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**FEEDBACK STABILIZATION: NONLINEAR
SOLUTIONS TO INHERENTLY NONLINEAR
PROBLEMS**

by

Andrew Richard Teel

Memorandum No. UCB/ERL M92/65

12 June 1992

COVER PAGE

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Andrew Richard Teel

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Feedback Stabilization: Nonlinear Solutions to Inherently Nonlinear Problems

by

Andrew Richard Teel

Abstract

Control strategies are developed for nonlinear systems that fail to satisfy differential geometric conditions for input-to-state linearizability under state feedback and change of coordinates.

The central part of this work is motivated primarily by a popular “ball and beam” laboratory experiment. For this example, the differential geometric conditions for input-to-state linearizability are not satisfied. Strategies have been developed previously to overcome this limitation in a neighborhood of an equilibrium manifold in order to achieve (approximate) tracking and local stabilization. However, the domains of attraction for these methods are very small.

Control strategies are presented for a general class of nonlinear systems, of which the “ball and beam” is an example, which result in arbitrarily large domains of attraction for both the small signal tracking problem and the stabilization problem. The main component of the approach is the use of saturation functions to limit the destabilizing effects that cannot be removed by geometric linearization techniques. One of the new elements of this work is the nesting of saturation functions to systematically isolate and diminish these destabilizing effects.

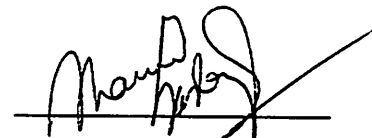
One can think of linear chain of integrator systems that are subject to “actuator constraints” as nonlinear systems that cannot be made to appear linear globally. The methodology of nested saturation functions provides new, simple globally stabilizing control laws for such systems.

In addition to developing methodologies for systems like the “ball and beam” and linear systems subject to “actuator constraints”, asymptotically stabilizing control strategies are developed for a class of nonholonomic control systems. These systems generically do not satisfy geometric conditions for input-to-state linearization. New, smooth time-varying and locally stabilizing control laws are developed based on previous work in the literature

on steering nonholonomic systems with sinusoids. Globally stabilizing strategies are then achieved by again introducing saturation functions.

Finally, results are presented that improve regions of feasibility for a recently developed nonlinear adaptive control scheme.

These different settings are used to argue for the desirability of tackling inherently nonlinear control problems with new, inherently nonlinear solutions. The case is made for continued research to develop powerful, specialized tools to add to the nonlinear control toolbox.

A handwritten signature in black ink, appearing to read 'Shankar Sastry', is written over a horizontal line.

S. Shankar Sastry

Chairman

Acknowledgements

I would like to express my sincere appreciation to Shankar Sastry who started me in my present field of research and allowed me the freedom to pursue the problems that interested me. I am grateful for the opportunities and direction he provided.

I have been greatly inspired by many researchers in addition to Professor Sastry. Although I do not know these people well personally, I am especially indebted to the work and inspiration of Professors Sontag and Sussmann, and Professors Byrnes, Isidori, and Willems. Their thoughts and results have inspired much of my research. In addition, I wish to express my gratitude to Professor Praly, Professor Kokotovic and Ioannis Kanelakopoulos for the friendly exchange of ideas and research challenges that have led to new developments in our field. I appreciate their commitment to excellence as well as their emerging friendships.

I am thankful for the opportunity to interact with the outstanding graduate students of the University of California at Berkeley. I would especially like to recognize recent co-authors, Raja Kadiyala, Greg Walsh and Richard Murray. Greg and Richard were gracious enough to allow me to join them in their research on nonholonomic control systems. In addition, I have learned much from my interactions with Matthew Berkemeier and A. K. Pradeep and have greatly enjoyed sharing cubicle space with Kris Pister and Neil Getz. Finally, I would like to thank former Berkeley student John Hauser whose ideas and examples have inspired me greatly.

I am grateful to Professors Pugh and Sanders who served on my qualifying committee as well as Professor Hedrick for serving on my dissertation committee. Further, I am especially thankful to Professor Desoer for serving on both of these committees as well as for his sage words of advice and encouragement.

I would also like to express my gratitude to NASA and the Army Research Office for supporting Professor Sastry's research and thus, in turn, supporting mine. Further, I would like to thank Dr. George Meyer of NASA-Ames for the opportunity to pursue research at his center during the summer of 1990 and for the continued interaction since that time.

