-hand side of this expression the first term is completely known if $\underline{z}(t_o)$ is known. But since $\underline{v}(t)$ is a process with independent increments, the second term is completely independent of anything that happened prior to t_o, so that all relevant information is contained in the first term, which is determined by $\underline{z}(t_o)$.

The question how to recover $\underline{z}(t)$ from $r_{(-\infty, t]}$ is easily solved. The white noise $\frac{dv(t)}{dt}$ can be regenerated by passing $r(t)$ through a filter with transfer function

$$\frac{1}{H(s)} = \frac{(s+2)(s+3)}{(s+1)} \quad (2.30)$$

Now by Eq. (2.24) $\underline{z}(t)$ can be thought of as being obtained by passing $\frac{dv(t)}{dt}$ through a filter with transfer matrix

$$\begin{pmatrix} s+2 & 0 \\ 0 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{s+2} \\ \frac{1}{s+3} \end{pmatrix} \quad (2.31)$$

Then $\underline{z}(t)$ can be recovered by passing $r(t)$ through a filter with transfer matrix

$$\begin{pmatrix} \frac{1}{s+2} \\ \\ \frac{1}{s+3} \end{pmatrix} \quad \frac{(s+2)(s+3)}{(s+1)} \quad = \quad \begin{pmatrix} \frac{s+3}{s+1} \\ \\ \frac{s+2}{s+1} \end{pmatrix} \tag{2. 32}$$

III. 3 <u>The Condition for an Optimal Controller</u>

Let Σ be a linear differential system with n-dimensional state $\underline{x}(t)$ as described in Section III. 1. Let $\{r(t)\}$ be a random process with k-dimensional state $\{\underline{z}(t)\}$ as described in Section III. 2.

When one is seeking for an optimal controller of the feedback type, say,

$$u(t) \;=\; F(\underline{x}, (t), \; \underline{z}(t), t) \tag{3. 1}$$

Theorems 3 and 2 provide a tool in the function μ_T for checking whether a given controller is optimal or not. The function μ_T was defined as

$$\mu_T(\underline{x}(t), \; \underline{z}(t), t) \;=\; E\left(\int_t^T h_{u^o}(\tau, t)\, w\,[\, y(\tau) - r(\tau)]\; dt \;\middle|\; \underline{x}(t), \underline{z}(t) \right) \tag{3. 2}$$

Theorem 3 in conjunction with Theorem 2 asserts that for a linear system a controller F is optimal if and only if

$$\mu_T(\underline{x}, \underline{z}, t) \quad \begin{cases} \geq 0 & \text{wherever} \quad F(\underline{x}, \underline{z}, t) \;=\; -\;\gamma(t) \\ = \theta & \text{wherever} \quad |F(\underline{x}, \underline{z}, t)| \;<\; \gamma(t) \\ \leq 0 & \text{wherever} \quad F(\underline{x}, \underline{z}, t) \;=\; +\;\gamma(t) \end{cases} \tag{3. 3}$$

almost surely with respect to the probability measure imposed by the random process $\{r(t)\}$ and the controller F on the random variables

$\underline{x}(t) = \underline{x}$ and $\underline{z}(t) = \underline{z}$.

Before proceeding to find an equation for μ_T, it is established that $\underline{x}(t)$ and $\underline{z}(t)$ jointly constitute a $(n+k)$-dimensional diffusion process. Upon combining the stochastic differential (2. 7) for $\underline{z}(t)$ with the system equation (1. 1), substituting (3. 1), it follows

$$\begin{aligned} d\underline{z}(t) &= \underline{p}(\underline{z}(t), t) \; dt + \underline{\Lambda}(\underline{z}(t), t) d\underline{v}(t) \\ d\underline{x}(t) &= (\underline{A}(t)\, \underline{x}(t) + \underline{b}(t) F(\underline{x}(t), \underline{z}(t), t)) \; dt \end{aligned} \tag{3. 4}$$

It is immediately recognized that (3. 4) defines a diffusion process $(\underline{x}(t), \underline{z}(t))$, and it is not difficult to derive that the backward Kolmogorov equation for the corresponding transition density function $f_{x, z}$ is

$$- \frac{\partial f_{x, z}(t, \underline{x}, \underline{z}; \tau, \underline{\xi}, \underline{\zeta})}{\partial t} = \mathcal{M} f_{x, z}(t, \underline{x}, \underline{z} \, ; \, \tau, \underline{\xi}, \underline{\zeta}) \tag{3. 5}$$

with

$$\mathcal{M} = \mathcal{L} + (\underline{A}(t)\underline{x} + \underline{b}(t) \;\; F(\underline{x}, \underline{z}, t))' \frac{\partial}{\partial \underline{x}} \tag{3. 6}$$

where $\mathcal{L}$ is the backward operator for f_z, as given by (2. 14).

In Section III. 1 an expression was given (Eq. 1. 7) for $h_u(\tau, t)$. Employing this in expression (3. 2) for the function μ_T, it is seen that μ_T can be written in the form

$$\mu_T(\underline{x}, \underline{z}, \underline{t}) = \underline{b}(t)' \int_t^T \underline{\phi}(\tau, t)' \underline{c}(\tau) \; E(w[\, y(\tau) - r(\tau)] \, | \underline{x}(t) = \underline{x}, \;\; \underline{z}(t) = \underline{z}) d\tau$$

$$+ \; k(t) \; w[\, y(t) - r(t)] \tag{3. 7}$$

When now all quantities are expressed in terms of the overall state $\underline{x}(t) = \underline{x}$, $\underline{z}(t) = \underline{z}$, and the conditional expectation is written out in terms of the transition density $f_{x,z}$, it follows that

$$\mu_T(\underline{x}, \underline{z}, t) =$$

$$\underline{b}(t)' \int_t^T \underline{\phi}(\tau, t)'\underline{c}(\tau)\, d\tau \int f_{x,z}(t, \underline{x}, \underline{z}; \tau, \underline{\xi}, \underline{\zeta})\, w[\underline{c}(\tau)'\,\underline{\xi} + k(\tau)F(\underline{\xi}, \underline{\zeta}, \tau) - \rho(\underline{\zeta}, \tau)]\, d\underline{\xi}\, d\underline{\zeta}$$

$$+ \; k(t)\, w\, [\underline{c}(t)'\underline{x} + k(t)F(\underline{x}, \underline{z}, t) - \rho\, (\underline{z}, t)] \qquad (3.8)$$

If now the abbreviation

$$w(\underline{x}, \underline{z}, t) = w\, [\underline{c}(t)'\underline{x} + k(t)F(\underline{x}, \underline{z}, t) \;\; -\rho(\underline{z}, t)] \qquad (3.9)$$

is introduced, (3. 8) can more compactly be written as

$$\mu_T(\underline{x}, \underline{z}, t) =$$

$$\underline{b}(t)' \int_t^T \underline{\phi}(\tau, t)'\underline{c}(\tau)\, d\tau \int f_{x,z}(t, \underline{x}, \underline{z}; \tau, \underline{\xi}, \underline{\zeta})\, w(\underline{\xi}, \underline{\zeta}, \tau)\, d\underline{\xi}\, d\underline{\zeta}$$

$$+ \; k(t)\, w(\underline{z}, \underline{x}, t) \qquad (3.10)$$

Now define a function $\underline{g}(\underline{x}, \underline{z}, t)$ as follows:

$$\underline{g}(\underline{x}, \underline{z}, t) = -\int_t^T \underline{\phi}(\tau, t)'\, \underline{c}(\tau)\, d\tau \int f_{x,z}(t, \underline{x}, \underline{z}; \tau, \underline{\xi}, \underline{\zeta})\, w(\underline{\xi}, \underline{\zeta}, \tau)\, d\underline{\xi}\, d\underline{\zeta} \qquad (3.11)$$

A partial differential equation for the function $\underline{g}(\underline{x}, \underline{z}, t)$ can be obtained by applying the operator $(\frac{\partial}{\partial t} + \mathcal{M})$ to it and using the facts

$$\left(\frac{\partial}{\partial t} + \mathcal{M}\right) f_{z,x}(t, x, z; \tau, \underline{\xi}, \underline{\zeta}) = 0 \tag{3.12}$$

$$f_{x,z}(t, \underline{x}, \underline{z}; t, \underline{\xi}, \underline{\zeta}) = \delta(\underline{x} - \underline{\xi})\, \delta(\underline{z} - \underline{\zeta}) \tag{3.13}$$

$$\frac{d}{dt} \underline{\phi}(\tau, t)' = -\underline{A}(t)' \underline{\phi}(\tau, t)' \tag{3.14}$$

It follows that

$$\left(\frac{\partial}{\partial t} + \mathcal{M}\right) \underline{g}(\underline{x}, \underline{z}, t) = - \underline{A}(t)' \underline{g}(\underline{x}, \underline{z}, t) + \underline{c}(t)\, w(\underline{x}, \underline{z}, t) \tag{3.15}$$

The appropriate terminal condition is

$$\underline{g}(\underline{x}, \underline{z}, T) = \underline{0} \tag{3.16}$$

and $\mu_T(\underline{x}, \underline{z}, t)$ can be obtained from $\underline{g}(\underline{x}, \underline{z}, t)$ by the relation

$$\mu_T(\underline{x}, \underline{z}, t) = - \underline{b}(t)' \underline{g}(\underline{x}, \underline{z}, t) + k(t)\, w(\underline{x}, \underline{z}, t) \tag{3.17}$$

The boundary conditions in $\underline{z}$ and $\underline{x}$ for $\underline{g}(\underline{x}, \underline{z}, t)$ follow from its definition (3.11) and are that as $\underline{x}$ and $\underline{z}$ go to $\pm \infty$ the function $\underline{g}(\underline{x}, \underline{z}, t)$ should not grow too fast.

The results of this section can be summarized as follows.

Theorem 6 : Let Σ be a linear, time-varying differential system described by Eq. (1.1). Let furthermore the random process $\{r(t)\}$ be of the type described in Section III. 2., such that its state $\{\underline{z}(t)\}$ is a k-dimensional diffusion process. Let F be an admissible feedback controller such that conditions R_1, R_2, and R_3 are satisfied. Then a necessary and sufficient condition for F to be T-optimal with respect to all admissible controllers satisfying R_1, R_2, and R_3 is that the function $\mu_T(\underline{x}, \underline{z}, t)$, which can be solved from Eqs. (3.15), (3.16), and (3.17), satisfies the following conditions

$$\mu_T(\underline{x}, \underline{z}, t) \quad \begin{cases} \geq 0 & \text{wherever} \quad F(\underline{x}, \underline{z}, t) = -\gamma(t) \\ = 0 & \text{wherever} \quad |F(\underline{x}, \underline{z}, t)| < \gamma(t) \\ \leq 0 & \text{wherever} \quad F(\underline{x}, \underline{z}, t) = +\gamma(t) \end{cases} \tag{3.17a}$$

almost surely with respect to the probability measure imposed by the random process $\{r(t)\}$ and the controller F upon the random variables $\underline{x}(t) = \underline{x}$ and $\underline{z}(t) = \underline{z}$ (for almost all $t \in [0, T]$).

For a given controller, Eq. (3.15) could be solved for $\underline{g}$; then from this μ_T can be obtained by formula (3.17). For later purposes, it will be useful to have a partial differential equation in μ_T itself. This equation can be derived very easily by comparing the expressions (3.15) and (3.17) with the state equations (1.12) which characterize the adjoint system Σ^*. It is seen that $\underline{g}$ takes the same position as the state $\underline{x}^*$, w takes the place of the input u^*, μ_T that of the output y^*, and that the operator $(\frac{\partial}{\partial t} + \mathcal{M})$ substitutes the operator $\frac{d}{dt}$. From this it can readily be seen that by eliminating $\underline{g}$ in exactly the same way from (3.15) and (3.17) as by which $\underline{x}^*$ can be eliminated from Eqs. (1.12), the function μ_T must satisfy a partial differential equation of the form

$$D^*\left(\frac{\partial}{\partial t} + \mathcal{M}\right) \mu_T(\underline{x}, \underline{z}, t) = N^* \left(\frac{\partial}{\partial t} + \mathcal{M}\right) w(\underline{x}, \underline{z}, t) \qquad (3.18)$$

where D^* and N^* are the polynomials with time-varying coefficients which occur in the scalar differential equation which characterizes the adjoint system. The appropriate terminal conditions that go with (3.18) can be formulated in terms of

$$\left(\frac{\partial}{\partial t} + \mathcal{M}\right)^i \mu_T(\underline{x}, \underline{z}, t)\Big|_{t=T}, \quad i = 0, 1, \ldots, (n-1) \qquad (3.19)$$

Equation (3.18) is not very useful as it stands because it will hardly ever be possible to solve it for μ_T. It will, however, be possible to use it to derive a property of the optimal controller for a certain subclass of systems, which will be done in the next section.

