STATE VARIABLES AND FEEDBACK THEORY

by

E. S. Kuh

Memorandum No. ERL-M232

1 December 1967

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

STATE VARIABLES AND FEEDBACK THEORY*

E. S. Kuh

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California
Berkeley, California

ABSTRACT

The state-space characterization of a linear time-invariant system can be viewed in terms of a general multiple loop feedback configuration. The return difference matrix and the null return difference matrix with respect to the $\underset{\sim}{A}$ matrix are derived and related to the poles and zeros of the transfer function. A useful formula for sensitivity of the transfer function with respect to an element of the $\underset{\sim}{A}$ matrix is also obtained.

1. INTRODUCTION

Bodes' feedback theory for single-loop systems was first generalized to the multiple-loop case by Sandberg [1, 2]. Further extension in terms of the return difference matrix and sensitivity was given by Kuh [3, 4]. The relation between general feedback theory and the state-space characterization of linear systems was first suggested by Kalman. He pointed out that the degree of a rational matrix which was crucial in its realization in terms of state-space characterization is related to the degree of a feedback system [5].

The present paper is intended to bring together further the general multiple-loop feedback theory and the state-space representation of linear time-invariant systems. The matrix signal flow graph is used to calculate and interpret the return difference matrix and the null return difference matrix. Based on these results we then derive a useful formula on sensitivity for the transfer function with respect to an element of the matrix $\underset{\sim}{A}$.

---

* The research reported herein was supported by the National Science Foundation under Grant GK-716 and the Joint Services Electronics Program (U.S. Army, U.S. Navy and U.S. Air Force) under Grant AF-AFOSR-139-67.

For simplicity, we restrict our study to single-input, single-output linear time-invariant systems. The state-space representation is

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{b}u \\ y &= \underset{\sim}{c}^t\,\underset{\sim}{x} + du \end{aligned} \tag{1}$$

where $\underset{\sim}{x}$, u and y are the state vector, the input and the output, respectively. The matrix $\underset{\sim}{A}$, the vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$, and the scalar d are the state-space parameters which characterize the system. The transfer function is

$$w(s) = \frac{\hat{y}}{\hat{u}} = d + \underset{\sim}{c}^t(s\underset{\sim}{1} - \underset{\sim}{A})^{-1}\underset{\sim}{b} \tag{2}$$

where s is the complex frequency variable, and the simbol hat (^) is used to denote the Laplace transform. The matrix signal flow graph representation of (1) is shown in Fig. 1, where the feedback loop is clearly indicated.

2. RETURN DIFFERENCE MATRICES

In feedback theory we always focus our attention to a particular entity in the system which is of special interest. In the signal flow graph of Fig. 1 we choose the entity to be the branch matrix $\underset{\sim}{A}$. The return difference matrix for the branch $\underset{\sim}{A}$, denoted by $\underset{\sim}{F}(\underset{\sim}{A})$, can be introduced as follows: Setting the input $\hat{u}$ zero and considering only the feedback loop, we break the loop at the input of the branch $\underset{\sim}{A}$ as shown in Fig. 2. We apply a vector signal $\underset{\sim}{g}$ and calculate the returned signal $\underset{\sim}{h}$. Clearly

$$\underset{\sim}{h} = \frac{1}{s}\underset{\sim}{A}\,\underset{\sim}{g} \tag{3}$$

The returned difference matrix $\underset{\sim}{F}(\underset{\sim}{A})$ is defined in terms of the difference between $\underset{\sim}{g}$ and $\underset{\sim}{h}$:

$$\underset{\sim}{F}(\underset{\sim}{A})\,\underset{\sim}{g} = \underset{\sim}{g} - \underset{\sim}{h} \tag{4}$$

From (3), we obtain

$$\underset{\sim}{F}(\underset{\sim}{A}) = \underset{\sim}{1} - \frac{1}{s}\underset{\sim}{A} = \frac{1}{s}(s\,\underset{\sim}{1} - \underset{\sim}{A}) \tag{5}$$

and

$$\det \underset{\sim}{F}(\underset{\sim}{A}) = \frac{Q(s)}{s^n} \tag{6}$$

where $Q(s)$ is the characteristic polynomial of the matrix $\underset{\sim}{A}$, and n is the order of $\underset{\sim}{A}$.

