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**ION ACOUSTIC WAVES AND  
SOLITONS IN BOUNDED PLASMAS**

by

Keith L. Cartwright

Memorandum No. UCB/ERL M99/67

15 December 1999

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**ELECTRONICS RESEARCH LABORATORY**

College of Engineering  
University of California, Berkeley  
94720

# **Ion Acoustic Waves and Solitons in Bounded Plasmas**

**Acknowledgments:** This research is supported by ONR grant number N00014-90-J-1198, ONR-AASERT 23057, AFOSR-AASERT F49620-94-1-0387, and AFOSR FDF49620-96-1-0154.

This is work done by K. L. Cartwright on ion acoustic waves and solitons during his graduate career at University of California-Berkeley not included in his PhD thesis. The topic of his thesis was on the time-independent and spectral content of crossed-field electron flow in diodes. The following research is published in the hopes that it will foster further research on the topic of ion acoustic waves in bounded plasmas.

## **Part A: Refraction and Reflection of Ion Acoustic Solitons by Space Charge Sheaths *ICOPS 1996, APS/DPP 1996, and Sendia 1997***

Experiments have shown that ion acoustic solitons tunnel through the space charge sheath in front of a grid without time delay. They are absorbed resonantly when the spatial width of the soliton is close to the characteristic gradient scale length of the sheath. The reflection and transmission coefficients found in these experiments have compared well with theory in the long wavelength limit. However, to achieve this comparison, two parameters were added that were not in the original theory. These parameters allow for the absorption of soliton energy by the space charge sheath. The goal of our numerical simulations, designed to reproduce the experimental results, is to uncover the mechanism of this energy loss and large speed of propagation through the sheath, as observed in the time-distance plots.

## **Part B: Non-quasi-neutral Theory of Ion-Acoustic Resonances in bounded Nonuniform Plasmas and Comparison with PIC Simulations *ICOPS 98***

A theory is developed of ion-acoustic resonances in an inhomogeneous plasma diode using a fluid plasma description. A 1d non-linear steady state model is first developed, to which linear perturbations are applied. Poisson's equation is included self-consistently, without making the assumption of quasineutrality. The resulting fifth order differential equation describing this inhomogeneous plasma does not have the singularity that occurs in the sheath region when quasineutrality is assumed. It is shown that both even and odd ion-acoustic resonances (standing waves) are confined within the plasma core; this implies reflection from the sheath. The wave does not reflect at the sheath edge, in fact the reflection position varies with frequency; the reflection location is approximately at the position where the group velocity (which is a function of frequency) is equal to the ion drift velocity.

To numerically investigate long time scale, low frequency phenomena, a hybrid electrostatic PIC algorithm is used in which the electrons reach thermodynamic equilibrium with the ions each time step, using the nonlinear Boltzmann relationship for the electrons with a PIC ion source term. The 1d non-linear steady state model is compared to the PIC simulations. Also, the mode structure of the ion-acoustic resonances are compared with the pure fluid model as well as using the steady state hybrid PIC spatial profiles of ion density and ion drift velocity along with the electron temperature, as the source for the linear mode analysis.

# Refraction and Reflection of Ion Acoustic Solitons by Space Charge Sheath

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Acknowledgments: This research is supported by ONR grant number N00014-90-J-1198 and ONR-AASERT 23057.

## Abstract

Experiments have shown[1],[2] that ion acoustic solitons tunnel through the space charge sheath in front of a grid without time delay. They are absorbed resonantly when the spatial width of the wave is close to the characteristic gradient scale length of the sheath. The reflection and transmission coefficients found in these experiments have compared well with theory in the long wavelength limit[3]. However, to achieve this comparison, two parameters were added that were not in the original theory. These parameters allow for the absorption of wave energy by the space charge sheath. The goal of our numerical simulations, designed to reproduce the experimental results, is to uncover the mechanism of this energy loss and large speed of propagation through the sheath.

## 1 Photo Ionization-Steady State

Electron Temperature (x=0)	.713 eV
Ion Temperature (x=0)	$1.28 \times 10^{-5}$ eV
Density (x=0)	$1 \times 10^{14} m^{-3}$
$\frac{Mass\ of\ Ion}{Mass\ of\ Electron}$	225
Background gas Pressure	5 mTorr
Electron Mean Free Path	$2.3 \times 10^{-4}$
	He elastic cross section
Ionization rate	$2.4 \times 10^{15} s^{-1}$
Energy of photon	4.51 eV
Transparency of grid	.90
Plasma Density	$1.28 \times 10^{14} m^{-3}$
Electron Plasma Frequency	$1.02 \times 10^8 sec^{-1}$
Ion Plasma Frequency	$6.79 \times 10^6 sec^{-1}$
Debye Length	$5 \times 10^{-4} m$
Number of Debye Lengths in system	108
Acoustic Speed	$2.36 \times 10^4 m/s$

## 2 Computer Model

Both boundaries are periodic. To maintain the plasma, ions - electron pairs are generated in the system at a constant rate uniformly across the system. This model would correspond to a plasma where UV light is responsible for the ionization. A grid placed at 0.03m which collects 10% of the plasma passing through. A diagram of the system is shown in Figure 1. The initial condition of this computer experiment is with the simulation region empty. The simulation was run until it reached steady state. Spatial profiles of the densities, potentials, and average velocity are shown in Figure 2. The elastic scattering is show in Figure 3. The electron and ion phase space is shown in Figure 4.

## 3 Self Excited Waves

A long time ( $\sim 12\tau_i$ ) average of the ions density shown in Figure 2 is subtracted from the instantaneous density to obtain Figure 5. This Figure show self excited waves moving in both directions in the bulk of the plasma.

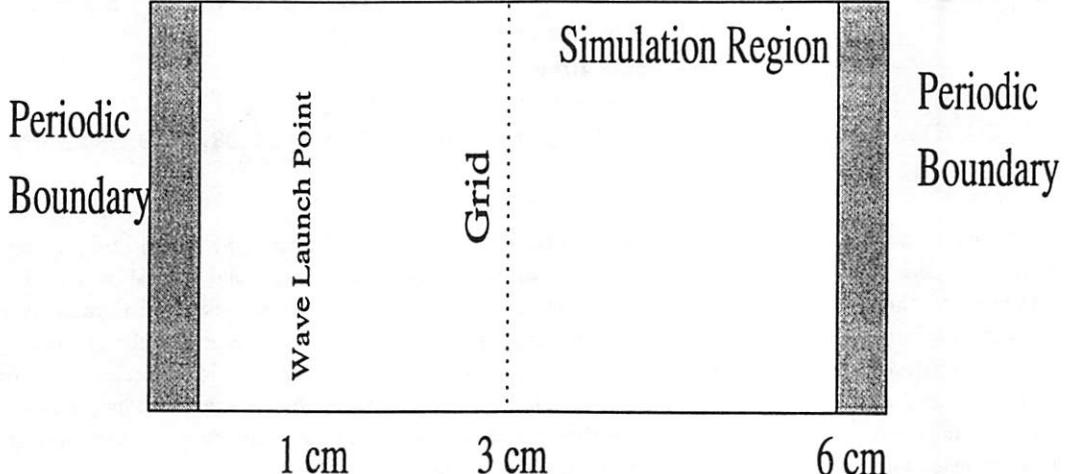


Figure 1: Diagram of 1d Simulation

## 4 Launching of a Wave

The waves are launched in this equilibrium state shown in Figure 2. To launch a wave, a wave is instantaneously added to the system by adding particles. Electron-ion pairs are added with modulation in velocity and in density with the center at 0.01 cm, see Figure 9. The wave is shown without the background plasma. The width of the modulation is varied to get the desired wave in the simulation. The density of the plasma is much larger than the wave which is launched. The ion density, velocity and potential perturbation are shown in Figure 6, 7 and 8.

## 5 Why do IAW Reflect from the Sheath

In the plasma bulk, ion acoustic waves may propagate. They are sustained by the charge neutrality of the plasma as a whole. In the sheath, however, an ion acoustic wave is no longer sustained by charge neutrality. In the sheath, electric potential energy is higher than the electron thermal energy (pressure) which drives the acoustic wave. In plasma, the ions and electrons oscillate in phase with little self E. However, the electric field in the sheath drives ions and electrons in opposite directions. Yamada and Raether [4] show that the direct influence of an electric field in a plasma causes a significant modification of the dispersion relation which should not be neglected. Yamada and Raether then neglect the electric field term because it leads to substantial simplification of the dispersion relation. The indirect effect of the electric field manifests itself in a modification of the unperturbed distribution functions. The unperturbed distribution functions are assumed to be drifting Maxwellians. Then we plan to calculate the path of a wave using Eikonal (WKB) theory, using this dispersion relation. The dispersion relation will need spatial profiles of the average velocity obtained from simulation.

## 6 Effects of the Width on Reflection and Transmission

The reflection and transmission coefficient,  $\gamma_r$  and  $\gamma_t$ , versus the wave width have been measured from the density perturbation plot. By keeping the wave amplitude constant, we have changed the wave width by varying the initial width. It is clearly seen from Figure 10 that the transmission rate of the wave decreases and reflection rate increase as the width of the wave is the order of the sheath width. This is also seen in the experiment by Nishida *et.al.* [2], as shown in Figure 11 from the same paper.