The people that have made my experience at Berkeley most enjoyable are, without a doubt, my road-tripping, card-playing comrades at East Hills Community Church. Their fellowship has made being in graduate school a real joy. We share a faith that is often

unduly maligned and misunderstood and which I would like to explain:

Your life can be modeled by a differential equation like

$$\dot{x} = \mu x^3 \quad (0.1)$$

where $\mu > 0$, because, after all, we all have finite escape time. The unknown constant μ accounts for the fact that some of us escape faster than others. Now, for purposes of analogy we make the following definitions:

$$\begin{aligned} +\infty &\doteq \text{“good”} \\ -\infty &\doteq \text{“bad”} \end{aligned}$$

The system was originally designed so that $x(0) > 0$. But due to a large *a priori* disturbance known as original sin, the initial condition has been irrevocably reset so that $x(0) < 0$. This is “bad”. Fortunately, there are some as yet unmodeled dynamics. In fact, your life is better modeled by

$$\begin{aligned} \dot{x} &= \mu x^3 + F(z)\Omega_\alpha(x) + W(w)\Phi(x) \\ \dot{z} &= u_1 \\ \dot{w} &= u_2 + z \end{aligned} \quad (0.2)$$

with $z(0) = 0$ and $w(0) = 0$. $F(z)$ is a monotone nondecreasing bump function with $F(z) = 0$ for all $z \leq 0$ and $F(z) = 1$ for all $z \geq \alpha$ for some arbitrarily small $\alpha > 0$. F is known as the “faith” function. The function W is known as the “good deeds” function and satisfies $W(w) = w$ for all $w \in \mathbb{R}$. The function Φ satisfies $\Phi(x) \equiv 0$. Observe that the z and w dynamics are completely controllable. $\Omega_\alpha(x)$ is the “restoring” function and satisfies $\Omega(x) + \mu x^3 > |x|^3$ for all $x \in \mathbb{R}$. We then have the following results:

Theorem 0.1 *For any $x(0) < 0$, $\exists u_1$ and a $T_L > 0$ associated to u_1 such that $x(T_L) = \text{“good”}$.*

Proof. We appeal to the results of [0]¹ which state, “for God so loved x that he gave the ‘restoring’ function Ω_α , that whosoever activates the ‘faith’ function F shall not go to ‘bad’ in finite time, but shall have everlasting life.” The result then follows from the fact that the ‘faith’ function can be completely activated using the control u_1 . \square

Theorem 0.2 *Let T_d be the finite escape time associated with the trajectory of (0.1) starting at $x(0) = x_o < 0$. If u_1 is chosen such that $z(t) \leq 0$ for all $t < T_d$, then the solution of (0.2) satisfies $x(T_d) = \text{“bad”}$.*

¹[0] John. The Gospel According to John. In *The Bible*. ch. 3, v. 16, 1st century A.D.

Proof. Again we appeal to the results of [0]² which state, “The ‘restoring’ function Ω_α is the way, the truth, and the life. x does not go to ‘good’ accept by the action of the ‘restoring’ function and hence by activating the ‘faith’ function.” \square

For more information, I refer the reader to the complete work of [0] which is an excellent monograph on the subject.

Although the above analysis would suggest otherwise, a relationship with God through Jesus Christ is very personal. It is this relationship that gives me the power to press on. Finally, I would like to thank my parents for their constant encouragement and my wife, Laura, who endured the rare lows and help me celebrate the abundant highs.

²[0] John. The Gospel According to John. In *The Bible*. ch. 14, v. 6, 1st century A.D.

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Chapter 1

Introduction

The intent of this dissertation is to present new, nonlinear solutions to control nonlinear systems which cannot be adequately controlled using existing methods. Of course, existing methods have provided a wonderful point of reference from which to work. For example, much work has been done in nonlinear control theory to determine conditions under which a nonlinear system can be exactly linearized using a nonlinear state feedback and coordinate transformation. In other words, the engineer is able to determine when he is dealing with a linear system that is simply masquerading as a nonlinear system. In this case, after the appropriate transforming feedback and coordinate change have been applied, linear control tools can be applied to achieve a desired response.

Further, we do not advocate a departure from the tools of geometric control theory which have been responsible for the development of many of the existing methods. These tools will continue to serve the control engineer well as coordinate-free conditions are generated for transforming nonlinear systems into normal forms that are short of being linear. The geometric conditions for transforming a system into pure feedback form as found in [Akhrif and Blankenship, 1988] and [Kanellakopoulos *et al.*, 1991], or the conditions for transforming a nonlinear system into a special partially linear normal form as found in [Byrnes and Isidori, 1991] are testimony that this is already taking place. We anticipate that geometric control theory will continue to work hand in hand with stability theory to establish useful normal forms and provide stabilizing solutions for systems in these normal forms.