Next we wish to derive the null return difference matrix. With reference to Fig. 3, we again open the loop at the input to the branch $\underset{\sim}{A}$ and feed in a vector signal $\underset{\sim}{g}$. In addition, we apply a special input $\hat{u}$ such that the output $\hat{y}$ is identically zero. Expressing $\hat{y}$ in terms of $\hat{u}$ and $\underset{\sim}{g}$ in Fig. 3, we have

$$\hat{y} = d\,\hat{u} + \frac{1}{s}\underset{\sim}{c}^t(\underset{\sim}{A}\underset{\sim}{g} + \underset{\sim}{b}\hat{u}) = 0 \tag{7}$$

From (7), we obtain

$$\hat{u} = \frac{-\underset{\sim}{c}^t\underset{\sim}{A}\underset{\sim}{g}}{sd + \underset{\sim}{c}^t\underset{\sim}{b}}$$

The returned signal $\underset{\sim}{h}$ with the presence of both $\hat{u}$ and $\underset{\sim}{g}$, is

$$\begin{aligned}\underset{\sim}{h} &= \frac{1}{s}(\underset{\sim}{A}\underset{\sim}{g} + \underset{\sim}{b}\hat{u}) \\ &= \frac{1}{s}\Big(\underset{\sim}{1} - \frac{\underset{\sim}{b}\underset{\sim}{c}^t}{sd + \underset{\sim}{c}^t\underset{\sim}{b}}\Big)\underset{\sim}{A}\underset{\sim}{g}\end{aligned} \tag{8}$$

We define the null return difference matrix $\underset{\sim}{F}^o(\underset{\sim}{A})$ as in Eq. (4) in terms of the difference of the returned signal h in Eq. (8) and the signal $\underset{\sim}{g}$. Thus, from (8), we have

$$\begin{aligned}\underset{\sim}{F}^o(\underset{\sim}{A}) &= \underset{\sim}{1} - \frac{1}{s}\Big(\underset{\sim}{1} - \frac{\underset{\sim}{b}\underset{\sim}{c}^t}{sd + \underset{\sim}{c}^t\underset{\sim}{b}}\Big)\underset{\sim}{A} \\ &\triangleq \underset{\sim}{1} - \frac{1}{s}\underset{\sim}{A}^o = \frac{1}{s}(s\,\underset{\sim}{1} - \underset{\sim}{A}^o)\end{aligned} \tag{9}$$

where

$$\underset{\sim}{A}^o \triangleq \Big(\underset{\sim}{1} - \frac{\underset{\sim}{b}\underset{\sim}{c}^t}{sd + \underset{\sim}{c}^t\underset{\sim}{b}}\Big)\underset{\sim}{A} \tag{10}$$

Now we are in a position to introduce the following theorem which represents a generlization of Blackmen's impedance formula

$$w(\underset{\sim}{A}) = w(\underset{\sim}{0})\,\frac{\det\underset{\sim}{F}^o(\underset{\sim}{A})}{\det\underset{\sim}{F}(\underset{\sim}{A})} \tag{11}$$

where $w(\underset{\sim}{0})$ is the transfer function under the condition that branch $\underset{\sim}{A}$ is zero. From Eq. (2) or from the signal flow graph of Fig. 1, we have

$$w(\underset{\sim}{0}) = d + \frac{\underset{\sim}{c}^t \underset{\sim}{b}}{s} = \frac{ds + \underset{\sim}{c}^t \underset{\sim}{b}}{s} \tag{12}$$

