CONTROLLABILITY, OBSERVABILITY AND STABILITY OF LINEAR SYSTEMS

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1. Introduction. Of the many types of stability which may be defined for dynamical systems, at least two are of special importance when the systems are linear. These are bounded-input bounded-output stability and exponential stability, defined below. The aim of this paper is to establish an equivalence between these two types of stability for a large class of linear time-variable systems.

The basic system description we shall consider is an impulse response matrix H which maps the system inputs u into the system outputs y via the formula

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(1) $$y(t) = \int_{t_0}^{t} H(t, \tau)\, u(\tau)\, d\tau ,$$

when the system is in the zero state at time t_0. An alternate description is provided by a set of state equations of the form

(2a) $$\dot{x} = Ax + Bu$$

(2b) $$y = Cx ,$$

where A, B and C are time-variable matrices, and x is the state vector associated with the coordinate basis used in setting up (2). The dimensions of the vectors x, u and y will be taken to be n, r and m, respectively. The well-known [1] relationship between the two representations is given by

(3) $$H(t, \tau) = C(t)\, \Phi(t, \tau)\, B(\tau) , \quad t \geq \tau$$

where Φ is the transition matrix of the homogeneous part of (2a).

A system is termed bounded-input bounded-output (BIBO) stable[1], if for any input u satisfying

(4) $$\|u(t)\| \leq c_1 , \quad \text{for all } t$$

with the system initially in the zero-state, there is an associated constant $c_2 = c_2(c_1)$ such that the output y of the system satisfies

(5) $$\|y(t)\| \leq c_2, \quad \text{for all } t$$

($\|\cdot\|$ denotes the Euclidean norm)

This type of stability, whose physical significance is obvious, may be very simply characterized in terms of the impulse response matrix [2-5].

Theorem 1. The system described by (1) is BIBO stable if and only if there exists a positive constant c_3 such that

(6) $$\int_{-\infty}^{t} \|H(t,\tau)\| \, d\tau \leq c_3, \quad \text{for all } t$$

In contrast to BIBO stability, which emphasises the external characteristics of a system, exponential stability emphasises the internal characteristics. The realization (2) of the impulse response matrix H is termed exponentially stable [6] (uniformly asymptotically stable [5]) if there exist positive constants c_4 and c_5 such that for any x satisfying the homogeneous part of (2a),

(7) $$\|x(t)\| \leq c_4 \|x(t_0)\| \, e^{-c_5(t-t_0)}$$

for all t_0 and for all $t \geq t_0$. A well known [5] criterion for exponential stability is given by the following theorem.

Theorem 2. The system realization (2) is exponentially stable if and only if there exist positive constants c_6 and c_7 such that

$$(8) \qquad ||\Phi(t,\tau)|| \leq c_6 e^{-c_7(t-\tau)},$$

for all τ and for all $t \geq \tau$.

It is important to note that exponential stability is a property determined solely by the matrix A in (2) by virtue of (8), while it is clear from (3) and (6) that BIBO stability reflects properties of A, B and C.

In the time-invariant case (A, B and C constant matrices), relations between the two types of stability are well known. Exponential stability implies BIBO stability, while BIBO stability, together with complete controllability [1] and complete observability [1] implies exponential stability. Unfortunately, no such simple and analogous statements can be made in the time-variable case. Indeed, as was observed by Kalman [7], and discussed in the next section, it is impossible to conclude the existence of any sort of relation between the two types of stability without further constraints on the realizations (2).

For certain special classes of time-variable systems, relationships between the two types of stability have been established. The best known of these results is that of Perron [8], [2]:

<u>Theorem 3.</u> Consider the system realization (2a) with the state x as output. If the matrices A and B are bounded, and if B contains an $n \times n$ submatrix $\tilde{B}$ with the property that

(9) $$|\det B(t)| \geq d > 0, \quad \text{for all } t$$

then the system is BIBO stable if and only if it is exponentially stable.

Recently [9], [10] it has been shown that in several classes of systems the two types of stability are equivalent with the major constraint (9) replaced by more meaningful and less restrictive constraints.