Observation: A possible approach for solving the optimal control problem seems the following. As was mentioned before, μ_T can be found by solving (3.15) for $\underline{g}$. This is to be done backwards since a terminal condition is given. By this same terminal condition, $\mu_T(\underline{x}, \underline{z}, T)$ is known. A choice for $F(\underline{z}, \underline{x}, T)$ could be made such that the optimality condition (3.17a) is satisfied. If it then is assumed that this controller is also optimal during a very short interval preceding T, it could be used to compute $\mu_T(\underline{x}, \underline{z}, T - \Delta T)$, with ΔT small, and on the basis of this $F(\underline{z}, \underline{x}, T-\Delta T)$ could be determined. Continuing in this fashion an iterative procedure could be developed to find the optimal controller.

III.4 Some Observations on the Character of the Optimal Controller

Although in general it seems very difficult to actually find optimal controllers, it is possible to make some remarks about their character in general. First the following lemma is proved.

Lemma 4 : Suppose that Σ is a linear differential system as described in Section III.1, which is both instantaneously controllable and instantaneously observable during some interval (t_o, t_1). Let n be the degree of the polynomial with time-varying coefficients D and m the degree of the polynomial N in the scalar differential equation for the system

$$D\left(\frac{d}{dt}\right) y(t) = N\left(\frac{d}{dt}\right) u(t) \tag{4.1}$$

Let $\{r(t)\}$ be a random process with the k-dimensional diffusion process $\{\underline{z}(t)\}$ as its state, as described in Section III. 2. Then if the weighting function W is quadratic and

$$m < \frac{n}{2} \quad , \tag{4. 2}$$

nowhere in $R^n \times R^k \times (t_o, t_1)$ can the function $\mu_T(\underline{x}, \underline{z}, t)$ vanish identically over some non-zero region.

Proof : The proof of the lemma is almost purely algebraic. Suppose that in some finite region S of $R^n \times R^k \times (t_o, t_1)$

$$\mu_T(\underline{x}, \underline{z}, t) \equiv 0 \tag{4.3}$$

Then since it was derived in the preceding section that the function μ_T has to satisfy Eq. (3.18), it follows that it must hold in S

$$N^*\left(\frac{\partial}{\partial t} + \mathcal{M}\right)w(\underline{x}, \underline{z}, t) = 0 \tag{4.4}$$

It will be shown that this cannot be satisfied for a n y controller if $m < \frac{n}{2}$, and therefore not for the optimal controller, either. At this point the proof will only be given for a quadratic weighting function, i. e.

$$W(e) = \frac{1}{2} e^2 \quad , \quad w(e) = e \tag{4.5}$$

The result may be extended to more general weighting functions, however.

To give the proof, it is first of all noted that if the degree of N is m, the degree of N^* is also m, by Lemma 3. This means that in $N^*\left(\frac{\partial}{\partial t} + \mathcal{M}\right)$ the highest degree in which the operator $\left(\frac{\partial}{\partial t} + \mathcal{M}\right)$ appears is m.

Secondly, the function $w(\underline{x}, \underline{z}, t)$ and the operator $\left(\frac{\partial}{\partial t} + \mathcal{M}\right)$ can be more explicitly written

$$w(\underline{x}, \underline{z}, t) = \underline{c}(t)'\underline{x} + k(t) \; F(\underline{x}, \underline{z}, t) - \rho(\underline{z}, t) \tag{4.6}$$

$$\frac{\partial}{\partial t} + \mathcal{M} = \frac{\partial}{\partial t} + \mathcal{L} + (\underline{A}(t)\underline{x} + \underline{b}(t)F(\underline{x}, \underline{z}, t))' \frac{\partial}{\partial \underline{x}} \tag{4.7}$$

where $\mathcal{L}$ is the operator as given by (2.14); it operates only on $\underline{z}$, and does not involve $F(\underline{x}, \underline{z}, t)$.

By applying $(\frac{\partial}{\partial t} + \mathcal{M})$ repeatedly to $w(\underline{x}, \underline{z}, t)$, it follows that

$$(\frac{\partial}{\partial t} + \mathcal{M})w(\underline{x}, \underline{z}, t) = \underline{c}_{(1)}(t)'\underline{x} + \underline{c}_{(0)}(t)'\underline{b}(t)\ F(\underline{z}, \underline{x}, t)$$

$$+ (\frac{\partial}{\partial t} + \mathcal{M})k(t)F(\underline{x}, \underline{z}, t) - (\frac{\partial}{\partial t} + \mathcal{L})\rho(\underline{z}, t) \tag{4.8}$$

More compactly, this can be written as

$$(\frac{\partial}{\partial t} + \mathcal{M})w = \underline{c}_{(1)}'\underline{x} + \underline{c}_{(0)}'\underline{b}\ F + (\frac{\partial}{\partial t} + \mathcal{M})kF - (\frac{\partial}{\partial t} + \mathcal{L})\rho \tag{4.9}$$

Continuing,

$$(\frac{\partial}{\partial t} + \mathcal{M})^2\ w = \underline{c}_{(2)}'\underline{x} + \underline{c}_{(1)}'\underline{b}\ F +$$

$$+ (\frac{\partial}{\partial t} + \mathcal{M})\ \underline{c}_{(0)}'\underline{b}\ F + (\frac{\partial}{\partial t} + \mathcal{M})^2\ kF - (\frac{\partial}{\partial t} + \mathcal{L})^2\rho \tag{4.10}$$

and finally,

$$(\frac{\partial}{\partial t} + \mathcal{M})^m\ w = \underline{c}_{(m)}'\underline{x} + \underline{c}_{(m-1)}'\underline{b}\ F +$$

$$+ (\frac{\partial}{\partial t} + \mathcal{M})\ \underline{c}_{(m-2)}'\underline{b}\ F + \ldots + (\frac{\partial}{\partial t} + \mathcal{M})^{m-1}\underline{c}_{(0)}'\underline{b}\ F$$

$$+ (\frac{\partial}{\partial t} + \mathcal{M})^m kF - (\frac{\partial}{\partial t} + \mathcal{L})^m\rho \tag{4.11}$$

It follows by inspection of these expressions that if N^* is of degree m, the function F will only occur in the equation

$$N^* \left(\frac{\partial}{\partial t} + \mathcal{M}\right) w = 0 \tag{4.12}$$

tagged with the coefficients

$$k(t), \ \underline{c}_{(i)}(t)' \underline{b}(t), \ i = 0, 1, \ldots, (m-1) \tag{4.13}$$

But the fact that the degree of N^* and N is m, means

$$a_{n-j}(t) = 0, \quad j = 0, 1, \ldots, (n-m-1) \tag{4.14}$$

which by Lemma 3 implies

$$k(t) = 0, \ \underline{c}_{(i)}(t)'\underline{b}(t) = 0, \quad i = 0, 1, \ldots, (n-m-2) \tag{4.15}$$

This, together with the previous statement, implies that the function F will not appear in Eq. (4.12) if

$$n-m-2 \geq m-1 \tag{4.16}$$

or, since m and n are integers, if

$$m < \frac{n}{2} \tag{4.17}$$

In fact, in this case Eq. (4.12) takes the form

$$\sum_{j=0}^{m} a_j^*(t)\underline{c}_{(j)}(t)' \underline{x} - \sum_{j=0}^{m} a_j^*(t) \left(\frac{\partial}{\partial t} + \mathcal{L}\right)^j \rho(\underline{z}, t) = 0 \tag{4.18}$$

If this were to be satisfied identically in S, one would need

$$\sum_{j=0}^{m} a_j^*(t) \ \underline{c}_{(j)}(t) = \underline{0} \tag{4.19}$$

during (t_o, t_1), which contradicts the assumed instantaneous observability of the system Σ. Thus it has been shown that there cannot be any such region S, which completes the proof that nowhere in $R^n \times R^k \times (t_o, t_1)$ the function $\mu_T(\underline{x}, \underline{z}, t)$ can vanish identically, when $m < \frac{n}{2}$.

The following comments are in order. The lemma suggests that if $m < \frac{n}{2}$ during (t_o, t_1) the optimal controller is of the bang-bang type, i.e., the input to the system during (t_o, t_1) is always either $+\gamma(t)$ or $-\gamma(t)$, since the function $\mu_T(\underline{x}, \underline{z}, t)$ is non-zero almost everywhere in the space $R^n \times R^k \times (t_o, t_1)$. This is not necessarily true, however, as will be explained in the following.

First of all, the following terminology is introduced. Call the subspace of $R^n \times R^k$ defined by

$$\mu_T(\underline{x}, \underline{z}, t) = 0 \quad , \quad t_o < t < t_1 \tag{4.20}$$

the <u>switching surface at time t</u> of the optimal controller. It follows from Theorem 3 that outside the switching surface the optimal control is either + or $-\gamma(t)$, but that on the switching surface the input may assume intermediate values. There are two possibilities:

i) The probability that the phase point $(\underline{x}, (t), \underline{z}(t))$ remains on the switching surface for any length of time after reaching it, is zero. In this case it will not affect the operation of the controller if the value of

$F(\underline{x}, \underline{z}, t)$ is changed on the switching surface to either $+\gamma(t)$ or $-\gamma(t)$. This means that bang-bang control is indeed optimal.

ii) The probability that the phase point $(\underline{x}(t), \underline{z}(t))$ stays on the switching surface for some time after it reaches it is greater than zero. In this case the optimal control is not necessarily bang-bang during the entire period (t_o, t_1). Just as in the deterministic control problem, this case will be referred to as the <u>singular case.</u>

To demonstrate that the singular case is not imaginary, the following example is given:

<u>Example of a singular case</u>: Let the system Σ be time-invariant with transfer function

$$H(s) = \frac{1}{s+2} \tag{4.21}$$

The state of this system is one-dimensional; it can be taken to be the output, $y(t)$. Lemma 4 applies to this case, since $m = 0$ and $n = 1$.

Furthermore, let $\{r(t)\}$ be a random process which is obtained by passing a bounded simple Markov process $q(t)$ through a linear time-invariant filter with transfer function $\frac{1}{s+1}$ (see Fig. 2)

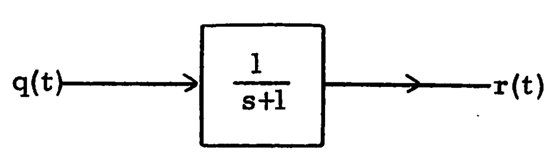

Fig. 2 Generation of the Process $\{r(t)\}$

The value of $q(t)$, which is the state of the process $\{q(t)\}$, can be recovered from $r(t)$ by the relation

$$q(t) = \dot{r}(t) + r(t) \tag{4.22}$$

Since the state of the linear filter $\frac{1}{s+1}$ can be taken to be the output $r(t)$, clearly the state of the random process $\{r(t)\}$ can in this case be taken to be the pair $(r(t), q(t))$. It is seen from relation (4.22) that one can, as well, choose the state to be

$$\underline{z}(t) = \mathrm{col}(r(t), \dot{r}(t)) \tag{4.23}$$

It was specified that the simple Markov process $\{q(t)\}$ was bounded, say, by B (one could for example take a Wiener process with elastic barriers at $-B$ and $+B$). If $\{q(t)\}$ is bounded by B, also $\{r(t)\}$ is bounded by B; hence by (4.22), $\{\dot{r}(t)\}$ is bounded by $2B$.

Thus a random process $\{r(t)\}$ has been obtained with state $\underline{x}(t) = (r(t), \dot{r}(t))$, such that $\{r(t)\}$ and $\{\dot{r}(t)\}$ are bounded by B and $2B$, respectively. Now consider a controller for the system that is considered given by

$$u(t) = r(t) + 2\,\dot{r}(t) \tag{4.24}$$

It is seen that this controller is in fact the inverse of the system, since it represents a linear filter between $r(t)$ and $u(t)$ with transfer function $(s+2)$. It is clear that if at some instant t_o the error

$$y(t_o) - r(t_o) = 0 \tag{4.25}$$

and if from t_o on the controller operates according to (4.24), the error will remain zero, no matter what is the behavior of $r(t)$ after t_o. Furthermore, because of the boundedness of $r(t)$ and $\dot{r}(t)$, the input $u(t)$ according to (4.24) will be bounded by $5B$.