Our focus has mainly been on stabilization issues although new normal forms have naturally arisen out of some of our work. Again, the existing literature has been the

inspiration for many of our results.

For example, the work of Hauser and co-workers [Hauser *et al.*, 1992] on the popular “ball and beam” experiment has been a central motivator. This system is an example of a system that cannot be exactly linearized, even locally, although its Jacobian linear approximation is controllable. The most applicable existing method for controlling this system is to use a feedback and coordinate transformation to make the system look approximately linear. Next, linear control tools are applied and finally, the nonlinear perturbations are incorporated into the performance analysis. The major limitation of this approach is that the performance is acceptable only in a small operating region. This limitation motivated the development of a new stabilizing design technique that involves the judicious use of saturation functions. The discussion of this approach as it applies to the “ball and beam” experiment is described in chapter 4. Using this approach we are able to achieve stabilization for the “ball and beam” on arbitrarily large regions of operation.

An offshoot of this research was to study the global stabilizability of linear systems subject to input saturation. Here, negative theorems in [Fuller, 1969] and [Sussmann and Yang, 1991] provided direction for establishing the power of using nesting saturation functions in a systematic manner. The negative theorem established that linear solutions were not sufficient, and in this sense the problem is inherently nonlinear. We discuss the nested saturation approach in chapter 2 as it applies to stabilization and tracking problems for linear null-controllable systems.

With the nested saturation technology in hand, we have a natural tool with which to achieve semi-global stabilization for a class of partially linear nonlinear systems. This class includes the cautionary examples of [Sussmann and Kokotovic, 1991] and [Byrnes and Isidori, 1991] which suggest existing high-gain methods are not sufficient. The stabilizing solution to this problem is discussed in chapter 3. We are encouraged that results of both chapters 2 and 3 have been further generalized by other authors (see [Yang *et al.*, 1992] and [Lin and Saberi, 1992b]) after studying our results. We include a statement of these advances for completeness.

The intuition developed for the “ball and beam” example is shown in chapter 4 to be applicable to a canonical example presented in [Kokotovic *et al.*, 1991] as a challenging control problem. In fact, we develop a stabilizing control strategy for a class of systems that includes this canonical example. We refer to these systems as *higher order feedforward systems* and discuss geometric conditions for obtaining this normal form.

For tracking (rather than stabilization) problems, we use chapter 5 to show that, in fact, solving the stabilization problem goes a long way in solving the associated tracking problem. We do this by combining the results of chapter 4 with the recently developed nonlinear regulator theory of [Byrnes and Isidori, 1990] to achieve small signal (approximate) tracking for the “ball and beam” example. The important feature is that we achieve basins of attraction much larger than for any existing methods.

An investigation of inherently nonlinear stabilization problems must include the study of nonholonomic control systems. Although these systems are controllable, even the local stabilization problem is difficult. In fact, the negative result of [Brockett, 1983] established that these systems cannot be asymptotically stabilized using *smooth static* state feedback. More recently, Coron provided a positive but nonconstructive result demonstrating that time-varying feedback was sufficient to stabilize these systems [Coron, 1992]. These results have encouraged the pursuit of explicit time-varying asymptotically stabilizing control laws for nonholonomic systems. In chapter 6, we develop locally and globally stabilizing control algorithms for a subclass of nonholonomic systems. We focus on nonholonomic systems in *chained form* developed by Murray and Sastry [Murray and Sastry, 1991a] and the diffeomorphically equivalent *power form* systems. Many of the results in chapter 6 are taken from joint work with Murray and Walsh in [Teel *et al.*, 1992].

Finally, in chapter 7, we return to the adaptive control problem we studied in [Teel *et al.*, 1991]. Recently, Kanellakopoulos and his coworkers have developed a nice solution for the class of nonlinear systems in pure feedback form. (See [Kanellakopoulos *et al.*, 1991] and [Krstic *et al.*, 1991] for example.) We combine these ideas with nonlinear error-based adaptive ideas (found in [Teel *et al.*, 1991] and [Pomet and Praly, 1989] for example) to produce an algorithm that has a potentially larger domain of feasibility.

Throughout this dissertation we illustrate the complementary roles that differential geometry and the methods of stability theory play in contributing powerful, specialized tools to the nonlinear control toolbox.