The proof of the theorem is straight forward and is omitted; it depends on the determinant identity

$$\det(\underset{\sim}{1} + \underset{\sim}{J}\underset{\sim}{G}) = \det(\underset{\sim}{1} + \underset{\sim}{G}\underset{\sim}{J}) \tag{13}$$

Eq. (11) has interesting interpretations. Consider the case d = 0,

$$\underset{\sim}{A}^o = \left(\underset{\sim}{1} - \frac{\underset{\sim}{b}\underset{\sim}{c}^t}{\underset{\sim}{c}^t \underset{\sim}{b}} \right) \underset{\sim}{A} \tag{14}$$

is a constant matrix. Thus in Eq. (9) the determinant of the null return difference matrix can be written as

$$\det \underset{\sim}{F}^o(\underset{\sim}{A}) = \frac{P(s)}{s^n} \tag{15}$$

where P(s) is the characteristic polynomial of the matrix $\underset{\sim}{A}^o$. The theorem as expressed by Eq. (11) becomes

$$w(\underset{\sim}{A}) = \frac{\underset{\sim}{c}^t \underset{\sim}{b}}{s} \quad \frac{P(s)}{Q(s)} \tag{16}$$

where P(s), the characteristic polynomial of $\underset{\sim}{A}^o$ in (14), gives the zero of the transfer function and Q(s), the characteristic polynomial of $\underset{\sim}{A}$, gives the poles of the transfer function †. These results check with that of Brockett which he obtained based on the concept of the inverse system [6].

---

†For the case $d \neq 0$, it is possible to derive an alternate formula by considering the branch $\frac{1}{s}\underset{\sim}{1}$ in the signal flow graph rather than the branch $\underset{\sim}{A}$ to be the entity of interest. In this case

$$w = d \, \frac{\det\left[\underset{\sim}{1} - \frac{1}{s}\left(\underset{\sim}{A} - \frac{\underset{\sim}{b}\underset{\sim}{c}^t}{d}\right)\right]}{\det\left[\underset{\sim}{1} - \frac{1}{s}\underset{\sim}{A}\right]}$$

3. RETURN DIFFERENCES FOR A GENERAL REFERENCE

The purpose of this section is to study the situation when only a portion of the matrix $\underset{\sim}{A}$ is of interest. This is the case if we are interested in the sensitivity of the system with respect to, say, an element a_{ij} of the matrix $\underset{\sim}{A}$. Thus we can decompose the matrix into two parts:

$$\underset{\sim}{A} = \underset{\sim}{A}' + \underset{\sim}{K} \tag{17}$$

where $\underset{\sim}{A}'$ represents the branch of interest and $\underset{\sim}{K}$ is called the reference matrix. Typically $\underset{\sim}{A}'$ may contain a single nonzero element, a_{ij}, then $\underset{\sim}{K}$ is the matrix $\underset{\sim}{A}$ under the condition $a_{ij} = 0$. We write

$$\underset{\sim}{K} = \underset{\sim}{A}\big|_{a_{ij}} = 0 \tag{18}$$

In Fig. 4 we redraw the signal flow graph of Fig. 1 but we split the branch $\underset{\sim}{A}$ into $\underset{\sim}{A}'$ and $\underset{\sim}{K}$. For ease in further reduction we insert two unity branches at the input and the output of the branch $\underset{\sim}{A}'$. Since we are interested now in the branch $\underset{\sim}{A}'$, we can redraw the signal flow graph of Fig. 4 by combining the branch $\underset{\sim}{K}$ and the branch $\frac{1}{s}\underset{\sim}{1}$ as shown in Fig. 5. The combined branch is $\underset{\sim}{J} = (s\,\underset{\sim}{1} - \underset{\sim}{K})^{-1}$. We may now use the signal flow graph of Fig. 5 to introduce the return difference matrix and the null return difference matrix by opening the feedback loop at the input to $\underset{\sim}{A}'$. The return difference matrix so obtained is clearly $\underset{\sim}{1} - \underset{\sim}{J}\underset{\sim}{A}' = \underset{\sim}{1} = (s\underset{\sim}{1} - \underset{\sim}{K})^{-1}\underset{\sim}{A}'$ and is called, by difinition, the return difference matrix with respect to the branch $\underset{\sim}{A}$ for the general reference $\underset{\sim}{K}$. We use the following notation