In the present paper, a much more general class of such systems is described, and it is shown that all previous classes are special cases.

2. Stability Difficulties of Linear Time-Variable Systems. Given a separable (i.e., realizable [1]) impulse response matrix

(10) $$H(t, \tau) = \Psi(t)\,\Theta(\tau),$$

where Ψ and Θ are $m \times n$ and $n \times r$ matrices, respectively, it is immediately possible to construct a realization which is Lyapunov stable, but not exponentially stable [1]:

(11a) $$\dot{x} = \Theta u$$

(11b) $$y = \Psi x$$

Actually, more than this is true. It is in fact possible to construct a realization of the form (2) with an essentially arbitrary $n \times n$ A matrix: with T any fundamental matrix solution [11] of $\dot{z} = Az$, it is easily verified that

(12a) $$\dot{x} = Ax + T\Theta u$$

(12b) $$y = \Psi T^{-1} x$$

is a realization of the impulse response matrix (10).

Accordingly, it is impossible to conclude anything concerning the internal stability of particular realizations of a BIBO stable impulse response matrix without some restrictions on the class of admissible realizations. Such a set of restrictions, motivated by physical as well as mathematical considerations will now be examined.

If (2) is to represent a practical physical system (e.g., an analog computer) then an immediate restriction is that the elements of the coefficient matrices A, B, and C be bounded functions of time. Consequently, it will be assumed that a constant K exists such that for all t

(13a) $$||A(t)|| \leq K$$

(13b) $$||B(t)|| \leq K$$

and

(13c) $$||C(t)|| \leq K .$$

Such a system will be termed a <u>bounded realization</u> of the impulse response matrix H.

From the known results for time-invariant systems, it is also clear that some type of controllability and observability conditions must be imposed on (2). As shown by the following example [7], however, complete controllability and observability does not suffice even in bounded realizations.

Example. Consider the system realization

$$\dot{x} = x + g(t)\,u$$

$$y = g(t)\,x ,$$

where $g(t) = e^{-2|t|}$. It is certainly completely controllable, completely observable and bounded for all t, yet it is simultaneously BIBO stable, and unstable in the sense of Lyapunov.

A more stringent but physically reasonable degree of controllability and observability does provide a connection between the two types of stability. These constraints will be discussed in the following section.

3. Uniform Complete Contollability and Observability. The concepts of complete controllability and complete observability are by now well known, as is their importance in correctly formulating systems problems. Less well known, however, are the ideas of uniform complete controllability and uniform complete observability introduced by Kalman [12] in order to guarantee the solution of certain time-variable quadratic variational problems.

The system representation (2) is termed <u>uniformly completely controllable</u> if any two of the following three conditions hold for some $\delta_c > 0$ (any two imply the third):[2]

(14) $$\alpha_1(\delta_c) I \leq M(s - \delta_c, s) \leq \alpha_2(\delta_c) I$$

(15) $$\alpha_3(\delta_c) I \leq \Phi(s - \delta_c, s) M(s - \delta_c, s) \Phi(s - \delta_c, s) \leq \alpha_4(\delta_c)$$

(16) $$\| \Phi(t, \tau) \| \leq \alpha_5(|t - \tau|), \text{ for all } t, \tau ,$$

where

(17) $$M(s - \delta_c, s) = \int_{s - \delta_c}^{s} \Phi(s, t) B(t) B'(t) \Phi'(s, t)\, dt ,$$

and the α_i are positive constants depending only on δ_c and $|t - \tau|$, respectively.

As shown by Kalman [12], these conditions imply that it is always possible to transfer a state x to the origin or the origin to a state x, in a time δ_c independently of the starting time. Moreover, the energy required to effect such a transition can never become arbitrarily large, nor can it become arbitrarily small.

The criteria for uniform complete controllability greatly simply if only bounded realizations are under consideration.

First note that (13a) is a sufficient condition for (16) [12]. Thus, for bounded realizations, one need only consider condition (14).

