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Abstract

Due to the strong experimental evidence that the traffic to be offered to future broadband networks will display long - range dependence, it is important to examine in more detail the possible implications that those new traffic models may have on the design and performance of those networks. In particular, one important question is whether the offered traffic preserves its long - range dependent nature after passing through the policing mechanism at the interface of the network. One of the proposed solutions for flow control in the context of the emerging ATM standard is the so-called leaky bucket scheme. In this note we consider a leaky bucket system with long - range dependent input traffic. We examine the departure process of the system and show that it, too, is long - range dependent for any token buffer size and infinite cell buffer size.

1 Introduction - Problem Formulation

Recent experimental studies of traffic to be carried by broadband networks have pointed out the possible importance of analyzing the performance of communication networks using traffic models with long - range dependence. Long - range dependence in network traffic has been reported, for instance, in [4], where statistical analysis of measurements of Ethernet traffic at Bellcore demonstrated its self - similar nature; in [3] long - range dependence has been established in variable bit rate video traffic generated by a number of different codecs;

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and in [7] the presence of long - range dependence in TELNET and other wide area network traffic was concluded.

Due to the strong experimental evidence that the traffic to be offered to future broadband networks will display long - range dependence, it is important to examine in more detail the possible implications that those new traffic models may have on the design and performance of those networks. In particular, one important question is whether the offered traffic preserves its long - range dependent nature after passing through the policing mechanism at the interface of the network. One of the proposed solutions for flow control in the context of the emerging ATM standard is the so-called *leaky bucket* scheme which is shown schematically in Figure 1. Fixed - size cells arrive in a buffer which for the subsequent analysis will be assumed to have infinite size. The departure of cells from the buffer is controlled by *tokens* that are stored in a buffer of fixed size C. An arriving cell can be transmitted only if it finds a token in the token buffer, in which case it is transmitted instantaneously by consuming a token. If the token buffer is empty, the cell has to wait for the generation of a new token. Time is assumed to be discrete and exactly one token is generated at the beginning of each unit of time. A stored cell is transmitted immediately upon the generation of a new token. Moreover, the outgoing capacity of the link is assumed to be at least C, so that it imposes

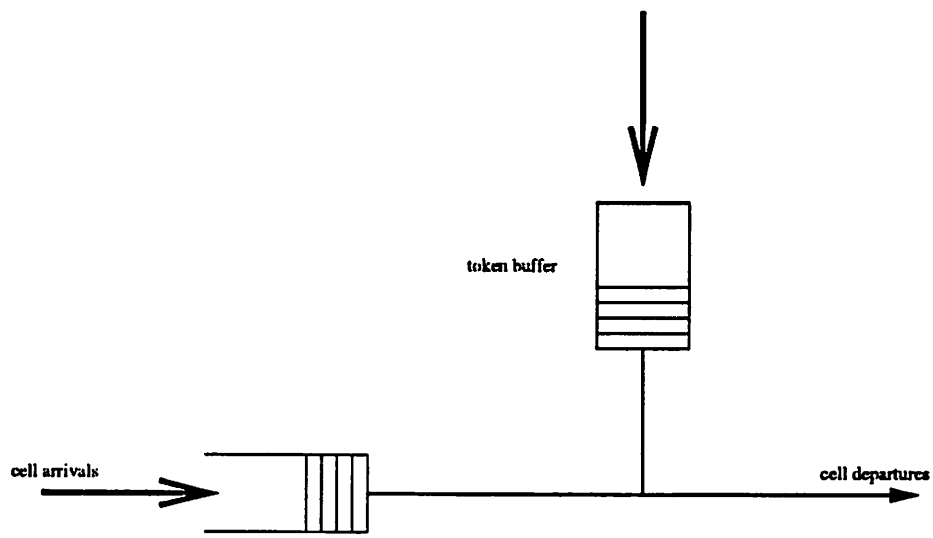

Figure 1: The leaky bucket scheme.

no limitations on the number of cells that can be transmitted instantaneously.