Now suppose that

$$\gamma(t) \geq 5B, \text{ for all } t \tag{4.26}$$

Then clearly the best control within the amplitude constraints on $u(t)$ is to let

$$F(\underline{x}(t), \underline{z}(t), t) = F(y(t), r(t), \dot{r}(t), t) = r(t) + 2\dot{r}(t) \tag{4.27}$$

as soon as zero error is reached, and to let the controller operate like this for the rest of the period $[0, T]$. The error will remain zero; hence the phase point $(y(t), r(t), \dot{r}(t))$ remains on the plane $y - r = 0$. Then this plane must be at least part of the switching surface since by Lemma 4 the control cannot be non-bang-bang on more than a subspace of $R^1 \times R^2$.

If the case $T \rightarrow \infty$ is considered, clearly the behavior of the controller until zero error is reached is of little importance as long as the probability that zero error is reached at some finite time is one. A good controller seems to be

$$F(y, r, \dot{r}, t) = \begin{cases} +\gamma(t) & \text{if} \quad r - y > 0 \\ r + 2\dot{r} & \text{if} \quad r - y = 0 \\ -\gamma(t) & \text{if} \quad r - y < 0 \end{cases} \tag{4.28}$$

Observation : The remarks which preceded the example were only valid for the case where $m < \frac{n}{2}$. There seems little reason, however, that they should not hold for other values of m, too. This is expressed as follows:

Conjecture : Lemma 4 and the subsequent discussion are valid for $m \leq n-1$.

No proof of this conjecture has been found to date. More substantiation is offered in the next section. That the conjecture need not be true for m = n is intuitively clear, since in this case $k(t) \neq 0$, which means that there is direct transmission between input and output of the system.

IV. OPTIMAL CONTROL OF LINEAR DIFFERENTIAL SYSTEMS WITH A QUADRATIC WEIGHTING FUNCTION AND NO AMPLITUDE CONSTRAINTS ON THE INPUT

When in the problem of the preceding section the amplitude constraint on the input is dropped, a solution for the optimal following problem can easily be obtained. One question that now enters into the picture, however, is that of the stability of the interconnection of system and controller. In Section IV.1, the optimization criterion is extended in order to include provisions for stability. The optimization problem is solved explicitly for finite T . in Section IV.2 the behavior and the properties of the solution as T approaches ∞ are studied for an illustrative example in Section IV. .

IV.1 Optimization Without Amplitude Constraints. Stability

Consider the problem outlined in the previous section, namely that of finding the optimal controller for a linear differential system Σ with n-dimensional state $\underline{x}(t)$, but now without the constraint on the amplitude of the input $u(t)$.

In this case an optimal feedback controller can easily be found. Consider the relation which expresses the output of the system in terms of the state and the input:

$$y(t) = \underline{c}(t)'\underline{x}(t) + k(t)\, u(t) \tag{1.1}$$

The error at any instant t is given by

$$e(t) = y(t) - r(t) = \underline{c}(t)'\underline{x}(t) + k(t)\, u(t) - r(t) \tag{1.2}$$

It is clear that if $k(t) \neq 0$ during some interval of time (t_o, t_1), the error can always be made identically zero during this interval by letting the input take on the following value:

$$u(t) = \frac{r(t) - \underline{c}(t)'\underline{x}(t)}{k(t)} \tag{1.3}$$

This breaks down, however, if $k(t) = 0$. In this case, the error can be made arbritrarily small at all times by considering the feedback controller

$$u(t) = K(\, r(t) - y(t)\,) \tag{1.4}$$

and choosing K large enough.

By considering some very elementary examples it is evident that the interconnection of controller and system as recommended above may be unstable, in the sense that as time progresses some or all of the state variables become unbounded. This is not surprising, since in the preceding development no provisions to prevent it were made.

It is not clear how one can introduce such provisions in a natural, non-artificial way, such as is done in the Wiener optimal filtering and control theory, where "realizability" is related to "stability" in an essential manner. In this investigation, the same artifact will be used as is usually employed in situations of this kind, namely, that of including in the optimization criterion a term which depends upon the magnitude of the state at the final instant T. It will be seen in IV. 3, however, that in a sense this approach is non-essential.

To get around the problem of instability, the problem of Section III is modified by not minimizing $\mathcal{E}(F, T)$ but instead the quantity

$$\mathcal{V}(F, T) = \mathcal{E}(F, T) + \frac{1}{2} \lambda E (||\underline{x}(T)||^2) \tag{1.5}$$

where as before $||\ .\ ||$ indicates the Euclidean norm of a vector, and

$$\mathcal{E}(F, T) = E(\frac{1}{T} \int_0^T W [y(t) - r(t)]\ dt) \tag{1.6}$$

The constant $\lambda > 0$ is a coefficient which determines the relative importance of the last term of $\mathcal{V}(F, T)$.

Consider any feedforward controller F which can be represented as

$$F \{r_{(-\infty, t]}\} = F^o \{r_{(-\infty, t]}\} + \varepsilon \bar{F} \{r_{(-\infty, t]}\} \tag{1.7}$$

where F^o is a supposedly optimal controller. Since no amplitude constraints are imposed, no particular conditions need be imposed upon $\bar{F}$ except that it be finite everywhere. For a differentiable system, it was derived in Section II. 3, that it holds for $\mathcal{E}(F, T)$

$$\mathcal{E}(F, T) = \mathcal{E}(F^o, T) + \varepsilon \frac{1}{T} \int_0^T E(\mu_T \{r_{(-\infty, t]}\} \bar{F}\{r_{(-\infty, t]}\}) \, dt$$

$$+ o(\varepsilon) \tag{1.8}$$

For the second term in (1. 5) a similar result can easily be obtained. Let $u^o_{(0, t]}$ and $\bar{u}_{(0, t]}$ be the inputs that are obtained from the controllers F^o and $\bar{F}$, respectively, and let $u_{(0, t]}$ be the input obtained from F. Then

$$u(t) = u^o(t) + \varepsilon \bar{u}(t) \tag{1.9}$$

It follows for $\underline{x}(T)$

$$\underline{x}(T) = \underline{\phi}(T, 0) \, \underline{x}(0) + \int_0^T \underline{\phi}(T, \tau) \, \underline{b}(\tau) \, u(\tau) \, d\tau \tag{1.10}$$

or

$$\underline{x}(T) = \underline{\phi}(T, 0) \, \underline{x}(0) + \int_0^T \underline{\phi}(T, \tau) \, \underline{b}(\tau) \, u^o(\tau) \, d\tau$$

$$+ \varepsilon \int_0^T \underline{\phi}(T, \tau) \, \underline{b}(\tau) \, \bar{u}(\tau) \, d\tau$$

$$= \underline{x}^o(T) + \varepsilon \int_0^T \underline{\phi}(T, \tau) \, \underline{b}(\tau) \, \bar{u}(\tau) \, d\tau \tag{1.11}$$

where $\underline{x}^o(T)$ indicates the state into the system has been brought by the optimal controller F^o as a result of a sample function $r_{(-\infty, t]}$. Now from this it can very easily be derived

$$E(\|\underline{x}(T)\|^2) = E(\|\underline{x}^o(T)\|^2) + 2\varepsilon E\left(\int_0^T \underline{b}(\tau)'\underline{\phi}(T,\tau)'\underline{x}^o(T)\,\overline{u}(\tau)\,d\tau\right) + o(\varepsilon) \quad (1.12)$$

Introduce the notation

$$\nu_T\{r_{(-\infty,t]}\} = E(\underline{b}(t)'\underline{\phi}(T,t)'\underline{x}^o(T) \mid r_{(-\infty,t]}) \quad (1.13)$$

and replace $\overline{u}(t) = \overline{F}\{r_{(-\infty,t]}\}$ in (1.12). It follows that

$$E(\|\underline{x}(T)\|^2) = E(\|\underline{x}^o(T)\|^2) + 2\varepsilon \int_0^T E(\nu_T\{r_{(-\infty,t]}\}\ \overline{F}\{r_{(-\infty,t]}\})dt + o(\varepsilon) \quad (1.14)$$

Using this in (1.5) together with (1.8), it is found that $\mathcal{V}(F,T)$ can be represented as

$$\begin{aligned}\mathcal{V}(F,T) &= \mathcal{E}(F^o,T) + \frac{1}{2}\lambda E(\|\underline{x}^o(T)\|^2)\\ &+ \varepsilon\frac{1}{T}\int_0^T E(\overline{F}\{r_{(-\infty,t]}\}\ \mu_T\{r_{(-\infty,t]}\})\,dt\\ &+ \varepsilon\lambda\int_0^T E(\overline{F}\{r_{(-\infty,t]}\}\ \nu_T\{r_{(-\infty,t]}\})\,dt\\ &+ o(\varepsilon) \end{aligned} \quad (1.15)$$

Now from this expression by the same arguments as used in the proofs of Theorems 1 and 2, it can be reasoned that a necessary and sufficient condition for F^o to be optimal is that

$$\tilde{\mu}_T \left\{ r_{(-\infty, t]} \right\} \triangleq \mu_T \left\{ r_{(-\infty, t]} \right\} + \lambda T \ \nu_T \left\{ r_{(-\infty, t]} \right\} \tag{1.16}$$

$$= 0 \tag{1.17}$$

almost surely with respect to the probability measure induced by the random process $\left\{ r(t) \right\}$. The condition takes this simple form since no amplitude constraints were imposed upon F^o.

As before, if a feedback controller $F(\underline{x}(t), \underline{z}(t), t)$ is used, $\mu_T \left\{ r_{(-\infty, t]} \right\}$ can be expressed as an instantaneous function of $\underline{x}(t)$ and $\underline{z}(t)$. It is not difficult to convince oneself that the same is true for $\nu_T \left\{ r_{(-\infty, t]} \right\}$. If one therefore redefines ν_T as follows,

$$\nu_T(\underline{x}(t), \underline{z}(t), t) = E \ (\underline{b}(t)' \underline{\phi}(T, t)' \underline{x}^o(T) \mid \underline{x}(t), \ \underline{z}(t) \ ) \tag{1.18}$$

condition (1.17) reads

$$\tilde{\mu}_T(\underline{x}(t), \underline{z}(t), t) = \mu_T(\underline{x}(t), \underline{z}(t), t) + \lambda \ T \ \nu_T(\underline{x} \ (t), \underline{z}(t), t) \tag{1.19}$$

$$= 0 \tag{1.20}$$

almost surely with respect to the probability measure imposed upon $\underline{x}(t)$ and $\underline{z}(t)$ by the random process $\left\{ r(t) \right\}$ and the controller F^o.

