A UNIFIED FRAMEWORK FOR SYNCHRONIZATION AND CONTROL OF DYNAMICAL SYSTEMS II — ARRAYS OF LINEARLY COUPLED SYSTEMS

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Abstract

In this paper, we extend the results in [Wu and Chua, 1994] and give sufficient conditions for an array of linearly coupled systems to synchronize. A typical result states that the array will synchronize if the nonzero eigenvalues of the coupling matrix have real parts that are negative enough. In particular, we show that the intuitive idea that strong enough mutual diffusive coupling will synchronize an array of identical cells is true in general. Sufficient conditions for synchronization for several coupling configurations will be considered. For coupling which leaves the array decoupled at the synchronized state, the cells each follow their natural uncoupled dynamics at the synchronized state. We illustrate this with an array of chaotic oscillators. Extensions of these results to general coupling are discussed.

1 Introduction

Recently, arrays of coupled systems has received much attention as they can exhibit many interesting phenomena, such as spatio-temporal chaos [Zhelenyak and Chua, 1994], autowaves [Perez-Munuzuri *et al.*, 1993b], and spiral waves [Perez-Munuzuri *et al.*, 1993a] and are important in modeling populations of interacting biological systems [Murray, 1989]. In addition, they have also been used in applications such as image processing [Chua and Yang, 1988b; Perez-Munuzuri *et al.*, 1993b]. It has been observed that coupling allow cells to synchronize to each other. For example, fireflies has been known to fire in unison, and this phenomena has been proved to occur in a group of integrate-and-fire cells [Mirollo and Strogatz, 1990]. In this paper, we analyze when individual systems in the array are synchronized in a strong sense, i.e. each system's trajectory tracks the trajectories of all other systems in the array. We give sufficient conditions for an array of identical cells to asymptotically synchronize. In particular, we extend the theory discussed in [Wu and Chua, 1994] to arrays of systems which are linearly coupled. It was argued in [Pérez-Villar *et al.*, 1993] that the role of diffusive coupling in a one-dimensional array of nonlinear active systems is always to stabilize the system, and this is proved for the case of a one-dimensional array of Chua's circuits for large enough homogeneous coupling in [Belykh *et al.*, 1993]. We prove that this is indeed the case in general for large diffusive coupling. In particular, we prove that systems with uniformly bounded Jacobians, with symmetric or mutual diffusive coupling that connects the entire array together, will synchronize for large enough coupling. This supports the intuitive idea that in a system of individuals, strong enough *cooperation* between individuals *will result* in synchronization among them. We illustrate this by means of resistively coupled Chua's oscillators. We discuss how coupling should be designed such that synchronization is preserved when some of the coupling are deleted or when the coupling is perturbed. We briefly discuss how these results can be extended to general additive coupling.

In most cases in the literature, the coupling is such that when the array is synchronized, the cells are decoupled. For this important subclass, at the synchronized state, the dynamics of the array reduce to that of a single cell. For example, if the uncoupled cells were chaotic, then after applying coupling which synchronizes the system, each cell will still be chaotic, although the array is synchronized.

In section 2 the state equations of the array that we will consider and the various notions of synchronization are defined. In section 3 we define the notations used and give some results from matrix theory which are useful in our discussions. Some of the classes of matrices that we encounter as coupling matrices are symmetric matrices, normal matrices, circulant matrices, irreducible matrices, and nonnegative matrices. In section 4 we state the main results of Lyapunov's direct method which we use to prove synchronization. In section 5 we consider synchronization of an array of identical systems and give an algorithm for proving asymptotic synchronization. In section 6 we consider a special case where the algorithm used to prove asymptotic synchronization in section 5 can be further simplified. This section contains the main results of this paper. Several coupling configurations are considered and these results are illustrated by means of an array of resistively coupled Chua's oscillators. In section 7 we simplify the analysis of the array by decomposing it into irreducible components. In section 8 we briefly discuss simplifications which are possible when the cells are arranged in a regular lattice with homogeneous coupling. In section 9, we give sufficient conditions for additive nonlinear coupling to make the synchronized state asymptotically stable.

2 Basic Framework

In this paper, we will denote scalar variables in lower case, matrices in bold type upper case, and vectors (or vector-valued functions) in bold type lower case. We consider an array of m *cells*, coupled linearly together, with each cell being a n-dimensional system. We assume that $m \geq 2$. The entire array is a system of nm ordinary differential equations. In particular, the state equations are:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1, t) + \textstyle\sum_{i=1}^m \mathbf{D}_{1,i}\mathbf{x}_i \\ &\vdots \\ \dot{\mathbf{x}}_m &= \mathbf{f}_m(\mathbf{x}_m, t) + \textstyle\sum_{i=1}^m \mathbf{D}_{m,i}\mathbf{x}_i \end{aligned} \tag{1}$$

where $\mathbf{x}_i \in \mathbb{R}^n$ and $\mathbf{D}_{i,j}$ are $n \times n$ real matrices. We denote

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_m \end{pmatrix}, \quad \mathbf{x}_i = \begin{pmatrix} x_{i,1} \\ \vdots \\ x_{i,n} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \mathbf{D}_{1,1} & \cdots & \mathbf{D}_{1,m} \\ \vdots & & \vdots \\ \mathbf{D}_{m,1} & \cdots & \mathbf{D}_{m,m} \end{pmatrix} \tag{2}$$

Then Eq. (1) can be written as

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1, t) \\ \vdots \\ \mathbf{f}_m(\mathbf{x}_m, t) \end{pmatrix} + \mathbf{D}\mathbf{x} \tag{3}$$

We also permute the state variables in the following way:

$$\tilde{\mathbf{x}}_i = \begin{pmatrix} x_{1,i} \\ x_{2,i} \\ \vdots \\ x_{m,i} \end{pmatrix}, \quad \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{\mathbf{x}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_n \end{pmatrix} \tag{4}$$

Then Eq. (1) can be written as

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}_1 &= \tilde{\mathbf{f}}_1(\tilde{\mathbf{x}},t) + \textstyle\sum_{i=1}^n \tilde{\mathbf{D}}_{1,i}\tilde{\mathbf{x}}_i \\ \vdots & \qquad \vdots \\ \dot{\tilde{\mathbf{x}}}_n &= \tilde{\mathbf{f}}_n(\tilde{\mathbf{x}},t) + \textstyle\sum_{i=1}^n \tilde{\mathbf{D}}_{n,i}\tilde{\mathbf{x}}_i \end{aligned} \tag{5}$$

where $\tilde{\mathbf{D}}_{i,j}$ are $m \times m$ real matrices. We will mainly be interested in arrays of identical systems, i.e. $\mathbf{f}_1 = \mathbf{f}_2 = \cdots = \mathbf{f}_m$.

We use the Euclidean norm on vectors in $\mathbb{R}^k$, although most of the results can be stated for other norms in $\mathbb{R}^k$ as well. The norm on $k \times k$ real-valued matrices will be the one induced by the norm in $\mathbb{R}^k$. We define $L = \{1, \cdots, m\}$. We assume the system of ordinary differential equations under consideration has a unique solution for all time and for each initial condition. We write $\mathbf{x}(t, \mathbf{x}_0, t_0)$ for the unique solution at time t where $\mathbf{x}_0$ is the initial conditions at time t_0. This will sometimes be simplified as $\mathbf{x}(t)$. Let S_α be the set of $\mathbf{x}$ such that $\|\mathbf{x}\| \le \alpha$. In the following, H^* will be a positive real number.

We define the system to be synchronized if the trajectories of all the cells approach each other:

Definition 1 *The system (1) is uniformly synchronized with respect to H^* if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $\|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)\| \le \delta(\epsilon)$ and $\mathbf{x}_i(t_0) \in S_{H^*}$ for all i, $j \in L$, then $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \le \epsilon$ for all $t \ge t_0$ and all i, $j \in L$.*

Definition 2 *The system (1) is uniformly asymptotically synchronized with respect to H^* if it is uniformly synchronized with respect to H^* and there exists $\delta > 0$ such that for all $\epsilon > 0$ there exists $T(\epsilon) \ge 0$ such that if*

$$\|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)\| \le \delta$$

and $\mathbf{x}_i(t_0) \in S_{H^*}$ for all i, $j \in L$ and $t \ge t_0 + T(\epsilon)$, then

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \le \epsilon$$

for all i, $j \in L$.

In the above definitions, the difference in the states between the cells goes to zero as $t \to \infty$. In the next definition, we allow for some synchronization error which can occur, for example, when the cells are not exactly identical.

Definition 3 *The system (1) is uniformly synchronized with respect to H^* with error bound ϵ if there exists $\delta > 0$ and $T \ge 0$ such that if*

$$\|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)\| \le \delta$$

and $\mathbf{x}_i(t_0) \in S_{H^*}$ for all i, $j \in L$, then $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \le \epsilon$ for all $t \ge t_0 + T$, and all i, $j \in L$.

We will say that the system (1) is uniformly (asymptotically) synchronized (with error bound ϵ) if it is uniformly (asymptotically) synchronized (with error bound ϵ) with respect to each $H^* > 0$.

3 Mathematical Preliminaries

For a real matrix $\mathbf{A}$, let $\mathbf{A}^T$ be the transpose of $\mathbf{A}$ and let $\chi_{\mathbf{A}}(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A})$ be the characteristic polynomial of $\mathbf{A}$. The i,j-th entry of $\mathbf{A}$ will be denoted $\mathbf{A}_{(i,j)}$. Note the parenthesis which differentiate it from $\mathbf{A}_{i,j}$ which denotes a matrix. We denote by $\mathbf{I}_n$ the $n \times n$ real identity matrix. In the following, R denotes a ring. We denote $M_{n \times m}(R)$ as the set of $n \times m$ matrices with entries in R. We will sometime need to use the subfield of $n \times n$ matrices $\mathcal{F}_k = \{\alpha \mathbf{I}_k : \alpha \in \mathbb{R}\}$. Note that $\mathcal{F}_1 = \mathbb{R}$. For matrices which have real-valued matrices as entries, we sometimes "expand" them and consider them as real-valued matrices depending on context. Circulant matrices will be denoted as:

$$\mathrm{circ}(a_0, a_1, \ldots a_n) = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_n & a_0 & a_1 & \\ & \ddots & \ddots & \ddots \\ a_1 & \cdots & a_n & a_0 \end{pmatrix}$$

Definition 4 *To each $n \times n$ matrix $\mathbf{A}$, we associate a directed graph $\Gamma_{\mathbf{A}}$, called the* matrix graph *of $\mathbf{A}$, as follows: there are n vertices in $\Gamma_{\mathbf{A}}$, with an edge from vertex j to vertex i if and only if $\mathbf{A}_{(i,j)}$ is nonzero.*

To make it easier to visualize the connection between the cells, it is useful to introduce a *directed interaction graph* $\Gamma_{\mathbf{D}}$, similar to that used in [Hirsch, 1989], by considering $\mathbf{D}_{i,j}$ as *entries* in the matrix $\mathbf{D}$, i.e., the nodes of $\Gamma_{\mathbf{D}}$ will be cells, with an edge from cell j to cell i if and only if $\mathbf{D}_{i,j} \neq 0$. Some examples of $\Gamma_{\mathbf{D}}$ we will consider are shown in Fig. 1.

We define the following classes of matrices which our coupling matrices $\mathbf{D}$ will generally fall under.

- $T_1(R, K) = \{$the set of matrices with entries in R such that the sum of the entries in each row is equal to K, where $K \in R.\}$
- $T_2(\epsilon) = \{$the set of matrices with real entries such that the sum of the entries in each row is equal to the real number $\epsilon.\}$
- $T_3(\epsilon) = \{$the set of matrices in $T_2(\epsilon)$ such that the off-diagonal elements are nonnegative.$\}$

It's clear that $T_1 \supset T_2 \supset T_3$. Coupling matrices in $T_3(0)$ correspond to diffusive or cooperative coupling. We need the following lemma for matrices in T_1:

Lemma 1 *Let $\mathbf{A}$ be a $n \times n$ matrix in $T_1(R, K)$. Then the $(n-1) \times (n-1)$ matrix $\mathbf{B}$ defined by $\mathbf{B} = \mathbf{CAG}$ satisfies $\mathbf{CA} = \mathbf{BC}$ where $\mathbf{C}$ is the $(n-1) \times n$ matrix*

$$\mathbf{C} = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & & \cdots & \\ & & & 1 & -1 \end{pmatrix} \tag{6}$$

$\mathbf{G}$ is the $n \times (n-1)$ matrix

$$\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ & & \ddots & & 1 \\ & & & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

and 1 is the multiplicative identity of R.

Proof Note that $\mathbf{CA}$ is in $T_1(R,0)$, since $\mathbf{A}$ is in $T_1(R,K)$. The $n \times n$ matrix $\mathbf{GC}$ is

$$\mathbf{GC} = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & \ddots & 0 & -1 \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{8}$$

Thus the first $n-1$ columns of $\mathbf{CAGC}$ is the same as those of $\mathbf{CA}$. The n-th column of $\mathbf{CAGC}$ is the negative of the sum of the first $n-1$ columns of $\mathbf{CA}$ which is the n-th column of $\mathbf{CA}$ since $\mathbf{CA}$ is in $T_1(R,0)$. So $\mathbf{CAGC} = \mathbf{CA}$. The matrix $\mathbf{B}$ can be written explicitly as $\mathbf{B}_{(i,j)} = \sum_{k=1}^{j} \mathbf{A}_{(i,k)} - \mathbf{A}_{(i+1,k)}$ for $i,j \in \{1,\cdots,n-1\}$. ■

We denote the map which maps $\mathbf{A}$ to $\mathbf{B}$ by S, i.e. the map S is defined as

$$\boxed{\mathbf{B} = S(\mathbf{A}) = \mathbf{CAG}}$$

For a $n \times n$ matrix $\mathbf{A}$ and $\alpha \in R$, $S(\mathbf{A} + \alpha \mathbf{I}_n) = S(\mathbf{A}) + \alpha \mathbf{I}_{n-1}$ since $\mathbf{CG} = \mathbf{I}_{n-1}$.

Definition 5 *A real square matrix* $\mathbf{A}$ *is called normal if* $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$.

The following lemmas summarize some simple properties of matrices in T_2 and T_3.