$$\underset{\sim}{F}_{\underset{\sim}{K}}(\underset{\sim}{A}) = \underset{\sim}{1} - (s\,\underset{\sim}{1} - \underset{\sim}{K})^{-1}\underset{\sim}{A}' \tag{19}$$

Clearly, if $\underset{\sim}{K}$ is zero, then (19) is reduced to the original return difference matrix $\underset{\sim}{F}(\underset{\sim}{A})$. It is also useful to point out that (19) can be written as

$$\begin{aligned}\underset{\sim}{F}_{\underset{\sim}{K}}(\underset{\sim}{A}) &= (s\,\underset{\sim}{1} - \underset{\sim}{K})^{-1}(s\,\underset{\sim}{1} - \underset{\sim}{K} - \underset{\sim}{A}') \\ &= (s\,\underset{\sim}{1} - \underset{\sim}{K})^{-1}(s\,\underset{\sim}{1} - \underset{\sim}{A}) \\ &= \underset{\sim}{F}(\underset{\sim}{K})^{-1}\,\underset{\sim}{F}(\underset{\sim}{A})\end{aligned} \tag{20}$$

where

$$\underset{\sim}{F}(\underset{\sim}{K}) = \underset{\sim}{1} - \frac{\underset{\sim}{K}}{s} \tag{21}$$

Similarly, using the signal flow graph of Fig. 4, we can introduce the null return difference matrix. Under such conditions the null return difference matrix is called the null return difference matrix with respect to the branch $\underset{\sim}{A}$ for the general reference $\underset{\sim}{K}$ and is denoted by

$F^o_K(A)$. Similar to the derivation of Eq. (9), we find

$$F^o_K(A) = 1 - \left[1 - \frac{(s1-K)^{-1} b c^t}{d + c^t (s1-K)^{-1} b} \right] (s1-K)^{-1} A' \tag{22}$$

It is straightforward to show that

$$\det F^o_K(A) = \frac{\det F^o(A)}{\det F^o(K)} \tag{23}$$

where

$$F^o(K) \triangleq 1 - \frac{1}{s} K^o = 1 - \frac{1}{s}\left(1 - \frac{b c^t}{ds + c^t b}\right) K \tag{24}$$

as in Eqs. (9) and (10).

4. SENSITIVITY

To obtain the sensitivity of the transfer function w with respect to an element a_{ij} of the matrix A, we recall the formula [4]

$$S^w_{a_{ij}} = \frac{1}{f(a_{ij})} - \frac{1}{f^o(a_{ij})} \tag{25}$$

where $f(a_{ij})$ is the sclalr return difference and $f^o(a_{ij})$ is the scalar null return difference with respect to the element a_{ij}. In the previous section we mentioned that we would choose A' to be the matrix with the only nonzero term a_{ij} in the i-th row and the j-th column. Because of the simplicity of the form of A', it is easily recognized that

$$\det F_K(A) = f(a_{ij}) \tag{26}$$

and

$$\det F^o_K(A) = f^o(a_{ij}) \tag{27}$$

Substituting (26) and (27) in (25) and using the formulas (20) and (23), and the fact $K = A|_{a_{ij}=0}$, we obtain

$$S^w_{a_{ij}} = \frac{\det F(A)|_{a_{ij}=0}}{\det F(A)} - \frac{\det F^o(A)|_{a_{ij}=0}}{\det F^o(A)} \tag{28}$$

This formula gives the sensitivities for the transfer function with respect to all elements of the matrix A. It is only necessary to calculate the determinants of the return difference matrix and the null return difference matrix of the system under the nominal condition and under the condition $a_{ij}=0$ to obtain the sensitivity of the transfer function with respect to a_{ij}.