### Time Average Spatial Profies

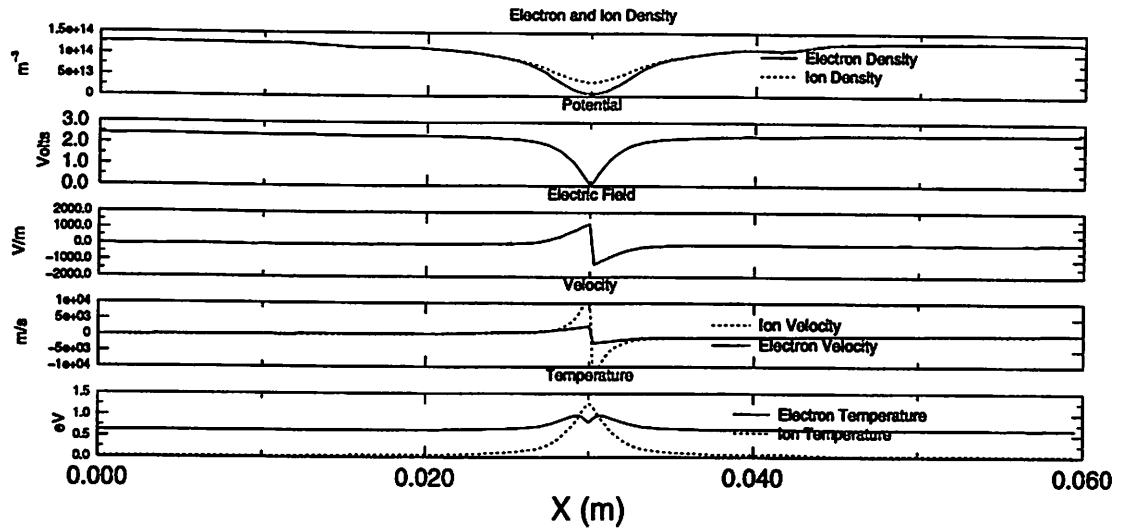


Figure 2: Spatial Profiles  
MCC Electron Elastic Scattering

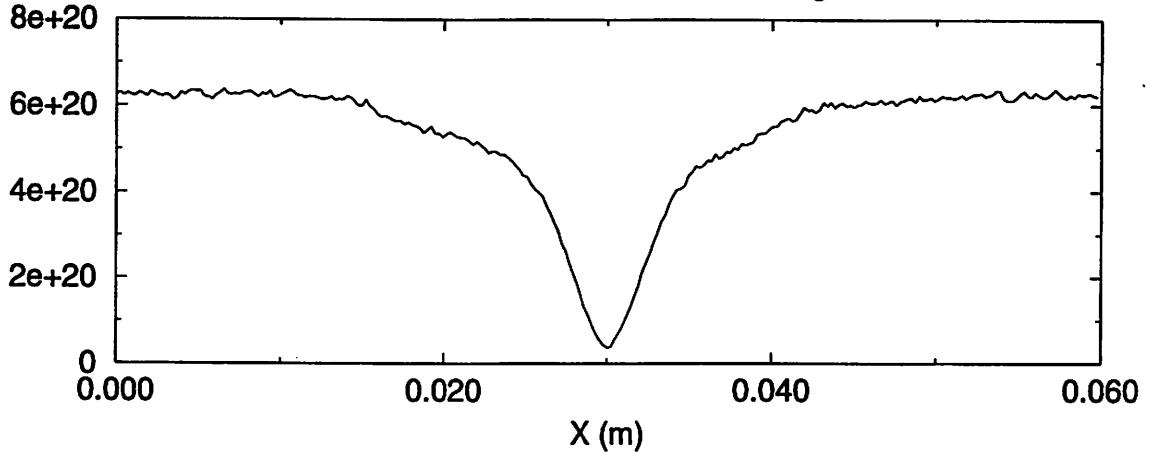


Figure 3: Elastic Scattering

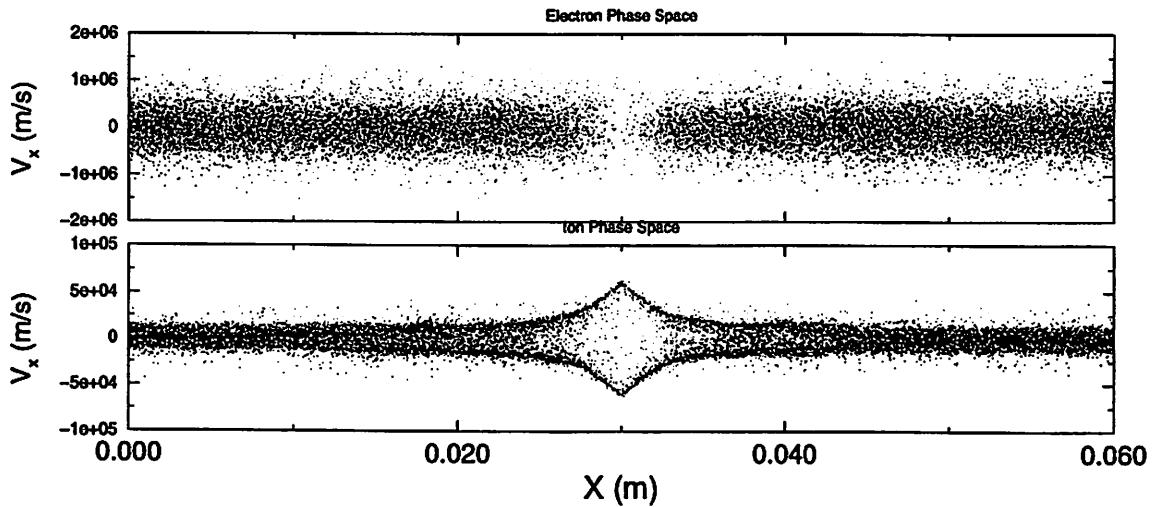


Figure 4: Electron and Ion Phase Space

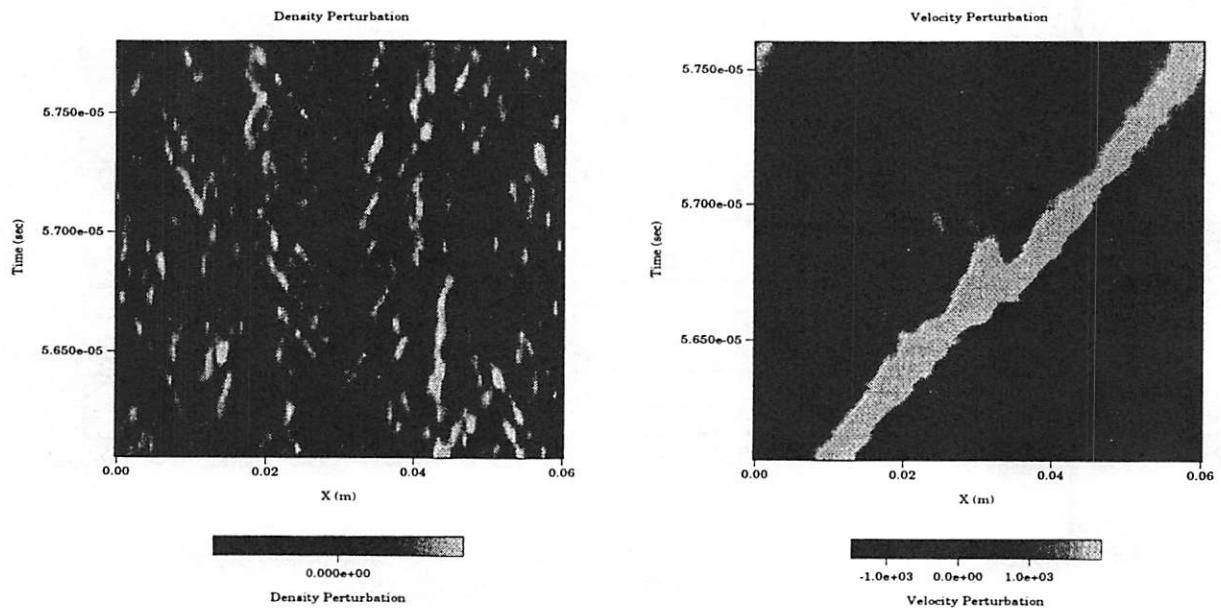


Figure 5: Time–distance plot of undriven fluctuations in the system

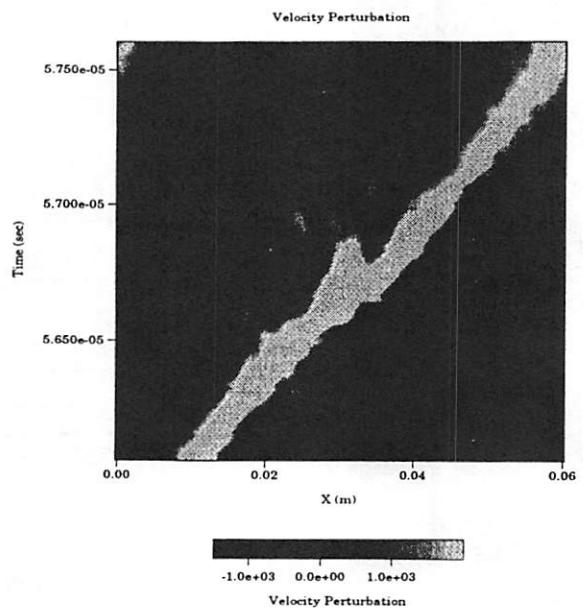


Figure 7: Time–distance velocity perturbation after a soliton is excited in the system.

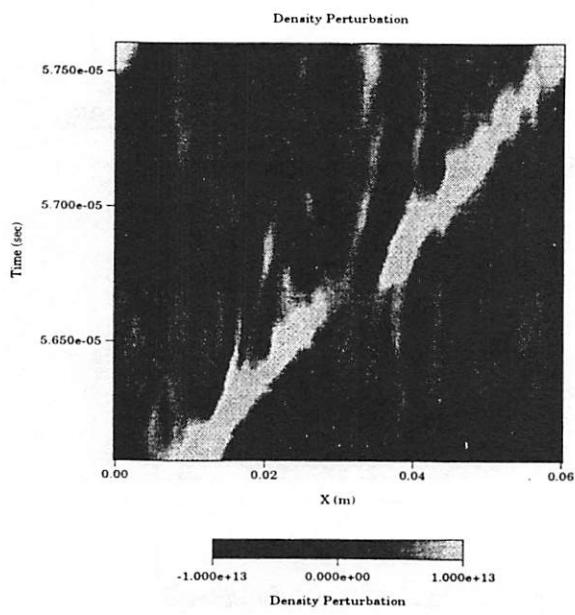


Figure 6: Time–distance plot of density perturbation after a soliton is excited in the system.

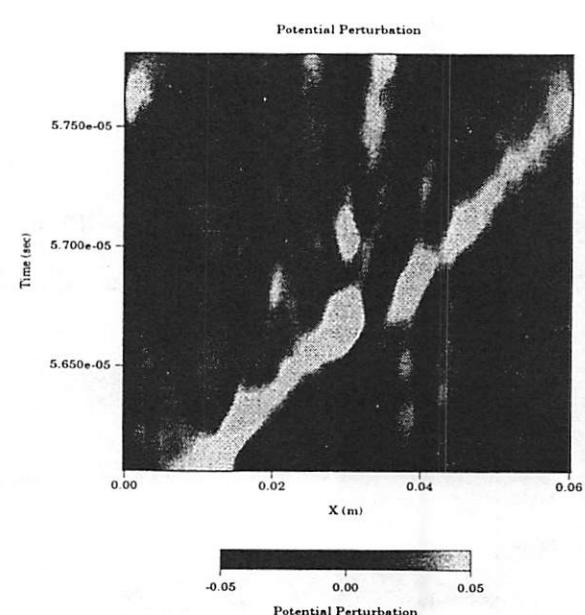


Figure 8: Time–distance potential perturbation after a soliton is excited in the system.

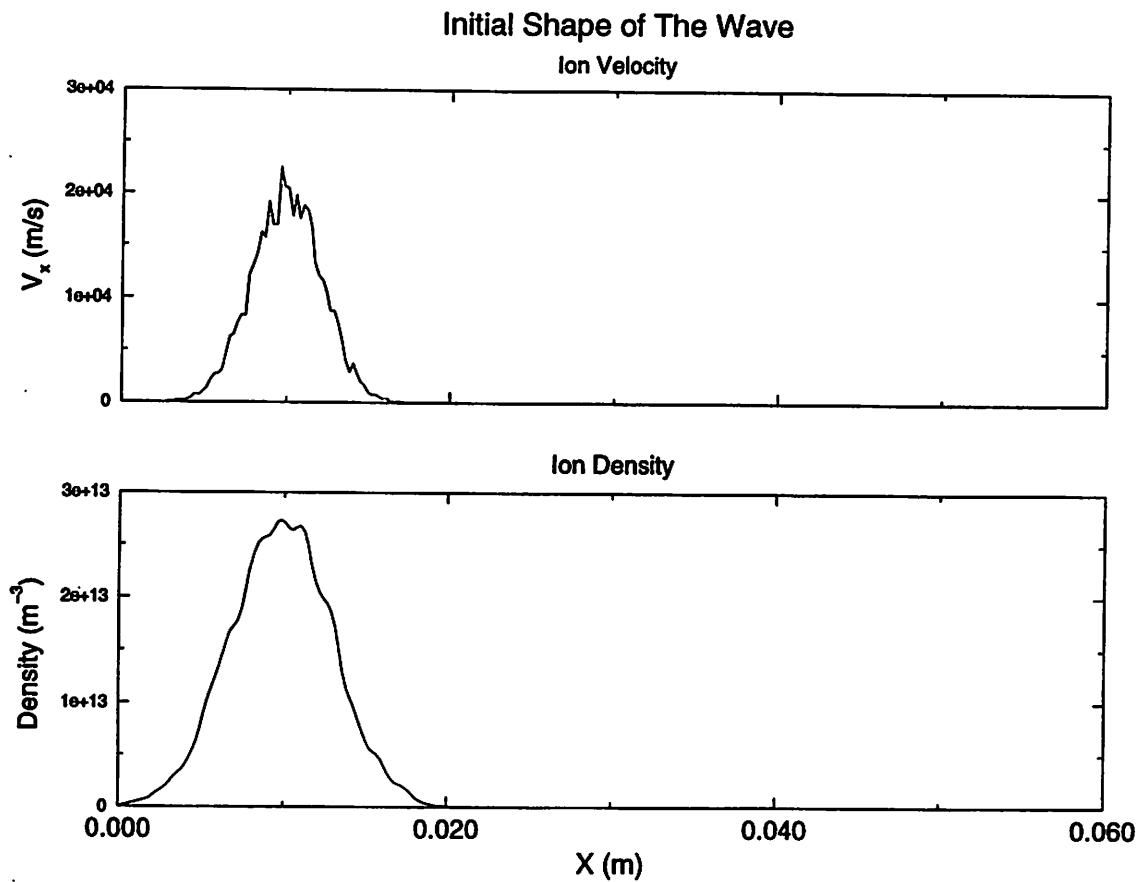


Figure 9: Ion velocity and density perturbation used to launch a soliton.

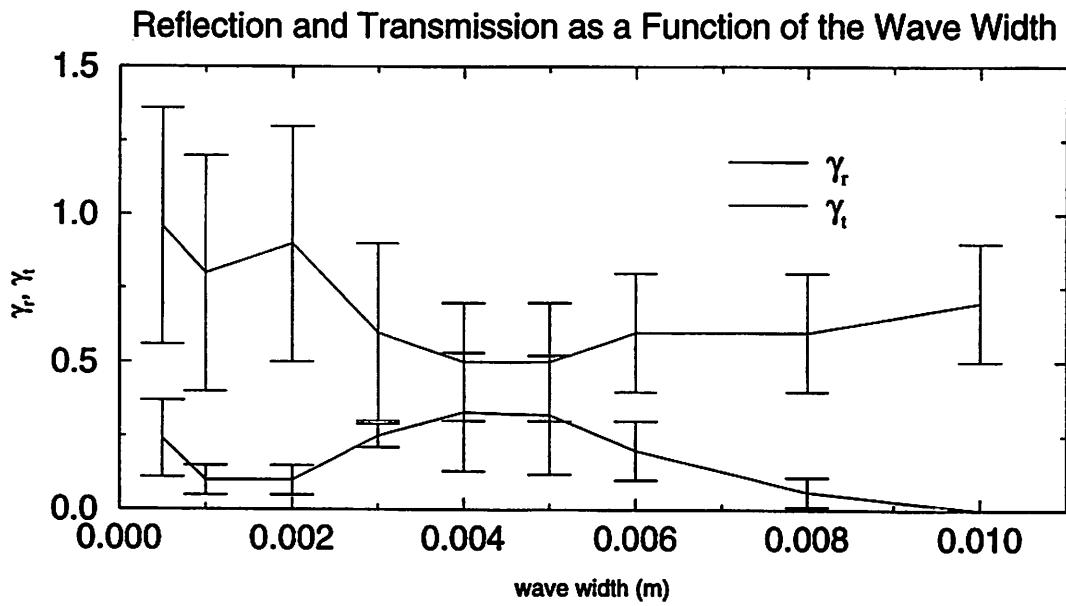


Figure 10: Reflections and transmission coefficient as a function of width of launched wave.

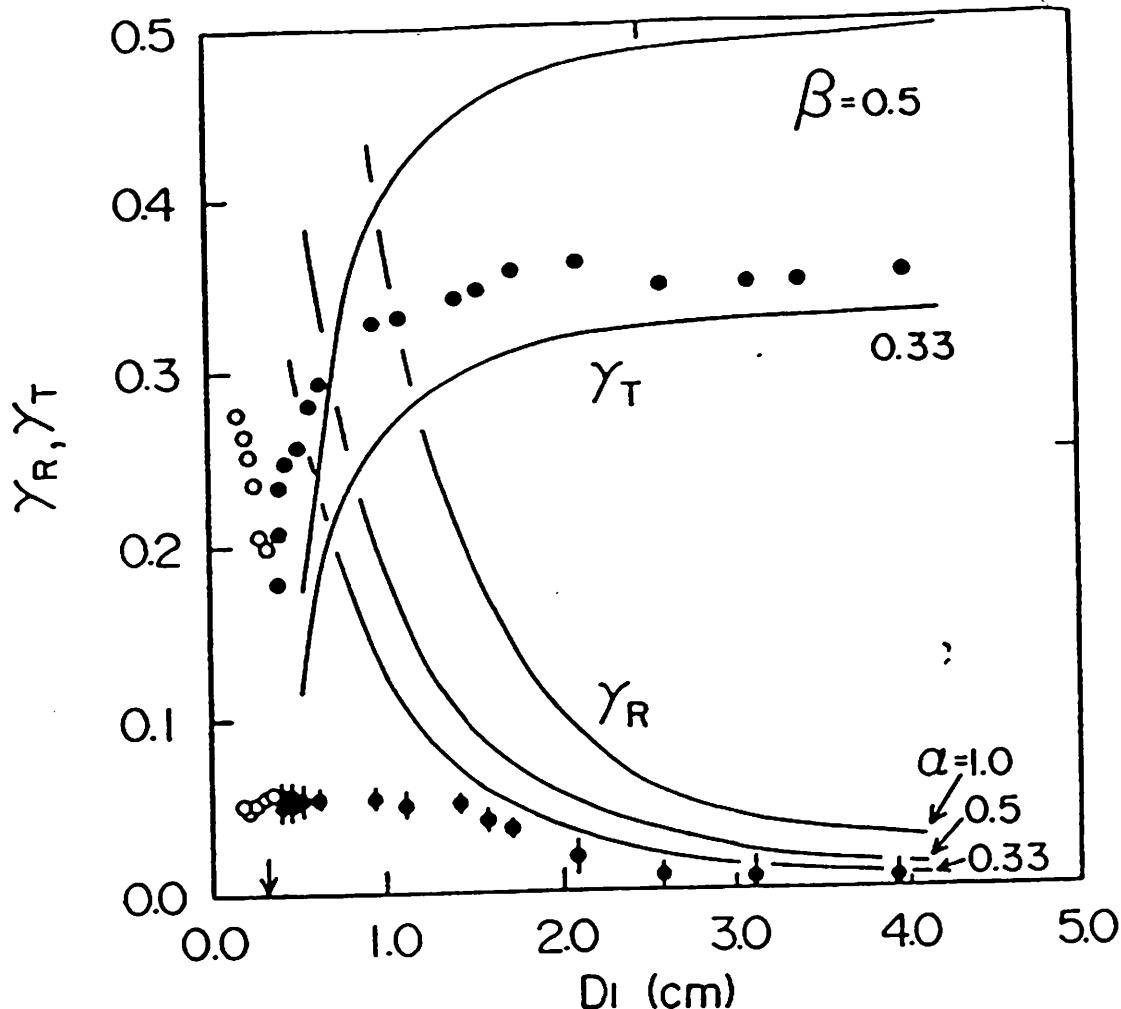


Figure 11: Reflections and transmission coefficient as a function of width of launched wave (circles) taken from Nishida *et.al.* Note at the characteristic scale length ( $D=0.3$  cm) the transmission coefficient is a minimum and reflection coefficient is a maximum.

## References

- [1] Y. Nishida. Reflection of a planar ion-acoustic soliton from a finite plane boundary. *Phys. Fluids*, 27(8):2176–2180, August 1984.
- [2] Y. Nishida K. Yoshida and T. Nagasawa. Refraction and reflection of ion acoustic solitons by space charge sheaths. *Phys. Fluids B*, 5(3):722–731, March 1993.
- [3] T. Watanabe C. Matsuoka and N. Yajima. Measurement of ion-rich sheath thickness by ion acoustic wave. *J. Plasma Physics*, 8(3):321–330, September 1972.
- [4] M. Yamada and M. Raether. Evolution of the ion-acoustic instability in a direct-current discharge plasma. *Phys. Fluids*, 18(3):361–368, March 1975.