Lemma 2 *If* $\mathbf{A}$ *is a matrix in* $T_2(\epsilon)$*, then the following holds:*

(2.1) ϵ *is an eigenvalue of* $\mathbf{A}$ *with eigenvector* $(1,1,\cdots,1)^T$.

(2.2) If $\mathbf{B} = S(\mathbf{A})$ *then* $\chi_{\mathbf{A}}(\lambda) = (\lambda - \epsilon)\chi_{\mathbf{B}}(\lambda)$. *Thus if* ϵ *is a eigenvalue of* $\mathbf{A}$ *of multiplicity* 1*, then* $\sigma(\mathbf{B})$ *(the spectrum of* $\mathbf{B}$*) is* $\sigma(\mathbf{A})\backslash\{\epsilon\}$.

If $\mathbf{A}$ *is in* $T_3(\epsilon)$*, then the following holds:*

(2.3) The real parts of all eigenvalues are less than or equal to ϵ *and all eigenvalues with real part equal to* ϵ *are real.*

(2.4) if $\mathbf{A}$ *is irreducible, then* ϵ *is an eigenvalue of multiplicity* 1.

Proof If $\mathbf{A}$ is a matrix in $T_2(\epsilon)$, (2.1) is clearly true. (2.2) follows from

$$\begin{pmatrix} & \mathbf{C} & \\ 0 & \cdots & 1 \end{pmatrix} (\lambda \mathbf{I} - \mathbf{A}) \begin{pmatrix} & 1 \\ \mathbf{G} & \vdots \\ & 1 \end{pmatrix} = \begin{pmatrix} \lambda \mathbf{I} - \mathbf{B} & 0 \\ & \vdots \\ \mathbf{w} & \lambda - \epsilon \end{pmatrix}$$

where $\mathbf{w}$ is the last row of $-\mathbf{AG}$. If $\mathbf{A}$ is in $T_3(\epsilon)$, then (2.3) follows from Gerschgorin's circle criterion. If $\mathbf{A} \in T_3(\epsilon)$ is irreducible, let α be such that $\alpha\mathbf{I} + \mathbf{A}$ is a nonnegative matrix. Then $\alpha + \epsilon$ is the maximal eigenvalue of $\alpha\mathbf{I} + \mathbf{A}$ (by Gerschgorin's circle criterion), and thus has multiplicity 1 [Minc, 1988, Theorem 4.3]. Thus ϵ is a eigenvalue of $\mathbf{A}$ with multiplicity 1. So (2.4) follows. ■

Lemma 3 *Let* $\mathbf{A}$, $\mathbf{B}$ *be matrices of the same size.*

(3.1) If $\mathbf{A} \in T_2(\epsilon)$ *and* $\mathbf{B} \in T_2(\delta)$*, then* $\mathbf{A} + \mathbf{B} \in T_2(\epsilon + \delta)$.

(3.2) If $\mathbf{AB} = \mathbf{BA}$ *and* $\mathbf{A} \in T_2(\epsilon)$ *with* ϵ *being an eigenvalue of* $\mathbf{A}$ *of multiplicity 1, then* $\mathbf{B} \in T_2(\delta)$ *for some* δ.

Proof (3.1) is trivial. Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Then $\mathbf{ABx} = \mathbf{BAx} = \epsilon\mathbf{Bx}$. So $\mathbf{Bx}$ is eigenvector of $\mathbf{A}$ with eigenvalue ϵ. Thus $\mathbf{Bx} = \delta\mathbf{x}$ for some δ. So $\mathbf{B} \in T_2(\delta)$. ■

Lemma 4 *Let* $\mathbf{A}$ *be a real normal matrix. Then the following holds:*

(4.1) If $\mathbf{A} + \mathbf{A}^T$ *is irreducible then* $\mathbf{A}$ *is irreducible.*

(4.2) If $\mathbf{A} \in T_2(\epsilon)$ *then* $\mathbf{A}^T \in T_2(\epsilon)$.

(4.3) If $\mathbf{A} + \mathbf{A}^T$ *is a matrix in* $T_2(\epsilon)$ *such that* ϵ *is an eigenvalue of multiplicity 1, then* $\mathbf{A}$ *and* $\mathbf{A}^T$ *are matrices in* $T_2(\frac{\epsilon}{2})$. *In particular, if* $\mathbf{A} + \mathbf{A}^T \in T_3(\epsilon)$ *is irreducible, then* $\mathbf{A}$ *and* $\mathbf{A}^T$ *are irreducible matrices in* $T_2(\frac{\epsilon}{2})$.

(4.4) If $\mathbf{A} \in T_3(\epsilon)$ *then* $\mathbf{A}$ *is irreducible* $\Leftrightarrow$ $\mathbf{A} + \mathbf{A}^T$ *is irreducible.*

Proof Suppose that $\mathbf{A}$ is normal and reducible. Then $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{P}^T \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \mathbf{P}$$

for some permutation matrix $\mathbf{P}$. Normality of $\mathbf{A}$ means that

$$\mathbf{P}^T \begin{pmatrix} \mathbf{B}^T\mathbf{B} & \mathbf{B}^T\mathbf{C} \\ \mathbf{C}^T\mathbf{B} & \mathbf{C}^T\mathbf{C} + \mathbf{D}^T\mathbf{D} \end{pmatrix} \mathbf{P} = \mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{P}^T \begin{pmatrix} \mathbf{B}\mathbf{B}^T + \mathbf{C}\mathbf{C}^T & \mathbf{C}\mathbf{D}^T \\ \mathbf{D}\mathbf{C}^T & \mathbf{D}\mathbf{D}^T \end{pmatrix} \mathbf{P}$$

Therefore $\mathbf{CC}^T = \mathbf{B}^T\mathbf{B} - \mathbf{BB}^T$. The diagonal elements of $\mathbf{B}^T\mathbf{B}$ are the inner products of the columns of $\mathbf{B}$. So the trace of $\mathbf{B}^T\mathbf{B}$ is the sum of the squares of the entries in $\mathbf{B}$. Similarly the trace of $\mathbf{BB}^T$ is the sum of the squares of the entries in $\mathbf{B}$. Therefore $\mathbf{CC}^T$ has zero trace. Since $\mathbf{CC}^T$ is symmetric positive semidefinite, this implies that $\mathbf{CC}^T = 0$. Since the diagonal elements of $\mathbf{CC}^T$ are the inner products of the rows of $\mathbf{C}$, this means that $\mathbf{C} = 0$. So $\mathbf{A} + \mathbf{A}^T$ is reducible, which proves (4.1). Since $\mathbf{A}$ is normal, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ with eigenvalue δ, then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^T$ with eigenvalue $\overline{\delta}$ [Gantmacher, 1960]. Applying this to the eigenvector $(1, 1, \ldots, 1)^T$ yields (4.2). Now suppose that $\mathbf{A} + \mathbf{A}^T$ is a matrix in $T_2(\epsilon)$ with ϵ being an eigenvalue of multiplicity 1. Since $(\mathbf{A} + \mathbf{A}^T)$ commute with both $\mathbf{A}$ and $\mathbf{A}^T$, by lemma 3, $\mathbf{A} \in T_2(\delta)$ and $\mathbf{A}^T \in T_2(\mu)$. By (4.2), it follows that $\delta = \mu$. Then by lemma 3, $\epsilon = 2\delta$, proving (4.3).

It's clear that if $\mathbf{A} \in T_3(\epsilon)$, then $\mathbf{A}$ being irreducible implies that $\mathbf{A} + \mathbf{A}^T$ is irreducible since $\mathbf{A}$ has no more off-diagonal nonzero entries than $\mathbf{A} + \mathbf{A}^T$, which proves (4.4). ■

We also define the following class of matrices:

- $M_1(k)$ are matrices $\mathbf{M}$ (not necessarily square) with entries in $\mathcal{F}_k$ such that each row of $\mathbf{M}$ contains zeros and exactly one $\alpha\mathbf{I}_k$ and one $-\alpha\mathbf{I}_k$ for some nonzero α.
- $M_2(k)$ are matrices $\mathbf{M}$ in $M_1(k)$ such that for any pair of indices i and j there exist indices $i_1, i_2, \ldots, i_l$ with $i_1 = i$ and $i_l = j$ such that for all $1 \le q < l$, $\mathbf{M}_{(p,i_q)} \neq 0$ and $\mathbf{M}_{(p,i_{q+1})} \neq 0$ for some p.

Matrices in $M_2(k)$ can be interpreted in the following way. For $\mathbf{M} \in M_1(k)$ construct a graph as follows: the number of vertices of the graph is the number of columns of $\mathbf{M}$, and the number of edges is the number of rows of $\mathbf{M}$. There is an edge between vertex j to vertex l if and only if $\mathbf{M}_{(i,j)} \neq 0$ and $\mathbf{M}_{(i,l)} \neq 0$ for some i. If $\mathbf{M} \in M_2(k)$, then this graph is connected. This also implies that the number of columns of $\mathbf{M} \in M_2(k)$ is at most one more than the number of rows of $\mathbf{M}$.

The following lemma relates these classes of matrices to the classes of matrices defined earlier.

Lemma 5 *If* $\mathbf{M}$ *is in* $M_1(k)$, *then for a positive integer* p, $(\mathbf{M}^T\mathbf{M})^p$ *is a symmetric matrix in* $T_1(\mathcal{F}_k, 0)$. *A symmetric matrix* $\mathbf{A}$ *is in* $T_3(0)$ *if and only if there exists* $\mathbf{M} \in M_1(1)$ *such that* $\mathbf{A} = -\mathbf{M}^T\mathbf{M}$. *A symmetric irreducible matrix* $\mathbf{A}$ *is in* $T_3(0)$ *if and only if there exists* $\mathbf{M} \in M_2(1)$ *such that* $\mathbf{A} = -\mathbf{M}^T\mathbf{M}$.

Proof Let $\mathbf{M} \in M_1(k)$. Clearly $(\mathbf{M}^T\mathbf{M})^p$ is symmetric for all nonnegative integers p. For $p > 0$

$$(\mathbf{M}^T\mathbf{M})^p \begin{pmatrix} \mathbf{I}_k \\ \vdots \\ \mathbf{I}_k \end{pmatrix} = 0$$

Since

$$\mathbf{M} \begin{pmatrix} \mathbf{I}_k \\ \vdots \\ \mathbf{I}_k \end{pmatrix} = 0$$

So $(\mathbf{M}^T\mathbf{M})^p \in T_1(\mathcal{F}_k, 0)$ if $p > 0$. Now $(\mathbf{M}^T\mathbf{M})_{(i,j)}$ is the inner product of columns i and j of $\mathbf{M}$. Let $\mathbf{M} \in M_1(1)$. Then the diagonal elements of $\mathbf{M}^T\mathbf{M}$ is greater than or equal to 0. If $i \neq j$, the only terms in the inner product of columns i and j of $\mathbf{M}$ is either 0 or some negative number, so the off-diagonal elements of $\mathbf{M}^T\mathbf{M}$ is less than or equal to 0. So $-\mathbf{M}^T\mathbf{M} \in T_3(0)$.

Let $\mathbf{A}$ be a symmetric matrix in $T_3(0)$. Construct $\mathbf{M}$ as follows. For each nonzero row of $\mathbf{A}$ we generate several rows of $\mathbf{M}$ of the same length as follows: for the i-th row of $\mathbf{A}$, if for each $i < j$ such that $\mathbf{A}_{(i,j)} = \alpha$ for some $\alpha > 0$, we add a row to $\mathbf{M}$ with with the i-th element being $\sqrt{\alpha}$, and the j-th element $-\sqrt{\alpha}$. We claim that this matrix $\mathbf{M}$ will do the trick. Certainly $\mathbf{M} \in M_1(1)$. $-\mathbf{A}_{(i,j)}$ is the inner product between the i-th column and the j-column of $\mathbf{M}$ since by construction, there is only one row of $\mathbf{M}$ with nonzero entries in both the i-th and j-th position, giving the appropriate result. From the construction of $\mathbf{M}$, it's clear that $\mathbf{A}$ is irreducible if and only if $\mathbf{M} \in M_2(1)$. ∎

The proof of lemma 5 gives another characterization of matrices in $M_2(k)$: a matrix $\mathbf{M}$ is in $M_2(k)$ if and only if $\mathbf{M} \in M_1(k)$ and $\mathbf{M}^T\mathbf{M}$ is irreducible.

4 Lyapunov's Direct Method

We will mainly use Lyapunov's direct method to prove uniformly asymptotical synchronization of system (1). We let $d(\mathbf{x})$ be a function which measure the distance between the various cells. In particular, we define $d(\mathbf{x})$ to have the following form:

$$\boxed{d(\mathbf{x}) = \|\mathbf{M}\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{M}^T\mathbf{M}\mathbf{x},\ \mathbf{M} \in M_2(n)} \tag{9}$$

where $\mathbf{M}$ is a $m \times m$ matrix in $M_2(n)$ (but considered as an $nm \times nm$ real-valued matrix).

Because of the assumptions on $\mathbf{M}$, the crucial property of $d(\mathbf{x})$ is that $d(\mathbf{x}) \to 0$ if and only if $\|\mathbf{x}_i - \mathbf{x}_j\| \to 0$ for all i and j.

One possible choice for $d(\mathbf{x})$ is

$$d(\mathbf{x}) = \sum_{i=1}^{m-1} \|\mathbf{x}_i - \mathbf{x}_{i+1}\|^2$$

which corresponds to

$$\mathbf{M} = \begin{pmatrix} \mathbf{I} & -\mathbf{I} & & & \\ & \mathbf{I} & -\mathbf{I} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{I} & -\mathbf{I} \end{pmatrix} \tag{10}$$

Definition 6 ([Vidyasagar, 1978]) *A function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to belong to class K if*

1. *$\alpha(\cdot)$ is continuous and nondecreasing,*
2. *$\alpha(0) = 0$,*
3. *$\alpha(p) > 0$ whenever $p > 0$.*

We assume that all Lyapunov functions we consider are continuous. For a Lyapunov function $V(t, \mathbf{x})$, the generalized (Dini) derivative along the trajectories of the system $\dot{\mathbf{x}} = \mathbf{f}_a(\mathbf{x}, t)$ is defined as:

$$D^+V(t, \mathbf{x}) = \lim \sup_{h \to 0+} \frac{1}{h} [V(t + h, \mathbf{x} + h\mathbf{f}_a(\mathbf{x}, t)) - V(t, \mathbf{x})]$$

Theorem 1 *Suppose that D is an open set such that if $\mathbf{x}_i(t_0) \in S_{H^*}$ for all i, then $\mathbf{x}(t, \mathbf{x}(t_0), t_0) \in D$ for all $t \geq t_0$. Suppose that a Lyapunov function $V(t, \mathbf{x})$, locally Lipschitzian in $\mathbf{x}$, exists on $\mathbb{R} \times D$ such that for all $t \geq t_0$ and $\mathbf{x} \in D$,*

$$a(d(\mathbf{x})) \leq V(t, \mathbf{x}) \leq b(d(\mathbf{x}))$$

where $a(\cdot)$ and $b(\cdot)$ are functions in class K. Suppose that there exists $\mu > 0$ such that for all $t \geq t_0$ and $d(\mathbf{x}) \geq \mu$,

$$D^+V(t, \mathbf{x}) \leq -c$$

for some constant $c > 0$ where $D^+V(t, \mathbf{x})$ is the generalized derivative of V along the trajectories of (1).