Furthermore, the right hand side of the inequality (14) is always satisfied since (13 b) and (16) imply

$$||M(s-\delta_c, s)|| \leq \int_{s-\delta_c}^{s} ||\Phi(s,t)\, B(t)||^2\, dt$$

$$\leq \int_{s-\delta_c}^{s} \alpha_5^2(|s-t|)\, K^2\, dt \leq \delta_c K^2 \alpha_5^2(\delta_c).$$

which in turn implies

$$M(s-\delta_c, s) \leq \delta_c K^2 \alpha_5^2(\delta_c)\, I \;,$$

since the Euclidean norm of a matrix is equal to its maximum eigenvalue. Hence, we have the following:

<u>Lemma 1.</u> A bounded realization (2) is uniformly completely controllable if and only if there exists $\delta_c > 0$ such that

(18) $$M(s-\delta_c, s) \geq \alpha_1(\delta_c)\, I \;,$$

or equivalently,[3]

(19) $$\det M(s-\delta_c, s) \geq \alpha_6(\delta_c)\,.$$

If (2) is a bounded realization, uniform complete controllability can also be redefined in the following useful way.

Theorem 4. A bounded realization (2) is uniformly completely controllable if and only if there exists $\delta_c > 0$ such that for every state $\xi \in R^n$ and for any time s, there exists an input u defined on $(s - \delta_c, s)$ such that

(i) if $x(s - \delta_c) = 0$ then $x(s) = \xi$

and

(ii) $||u(t)|| \leq \gamma(\delta_c, \xi)$ for all $t \in (s - \delta_c, s)$,

where $\gamma(\delta_c, \xi)$ is a finite positive number.

Proof. If the system (2) is uniformly completely controllable, then the input

$$u(t) = B'(t)\Phi'(s,t)M^{-1}(s - \delta_c, s)\xi$$

will transfer the system from the zero state at time $s - \delta_c$ to the state ξ at time s. From (13b), (14) and (16) it is clear that a constant γ independent of s and t exists such that $||u(t)|| < \gamma$ for all $t \in (s - \delta_c, s)$.

The converse will be established by contradiction. If the system is not uniformly completely controllable, then Lemma 1 implies that for each $\delta > 0$ and for any $\alpha > 0$ there is vector $\lambda \in R^n$, with $||\lambda|| = 1$, such that for some s, $\lambda' M(s - \delta, s)\lambda < \alpha$

or equivalently, for some s ,

$$(20) \qquad \int_{s-\delta}^{s} ||\lambda' \Phi(s,\tau) B(\tau)||^2 d\tau < \alpha$$

Suppose that a bounded control u exists which transfers the zero state at time $s - \delta$ to the state λ at time s. Then,

$$\lambda = \int_{s-\delta}^{s} \Phi(s,\tau) B(\tau) u(\tau) d\tau ,$$

which implies

$$||\lambda||^2 \leq \int_{s-\delta}^{s} ||\lambda' \Phi(s,\tau) B(\tau) u(\tau)|| d\tau ,$$

and by the Schwarz inequality,

$$(21) \qquad ||\lambda||^2 \leq \left[\int_{s-\delta}^{s} ||\lambda' \Phi(s,\tau) B(\tau)||^2 d\tau\right]^{1/2} \left[\int_{s-\delta}^{s} ||u(\tau)||^2 d\tau\right]^{1/2}$$

If $||u(t)|| < \gamma$ for all $t \in (s-\delta, s)$ and for all s then (20) and (21) imply that for some s, $\gamma \sqrt{\alpha\delta} \geq 1$, a contradiction, since α can be made arbitrarily small. This completes the proof.

The dual [1] of uniform complete controllability is uniform complete observability and it has a similar definition in terms of the matrix

$$(22) \qquad W(s-\delta_0, s) = \int_{s-\delta_0}^{s} \Phi'(t, s-\delta_0) C'(t) C(t) \Phi(t, s-\delta_0) dt .$$

The realization (2) is termed uniformly completely observable if any two of the following three conditions hold for some $\delta_0 > 0$ (again, any two imply the third):

(23) $$\beta_1(\delta_0) I \leq W(s-\delta_0, s) \leq \beta_2(\delta_0) I$$

(24) $$\beta_3(\delta_0) I \leq \Phi'(s-\delta_0, s) W(s-\delta_0, s) \Phi(s-\delta_0, s) \leq \beta_4(\delta_0) I$$

(25) $$\| \Phi(t,\tau) \| \leq \beta_5(|t-\tau|), \quad \text{for} \quad t,\tau$$

By appealing to the duality theorem of Kalman [1], results similar to Lemma 1 and Theorem 4 may be obtained and need not be stated explicitly here.