We assume that the cell arrival process belongs to a class of discrete - time long - range dependent traffic models which includes as a special case the one proposed in [5]. In an arrival model of this class a number of sessions are initiated at the beginning of each unit of time, which is a Poisson random variable with parameter λ. Each of those sessions consists of a random number τ of cells which has finite mean, infinite variance and a regularly varying tail, i.e. $P(\tau > k) \sim k^{-\alpha}L(k)$, where $1 < \alpha < 2$ and $L(\cdot)$ is a slowly varying function. Once a session is initiated, it generates one cell at the beginning of each unit of time until its termination. Finally, the stability condition $\lambda E\tau < 1$ is assumed to hold. This means that the departure process is stationary and ergodic according to the results in [6].

In this note we consider a leaky bucket system with the arrival process having a long - range dependent behaviour as discussed above. We examine the departure process of the system and show that it, too, is long - range dependent.

2 Statement of Results

We first examine the departure process in the case $C = 1$. In this case the leaky bucket system is equivalent to a single - server queue that is served at a constant rate equal to one cell per unit time. Thus we have one of the cases analyzed in [1], namely the one where the service rate s does not exceed the cell generation rate R (in our case $s = R = 1$). We prove the following:

Lemma 1. *The departure process of the leaky bucket system in the case $C = 1$ is long - range dependent.*

Proof

We recall that a necessary and sufficient condition for a stationary second-order stochastic process to be long - range dependent is that the sum of the absolute values of its covariances be infinite, i.e.

$$\sum_{m=1}^{\infty} \mid r(m) \mid = \infty, \tag{1}$$

where $r(m)$ the covariance function of the process.

If we denote the departure process by $\{d^{(1)}(k)\}$, then it is obvious that $d^{(1)}(k) \in \{0, 1\}$, $\forall k$. Moreover, the expected value of the departure process is equal to that of the input process in stationarity, since the stability condition holds, i.e.

$$E[d^{(1)}(k)] = \lambda E\tau. \tag{2}$$

The covariance of the departure process is:

$$\begin{aligned} r^{(1)}(m) &= E[(d^{(1)}(k) - \lambda E\tau)(d^{(1)}(k+m) - \lambda E\tau)] \\ &= (\lambda E\tau)^2 P(d^{(1)}(k+m) = 0, d^{(1)}(k) = 0) \\ &\quad -(\lambda E\tau)(1 - \lambda E\tau) P(d^{(1)}(k+m) = 0, d^{(1)}(k) = 1) \\ &\quad -(\lambda E\tau)(1 - \lambda E\tau) P(d^{(1)}(k+m) = 1, d^{(1)}(k) = 0) \\ &\quad +(1 - \lambda E\tau)^2 P(d^{(1)}(k+m) = 1, d^{(1)}(k) = 1). \end{aligned}$$

If we define $P^m_{i|j} \triangleq P(d^{(1)}(k+m) = i \mid d^{(1)}(k) = j)$ and take into account that because of (2) we have $P(d^{(1)}(k) = 0) = 1 - \lambda E\tau$, $P(d^{(1)}(k) = 1) = \lambda E\tau$, we get from the above relation:

$$r^{(1)}(m) = (\lambda E\tau)^2(1 - \lambda E\tau) P^m_{0|0} - (\lambda E\tau)^2(1 - \lambda E\tau) P^m_{0|1}$$

$$
\begin{aligned}
&-(\lambda E\tau)(1-\lambda E\tau)^2 P^m_{1|0} + (\lambda E\tau)(1-\lambda E\tau)^2 P^m_{1|1} \\
&= (\lambda E\tau)(1-\lambda E\tau)(P^m_{0|0} - P^m_{0|1}) \quad (3)\\
&= (\lambda E\tau)(1-\lambda E\tau)(P^m_{1|1} - P^m_{1|0}) \quad (4)
\end{aligned}
$$

In the above we used the fact that $P^m_{0|0} + P^m_{1|0} = P^m_{0|1} + P^m_{1|1} = 1$. From equation (3) it is obvious that $r^{(1)}(m)$ is positive, since $0 < \lambda E\tau < 1$ and $P^m_{0|0} > P^m_{0|1}$ which can be easily seen from the definition of $P^m_{i|j}$.