If there exists $\delta > 0$ such that $a(\delta) > b(\mu)$, then for each $\mathbf{x}(t_0) \in S_{H^*}$ there exists $t_1 \geq t_0$ such that for all $t \geq t_1$,

$$d(\mathbf{x}(t, \mathbf{x}(t_0), t_0)) \leq \delta$$

Furthermore, if $d(\mathbf{x}(t_0)) \leq \mu$ then

$$d(\mathbf{x}(t, \mathbf{x}(t_0), t_0)) \leq \delta$$

for all $t \geq t_0$.

Theorem 2 *Suppose that D is an open set such that if $\mathbf{x}_i(t_0) \in S_{H^*}$ for all i then $\mathbf{x}(t, \mathbf{x}(t_0), t_0) \in D$ for all $t \geq t_0$. Suppose that a Lyapunov function $V(t, \mathbf{x})$, locally Lipschitzian in $\mathbf{x}$, exists on $\mathbb{R} \times D$ such that for all $t \geq t_0$, $\mathbf{x} \in D$,*

$$a(d(\mathbf{x})) \leq V(t, \mathbf{x}) \leq b(d(\mathbf{x}))$$

where $a(\cdot)$ and $b(\cdot)$ are in class K, and for all $t \geq t_0$,

$$D^+V(t, \mathbf{x}) \leq -c(d(\mathbf{x}))$$

for some function $c(\cdot)$ in class K where $D^+V(t, \mathbf{x})$ is the generalized derivative of V along the trajectories of (1). Then the system (1) is uniformly asymptotically stable *with respect to H^*.*

The proofs of these two theorems are similar to the proofs of theorems 1 and 2 in [Wu and Chua, 1994]. The following theorems concern perturbed systems.

Theorem 3 *Consider the system*

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1, t) \\ \vdots \\ \mathbf{f}_m(\mathbf{x}_m, t) \end{pmatrix} + \mathbf{D}\mathbf{x} + \mathbf{e}(t) \tag{11}$$

Suppose that D is an open set such that if $\mathbf{x}_i(t_0) \in S_{H^*}$ for all i then $\mathbf{x}(t, \mathbf{x}(t_0), t_0) \in D$ for all $t \geq t_0$. Suppose that a Lyapunov function $V(t, \mathbf{x})$, uniformly Lipschitzian in $\mathbf{x}$, exists on $\mathbb{R} \times D$ such that for all $t \geq t_0$, $\mathbf{x} \in D$,

$$a(d(\mathbf{x})) \leq V(t, \mathbf{x}) \leq b(d(\mathbf{x}))$$

where $a(\cdot)$ and $b(\cdot)$ are in class K, and for all $t \geq t_0$,

$$D^+V(t, \mathbf{x}) \leq -c(d(\mathbf{x}))$$

for some function $c(\cdot)$ in class K where $D^+V(t, \mathbf{x})$ is the generalized derivative of V along the trajectories of (1). Let $\mu > 0$ be such that there exists $\epsilon > 0$ such that $a(\epsilon) > b(\mu)$. If $\|\mathbf{e}(t)\| \leq \frac{c(\mu)}{M} - \delta$ for all $t \geq t_0$ and some $\delta > 0$, then the system (11) is uniformly synchronized with respect to H^* with error bound ϵ, where M is a Lipschitz constant of V. Furthermore, if $d(\mathbf{x}(t_0)) \leq \mu$, then $d(\mathbf{x}(t, \mathbf{x}(t_0), t_0)) \leq \epsilon$ for all $t \geq t_0$.

Note that Eq. (11) is the state equation (3) with a perturbation term $\mathbf{e}(t)$.

Proof The generalized derivative $D^+V(t, \mathbf{x})$ along the trajectories of system Eq. (11) satisfies

$$D^+V(t, \mathbf{x}) \leq -c(d(\mathbf{x})) + M\|\mathbf{e}(t)\|$$

and the result follows from theorem 1. ■

Corollary 1 *Consider the system (11). Suppose that D is an open set such that if $\mathbf{x}_i(t_0) \in S_{H^*}$ for all i then $\mathbf{x}(t, \mathbf{x}(t_0), t_0) \in D$ for all $t \geq t_0$. Suppose that a Lyapunov function $V(t, \mathbf{x})$, uniformly Lipschitzian in $\mathbf{x}$, exists on $\mathbb{R} \times D$ such that for all $t \geq t_0$, $\mathbf{x} \in D$,*

$$a(d(\mathbf{x})) \leq V(t, \mathbf{x}) \leq b(d(\mathbf{x}))$$

where $a(\cdot)$ and $b(\cdot)$ are in class K, and for all $t \geq t_0$,

$$D^+V(t, \mathbf{x}) \leq -c(d(\mathbf{x}))$$

for some function $c(\cdot)$ in class K where $D^+V(t, \mathbf{x})$ is the generalized derivative of V along the trajectories of (1). If $\|\mathbf{e}(t)\| \to 0$ as $t \to \infty$, then for $\mathbf{x}_i(t_0) \in S_{H^*}$, $d(\mathbf{x}(t, \mathbf{x}(t_0), t_0) \to 0$ as $t \to \infty$. If in addition, $\mathbf{e}$ depends on $\mathbf{x}$ and $\sup_{t \geq t_0} \|\mathbf{e}(\mathbf{x}, t)\| \to 0$ as $d(\mathbf{x}) \to 0$, then the system (11) is uniformly asymptotically synchronized with respect to H^*.

In this paper, the Lyapunov function used has a quadratic form. In this case the requirement that V is uniformly Lipschitzian can be removed.

Definition 7 *A array of cells coupled in the form (1) is said to belong to class Q^+ if there exists a Lyapunov function $V(t, \mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{M}^T\mathbf{V}\mathbf{M}\mathbf{x}$ with symmetric positive definite $\mathbf{V}$, such that*

$$D^+V(t, \mathbf{x}) \leq -\gamma d(\mathbf{x})$$

for some constant $\gamma > 0$, where $D^+V(t, \mathbf{x})$ is the derivative of V along the trajectories of (1). The array belongs to class Q if the term $-\gamma d(\mathbf{x})$ in the above definition is replaced by $-c(d(\mathbf{x}))$ for some function c in class K.

Theorem 4 *Consider the system (11). Suppose that (1) belongs to class* Q^+. *Let* $\mu > 0$ *be such that there exists* $\epsilon > 0$ *such that* $a(\epsilon) > b(\mu)$. *If* $\|\mathbf{e}(t)\| \leq \frac{\gamma\sqrt{\mu}}{\|\mathbf{VM}\|} - \delta$ *for all* $t \geq t_0$ *and some* $\delta > 0$, *then the system (11) is uniformly synchronized with respect to* H^* *with error bound* ϵ. *Furthermore, if* $d(\mathbf{x}(t_0)) \leq \mu$, *then* $d(\mathbf{x}(t, \mathbf{x}(t_0), t_0)) \leq \epsilon$ *for all* $t \geq t_0$. *The constant* $\gamma > 0$ *is as defined in definition 7.*

Proof The derivative $D^+V(t, \mathbf{x})$ along the trajectories of system Eq. (11) satisfies

$$D^+V(t,\mathbf{x}) \leq -\gamma d(\mathbf{x}) + \mathbf{x}^T\mathbf{M}^T\mathbf{VMe}(t) \leq -\gamma d(\mathbf{x}) + \|\mathbf{VM}\|\sqrt{d(\mathbf{x})}\|\mathbf{e}(t)\| \leq \sqrt{d(\mathbf{x})}(-\gamma\sqrt{d(\mathbf{x})} + \|\mathbf{VM}\|\|\mathbf{e}(t)\|)$$

and the result follows from theorem 1. ∎

Corollary 2 . *Consider the system (11). Suppose that (1) belongs to class* Q^+. *If* $\|\mathbf{e}(t)\| \to 0$ *as* $t \to \infty$, *then for* $\mathbf{x}_i(t_0) \in S_{H^*}$, $d(\mathbf{x}(t, \mathbf{x}(t_0), t_0)) \to 0$ *as* $t \to \infty$. *If in addition,* $\mathbf{e}$ *depends on* $\mathbf{x}$ *and* $\sup_{t \geq t_0} \|\mathbf{e}(\mathbf{x}, t)\| \to 0$ *as* $d(\mathbf{x}) \to 0$, *then the system (11) is uniformly asymptotically synchronized with respect to* H^*.

Theorem 5 (Converse Theorem) *Consider the system (1). Suppose that the* $\mathbf{f}_i$*'s are uniformly Lipschitz continuous, i.e.,* $\|\mathbf{f}_i(\mathbf{x}_i, t) - \mathbf{f}_i(\mathbf{y}_i, t)\| \leq M_i\|\mathbf{x}_i - \mathbf{y}_i\|$ *for all* $\mathbf{x}$, $\mathbf{y}$ *and* $t \geq t_0$. *If system (1) is uniformly asymptotically synchronized, then given* $d(\mathbf{x})$ *there exists* $\rho > 0$ *and a continuous function* $V(\mathbf{x}, t)$ *defined for* $d(\mathbf{x}) \leq \rho$ *and* $t \geq t_0$ *and Lipschitzian in* $\mathbf{x}$ *such that*

$$a(d(\mathbf{x})) \leq V(\mathbf{x}, t) \leq b(d(\mathbf{x}))$$

and

$$D^+V(\mathbf{x}, t) \leq -c(d(\mathbf{x}))$$

for $d(\mathbf{x}) \leq \rho$ *and* $t \geq t_0$ *where* $a(\cdot)$, $b(\cdot)$ *and* $c(\cdot)$ *are functions in class* K.

Proof See [Lakshmikantham and Liu, 1993, Corollary 1.4.1]. ∎

5 Synchronizing Arrays of Identical Dynamical Systems

In [Wu and Chua, 1994], asymptotical synchronization is related to asymptotical stability of related systems. In this section we extend this idea to arrays of cells by relating synchronization of the array to the amount of linear feedback required to asymptotically stabilize a cell.

In particular, for the results in this paper concerning arrays of identical dynamical systems (cells) which synchronize, the main requirement is that for a cell $\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t)$, there exists a matrix $\mathbf{T}$ such that $\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t) - \mathbf{T}\mathbf{x}_1 + \eta(t)$ is uniformly asymptotically stable for all $\eta(t)$. In [Wu and Chua, 1994] it was shown that for systems with a uniformly bounded Jacobian, it is possible to find such $\mathbf{T}$. In particular, the following holds:

Lemma 6 *If* $\mathbf{f}_1$ *is continuously differentiable and the Jacobian* $\mathbf{D_x f}_1(\mathbf{x}_1, t)$ *is uniformly bounded, i.e. there exists a constant* M *such that* $\|\mathbf{D_x f}_1(\mathbf{x}_1, t)\| \leq M$ *for all* $\mathbf{x}_1$ *and* t, *then there exists a diagonal matrix* $\mathbf{T}$ *such that for* $V(\mathbf{x}_1, \mathbf{y}_1) = \frac{1}{2}\|\mathbf{x}_1 - \mathbf{y}_1\|^2$, $\dot{V} \leq -c(\|\mathbf{x}_1 - \mathbf{y}_1\|)$, *where* $c(\cdot)$ *is in class* K *and* $\dot{V}$ *is the derivative of* V *along the trajectories of*

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1, t) - \mathbf{T}\mathbf{x}_1 + \eta(t) \\ \dot{\mathbf{y}}_1 &= \mathbf{f}_1(\mathbf{y}_1, t) - \mathbf{T}\mathbf{y}_1 + \eta(t) \end{aligned}$$

for $\eta(t)$ *continuous.*

In many systems studied, the linear coupling is such that when the states of the cells are identical to each other, the cells are decoupled. This implies that the matrix $\mathbf{D}$ satisfies $\sum_{j=1}^{m} \mathbf{D}_{i,j} = 0$ for each $i \in L$. This motivates us to define the classes of matrices in section 3. Throughout this paper we will assume that $\sum_{j=1}^{m} \mathbf{D}_{i,j} = 0$ or if we consider $\mathbf{D}_{i,j}$ as entries of $\mathbf{D}$,

$$\boxed{\mathbf{D} \in T_1(M_{n\times n}(\mathbb{R}), 0)} \tag{12}$$

For this case, it means that at the synchronized state, the dynamics of the cells are the same as those of an individual uncoupled cell.

Consider the case of an array of identical systems, i.e., $\mathbf{f}_i = \mathbf{f}$ for all i. Assume that there exists $n \times n$ matrices $\mathbf{V}$ and $\mathbf{T}$ such that $\mathbf{V}$ is symmetric positive definite, and for all t

$$\boxed{(\mathbf{x}-\mathbf{y})^T\mathbf{V}(\mathbf{f}(\mathbf{x},t) - \mathbf{T}\mathbf{x} - \mathbf{f}(\mathbf{y},t) + \mathbf{T}\mathbf{y}) \leq -c(\|\mathbf{x}-\mathbf{y}\|)} \tag{13}$$

for $c(\cdot)$ a function in class K. This implies that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) - \mathbf{T}\mathbf{x}$ is uniformly asymptotically stable (see [Wu and Chua, 1994]), with $V(\mathbf{x},\mathbf{y}) = \frac{1}{2}(\mathbf{x}-\mathbf{y})^T\mathbf{V}(\mathbf{x}-\mathbf{y})$ being the corresponding Lyapunov function. As lemma 6 indicates, for $\mathbf{f}$ with uniformly bounded Jacobian, diagonal $\mathbf{V}$ and $\mathbf{T}$ can be found.