It will be shown in the following sections that under the constraints of uniform complete controllability and uniform complete observability, BIBO and exponential stability are equivalent in bounded realizations.

4. Application of Uniform Complete Controllability. If the system (2a) is BIBO stable with x considered as the output, it will be said that the system is bounded-input, bouded-state (BIBS) stable. The following theorem establishes a connection between BIBS stability and exponential stability in uniformly completely controllable, bounded realizations.

Theorem 5. If (2a) is bounded and uniformly completely controllable, then it is BIBS stable if and only if it is exponentially stable.

Proof. It is well known, and straightforward to show [5] that if B is bounded, then exponential stability implies BIBS stability.

Let λ be any unit norm vector in R^n. It follows from Theorem 4 that if (2a) is uniformly completely controllable and bounded there exists a $\delta_c > 0$ such that for all s, an input u exists which satisfies

$$(26) \qquad \lambda = \int_{s-\delta_c}^{s} \Phi(s,\tau)\, B(\tau)\, u(\tau)\, d\tau ,$$

and $||u(\tau)|| \leq \gamma_1(\delta_c)$ for $t \in (s-\delta_c, s)$. Multiplying both sides of (26) by $\Phi(t,s)$ yields the inequality

$$(27) \qquad ||\Phi(t,s)\lambda|| \leq \gamma_1 \int_{s-\delta_c}^{s} ||\Phi(t,\tau)\, B(\tau)||\, d\tau .$$

Integrating both sides of (27) from an arbitrary t_0 to t then gives the relationship

$$(28) \qquad \int_{t_0}^{t} ||\Phi(t,s)\,\lambda||\, ds \leq \gamma_1 \int_{t_0}^{t} \left\{ \int_{s-\delta_0}^{s} ||\Phi(t,\tau)\, B(\tau)||\, d\tau \right\} ds$$

Letting $r = \tau - s + \delta_c$, and interchanging the order of integration on the right hand side of (28) it is clear that

(29) $$\int_{t_0}^{t} ||\Phi(t,s)\lambda||\, ds \le \gamma_1 \int_0^{\delta}\left\{\int_{t_0}^{t} ||\Phi(t, s+r-\delta_c)\, B(s+r-\delta_c)||\, ds\right\} dr$$

$$= \gamma \int_0^{\delta_c}\left\{\int_{t_0+r-\delta_c}^{t+r-\delta_c} ||\Phi(t,\tau)\, B(\tau)||\, d\tau\right\} dr\ .$$

For $0 \le r \le \delta_c$,

(30) $$\int_{t_0+r-\delta_c}^{t+r-\delta_c} ||\Phi(t,\tau)\, B(\tau)||\, d\tau \le \int_{t_0+r-\delta_c}^{t} ||\Phi(t,\tau)\, B(\tau)||\, d\tau\ ,$$

and since (2a) is assumed BIBS stable, there exists (by Theorem 1) a finite constant γ_2 such that

(31) $$\int_{-\infty}^{t} ||\Phi(t,\tau)\, B(\tau)||\, d\tau \le \gamma_2, \text{ for all } t$$

Equations (29) - (31) imply that for all t,

(32) $$\int_{-\infty}^{t} ||\Phi(t,s)\lambda||\, ds \le \gamma_1 \gamma_2 \delta_c\ .$$

Hence, if the supremum of (32) over all $||\lambda|| = 1$ is taken, the bound

(33) $$\int_{-\infty}^{t} ||\Phi(t,s)||\, ds \le \gamma_1 \gamma_2 \delta_c$$

is obtained. But (33) together with the bound (13a) on A suffices to imply exponential stability [8], [2]. This completes the proof.