Chapter 2

Global Control Problems for Linear Systems with Bounded Controls

For linear systems subject to “input saturation”, the global stabilization problem is inherently nonlinear. By this we mean that, for all but the simplest of cases, a linear feedback will not suffice. In this chapter, we present a new, nonlinear solution to the problem of global stabilizing multiple integrators with bounded controls. The generalization of our result to all linear null-controllable¹ systems, developed by Sontag, Sussmann and Yang in [Sontag and Yang, 1991], [Yang *et al.*, 1992], is also mentioned. We discuss how our solution can be applied to the tracking problem for a certain class of trajectories. This includes a discussion of linear regulator theory with “input saturation”. Also, emerging performance issues are discussed.

2.1 Introduction

The study of linear systems subject to “input saturation” has a rich history. (See, for example, [Anderson and Moore, 1971].) One of the primary focuses of this research has been to study the effects of a saturating input when applying a linear control law. For

¹A linear time-invariant system is said to be (globally) null-controllable (with respect to some constraint set) if, given any initial condition, there exists a control, which takes values in the constraint set, that steers the system to the zero state in some finite time.

example, much of the rich literature of the 1950's and 1960's on the problem of absolute stability (see [Aizerman and Gantmacher, 1964], [Narendra and Taylor, 1973], or [Popov, 1973]) was motivated by this problem.

It has been shown that only linear, stabilizable systems having no open-loop eigenvalues with positive real part can be globally asymptotically stabilize using a bounded control (see theorem 2.1.) Therefore, when this condition does not hold, it is natural to study domains of attraction for open-loop unstable systems with saturating linear feedbacks. See, for example, [Kosut, 1983], [Krikelis and Barkas, 1984], [Gutman and Hagander, 1985] and [Dolphus and Schmitendorf, 1991].

Concerning the global stabilization problem, some authors have ignored the natural open-loop eigenvalue constraint to propose algorithms that have no hope of converging ([Chen and Wang, 1988]) except in the simplest of cases: open-loop systems that are stable or have *simple* $j\omega$ -axis eigenvalues. This approach has spread to the discrete-time literature ([Chou, 1991]) even though the analogous natural open-loop eigenvalue condition has been established (see, for example, [Sontag, 1984], [Ma, 1991]).

We propose to study the interesting problems that remain in the global stabilization of linear systems subject to "input saturation" motivated by the following two results:

Theorem 2.1 ([Sontag and Sussmann, 1990]) *Given the system*

$$\dot{x} = Ax + B\sigma(u) \tag{2.1}$$

where $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is bounded, globally Lipschitz, and invertible in a neighborhood of the origin, there exists a globally stabilizing control $u = k(x)$ if and only if (2.1) is asymptotically null-controllable.

Asymptotic null-controllability is equivalent to the conditions that the pair (A, B) is stabilizable and all the eigenvalues of A are located in the closed left-half complex plane. For a discussion of this notion, see [Schmitendorf and Barmish, 1980] and [Sontag, 1984].

Theorem 2.2 ([Fuller, 1969],[Sussmann and Yang, 1991]) *Suppose the system (2.1) is a chain of integrators of length n where $n \geq 3$. Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be bounded with $s\sigma(s) > 0$ for $s \neq 0$ and with both limits $\lim_{s \rightarrow \pm\infty}$ existing and nonzero. Then there does not exist a linear functional $h(x)$ such that the control $u = h(x)$ globally stabilizes (2.1).*

The implications of these theorems are straightforward. Given that the linear system is stabilizable, the problem can be solved if and only if the eigenvalues of the linear system are in the closed left-half of the complex plane, and, even then, only the simplest cases can be handled with linear feedback.

In [Sontag and Sussmann, 1990], a complicated induction procedure was outlined to generate a globally stabilizing control for all linear null-controllable systems. In this chapter we develop a far more explicit and straightforward construction, specialized to linear chain of integrator systems. The procedure has recently been extended in [Sontag and Yang, 1991] to apply to all linear null-controllable systems.

2.2 Global Stabilization

We start with the following definition:

Definition 2.1 *Given two positive constants L, M with $L \leq M$, a function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a linear saturation for (L, M) if it is a continuous, nondecreasing function satisfying*

1. $\sigma(s) = s$ when $|s| \leq L$
2. $|\sigma(s)| \leq M$ for all $s \in \mathbb{R}$.

In the subsequent control design, one can choose arbitrarily smooth functions out of this class. Now consider the linear system consisting of multiple integrators:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_n &= u \end{aligned} \tag{2.2}$$

We are searching for a bounded control that will globally asymptotically stabilize (2.2). We now present our main results.