For a matrix $\mathbf{M} \in M_2(n)$, we consider the Lyapunov function candidate

$$V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{M}^T \begin{pmatrix} \mathbf{V} & & & \\ & \mathbf{V} & & \\ & & \ddots & \\ & & & \mathbf{V} \end{pmatrix} \mathbf{M}\mathbf{x} = \frac{1}{2}\mathbf{x}^T\mathbf{U}\mathbf{x} \tag{14}$$

where

$$\mathbf{U} = \mathbf{M}^T \begin{pmatrix} \mathbf{V} & & & \\ & \mathbf{V} & & \\ & & \ddots & \\ & & & \mathbf{V} \end{pmatrix} \mathbf{M} = \begin{pmatrix} \mathbf{V} & & & \\ & \mathbf{V} & & \\ & & \ddots & \\ & & & \mathbf{V} \end{pmatrix} \mathbf{M}^T\mathbf{M}$$

since the entries of $\mathbf{M} \in M_1(n)$ has the form $\alpha\mathbf{I}_n$.

Clearly $a(d(\mathbf{x})) \leq V(\mathbf{x}) \leq b(d(\mathbf{x}))$ for some functions $a(\cdot)$ and $b(\cdot)$ in class K since $d(\mathbf{x}) = \mathbf{x}^T\mathbf{M}^T\mathbf{M}\mathbf{x}$ and $\mathbf{V} > 0$. The derivative of V along the trajectories of (1) is

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T\mathbf{U} \begin{pmatrix} \mathbf{f}(\mathbf{x}_1,t) - \mathbf{T}\mathbf{x}_1 \\ \mathbf{f}(\mathbf{x}_2,t) - \mathbf{T}\mathbf{x}_2 \\ \vdots \\ \mathbf{f}(\mathbf{x}_m,t) - \mathbf{T}\mathbf{x}_m \end{pmatrix} + \mathbf{x}^T\mathbf{U}\left[\mathbf{D} + \begin{pmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T} \end{pmatrix}\right]\mathbf{x} \tag{15}$$

The first term is less than $-c(d(\mathbf{x}))$ for some function c in class K by our assumptions. If $\mathbf{M}$ is as defined in Eq. (10), then by lemma 1, the second term can be written as

$$\mathbf{x}^T\mathbf{M}^T \begin{pmatrix} \mathbf{V} & & & \\ & \mathbf{V} & & \\ & & \ddots & \\ & & & \mathbf{V} \end{pmatrix} \mathbf{B}\mathbf{M}\mathbf{x}$$

for the matrix $\mathbf{B}$ defined as

$$\mathbf{B} = S\left(\mathbf{D} + \begin{pmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T} \end{pmatrix}\right) \tag{16}$$

So the conditions of theorem 2 are satisfied if the matrix $\mathbf{U}\left[\mathbf{D} + \begin{pmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T} \end{pmatrix}\right]$ or the matrix $\begin{pmatrix} \mathbf{V} & & & \\ & \mathbf{V} & & \\ & & \ddots & \\ & & & \mathbf{V} \end{pmatrix}\mathbf{B}$ is negative semidefinite.

Thus the general algorithm for proving that an array of identical system (Eq. (1)) is uniformly asymptotically synchronized is as follows:

Step 1: Find $n \times n$ matrices $\mathbf{T}$ and $\mathbf{V}$ with $\mathbf{V}$ symmetric positive definite such that condition (13) is satisfied and thus $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) - \mathbf{T}\mathbf{x}$ is asymptotically stable with Lyapunov function $V(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(\mathbf{x} - \mathbf{y})^T\mathbf{V}(\mathbf{x} - \mathbf{y})$.

Step 2: If the matrix

$$\mathbf{H}_1 = \mathbf{U}\left[\mathbf{D} + \begin{pmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T} \end{pmatrix}\right]$$

is negative semidefinite, then system (1) is uniformly asymptotically synchronized.

By choosing $\mathbf{M}$ as in Eq. (10) step 2 can be replaced by:

Step 2a: Use lemma 1 to construct $\mathbf{B}$ from $\mathbf{T}$ and $\mathbf{D}$ as in Eq. (16). If the matrix

$$\mathbf{H}_2 = \begin{pmatrix} \mathbf{V} & & & \\ & \mathbf{V} & & \\ & & \ddots & \\ & & & \mathbf{V} \end{pmatrix}\mathbf{B}$$

is negative semidefinite, then system (1) is uniformly asymptotically synchronized.

Note that this is a sufficient condition for synchronization; failure of the test (i.e. the matrix in step 2 is not negative semidefinite) does not imply that the system will not synchronize.

6 V, T and $\mathbf{D}_{i,j}$ are diagonal

We consider this case since we can make some more simplifications to the above algorithm. Furthermore, by lemma 6, a diagonal $\mathbf{V}$ and $\mathbf{T}$ can always be found for systems with uniformly bounded Jacobians.

The condition of $\mathbf{D}_{i,j}$ being diagonal implies that in Eq. (5) $\tilde{\mathbf{D}}_{i,j} = 0$ for $i \neq j$. Note that since $\mathbf{D} \in T_1(M_{n\times n}(\mathbb{R}), 0)$, $\tilde{\mathbf{D}}_{i,i} \in T_2(0)$ for all i. We rearrange the state equations as in Eqs. (4), (5). Let us write

$$\mathbf{V} = \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_n \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$$

Note that since $\mathbf{V}$ is positive definite, $v_i > 0$ for all i. We denote $\tilde{\mathbf{M}}$ as the real-valued matrix obtained from $\mathbf{M}$ by replacing $\mathbf{I}$ by 1 (i.e. by identifying $\mathcal{F}_n$ with $\mathbb{R}$).

Then the second term in Eq. (15) can be written as

$$\begin{aligned}
&\mathbf{x}^T\mathbf{U}\left[\mathbf{D} + \begin{pmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T} \end{pmatrix}\right]\mathbf{x} \\
&= \tilde{\mathbf{x}}^T \begin{pmatrix} \tilde{\mathbf{M}}^T v_1 \mathbf{I}\tilde{\mathbf{M}} & & & \\ & \tilde{\mathbf{M}}^T v_2 \mathbf{I}\tilde{\mathbf{M}} & & \\ & & \ddots & \\ & & & \tilde{\mathbf{M}}^T v_n \mathbf{I}\tilde{\mathbf{M}} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{D}}_{1,1} + t_1\mathbf{I} & & & \\ & \tilde{\mathbf{D}}_{2,2} + t_2\mathbf{I} & & \\ & & \ddots & \\ & & & \tilde{\mathbf{D}}_{n,n} + t_n\mathbf{I} \end{pmatrix} \tilde{\mathbf{x}} \\
&= \sum_{i=1}^{n} v_i \tilde{\mathbf{x}}_i^T \tilde{\mathbf{M}}^T \tilde{\mathbf{M}} \left(\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I}\right) \tilde{\mathbf{x}}_i
\end{aligned}$$

If $\mathbf{M}$ is as in Eq. (10), then since $\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I} \in T_2(t_i)$, this term can be written as $\sum_{i=1}^{n} v_i \tilde{\mathbf{x}}_i^T \tilde{\mathbf{M}}^T \mathbf{B}_i \tilde{\mathbf{M}} \tilde{\mathbf{x}}_i$ where $\mathbf{B}_i = S(\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I})$. Thus the conditions of theorem 2 is satisfied if these quadratic forms are negative semidefinite.

Theorem 6 *Given the matrices are defined above, system (1) is uniformly asymptotically synchronized if for some $\tilde{\mathbf{M}} \in M_2(1)$, $\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}\left(\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I}\right)$ is negative semidefinite for all i, or if $\mathbf{B}_i = S(\tilde{\mathbf{D}}_{i,i}) + t_i\mathbf{I}$ is negative semidefinite for all i.*

6.1 Normal $\tilde{\mathbf{D}}_{i,i}$

When $\tilde{\mathbf{D}}_{i,i}$ is normal (definition 5) and of the same form, we have the following result which prove synchronization by only considering the eigenvalues of $\tilde{\mathbf{D}}_{i,i}$. Two important subclass of normal matrices are the symmetric matrices and the circulant matrices. Symmetric coupling matrices correspond to mutual coupling while circulant coupling matrices correspond to homogeneous or space-invariant coupling when the cells are arranged in a ring.

Theorem 7 *Suppose that $\tilde{\mathbf{D}}_{i,i}$ is either zero or a normal matrix such that $\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T$ is an irreducible matrix in $T_3(0)$. Assume also that all the $\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T$ are of the same form, i.e. there exists $\alpha_{i,j}$ for each i, j such that $\alpha_{i,j}(\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T) = \tilde{\mathbf{D}}_{j,j} + \tilde{\mathbf{D}}_{j,j}^T$. Then the system (1) is uniformly asymptotically synchronized if*

- *the real parts of the eigenvalues of $\tilde{\mathbf{D}}_{i,i}$ which do not lie on the imaginary axis are less than or equal to $-t_i$[1],*

[1] Or equivalently, since $\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T \in T_3(0)$ is irreducible, the least negative real part of the nonzero eigenvalues of $\tilde{\mathbf{D}}_{i,i}$ is less than or equal to $-t_i$.

- *For* $\tilde{\mathbf{D}}_{i,i} = 0,\ t_i \le 0$.

Proof For nonzero $\tilde{\mathbf{D}}_{i,i}$, by lemma 5, there exists $\mathbf{M}$ in $M_2(n)$ such that $\tilde{\mathbf{M}}^T\tilde{\mathbf{M}} = -\alpha_i(\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T)$ for some $\alpha_i > 0$, where $\mathbf{M}$ can be chosen to be independent of i. This $\mathbf{M}$ will be the matrix used in defining $d(\mathbf{x})$ (Eq. (9)). Let us denote $\tilde{\mathbf{G}}_{i,i} = (\tilde{\mathbf{D}}_{i,i}+\tilde{\mathbf{D}}_{i,i}^T)$. A real matrix $\mathbf{A}$ is negative semidefinite if and only if $(\mathbf{A}+\mathbf{A}^T)$ is negative semidefinite. So to test whether $\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}\left(\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I}\right)$ is negative semidefinite we construct

$$\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}\left(\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I}\right) + \left(\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}\left(\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I}\right)\right)^T$$

which is equal to

$$\begin{aligned}
&= (\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}) + 2t_i\tilde{\mathbf{M}}^T\tilde{\mathbf{M}} \\
&= -\alpha_i((\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T)\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T(\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T)) + 2t_i\tilde{\mathbf{M}}^T\tilde{\mathbf{M}} \\
&= -\alpha_i(\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T)^2 + 2t_i\tilde{\mathbf{M}}^T\tilde{\mathbf{M}} \\
&= -\alpha_i((\tilde{\mathbf{G}}_{i,i})^2 + 2t_i\tilde{\mathbf{G}}_{i,i}) = -\alpha_i\tilde{\mathbf{G}}_{i,i}(\tilde{\mathbf{G}}_{i,i} + 2t_i\mathbf{I})
\end{aligned}$$

using the normality of $\tilde{\mathbf{D}}_{i,i}$.

The matrix $-\alpha_i\tilde{\mathbf{G}}_{i,i}(\tilde{\mathbf{G}}_{i,i}+2t_i\mathbf{I})$ being negative semidefinite is equivalent to $\tilde{\mathbf{G}}_{i,i}(\tilde{\mathbf{G}}_{i,i}+2t_i\mathbf{I})$ being positive semidefinite. Since $\tilde{\mathbf{G}}_{i,i}(\tilde{\mathbf{G}}_{i,i} + 2t_i\mathbf{I})$ is symmetric, this is equivalent to all the eigenvalues of $\tilde{\mathbf{G}}_{i,i}(\tilde{\mathbf{G}}_{i,i} + 2t_i\mathbf{I})$ being nonnegative. By lemma 2, all eigenvalues of $\tilde{\mathbf{G}}_{i,i}$ are nonpositive, so by the spectral mapping theorem, the eigenvalues are $\tilde{\mathbf{G}}_{i,i}(\tilde{\mathbf{G}}_{i,i} + 2t_i\mathbf{I})$ are nonnegative if and only if $-2t_i \ge$ largest nonzero eigenvalue of $\tilde{\mathbf{G}}_{i,i}$. Since $\tilde{\mathbf{D}}_{i,i}$ is normal, the eigenvalues of $\frac{1}{2}\tilde{\mathbf{G}}_{i,i}$ are just the real parts of the eigenvalues of $\tilde{\mathbf{D}}_{i,i}$[Gantmacher, 1960]. Note that since $\tilde{\mathbf{G}}_{i,i}$ is a $m \times m$ irreducible matrix in $T_3(0)$, where $m \ge 2$, there exists by lemma 2 a nonzero eigenvalue of $\tilde{\mathbf{G}}_{i,i}$. If $\tilde{\mathbf{D}}_{i,i} = 0$ then $\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}(\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I}) = t_i\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}$ is negative semidefinite if $t_i \le 0$. Thus the result follows from theorem 6. ■

It follows from lemma 4 that to satisfy the conditions of theorem 7, the nonzero $\tilde{\mathbf{D}}_{i,i}$ should be irreducible and in $T_2(0)$ such that $\tilde{\mathbf{D}}_{i,i}^T$ is also in $T_2(0)$. There are examples where $\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T$ is irreducible, but $\tilde{\mathbf{D}}_{i,i}$ is reducible such that the array is not synchronized. For example, the coupling matrix

$$\tilde{\mathbf{D}}_{i,i} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is reducible. The first and third cell does not receive coupling from other cells and thus operates autonomously. Therefore they will not synchronize to each other, when the cells are chaotic systems exhibiting sensitive dependence on initial conditions.

For symmetric matrices theorem 7 reduces to

Corollary 3 *Suppose that $\tilde{\mathbf{D}}_{i,i}$ is either zero or a symmetric irreducible matrix in $T_3(0)$ and of the same form, i.e. there exists $\alpha_{i,j}$ for each i, j such that $\alpha_{i,j}\tilde{\mathbf{D}}_{i,i} = \tilde{\mathbf{D}}_{j,j}$. Then the system (1) is uniformly asymptotically synchronized if*

- *The largest non-zero eigenvalue*[2] *of nonzero $\tilde{\mathbf{D}}_{i,i}$ is less than or equal to $-t_i$,*
- *For* $\tilde{\mathbf{D}}_{i,i} = 0,\ t_i \le 0$.