5. EXAMPLE

Let the single-input single-output system be given by

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} \underset{\sim}{x}$$

Let us calculate the sensitivity of the system with respect to the term $a_{21} = -1$. For convenience we write the matrix $\underset{\sim}{A}$ in terms of the element a_{21} as

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 1 \\ a_{21} & 2 \end{pmatrix}$$

$$\underset{\sim}{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \underset{\sim}{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The following calculation is easily checked:

$$\underset{\sim}{F}(\underset{\sim}{A}) = \frac{1}{s} \begin{pmatrix} s-1 & -1 \\ -a_{21} & s-2 \end{pmatrix}$$

$$\det \underset{\sim}{F}(\underset{\sim}{A}) = \frac{1}{s^2}(s^2 - 3s + 2 - a_{21})$$

$$\underset{\sim}{A}^{o} = \left(\underset{\sim}{1} - \frac{\underset{\sim}{b}\underset{\sim}{c}^t}{\underset{\sim}{c}^t\underset{\sim}{b}} \right) \underset{\sim}{A} = \frac{1}{2} \begin{pmatrix} 1-a_{21} & -1 \\ -1+a_{21} & 1 \end{pmatrix}$$

$$\underset{\sim}{F}^{o}(\underset{\sim}{A}) = \frac{1}{s} \begin{pmatrix} s - \frac{1-a_{21}}{2} & \frac{1}{2} \\ \frac{1-a_{21}}{2} & s - \frac{1}{2} \end{pmatrix}$$

$$\det \underset{\sim}{F}^{o}(\underset{\sim}{A}) = \frac{1}{s} \left(s - \frac{2-a_{21}}{2}\right)$$

Thus

$$S^{w}_{a_{21}} = \frac{\det \underset{\sim}{F}(\underset{\sim}{A})\,|_{a_{21}=0}}{\det \underset{\sim}{F}(\underset{\sim}{A})} - \frac{\det \underset{\sim}{F}^{o}(\underset{\sim}{A})\,|_{a_{21}=0}}{\det \underset{\sim}{F}^{o}(\underset{\sim}{A})}$$

$$= \frac{s^2 - 3s + 2}{s^2 - 3s + 3} - \frac{s - 1}{s - \frac{3}{2}}$$

We can also use the information of the determinants to write the transfer function immediately. From (12) we have

$$w(\underset{\sim}{0}) = \frac{\underset{\sim}{c}^t \underset{\sim}{b}}{s} = \frac{2}{s}$$

Thus from (11), we have

$$w(\underset{\sim}{A}) = \frac{2}{s} \quad \frac{\frac{1}{s}\left(s - \frac{2-a_{21}}{2}\right)}{\frac{1}{s^2}\left(s^2 - 3s + 2 - a_{21}\right)} = \frac{2s-3}{s^2-3s+3}$$

6. CONCLUSION

In this paper we have employed the feedback theory to the state equations. We have found some siginificances of the return difference matrix and the null return difference matrices. In particular, the determinants contain information of the nonzero poles and zeros of the transfer function and the sensitivities.

7. REFERENCES

1. Bode, H. W., Network Analysis and Feedback Amplifier Design, D. Van Nostrand, New York, 1945.

2. Sandberg, I. W., "Linear Multi-Loop Feedback Systems," Bell System Tech. J., Vol. 42, No. 2, pp. 355-382, 1963.

3. Kuh, E. S., "Some Results in Linear Multiple-Loop Feedback Systems," Proc. Allerton Conf. on Circuit and System Theory, Vol. 1, pp. 471-487, 1963.

4. Kuh, E. S., and Rohrer, R. A., Theory of Linear Active Networks, Holden Day, San Francisco, 1967.

5. Kalman, R. E., "Irreducible Realizations and The Degree of a Matrix of Rational Functions," SIAM J. Vol. 13, No. 2, pp. 520-544, June 1965.

