SOME RESULTS CONCERNING THE ZERO-CROSSINGS OF GAUSSIAN NOISE

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1. Introduction

Let $x(t)$ be a zero-mean stationary Gaussian process, with covariance function of the form

$$(1) \qquad Ex(t)\, x(t+\tau) = \rho(\tau) = 1 - \frac{\tau^2}{2} + \frac{a}{6}\, |\tau|^3 + 0(\tau^4)\,.$$

Let ξ be a random variable denoting the interval between two successive zeros of $x(t)$. The problem of finding the probability distribution of ξ is of considerable interest and remains largely unsolved. (For further references and a more detailed discussion, see Refs. 1 and 2.) In this paper we present some explicit results concerning a zero-mean Gaussian process with covariance function that is a special case of (1), $\left(a = \frac{4}{\sqrt{3}}\right)$.

Let $F(t) = \text{Prob}\,(\xi \le t)$ and $q(t) = \dfrac{dF(t)}{dt}$. The principal results of this paper are that for a zero-mean Gaussian process with covariance function given by

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(2) $$\rho(\tau) = \frac{3}{2}\, e^{-\frac{|\tau|}{\sqrt{3}}} \left(1 - \frac{1}{3}\, e^{-\frac{2}{\sqrt{3}}|\tau|}\right),$$

$F(t)$ and $q(t)$ can be expressed explicitly in terms of complete elliptic integrals. These results appear as (24) and (25) below.

It has been known for some time that for zero-mean Gaussian processes with covariance functions of the form given by (1), $q(0^+) = Ca$. Longuet-Higgins has given various bounds for C, the best ones being [3]

$$\frac{1.1556}{6} < C < \frac{1.158}{6}$$

The results of this paper suffice to show that in fact

(3) $$C = \left(\frac{37}{32}\right)\frac{1}{6} = \frac{1.15625}{6}$$

2. Some Preliminary Relationships

Let $x(t)$ be a zero-mean Gaussian process with covariance function given by (2). It is assumed that a separable version is being considered. Then $x(t)$ is almost surely differentiable, and we shall denote its derivative by $\dot{x}(t)$ $\left(\dot{x}(t) = \frac{dx(t)}{dt}\right)$. Now, let $\tau(y_0)$ be defined by

(4) $$\tau(y_0) = \min\{t; t > 0,\ x(t) = 0 \,|\, x(0) = 0,\ \dot{x}(0) = y_0\}$$

where the condition $x(0) = 0$ is understood to be in the horizontal window sense [4]. Now, let

(5) $$\varphi(y_0, t) = \text{Prob}\{\tau(y_0) > t\}$$

and

(6) $$p_h(y_0)\,dy_0 = \text{Prob}\,\{\dot{x}(0) \in (y_0, y_0 + dy_0) \mid x(0) = 0\}$$

In (6) the conditioning is again in the horizontal window sense. Then, $F(t)$ can be expressed as

(7) $$F(t) = 1 - \text{Prob}\,(\xi > t)$$

$$= 1 - \int_{-\infty}^{\infty} p_h(y_0)\,\varphi\,(y_0, t)\,dy_0\,.$$

Now, $p_h(y_0)$ can be derived as in Ref. 4. For the process being considered, we have

(8) $$p_h(y_0) = \frac{|y_0|}{2}\; e^{-\frac{1}{2} y_0^2}\,.$$

Therefore,

(9) $$F(t) = 1 - \int_0^{\infty} y_0\, e^{-\frac{1}{2} y_0^2}\,\varphi\,(y_0, t)\,dy_0\,,$$

where we have used the symmetry $\varphi(y_0, t) = \varphi\,(-y_0, t)$

3. A Representation of $x(t)$

Let $\eta(t)$ be a standard Brownian motion $(E\eta^2(t) = t)$. Define $z(t)$ by

(10) $$z(t) = \int_0^t \eta(s)\,ds, \qquad t \geq 0$$

The covariance function of $z(t)$ is given by

$$(11) \qquad R_z(s,t) = E\,z(s)\,z(t) = \frac{1}{2} s^2 t - \frac{1}{6} s^3, \qquad t \geq s$$