Consider now equation (4). It is obvious that $d^{(1)}(k) = 0(1)$ means that the cell buffer at the time instant k is empty (nonempty). We will henceforth refer to cells belonging to sessions that have been initiated at or before time k as cells of type 1, while cells belonging to sessions that arrived after time k will be referred to as cells of type 2. Assume that beginning at time $k+1$ priority is given to cells of type 2 over cells of type 1 in the case of the initially nonempty cell buffer. If at some point there are no cells of type 2 in the system, then the service of any remaining cells of type 1 may be resumed. Since the arrival process after time $k+1$ is independent of the past, we may assume that it is pathwise the same for both the system with the initially empty cell buffer and the one with the initially nonempty buffer. Then it follows easily that we get the following relation:

$$
P^m_{1|1} - P^m_{1|0} = P\{d^{(1)}(k+m) = 1 \textit{ and the cell departing at } k+m \textit{ is of type } 1 \mid d^{(1)}(k) = 1\} \quad (5)
$$

Let Z^*_k denote the remaining number of cells at time k that it is already known will have to be handled by the system. Thus Z^*_k consists of the cells that are in the cell buffer immediately before time k as well as the cells of the already existing sessions that have not yet arrived in the system. Moreover, let ϕ_k be the total number of cells that are brought in by the sessions that are initiated at time k. Then the number of cells of type 1 in the system immediately before time $k+1$ is obviously equal to $N_1 = (Z^*_k + \phi_k - 1)^+$, where we take into account that in the time slot $[k, k+1)$ at most one cell of type 1 leaves the system. In the above $x^+ = max\{0, x\}$. The random variables Z^*_k and ϕ_k are assumed to have their stationary distribution in the analysis that follows.

According to the above discussion in order to show that the departure process is long - range dependent, it suffices to show that

$$
\sum_{m=1}^{\infty} P\{d^{(1)}(k+m) = 1 \textit{ and the cell departing at } k+m \textit{ is of type } 1 \mid d^{(1)}(k) = 1\} = \infty \quad (6)
$$

We have:

$$
\sum_{m=1}^{\infty} P\{d^{(1)}(k+m) = 1 \textit{ and cell of type } 1 \mid d^{(1)}(k) = 1\} =
$$

$$
\sum_{m=1}^{\infty}\sum_{i=1}^{\infty} P\{d^{(1)}(k+m) = 1 \textit{ and cell of type } 1 \mid N_1 = i, d^{(1)}(k) = 1\} P\{N_1 = i \mid d^{(1)}(k) = 1\} \quad (7)
$$

Since all terms are nonnegative we may change the order of summation so that we get:

$$\sum_{m=1}^{\infty} P\{d^{(1)}(k+m)=1 \text{ and cell of type } 1 \mid d^{(1)}(k)=1\} =$$
$$\sum_{i=1}^{\infty} P\{N_1=i \mid d^{(1)}(k)=1\} \sum_{m=1}^{\infty} P\{d^{(1)}(k+m)=1 \text{ and cell of type } 1 \mid N_1=i,\ d^{(1)}(k)=1\} \tag{8}$$

For the inner summation we have:

$$\begin{aligned}
&\sum_{m=1}^{\infty} P\{d^{(1)}(k+m)=1 \text{ and cell of type } 1 \mid N_1=i,\ d^{(1)}(k)=1\} = \\
&\quad \sum_{m=1}^{\infty} E[I\{d^{(1)}(k+m)=1 \text{ and cell of type } 1\} \mid N_1=i,\ d^{(1)}(k)=1] = \\
&\quad E[\sum_{m=1}^{\infty} I\{d^{(1)}(k+m)=1 \text{ and cell of type } 1\} \mid N_1=i,\ d^{(1)}(k)=1]
\end{aligned} \tag{9}$$

where in the above we used the fact that we can exchange the limit and the expectation according to the conditional version of the monotone convergence theorem [9]. We also used the conventional notation that $I\{A\}$ is the indicator function of the event A. Now we observe that since the stability condition $\lambda E\tau < 1$ holds, the cell buffer will empty in finite time with probability 1 [6]. This means that on the event $N_1 = i$

$$\sum_{m=1}^{\infty} I\{d^{(1)}(k+m)=1 \text{ and cell of type } 1\} = i \quad w.p.\ 1 \tag{10}$$