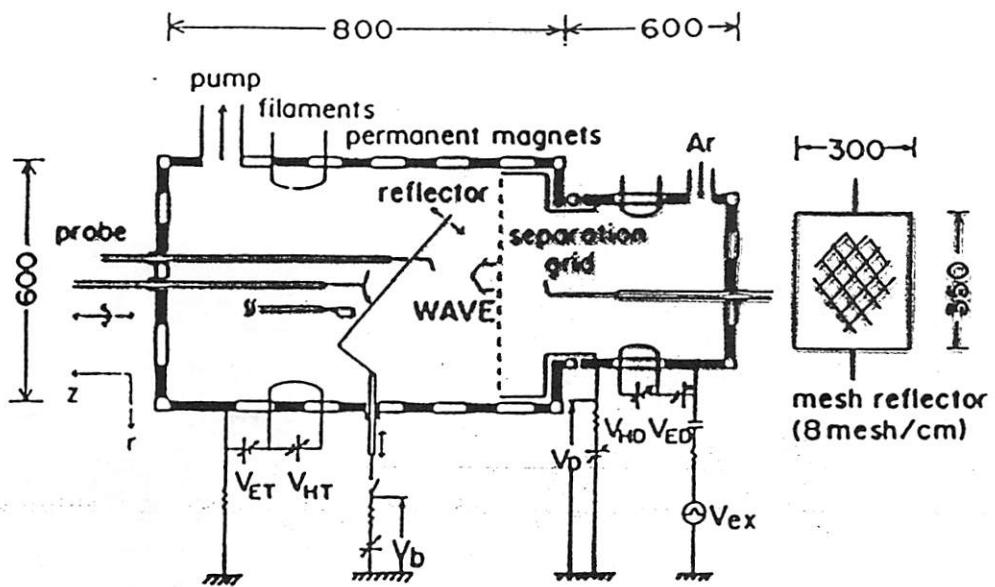
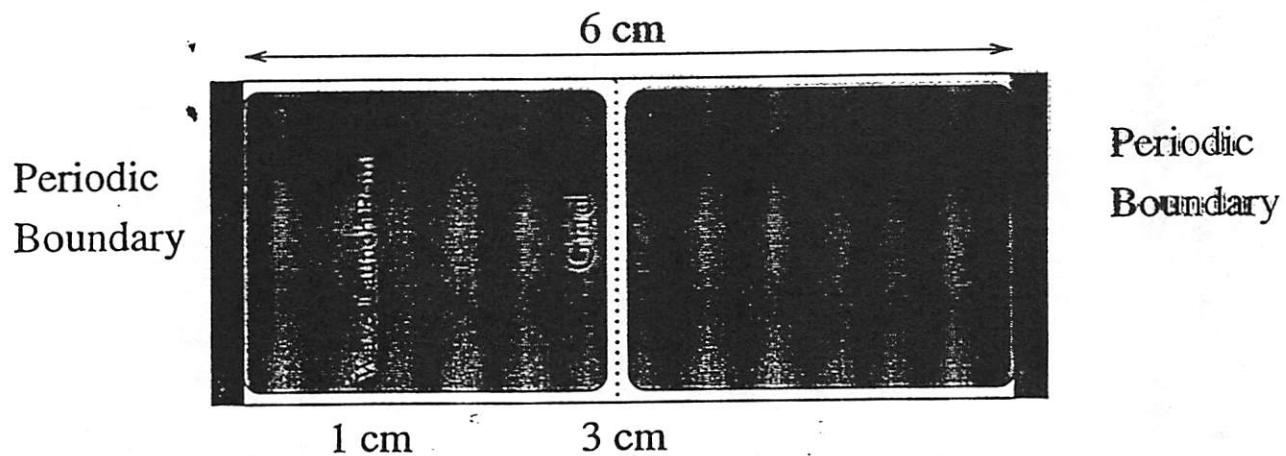


FIG. 1. A schematic drawing of the experimental apparatus.

Yasushi Nishida, Kazuhiko Yoshida, and Takeshi Nagasawa  
 Phys Fluids B 5(3) March 1993 p 722



## Grid-Photo Ionization-Steady State

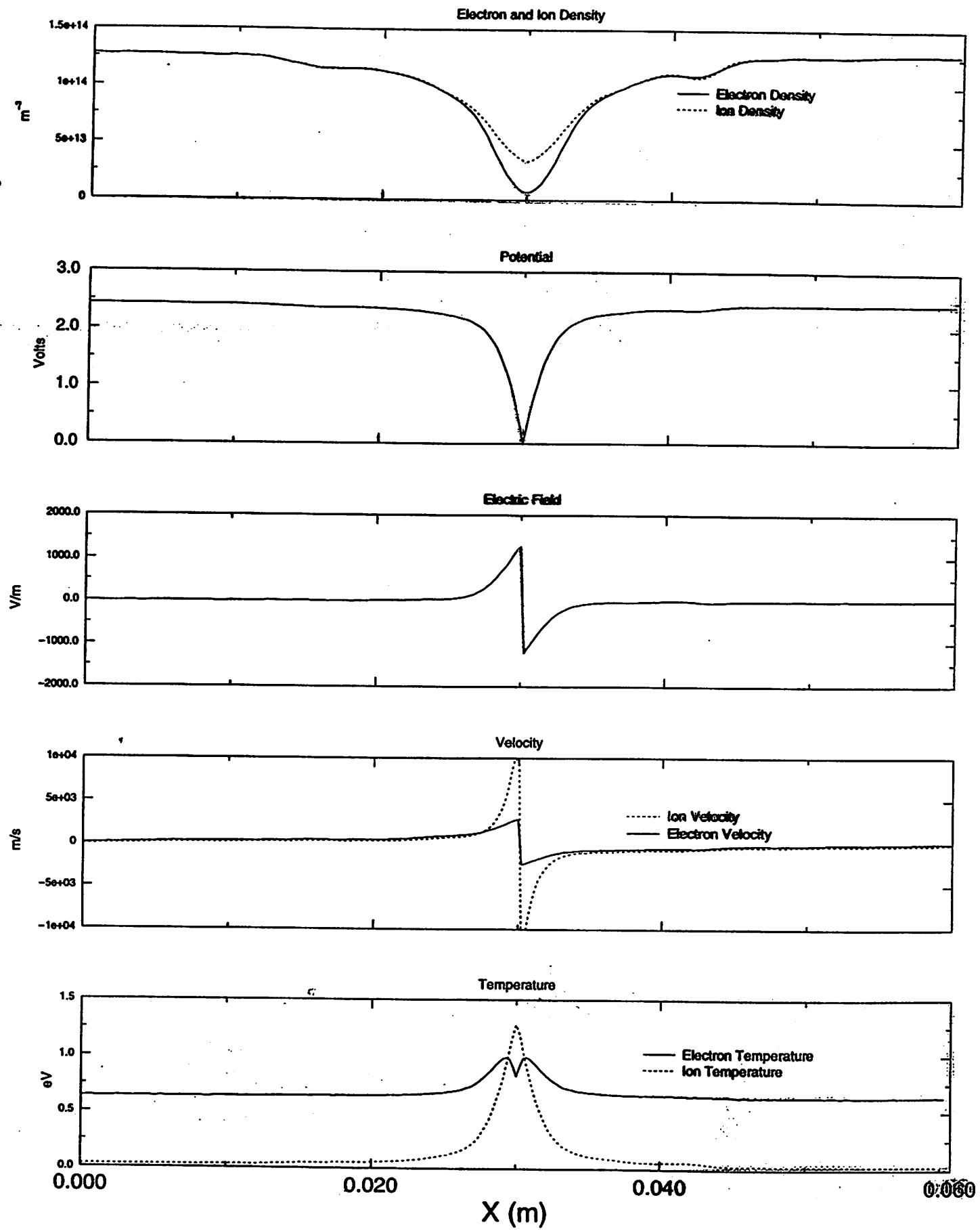
### Input Parameters

Energy of photon	4.51 eV
Ionization rate	$2.4 \times 10^{15} \text{ s}^{-1}$
Background gas Pressure	5 mTorr
	He cross section
Transparency of grid	.90
$\frac{\text{Mass of Ion}}{\text{Mass of Electron}}$	225

### Resulting Plasma

Electron Temperature (x=0)	.713 eV
Ion Temperature (x=0)	0.029 eV
Density (x=0)	$1.28 \times 10^{14} \text{ m}^{-3}$
Electron Mean Free Path (elastic)	.1 m
Electron Plasma Frequency	$1.02 \times 10^8 \text{ sec}^{-1}$
Ion Plasma Frequency	$6.79 \times 10^6 \text{ sec}^{-1}$
Debye Length	$5.5 \times 10^{-4} \text{ m}$
Number of Debye Lengths in system	108
Acoustic Speed	$2.36 \times 10^4 \text{ m/s}$

## Time Average Spatial Profiles



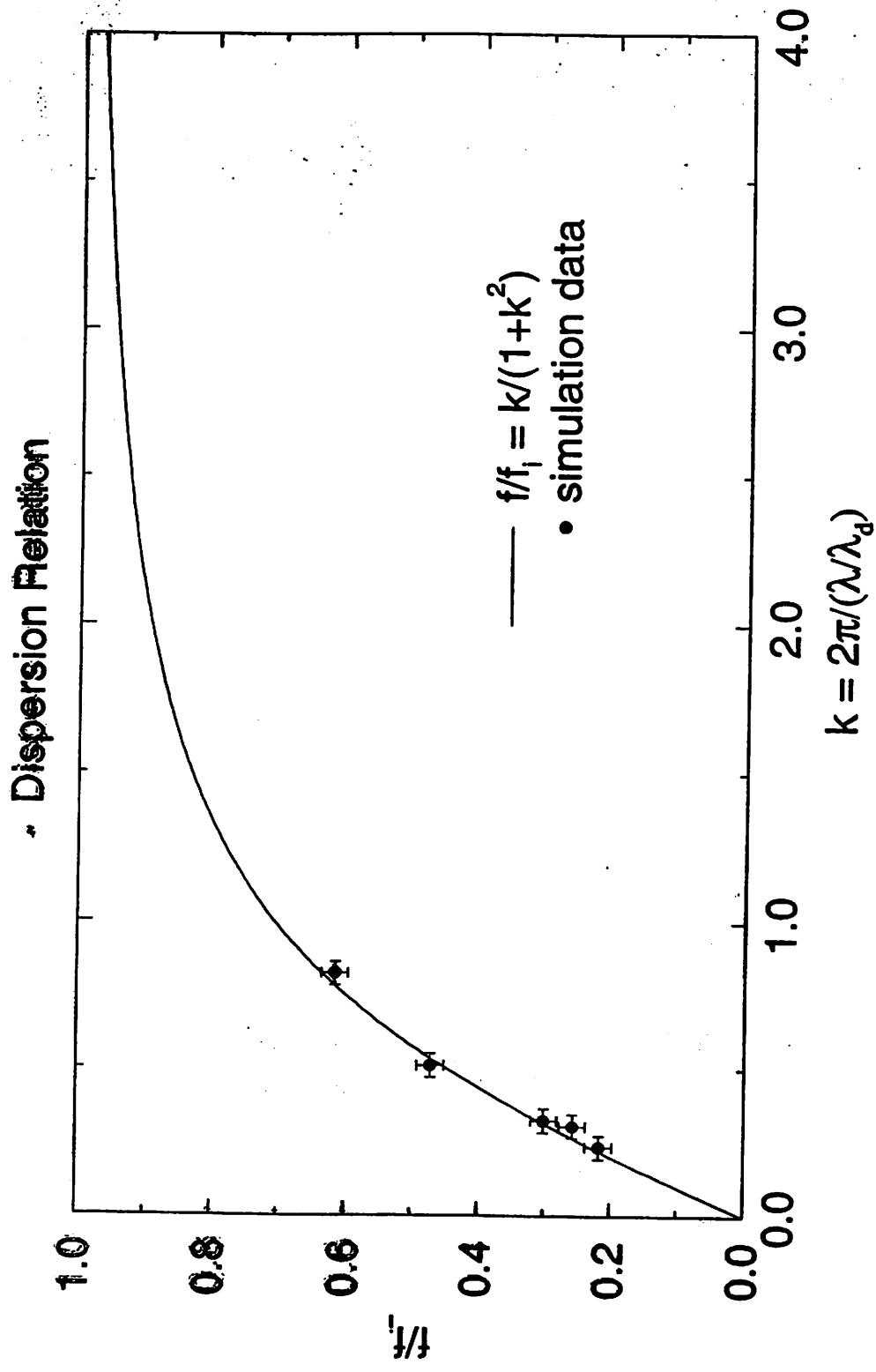
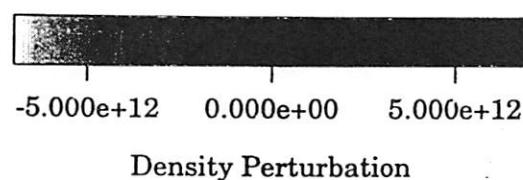
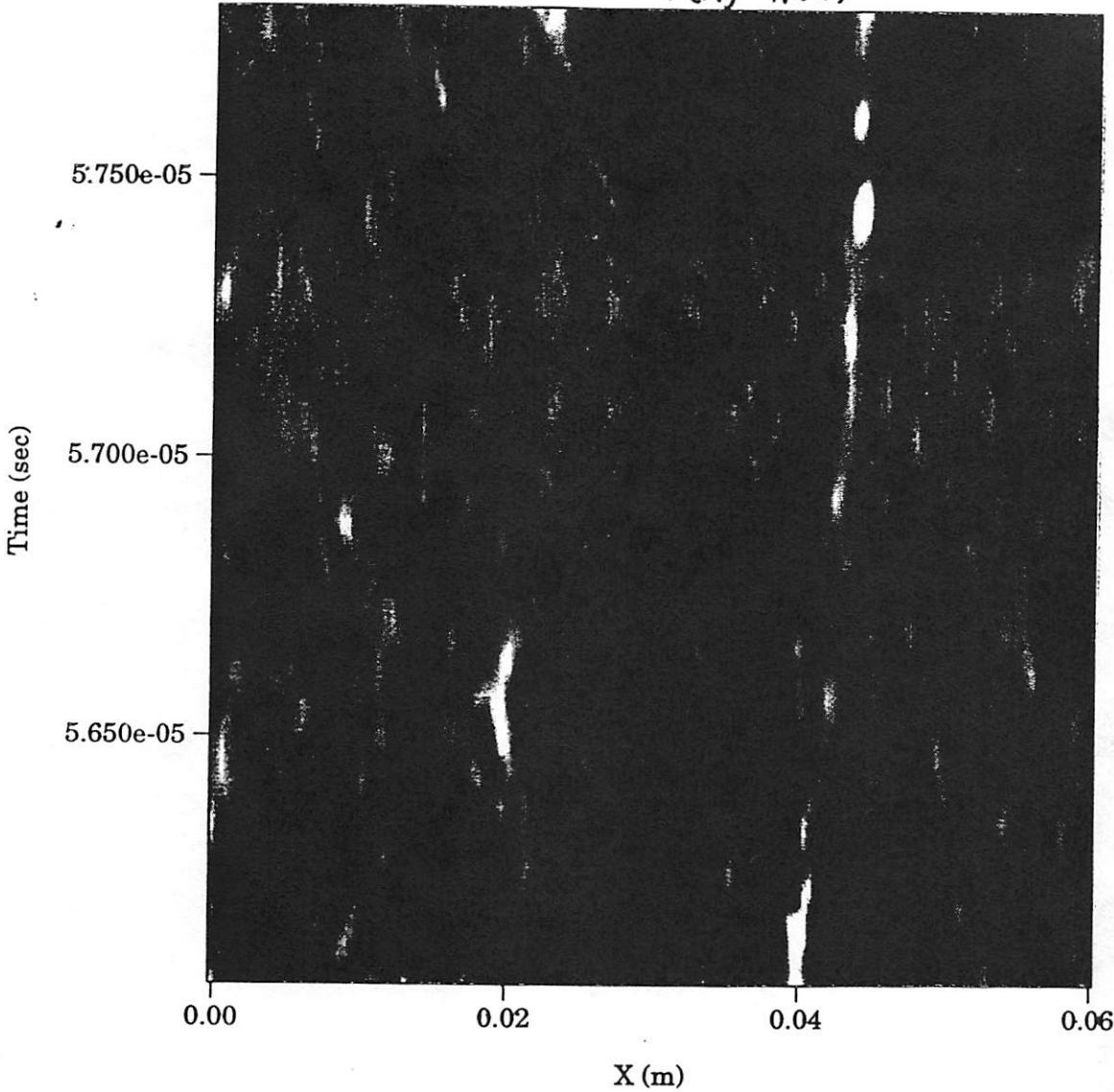