Theorem 2.3 *There exist linear functions $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ such that, for any set of positive constants $\{(L_i, M_i)\}$ where $L_i \leq M_i$ for $i = 1, \dots, n$ and $M_i < \frac{L_{i+1}}{2}$ for $i = 1, \dots, n-1$, and for any set of functions $\{\sigma_i\}$ that are linear saturations for $\{(L_i, M_i)\}$, the bounded control*

$$u = -\sigma_n(h_n(x) + \sigma_{n-1}(h_{n-1}(x) + \dots + \sigma_1(h_1(x))) \dots)$$

results in global asymptotic stability for the system (2.2).

Proof. Consider the linear coordinate transformation $y = Tx$ which transforms (2.2) into $\dot{y} = Ay + Bu$ where A and B are given by

$$A = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

The recursive nature involved yields a transformation characterized by

$$y_{n-i} = \sum_{j=0}^i \binom{i}{j} x_{n-j}$$

where

$$\binom{i}{j} = \frac{i!}{j!(i-j)!}$$

The inverse of the transformation is characterized by

$$x_{n-i} = \sum_{j=0}^i (-1)^{i+j} \binom{i}{j} y_{n-j}$$

A suitable control law is

$$u = -\sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)) \cdots) \quad (2.3)$$

which yields the closed loop system

$$\begin{aligned} \dot{y}_1 &= y_2 + \cdots + y_n - \sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)) \cdots) \\ \dot{y}_2 &= y_3 + \cdots + y_n - \sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)) \cdots) \\ &\vdots \\ \dot{y}_{n-1} &= y_n - \sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)) \cdots) \\ \dot{y}_n &= -\sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)) \cdots) \end{aligned} \quad (2.4)$$

We begin by considering the evolution of the state y_n . Consider the Lyapunov function $V_n = y_n^2$. The derivative of V_n is given by

$$\dot{V}_n = -2y_n[\sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)) \cdots)]$$

From definition 1, condition 1 applied to σ_n and condition 3 applied to σ_{n-1} coupled with the fact that $M_{n-1} < \frac{L_n}{2}$, we see that $\dot{V}_n < 0$ for all $y_n \notin Q_n = \{y_n : |y_n| \leq \frac{L_n}{2}\}$. In fact, \dot{V}_n

is negative and bounded away from zero since L_n and M_{n-1} are constants. Consequently, y_n enters Q_n in finite time and remains in Q_n thereafter. Further, because the right-hand side of (2.4) is globally Lipschitz, the remaining states y_1, \dots, y_{n-1} remain bounded for any finite time.

Now consider the evolution of the state y_{n-1} . First observe that after y_n has entered Q_n , the argument of σ_n is bounded as

$$\begin{aligned} |y_n + \sigma_{n-1}(y_{n-1} + \dots + \sigma_1(y_1)) \dots| &\leq \frac{L_n}{2} + M_{n-1} \\ &\leq L_n \end{aligned}$$

Consequently, after y_n enters Q_n , σ_n operates in its linear region from condition 2 of definition 1. Then the evolution of y_{n-1} is given by

$$\dot{y}_{n-1} = -\sigma_{n-1}(y_{n-1} + \dots + \sigma_1(y_1)) \dots$$

Using the same argument as for y_n we can show that y_{n-1} enters an analogous set Q_{n-1} in finite time and remains in Q_{n-1} thereafter. Again, all of the remaining states stay bounded. This procedure can be continued to show that after some finite time the argument of every function σ_i has entered the region where the function is linear. After this finite time, the closed loop equations have the form

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= -y_1 - y_2 \\ &\vdots \\ \dot{y}_n &= -y_1 - y_2 - \dots - y_n \end{aligned}$$

Clearly, the dynamics, after the prescribed finite time, are exponentially stable. \square

The number of saturation functions required can be decreased by stabilizing the states in pairs rather than one at a time. We employ a slightly more restrictive class of linear saturation functions.

Definition 2.2 *Given two positive constants L, M with $L \leq M$ a function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a simple linear saturation for (L, M) if it is a continuous, nondecreasing function satisfying*

1. $\sigma(s) = s$ when $|s| \leq L$
2. $s[\sigma(s) - s] \geq 0$ when $|s| \leq M$

3. $|\sigma(s)| = M$ when $|s| \geq M$.