Therefore, from (9) and (10) we immediately get that the value of the inner summation is equal to i, so that we have from equation (8):

$$\begin{aligned}
\sum_{m=1}^{\infty} P\{d^{(1)}(k+m)=1 \text{ and cell of type } 1 \mid d^{(1)}(k)=1\} &= \sum_{i=1}^{\infty} iP\{N_1=i \mid d^{(1)}(k)=1\} \\
&= E[N_1 \mid d^{(1)}(k)=1]
\end{aligned} \tag{11}$$

But from the definition of N_1 it is obvious that

$$\begin{aligned}
E[N_1 \mid d^{(1)}(k)=1] &= E[(Z_k^* + \phi_k - 1)^+ \mid d^{(1)}(k)=1] \\
&= E[Z_k^* + \phi_k - 1 \mid d^{(1)}(k)=1] \\
&\geq E[Z_k^*] - 1
\end{aligned} \tag{12}$$

But from Lemma 3.2 of [1] we have that

$$P(Z_k^* > x) \sim x^{1-\alpha} L(x) \tag{13}$$

where $1 < \alpha < 2$ is the parameter of the regularly varying tail of the session duration and $L(\cdot)$ is a slowly varying function. From this result we immediately conclude that $E[Z_k^*] = \infty$ which means that $\sum_{m=1}^{\infty} r^{(1)}(m) = \infty$ so that the departure process in the case $C = 1$ is indeed long - range dependent. $\square$

We will now consider the case of $C > 1$. In accordance with the notation used above we denote by $\{d^{(C)}(k)\}$ the departure process of the leaky bucket system in the case that the token buffer size is C. In what follows we will refer to the leaky bucket system with token buffer size equal to C (1) as the C-system (1-system). We will first need the following:

Lemma 2. *For every sample path of the arrival process the following inequality holds:*

$$d^{(1)}(k)+d^{(1)}(k+1)+\ldots+d^{(1)}(k+m) \leq d^{(C)}(k)+d^{(C)}(k+1)+\ldots+d^{(C)}(k+m)+(C-1), \quad (14)$$

for any selection of k and $m \geq 0$.

Proof

In what follows we will consider the evolution of the two systems under the assumption that they are driven by the same sample path of the arrival process. We will use the pathwise construction of the departure process of a leaky bucket system presented in [2]. Let $X^{(C)}(k)$, ($Y^{(C)}(k)$) denote the number of cells (tokens) in the cell (token) buffer immediately after time k in the system with token buffer size C. We assume that cells arrive exactly at time k so that the quantities $X^{(C)}(k)$, $Y^{(C)}(k)$ give the number of cells or tokens after the maximum possible number of cells has been served at time k. Obviously, $0 \leq X^{(C)}(k) < \infty$ and $0 \leq Y^{(C)}(k) \leq C$. We may clearly assume that $X^{(C)}(k)Y^{(C)}(k) = 0$ for all k, so that the system can be described by the parameter $Z^{(C)}(k) = X^{(C)}(k) - Y^{(C)}(k)$. If $Z^{(C)}(k) > 0$ there are $Z^{(C)}(k)$ cells and no tokens, if $-C \leq Z^{(C)}(k) < 0$ there are $-Z^{(C)}(k)$ tokens and no cells, and if $Z^{(C)}(k) = 0$ then there are neither cells nor tokens. Let $a(k)$ denote the number of cells arriving into the leaky bucket at time k. Then it is easy to see that $Z^{(C)}(k)$ is given by the recursion

$$Z^{(C)}(k+1) = Z^{(C)}(k) + a(k+1) - I\{Z^{(C)}(k) \geq -C+1\} \quad (15)$$

If we now define the quantity $W(k) = Z^{(C)}(k) + C = X^{(C)}(k) - Y^{(C)}(k) + C$, we get the following recursion for $W(k)$ from (15):

$$W(k+1) = W(k) + a(k+1) - I\{W(k) > 0\} = (W(k)-1)^+ + a(k+1) \quad (16)$$

Note that $W(k) \geq 0$, $\forall k$. Relation (16) is a familiar recursion from basic queueing theory [8] and from [6] we know that, since the stability condition $\lambda E\tau < 1$ holds, there exists a stationary and ergodic random process that satisfies the recursion (16). Note that the distribution of W does not depend on the choice of the token buffer size C. Nevertheless, the distributions of $Z^{(C)} = W - C$, $X^{(C)} = max\{W - C, 0\}$ and $Y^{(C)} = max\{C - W, 0\}$ do depend on C, as they should.