Therefore, the normalized covariance function is given by

$$(12) \qquad \rho_z(s,t) = \frac{R_z(s,t)}{\sqrt{R_z(s,s)\,R_z(t,t)}} = \frac{3}{2}\sqrt{\frac{s}{t}} - \frac{1}{2}\left(\frac{s}{t}\right)^{3/2}, \quad t \geq s.$$

As before, let $x(t)$ be a zero-mean Gaussian process with covariance function given by (2). Comparing (2) and (12), we see that $x(t)$ must have the same probability laws as $\sqrt{3}\, e^{-\frac{1}{\sqrt{3}}t}\, z\left(e^{\frac{2}{\sqrt{3}}t}\right)$. From (10) this means that $x(t)$ has the representation

$$(13) \qquad x(t) = \sqrt{3}\, e^{-\frac{1}{\sqrt{3}}t} \int_0^{e^{\frac{2}{\sqrt{3}}t}} \eta(s)\, ds,$$

where $\eta(s)$ is again a standard Brownian motion. Furthermore, we can rewrite (13) as

$$(14) \qquad x(t) = \sqrt{3}\, e^{-\frac{1}{\sqrt{3}}t} \int_0^1 \eta(s)\, ds + \sqrt{3}\, e^{-\frac{1}{\sqrt{3}}t} \int_1^{e^{\frac{2}{\sqrt{3}}t}} \eta(s)\, ds$$

$$= \sqrt{3}\, e^{-\frac{1}{\sqrt{3}}t} \int_0^1 \eta(s)\, ds + \sqrt{3}\, e^{-\frac{1}{\sqrt{3}}t} \left(e^{\frac{2}{\sqrt{3}}t} - 1\right) \eta(1)$$

$$+ \sqrt{3}\, e^{-\frac{1}{\sqrt{3}}t} \int_1^{e^{\frac{2}{\sqrt{3}}t}} [\, \eta(s) - \eta(1)\,]\, ds$$

Now, we note that

$$x(0) = \sqrt{3} \int_0^1 \eta(s)\, ds \tag{15}$$

and

$$\dot{x}(0) = 2\eta(1) - \int_0^1 \eta(s)\, ds \,. \tag{16}$$

We further note that $\eta(s)$ being a Brownian motion, $\eta(s) - \eta(1)$ and $\eta(s-1)$ are identical in law. Thus, $x(t)$ can be written as

$$\begin{aligned} x(t) = {} & \sqrt{3}\, e^{-\frac{1}{\sqrt{3}}t} x(0) + \frac{1}{2} e^{-\frac{1}{\sqrt{3}}t}\left(e^{\frac{2}{\sqrt{3}}t} - 1\right) x(0) \\ & + \frac{\sqrt{3}}{2} e^{-\frac{1}{\sqrt{3}}t}\left(e^{\frac{2}{\sqrt{3}}t} - 1\right) \dot{x}(0) \\ & + \sqrt{3}\, e^{-\frac{1}{\sqrt{3}}t} \int_0^{e^{\frac{2}{\sqrt{3}}t} - 1} \eta(s)\, ds, \end{aligned} \tag{17}$$

where $\eta(s)$ is again a standard Brownian motion. (Note that the $\eta(s)$ in (17) and the $\eta(s)$ in (13) through (16) are not the same except in law.)