For circulant matrices theorem 7 reduces to

[2] Which is the nonzero eigenvalue smallest in magnitude since $\tilde{\mathbf{D}}_{i,i} \in T_3(0)$ is irreducible.

Corollary 4 *Suppose that $\tilde{\mathbf{D}}_{i,i}$ is either zero or a circulant matrix such that $\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T$ is an irreducible matrix in $T_3(0)$ and of the same form, i.e. there exists $\alpha_{i,j}$ for each i, j such that $\alpha_{i,j}(\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T) = \tilde{\mathbf{D}}_{j,j} + \tilde{\mathbf{D}}_{j,j}^T$. Then the system (1) is uniformly asymptotically synchronized if*

- *For nonzero $\tilde{\mathbf{D}}_{i,i} = circ(a_0, a_1, \cdots, a_n)$, the nonzero elements of the Discrete Fourier Transform of $(a_0, a_1, \cdots, a_n)$ has real parts less than or equal to $-t_i$,*
- *For $\tilde{\mathbf{D}}_{i,i} = 0$, $t_i \leq 0$.*

Proof Follows from the fact that the eigenvalues of circulant matrices is just the Discrete Fourier Transform of the first row [Davis, 1979]. ■

The condition that $\tilde{\mathbf{D}}_{i,i}$ are of the same form implies that the coupling patterns between the i-th state variables of the cells are the same regardless of i (unless $\tilde{\mathbf{D}}_{i,i} = 0$). Note that $\tilde{\mathbf{D}}_{i,i}$ does not have to be in $T_3(0)$. We only require that $\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T$ is in $T_3(0)$. However, when $\tilde{\mathbf{D}}_{i,i}$ is irreducible and in $T_3(0)$, there exists a nonzero eigenvalue of $\tilde{\mathbf{D}}_{i,i}$, and we can show the following general result:

Corollary 5 *Suppose $\mathbf{f}$ is continuously differentiable and the Jacobian $\mathbf{D_x f}(\mathbf{x}, t)$ is uniformly bounded. Suppose that $\mathbf{D}'$ is a normal matrix such that $\mathbf{D}' + \mathbf{D}'^T$ is an irreducible matrix in $T_1(\mathcal{F}_n, 0)$ with the off-diagonal entries being $\alpha\mathbf{I}_n$ for $\alpha \geq 0$. Then there exists $\beta^* > 0$ such that the system (1) is uniformly asymptotically synchronized if $\mathbf{D} = \beta\mathbf{D}'$ for $\beta > \beta^*$.*

This corollary applied to symmetric coupling matrices says that in general, mutual diffusive coupling which connects the whole array together will synchronize the array if the coupling is large enough. The property that off-diagonal elements of $\tilde{\mathbf{D}}_{i,i}$ are positive can be considered as a form of cooperative coupling. So we can also say that strong enough mutual cooperation results in synchronization among cells. For circulant coupling matrices this means that strong enough homogeneous diffusive coupling of cells arranged in a ring will synchronize the array.

6.2 Various coupling configurations

Next we will consider various coupling configurations. Depending on the coupling configuration, different methods will be used to prove synchronization. In most cases, theorem 7 and corollary 3 will be used. In other cases, theorem 6 is used and $\mathbf{B}_i = S(\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I})$ is constructed and checked for negative semidefiniteness. To be able to use theorem 7, we will assume that the form of $\tilde{\mathbf{D}}_{i,i}$ is independent of i.

Recall that $\tilde{\mathbf{D}}_{i,i}$ is an $m \times m$ matrix in $T_2(0)$. For the case of diffusive coupling, the matrices $\tilde{\mathbf{D}}_{i,i}$ belongs to the class $T_3(0)$. We show the matrix graphs of the various $\tilde{\mathbf{D}}_{i,i}$ we consider in Fig. 1. The matrix graph of $\tilde{\mathbf{D}}_{i,i}$ can be considered as the directed interaction graph between the i-th state of each cell.

6.2.1 Tridiagonal coupling, periodic boundary condition

Consider the coupling matrix $\tilde{\mathbf{D}}_{i,i}$ being equal to

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -a-b & a & & & b \\ b & -a-b & a & & \\ & \ddots & & \ddots & \ddots \\ & & & & \\ a & & & b & -a-b \end{pmatrix}$$

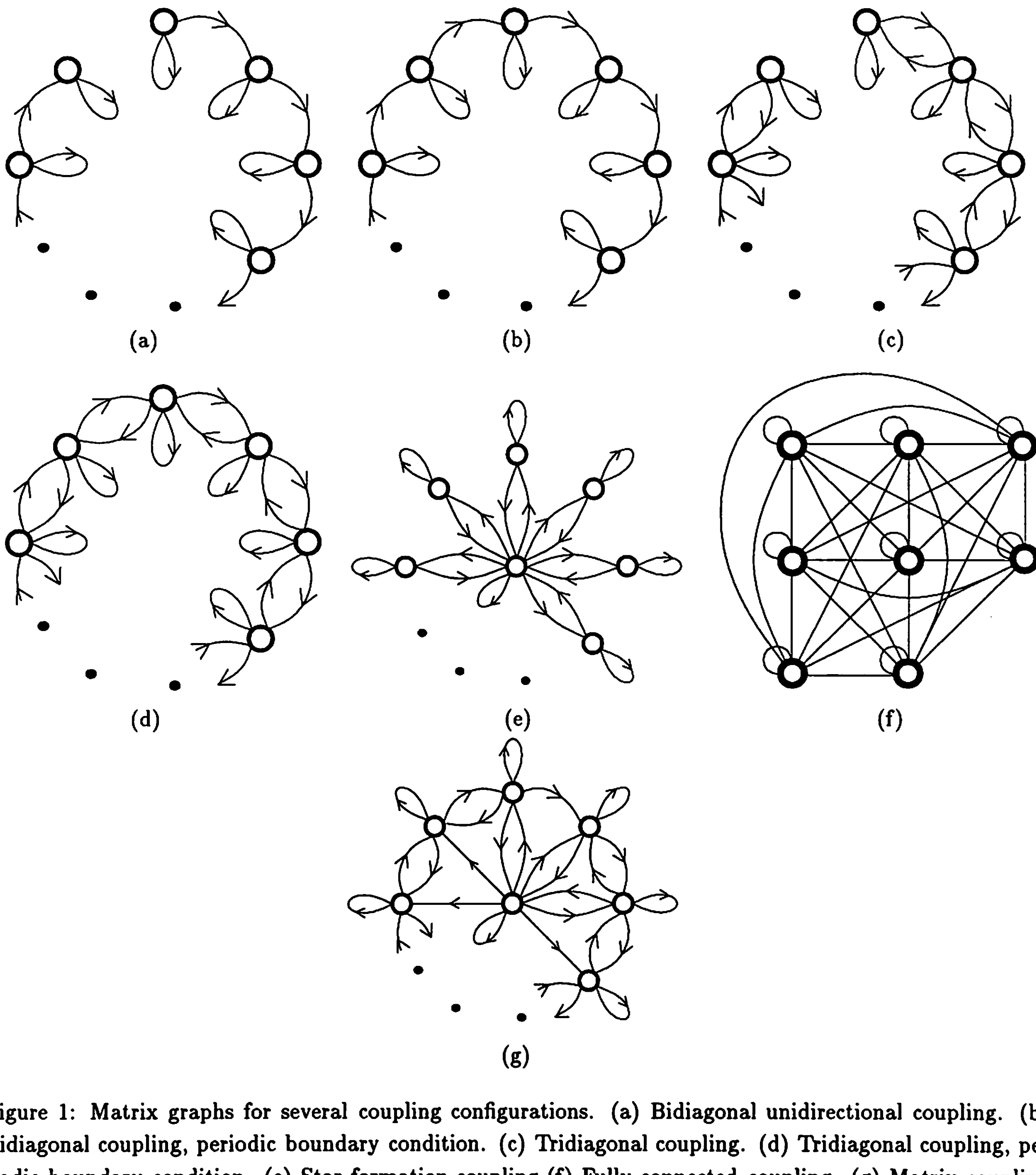

Figure 1: Matrix graphs for several coupling configurations. (a) Bidiagonal unidirectional coupling. (b) Bidiagonal coupling, periodic boundary condition. (c) Tridiagonal coupling. (d) Tridiagonal coupling, periodic boundary condition. (e) Star formation coupling (f) Fully connected coupling. (g) Matrix coupling corresponding to coupling matrix (26).

for all i. The matrix $\tilde{\mathbf{D}}_{i,i}$ is circulant and therefore normal. The real parts of the eigenvalues of $\tilde{\mathbf{D}}_{i,i}$ is equal to the eigenvalues of

$$\frac{1}{2}(\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T) = d_i \begin{pmatrix} -a-b & \frac{1}{2}(a+b) & & & \frac{1}{2}(a+b) \\ \frac{1}{2}(a+b) & -a-b & \frac{1}{2}(a+b) & & \\ & \ddots & \ddots & \ddots & \\ \frac{1}{2}(a+b) & & & \frac{1}{2}(a+b) & -a-b \end{pmatrix}$$

which is equal to the Discrete Fourier Transform of the first row:

$$\mathrm{Re}(\sigma(\tilde{\mathbf{D}}_{i,i})) = \sigma(\frac{1}{2}(\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T)) = \left\{ -2d_i(a+b)\sin^2\left(\frac{\pi k}{m}\right) : k = 0, \ldots, m-1 \right\}$$

For $d_i(a+b) > 0$, the largest (least negative) nonzero eigenvalue is $-2d_i(a+b)\sin^2(\frac{\pi}{m})$. So by theorem 7 the array will uniformly asymptotically synchronize if $d_i(a+b) > 0$ and

$$\boxed{2d_i(a+b)\sin^2(\tfrac{\pi}{m}) \geq t_i} \tag{17}$$

for all i. Note that as the number of cells m increases, $\sin^2(\frac{\pi}{m})$ decreases to zero, so presumably more coupling is required to synchronize the system.

Two special cases are given below:

1. (Tridiagonal symmetric coupling, periodic boundary condition) Let $\tilde{\mathbf{D}}_{i,i}$ be

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix}$$

for all i. The array will uniformly asymptotically synchronize if $d_i > 0$ and

$$\boxed{4d_i\sin^2(\tfrac{\pi}{m}) \geq t_i} \tag{18}$$

for all i.

2. (Bidiagonal coupling, periodic boundary condition) Let $\tilde{\mathbf{D}}_{i,i}$ be

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ 1 & & & & -1 \end{pmatrix}$$

The array will uniformly asymptotically synchronize if $d_i > 0$ and

$$\boxed{2d_i\sin^2(\tfrac{\pi}{m}) \geq t_i} \tag{19}$$

for all i.

6.2.2 Bidiagonal Unidirectional Coupling

Consider the coupling matrix $\tilde{\mathbf{D}}_{i,i}$ being equal to

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} 0 & & & & \\ 1 & -1 & & & \\ & \ddots & \ddots & & \\ & & & 1 & -1 \end{pmatrix}$$

for all i. Because the coupling is unidirectorial, the case for an array of three cells will generalize to the general case. The system state equations for three cells are

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1, t) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2, t) + \mathbf{D}_{2,2}\mathbf{x}_2 + \mathbf{D}_{2,1}\mathbf{x}_1 = \tilde{\mathbf{f}}(\mathbf{x}_2, t) \\ \dot{\mathbf{x}}_3 &= \mathbf{f}(\mathbf{x}_3, t) + \mathbf{D}_{3,3}\mathbf{x}_3 + \mathbf{D}_{3,2}\mathbf{x}_2 \end{aligned} \tag{20}$$

Now suppose that $d_i > t_i$ for all i. Then there exists $\gamma > 0$ such that

$$(\mathbf{x}-\mathbf{y})^T\mathbf{V}\left(\mathbf{f}(\mathbf{x},t) - \mathbf{f}(\mathbf{y},t) - \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}(\mathbf{x}-\mathbf{y})\right) \leq -\gamma\|\mathbf{x}-\mathbf{y}\|^2$$

Then $\mathbf{x}_2 \longrightarrow \mathbf{x}_1$ as $t \longrightarrow \infty$ since cell 1 and cell 2 are synchronized. Thus $\mathbf{f}(\mathbf{x}_2, t) - \tilde{\mathbf{f}}(\mathbf{x}_2, t) \longrightarrow 0$ as $t \longrightarrow \infty$ and by corollary 2 $\mathbf{x}_3 \longrightarrow \mathbf{x}_2$ as $t \longrightarrow \infty$.

As $\mathbf{x}_1 \longrightarrow \mathbf{x}_2$, $\sup_{t \geq t_0} \|\mathbf{f}(\mathbf{x}_2, t) - \tilde{\mathbf{f}}(\mathbf{x}_2, t)\| \longrightarrow 0$, so by corollary 2 the array will uniformly asymptotically synchronize if

$$\boxed{d_i > t_i} \tag{21}$$

for all i. Note that this condition does not depend on m.

6.2.3 Tridiagonal Coupling

Consider the coupling matrix $\tilde{\mathbf{D}}_{i,i}$ being equal to

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -a & a & & & \\ b & -a-b & a & & \\ & \ddots & \ddots & \ddots & \\ & & b & -a-b & a \\ & & & b & -b \end{pmatrix}$$

for all i.

The $m \times m$ matrix $\frac{1}{2}(S(\tilde{\mathbf{D}}_{i,i}) + S(\tilde{\mathbf{D}}_{i,i})^T)$ is

$$d_i \begin{pmatrix} -a-b & \frac{1}{2}(a+b) & & & \\ \frac{1}{2}(a+b) & -a-b & \frac{1}{2}(a+b) & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2}(a+b) & -a-b & \frac{1}{2}(a+b) \\ & & & \frac{1}{2}(a+b) & -a-b \end{pmatrix}$$

which is Toeplitz and has eigenvalues [Trench, 1985]

$$\left\{d_i\left(-a-b+|a+b|\cos\left(\frac{k\pi}{m}\right)\right) : k=1,\ldots,m-1\right\}$$

The largest eigenvalue is $-a-b+|a+b|\cos\left(\frac{\pi}{m}\right)$, so by theorem 6 the array will uniformly asymptotically synchronize if

$$\boxed{d_i\left(-a-b+|a+b|\cos\left(\tfrac{\pi}{m}\right)\right) \le -t_i}$$

for all i. Note that when d_i and t_i are positive, $a+b$ need to be positive for this condition to be satisfied.