Figure 9: Dispersion Relation

*Ion* Density Perturbation  
 $n(x) - \bar{n}(x)$



Density Perturbation

## Ion-Acoustic Solitons

With the normalization of  $n = n_i/n_0$ ,  $u = u_i/u_s$ ,  $x = x/\lambda_d$ ,  $\tau = v_s t/\lambda_d$ , and  $\Phi = e\phi/kT_e$  the fluid equations of continuity and momentum are:

$$\frac{\partial n}{\partial \tau} + \frac{\partial(nu)}{\partial x} = 0 \quad (1)$$

$$\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} = -\frac{\partial \Phi}{\partial x}. \quad (2)$$

Using the Boltzmann relation for the electron density,  $n_e = \exp(\Phi)$ , Poisson's equation is:

$$\frac{\partial \Phi^2}{\partial x^2} = e^\Phi - n. \quad (3)$$

Using the dispersion relationship derived from linear perturbation the phase of an ion acoustic waves is:

$$kx - \omega\tau \simeq (\omega + \frac{1}{2}\omega^3)x - \omega\tau = \omega(x - \tau) + \frac{1}{2}\omega^3x. \quad (4)$$

Introducing new variables  $\eta = \omega(x - \tau)$ ,  $\xi = \omega^3x$ , and a smallness parameter  $\epsilon = \omega^2$  so that:

$$kx - \omega\tau \simeq \epsilon^{\frac{1}{2}}(x - \tau) + \frac{1}{2}\epsilon^{\frac{3}{2}}x = \eta + \frac{1}{2}\xi. \quad (5)$$

expanding to second order in  $\epsilon$ ,

$$n = 1 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \dots \quad (6)$$

$$u = \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \dots \quad (7)$$

$$\Phi = \epsilon \Phi^{(1)} + \epsilon^2 \Phi^{(2)} + \dots \quad (8)$$

To first order,

$$\frac{\partial n^{(1)}}{\partial \eta} = \frac{\partial u^{(1)}}{\partial \eta} = \frac{\partial \Phi^{(1)}}{\partial \eta} \quad (9)$$

which leads to  $n^{(1)} = u^{(1)} = \phi^{(1)} \equiv \phi$ . Then to second order,

$$\frac{\partial n^{(2)}}{\partial \eta} - \frac{\partial u^{(2)}}{\partial \eta} = \frac{\partial u^{(1)}}{\partial \xi} + \frac{\partial(n^{(1)}u^{(1)})}{\partial \eta} = \phi_\xi + 2\phi\phi_\eta \quad (10)$$

$$\frac{\partial u^{(2)}}{\partial \eta} - \frac{\partial \Phi^{(2)}}{\partial \eta} = \frac{\partial u^{(1)}}{\partial \xi} + u^{(1)} \frac{\partial u^{(1)}}{\partial \eta} = \phi_\xi + \phi\phi_\eta \quad (11)$$

$$\Phi^{(2)} - n^{(2)} = \frac{\partial^2 \Phi^{(1)}}{\partial \eta} - \frac{(\Phi^{(1)})^2}{2} = \phi_{\eta\eta} - \frac{1}{2}\phi^2. \quad (12)$$

To eliminate the second order terms, differentiate the last equation with respect to  $\eta$ . The addition of all three equations results in.

$$\phi_\xi + \phi\phi_\eta + \frac{1}{2}\phi_{\eta\eta\eta} = 0 \quad (13)$$

This is the Korteweg-de Vries (KdV) equation with  $a = 1$  and  $b = \frac{1}{2}$  having the solution:

$$\phi = A \cdot \operatorname{Sech}^2(\kappa(\eta - U\xi)) \quad (14)$$

**Soliton**

In terms of the  $x, \tau$  variables:

$$\begin{aligned} n, u, \Phi &= 3(u-1) \operatorname{Sech}^2\left(\sqrt{\frac{u-1}{2}}(x-u\tau)\right) \\ &= A \operatorname{Sech}^2((x-u\tau)/w) \end{aligned}$$

where  $u$  is the speed of the soliton (normalized to  $v_s$ ). Singh and Dahiya [1] showed that  $Aw^2$  is a constant with respect to a slowly varying background density (finite ion temperature). In the simulation  $Aw^2$  is a constant to 5 % for all three waves when they are more than .005 away from the grid.

## Reflection and Transmission Relation

The conservation of energy for the incident, transmitted and reflected solitons can be written as [12]:

$$A_i^2 w_i = A_t^2 w_t + A_r^2 w_r \quad (22)$$

Since the three waves are KdV solitons, each component satisfies the relation:

$$A_i w_i^2 = A_t w_t^2 = A_r w_r^2 = \text{const} \quad (23)$$

From these two equation:

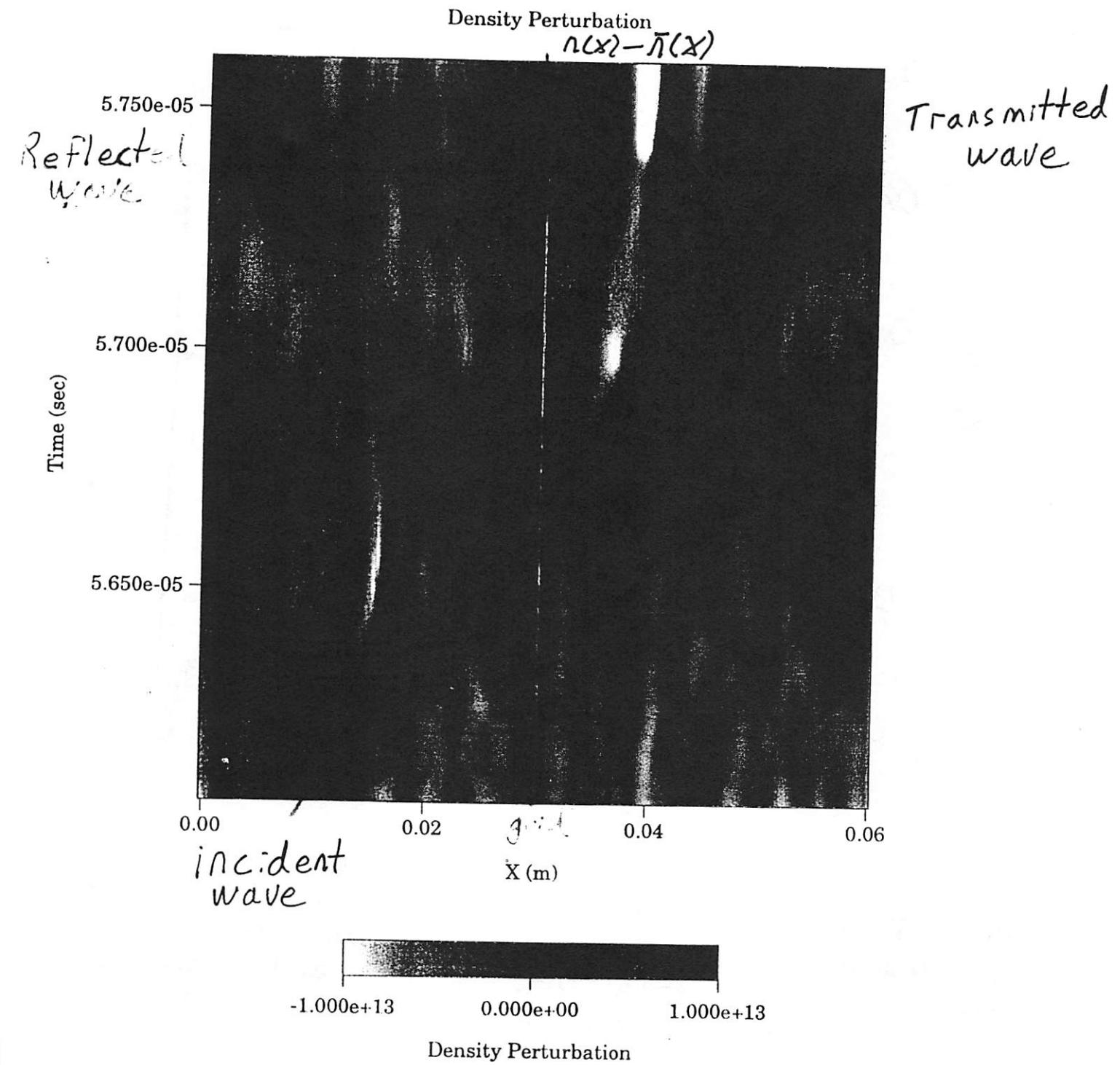
$$A_i^2 w_i = A_t^2 \sqrt{A_i/A_t} w_i + A_r^2 \sqrt{A_i/A_r} w_i \quad (24)$$

or

$$1 = \gamma_t^{\frac{3}{2}} + \gamma_r^{\frac{3}{2}} \quad (25)$$

where

$$\gamma_t = A_t/A_i \text{ and } \gamma_r = A_r/A_i \quad (26)$$



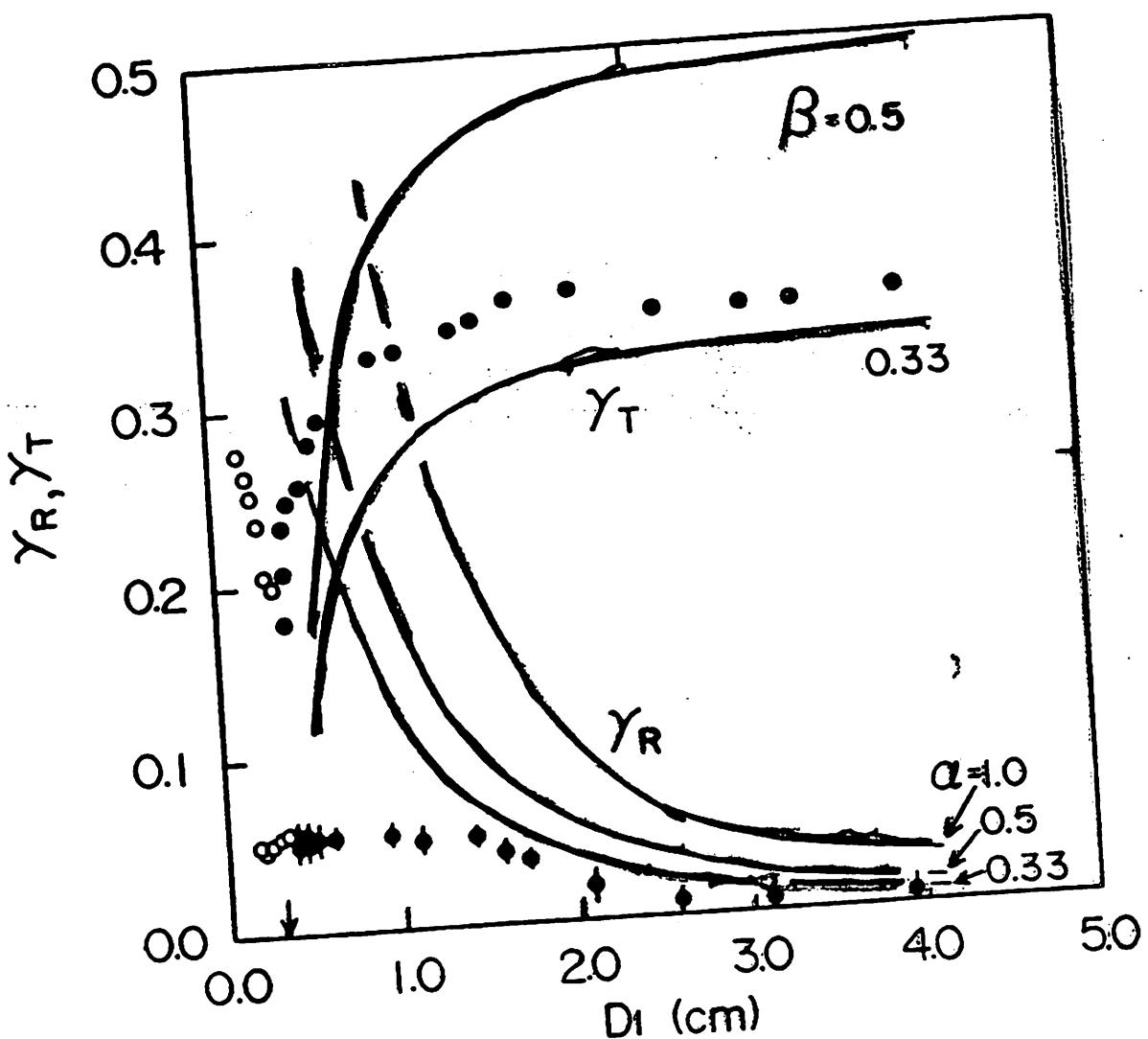


FIG. 6. The reflection and transmission rate as a function of the wave width  $D$  at the incident angle  $\theta_i=0$  (closed circles). Open circles are depicted from Fig. 7. An arrow ( $D=0.3$  cm) indicates the characteristic scale length of the density gradient in the transition layer of the space charge sheath.

Nishida, Yoshida, and Nagusawa  
Phys Fluids B 5(3) March 1993

Theoretical Model with  $\alpha=\beta=1$  (no loss)  
Watanabe, Karamori, and Yajima, J. Phys. Soc. Jpn.  
58 1273, 1989

Reflection and Transmission as a Function of the Wave Width

$$\gamma_t = A_t / A_i \text{ and } \gamma_r = A_r / A_i$$

$$1 = \gamma_t^{\frac{3}{2}} + \gamma_r^{\frac{3}{2}}$$

$$\gamma_r$$
  
$$\gamma_t$$

1.0

0.5

0.0

$\lambda - \lambda'$

wave width (m)

0.010

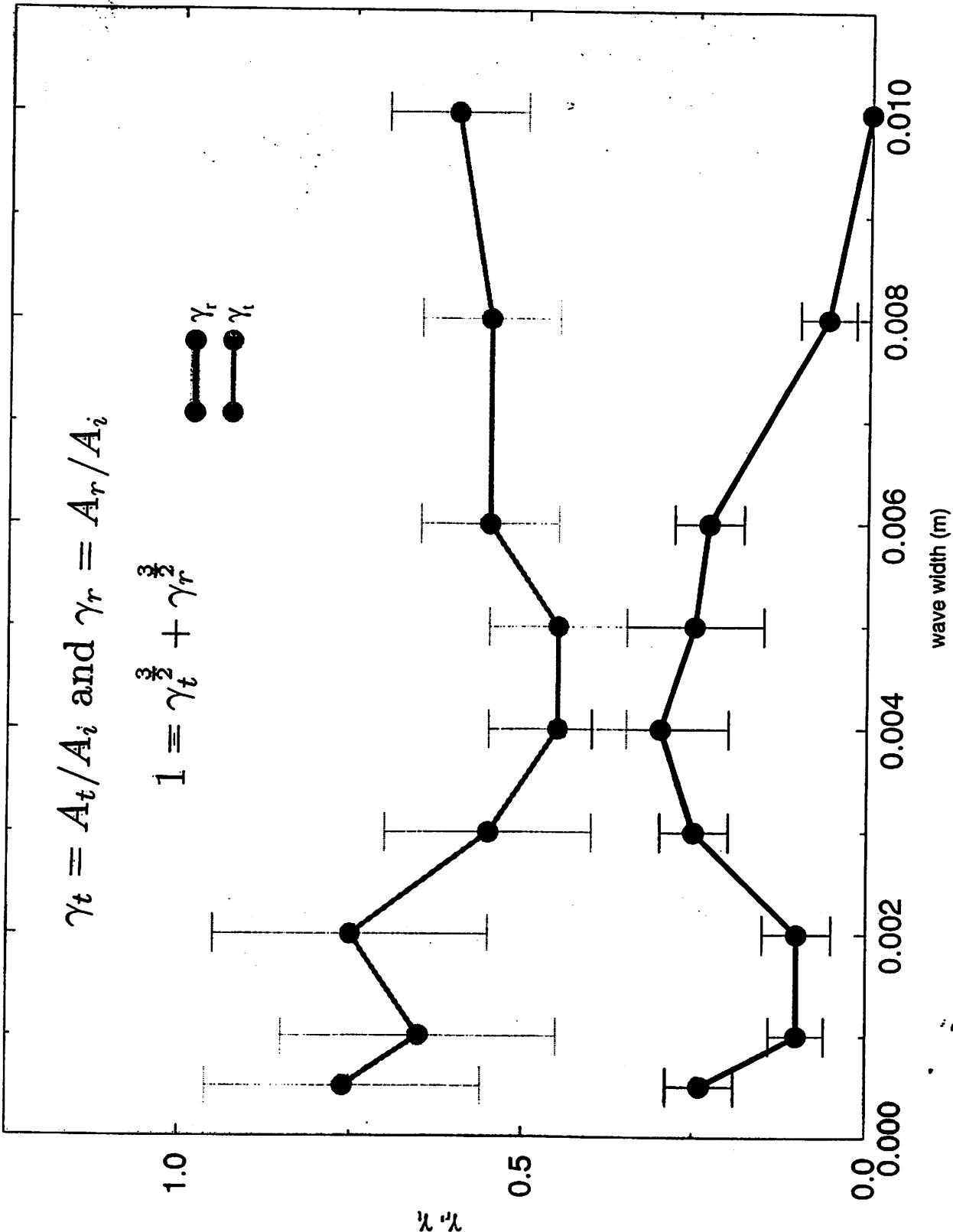
0.008

0.006

0.004

0.002

0.000



## Future Work

1. Reflection of Solitons from a Grid.
  - Account for loss mechanism used in theory (collisions, other waves ...)
2. Undriven Small Amplitude Standing waves.
  - To find out what causes the reflection (frequency dependent).
  - To find out why Eikonal comparisons worse than single wavelength comparison (drifting dispersion error).
3. Would like to drive the system to test for coupling (parametric with electron waves and nonlinear coupling with ion waves).
  - To find out why eigenfrequency were not found (External circuit?)