<u>5. Application of Uniform Complete Observability and Main Result.</u> To complement the result of the previous section, it will now be shown that BIBO stability is equivalent to BIBS stability in uniformly completely observable, bounded realizations.

<u>Theorem 6.</u> If (2) is bounded and uniformly completely observable, then it is BIBO stable if and only if it is BIBS stable.

<u>Proof.</u> Suppose that BIBO stability does not imply BIBS stability, i.e., there exists a bounded input u which produces both a bounded output and an unbounded state. Then, corresponding to an arbitrary positive number N, there is a value of time $s - \delta_0$ for which

$$\|x(s - \delta_0)\| > N . \tag{34}$$

Set u equal to zero over the interval $(s - \delta_0, s)$. Then the output y over this interval is given by

$$y(t) = C(t)\Phi(t, s - \delta_0)x(s - \delta_0) .$$

Consequently, using (23) and (34)

$$\int_{s-\delta_0}^{s} y'(t)y(t)\,dt = x'(s - \delta_0)W(s - \delta_0, s)x(s - \delta_0) \geq \beta_1(\delta_0)N^2 .$$

Hence, at some point t in $(s-\delta_0, s)$

$$\|y(t)\| > \sqrt{\frac{\beta_1(\delta_0)}{\delta_0}}\, N . \tag{35}$$

Since N is arbitrary, while u is bounded,(35) contradicts the assumption of BIBO stability.

To prove the converse, it suffices to observe that in the class of systems under consideration, $y(t) = C(t)\, x(t)$ and $C(t)$ is a bounded matrix.

Following immediately from Theorems 5 and 6 is the main result:

Theorem 7. If (2) is bounded, uniformly completely controllable and uniformly completely observable, then it is BIBO stable if and only if it is exponentially stable.

A valid question at this point is whether the boundedness constraint of Theorem 7 is essential to the conclusion. It is clear that the constraint on the matrix A can be relaxed since (16) holds under somewhat weaker conditions [12] than (13a). However, as shown by the following example, the constraints on B and C are essential.

Example 2. Consider the system realization

$$\dot{x} = -x + u$$

$$y = g(t)\, x ,$$

where $g(t) = k$ for $t \in (k, k+\frac{1}{k})$ $(k=1, 2, \cdots)$ and is zero elsewhere. It is easily verified that this system is uniformly completely controllable and observable, yet it is simultaneously exponentially stable and BIBO unstable.

6. Classes of Uniformly Completely Controllable Systems.

In this section, it will be shown that several broad classes of system structures have the uniform complete controllability property. Included in this development are all classes of systems for which it has previously been established that BIBO and exponential stability are equivalent.

The first such class to be considered is that treated in Theorem 3.

Theorem 8. If the matrices A and B satisfy (13) and if B contains an $n \times n$ submatrix $\tilde{B}$ with the property that

$$|\det \tilde{B}(t)| > d > 0 \quad \text{for all } t \tag{36}$$

then (2a) is uniformly completely controllable.

Proof. Under the hypothesis of the theorem it is clear that

$$\det[\Phi(s,\tau)B(\tau)B'(\tau)\Phi'(s,\tau)] \geq d^2 [\det \Phi(s,\tau)]^2 \tag{37}$$

Also, (13a) implies that $|\mathrm{tr}\, A(t)| < K_1$ for all t. Using the relationship [5]

$$\det \Phi(s,\tau) = \exp \int_{\tau}^{s} \mathrm{tr}\, A(t)\, dt,$$

it can be shown that for all $\delta > 0$

$$(38) \qquad \det \Phi(s,\tau) \geq e^{-K_1 \delta}, \quad \text{for} \quad \tau \epsilon (s-\delta, s) .$$

Consequently, if λ is an arbitrary constant vector, (37) and (38) imply

$$\lambda' M(s-\delta, s) \lambda \geq \delta d^2 e^{-2K_1 \delta} ||\lambda||^2$$

Hence, by Lemma 1 the system is uniformly completely controllable.

Observe that Theorem (8) together with Theorem (5) implies Theorem (3).