When $a=0$ and $b=1$, we obtain the coupling of Sec. 6.2.2. But the condition obtain here is more conservative than that obtained in Sec. 6.2.2.

When $a=b=1$, we obtain the coupling

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix}$$

for all i.

This case was studied in [Fujisaka and Yamada, 1983; Pérez-Villar *et al.*, 1993; Belykh *et al.*, 1993]. By the result above, the array will uniformly asymptotically synchronize if

$$\boxed{4d_i\sin^2\left(\tfrac{\pi}{2m}\right) \ge t_i} \tag{22}$$

for all i. Note that as m increases, $\sin^2(\frac{\pi}{2m})$ decreases to zero, so more coupling is required to synchronize the system.

6.2.4 Fully Connected Coupling

Consider the coupling matrix $\tilde{\mathbf{D}}_{i,i}$ being equal to

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -m+1 & 1 & 1 & \cdots & 1 \\ 1 & -m+1 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & 1 & \cdots & -m+1 & 1 \\ 1 & 1 & \cdots & 1 & -m+1 \end{pmatrix}$$

for all i.

We calculate the corresponding matrices $\mathbf{B}_i = S(\tilde{\mathbf{D}}_{i,i} + t_i\mathbf{I})$, which is

$$\mathbf{B}_i = d_i \begin{pmatrix} -m+\frac{t_i}{d_i} & & & & \\ & -m+\frac{t_i}{d_i} & & & \\ & & \ddots & & \\ & & & -m+\frac{t_i}{d_i} & \\ & & & & -m+\frac{t_i}{d_i} \end{pmatrix}$$

Thus $\mathbf{B}_i$ is negative semidefinite if $d_i m \ge t_i$. By theorem 6 the array will uniformly asymptotically synchronize if

$$\boxed{d_i m \ge t_i} \tag{23}$$

for all i.

Note that for $d_i > 0$, as m get larger this condition will surely be satisfied for fixed t_i and d_i, in contrast to cases 6.2.1, 6.2.3 where for fixed t_i, d_i needs to be increased for large m for the matrix to be negative semidefinite. This makes sense as for fixed d_i, there is more coupling as m gets larger.

6.2.5 Star Formation

Consider the coupling matrix $\tilde{\mathbf{D}}_{i,i}$ being equal to

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -(m-1) & 1 & \cdots & 1 \\ 1 & -1 & & \\ \vdots & & \ddots & \\ 1 & & & -1 \end{pmatrix}$$

for all i. The m eigenvalues of $\tilde{\mathbf{D}}_{i,i}$ are $\{0, -md_i, -d_i, \ldots, -d_i\}$, so that the system will uniformly asymptotically synchronize if $d_i > 0$ and

$$\boxed{\begin{array}{rl} 2d_i \geq t_i, & \text{if } m = 2 \\ d_i \geq t_i, & \text{if } m > 2 \end{array}} \tag{24}$$

for all i. Note that as in case 6.2.2 this condition does not depend on m (except when $m = 2$), as contrasted to 6.2.1, 6.2.3 and 6.2.4.

6.2.6 Another formation

From the proof of theorem 7 we see that if we choose $\tilde{\mathbf{D}}_{i,i} = -d_i(\tilde{\mathbf{M}}^T\tilde{\mathbf{M}})^p$, then we can also prove synchronization if d_i is large enough. For example, let us choose

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -6 & 4 & -1 & 0 & \cdots & -1 & 4 \\ 4 & -6 & 4 & -1 & 0 & \cdots & -1 \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & & & & & \\ 4 & -1 & 0 & \cdots & -1 & 4 & -6 \end{pmatrix}$$

for all i.

Then $\tilde{\mathbf{D}}_{i,i} = -d_i(\tilde{\mathbf{M}}^T\tilde{\mathbf{M}})^2$ where $\tilde{\mathbf{M}}$ is defined as Eq. (6). Using a similar argument as in the proof of theorem 7, the array will uniformly asymptotically synchronize if

$$\boxed{16d_i \sin^4(\tfrac{\pi}{m}) \geq t_i} \tag{25}$$

for all i.

The cases 6.2.1 (special case 1), 6.2.3, and 6.2.4 were considered in [Fujisaka and Yamada, 1983]. For the cases 6.2.3 and 6.2.4, $\mathbf{B}_i$ is symmetric, so negative semidefiniteness of $\mathbf{B}_i$ follows if all its eigenvalues are nonpositive. If $\mathbf{B}_i$ is normal, then $\mathbf{B}_i$ is negative semidefinite if and only if all the eigenvalues has nonpositive real parts [Gantmacher, 1960]. By lemma 2, the eigenvalues of $\mathbf{B}_i$ are the nonzero eigenvalues of $\tilde{\mathbf{D}}_{i,i}$ incremented by t_i, assuming that 0 is an eigenvalue of multiplicity 1 of $\tilde{\mathbf{D}}_{i,i}$. As all nonzero eigenvalues of $\tilde{\mathbf{D}}_{i,i}$ have negative real parts, there exists a positive nonzero $\frac{t_i}{d_i}$ such that $\mathbf{B}_i$ is negative semidefinite. These observations can be summarized as follows:

Theorem 8 *Let $\tilde{\mathbf{D}}_{i,i} \in T_2(0)$ and $S(\tilde{\mathbf{D}}_{i,i})$ is normal for all i. The system (1) is uniformly asymptotically synchronized if*

- *For $\tilde{\mathbf{D}}_{i,i}$ such that 0 is an eigenvalue of multiplicity $k \geq 2$, all eigenvalues of $\tilde{\mathbf{D}}_{i,i}$ has real parts less than or equal to $-t_i$.*

- *Otherwise, all non-zero eigenvalues of $\tilde{\mathbf{D}}_{i,i}$ has real part less than or equal to $-t_i$*

Corollary 6 *Suppose $\mathbf{f}(\mathbf{x},t)$ is continuously differentiable and has a uniformly bounded Jacobian. Given a coupling configuration $\mathbf{D}'$ in $T_1(M_{n\times n}(\mathbb{R}),0)$ such that the entries are diagonal matrices. Suppose that the corresponding $\tilde{\mathbf{D}}_{i,i}$ are such that $\tilde{\mathbf{D}}_{i,i} \in T_3(0)$ and irreducible for all i and $S(\tilde{\mathbf{D}}_{i,i})$ is normal for all i. Then there exists $\tilde{\beta} > 0$ such that Eq. (1) with coupling matrix $\mathbf{D} = \beta\mathbf{D}'$ is uniformly asymptotically synchronized for $\beta > \tilde{\beta}$.*

Note that in these two results, in contrast to theorem 7, we do not require that all $\tilde{\mathbf{D}}_{i,i}$ be of the same form and in theorem 8 we do not require that $\tilde{\mathbf{D}}_{i,i} + \tilde{\mathbf{D}}_{i,i}^T \in T_3(0)$.

This proposition roughly says that for coupling configurations with results in normal $S(\tilde{\mathbf{D}}_{i,i})$'s, strong enough dissipative coupling will synchronize the array.

For example, consider the following rather contrived coupling configuration:

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -1+\epsilon & 1-\epsilon & & 1 & -1 \\ 1+\epsilon & -2 & 1-\epsilon & & \\ \epsilon & 1 & -2 & 1-\epsilon & \\ \vdots & & \ddots & \ddots & \ddots \\ -1+2\epsilon & 1-\epsilon & & 1 & -1-\epsilon \end{pmatrix} \tag{26}$$

for all i. The corresponding matrix graph is shown in Fig. 1(g).

$$S(\tilde{\mathbf{D}}_{i,i}) = d_i \begin{pmatrix} -2 & 1-\epsilon & & & 1 \\ 1 & -2 & 1-\epsilon & & \\ & \ddots & \ddots & \ddots & \\ 1-\epsilon & & & 1 & -2 \end{pmatrix}$$

which is circulant. Its eigenvalues are

$$\left\{ d_i \left[-2(2-\epsilon)\sin^2\left(\frac{k\pi}{m}\right) - \epsilon - i\epsilon \sin\left(\frac{2k\pi}{m}\right) \right] : k = 0, \ldots, m-1 \right\}$$

For $d_i(2-\epsilon) \geq 0$, the real parts of all the eigenvalues are less than $-d_i\epsilon$. So system (1) will uniformly asymptotically synchronize if

$$\boxed{\epsilon d_i \geq t_i \text{ and } d_i(2-\epsilon) \geq 0}$$

for all i.

Given these theorems, we can design coupling matrices which will guarantee asymptotical synchronization. A possible design criteria would be robustness of synchronization. For example, synchronization should be preserved if some of the coupling is deleted or if the coupling matrix is perturbed. Theorem 6 only puts a constraint on the eigenvalues of $S(\tilde{\mathbf{D}}_{i,i}) + S(\tilde{\mathbf{D}}_{i,i})^T$. Thus we can design robust coupling matrices by placing the eigenvalues of $S(\tilde{\mathbf{D}}_{i,i}) + S(\tilde{\mathbf{D}}_{i,i})^T$ far to the left of the $-t_i$, so that a small perturbation of $\tilde{\mathbf{D}}_{i,i}$ which keeps $\tilde{\mathbf{D}}_{i,i}$ in T_2 will not destroy synchronization of the array[3].

[3] The requirement that $\tilde{\mathbf{D}}_{i,i} \in T_2$ might seem too strict, but it is very natural in resistively coupled circuits because Kirchhoff Current Law guarantees that the coupling matrices will be in $T_2(0)$. See Sec. 6.2.7.

For example, consider the case of fully connected coupling (Sec. 6.2.4). Suppose one of the coupling is removed, i.e., the resulting coupling matrix is:

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -m+2 & 0 & 1 & \cdots & 1 \\ 1 & -m+1 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & 1 & \cdots & -m+1 & 1 \\ 1 & 1 & \cdots & 1 & -m+1 \end{pmatrix}$$

Calculating $S(\tilde{\mathbf{D}}_{i,i})$ we find that to ensure synchronization when one coupling weight is removed from a fully connected configuration, we need

$$\boxed{d_i(m-1) \geq t_i}$$

for all i. If the symmetric coupling weight is also removed, i.e.

$$\tilde{\mathbf{D}}_{i,i} = d_i \begin{pmatrix} -m+2 & 0 & 1 & \cdots & 1 \\ 0 & -m+2 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & 1 & \cdots & -m+1 & 1 \\ 1 & 1 & \cdots & 1 & -m+1 \end{pmatrix}$$

then

$$\boxed{d_i(m-2) \geq t_i}$$

for all i is required to ensure synchronization.

6.2.7 Example: Chua's Oscillator

We illustrate the above results using Chua's oscillator [Chua *et al.*, 1993] as a cell. Chua's oscillator is a system which for some parameter values exhibits chaotic behavior. The circuit diagram of Chua's oscillator is shown in Fig. 2.

The state equations for Chua's oscillator are:

$$\begin{aligned} \frac{dv_1}{dt} &= \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)] \\ \frac{dv_2}{dt} &= \frac{1}{C_2}[G(v_1 - v_2) + i_3] \\ \frac{di_3}{dt} &= -\frac{1}{L}(v_2 + R_0 i_3) \end{aligned} \tag{27}$$

where $G = \frac{1}{R}$ and

$$f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)\{|v_1 + E| - |v_1 - E|\} \tag{28}$$

It was shown in [Wu and Chua, 1994] that a diagonal $\mathbf{V}$ and $\mathbf{T}$ exists when C_1, C_2, R_0, R, L are positive. In this case t_1 can be chosen be any number strictly larger than $\frac{1}{C_1}\max(-G_a, -G_b)$. The values t_2 and t_3 can be chosen to be 0. This means that only the variables v_1 between the cells needs to be coupled to achieve synchronization. Translating corollary 5 to an array of Chua's oscillators we obtain the following result:

Theorem 9 *Let C_1, C_2, R_0, R, L be positive. Let m Chua's oscillators be coupled via linear resistors by connecting nodes 1 (see Fig. 2) of two Chua's oscillators with a linear resistor of conductance $G > 0$. If all the Chua's oscillators are coupled to each other, either directly or indirectly, then for large enough G, the array of Chua's oscillators will uniformly asymptotically synchronize.*

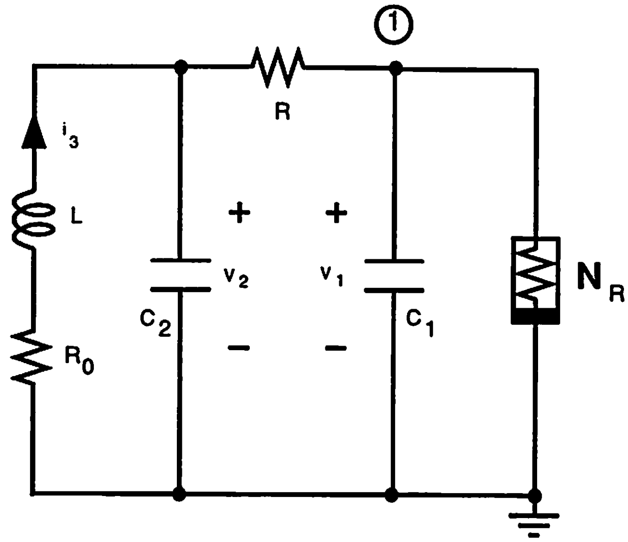

Figure 2: Chua's oscillator

Proof We have $\tilde{\mathbf{D}}_{2,2} = \tilde{\mathbf{D}}_{3,3} = 0$ and $t_2 = t_3 = 0$. By Kirchhoff's Current Law, the coupling matrix $\tilde{\mathbf{D}}_{1,1}$ is a symmetric matrix in $T_2(0)$, and lies in $T_3(0)$ if $G > 0$. The result then follows from corollary 3 and lemma 2. We have assumed that the datum nodes of the Chua's oscillators are chosen to be the same node and are connected together. ■

Note that the coupling configuration can be arbitrary. The only requirement is that it connects the whole array together. This theorem extends the result in [Belykh *et al.*, 1993] which only considers the case of tridiagonal coupling (sec. 6.2.3).