## Conclusion

1. Reflection of Solitons from a Grid.
  - Qualitative agreement of Transmission Minimum and Reflection Maximum with experiment and theory.
2. Undriven Small Amplitude Standing waves.
  - Agreement with experiment that standing waves are formed (implies reflection).
  - Did not find eigenfrequency as in experiments, but a continuous spectrum.
  - Qualitative comparison that the trapped wave is close to a single wavelength wave.
  - Wave does not end at sheath edge or the same place for all frequency (assumed in some experimental results).
  - Clear reason for reflection has not been found.

# Nonquasineutral Theory of Ion-Acoustic Resonances in Bounded Nonuniform

## Plasmas and Comparison with PIC Simulations

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A theory is developed of ion-acoustic resonances in an inhomogeneous plasma diode using a fluid plasma description. A 1d non-linear steady state model is first developed, to which linear perturbation ions are applied. Poisson's equation is included self-consistently, without making the assumption of quasineutrality. The resulting fifth order differential equation describing the wave does not have the singularity that occurs in the sheath region when quasineutrality is assumed. It is shown that both even and odd ion-acoustic resonances are confined within the plasma core. To numerically investigate long time scale, low frequency phenomena, a hybrid electrostatic PIC algorithm is used in which the electrons reach thermodynamic equilibrium with the ions each time step, using the nonlinear Boltzmann relationship for the electrons with a PIC ion source term. The 1d non-linear steady state model is compared to the PIC simulations. Also, the mode structure of the ion-acoustic resonances are compared with the pure fluid model as well as using the steady state hybrid PIC spatial profiles,  $n_e(x)$ ,  $n_i(x)$ , and  $u_i(x)$ , as the source for the linear mode analysis.

## Outline

1. 1d Model
2. PIC-Boltzmann Results
3. Fluid Theory
4. Comparison with Experiments [1-10] from Literature
5. Conclusion

# Boltzmann Electrons–Photo Ionization–Steady State

## Input Parameters

Energy of photon	4.51 eV + ionization
Ionization rate	$1.8 \times 10^{19} \text{ s}^{-1}$
Background gas Pressure	3 mTorr
	He cross section
$\frac{\text{Mass of Ion}}{\text{Mass of Electron}}$	225

## Resulting Plasma

Electron Temperature (fixed)	1.00 eV
Ion Temperature	.028 eV
Density	$9.2 \times 10^{13} \text{ m}^{-3}$
Electron Plasma Frequency	86 MHz
Ion Plasma Frequency	5.7 MHz
Debye Length	$7.76 \times 10^{-4} \text{ m}$
Number of Debye Lengths in system	155
Acoustic Speed	$2.79 \times 10^4 \text{ m/s}$

## Fluid Equations

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = S \quad (1)$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = S \quad (2)$$

$$m \left( \frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i \right) = e \mathbf{E} \quad (3)$$

$$m n_e \left( \frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{u}_e \cdot \nabla) \mathbf{u}_e \right) = -e n_e \mathbf{E} - k T \nabla n \quad (4)$$

$$\epsilon_0 \nabla \mathbf{E} = e(n_i - n_e) \quad (5)$$

## Normalizations

$$n_i = n_0 \tilde{n}_i, \quad n_e = n_0 \tilde{n}_e \quad (6)$$

$$\mathbf{u}_i = \sqrt{\frac{kT}{m_i}} \tilde{\mathbf{u}}_i, \quad \mathbf{u}_e = \sqrt{\frac{kT}{m_e}} \tilde{\mathbf{u}}_e \quad (7)$$

$$\mathbf{E} = \sqrt{\frac{n_0 k T}{\epsilon_0}} \tilde{\mathbf{E}}, \quad \phi = \frac{k T}{e} \tilde{\phi} \quad (8)$$

$$\mathbf{x} = \sqrt{\frac{k T}{n_0 \epsilon_0 e^2}} \tilde{\mathbf{x}}, \quad t = \tilde{t} / \sqrt{\frac{n_0 e^2}{\epsilon_0 m_i}} \quad (9)$$

$$S = n_0 \omega_{pi} \tilde{S}, \quad \alpha = \sqrt{m_e/m_i} \quad (10)$$

## Normalized Fluid Equations

$$\frac{\partial \tilde{n}_i}{\partial t} + \nabla \cdot (\tilde{n}_i \tilde{\mathbf{u}}_i) = \tilde{S} \quad (11)$$

$$\alpha \frac{\partial \tilde{n}_e}{\partial t} + \nabla \cdot (\tilde{n}_e \tilde{\mathbf{u}}_e) = \alpha \tilde{S} \quad (12)$$

$$\frac{\partial \tilde{\mathbf{u}}_i}{\partial t} + \tilde{\mathbf{u}}_i \cdot \nabla \tilde{\mathbf{u}}_i = \tilde{\mathbf{E}} \quad (13)$$

$$\tilde{n}_e \left( \alpha \frac{\partial \tilde{\mathbf{u}}_e}{\partial t} + \tilde{\mathbf{u}}_e \cdot \nabla \tilde{\mathbf{u}}_e \right) = -\tilde{n}_e \tilde{\mathbf{E}} - \nabla \tilde{n}_e \quad (14)$$

$$\nabla \tilde{\mathbf{E}} = \tilde{n}_i - \tilde{n}_e \quad (15)$$

Look for solutions of the form:

$$A(x, y, t) = A_0(x) + A_1(x)e^{-i\omega t}. \quad (16)$$

$A \Rightarrow n_e, n_i, \mathbf{U}_e, \mathbf{U}_i$ , and  $\mathbf{E}$ .

Which gives the DC equations:

$$(\mathbf{u}_{i0} \cdot \nabla) \mathbf{u}_{i0} = \mathbf{E}_0 \Rightarrow \nabla \left( \frac{1}{2} \mathbf{u}_{i0}^2 \right) = \mathbf{E}_0 \quad (17)$$

$$n_{e0} (\mathbf{u}_{e0} \cdot \nabla) \mathbf{u}_{e0} = -\mathbf{E}_0 n_{e0} - \nabla n_{e0} \Rightarrow \nabla \left( \frac{1}{2} \mathbf{u}_{e0}^2 \right) = -\mathbf{E}_0 - \nabla \ln(n_{e0}) \quad (18)$$

$$\nabla(\mathbf{u}_{i0} n_{i0}) = S \quad (19)$$

$$\nabla(\mathbf{u}_{e0} n_{e0}) = 0 \Rightarrow \mathbf{u}_{e0} = 0 \quad (20)$$

$$\nabla^2 \phi_0 = n_{e0} - n_{i0} \quad (21)$$

The DC fluid equations can be combined to get one equation:

$$\frac{\partial^2 \phi_0}{\partial x^2} = e^{\phi_0} - \frac{Sx}{\sqrt{-2\phi_0}}. \quad (22)$$

The second term on the RHS has a singularity at  $x = 0$ ,  $\phi_0 = 0$ .

1. This singularity can be overcome by Taylor expanding  $\phi_0$  around 0.
2. This equation does not have a singularity at the sheath edge.
3.  $n_{i0}(x) > n_{e0}(x)$  In the limit that  $L \rightarrow \infty$ ,  $n_{i0}(x) = n_{e0}(x)$ .

## 2d Linear Perturbation Equations

ion momentum:

$$(-i\omega + \frac{\partial u_{i0}}{\partial x})u_{i1x} + u_{i0}\frac{\partial u_{i1x}}{\partial x} + = E_{1x} \quad (23)$$

$$-i\omega u_{i1y} + u_{i0}\frac{\partial u_{i1y}}{\partial x} = E_{1y} \quad (24)$$

electron momentum:

$$-E_0 n_{e1} - n_{e0} E_{1x} - \frac{\partial n_{e1}}{\partial x} = 0, \quad -n_{e0} E_{1y} - ik_y n_{e1} = 0 \quad (25)$$

ion continuity:

$$(-i\omega + \frac{\partial u_{i0}}{\partial x})n_{i1} + u_{i0}\frac{\partial n_{i1}}{\partial x} + \frac{\partial n_{i0}}{\partial x}u_{i1x} + n_{i0}(\frac{\partial u_{i1x}}{\partial x} + ik_y u_{i1y}) = 0 \quad (26)$$

electron continuity

$$\frac{\partial u_{e0}}{\partial x}u_{e1x} + n_{e0}(\frac{\partial u_{e1x}}{\partial x} + ik_y u_{e1y}) = 0 \quad (27)$$

Poisson:

$$\frac{\partial E_{1x}}{\partial x} + ik_y E_{1y} = n_{i1} - n_{e1} \quad (28)$$

## 1d Linear Perturbation Equations

ion momentum:

$$(-i\omega + \frac{\partial u_{i0}}{\partial x})u_{i1} + u_{i0}\frac{\partial u_{i1}}{\partial x} + = E_1 \quad (29)$$

electron momentum:

$$E_0 n_{e1} + n_{e0} E_1 + \frac{\partial n_{e1}}{\partial x} = 0, \quad (30)$$

ion continuity:

$$(-i\omega + \frac{\partial u_{i0}}{\partial x})n_{i1} + u_{i0}\frac{\partial n_{i1}}{\partial x} + \frac{\partial n_{i0}}{\partial x}u_{i1} + n_{i0}\frac{\partial u_{i1}}{\partial x} = 0 \quad (31)$$

electron continuity

$$\frac{\partial u_{e0}}{\partial x}u_{e1} + n_{e0}\frac{\partial u_{e1}}{\partial x} = 0 \quad (32)$$

Poisson:

$$\frac{\partial E_1}{\partial x} = n_{i1} - n_{e1} \quad (33)$$

## Limiting Cases

$k = 0 \Rightarrow 1d$ , all gradients are zero.

1.  $u_{i0} = 0, E_0 = 0$ , and  $n_{e0} = n_{i0} = n$ .
2.  $n_{e0} = n_{i0} = n$  and  $E_0 = 0$ .
3.  $n_{e0} = n_{i0} = 1$  and  $u_{i0} = 0$ .

## Changing Density

$$\omega = \frac{\pm k}{\sqrt{1 + k^2/n}} \quad (34)$$

## Drift Velocity

$$\omega = k(u_{i0} \pm \frac{1}{\sqrt{1 + k^2/n}}) \quad (35)$$

To find  $k$  at the turning point set  $v_g = 0$ :

$$v_g = u_{i0} \pm \frac{1}{(1 + k^2/n)^{\frac{3}{2}}} \quad (36)$$

Plug this back into the dispersion relation:

$$\omega = \pm \sqrt{n(1 - u_{i0}^{2/3}) + u_{i0} \sqrt{n(1 - u_{i0}^{-2/3})}} \quad (37)$$

## Electric Field

$$\omega^2 = \frac{(kE_0)^2 + k^2(k^2 + 1) - ikE_0}{(kE_0)^2 + (k^2 + 1)^2} \quad (38)$$

## Experimental Observation of Ion Acoustic Resonances

Experiment	Plasma Type	Nature of the Observation	Gases	Fundamental Mode	Q
	Noise	Excited Resonance			
<b>Cylindrical Geometry</b>					
Crawford et al. [1, 2]	dc	X	X	Hg	$J'_0(k\alpha)$
Stern et al. [3, 4]	dc		X (NL)	He, Hg	$J'_1(k\alpha)$
Lisitano et al. [5]	rf	X		H, He, Ar, Xe	$J'_0(k\alpha)$
Demokan et al. [6]	dc		X (NL)	Hg	$J'_1(k\alpha)$
Weynants et al. [7]	dc	X	X	Ar, Xe, Hg	$J'_0(k\alpha)$
<b>Planar Geometry</b>					
Alexandrov et al. [8]	dc	X		He, Ne, Hg	$\cos(k\alpha)$
Asmussen et al. [9]	rf		X (NL)	Air	$\cos(k\alpha)$
Kuzovnikov et al. [10]	dc	X		He	$\cos(k\alpha)$
					?

## Conclusions

1. “trapped” modes fit linear fluid theory.
2. Edge perturbation are not explained by linear fluid theory.

## Future Work

1. Would like to drive the system to test for coupling (parametric with electron waves and nonlinear coupling with ion waves).
2. Find saturation amplitude of IAW.
3. Extend to 2d.

## PIC-Boltzmann Simulation Algorithm

Using the Boltzmann relationship for the electron density, Poisson's equation may be cast in the following form:

$$N(\phi(\mathbf{x})) = \nabla^2 \phi(\mathbf{x}) + (\rho_{PIC}(\mathbf{x}) - e n_{e0} e^{e\phi(\mathbf{x})/kT}) / \epsilon_0 \quad (39)$$

$$\text{where we iterate towards } N(\phi(\mathbf{x})) \Rightarrow 0, \quad (40)$$

$\rho_{PIC}(\mathbf{x})$  is the charge density of all the PIC species.  $T$  is given and  $n_{e0}$  still need to be determined.

## Determining $n_{e0}$

Integrating the continuity equation for the Boltzmann electrons over the system volume produces:

$$\frac{dN_B}{dt} + \int_{\text{surface}} (n\mathbf{u}) \cdot d\mathbf{S} = G - L. \quad (41)$$

$G$  and  $L$  are volumetric ionization and recombination, respectively. The second term is the electron flux through the system boundaries. Recombination is neglected.