It is easy to see that the departure process from the leaky bucket can be constructed from a sample path of the W process as shown in Figure 2. In order to determine the number of

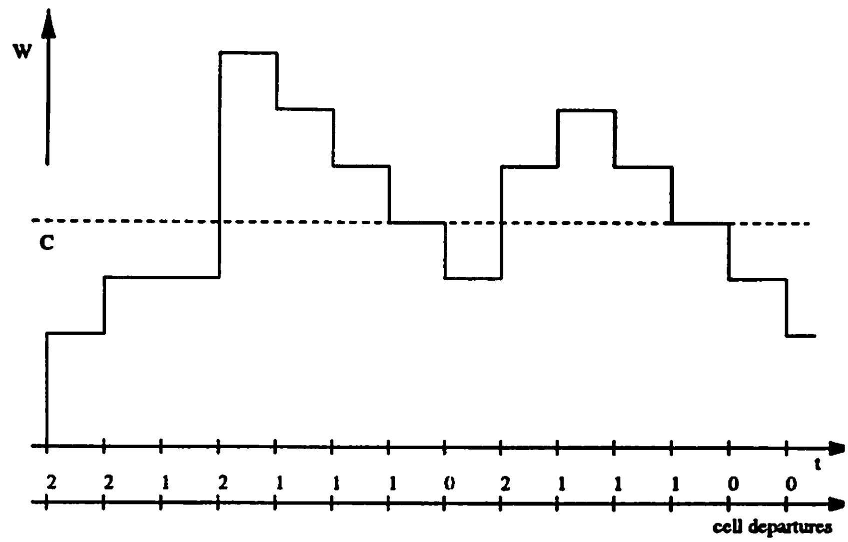

Figure 2: Construction of the departure process. The numbers on the axis labeled 'cell departures' give the number of cells leaving the system at each time instant for the specific realization of the process W.

cells $d^{(C)}(k)$ that leave the C-system at time k we consider the following cases in terms of the transition $W(k-1) \to W(k)$:
a) $W(k-1) = 0$: In that case the number of departing cells is $d^{(C)}(k) = min\{W(k), C\}$.
b) $0 < W(k-1) \leq C$: In that case the number of departing cells is $d^{(C)}(k) = min\{W(k), C\} - W(k-1) + 1$.
c) $W(k-1) > C$: In that case there is exactly one departure at time k due to the arriving token, i.e. $d^{(C)}(k) = 1$.
It is obvious that the above cases hold for any token buffer size and therefore also for $C = 1$. This means that the departure processes for a whole range of systems, corresponding to different values of C, can be read off directly from the sample path of W. Note that for $W(k-1) > C$ we can only have one departure at any time instant.

Let $d_{k,m}^{(C)} = d^{(C)}(k) + \ldots + d^{(C)}(k+m)$ be the number of departing cells from the system with token buffer of size C during the interval $[k, k+m]$, $m > 0$. Also let $\Delta d_{k,m} = d_{k,m}^{(1)} - d_{k,m}^{(C)}$ be the difference in the number of departing cells from the two systems in the same interval. We are interested in an upper bound of the quantity $\Delta d_{k,m}$. Using the above results (a) - (c) we have the following cases:
- $W(k-1) \in \{0,1\}$ and $W(k+m) \in \{0,1\}$: Then $\Delta d_{k,m} = 0$.
- $W(k-1) \in \{0,1\}$ and $W(k+m) \in \{2, \ldots, C\}$: Then $\Delta d_{k,m} = 1 - W(k+m)$.
- $W(k-1) \in \{0,1\}$ and $W(k+m) > C$: Then $\Delta d_{k,m} = 1 - C$.