4. The Distribution of Intervals Between Zeros

For a standard Brownian motion $\eta(s)$, define σ by

$$(18) \qquad \sigma = \min \{t ; t > 0, \ t + \int_0^t \eta(s)\, ds = 0\} .$$

In a very interesting paper McKean [5] has obtained explicit expressions concerning the distribution of σ. Specifically, he has shown [5, Sec. 3, 6] that

$$(19) \qquad f(y, t)\, dy\, dt = \text{Prob}\, \{\sigma \in (t, t+dt), \ [\eta(\sigma)+1] \in (-y, -y+dy)\}$$

$$= dy\, dt \ \frac{3}{\sqrt{2}\,\pi} \ \frac{y}{t^2} \ e^{-\frac{2}{t}(1-y+y^2)} \int_0^{\frac{4y}{t}} \frac{e^{-\frac{3\theta}{2}}}{\sqrt{\pi\theta}}\, d\theta ,$$

$$y > 0 .$$

Now, $\tau(y_0)$ as defined by (4) can be related to σ as given by (18), through (17). In what follows, we make free use of the fact that $\eta(t)$ and $c\eta\left(\frac{t}{c^2}\right)$ have the same law when $\eta(t)$ is a standard Brownian motion and $c > 0$. While $\eta(t)$ always denotes a standard Brownian motion in the following derivation, $\eta(t)$ from one line to the next need not be the same except in law. Let $g(t) = e^{\frac{2}{\sqrt{3}}t} - 1$, and $g^{-1}(t) = \frac{\sqrt{3}}{2} \ln(1+t)$. Then, from (4) and (17) we have

$$(20) \qquad \tau(y_0) = \min \{t ; t > 0, \ \frac{y_0}{2}\, g(t) + \int_0^{g(t)} \eta(s)\, ds = 0 \}$$

$$= \min \{ g^{-1}(t) \,;\, t > 0 \,,\ \frac{y_0}{2}\, t + \int_0^t \eta(s)\, ds = 0 \}$$

$$= \min \{ g^{-1}(t) \,;\, t > 0,\ \frac{y_0}{2}\, t + \int_0^t c\eta\left(\frac{s}{c^2}\right) ds = 0 \}$$

$$= \min \{ g^{-1}(t) \,;\, t > 0,\ \frac{y_0}{2}\ t + c^3 \int_0^{\frac{t}{c^2}} \eta(s)\, ds = 0 \}$$

$$= \min \{ g^{-1}(t) \,;\, t > 0,\ \frac{y_0}{2c}\left(\frac{t}{c^2}\right) + \int_0^{\frac{t}{c^2}} \eta(s)\, ds = 0 \}$$

$$= \min \{ g^{-1}(t) \,;\, t > 0 , \frac{4t}{y_0^2} + \int_0^{\frac{4t}{y_0^2}} \eta(s)\, ds = 0 \}$$

$$= \min \{ g^{-1}\left(\frac{y_0^2 t}{4}\right) \,;\, t > 0 \,, t + \int_0^t \eta(s)\, ds = 0 \}$$

$$= g^{-1}\left(\frac{y_0^2}{4}\,\sigma\right) = \frac{\sqrt{3}}{2}\ \ln\left(1 + \frac{y_0^2 \sigma}{4}\right) .$$

Therefore, from (5), (19) and (20) we have

$$\text{(21)} \qquad \varphi(y_0, t) = \text{Prob}\ \{ g^{-1}\left(\frac{y_0^2}{4}\ \sigma\right) > t \}$$

$$= \text{Prob}\ \{ \sigma > \frac{4}{y_0^2}\ g(t) \}$$

$$= \int_{\frac{4}{y_0^2}\, g(t)}^{\infty} ds \int_0^{\infty} dy\, f(y, s)$$

where f(y, s) is given by (19). It follows from (9) that

$$F(t) = 1 - \int_0^{\infty} y_0\, e^{-\frac{1}{2} y_0^2}\, \varphi(y_0, t)\, dy_0 \tag{22}$$

$$= 1 - \int_{g(t)}^{\infty} ds \int_0^{\infty} dy_0 \int_0^{\infty} dy\, \frac{4}{y_0}\, e^{-\frac{1}{2} y_0^2}\, f\left(y, \frac{4s}{y_0^2}\right)$$

and

$$q(t) = \dot{g}(t) \int_0^{\infty} dy_0 \int_0^{\infty} dy\, \frac{4}{y_0}\, e^{-\frac{1}{2} y_0^2}\, f\left(y, \frac{4g(t)}{y_0^2}\right) \tag{23}$$