Let us now consider 7 Chua's oscillators coupled as in Fig. 3. The Chua's oscillators are arranged in a random manner to illustrate that they do not have to be arranged in a regular grid with regular coupling. The value G_i indicates the conductance of the corresponding linear coupling resistor. The coupling matrices are $\tilde{\mathbf{D}}_{2,2} = \tilde{\mathbf{D}}_{3,3} = 0$ and

$$\tilde{\mathbf{D}}_{1,1} = \frac{1}{C_1}\begin{pmatrix} -G_1-G_4-G_5 & G_1 & G_4 & 0 & 0 & G_5 & 0 \\ G_1 & -G_1-G_2-G_6 & G_2 & 0 & G_6 & 0 & 0 \\ G_4 & G_2 & -G_4-G_2-G_3-G_7-G_8 & G_3 & G_7 & 0 & G_8 \\ 0 & 0 & G_3 & -G_3-G_{10} & 0 & G_{10} & 0 \\ 0 & G_6 & G_7 & 0 & -G_6-G_7-G_9 & 0 & G_9 \\ G_5 & 0 & 0 & G_{10} & 0 & -G_5-G_{10} & 0 \\ 0 & 0 & G_8 & 0 & G_9 & 0 & -G_8-G_9 \end{pmatrix}$$

The parameter values of Chua's oscillator are chosen as $C_1 = 5.56\text{nF}, C_2 = 50\text{nF}, R = 1428\Omega, L = 7.1\text{mH}$, $R_0 = 1\Omega$, $G_a = -0.8\text{mS}$, $G_b = -0.5\text{mS}$, and $E = 1\text{V}$. For these parameter values, the uncoupled circuit operates in the chaotic region. For the conductances values $G_1 = 0.71\text{mS}$, $G_2 = 0.69\text{mS}$, $G_3 = 1.04\text{mS}$, $G_4 = 0.81\text{mS}$, $G_5 = 0.67\text{mS}$, $G_6 = 0.65\text{mS}$, $G_7 = 0.63\text{mS}$, $G_8 = 0.98\text{mS}$, $G_9 = 1.02\text{mS}$, $G_{10} = 0.98\text{mS}$, the eigenvalues of $\tilde{\mathbf{D}}_{1,1}$ are

$$\frac{10^{-3}}{C_1} \times \{0, -0.819, -1.606, -2.456, -2.811, -3.480, -5.188\}$$

The least negative nonzero eigenvalue is strictly less than $\frac{\min(G_a,G_b)}{C_1}$, and the array will thus uniformly asymptotically synchronize by corollary 3.

Let us consider another set of coupling conductance values. For the conductance values $G_1 = -0.31\text{mS}$, $G_2 = 2.49\text{mS}$, $G_3 = 3.74\text{mS}$, $G_4 = 3.21\text{mS}$, $G_5 = -0.27\text{mS}$, $G_6 = -0.23\text{mS}$, $G_7 = 1.53\text{mS}$, $G_8 = 1.48\text{mS}$,

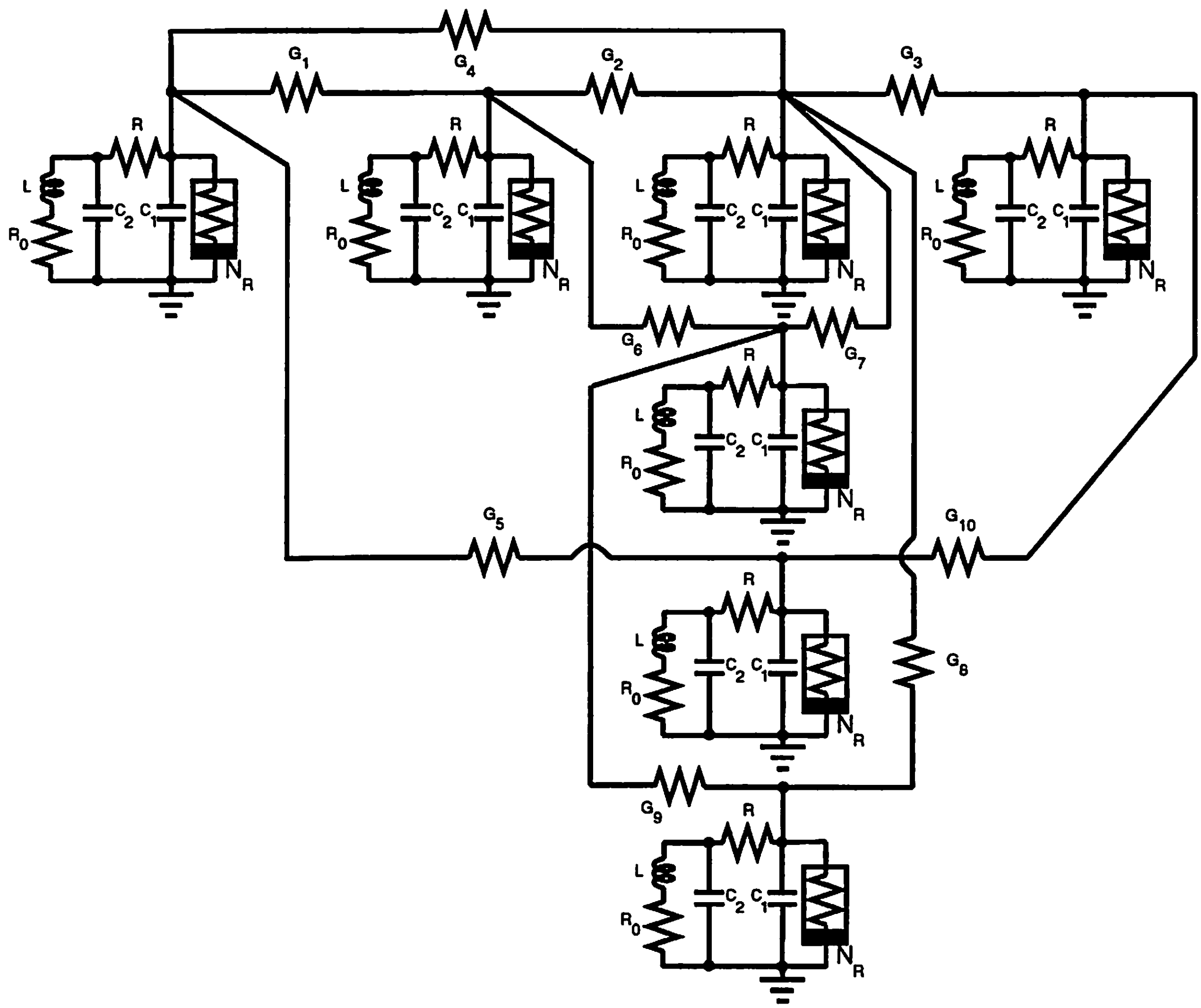

Figure 3: An array of 7 linearly coupled Chua's oscillators. The values G_i's indicate the conductances of the corresponding linear coupling resistors. The parameter values of Chua's oscillator are chosen as $C_1 = 5.56\text{nF}$, $C_2 = 50\text{nF}$, $R = 1428\Omega$, $L = 7.1\text{mH}$, $R_0 = 1\Omega$, $G_a = -0.8\text{mS}$, $G_b = -0.5\text{mS}$, and $E = 1\text{V}$.

$G_9 = 0.62\text{mS}$, $G_{10} = 2.18\text{mS}$, the matrix $\tilde{\mathbf{D}}_{i,i}$ contains negative off-diagonal elements and thus we cannot use corollary 3. We calculate $\mathbf{B}_1$ and use theorem 6 instead. The eigenvalues of $\frac{1}{2}(S(\tilde{\mathbf{D}}_{1,1}) + S(\tilde{\mathbf{D}}_{1,1})^T)$ are

$$\frac{10^{-3}}{C_1} \times \{-0.845, -0.931, -2.198, -2.558, -5.980, -16.367\}$$

All the eigenvalues are strictly less than $\frac{\min(G_a,G_b)}{C_1}$. This implies that $\mathbf{B}_1$ is negative semidefinite, and the array will uniformly asymptotically synchronize by theorem 6 (note that $\mathbf{B}_2 = \mathbf{B}_3 = 0$).

Recall that as the cells are synchronized, they are decoupled. Therefore each Chua's oscillator will oscillate chaotically if the synchronized state is in the basin of attraction of the chaotic attractor. But the chaotic attractor in Chua's oscillator for this set of parameter values is not a global attractor. In fact, for large enough initial conditions, the trajectories will become unbounded. Therefore it is possible that in the array the trajectories of the cells still become unbounded, even though the differences between the cells go to 0.

7 Decomposing Reducible Coupling Matrices

Any reducible matrix can be permuted (i.e. is similar to via a permutation matrix) to a block triangular matrix such that the diagonal blocks are irreducible. This corresponds to a cascade of irreducible components. We say a directed graph is strongly connected if for all i and j there is a directed path from vertex i to vertex j. In terms of the matrix graph, we can decompose any directed graph into subgraphs forming the vertices of an acyclic graph. These subgraphs are the strongly connected components or maximally strongly connected subgraphs of the graph. The vertices of the acyclic graph with indegree 0 are strongly connected components whose cells are not influenced by the cells in other strongly connected components. Thus we can analyze these components along with the strongly connected components they influence, while ignoring the rest of the system. Then if these components have been shown to synchronize, we can "collapse" the dynamics of the synchronized cells into that of one cell. In the irreducible components that they influence, their dependence is deleted from the state equations and replaced by coupling of a single cell. This is made more precise in the following theorems. We will only state the theorems for the case of 3 strongly connected components, but the general case follows easily from this case.

Lemma 7 *If (1) is in class Q^+, then*

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1, t) + \textstyle\sum_{i=1}^m \mathbf{D}_{1,i}\mathbf{x}_i + \mathbf{g}(t) \\
&\vdots \\
\dot{\mathbf{x}}_m &= \mathbf{f}_m(\mathbf{x}_m, t) + \textstyle\sum_{i=1}^m \mathbf{D}_{m,i}\mathbf{x}_i + \mathbf{g}(t)
\end{aligned}$$

is also in class Q^+.

Proof Use the same Lyapunov function. ∎

Theorem 10 *Let $\mathbf{x}$ in (1) be partitioned as*

$$\mathbf{c}_1 = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_a \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} \mathbf{x}_{a+1} \\ \vdots \\ \mathbf{x}_b \end{pmatrix}, \mathbf{c}_3 = \begin{pmatrix} \mathbf{x}_{b+1} \\ \vdots \\ \mathbf{x}_m \end{pmatrix}$$

where $\mathbf{c}_i$'s are irreducible components which are coupled in a cascade as follows

$$\dot{\mathbf{c}}_1 = \mathbf{f}_{\mathbf{c}_1}(\mathbf{c}_1,t) + \begin{pmatrix} \mathbf{D}_{1,1} & \cdots & \mathbf{D}_{1,a} \\ \vdots & & \vdots \\ \mathbf{D}_{a,1} & \cdots & \mathbf{D}_{a,a} \end{pmatrix} \mathbf{c}_1 \tag{29}$$

$$\dot{\mathbf{c}}_2 = \mathbf{f}_{\mathbf{c}_2}(\mathbf{c}_2,t) + \begin{pmatrix} \mathbf{D}_{a+1,1} & \cdots & \mathbf{D}_{a+1,a} \\ \vdots & & \vdots \\ \mathbf{D}_{b,1} & \cdots & \mathbf{D}_{b,a} \end{pmatrix} \mathbf{c}_1 + \begin{pmatrix} \mathbf{D}_{a+1,a+1} & \cdots & \mathbf{D}_{a+1,b} \\ \vdots & & \vdots \\ \mathbf{D}_{b,a+1} & \cdots & \mathbf{D}_{b,b} \end{pmatrix} \mathbf{c}_2 \tag{30}$$

$$\begin{aligned} \dot{\mathbf{c}}_3 &= \mathbf{f}_{\mathbf{c}_3}(\mathbf{c}_3,t) + \begin{pmatrix} \mathbf{D}_{b+1,1} & \cdots & \mathbf{D}_{b+1,a} \\ \vdots & & \vdots \\ \mathbf{D}_{m,1} & \cdots & \mathbf{D}_{m,a} \end{pmatrix} \mathbf{c}_1 + \begin{pmatrix} \mathbf{D}_{b+1,a+1} & \cdots & \mathbf{D}_{b+1,b} \\ \vdots & & \vdots \\ \mathbf{D}_{m,a+1} & \cdots & \mathbf{D}_{m,b} \end{pmatrix} \mathbf{c}_2 \\ &+ \begin{pmatrix} \mathbf{D}_{b+1,b+1} & \cdots & \mathbf{D}_{b+1,m} \\ \vdots & & \vdots \\ \mathbf{D}_{m,b+1} & \cdots & \mathbf{D}_{m,m} \end{pmatrix} \mathbf{c}_3 \end{aligned} \tag{31}$$

Suppose that system (29) is in class Q^+, and the following two systems

$$\begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{c}}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1,t) \\ \mathbf{f}_{\mathbf{c}_2}(\mathbf{c}_2,t) \end{pmatrix} + \begin{pmatrix} 0 & & \cdots & 0 \\ \sum_{i=1}^{a} \mathbf{D}_{a+1,i} & \mathbf{D}_{a+1,a+1} & \cdots & \mathbf{D}_{a+1,b} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{a} \mathbf{D}_{b,i} & \mathbf{D}_{b,a+1} & \cdots & \mathbf{D}_{b,b} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{c}_2 \end{pmatrix} \tag{32}$$

$$\begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{c}}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1,t) \\ \mathbf{f}_{\mathbf{c}_3}(\mathbf{c}_3,t) \end{pmatrix} + \begin{pmatrix} 0 & & \cdots & 0 \\ \sum_{i=1}^{b} \mathbf{D}_{b+1,i} & \mathbf{D}_{b+1,b+1} & \cdots & \mathbf{D}_{b+1,m} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{b} \mathbf{D}_{m,i} & \mathbf{D}_{m,b+1} & \cdots & \mathbf{D}_{m,m} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{c}_3 \end{pmatrix} \tag{33}$$

are also in Q^+, then system (1) in the form of Eqs. (29-31) is uniformly asymptotically synchronized.