Two classes of single-input systems (let $B = b$ in (2)) will now be considered. The first may be defined in terms of the controllability matrix [13]

$$Q_c = [p_0, p_1 \cdots p_{n-1}] ,$$

where

$$p_{k+1} = - A\, p_k + \dot{p}_k ; \; p_0 = b .$$

This matrix yields a sufficient criterion for uniform complete controllability which does not require calculation of the transition matrix.

Theorem 9: If (2a) is a bounded, single-input realization and Q_c is a Lyapunov transformation[4] [5], then the system is uniformly completely controllable.

Proof. Let λ be an arbitrary constant vector, and let $g(s,\tau) = \lambda'\Phi(s,\tau)b(\tau)$. Also, let

$$\overline{M}(s-\delta, s) = \int_{s-\delta}^{s} \Phi(s,\tau)Q_c(\tau)Q_c'(\tau)\Phi'(s,\tau)\,d\tau$$

It may be shown by a simple induction argument that

$$\frac{\partial^i}{\partial\tau^i} g(s,\tau) = \lambda'\Phi(s,\tau)\,p_i(\tau)$$

so that

$$\lambda'\overline{M}(s-\delta,s)\lambda = \sum_{i=0}^{n-1} \int_{s-\delta}^{s} \left[\frac{\partial^i}{\partial\tau^i} g(s,\tau)\right]^2 d\tau \tag{39}$$

It is also a simple matter to show [9] that for all s, each element of $\Phi(s,t)b(t)$, and hence $g(s,t)$, is a solution of the differential equation

$$z^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t)\, z^{(i)}(t) = 0\,, \tag{40}$$

where,

$$[a_0 a_1 \cdots a_{n-1}]' = -Q_c^{-1} p_n \tag{41}$$

By virtue of the assumptions on A and Q_c, the coefficients $a_i(t)$ are bounded for all t, so that the following lemma, which is a special case of a theorem proved in [9], is applicable.

<u>Lemma 2.</u> If for each s, $g(s,t)$ is a solution of an equation of the form (40) with bounded coefficients, then for each $\delta > 0$, there exists a positive constant K_1 such that

(42) $$\int_{s-\delta}^{s} \left[\frac{\partial^i}{\partial \tau^i} g(s,\tau) \right]^2 d\tau \leq K_1 \int_{s-\delta}^{s} g^2(s,\tau)\, d\tau \,, \text{ for } 1 \leq i \leq n.$$

for $1 \leq i \leq n$.

From (39) and (42), therefore, it follows that

(43) $$\lambda' M(s-\delta, s)\lambda \geq \frac{1}{nK_1} \lambda' \overline{M}(s-\delta, s)\lambda \,.$$

If Q_c is identified with $\widetilde{B}$ in Theorem 8 it is clear that a constant $K_2(\delta)$ exists such that

(44) $$\lambda' \overline{M}(s-\delta, s)\lambda \geq K_2 \, ||\lambda||^2 \,.$$

Thus, (43) and (44) imply

$$\lambda' M(s-\delta, s)\lambda \geq \frac{K_2}{nK_1} \, ||\lambda||^2 \,,$$

which completes the proof.

It should be noted that a canonical representation for the class of systems satisfying the criterion of Theorem 9 is

$$(45) \qquad A = -\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the coefficients a_i are given by (41), and can be obtained by the transformation of coordinates $z = Q_c^{-1} x$.

From Theorems 9 and 5 it follows that BIBS and exponential stability are equivalent in all systems having the canonical representation (45). (a result also established in [9])

Before deriving the second class of single-input, uniformly completely controllable systems it is convenient to establish the following lemma, which is a generalization of a result of Brockett's for constant systems [14].

Lemma 3. Uniform complete controllability in a bounded realization (2) is invariant under state-variable feedback of the form

$$(46) \qquad u(t) = G(t)\,x(t) + r(t)$$

where $||G(t)|| \leq K_1$ for all t, and r is the input to the closed loop system.