6. Brockett, R. W., "Poles, Zeros, and Feedback: State Space Interpretation," IEEE Trans., Vol. AC-10, No. 2, pp. 129-135, April 1965.

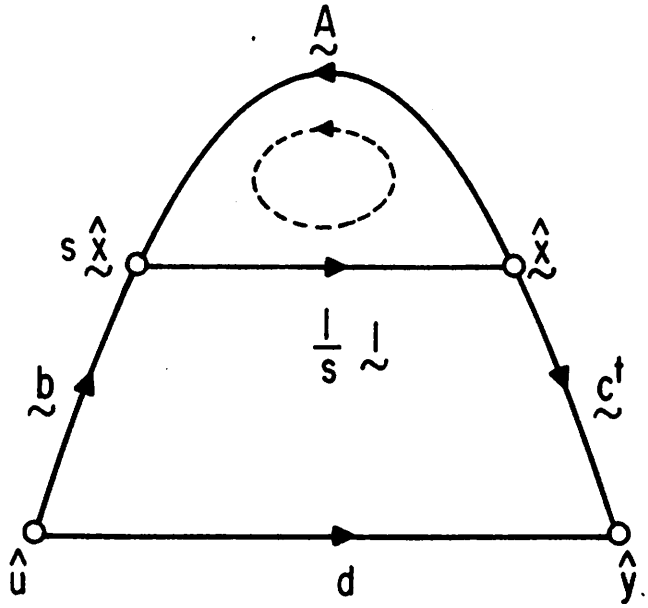


Fig. 1. Matrix signal flow graph representation of the state-space characterization of linear time-invariant system.

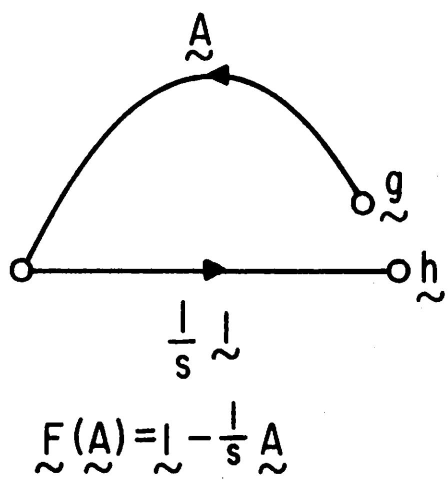
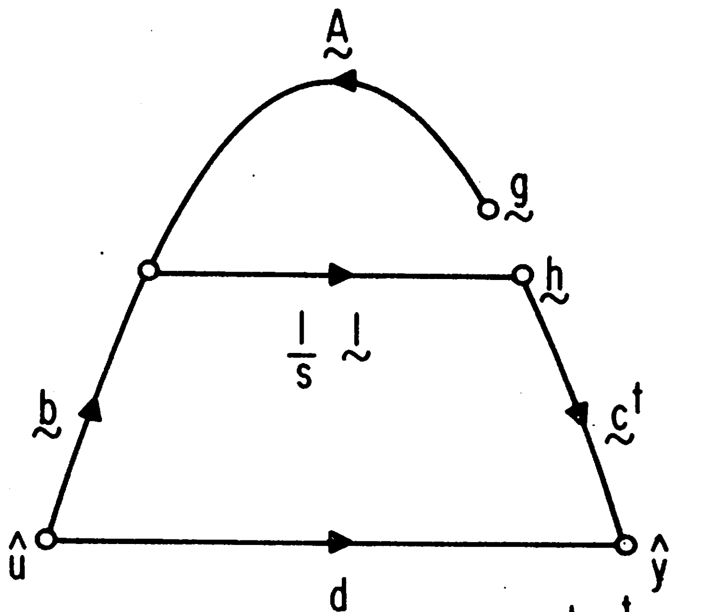