The function $\nu_T(\underline{x}, \underline{z}, t)$ can be written as

$$\nu_T(\underline{x}, \underline{z}, t) = \underline{b}(t)' \ \underline{\phi} \ (T, t)' \int f_{x, z} \ (t, \underline{x}, \underline{z}; T, \underline{\xi}, \underline{\zeta}) \ \underline{\xi} \ d\underline{\xi} \ d\underline{\zeta} \tag{1.21}$$

where $f_{x, z}$ is the transition density function of the diffusion process $(\underline{x}(t), \underline{z}(t) \ )$ induced by the controller F^o. Define a function $\underline{h}(\underline{x}, \underline{z}, t)$ as follows:

$$\underline{h}(\underline{x}, \underline{z}, t) = - \underline{\phi}\,(T, t)' \int f_{x, z}\,(t, \underline{x}, \underline{z}; T, \underline{\xi}, \underline{\zeta})\ \underline{\xi}\ d\underline{\xi}\ d\underline{\zeta} \tag{1.22}$$

It is related to $\nu_T(\underline{x}, \underline{z}, t)$ by

$$\nu_T(\underline{x}, \underline{z}, t) = - \underline{b}(t)'\ \underline{h}(\underline{x}, \underline{z}, t) \tag{1.23}$$

By applying the operator $(\frac{\partial}{\partial t} + \mathcal{M})$ to $\underline{h}$, it follows (with the use of the relations III. 3.12, III. 3.13 and III. 3.14) that

$$(\frac{\partial}{\partial t} + \mathcal{M})\ \underline{h}(\underline{x}, \underline{z}, t) = - \underline{A}(t)'\underline{h}(\underline{x}, \underline{z}, t) \tag{1.24}$$

The terminal condition is

$$\underline{h}(\underline{x}, \underline{z}, T) = - \underline{x} \tag{1.25}$$

It was found in Section III. 3 that the function μ_T can be expressed in terms of a function $\underline{g}$ as follows:

$$\mu_T(\underline{x}, \underline{z}, t) = -\underline{b}(t)'\underline{g}(\underline{x}, \underline{z}, t) + k(t)\ w(\underline{x}, \underline{z}, t) \tag{1.26}$$

where $\underline{g}$ is the solution of

$$\begin{cases} (\frac{\partial}{\partial t} + \mathcal{M})\ \underline{g}(\underline{x}, \underline{z}, t) = - \underline{A}(t)'\underline{g}(\underline{x}, \underline{z}, t) + \underline{c}(t)\ w(\underline{x}, \underline{z}, t) & (1.27) \\ \underline{g}(\underline{x}, \underline{z}, T) = \underline{0} & (1.28) \end{cases}$$

Now define a function $\underline{\tilde{g}}$ by

$$\underline{\tilde{g}}(\underline{x}, \underline{z}, t) = \underline{g}(\underline{x}, \underline{z}, t) + \lambda T\underline{h}(\underline{x}, \underline{z}, t) \tag{1.29}$$

By adding Eqs. (1. 27) and λT(1. 24), and (1. 28) and λT(1. 25), it follows that $\underline{\tilde{g}}$ satisfies

$$\left(\frac{\partial}{\partial t} + \mathcal{M}\right) \tilde{\underline{g}}(\underline{x}, \underline{z}, t) = - \underline{A}(t)'\tilde{\underline{g}}(\underline{x}, \underline{z}, t) + \underline{c}(t) \; w(\underline{x}, \underline{z}, t) \tag{1.30}$$

$$\tilde{\underline{g}}(x, z, T) = - \lambda \, T \, \underline{x} \tag{1.31}$$

Furthermore, in terms of $\tilde{\underline{g}}$ the condition (1.20) for the optimal controller takes the form

$$\tilde{\mu}_T(\underline{x}, \underline{z}, t) = -\underline{b}(t)'\tilde{\underline{g}}(\underline{x}, \underline{z}, t) + k(t) \; w(\underline{x}, \underline{z}, t) \tag{1.32}$$

$$= 0 \tag{1.33}$$

where (1.32) was obtained by adding (1.26) and λ T(1.23).

The results of this section can be summarized as follows:

Theorem 7 : Let Σ be a linear, time-varying differential system described by Eq. (III.1.1). Furthermore let the random process $\{r(t)\}$ be of the type described in Section III. 2, such that the state $\{\underline{z}(t)\}$ is a k-dimensional diffusion process. Let F^o be a feedback controller such that conditions R_1, R_2, and R_3 are satisfied. Then a necessary and sufficient condition for F^o to minimize the quantity

$$\mathcal{V}(F, T) = \frac{1}{T} \int_o^T E\left(W[y(t) - r(t)]\right) dt + \frac{1}{2} \lambda \, E(|| \, \underline{x}(T) \, ||^2) \tag{1.34}$$

is that

$$\tilde{\mu}_T(\underline{x}, \underline{z}, t) = 0 \tag{1.35}$$

almost surely with respect to the probability measure imposed by the random process $\{r(t)\}$ and the controller F^o upon the random variables

$\underline{x}(t) = \underline{x}$ and $\underline{z}(t) = \underline{z}$, and for almost all t.

Upon comparing this theorem with Theorem 6 it is interesting to note that the only effect of introducing the constraint on the terminal state is that of modifying the terminal condition for the function $\tilde{g}(\underline{x}, \underline{z}, t)$, which is the solution of the same equation as the function $g(\underline{x}, \underline{z}, t)$.

In the next section the problem will be completely solved for the case of a quadratic weighting function.

IV. 2 Solution of the Optimization Problem for a Quadratic Weighting Function

In this section the following specialization of the general problem of this report is considered.

. The system Σ is an n-dimensional (time-varying) differential system as described in Section III. 1.

. The random process $\{r(t)\}$ has the k-dimensional diffusion process $\{\underline{z}(t)\}$ as its state.

. The weighting function W is quadratic, i. e., $W(e) = \frac{1}{2} e^2$, hence $w(e) = e$.

. There are no amplitude constraints on the input u(t).

. The interconnection of controller and system is to be stable in the sense that $\|\underline{x}(t)\|$ remains finite as $t \rightarrow \infty$ with probability one.

. The optimization criterion is that of minimizing the quantity

$$\mathcal{E}(F, T) = \frac{1}{T} E\left(\int_0^T W[y(t) - r(t)] \, dt \right) \tag{2.1}$$

with particular interest in the case where $T = \infty$.

Since no method seems available to solve the problem which is outlined above directly, instead the problem is considered which is obtained by replacing the last two entries of the list by the following single entry:

. The optimization criterion is that of minimizing the quantity

$$\mathcal{V}(F, T) = \mathcal{E}(F, T) + \frac{1}{2} \lambda \, E\left(|| \underline{x}(T) ||^2 \right) \tag{2.2}$$

It is noted that actually the original problem (specified by the list on page 76) makes little sense if T is finite, since then the behavior of the system after time T is of little interest and hence also the stability is of little concern. It is still to be seen, however, when the modified problem makes sense for $T \longrightarrow \infty$.

Consider now the modified problem. It is claimed that the optimal controller takes the form

$$F^o(\underline{x}, \underline{z}, t) = \underline{\varphi}_1(t)'\underline{x} + \varphi_2(\underline{z}, t) \tag{2.3}$$

where $\underline{\varphi}_1(t)$ is a column vector and $\varphi_2(\underline{z}, t)$ a scalar function.

This claim will now be proved. Suppose that the controller given by (2.3) is used. Then the interconnection of controller and system is described by the set of equations

$$\underline{\dot{x}}(t) = (\underline{A}(t) + \underline{b}(t)\ \underline{\varphi}_1(t)')\ \underline{x}(t) + \underline{b}(t)\ \varphi_2(\underline{z}(t), t) \tag{2.4}$$

$$y(t) = (\underline{c}(t) + k(t)\,\underline{\varphi}_1(t)\,)'\underline{x}(t) + k(t)\ \varphi_2(\underline{z}(t), t) \tag{2.5}$$

The solution of equation (2.4) at some time $\tau \geq t$ can be put in the form

$$\underline{x}(\tau) = \underline{\chi}(\tau, t)\ \underline{x}(t) + \int_t^{\tau} \underline{\chi}(\tau, \sigma)\ \underline{b}(\sigma)\ \varphi_2(\underline{z}(\sigma), \sigma)\ \ d\sigma \tag{2.6}$$

where $\underline{\chi}(\tau, t)$ is the transition matrix connected with the matrix $(\underline{A}(t) + \underline{b}(t)\ \underline{\varphi}_1(t)')$. It can be seen from this expression that the conditional expectation of $\underline{x}(\tau)$ given $\underline{x}(t) = \underline{x}$ must be linear in $\underline{x}$, since in (2.6) only the first term on the right-hand side depends on $\underline{x}(t) = \underline{x}$.

Consider now the function $\underline{\tilde{g}}(x, z, t)$, which by Theorem 7 is of importance in solving the optimization problem. By its definition (IV.1.29) and the definitions of $\underline{g}$ and $\underline{h}$, one can write

$$\underline{\tilde{g}}(\underline{x}, \underline{z}, t) = - \int_t^T \underline{\phi}(\tau, t)'c(\tau)\ \ E(w[\,y(\tau)-r(\tau)]\,|\underline{x}(t)=\underline{x}, \underline{z}(t)=\underline{z})\ d\tau)$$

$$- \lambda\ T\ \underline{\phi}(T, t)'\ E(\underline{x}(T)\ |\ \underline{x}(t)=\underline{x},\ \ \underline{z}(t)=\underline{z}) \tag{2.7}$$

By assumption, $w(e) = e$ and is therefore linear. By (2.5), $y(\tau)$ is linear in $\underline{x}(\tau)$. Then it follows from the observation following Eq. (2.6) that both conditional expectations which occur in the expression for $\underline{\tilde{g}}(x, z, t)$ are linear in $\underline{x}$. Hence, $\underline{\tilde{g}}(x, z, t)$ must be linear in $\underline{x}$, and it must be possible to put it in the form

$$\tilde{\underline{g}}(\underline{x}, \underline{z}, t) = \underline{G}_1(t)\,\underline{x} + \underline{g}_2(\underline{z}, t) \tag{2.8}$$

where $\underline{G}_1(t)$ is a time-varying matrix and $\underline{g}_2(\underline{z}, t)$ is a vector-valued function of $\underline{z}$ and t alone. One can now substitute (2. 8) into Eqs. (1. 30)and (1. 31) from which $\tilde{\underline{g}}(\underline{x}, z, t)$ is to be solved. Taking into account the linearity of w(e), it follows from (1. 30) that

$$\dot{\underline{G}}_1(t)\,\underline{x} + \underline{G}_1(t)\Big(\underline{A}(t)\underline{x} + \underline{b}(t)\,\underline{\varphi}_1(t)'\underline{x} + \underline{b}(t)\varphi_2(\underline{z}, t)\Big) + \frac{\partial \underline{g}_2(\underline{z}, t)}{\partial t} + \mathcal{L}\,\underline{g}_2(\underline{z}, t)$$

$$= -\underline{A}(t)'\underline{G}_1(t)\,\underline{x} - \underline{A}(t)'\underline{g}_2(\underline{z}, t) + \underline{c}(t)\Big(\underline{c}(t)'\underline{x} + k(t)\,\underline{\varphi}_1(t)'\,\underline{x} + k(t)\,\varphi_2(\underline{z}, t) - \rho(\underline{z}, t)\Big) \tag{2.9}$$

and from (1. 31) that

$$\underline{G}_1(T)\,\underline{x} + \underline{g}_2(\underline{z}, T) = -\lambda\, T\,\underline{x} \tag{2.10}$$

Since (2. 9) and (2.10) have to hold identically in $\underline{x}$ and $\underline{z}$, they can be separated and the following equations are obtained (for brevity the explicit time-dependence of $\underline{G}_1$, $\underline{A}$, $\underline{b}$, etc., is dropped)

$$\dot{\underline{G}}_1 + \underline{G}_1\underline{A} + \underline{A}'\,\underline{G}_1 + \underline{G}_1\,\underline{b}\,\underline{\varphi}_1' = \underline{c}\,(\underline{c}' + k\,\underline{\varphi}_1') \tag{2.11}$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L} + \underline{A}'\right)\underline{g}_2(\underline{z}, t) = \underline{c}\,(k\,\varphi_2(\underline{z}, t) - \rho(\underline{z}, t)) - \underline{G}_1\,\underline{b}\,\varphi_2(\underline{z}, t) \tag{2.12}$$

$$\underline{G}_1(T) = -\lambda T \underline{I} \tag{2.13}$$

$$\underline{g}_2(\underline{z}, T) = \underline{0} \tag{2.14}$$

For a given controller, the matrix function $\underline{G}_1$ can be solved from (2.11) together with terminal condition (2.13). If $\underline{G}_1$ is given, Eq. (2.12) constitutes a linear partial differential equation in the function $\underline{g}_2(\underline{z}, t)$; it has as its terminal condition (2.14) and as boundary condition that as $\underline{z}$ goes to $\pm\infty$, $\underline{g}_2$ should not grow too fast.