The flux to a wall for a Maxwell-Boltzmann distribution is given by:

$$\Gamma_{wall} = \sqrt{\frac{kT}{2\pi m_e}} n_{e0} e^{e\phi_w/kT}, I_{wall} = \int_{surface} \Gamma_{wall} dS \quad (42)$$

The change in the number of Boltzmann electrons can be written as:

$$\Delta N_B = -I_{wall} \Delta t + G \Delta t. \quad (43)$$

This change is then updated each time step, so that the total number of Boltzmann electrons are related to  $n_{e0}$  by

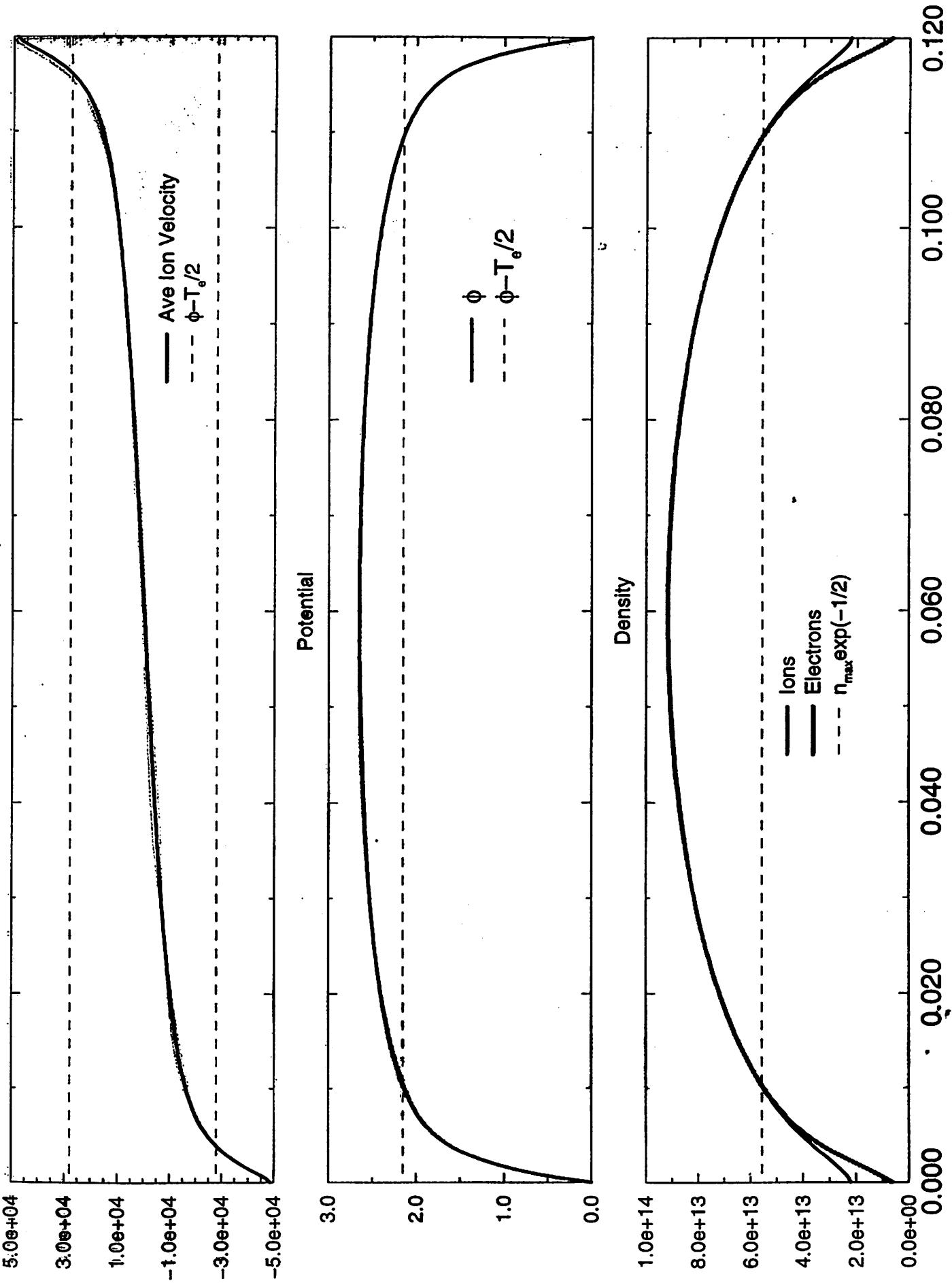
$$N_B = n_{e0} \int_V e^{e\phi(\mathbf{x})/kT} dV. \quad (44)$$

The initial value of  $N_B$  is  $\frac{1}{e} \int \rho_{PIC}(\mathbf{x}) dV$  for a charge neutral system.

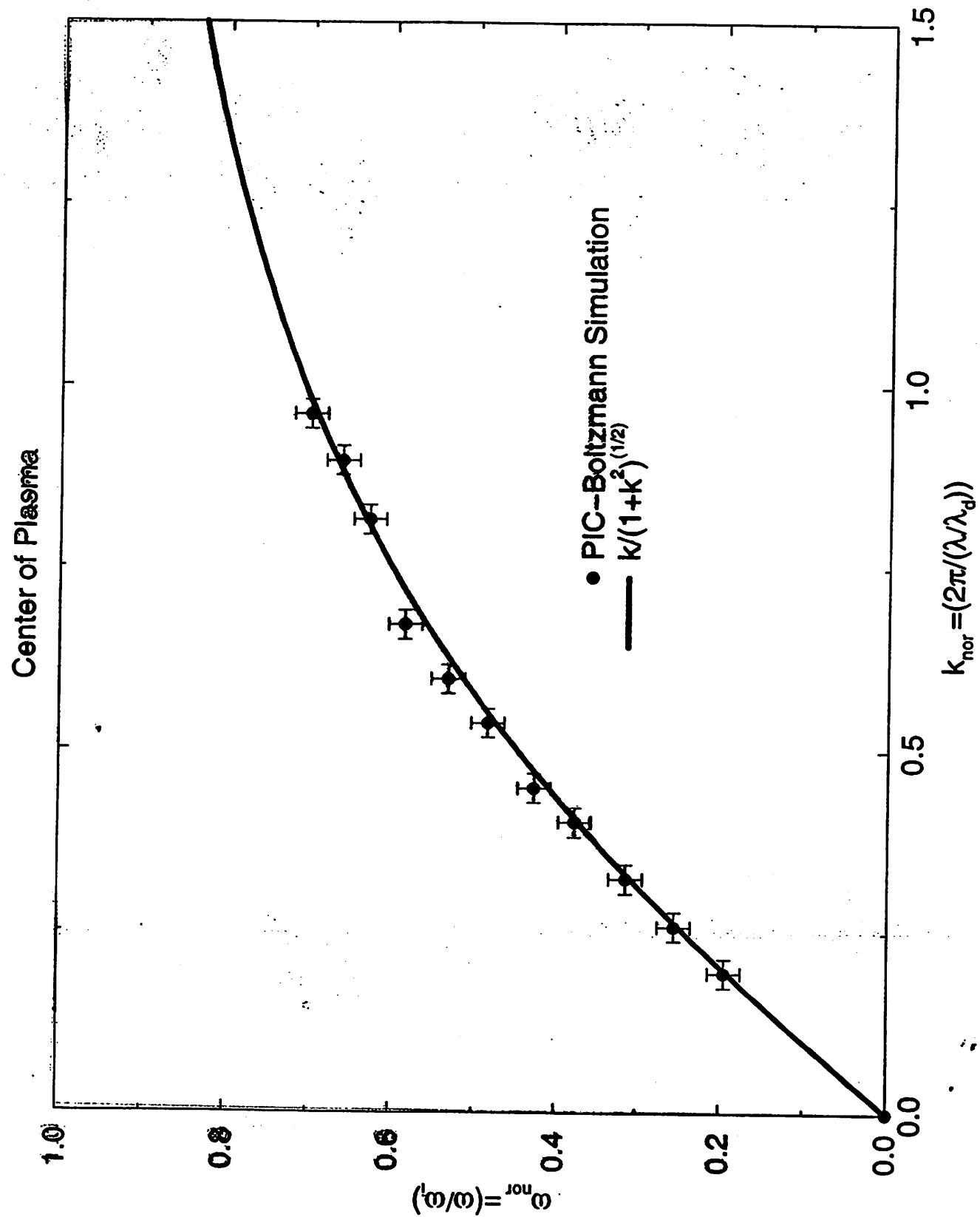
## References

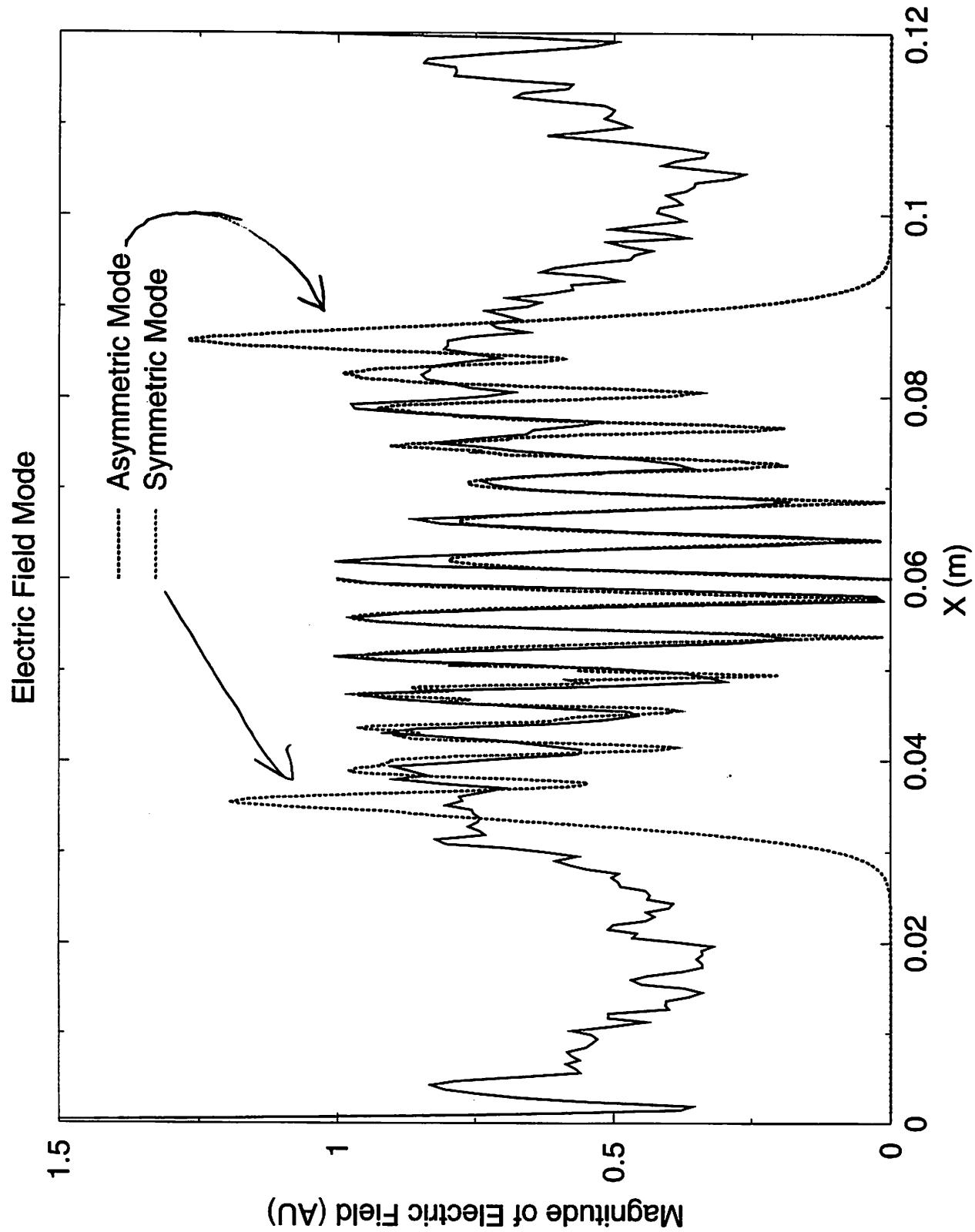
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## Boltzmann Electron Model



# Dispersion Relation





## Reflection of Ion Acoustic Waves and Solitons by Space Charge Sheath

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Committee:

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C. K. Birdsall, EECS advisor

J. Fajans, Physics advisor

W. Kunkel

M. Lieberman

## Introduction

1. Reflection and Transmission of Solitons from a Grid.
  - Experimental Results and Theoretical Explanation
    - Simulation Model
    - Simulation Results
2. Undriven Small Amplitude Standing waves.
  - Experimental Results
  - Lack of Theoretical Explanation
    - Simulation Model
3. Conclusions and Future Work

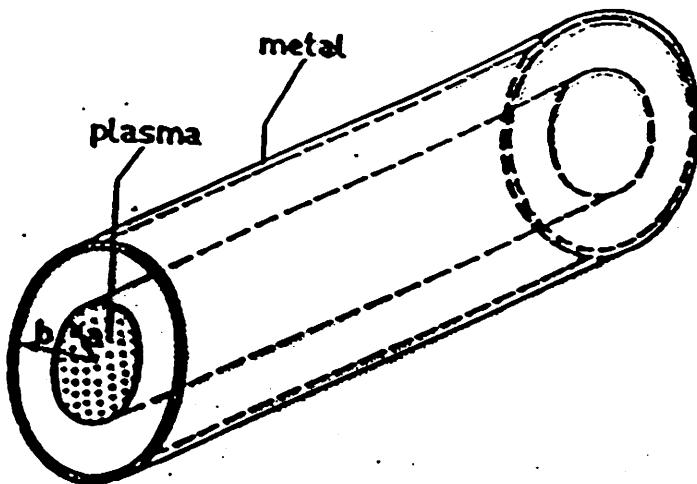


FIG. 1. Plasma column inside a cylindrical wave-guide.

Bengt Anderson & Peter Weissglas  
Physics of Fluids Vol 9, num 2 Feb 1966 p 271

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S. Watanabe, O. Ishikara and H. Tanaka

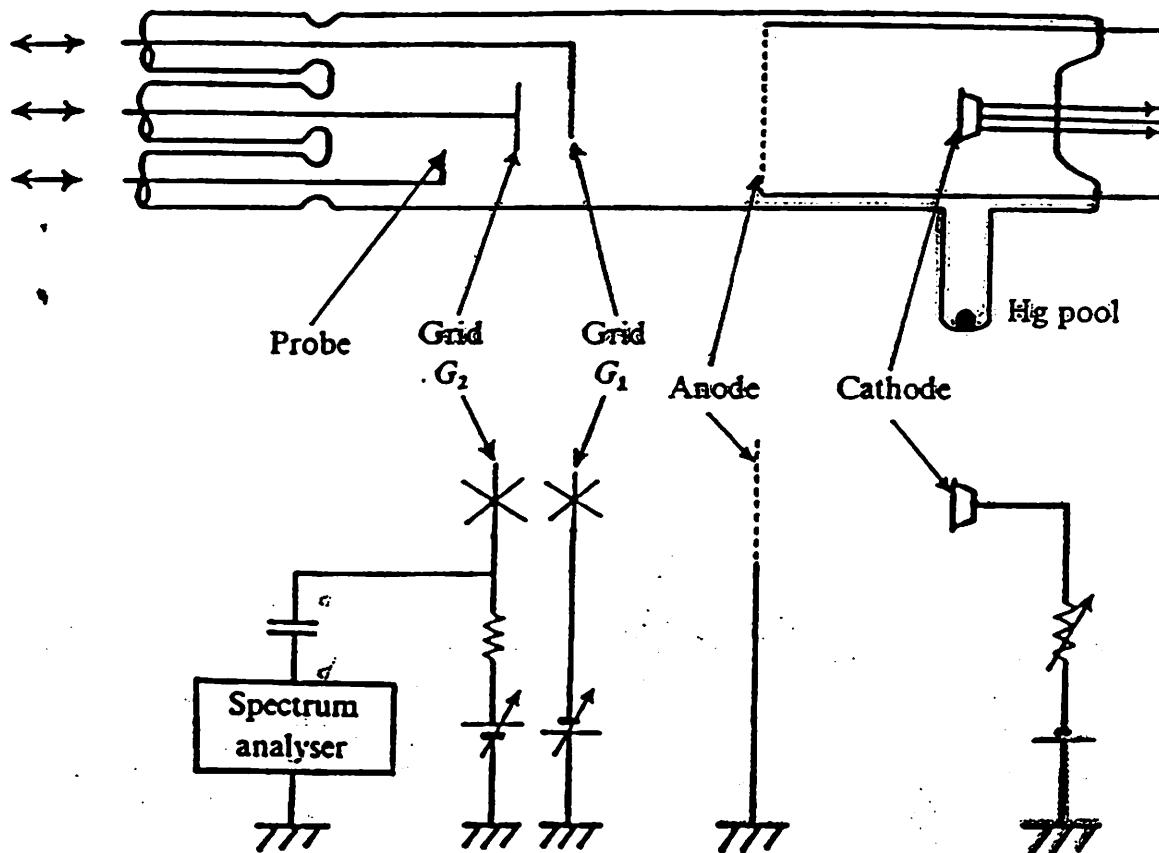


FIGURE 1. Experimental apparatus.