- $W(k-1) \in \{2, \ldots, C\}$ and $W(k+m) \in \{0,1\}$: Then $\Delta d_{k,m} = W(k-1) - 1$.
- $W(k-1) \in \{2, \ldots, C\}$ and $W(k+m) \in \{2, \ldots, C\}$: Then $\Delta d_{k,m} = W(k-1) - W(k+m)$.
- $W(k-1) \in \{2, \ldots, C\}$ and $W(k+m) > C$: Then $\Delta d_{k,m} = W(k-1) - C$.
- $W(k-1) > C$ and $W(k+m) \in \{0,1\}$: Then $\Delta d_{k,m} = C - 1$.
- $W(k-1) > C$ and $W(k+m) \in \{2, \ldots, C\}$: Then $\Delta d_{k,m} = C - W(k+m)$.
- $W(k-1) > C$ and $W(k+m) > C$: Then $\Delta d_{k,m} = 0$.

We see that the greatest value of $\Delta d_{k,m}$ is achieved for any interval $[k, k+m]$ with $W(k-1) \geq C$ and $W(k+m) \leq 1$ and for those intervals we get $\Delta d_{k,m} = C - 1$. Hence, $d^{(1)}_{k,m} - d^{(C)}_{k,m} \leq C - 1$, $\forall m > 0, k$ and the proof is complete. □

Now we can prove the following:

Theorem *The departure process of the leaky bucket system is long - range dependent for any token buffer size C.*

Proof
We have shown the result for $C = 1$. To prove the result in the case $C > 1$ we proceed as follows: Both sides of (14) are nonegative, so we may square both sides and the inequality will still hold. If we also take expectations on both sides and subtract the quantity $E[d^{(1)}(k)]^2 = E[d^{(C)}(k)]^2 = (\lambda E\tau)^2$ from each expectation term on both sides we get the following relation:

$$(m+1)r^{(1)}(0) + 2\sum_{i=1}^{m}(m-i+1)r^{(1)}(i) \leq$$
$$(m+1)r^{(C)}(0) + 2\sum_{i=1}^{m}(m-i+1)r^{(C)}(i) + 2(m+1)C\lambda E\tau + C^2 \tag{17}$$

where in the above we took into account that the output process is stationary. Let $\Sigma^{(C)}_0(l) \triangleq \sum_{i=0}^{l}\left|r^{(C)}(i)\right|$ and $\Sigma^{(1)}_1(l) \triangleq \sum_{i=1}^{l} r^{(1)}(i)$. Also let $d \triangleq C\lambda E\tau$.
Since $r^{(1)}(0), r^{(C)}(0) \geq 0$ and $2C\lambda E\tau < C^2$ for $C \geq 2$ we get from (17) after a few simple algebraic manipulations.

$$\sum_{l=1}^{m}(\Sigma^{(1)}_1(l) - d) \leq \sum_{l=0}^{m}\Sigma^{(C)}_0(l) \tag{18}$$

In order to show that $\{d^{(C)}(k)\}$ is long - range dependent, it suffices to show that for every $M > 0$ there exists some integer L, such that

$$\Sigma^{(C)}_0(l) > M, \quad \forall l \geq L \tag{19}$$

Note that since $\Sigma^{(C)}_0(l)$ is monotone increasing in l, then if (19) holds for some index L, it will definitely hold for all indices $l \geq L$. To show (19) we may argue by contradiction. Suppose that for some M_0 there is no integer L, such that $\Sigma^{(C)}_0(L) > M_0$. But, since

$\lim_{l\to\infty} \Sigma_1^{(1)}(l) = \infty$, this means that there exists some integer K, such that $\Sigma_1^{(1)}(l) - d > M_0 > \Sigma_0^{(C)}(l)$, $\forall l \geq K$. This obviously contradicts the inequality in (18) and therefore we must have that (19) holds or equivalently that $\{d^{(C)}(k)\}$ is long - range dependent. □

3 Concluding Remarks

We have studied the departure process of a leaky bucket system in an ATM network fed by a class of proposed models for long - range dependent input traffic. We established the fact that the departure process is long - range dependent for any token buffer size and infinite cell buffer size.

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