The optimality condition, as given by Eq. (IV.1.35) of Theorem 7 reads in terms of the functions $\underline{G}_1$ and $\underline{g}_2$ as follows:

$$0 = \tilde{\mu}_T(\underline{x}, \underline{z}, t) = -\underline{b}' \tilde{\underline{g}}(x, z, t) + k\, w(x, z, t)$$

$$= -\underline{b}'\underline{G}_1\underline{x} - \underline{b}'\, \underline{g}_2(\underline{z}, t)$$

$$+ k\,(\underline{c}'\underline{x} + k\underline{\varphi}_1'\underline{x} + k\varphi_2(\underline{z}, t) - \rho(\underline{z}, t)) \tag{2.15}$$

which can be separated as follows

$$\underline{0}' = -\underline{b}'\, \underline{G}_1 + k(\underline{c}' + k\underline{\varphi}_1') \tag{2.16}$$

and

$$0 = -\underline{b}'\, \underline{g}_2(\underline{z}, t) + k(k\,\varphi_2(\underline{z}, t) - \rho(\underline{z}, t)) \tag{2.17}$$

From Eq. (2.16) the function $\underline{\varphi}_1(t)$ can be expressed in terms of $\underline{G}_1(t)$. Similarly, from (2.17), $\varphi_2(\underline{z}, t)$ can be expressed in terms of $\underline{g}_2(\underline{z}, t)$. The so-found relations can be substituted into Eqs. (2.11) and

(2.12) to yield the following equations for the $\underline{G}_1$ and $\underline{g}_2$ corresponding to the optimal controller:

$$\dot{\underline{G}}_1 + \underline{G}_1 (\underline{A} - \frac{1}{k} \underline{b}\, \underline{c}') + (\underline{A} - \frac{1}{k} \underline{b}\, \underline{c}')' \underline{G}_1 + \frac{1}{k^2} \underline{G}_1 \underline{b}\, \underline{b}' \underline{G}_1 = \underline{0} \tag{2.18}$$

and

$$\left(\frac{\partial}{\partial t} + \mathcal{L} + \underline{A}' - \frac{1}{k} \underline{c}\, \underline{b}' + \frac{1}{k^2} \underline{G}_1 \underline{b}\, \underline{b}'\right) \underline{g}_2(\underline{z}, t) = - \frac{1}{k} \underline{G}_1 \underline{b}\, \rho(\underline{z}, t) \tag{2.19}$$

The terminal conditions remain

$$\underline{G}_1(T) = - \lambda\, T\, \underline{I} \tag{2.20}$$

$$\underline{g}_2(\underline{z}, T) = \underline{0} \tag{2.21}$$

When from these equations the functions $\underline{G}_1$ and $\underline{g}_2$ are solved, they can be substituted into (2.16) and (2.17) to yield the optimal controller.

Equation (2.18) is a homogeneous matrix Ricatti differential equation and can be transformed into a linear matrix differential equation by the substitution

$$\underline{G}_1(t) = \underline{H}(t)^{-1} \tag{2.22}$$

After the solution has been obtained, it will be verified that it is nonsingular, so that the substitution is allowed. The following equation for $\underline{H}$ is obtained:

$$-\dot{\underline{H}} + (\underline{A} - \frac{1}{k}\underline{b}\,\underline{c}')\,\underline{H} + \underline{H}(\underline{A} - \frac{1}{k}\,\underline{b}\,\underline{c}')' = -\frac{1}{k^2}\,\underline{b}\,\underline{b}' \tag{2.23}$$

$$\underline{H}(T) = -\frac{1}{\lambda T}\,\underline{I} \tag{2.24}$$

It is observed that the matrix $\underline{A} - \frac{1}{k}\,\underline{b}\,\underline{c}'$, which occurs in (2.18) and (2.23), characterizes the system which is obtained from the original system

$$\begin{aligned} \dot{\underline{x}} &= \underline{A}\,\underline{x} + \underline{b}\,u \\ y &= \underline{c}'\underline{x} + k\,u \end{aligned} \tag{2.25}$$

by using an input

$$u = \frac{v - \underline{c}'\underline{x}}{k} \tag{2.26}$$

where v is a new input. This new system is described by

$$\begin{aligned} \dot{\underline{x}} &= (\underline{A} - \frac{1}{k}\,\underline{b}\,\underline{c}')\,\underline{x} + \frac{1}{k}\,\underline{b}\,v \\ y &= v \end{aligned} \tag{2.27}$$

It is noted that this new system has ideal transfer between the input v and the output y. One would choose this to be the optimally controlled system when no requirements are imposed on the stability. The system described by (2.27) will be called the <u>derived system</u> corresponding to the system (2.25). It will play a role of some importance in the following.

Suppose now that the transition matrix of the derived system is given by $\underline{\psi}(t, \tau)$; i.e., $\underline{\psi}(t, \tau)$ is the solution of

$$\frac{d}{dt}\,\underline{\psi}(t,\tau) = (\underline{A} - \frac{1}{k}\,\underline{b}\,\underline{c}')\,\underline{\psi}(t,\tau) \tag{2.28}$$

$$\underline{\psi}(\tau,\tau) = \underline{I}$$

It can be derived or checked that with the aid of this transition matrix the solution of (2.23) satisfying (2.24) is

$$\underline{H}(t,T) = -\int_t^T \underline{\psi}(t,\tau)\,\underline{b}(\tau)\,\frac{1}{k(\tau)^2}\,b(\tau)'\underline{\psi}(t,\tau)'\,d\tau$$

$$-\frac{1}{\lambda T}\,\underline{\psi}(t,T)\underline{\psi}(t,T)' \tag{2.29}$$

It is noted that of the two terms of the right-hand side of (2.29) the first is negative semi-definite and the second negative-definite, since $\underline{\psi}(t,T)$ is nonsingular for all t and T, so that it follows that $\underline{H}(t,T)$ is negative-definite and hence nonsingular.

By inverting this matrix, the matrix $\underline{G}_1(t)$ can now be found and from this the feedback part of the optimal controller:

$$\underline{\varphi}_1(t)' = \frac{1}{k(t)^2}\,\underline{b}(t)'\underline{H}(t,T)^{-1} - \frac{1}{k(t)}\,\underline{c}(t)' \tag{2.30}$$

If a controller with this $\underline{\varphi}_1(t)'$ is used, the interconnection of controller and system is described by

$$\dot{\underline{x}} = (\underline{A} - \frac{1}{k}\,\underline{b}\,\underline{c}' + \frac{1}{k^2}\,\underline{b}\,\underline{b}'\,\underline{H}^{-1})\,\underline{x} + \frac{1}{k}\underline{b}\;\varphi_2(\underline{z},t)$$

$$y = \frac{1}{k}\,\underline{b}'\,\underline{H}^{-1}\,\underline{x} + k\,\varphi_2(\underline{z},t) \tag{2.31}$$

Suppose that the transition matrix of this system is $\underline{\chi}(t, \tau)$; i.e., $\underline{\chi}(t, \tau)$ is the solution of

$$\frac{d}{dt} \underline{\chi}(t, \tau) = (\underline{A} - \frac{1}{k} \underline{b}\, \underline{c}' + \frac{1}{k^2} \underline{b}\, \underline{b}' \underline{H}^{-1}) \underline{\chi}(t, \tau) \tag{2.32}$$

$$\underline{\chi}(\tau, \tau) = \underline{I}$$

It can be derived or checked that with the aid of the transition matrix $\underline{\chi}(t, \tau)$ the solution of Eq. (2.19) for $\underline{g}_2(\underline{z}, t)$ can be written in the form

$$\underline{g}_2(\underline{z}, t) = \int_0^T \underline{\chi}(\tau, t)' \frac{1}{k(\tau)} \underline{H}(\tau, t)^{-1} \underline{b}(\tau)\, d\tau \int f_z(t, \underline{z}; \tau, \underline{\xi})\, \rho(\underline{\xi}, \tau)\, d\underline{\xi} \tag{2.33}$$

where, as before, f_z is the transition probability density function of the diffusion process $\{\underline{z}(t)\}$.

By Eq. (2.17), the feedforward part of the optimal controller $\varphi_2(\underline{z}, t)$ is expressed in terms of $\underline{g}_2(\underline{z}, t)$ by the relation

$$\varphi_2(\underline{z}, t) = \frac{1}{k^2} \underline{b}' \underline{g}_2(\underline{z}, t) + \frac{1}{k} \rho(\underline{z}, t) \tag{2.34}$$

By combining the results obtained so far, it follows that the optimal controller can be represented by

$$u^o(t) = (\frac{1}{k^2} \underline{b}' \underline{H}^{-1} - \frac{1}{k} \underline{c}') \underline{x} + \frac{1}{k^2} \underline{b}' \underline{g}_2(\underline{z}, t) + \frac{1}{k} \rho(\underline{z}, t) \tag{2.35}$$

or, equivalently,

$$u^o(t) = \frac{1}{k} (r(t) - \underline{c}'\underline{x}(t)) + \frac{1}{k^2} \underline{b}' \underline{H}^{-1} \underline{x}(t) + \frac{1}{k^2} \underline{b}' \underline{g}_2(\underline{z}(t), t) \tag{2.36}$$

It is observed that the first two terms of this expression represent precisely the control which would be optimal if no constraints were placed on the terminal state. The last two terms constitute a correction to this. It is furthermore observed that the function $\underline{g}_2(\underline{z}, t)$ also can be written as

$$\underline{g}_2(\underline{z}(t), t) = \int_t^T \underline{\chi}(\tau, t)' \frac{1}{k(\tau)} \underline{H}(\tau, T)^{-1} \underline{b}(\tau) E(r(\tau) \mid r_{(-\infty, t]}) d\tau \tag{2.37}$$

so that the optimal input $u^o(t)$ is entirely determined by $\underline{x}(t)$ and the function $E(r(\tau)|r_{(-\infty, t]})$ for $\tau \geq t$. In fact, the control is linear in these quantities.

It seems likely that this result is not only valid for random processes which have a finite-dimensional diffusion process as their state, but for a much larger class of random processes.

It is observed that the characteristics of the random process $\{r(t)\}$ only enter into the optimal solution by way of the conditional expectation function $E(r(\tau) \mid r_{(-\infty, t]})$. The feedback part of the controller is completely independent of the characteristics of the random process. It seems likely that the solution of the stochastic problem is closely related to the solution of comparable deterministic problems.

In the next section, some attention will be given to the behavior of the solutions as T approaches ∞.

IV. 3 Optimal Stable Solution; An Example

It is of considerable interest to investigate the behavior of the solution obtained in the previous section as $T \to \infty$, but since this question involves a number of complications, the treatment will here be limited to an example which exhibits some of the typical features of this type of problem.

<u>Example</u> : Consider the system which is described by

$$
\begin{aligned}
\dot{x} &= -x + u \\
y &= x + ku
\end{aligned}
\tag{3.1}
$$

with $k \neq 0$ = constant. This system is complete controllable, completely observable (Kalman[6]) and uniformly asymptotically stable. It has the frequency transfer function

$$
H(s) = \frac{ks + (k+1)}{s+1} \tag{3.2}
$$

It is noted that if $\frac{k+1}{k} > 0$, the system has a stable inverse, so that it is to be expected that the optimal solution will be related to this inverse.

Furthermore let the random process $\{r(t)\}$ be a simple Gaussian Markov process with time constant θ and variance σ^2. Of this process the state is $z(t) = r(t)$ and the conditional expectation

$$
E(\, r(\tau) \mid r_{(-\infty,\, t]}) = r(t)\; e^{-\frac{\tau - t}{\theta}} , \quad \tau \geq t \tag{3.3}
$$

The <u>derived system</u> is characterized by

$$
\begin{aligned}
\dot{x} &= (-1 - \tfrac{1}{k})\, x + \tfrac{1}{k}\, v \\
y &= v
\end{aligned}
\tag{3.4}
$$

It is observed that the stability of the derived system is determined precisely by the sign of $\frac{k+1}{k}$. The transition matrix of the derived system is

$$\psi(t, \tau) = e^{-\frac{k+1}{k}(t-\tau)} \tag{3.5}$$

With the aid of this the expression for $H(t, T)$ can readily be integrated; it is found to be

$$H(t,T) = \frac{1}{2k(k+1)}\left(1 - e^{2\frac{k+1}{k}(T-t)}\right) - \frac{1}{\lambda T}\, e^{2\frac{k+1}{k}(T-t)} \tag{3.6}$$

Using this result the optimal controller can easily be found. Now as T approaches ∞ there are two possibilities:

i) $\frac{k+1}{k} > 0$. Then as $T \to \infty$ $H(t,T) \to -\infty$ and hence

$$H(t, T)^{-1} \longrightarrow 0$$

ii) $\frac{k+1}{k} < 0$. Then as $T \to \infty$ $H(t, T) \to \frac{1}{2k(k+1)}$ and hence

$$H(t, T)^{-1} \longrightarrow 2k(k+1)$$

Note that if $\frac{k+1}{k} = 0$, i.e., $k = -1$,

$$H(t, T) = -\frac{T-t}{k^2} - \frac{1}{\lambda T} \tag{3.7}$$

so that in this case

$$\lim_{T \to \infty} H(t, T)^{-1} = 0 \tag{3.8}$$

which means that this case can be included in both (i) and (ii).