J. Plasma Physics (1972) Vol. 8, part 3 pp 321-330

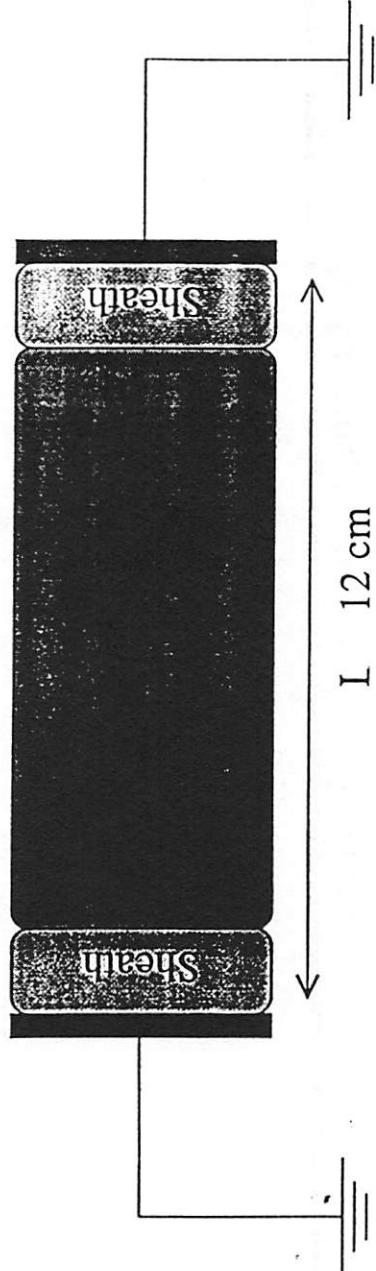
# Boltzmann Electrons–Photo Ionization–Steady State

## Input Parameters

Energy of photon	4.51 eV
Ionization rate	$2.16 \times 10^{16} \text{ s}^{-1}$
Background gas Pressure	3 mTorr
	He cross section
$\frac{\text{Mass of Ion}}{\text{Mass of Electron}}$	225

## Resulting Plasma

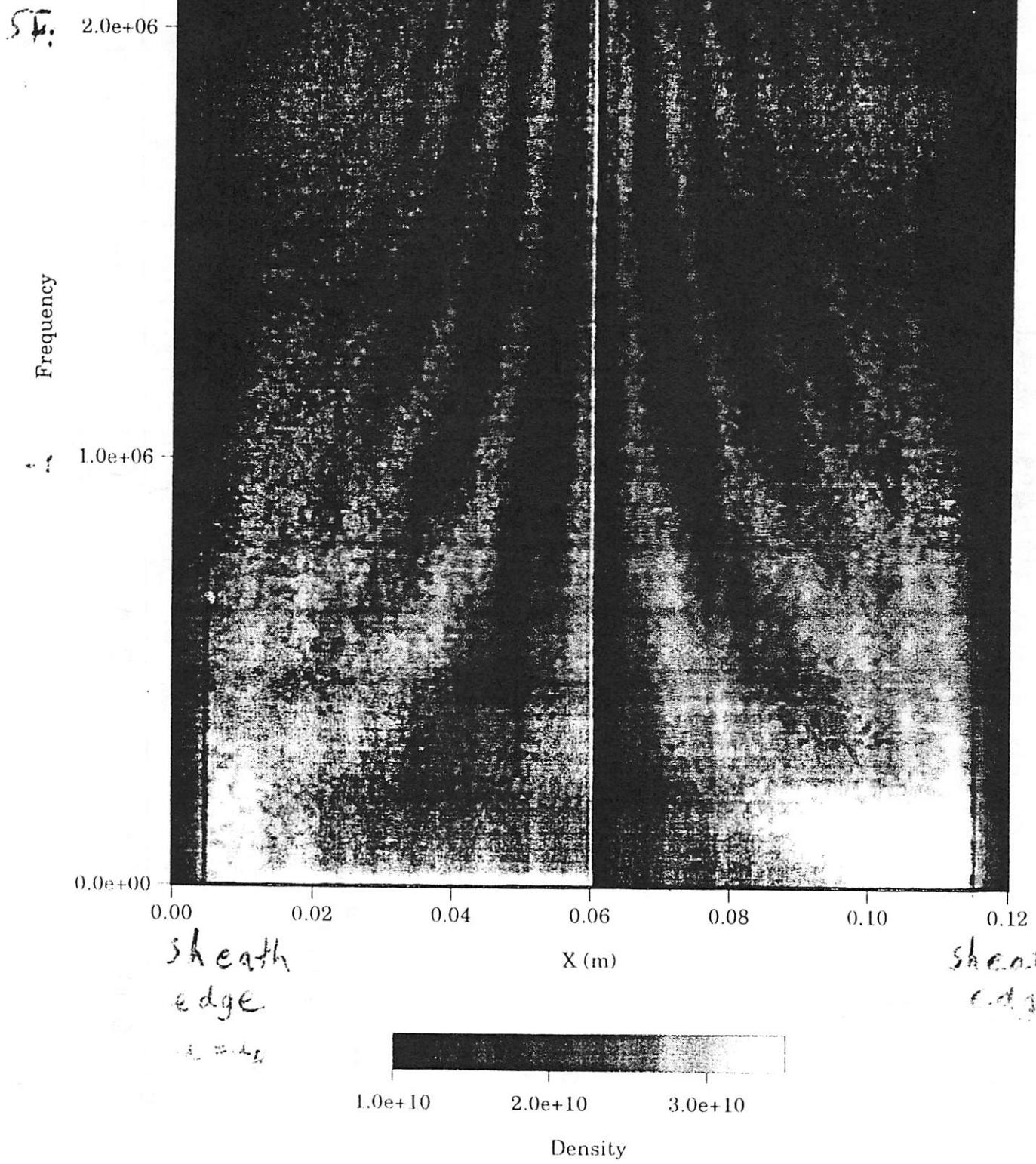
Electron Temperature	1.00 eV
Ion Temperature	.028 eV
Density	$9.2 \times 10^{13} \text{ m}^{-3}$
Electron Plasma Frequency	86 MHz
Ion Plasma Frequency	5.7 MHz
Debye Length	$7.76 \times 10^{-4} \text{ m}$
Number of Debye Lengths in system	155
Acoustic Speed	$2.79 \times 10^4 \text{ m/s}$



## Experimental Observation of Ion Acoustic Resonances

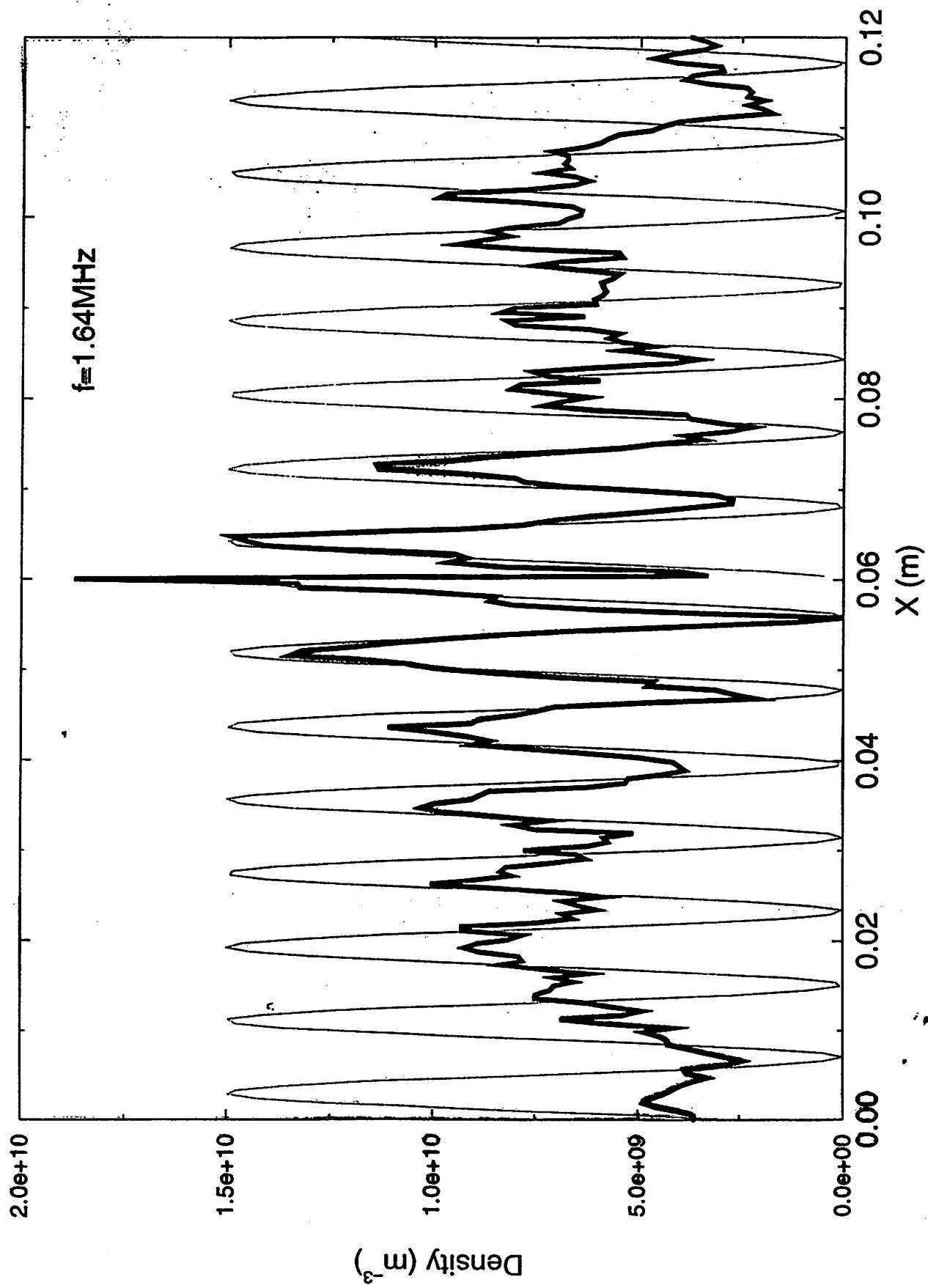
Experiment	Plasma Type	Nature of the Observation	Gases	Fundamental Mode	$Q$
	Noise	Excited Resonance			
<b>Cylindrical Geometry</b>					
Crawford et al. [2, 3]	dc	X	Hg	$J'_0(k\alpha)$	0.85
Stern et al. [4, 5]	dc	X	He, Hg	$J'_1(k\alpha)$	0.85
Lisitano et al. [6]	rf	X	H, He, Ar, Xe	$J'_0(k\alpha)$	0.3-0.65
Demokan et al. [7]	dc	X	Hg	$J'_1(k\alpha)$	1.13
Weynants et al. [8]	dc	X	Ar, Xe, Hg	$J'_0(k\alpha)$	.0.92-1.15
<b>Planar Geometry</b>					
Alexandrov et al. [9]	dc	X	He, Ne, Hg	$\cos(k\alpha)$	1.04-1.25
Asmussen et al. [10]	rf	X (NL)	Air	$\cos(k\alpha)$	.88
Kuzovnikov et al. [11]	dc	X	He	$\cos(k\alpha)$	?

$\text{Sym FFT} \left[ \frac{1}{\pi} (\delta(x) + \delta(1-x)) \right]$  Fourier Transform of Ion Density  
 $\text{asym FFT} \left[ \frac{1}{\pi} (\delta(x) - \delta(1-x)) \right]$



# Wavelength Comparison of sym-asym modes

Boltzmann Model



## Eikonal

The Eikonal form is:

$$\phi(x, t) = \tilde{\phi}(x) e^{-i\omega t} e^{i\Theta(x)} \quad (17)$$

Hamilton Ray Equations are:

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial\omega}{\partial\mathbf{x}}, \quad \frac{d\mathbf{x}}{dt} = \frac{\partial\omega}{\partial\mathbf{k}}, \quad \frac{d\omega}{dt} = \frac{\partial\omega}{\partial t} \quad (18)$$

The phase is:

$$\Theta(x) = \int_0^x k(x'; \omega) dx' \quad (19)$$

The amplitude-transport equation:

$$\frac{\partial}{\partial x} (\mathbf{v}_g |\tilde{\phi}(\mathbf{x})|^2) = 0 \quad (20)$$

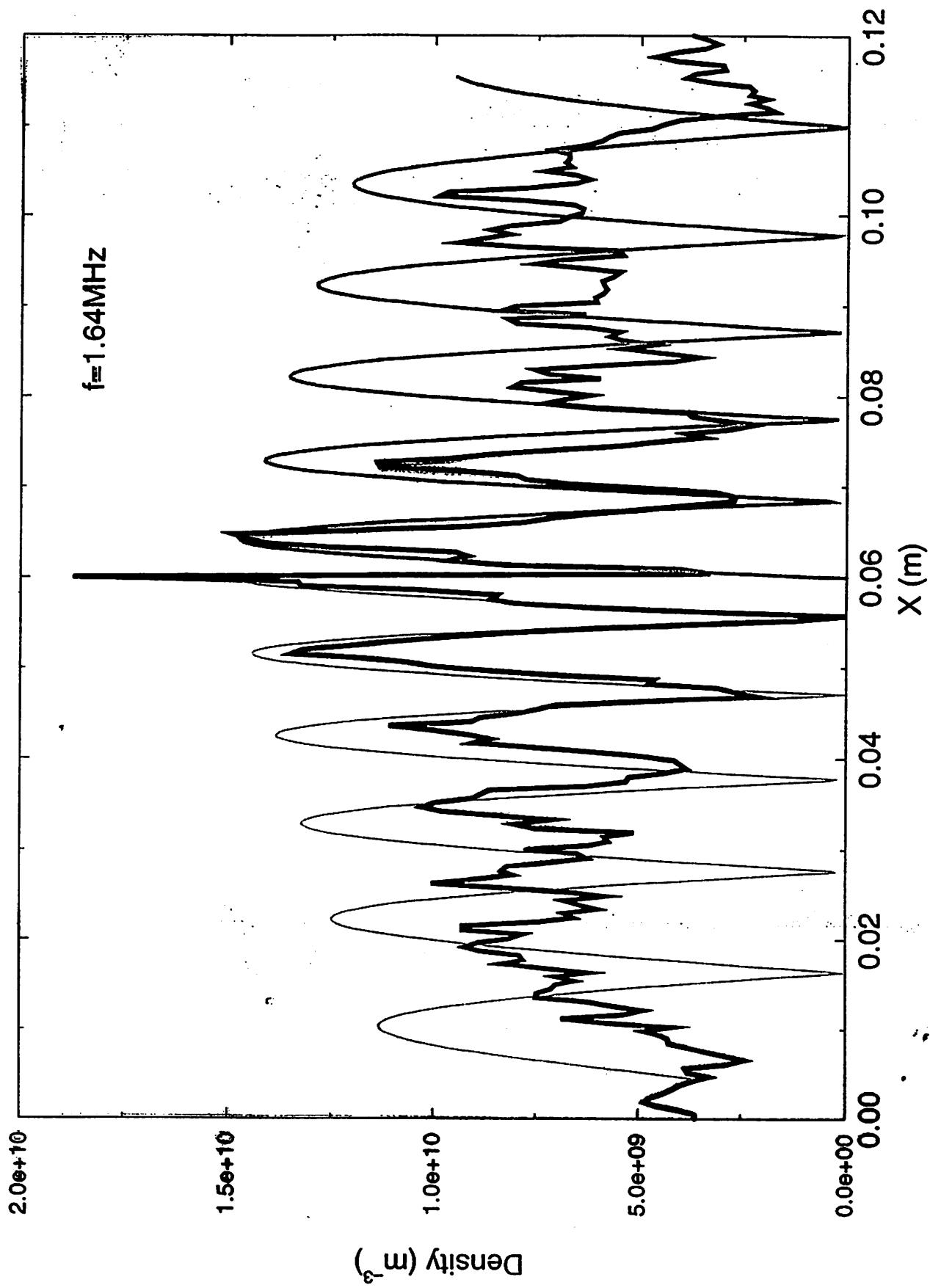
The ion acoustic dispersion relation that is used.

$$\omega = k(v(x) \pm \frac{1}{\sqrt{1 + n(x)k^2}}) \quad (21)$$

with  $v(x)$  and  $n(x)$  taken from the simulation.