With the substitution of (19), the integrals in (22) and (23) can be evaluated. The results are (See appendix)

$$F(t) = 1 - \frac{3}{2\pi} \left\{ \frac{[1 - 2r^2(t)]^{3/2}}{3 - 2r^2(t)}\, \Pi_1 \left(-\frac{3}{4} + \frac{1}{2} r^2(t), r(t) \right) + \frac{2\sqrt{1 - 2r^2(t)}}{3 - 2r^2(t)}\, K(r(t)) \right\} \tag{24}$$

and

$$(25) \qquad q(t) = \frac{\sqrt{3}}{4\pi}\left\{ \frac{[1-2r^2(t)]^{1/2}}{[1-r^2(t)][1+2r^2(t)]} E(r(t)) \right.$$

$$+ \frac{[1-2r^2(t)]^{1/2}}{[3-2r^2(t)]}\left[\frac{K(r(t))-E(r(t))}{r^2(t)}\right]$$

$$\left. + \frac{8[1-2r^2(t)]^{3/2}}{[3-2r^2(t)]^2[1+2r^2(t)]}\left[\Pi_1\left(-\frac{3}{4}+\frac{1}{2}r^2(t), r(t)\right) - K\left(r(t)\right)\right]\right\}$$

where

$$(26) \qquad r(t) = \left[\frac{1}{2}\left(1-e^{-\frac{1}{\sqrt{3}}t}\right)\right]^{1/2}$$

and

$$(27) \qquad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

$$(28) \qquad K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\varphi}}\, d\varphi$$

$$(29) \qquad \Pi_1(\nu, k) = \int_0^{\frac{\pi}{2}} \frac{1}{[1+\nu\sin^2\varphi]\sqrt{1-k^2\sin^2\varphi}}\, d\varphi$$

are complete elliptic integrals.

It is easy to see from (25) that

$$q(0^+) = \left(\frac{37}{32}\right) \frac{1}{6} \left(\frac{4}{\sqrt{3}}\right), \tag{30}$$

which verifies (3), since (2) corresponds to $a = \frac{4}{\sqrt{3}}$. Further, $q(t) \to 0$ exponentially as $t \to \infty$. In fact,

$$\lim_{t \to \infty} e^{\frac{1}{2\sqrt{3}} t} q(t) = \frac{\sqrt{3}}{4\pi} K\left(\frac{1}{\sqrt{2}}\right) \tag{31}$$

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APPENDIX

Let $\psi(t)$ be defined by

$$(A\text{-}1) \qquad \psi(t) = \int_0^\infty \int_0^\infty \frac{4}{y_0} e^{-\frac{1}{2} y_0^2} f\left(y, \frac{4t}{y_0^2}\right) dy_0\, dy ,$$

where $f\left(y, \frac{4t}{y_0^2}\right)$ can be found from (19) to be

$$(A\text{-}2) \qquad f\left(y, \frac{4t}{y_0^2}\right) = \frac{3}{\sqrt{2\pi}} \frac{y_0^4 y}{16 t^2} e^{-\frac{y_0^2}{2t}(1-y+y^2)} \int_0^{\frac{y_0^2 y}{t}} \frac{e^{-\frac{3\theta}{2}}}{\sqrt{\pi\theta}} d\theta$$

Substituting (A-2) in (A-1) and letting $r = \frac{y_0}{2\sqrt{t}}$, we have

$$(A\text{-}3) \qquad \psi(t) = \frac{12}{\pi\sqrt{2}} \int_0^\infty \int_0^\infty y r^3 e^{-2r^2(1-y+y^2)} e^{-2tr^2} \left[\int_0^{4yr^2} \frac{e^{-\frac{3\theta}{2}}}{\sqrt{\pi\theta}} d\theta\right] dy\, dr .$$

$$= -\frac{6}{\pi\sqrt{2}} \frac{d}{dt} \left\{ \int_0^\infty \int_0^\infty yr\, e^{-2r^2(t+1-y+y^2)} \right.$$

$$\left. \left[\int_0^{4yr^2} \frac{e^{-\frac{3\theta}{2}}}{\sqrt{\pi\theta}}\, d\theta \right] dy\, dr \right\}$$

$$= -\frac{3}{2\pi} \frac{d}{dt} \left\{ \int_0^\infty \frac{y^{3/2}\, dy}{[t+1-y+y^2]\sqrt{t+(1+y)^2}} \right\}$$