The controller which is obtained by replacing $\underline{H}^{-1}$ in equation (2.30) by the limits as given above is characterized by

$$\begin{aligned} &\text{i)} \quad \varphi_1(t) = -\frac{1}{k} \\ &\text{ii)} \quad \varphi_1(t) = 2 + \frac{1}{k} \end{aligned} \tag{3.9}$$

The transition matrix of the interconnection of system and these controllers can be solved from (2.32); it is found that

$$\text{i)} \qquad \chi(t,\tau) = \psi(t,\tau) = e^{-\frac{k+1}{k}(t-\tau)} \tag{3.10}$$

$$\text{ii)} \qquad \chi(t,\tau) = \psi(\tau,t) = e^{\frac{k+1}{k}(t-\tau)} \tag{3.11}$$

Using these transition matrices in Eq. (2.33) and (2.34), the feed-forward part of the controller is found to be

$$\text{i)} \ \varphi_2(z,t) = \frac{1}{k} z \tag{3.12}$$

$$\text{ii)} \ \varphi_2(z,t) = \frac{1}{k} \frac{k + (k+1)\theta}{k - (k+1)\theta} z \tag{3.13}$$

The controllers so obtained can now be written in the form

$$\text{i)} \ \frac{k+1}{k} \geq 0 \qquad u^o(t) = \frac{r(t) - y(t)}{k} \tag{3.14}$$

$$\text{ii)} \ \frac{k+1}{k} \leq 0 \qquad u^o(t) = \left(2+\frac{1}{k}\right) x(t) + \frac{1}{k} \frac{k + (k+1)\,\theta}{k - (k+1)\,\theta} r(t) \tag{3.15}$$

It is seen that the control used in case (i) is precisely the one which gives the derived system, so that ideal control is achieved, since the error is zero at all times. Moreover, this system is stable if $\frac{k+1}{k} > 0$.

The system given under (ii) is also stable if $\frac{k+1}{k} < 0$, as can be seen from the transition matrix given by (3.11), but its optimality remains to be investigated. This will be accomplished with the aid of Theorem 5. For this theorem it is required to find the function μ_T and its limit as $T \rightarrow \infty$. To find this function, the whole machinery that was developed for the function $\tilde{\mu}_T$, but keeping $\lambda = 0$, can be used. From the results in Section IV.1 it follows that μ_T can be solved from

$$\mu_T(\underline{x}, \underline{z}, t) = -\underline{b}' \, \underline{G}_1 \underline{x} - \underline{b}' \, \underline{g}_2(\underline{z}, t) + k \, (\underline{c}' \, \underline{x} + k \underline{\varphi}_1' \, \underline{x} + k\varphi_2(\underline{z}, t) - \rho(\underline{z}, t)) \tag{3.16}$$

where $\underline{G}_1$ and $\underline{g}_2$ are to be solved from

$$\dot{\underline{G}}_1 + \underline{G}_1\underline{A} + \underline{A}+\underline{G}_1 + \underline{G}_1 \, \underline{b} \, \underline{\varphi}_1' = \underline{c} \, (\underline{c} + k \, \underline{\varphi}_1') \tag{3.17}$$

$$(\frac{\partial}{\partial t} + \mathcal{L} + \underline{A}') \, \underline{g}_2(\underline{z}, t) = \underline{c} \, (k\varphi_2(\underline{z}, t) - \rho(\underline{z}, t)) - \underline{G}_1 \, \underline{b} \, \varphi_2(\underline{z}, t) \tag{3.18}$$

$$\underline{G}_1(T) = \underline{0} \tag{3.19}$$

$$\underline{g}_2(\underline{z}, t) = \underline{0} \tag{3.20}$$

Making the appropriate substitutions, these equations can be solved.

For the controller given by (3.15) it is found that

$$\mu_T(x, z, t) = 2k(k+1) \, e^{\frac{T-t}{k}} \, x + \frac{2k(k+1)\theta}{k - (k+1)\theta} \, e^{\frac{T-t}{k}} \, z \tag{3.21}$$

It follows immediately that as $T \to \infty$

$$\mu_T(x, z, t) \longrightarrow 0 \tag{3.22}$$

<u>pointwise</u>, provided $\frac{1}{k} < 0$, which is implied by $1 + \frac{1}{k} < 0$.

Theorem 5, however, requires that

$$\mu_T \left\{ r_{(-\infty, t]} \right\} \longrightarrow \mu_\infty \left\{ r_{(-\infty, t]} \right\} \tag{3.23}$$

in the sense that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E(|\mu_T\{r_{(-\infty, t]}\} - \mu_\infty\{r_{(-\infty, t]}\}|^2) dt = 0 \tag{3.24}$$

This means that in order to prove that $\mu_\infty\{r_{(-\infty, t]}\} = 0$, one has to demonstrate

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E(|\mu_1(t)\, x(t) + \mu_2(t)\, z(t)|^2)\, dt = 0 \tag{3.25}$$

It can be shown that this is indeed so for the controller given under (ii), provided $1 + \frac{1}{k} \leq 0$. The proof of this is not difficult but requires some labor. It is interesting to note that the convergence of $\mu_T\{r_{(-\infty, t]}\}$ to zero is pointwise correct for $0 < 1 + \frac{1}{k} < 1$, but not in the sense required by (3.24).

Before applying Theorem 5 in all its detail, it is helpful to make a definition and some comments. A controller F will be called a stable controller if the quantity

$$E(|F|^2) \tag{3.26}$$

remains bounded for all t. Furthermore the system described by

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\,u \tag{3.27}$$

$$y = \underline{c}'\underline{x} + k\,u \tag{3.28}$$

will be called stable if the quantities

$$E(|\underline{x}(t)|^2) \tag{3.29}$$

and

$$E(|y(t)|^2) \tag{3.30}$$

remain bounded for all t, for all random inputs for which $E(|u(t)|^2)$ remains bounded for all t. Here the <u>norm</u> of a vector $\underline{a} = \mathrm{col}(a_1, \ldots a_n)$ (not Euclidean norm) is defined as

$$|\underline{a}| = \sum_{i=1}^{n} |a_i| \tag{3.31}$$

It is clear from these definitions that if both the controller and the system are stable, the interconnection of system and controller is stable in the sense that $E(|\underline{x}(t)|^2)$ and $E(|y(t)|^2)$ remain bounded for all t.

It is not difficult to check that the system considered in the example is stable in the sense required here, and that also the controller given by (3.15) is stable <u>provided</u> $\frac{k+1}{k} < 0$. If $\frac{k+1}{k} = 0$, however, the transition matrix of the interconnection of system and controller can be seen to be $\chi(t, \tau) = 1$, which means that the controller cannot be stable.

When one now returns to the problem of deciding on the optimality of controller (3.15), and Theorem 5 is invoked, it is first of all observed that the conditions R_1, R_2, and R_3 in this case amount to the requirement that

$$E(|y(t) - r(t)|^2) \tag{3.32}$$

be finite for all t. If $\frac{k+1}{k} \leq 0$, this condition is fulfilled by F^o and by all stable controllers. Furthermore, it can be checked that the quantity $\lim_{T \to \infty} \mathcal{E}(F^o, T)$ exists.

Then, since as was checked, $\mu_\infty \{r_{(-\infty, t]}\} = 0$, the theorem asserts that F^o is optimal with respect to all controllers F for which the following three quantities exist and are finite:

$$\lim_{T \to \infty} \mathcal{E}(F, T) \tag{3.33}$$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E(|F\{r_{(-\infty, t]}\} - F^o \{r_{(-\infty, t]}\}|^2) \, dt \tag{3.34}$$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E(\mu_\infty \{r_{(-\infty, t]}\} (F\{r_{(-\infty, t]}\} - F^o\{r_{(-\infty, t]}\})) \, dt \tag{3.35}$$

The third quantity presents no problems, because $\mu_\infty = 0$; and therefore it is 0. The second quantity is finite if F^o is stable and F is stable, since

$$\left[\lim_{T \to \infty} \frac{1}{T} \int_0^T E(|F - F^o|^2) \, dt\right]^{\frac{1}{2}} \leq \left[\lim_{T \to \infty} \frac{1}{T} \int_0^T E(|F|^2) \, dt\right]^{\frac{1}{2}} + \left[\lim_{T \to \infty} \frac{1}{T} \int_0^T E(|F^o|^2) dt\right]^{\frac{1}{2}} \tag{3.36}$$

The first quantity is finite for all stable controllers; those controllers F for which the limit does not exist will be ruled out.

One can therefore conclude from the theorem that if the controller F^o is stable, it is optimal with respect to all stable controllers F for which $\lim_{T \to \infty} \mathcal{E}(F, T)$ exists. From this it follows that the controller given by (3.15) is indeed the optimal stable controller provided $\frac{k+1}{k} < 0$.

If $\frac{k+1}{k} = 0$, however, the resulting controller (obtained from either case (i) or (ii)), is optimal in the sense that it produces zero error at all times, but it is <u>not</u> stable in the required sense. It appears that in this case an optimal stable controller does not exist.

The connection of the solutions obtained here with the <u>Wiener optimal solution</u> follows.

When by taking the Laplace transform of (3.14), (3.15) and (3.1) the Laplace transform of u(t) is expressed in terms of the Laplace transform of z(t) = r(t), it is found that in the cases (i) and (ii) the respective relations are

$$\text{a)}\quad 1 + \frac{1}{k} \geq 0 \qquad U(s) = \frac{s+1}{ks + k + 1}\, Z(s) \tag{3.37}$$

$$\text{b)}\quad 1 + \frac{1}{k} < 0 \qquad U(s) = \frac{k+(k+1)\theta}{k-(k+1)\theta}\, \frac{s+1}{ks-k-1}\, Z(s) \tag{3.38}$$

where $U(s)$ and $Z(s)$ represent the Laplace transforms of u(t) and z(t), respectively. The filters which are represented by these relations are precisely the filters that would have been obtained according to the Wiener optimization procedure for the cases $1 + \frac{1}{k} \geq 0$ and $1 + \frac{1}{k} < 0$.

It is noted, however, that the nature of the controller that is obtained by the Wiener optimization procedure is different from the one obtained here, in that the Wiener solution is essentially a feedforward filter and the present solution is in the form of a feedback controller. The feedback solution has a feedforward version, which can be represented in terms of the same linear filter as the Wiener solution, plus an initial condition for this filter, depending upon the initial state of the system and random process. The Wiener solution assumes zero initial conditions of the same filter. But if the optimal controller which is obtained is stable, the effect of any initial condition vanishes with time so that no essential difference in operation over the infinite time interval $[0, \infty)$ results.

Additional observations

Observation 1 : The solution for the optimization problem as outlined in the preceding section clearly breaks down if $k(t) = 0$. If in the example that was worked out in this section one lets $k \longrightarrow 0$, the optimal solution $\varphi_1(t)$ as given by (3.9) behaves asymptotically in one of the two following ways:

i) $k \downarrow 0$. Then $\varphi_1(t) \sim -\frac{1}{k}$, $0 \leq t < T$ (3.39)

ii) $k \uparrow 0$. Then $\varphi_1(t) \sim \frac{1}{k}$., $0 \leq t < T$ (3.40)

It can be verified that the corresponding asymptotic behavior of $\varphi_2(z, t)$ is given by

i) $k \downarrow 0$. Then $\varphi_2(z,t) \sim \frac{1}{k} z$ (3.41)

ii) $k \uparrow 0$. Then $\varphi_2(z,t) \sim -\frac{1}{k} z$ (3.42)

By combining these two results, it is conjectured that the optimal controller for the system described by setting $k = 0$ in (3.1), namely

$$\begin{aligned} \dot{x} &= -x + u \\ y &= x \end{aligned} \qquad (3.43)$$

can be approximated arbitrarily closely by a feedback controller of the form

$$F(x, z, t) = K(-x + z) \qquad (3.44)$$

with $K \longrightarrow +\infty$.

This phenomenon is typical for systems with $k(t) = 0$. In general, properly speaking, no optimal feedback controller (stable or not) exists, but optimal (stable) control can be achieved arbitrarily closely by a feedback controller of the type

$$F(\underline{x}, \underline{z}, t) = K\,(\underline{\psi}_1(t)'\,\underline{x} + \psi_2(\underline{z}, t)) \qquad (3.45)$$

with $K \longrightarrow +\infty$. (No rigorous proof of this statement has been obtained to date).