Proof Follows from corollary 2, and noting that when the array is synchronized, the cells are decoupled. ■

Theorem 11 *In Eqs. (29-31), assume that $\sum_{i=1}^{a} \mathbf{D}_{a+1,i} = \sum_{i=1}^{a} \mathbf{D}_{a+2,i} = \cdots = \sum_{i=1}^{a} \mathbf{D}_{b,i}$ and that $\sum_{i=1}^{b} \mathbf{D}_{b+1,i} = \sum_{i=1}^{b} \mathbf{D}_{b+2,i} = \cdots = \sum_{i=1}^{b} \mathbf{D}_{m,i}$, then system (29-31) is uniformly asymptotically synchronized if (29) and the following two subarrays of systems are in class Q^+ :*

$$\begin{aligned} \dot{\mathbf{c}}_2 &= \mathbf{f}_{\mathbf{c}_2}(\mathbf{c}_2,t) + \begin{pmatrix} \mathbf{D}_{a+1,a+1} & \cdots & \mathbf{D}_{a+1,b} \\ \vdots & & \vdots \\ \mathbf{D}_{b,a+1} & \cdots & \mathbf{D}_{b,b} \end{pmatrix} \mathbf{c}_2 \\ \dot{\mathbf{c}}_3 &= \mathbf{f}_{\mathbf{c}_3}(\mathbf{c}_3,t) + \begin{pmatrix} \mathbf{D}_{b+1,b+1} & \cdots & \mathbf{D}_{b+1,m} \\ \vdots & & \vdots \\ \mathbf{D}_{m,b+1} & \cdots & \mathbf{D}_{m,m} \end{pmatrix} \mathbf{c}_3 \end{aligned}$$

and the following three cell array

$$\dot{\mathbf{x}}_1 = \mathbf{f}(\mathbf{x}_1,t)$$

$$
\begin{aligned}
\dot{\mathbf{x}}_{a+1} &= \mathbf{f}(\mathbf{x}_{a+1},t)+\sum_{i=1}^{a}\mathbf{D}_{a+1,i}\mathbf{x}_1+\sum_{i=a+1}^{b}\mathbf{D}_{a+1,i}\mathbf{x}_{a+1} \\
\dot{\mathbf{x}}_{b+1} &= \mathbf{f}(\mathbf{x}_{b+1},t)+\sum_{i=1}^{b}\mathbf{D}_{b+1,i}\mathbf{x}_1+\sum_{i=b+1}^{m}\mathbf{D}_{b+1,i}\mathbf{x}_{b+1}
\end{aligned}
$$

is in class Q^+.

Proof From lemma 7, all the cells in partition $\mathbf{c}_2$ are synchronized to each other. Similarly for partition $\mathbf{c}_3$. The result then follows from theorem 10 and corollary 2. ■

Note that in this case the irreducible components are analyzed separately, and when within these components the cells synchronize, then these components are "collapsed" into single cells and the resulting system has a signed interaction graph which is the acyclic graph mentioned earlier. This is then analyzed further by collapsing synchronized cells into a single cell. At this stage we only need to consider two cell arrays in master-slave (i.e. reducible coupling) configuration. We can collapse synchronized cells into a single cell because at the synchronized state, the cells are decoupled, so that the dynamics is reduced to that of one (uncoupled) cell.

8 Linear Coupling in Regular Arrays

Recall that in order to apply theorem 7, we require that the coupling matrix $\bar{\mathbf{D}}_{i,i}$ be irreducible, and we need to calculate its eigenvalues. In some applications the cells are arranged in a regular grid as in a Cellular Neural Network (CNN) [Chua and Yang, 1988a; Chua and Yang, 1988b] with space-invariant (or homogeneous) coupling templates, also called cloning templates. Several coupling configurations in Sec. 6.2 are of this type. In cases where the boundary conditions are Toeplitz, irreducibility of the coupling matrix can be determined using the irreducibility tests in [Chua and Wu, 1992]. If the boundary conditions are periodic, then the eigenvalues can be calculated as multidimensional discrete fourier transforms of the cloning template [Shi, 1994].

The vector field of Chua's oscillator is odd-symmetric. Therefore the vector field is invariant under the transformation $\mathbf{x}_i \longrightarrow -\mathbf{x}_i$. For certain cloning templates which contains negative off center elements, state transformations in [Chua and Wu, 1992] can be used to transform the cloning templates to templates with nonnegative off center elements. Synchronization for these transformed templates implies that the cells are also synchronized in the original system, although some of the cells are synchronized to the negative of other cells.

9 General Additive Coupling

So far, we have only considered linear coupling among cells. In this section, we give a brief analysis of general nonlinear additive coupling, i.e. coupling terms which are added to the vector fields. Again we consider an array of *m cells*, coupled together with ***additive*** coupling, with each cell being a n-dimensional system. The entire array is a system of nm ordinary differential equations:

$$
\begin{aligned}
\dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1,t)+\mathbf{g}_1(\mathbf{x}_1,\ldots,\mathbf{x}_m,t) \\
\vdots &= \vdots \\
\dot{\mathbf{x}}_m &= \mathbf{f}_m(\mathbf{x}_m,t)+\mathbf{g}_m(\mathbf{x}_1,\ldots,\mathbf{x}_m,t)
\end{aligned} \tag{34}
$$

where $\mathbf{x}_i \in \mathbb{R}^n$. We write $\mathbf{D}_{i,j} = \frac{\partial \mathbf{g}_i}{\partial \mathbf{x}_j}$. Thus $\mathbf{D}_{i,j} = \mathbf{D}_{i,j}(\mathbf{x},t)$ is a $n \times n$ matrix which in this case depends on both $\mathbf{x}$ and t. We denote

$$\mathbf{D}(\mathbf{x},t) = \begin{pmatrix} \mathbf{D}_{1,1} & \cdots & \mathbf{D}_{1,m} \\ \vdots & & \vdots \\ \mathbf{D}_{m,1} & \cdots & \mathbf{D}_{m,m} \end{pmatrix}$$

Using the notation of Sec. 2, Eq. (34) can be written as

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1,t) + \mathbf{g}_1(\mathbf{x},t) \\ \vdots \\ \mathbf{f}_m(\mathbf{x}_m,t) + \mathbf{g}_m(\mathbf{x},t) \end{pmatrix}$$

Definition 8 *A function $\mathbf{g}(\mathbf{x}_1,\ldots,\mathbf{x}_k,t)$ is separable with respect to $\mathbf{x}_1,\ldots,\mathbf{x}_k$ if it can be written as*

$$\mathbf{g}(\mathbf{x}_1,\ldots,\mathbf{x}_k,t) = \mathbf{g}_1(\mathbf{x}_1,t) + \cdots + \mathbf{g}_k(\mathbf{x}_k,t)$$

Definition 9 *A function $\mathbf{g}(\mathbf{x}_1,\ldots,\mathbf{x}_k,t)$ belong to class S if $\mathbf{g}(\mathbf{x}_c,\ldots,\mathbf{x}_c,t) = 0$ for all $\mathbf{x}_c$ and all t.*

Note that a separable function $\mathbf{g}(\mathbf{x}_1,\ldots,\mathbf{x}_k,t) = \mathbf{g}_1(\mathbf{x}_1,t) + \cdots + \mathbf{g}_k(\mathbf{x}_k,t)$ belong to class S if and only if $\mathbf{g}_1 + \cdots + \mathbf{g}_k = 0$.

Lemma 8 *If $\mathbf{g}(\mathbf{x}_1,\ldots,\mathbf{x}_k,t)$ is in class S and differentiable, then $\sum_{i=1}^k \frac{\partial \mathbf{g}}{\partial \mathbf{x}_i} = 0$ for all t when $\mathbf{x}_1 = \cdots = \mathbf{x}_k$.*

Proof Define $h(\mathbf{x}) = (\mathbf{x},\ldots,\mathbf{x})^T$. Then $\mathbf{g}(h(\mathbf{x}),t) = 0$ for all $\mathbf{x}$ and all t. Thus for fixed t, $D\mathbf{g}(h(\mathbf{x}),t) \cdot Dh(\mathbf{x}) = 0$. Since $Dh(\mathbf{x}) = (1,\ldots,1)^T$, the result follows. ■

Let us assume that we have identical cells, ($\mathbf{f}_1 = \cdots = \mathbf{f}_m$). Let us also assume that the subspace $\mathcal{M} = \{\mathbf{x} : \mathbf{x}_i = \mathbf{x}_j \text{ for all } i \text{ and } j\} = \{\mathbf{x} : d(\mathbf{x}) = 0\}$ is invariant; i.e. synchronization is possible. This implies that $\mathbf{g}_1(\mathbf{x},t) = \cdots = \mathbf{g}_m(\mathbf{x},t)$ for all $\mathbf{x} \in \mathcal{M}$. If we also assume that the cells are decoupled in $\mathcal{M}$, then this means that $\mathbf{g}_i$ is in class S. We also assume that $\mathbf{g}_i$ is differentiable. Then by lemma 8 this means that $\mathbf{D}(\mathbf{x},t)$ as defined above is in $T_2(0)$ for $\mathbf{x} \in \mathcal{M}$ and all t. We want to find conditions for which the invariant subspace $\mathcal{M}$ is asymptotically stable. Assume that $\mathbf{V}$ and $\mathbf{T}$ are as defined in Sec. 5. For the Lyapunov function $V(\mathbf{x})$ as in Eq. (14), with $\mathbf{M}$ as in Eq. (10), the derivative along the trajectories of (34) is

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T\mathbf{U} \begin{pmatrix} \mathbf{f}(\mathbf{x}_1,t) - \mathbf{T}\mathbf{x}_1 \\ \mathbf{f}(\mathbf{x}_2,t) - \mathbf{T}\mathbf{x}_2 \\ \vdots \\ \mathbf{f}(\mathbf{x}_m,t) - \mathbf{T}\mathbf{x}_m \end{pmatrix} + \mathbf{x}^T\mathbf{U} \left[\begin{pmatrix} \mathbf{g}_1(\mathbf{x},t) \\ \mathbf{g}_2(\mathbf{x},t) \\ \vdots \\ \mathbf{g}_m(\mathbf{x},t) \end{pmatrix} + \begin{pmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T} \end{pmatrix} \mathbf{x} \right] \tag{35}$$

Using arguments as in Sec. 5, the subspace $\mathcal{M}$ is asymptotically stable if the second term of Eq. (35) is less than or equal to zero for all $\mathbf{x}$ such that $d(\mathbf{x}) \leq \epsilon$, i.e. initial states located near the subspace $\mathcal{M}$ will converge towards $\mathcal{M}$. Given $\mathbf{x}$, we define $\mathbf{x}' = (\mathbf{x}_1,\ldots,\mathbf{x}_1)^T \in \mathcal{M}$. By the Mean Value Theorem, $\mathbf{g}_i(\mathbf{x},t) = \mathbf{g}_i(\mathbf{x}',t) + \int_0^1 D\mathbf{g}_i(\mathbf{x}' + s(\mathbf{x}-\mathbf{x}'),t)ds(\mathbf{x}-\mathbf{x}')$. Therefore

$$\mathbf{g}_i(\mathbf{x},t) - \mathbf{g}_j(\mathbf{x},t) = \int_0^1 (D\mathbf{g}_i - D\mathbf{g}_j)(\mathbf{x}' + s(\mathbf{x}-\mathbf{x}'),t)ds(\mathbf{x}-\mathbf{x}')$$

and

$$\mathbf{M} \begin{pmatrix} \mathbf{g}_1(\mathbf{x},t) \\ \mathbf{g}_2(\mathbf{x},t) \\ \vdots \\ \mathbf{g}_m(\mathbf{x},t) \end{pmatrix} = \int_0^1 \mathbf{M}\mathbf{D}(\mathbf{x}' + s(\mathbf{x}-\mathbf{x}'),t)ds(\mathbf{x}-\mathbf{x}') \tag{36}$$

Noting that $(\mathbf{x}-\mathbf{x}')^T\mathbf{M}^T = \mathbf{x}^T\mathbf{M}^T$, the second term of Eq. (35) can be written as

$$(\mathbf{x}-\mathbf{x}')^T\mathbf{M}^T \begin{pmatrix} \mathbf{V} & & & \\ & \mathbf{V} & & \\ & & \ddots & \\ & & & \mathbf{V} \end{pmatrix} \mathbf{M} \left[\int_0^1 \mathbf{D}(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}'),t)ds + \begin{pmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T} \end{pmatrix} \right] (\mathbf{x}-\mathbf{x}')$$

So if for all t and for all $\mathbf{x}$ such that $d(\mathbf{x}) \leq \epsilon$,

$$\mathbf{U} \left[\mathbf{D}(\mathbf{x},t) + \begin{pmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T} \end{pmatrix} \right] \tag{37}$$

is negative semidefinite then we have uniform asymptotical synchronization for initial states close enough to $\mathcal{M}$.

Let us now assume that $\mathbf{D}(\mathbf{x},t) \in T_1(M_{n\times n}(\mathbb{R}), K)$ for all t and for all $\mathbf{x}$ such that $d(\mathbf{x}) \leq \epsilon$. Then Eq. (36) can be written as

$$\int_0^1 \mathbf{M}\mathbf{D}(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}'),t)ds(\mathbf{x}-\mathbf{x}') = \int_0^1 S(\mathbf{D}(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}'),t))ds\mathbf{M}(\mathbf{x}-\mathbf{x}')$$

Let $\mathbf{B}(\mathbf{x},t)$ be defined as

$$\mathbf{B}(\mathbf{x},t) = S\left(\mathbf{D}(\mathbf{x},t) + \begin{pmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T} \end{pmatrix} \right)$$

If

$$\begin{pmatrix} \mathbf{V} & & & \\ & \mathbf{V} & & \\ & & \ddots & \\ & & & \mathbf{V} \end{pmatrix} \mathbf{B}(\mathbf{x},t) \tag{38}$$

is negative semidefinite for for all t and all $\mathbf{x}$ such that $d(\mathbf{x}) \leq \epsilon$ then we have asymptotical synchronization for initial states close enough to $\mathcal{M}$. Thus Eqs. (37-38) give sufficient conditions for the synchronized state to be locally asymptotically stable. Similar simplification as in Sec. 6 can be made when $\mathbf{V}$, $\mathbf{T}$, and $\mathbf{D}_{i,j}$ are diagonal.

10 Conclusions

In this paper we give sufficient conditions for which an array of identical dynamical systems, linearly coupled, will synchronize. This allows us to prove the intuitive idea that strong enough cooperative coupling will synchronize an array of identical cells in general. Because the coupling pattern does not need to be regular, we can design robust coupling matrices that ensure synchronization under perturbation of the coupling.

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