<u>Proof.</u> Let (2) be uniformly completely controllable. Then by Theorem 4 there is a $\delta > 0$ and an input u_1 which takes $x(s-\delta) = 0$ to $x(s) = \xi$, such that

$$(47) \qquad \|u_1(t)\| \leq \gamma(\delta, \xi)$$

for all $t \epsilon (s-\delta, s)$ and for all x. It is readily verified that if $r_1(t) = u_1(t) - G(t)x_1(t)$ is the input to the closed loop system, where x_1 is the trajectory in the open loop system due to u_1, then $z_1(s-\delta) = 0$ and $z_1(s) = \xi$, where z_1 is the trajectory of the closed loop system due to r_1 (in fact, $z_1(t) = x_1(t)$ for all $t \epsilon (s-\delta, s)$). Furthermore, for all $t \epsilon (s-\delta, s)$,

$$\|r_1(t)\| \leq \|u_1(t)\| + \|G(t)\| \int_{s-\delta}^{s} \|\Phi(t,\tau) B(\tau) u_1(\tau)\| \, d\tau .$$

Hence, by (13b), (16) and (47)

$$\|r_1(t)\| \leq \left(1 + K_1 K \int_{s-\delta}^{s} \|\Phi(t,\tau)\| \, d\tau \right) \gamma$$

$$\leq (1 + K_1 K \delta \alpha_5) \gamma .$$

It follows from Theorem 4, therefore, that the closed loop system is uniformly completely controllable.

The converse follows by a similar argument.

Theorem 10. The (phase-variable) canonical form

$$(48) \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where the coefficients $a_i(t)$ are bounded for all t, is uniformly completely controllable.

Proof. The proof follows immediately from Lemma 3 and the observation that (48) can be represented as a constant completely controllable system with state variable feedback of the form

$$u = -[a_0\, a_1\, a_2 \cdots a_{n-1}]\, x + r ,$$

Theorem 10 implies that BIBS and exponential stability are equivalent in systems represented in phase-variable canonical form (this result was established previously in [10]). It should be noted that any representation which can be transformed to this form via a Lyapunov transformation also has this property. A general method for calculating a transformation to phase-variable form was given in [15], and it is clear from the form of this transformation that with some additional constraints on the derivatives of the matrices A and b, the classes of systems considered in Theorems 9 and 10 are equivalent. Without such constraints, however, they are distinct.

An interesting corollary to Theorems 9 and 10 is the following result for n th order differential equations.

Corollary. Consider the equation

$$y^{(n)} + \sum_{i=0}^{n-1} a_i y^{(i)} = u \tag{49}$$

where the $a_i(t)$ are bounded for all $t (0 \leq i \leq n-1)$. The system represented by (49) is BIBO stable if and only if there exists positive constants c_1 and c_2 such that for any solution y of the homogeneous part of (49)

$$|y^{(i)}(t)| \leq c_1 ||\bar{y}(t_0)|| e^{-c_2(t-t_0)} \tag{50}$$

for all $t \geq t_0$, where $\bar{y} = [y \; y^{(1)} \cdots y^{(n-1)}]$.

Proof. If we let $x_i = y^{(i)}$ $(0 \leq i \leq n-1)$ then (50) has the state representation (48), with

$$y = [1 \quad 0 \cdots 0] x .$$

From Theorem (10), this representation is uniformly completely controllable and from the dual version of Theorem 10, it is uniformly completely observable. Hence, by Theorem 6, the two types of stability are equivalent.

A weaker version of this result was established by Kaplan ([16] Chapter 8, Theorem (25)), who showed that if the impulse response function of (49) is exponentially bounded then the system is stable in the sense (50).

Finally, we shall consider the class of periodic systems (A, B, C periodic with the same period). It appears to be known [17] to researchers in stability theory that BIBO and exponential stability are equivalent in completely controllable and observable periodic realizations. However, the authors are not aware of any proof of this result in the literature. A simple proof is provided by the following theorem which establishes an equivalence between complete and uniform complete controllability (observability) in periodic systems.

<u>Theorem 11.</u> If (2) is a periodic realization, then it is uniformly completely controllable (observable) if and only if it is completely controllable (observable).