Now, let H(t) be defined by

$$\text{(A-4)} \qquad H(t) = \frac{3}{2\pi} \int_0^\infty \frac{y^{3/2}\, dy}{[t+1-y+y^2]\sqrt{t+(1+y)^2}}$$

Then, we have

$$\text{(A-5)} \quad \psi(t) = -\frac{d}{dt} H(t)$$

$$= \frac{3}{2\pi} \int_0^\infty \frac{y^{3/2}\, dy}{(t+1-y+y^2)\sqrt{t+(1+y)^2}} \left\{ \frac{1}{(t+1-y+y^2)} \right.$$

$$\left. + \frac{1}{2} \frac{1}{[t+(1+y)^2]} \right\}$$

From (22) and (23) it is easily seen that

(A-6)
$$F(t) = -\int_{g(t)}^{\infty} \psi(s)\, ds$$
$$= 1 - H(g(t))$$

and

(A-7)
$$q(t) = \dot{g}(t)\, \psi(g(t)) \qquad \left(g(t) = e^{\frac{2}{\sqrt{3}}t} - 1\right)$$

Proceeding to evaluate H(t), we make a change in the variable of integration in (A-4)

(A-8)
$$y = \sqrt{1+t}\left(\frac{1-\cos\varphi}{1+\cos\varphi}\right)$$

The result is

(A-9)
$$H(t) = \frac{3}{8\pi}\, \frac{1}{(1+t)^{1/4}} \int_0^{\pi} \frac{(1-\cos\varphi)^2\, d\varphi}{[1-\nu(t)\sin^2\varphi]\sqrt{1-k^2(t)\sin^2\varphi}}$$

with

(A-10)
$$\nu(t) = \frac{1}{2} + \frac{1}{4\sqrt{1+t}}$$

(A-11)
$$k^2(t) = \frac{1}{2} - \frac{1}{2\sqrt{1+t}}$$

Therefore,

$$\text{(A-12)} \qquad H(t) = \frac{3}{8\pi} \frac{1}{(1+t)^{1/4}} \int_0^{\pi} \frac{(1+\cos^2\varphi - 2\cos\varphi)d\varphi}{[1-\nu(t)\sin^2\varphi]\sqrt{1-k^2(t)\sin^2\varphi}}$$

$$= \frac{3}{4\pi} \frac{1}{(1+t)^{1/4}} \int_0^{\pi/2} \frac{(1+\cos^2\varphi)\, d\varphi}{[1-\nu(t)\sin^2\varphi]\sqrt{1-k^2(t)\sin^2\varphi}}$$

$$= \frac{3}{4\pi} \frac{1}{(1+t)^{1/4}} \left\{ [2 - \frac{1}{\nu(t)}] \int_0^{\pi/2} \frac{d\varphi}{[1-\nu(t)\sin^2\varphi]\sqrt{1-k^2(t)\sin^2\varphi}} \right.$$

$$\left. + \frac{1}{\nu(t)} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2(t)\sin^2\varphi}} \right\}$$

$$= \frac{3}{4\pi(1+t)^{1/4}} \left\{ [2 - \frac{1}{\nu(t)}] \Pi_1 \left(-\nu(t), k(t)\right) + \frac{1}{\nu(t)} K(k(t)) \right\}$$

Using (A-12) in (A-6) yields (24).