Observation 2 : The preceding observation strengthens the conjecture made in Section III.4; namely, that if $k(t) = 0$, bang-bang type control is optimal when the input amplitude is constrained, since for the non-constrained case, very large input amplitudes seem to be required.

Finally, Observation 1 suggests what might be a suboptimal controller for the constrained case. Consider the system

$$\begin{aligned} \dot{\underline{x}} &= \underline{A}\,\underline{x} + \underline{b}\,u \\ y &= \underline{c}'\underline{x} + ku \end{aligned} \tag{3.46}$$

with k = constant ≠ 0. Suppose that for this system an optimal (stable) controller is obtained, which as $k \downarrow 0$ behaves as

$$F(\underline{x}, \underline{z}, t) \sim \frac{1}{k}\,(\underline{\psi}_1(t)'\underline{x} + \psi_2(\underline{z}, t)) \tag{3.47}$$

with $\underline{\psi}_1$ and ψ_2 independent of k.

Then if the function $\gamma(t)$ by which the input amplitude is constrained is large enough to allow "reasonably good" control of the system given by

$$\begin{aligned} \dot{\underline{x}} &= \underline{A}\,\underline{x} + \underline{b}\,u \\ y &= \underline{c}'\,\underline{x} \end{aligned} \tag{3.48}$$

in following the random process $\{r(t)\}$ (i.e. the rates of change in $\underline{x}(t)$ and its derivatives that can be effected by applying inputs bounded by $\gamma(t)$ are not much inferior to those needed for optimal control without amplitude constraints) then it can be expected that the controller

$$F(\underline{x}, \underline{z}, t) = \gamma(t)\ \operatorname{sign}\,(\underline{\psi}_1(t)'\,\underline{x} + \psi_2(\underline{z}, t)) \tag{3.49}$$

is a reasonably good controller for the system (3.48). Moreover, this controller satisfies the constraint of boundedness of the input amplitude; hence, it can be considered as a candidate for a suboptimal controller for the system (3.48), with input amplitude constraints.

V. CONCLUSIONS

It should be clear, after reading this report, that the optimal following problem has in general not been solved. Although in Section II some theorems concerning necessary and sufficient conditions for optimal controllers are given, not much progress has been made to finding analytical or numerical solutions.

For linear systems and random processes with finite-dimensional, diffusion-type states, the analysis has been pushed to a point in Section III where it might be possible to devise numerical iterative procedures for finding an optimal controller (see observation at the end of Section III. 3). Some remarks about the general nature of optimal controllers can be made.

A complete solution has been obtained in Section IV for the case of a linear, finite-dimensional differential system, a quadratic weighting function and without amplitude constraints on the input. This solution exhibits some interesting features which deserve further investigation. The most striking phenomenon is the fact that the optimal controller is to a remarkable extent independent of the characteristics of the random process. A question of interest that was brought up in Section IV is the stability problem, which also merits continued attention.

The solutions which were obtained in Section IV for the case of an unconstrained input instigate some speculations about sub-optimal solutions for the constrained case.

The problem which is solved in Section IV includes as a special case the Wiener problem.

VI. APPENDIX 1. ESTIMATION OF CERTAIN TERMS

It is required to prove that the following two terms which occur in equation (I. 2.13) can be replaced by $o(\varepsilon)$:

$$I_1 = E\left(\frac{1}{T} \int_o^T o(\varepsilon;t) \; w[y^o(t) - r(t)] \; dt \right) \tag{A.1.1}$$

$$I_2 = E\left(\frac{1}{T} \int_o^T o'[\varepsilon H_{u^o} \bar{u}(t) + o(\varepsilon;t)] \; dt \right) \tag{A.1.2}$$

The following series of approximations can be made for I_1:

$$|I_1| \leq \frac{1}{T} \int_o^T E|\, o(\varepsilon;t) \, w[y^o(t) - r(t)] \,|\, dt$$

$$\leq \frac{1}{T} \int_o^T (E(o(\varepsilon;t))^2)^{1/2} \; (E(w[y^o(t) - r(t)])^2)^{1/2} \, dt \tag{A.1.3}$$

By the requirements imposed by the definition of differentiability of the system Σ, the limit

$$\lim_{\varepsilon \to 0} \frac{0(\varepsilon;t)}{\varepsilon} = 0 \tag{A.1.4}$$

is uniform in $u^o_{(0,t]}$, $\bar{u}_{(0,t]}$ and t. This means that for a given δ it is possible to find an ε_m such that for $0 < \varepsilon < \varepsilon_m$ and for all $u^o_{(0,t]}$, $\bar{u}_{(0,t]}$ and t,

$$\left|\frac{o(\varepsilon;t)}{\varepsilon}\right| \leq \delta \tag{A.1.5}$$

or, equivalently,

$$|o(\varepsilon;t)|^2 \leq \delta^2 \varepsilon^2 \tag{A.1.6}$$

Because of the uniform validity of this statement it follows immediately that

$$(E|o(\varepsilon;t)|^2)^{1/2} < \delta\varepsilon \tag{A.1.7}$$

for all t. By condition R_2, the second factor of the integrand of the last member of (A.1.3) is bounded, say by B. It follows

$$|I_1| \leq B\delta\varepsilon \tag{A.1.8}$$

This implies that I_1 is of order ε, since for each given $\delta'=\delta B$ there exists an ε_m such that for $0 < \varepsilon < \varepsilon_m$

$$|I_1| \leq \delta'\varepsilon \tag{A.1.9}$$

The second term of interest is I_2. In expression (A.1.2), the quantity o' is defined by the relation

$$W(x+y) = W(x) + yw(x) + o'(y) \tag{A.1.10}$$

It follows from Taylor's theorem

$$o'(y) = \int_0^y (y-\eta)\, w'(x+\eta)\, d\eta \tag{A.1.11}$$

Referring to Eq. (I.2.9) of the text, it follows that

$$o'[\varepsilon H_{u^o}\bar{u}(t) + o(\varepsilon;t)] = \int_0^{(\varepsilon H_{u^o}\bar{u}(t) + o(\varepsilon;t))} [\varepsilon H_{u^o}\bar{u}(t) + o(\varepsilon;t)-\eta] \cdot w'[y^o(t)-r(t)+\eta]\, d\eta \tag{A.1.12}$$

By assumption of the boundness of the system and hence also of the variational system, the term $H_{u^o}\bar{u}(t)$ is bounded, say by B'. For ε small enough, also

$$\frac{o(\varepsilon\,;t)}{\varepsilon} \qquad \text{(A.1.13)}$$

is bounded, say by B''. Thus it follows that

$$|\varepsilon H_{u^o}\bar{u}(t) + o(\varepsilon\,;t)| \leq (B' + B'')\varepsilon = B\varepsilon \qquad \text{(A.1.14)}$$

where $B=B' + B''$ is not the same as in (A.1.8). By this inequality, o' as given in (A.1.12) can be approximated as follows:

$$|o'[\varepsilon H_{u^o}\bar{u}(t) + o(\varepsilon\,;t)]|$$

$$\leq \int_0^{B\varepsilon} |\varepsilon H_{u^o}\bar{u}(t) + o(\varepsilon\,;t) - \eta|\,|w'[y^o(t)-r(t)+\eta]|\,d\eta \qquad \text{(A.1.15)}$$

$$\leq \int_0^{B\varepsilon} 2\,\varepsilon\,B|w'[y^o(t)-r(t)+\eta]|\,d\eta \qquad \text{(A.1.16)}$$

By restriction R_3, the expectation of the second factor of the integrand of (A.1.16) is bounded uniformly in η, say by C. It follows that

$$E\left(|o'[\varepsilon H_{u^o}\bar{u}(t) + o(\varepsilon;t)]|\right) \leq 2\,\varepsilon^2 B^2 C \qquad \text{(A.1.17)}$$

Upon using this in (A.1.2) it follows very easily that I_2 is estimated by the same term

$$|I_2| \leq 2 \varepsilon^2 B^2 C \qquad \text{(A.1.18)}$$

which immediately implies that I_2 is of order ε. This completes the proof.

VII. APPENDIX 2. PART OF PROOF OF LEMMA 2

The proof that $\underline{c}_{(i)}(t)'\underline{b}(t) = 0$, $i = 0, 1, \ldots, (p-1)$ during the interval (t_o, t_1) implies $\underline{c}(t)'\underline{b}_{(j)}(t) = 0$, $j = 0, 1, \ldots, (p-1)$, during the same interval, will be given by induction on j. The assertion is true for $j = 0$, since $\underline{b}_{(0)}(t) = -\underline{b}(t)$ and $\underline{c}(t) = \underline{c}_{(0)}(t)$.

Assume that the assertion is true for $j = k-1$. Consider

$$o = \underline{c}_{(k)}(t)'\underline{b}(t) = \underline{b}(t)'\left(\frac{d}{dt} + \underline{A}'\right)^k \underline{c}(t) \tag{A. 2.1}$$

Now let $\varphi(t)$ be a testing function which is at least k times differentiable and which vanishes outside the open interval (t_o, t_1). Multiply (A. 2.1) by $\varphi(t)$ and integrate over the interval (t_o, t_1). Since $\varphi(t)$ and all its derivatives must vanish at the endpoints of the interval, it follows by partial integration that

$$0 = \int \varphi(t) \quad \underline{b}(t)' \left(\frac{d}{dt} + \underline{A}'\right)^k \underline{c}(t) \; dt \tag{A. 2. 2}$$

$$= \int \underline{c}(t)' \left(\left(-\frac{d}{dt} + \underline{A}\right)^k \underline{b}(t) \; \varphi(t) \right) dt \tag{A. 2.3}$$

which can easily be checked with the use of the binomial expansion.

For the integrand of (A. 2. 3) it can now be written

$$\underline{c}(t)' \left(\left(-\frac{d}{dt} + \underline{A}\right)^k \underline{b}(t)\,\varphi(t)\right)$$

$$= \underline{c}(t)' \left(\left(-\frac{d}{dt} + \underline{A}\right)^{k-1} \left(-\frac{d}{dt} + \underline{A}\right) \underline{b}(t)\,\varphi(t)\right) \tag{A. 2.4}$$

$$= (\underline{c}(t)'(-\frac{d}{dt} + \underline{A})^{k}\underline{b}(t))\varphi(t) - (\underline{c}(t)'(-\frac{d}{dt} + \underline{A})^{k-1}\underline{b}(t))\frac{d\varphi(t)}{dt}$$

(A. 2. 5)

Since by the induction hypothesis

$$\underline{c}(t)'\underline{b}_{(k-1)}(t) = (-1)^{k}\,\underline{c}(t)'(-\frac{d}{dt} + \underline{A})^{k-1}\,\underline{b}(t) = 0 \qquad \text{(A. 2. 6)}$$

the last term in (A. 2. 5) vanishes and one obtains for the integrand of (A. 2. 3) the expression

$$(\underline{c}(t)'(-\frac{d}{dt} + \underline{A})^{k}\,\underline{b}(t))\varphi(t) = (-1)^{k+1}\,\underline{c}(t)'\underline{b}_{(k)}(t)\ \varphi(t) \qquad \text{(A. 2. 7)}$$

Since the integral (A. 2. 3) vanishes, it follows that

$$\int \underline{c}(t)'\ \underline{b}_{(k)}(t)\ \varphi(t)\ dt = 0 \qquad \text{(A. 2. 8)}$$

This holds for all testing functions $\varphi(t)$. Since by assumption both $\underline{c}(t)$ and $\underline{b}_{(k)}(t)$ are continuous, it follows that

$$\underline{c}(t)'\underline{b}_{(k)}(t) = 0 \qquad \text{(A. 2. 9)}$$

for t in the interval (t_o, t_1). This completes the proof by induction.

VIII. APPENDIX 3 : DERIVATION OF BACKWARD KOLMOGOROV EQUATION

Since the diffusion process $\{\underline{z}(t)\}$ is a Markov process, the Chapman-Kolmogorov equation can be invoked. It follows that one can write

$$f_z(t, \underline{z};\ \tau, \underline{\zeta}) = \int f_z(t, \underline{z};\ t + \Delta t,\ \underline{\xi})\ f_z(t + \Delta t,\ \underline{\xi};\ \tau, \underline{\zeta})\ d\,\underline{\xi} \qquad \text{(A. 3.1)}$$

where $\Delta t > 0$ will be taken small. For the time being scalar notation will be used. Subscripts will indicate the number of the component, i.e., z_i is the ith component of the vector $\underline{z}$; Q_{ij} is the i, jth element of the matrix $\underline{Q}$, etc.