Fig. 2. Interpretation of the return difference matrix:

$$F(A)\,g = g - h$$



$$\underset{\sim}{F}^{0}(\underset{\sim}{A}) = \underset{\sim}{1} - \frac{1}{s}\underset{\sim}{A}^{0} = \underset{\sim}{1} - \frac{1}{s}\left(\underset{\sim}{1} - \frac{\underset{\sim}{b}\,\underset{\sim}{c}^{t}}{sd + \underset{\sim}{c}^{t}\underset{\sim}{b}}\right)\underset{\sim}{A}$$

Fig. 3. Interpretation of the null return difference matrix: $\underset{\sim}{F}^{o}(\underset{\sim}{A})\,\underset{\sim}{g} = \underset{\sim}{g} - \underset{\sim}{h}$ under the condition that $\hat{u}$ is adjusted such that $\hat{y} = 0$.

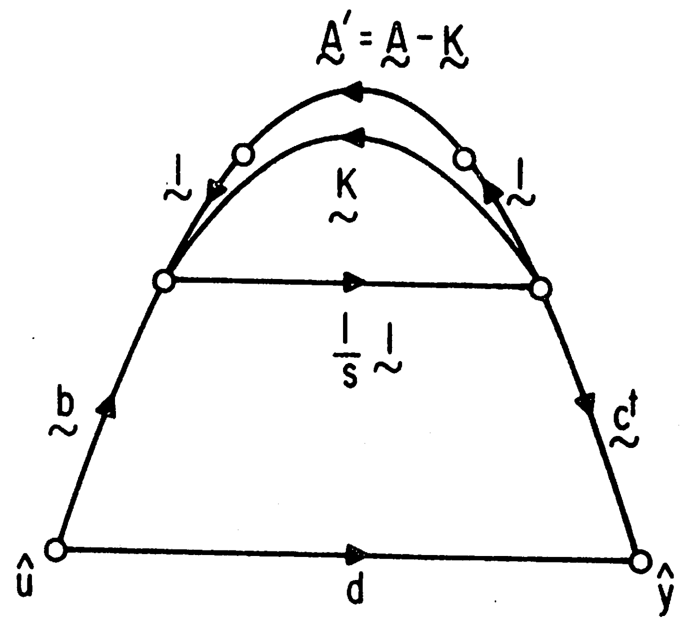


Fig. 4. The branch $\underset{\sim}{A}$ is splitted into $\underset{\sim}{A}'$ and $\underset{\sim}{K}$.

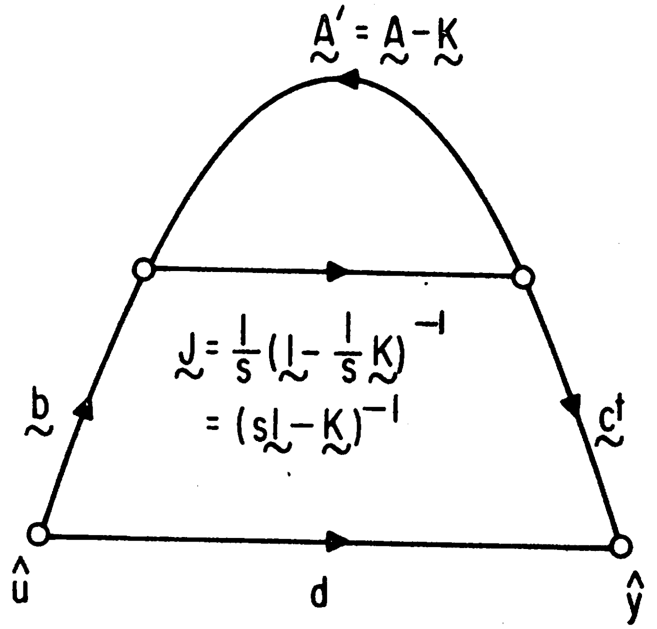


Fig. 5. The signal flow graph of Fig. 4 is redrawn to emphasize the effect of $\underset{\sim}{A}'$.