The function $\psi(t)$ can be found by differentiating $H(t)$. However, it is somewhat simpler to proceed directly from (A-5). Making the change in variable of integration (A-8) in (A-5), we find

$$\text{(A-13)} \qquad \psi(t) = \frac{3}{32\pi} \frac{1}{(1+t)^{5/4}} \int_0^{\pi} \frac{\sin^4\varphi}{[1-\nu(t)\sin^2\varphi]\sqrt{1-k^2(t)\sin^2\varphi}}$$

$$\left\{ \frac{1}{1-\nu(t)\sin^2\varphi} + \frac{1}{2} \frac{1}{[1-k^2(t)\sin^2\varphi]} \right\} d\varphi \; .$$

Changing variables a second time ($z = \sin^2 \varphi$), we obtain

$$\text{(A-14)} \qquad \psi(t) = \frac{3}{32\pi} \frac{1}{(1+t)^{5/4}} \int_0^1 \frac{z^2}{(1-\nu z)\sqrt{z(1-z)(1-k^2 z)}}$$

$$\left\{ \frac{1}{1-\nu z} + \frac{1}{2} \frac{1}{(1-k^2 z)} \right\} dz$$

where $\nu = \nu(t)$, $k^2 = k^2(t)$ are given by (A-10) and (A-11). Equation (A-15) can be rewritten by partial fraction expansion as

$$\text{(A-16)} \quad \psi(t) = \frac{3}{32\pi} \frac{1}{(1+t)^{5/4}} \int_0^1 \frac{1}{\sqrt{z(1-z)(1-k^2 z)}} \left\{ \frac{1}{\nu^2} \left[1 - \frac{2}{(1-\nu z)} \right. \right.$$

$$\left. \left. + \frac{1}{(1-\nu z)^2} \right] + \frac{1}{2k^2 \nu} \left[1 + \frac{k^2}{\nu - k^2} \frac{1}{(1-\nu z)} - \left(\frac{\nu}{\nu - k^2} \right) \frac{1}{1-k^2 z} \right] dz \right\}$$

To proceed further, we note that

$$\text{(A-17)} \quad \frac{1}{(1-\nu z)^2 \sqrt{z(1-z)(1-k^2 z)}} = \frac{\nu^2}{(\nu-1)(\nu-k^2)} \frac{d}{dz} \left[\frac{\sqrt{z(1-z)(1-k^2 z)}}{(1-\nu z)} \right]$$

$$+ \left[1 - \frac{1}{2} \frac{(\nu^2 - k^2)}{(\nu - k^2)(\nu - 1)} \right] \frac{1}{(1-\nu z)\sqrt{z(1-z)(1-k^2 z)}}$$

$$+ \frac{1}{2(\nu-1)\sqrt{z(1-z)(1-k^2 z)}} - \frac{\nu}{2(\nu-1)(\nu-k^2)} \sqrt{\frac{1-k^2 z}{z(1-z)}}$$

and

$$\text{(A-18)} \qquad \frac{1}{(1-k^2 z)\sqrt{z(1-z)(1-k^2 z)}} = \frac{2k^2}{k^2-1} \frac{d}{dz} \sqrt{\frac{z(1-z)}{1-k^2 z}}$$

$$- \frac{1}{(k^2-1)} \sqrt{\frac{1-k^2 z}{z(1-z)}}$$

Using (A-17) and (A-18) in (A-16) and simplifying the result (including the transformation $z = \sin^2 \varphi$), we obtain

$$\text{(A-19)} \qquad \psi(t) = \frac{3}{16\pi(1+t)^{5/4}} \left\{ \frac{2\nu(t)-1}{2\nu^2(t)[1-\nu(t)]} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2(t)\sin^2\varphi}} \left[\frac{1}{1-\nu(t)\sin^2\varphi} - 1 \right] \right.$$

$$+ \frac{1}{2\nu(t)k^2(t)} \int_0^{\frac{\pi}{2}} \left[\frac{1}{\sqrt{1-k^2(t)\sin^2\varphi}} - \sqrt{1-k^2(t)\sin^2\varphi} \right] d\varphi$$

$$\left. + \frac{1}{2(1-\nu(t))(1-k^2(t))} \int_0^{\frac{\pi}{2}} \sqrt{1-k^2(t)\sin^2\varphi} \; d\varphi \right\}$$

Combining (A-7) and (A-19) yields (25).