<u>Proof.</u> If (2) is completely controllable, there must exist a finite $\sigma > 0$ such that $M(0,\sigma) \geq \epsilon I > 0$. Let k be a positive integer such that $kT > \sigma$, where T is the period of the matrices A, B and C. Clearly, for $s \in (kT, 2kT)$, $M(s-2kT, s) \geq \epsilon I$. It is easily verified, however, that $M(s-2kT, s)$ is periodic in s with period T. Hence, $M(s-2kT, s) \geq \epsilon I$

for all s. By Theorem 3, therefore, (2) is uniformly completely controllable. Since the converse is obviously true, this completes the proof.

In conclusion, we note that Theorems 9, 10 and 11 are applicable to the synthesis of impulse response matrices. Under appropriate conditions [18] H can be realized as a member of one of the classes discussed above. Thus the internal stability of the corresponding physical realizations is guaranteed, if H represents a BIBO stable system.

FOOTNOTES

1. The way in which this type of stability is defined here is also referred to as zero-state BIBO stability [5].

2. If A and B are symmetric matrices, $A > B$ $(A \geq B)$ means $A - B$ is positive (nonnegative) definite.

3. Equations (18) and (19) are equivalent because of the uniform bound on the maximum eigenvalue of $M(s - \delta_c, s)$.

4. For time-invariant systems, this condition on Q_c is equivalent to complete controllability.

REFERENCES

[1] R. E. Kalman, "Mathematical description of linear dynamical systems," J. SIAM Control, vol. 1, 1963, pp. 152-192.

[2] R. E. Kalman and J. E. Bertram, "Control system analysis and design via the 'second method' of Lyapunov. I. Continuous-time systems," J. Basic Engr., vol. 82 D, 1960, pp. 371-393.

[3] D. C. Youla, "On the stability of linear systems," IEEE Trans. on Circuit Theory, vol. CT-10, June 1963, pp. 276-279.

[4] C. A. Desoer and A. J. Thomasian, "A note on zero-state stability of linear systems," Proc. 1st Allerton Conf. on Circuit and System Theory, 1963, pp. 50-52.

[5] L. A. Zadeh and C. A. Desoer, Linear System Theory, McGraw-Hill Book Co., New York, 1963.

[6] N. P. Bhatia, "On exponential stability of linear differential systems," J. SIAM Control, vol. 2, 1965, pp. 181-191.

[7] R. E. Kalman, "On the stability of time-varying linear systems," IRE Trans. on Circuit Theory, vol. CT-9, 1962, pp. 420-422.

[8] O. Perron, "Die stabilitatsfrage ber differential gleichungen," Mathematische Zeitschrft, vol. 32, 1930, pp. 703-728.

[9] L. H. Haines and L. M. Silverman, "Internal and external stability of linear systems," ERL Tech. Memorandum M-204, University of California, Berkeley; March 1967.

[10] B. D. O. Anderson, "Stability properties of linear systems in phase-variable form," in preparation.

[11] E. A. Coddington and N. Levenson, Theory of Ordinary Differential Equations, McGraw-Hill Book Co., New York, 1955.

[12] R. E. Kalman, "Contributions to the theory of optimal control," Bol. Soc. Mat. Mex., 1960, pp. 102-119.

[13] L. M. Silverman and H. E. Meadows, "Controllability and observability in time-variable linear systems," J. SIAM Control, vol. 5, 1967, pp. 64-73.

[14] R. W. Brockett, "Poles, zeros and feedback; state space interpretation," IEEE Trans. on Automatic Control, vol. AC-10, 1965, pp. 129-134.

[15] L. M. Silverman, "Transformation of time-variable systems to canonical (phase-variable) form," IEEE Trans. on Automatic Control, vol. AC-11, 1966, pp. 300-303.

[16] W. Kaplan, Operational Methods for Linear Systems, Addison Wesley Book Co., Massachusetts, 1962.

[17] R. W. Brockett, "The status of stability theory for deterministic systems," IEEE Trans. on Automatic Control, vol. AC-11, 1966, pp. 596-606.

[18] L. M. Silverman, "Stable realization of impulse response matrices," Proc. IEEE International Convention, March 1967.