# Eikonal Comparison of sym-asym modes

Boltzmann Model



Ion Density Perturbation  
 $n_i(x) - \bar{n}_i(x)$

Time (sec)

$T = 115$

5.0e-05

$T = 57$

4.0e-05

$T = 0$

3.0e-05

0.00 0.02 0.04 0.06 0.08 0.10 0.12

X (m)

Stability  
c. 1.5C  
(1.5 times)

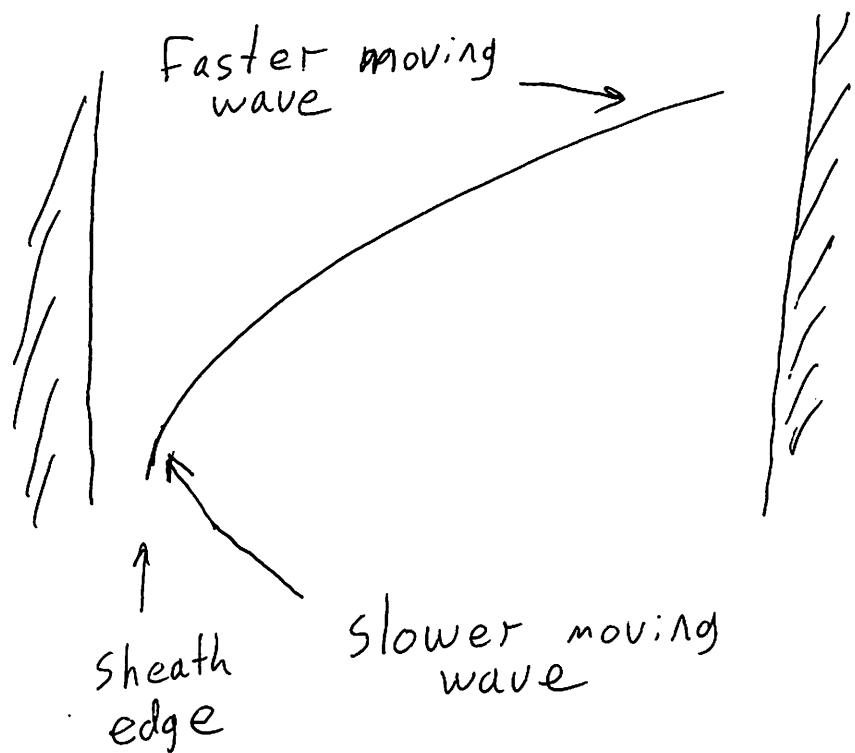
Stability  
c. 1.5C



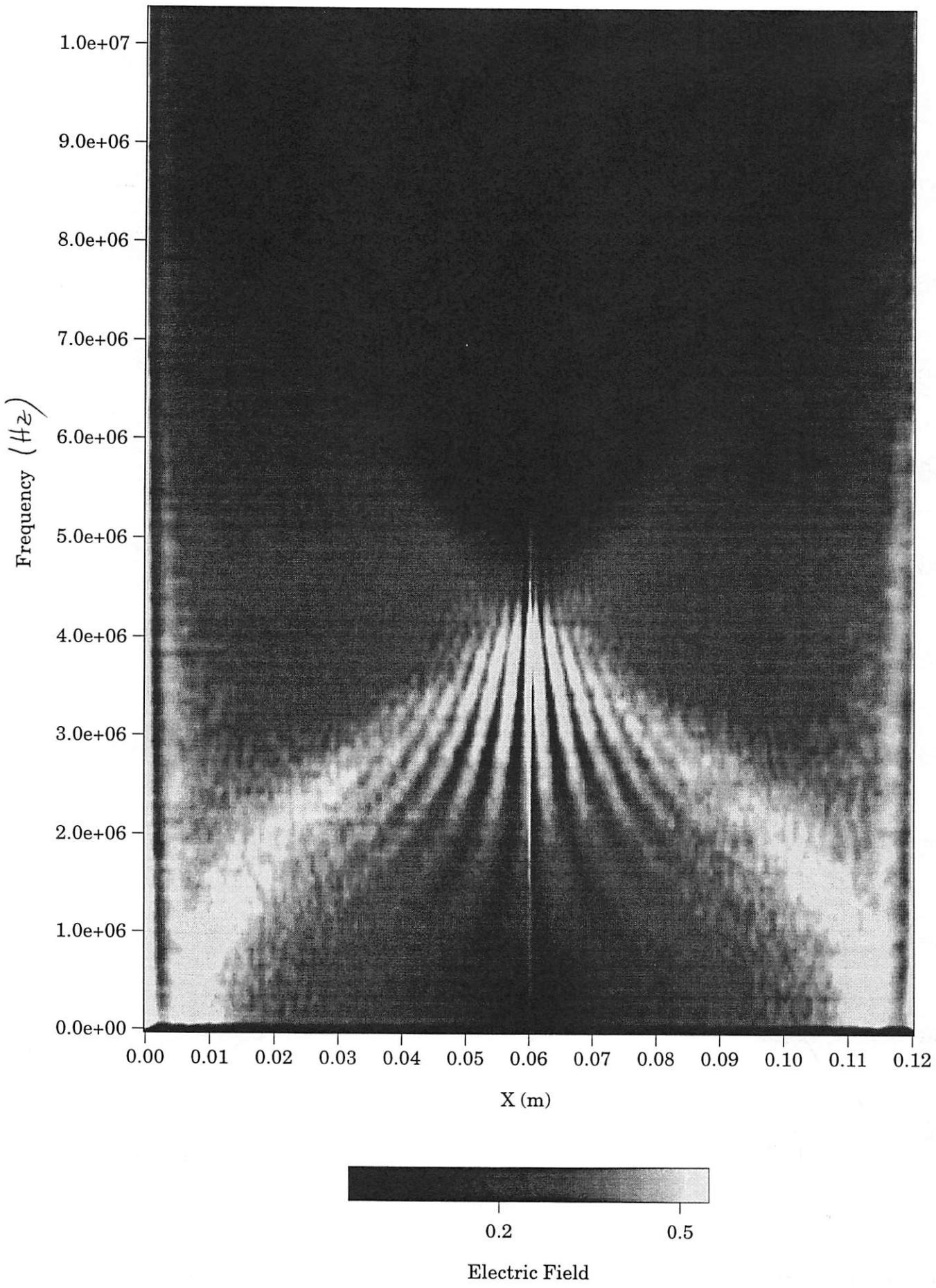
Density ( $m^{-3}$ )

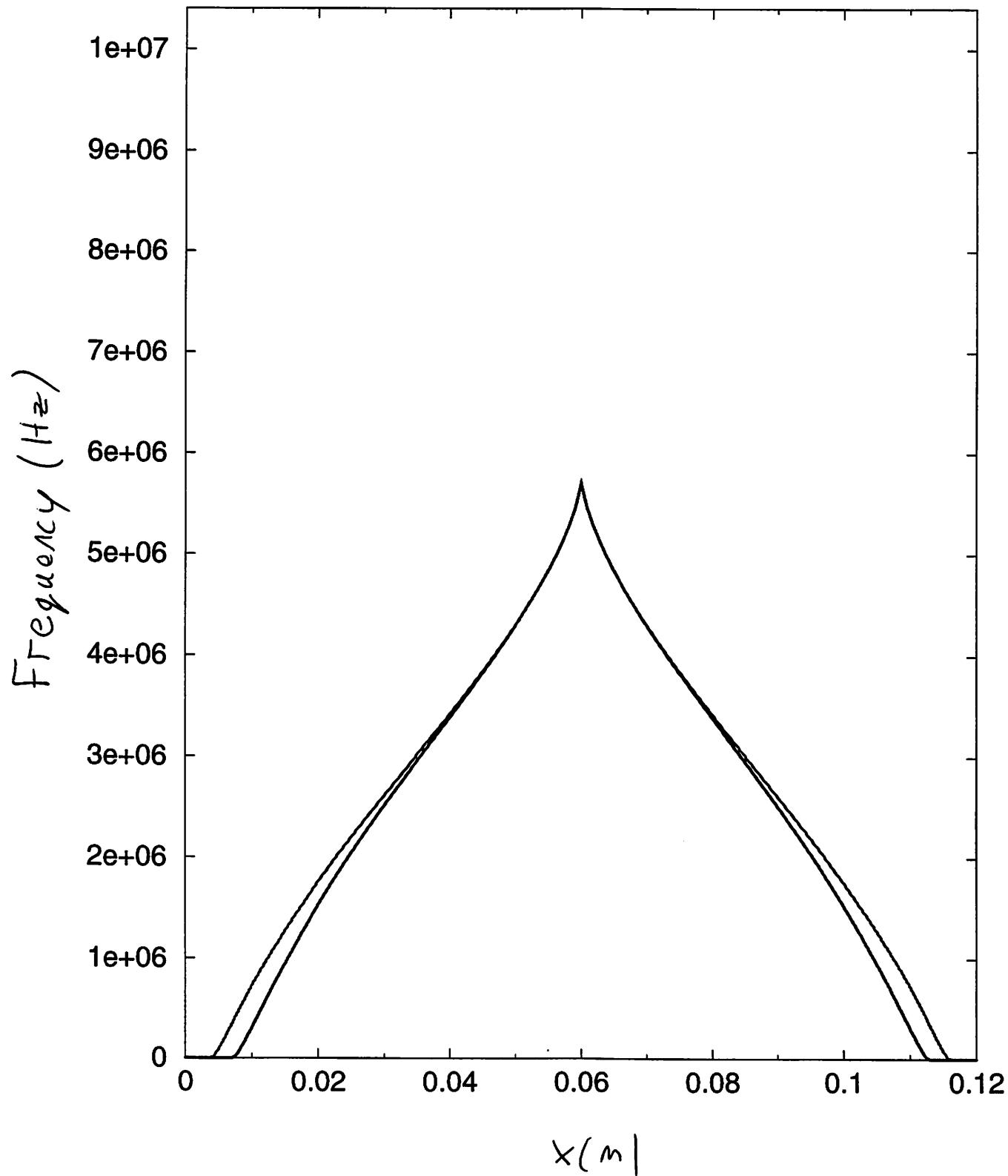
# Caption for the Figure entitled "Ion Density Perturbation"

The speed of the ion acoustic wave is equal to  $1/\text{slope}$  in the time-distance plot. Note that near the sheath the wave moving into the bulk is slowed down by the drifting ions and the wave increases in speed moving toward the wall by the drifting ions.

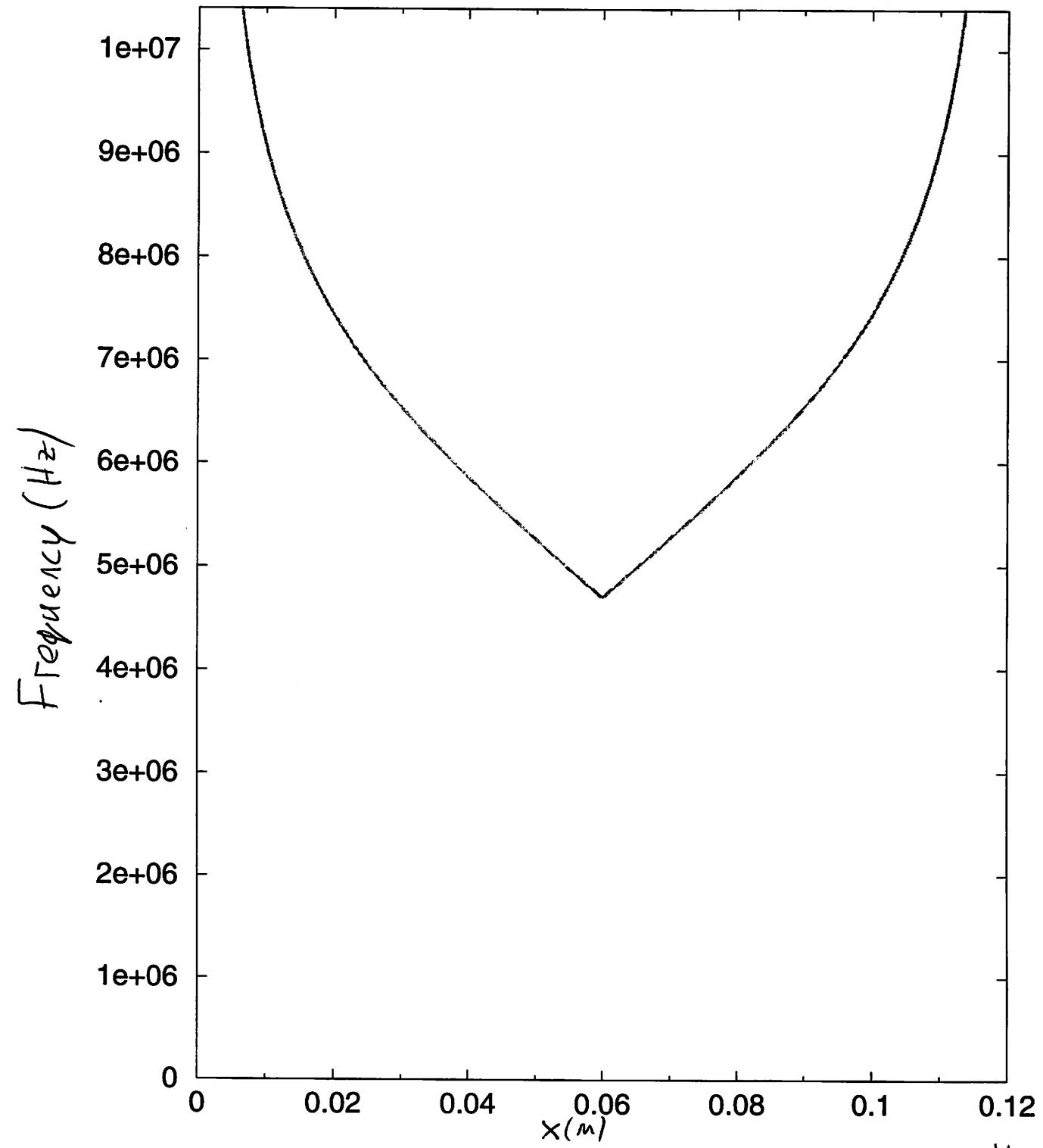


### Fourier Transform of Electric Field





The Frequency at which the group velocity is equal to the ion drift velocity,



The ion plasma frequency Doppler-shifted by the ion drift velocity.

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