Now suppose that the necessary derivatives of $f_z(t, \underline{z};\tau, \underline{\zeta})$ with respect to its first two arguments exist, so that the following Taylor expansion can be made

$$\begin{aligned} f_z(t + \Delta t, \underline{\xi};\ \tau, \underline{\zeta}) = \quad & f_z(t, \underline{z};\tau, \underline{\zeta}) + \Delta t\ \frac{\partial f_z(t, \underline{z};\tau, \underline{\zeta})}{\partial t} \\ & + (\xi_1 - z_1)\ \frac{\partial f_z(t, \underline{z};\tau, \underline{\zeta})}{\partial z_1} \quad + \\ & + \ldots\ldots\ldots\ldots \\ & + (\xi_k - z_k)\ \frac{\partial f_z(t, \underline{z};\tau, \underline{\zeta})}{\partial z_k} + \\ & + \frac{1}{2}\ (\xi_1 - z_1)^2\ \frac{\partial^2 f_z(t, \underline{z};\tau, \underline{\zeta})}{\partial z_1^{\,2}} \quad + \end{aligned}$$

$$+ \frac{1}{2}(\xi_1 - z_1)(\xi_2 - z_2) \frac{\partial^2 f_z(t, \underline{z}; \tau, \underline{\xi})}{\partial z_1 \, \partial z_2} +$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ \frac{1}{2}(\xi_k - z_k)^2 \frac{\partial^2 f_z(t, \underline{z}; \tau, \underline{\xi})}{\partial z_k^2}$$

$$+ \ldots\ldots\ldots\ldots \qquad \text{(A. 3. 2)}$$

For brevity, the arguments of the partial derivatives of the function f_z will be dropped. With the aid of (A. 3. 2), Eq. (A. 3.1) can be written as follows

$$f_z(t, \underline{z}; \tau, \underline{\xi}) =$$

$$E \Big\{ f_z(t, \underline{z}; \tau, \underline{\xi}) + \Delta t \frac{\partial f_z}{\partial t} +$$

$$+ (z_1(t+\Delta t) - z_1(t)) \frac{\partial f_z}{\partial z_1}$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ (z_k(t+\Delta t) - z_k(t)) \frac{\partial f_z}{\partial z_k}$$

$$+ \frac{1}{2}(z_1(t+\Delta t) - z_1(t))^2 \frac{\partial^2 f_z}{\partial z_1^2}$$

$$+ \frac{1}{2} (z_1(t+\Delta t) - z_1(t)) (z_2(t+\Delta t) - z_2(t)) \frac{\partial^2 f_z}{\partial z_1 \partial z_2}$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ \frac{1}{2} (z_k(t+\Delta t) - z_k(t))^2 \frac{\partial^2 f_z}{\partial z_k^2}$$

$$+ \ldots\ldots\ldots\ldots \Bigg|_{\underline{z}(t) = \underline{z}} \Bigg\} \tag{A.3.3}$$

It is observed that the term $f_z(t, \underline{z}; \tau, \underline{\zeta})$ can be canceled on both sides. Now no matter whether the diffusion process is defined according to the hypotheses H_1, H_2 and H_3 or by the use of the stochastic differential equation (II. 2. 7), the limits (II. 2. 5) and (II. 2. 6) exist and are as given. By dividing (A. 3. 3) by Δt, letting $\Delta t \downarrow 0$, and using these limits, it follows that

$$0 = \frac{\partial f_z}{\partial t} + p_1(\underline{z}, t) \frac{\partial f_z}{\partial z_1} + \ldots . + p_k(\underline{z}, t) \frac{\partial f_z}{\partial z_k} +$$

$$+ \frac{1}{2} Q_{11}(\underline{z}, t) \frac{\partial^2 f_z}{\partial z_1^2} + \frac{1}{2} Q_{12}(\underline{z}, t) \frac{\partial^2 f_z}{\partial z_1 \partial z_2} + \ldots\ldots$$

$$+ \ldots\ldots + \frac{1}{2} Q_{kk}(\underline{z}, t) \frac{\partial^2 f_z}{\partial z_k^2} \tag{A.3.4}$$

It can be proved that the remaining terms in (A. 3. 3) indeed go to zero upon division by Δt. Equation (A. 3. 4) is the backward equation sought. It can be put in vector notation and then reads as follows:

$$0 = \frac{\partial f_z}{\partial t} + \left(\underline{\pi}(\underline{z}, t)' \frac{\partial}{\partial \underline{z}} + \frac{1}{2} \frac{\partial}{\partial \underline{z}}' \underline{Q}(\underline{z}, t) \frac{\partial}{\partial \underline{z}} \right) f_z \tag{A.3.5}$$

where the vector function $\underline{\pi}(\underline{z}, t) = \underline{p}(\underline{z}, t) - \frac{1}{2} \left(\frac{\partial}{\partial \underline{z}}' \underline{Q} \right)'$ (A.3.6)

The forward equation will not be derived, since it will not be needed. Its derivation can be found in the literature.

IX. APPENDIX 4 : FULLER'S PROBLEM

Fuller[10] has studied the problem of finding an optimal controller for the case when the system is characterized by

$$\dot{x} = b\,u$$

$$y = x \quad , \quad b > 0 \tag{A.4.1}$$

(i.e., the system is a simple integrator), and the random process is the Wiener process. The Wiener process is a one-dimensional diffusion process $r(t) = z(t)$, where each sample function is a solution of

$$dz(t) = N_o^{\frac{1}{2}}\, dv(t) \tag{A.4.2}$$

It is claimed that the controller

$$F(x, z, t) = \gamma \;\; \text{sign}\,(z-x) \tag{A.4.3}$$

is an optimal solution, for a weighting function W which is symmetric about 0, $\gamma(t) = \gamma$ = constant, and $T = \infty$.

The assertion will be verified by computing the function $\mu_T(x, z, t)$ and from this $\mu_\infty(x, z, t)$, and comparing the latter against $F(x, z, t)$. It is known from Section III.3 that the function μ_T can be obtained from the function $g(x, z, t)$, which is the solution of Eq. (III.3.15), which in this case reads

$$\left(\frac{\partial}{\partial t} + \mathcal{M}\right) \quad g(x, z, t) = w(x-z) \tag{A.4.4}$$

The terminal condition is

$$g(x, z, T) = 0, \tag{A.4.5}$$

and μ_T is expressed in terms of g as follows

$$\mu_T(x, z, t) = -b\, g(x, z, t) \tag{A.4.6}$$

The operators $\mathcal{M}$ and $\mathcal{L}$ are given by

$$\mathcal{M} = \mathcal{L} + b \; F(x, z, t) \; \frac{\partial}{\partial x} \tag{A.4.7}$$

$$\mathcal{L} = \frac{1}{2} \; N_o \; \frac{\partial^2}{\partial z^2} \tag{A.4.8}$$

The following is now observed. By the use of controller (A.4.3) the following relations hold, where (A.4.9) is obtained by putting F into (A.4.1):

$$dx(t) = b \; \gamma \; \text{sign}[\, z(t) - x(t)] \; dt \tag{A.4.9}$$

$$dz(t) = N_o^{\frac{1}{2}} \, dv(t) \tag{A.4.10}$$

Subtracting, it follows

$$d(x-z)(t) = b\gamma \; \text{sign}[\, z(t) - x(t)] \; dt - N_o^{\frac{1}{2}} \; dv(t) \tag{A.4.11}$$

It follows immediately that $e(t) = x(t) - z(t)$ is a Markov process. From this it is concluded that (from the definition of g (II.3.11)

$$g(x, z, t) = - \int_t^T E(w[\, e(\tau)] \mid z(t) = z, \; x(t) = x) \, d\tau \tag{A.4.12}$$

is a function that depends upon $x - z = \eta$ alone; therefore one can write

$$g(x, z, t) = \hat{g}(x-z, t) \tag{A.4.13}$$

Upon using this in (A.4.4) and by applying (A.4.7) and (A.4.8), it follows that g satisfies the partial differential equation

$$\frac{\partial \hat{g}(\eta, t)}{\partial t} + \frac{1}{2} N_o \frac{\partial^2 \hat{g}(\eta, t)}{\partial \eta^2} - b\gamma \; \mathrm{sign}(\eta) \frac{\partial \hat{g}(\eta, t)}{\partial \eta} = w(\eta) \tag{A.4.14}$$

with the terminal condition

$$\hat{g}(\eta, T) = 0 \tag{A.4.15}$$

Equation (A.4.14) could be solved, but the main interest is in the stationary solution which is presumably obtained as $T \longrightarrow \infty$. If it is assumed that as $T \longrightarrow \infty$, the solution of (A.4.14) with terminal condition (A.4.15) becomes independent of T and constant, and consequently the function $\hat{g}(\eta, t)$ is replaced by a function $\bar{g}(\eta)$, it follows that this function should be the solution of the ordinary differential equation

$$\frac{1}{2} N_o \frac{d^2 \bar{g}(\eta)}{d\eta^2} - b\gamma \; \mathrm{sign}(\eta) \frac{d\bar{g}(\eta)}{d\eta} = w(\eta) \tag{A.4.16}$$

The general solution of this equation is

$$
\bar{g}(\eta) = \begin{cases} +\dfrac{2}{N_o}\displaystyle\int_0^{\eta} W(y)\ \exp\left(\dfrac{2b\gamma}{N_o}(\eta-y)\right) dy + c\,\exp\left(\dfrac{2b\gamma}{N_o}\eta\right) + c' , & \eta \geq 0 \\ -\dfrac{2}{N_o}\displaystyle\int_{\eta}^{o} W(y)\ \exp\left(\dfrac{-2b\gamma}{N_o}(\eta-y)\right) dy + c_1\,\exp\left(-\dfrac{2b\gamma}{N_o}\eta\right) + c'_1 & \eta \leq 0 \end{cases}
$$

(A. 4.17)

Since $W(e)$ is even, $w(e)$ must be odd, and also $\bar{g}(\eta)$ must be odd. Since $\bar{g}(\eta)$ is also continuous at $\eta = 0$, it follows that $\bar{g}(0) = 0 = = c + c' = c_1 + c_1'$. From the fact that $\dfrac{d\bar{g}(\eta)}{d\,\eta}$ is continuous at $\eta = 0$, it follows that $c = - c_1$. Now the boundary conditions on $\bar{g}(\eta)$ do not allow it to go to ∞ exponentially fast, so one must have

$$
c = - \frac{2}{N_o} \int_o^{\infty} W(y)\ \exp\left(- \frac{2b\gamma}{N_o} y\right) dy \qquad \text{(A.4.18)}
$$

and

$$
c' = \frac{2}{N_o} \int_{-\infty}^{o} W(y)\ \exp\left(\frac{2b\gamma}{N_o} y\right) dy \qquad \text{(A.4.19)}
$$

By using all these facts, and manipulating expression (A. 4.17) somewhat, it can be shown that the function μ_∞, in which the main interest is, and which is given by

$$
\mu_\infty (x, z, t) = - b\,\bar{g}(x-z) \qquad \text{(A.4.20)}
$$

can be expressed as

$$\mu_\infty(x,z,t) = \begin{cases} \dfrac{2b}{N_o} \displaystyle\int_0^\infty [W(x-z+y) - W(y)] \exp\left(-\frac{2b\gamma}{N_o} y\right) dy, & x-z \geq 0 \\ -\dfrac{2b}{N_o} \displaystyle\int_{-\infty}^0 [W(x-z+y) - W(y)] \exp\left(+\frac{2b\gamma}{N_o} y\right) dy, & x-z \leq 0 \end{cases} \tag{A.4.21}$$

It still remains to verify that $\mu_T(x, z, t)$ actually converges to this function in the sense required by Theorems 4 and 5. Since solutions of partial differential equations of the type of (A.4.14) converge exponentially, there should not be any problem about this, however.

It finally can be verified that, indeed,

$$\mu_\infty(x,z,t) \begin{cases} \geq 0 \text{ wherever } F(x,z,t) = -\gamma \\ \leq 0 \text{ wherever } F(x,z,t) = +\gamma \end{cases} \tag{A.4.22}$$

so that it can be concluded from Theorem 5 that the controller which is considered is optimal.

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