Where before we needed n saturation functions, now we need one function for each pair of states. If the dimension of the state space is odd, we will need one additional saturation function for the additional state. Accordingly, define $\tilde{n} = n/2$ if n is even and $\tilde{n} = (n+1)/2$ if n is odd.

Theorem 2.4 *There exist linear functions $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for any set of positive constants $\{(L_i, M_i)\}$ where $L_i \leq M_i$ for $i = 1, \dots, \tilde{n}$ and $M_i < \frac{L_{i+1}}{1+\sqrt{2}}$ for $i = 1, \dots, \tilde{n}-1$, and for any set of functions $\{\sigma_i\}$ that are simple linear saturations for $\{(L_i, M_i)\}$, the bounded control*

$$u = -\sigma_{\tilde{n}}(h_{\tilde{n}}(x) + \sigma_{\tilde{n}-1}(h_{\tilde{n}-1}(x) + \dots + \sigma_1(h_1(x))) \dots)$$

results in global asymptotic stability for the system (2.2).

Proof. Consider the same coordinate change as in the proof of the previous theorem. We will proceed in a similar manner as before, this time showing that the states y_{n-1}, y_n enter within finite time and thereafter remain in a region where the function $\sigma_{\tilde{n}}$ is linear. Since the differential equation is globally Lipschitz, the remaining states y_1, \dots, y_{n-2} remain bounded. With $\sigma_{\tilde{n}}$ operating in its linear region we can iterate to show that y_{n-3}, y_{n-2} enter and remain in a region where $\sigma_{\tilde{n}-1}$ is linear. Eventually, this leads to the conclusion that after some finite time, the closed loop equations have the form

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= -y_1 - y_2 \\ &\vdots \\ \dot{y}_n &= -y_1 - y_2 - \dots - y_n \end{aligned}$$

which is an exponentially stable linear system.

Consider the dynamics of y_{n-1}, y_n :

$$\begin{aligned} \dot{y}_{n-1} &= y_n - \sigma_{\tilde{n}}(y_{n-1} + y_n + \sigma_{\tilde{n}-1}(\cdot)) \\ \dot{y}_n &= -\sigma_{\tilde{n}}(y_{n-1} + y_n + \sigma_{\tilde{n}-1}(\cdot)) \end{aligned} \tag{2.5}$$

To show that y_{n-1}, y_n enters a sufficiently small neighborhood of the origin we use the following Lyapunov-like function:

$$W(y_{n-1}, y_n) = \frac{1}{2}y_{n-1}^2 + \frac{1}{2}y_n^2 \tag{2.6}$$

This positive definite function is only a Lyapunov-*like* function because there will be points in the state space where $\dot{W} > 0$. However, we will show that the integral of \dot{W} is negative over known closed form solutions of (2.5) in the region where $\sigma_{\bar{n}}$ is saturated. Further, when it is possible that $\sigma_{\bar{n}}$ is not saturated, W is strictly decreasing (outside a neighborhood of the origin.)

Consider the following regions of the state space:

$$\begin{aligned} \text{region I:} & \quad y_{n-1} + y_n > M_{\bar{n}} + M_{\bar{n}-1} \\ \text{region II:} & \quad y_{n-1} + y_n < -M_{\bar{n}} - M_{\bar{n}-1} \\ \text{region III:} & \quad |y_{n-1} + y_n| \leq M_{\bar{n}} + M_{\bar{n}-1} \end{aligned}$$

We begin by showing that any bounded initial condition in region I yields a trajectory that enters region III in finite time. Observe that in region I (2.5) is given by

$$\begin{aligned} \dot{y}_{n-1} &= y_n - M_{\bar{n}} \\ \dot{y}_n &= -M_{\bar{n}} \end{aligned} \tag{2.7}$$

Consequently, the closed form solution of the trajectories in region I are given by

$$\begin{aligned} y_{n-1}(t) &= y_{n-1}(t_o) + y_n(t_o)t - M_{\bar{n}}\frac{t^2}{2} - M_{\bar{n}}t \\ y_n(t) &= y_n(t_o) - M_{\bar{n}}t \end{aligned}$$

(For purposes of integration, we have set $t_o = 0$. This can be done since (2.7) is time-invariant.) Now, we have

$$y_{n-1}(t) + y_n(t) = y_{n-1}(t_o) + y_n(t_o) - 2M_{\bar{n}}t + y_n(t_o)t - M_{\bar{n}}\frac{t^2}{2}$$

We assume that

$$y_{n-1}(t_o) + y_n(t_o) > M_{\bar{n}} + M_{\bar{n}-1}$$

and we solve for a t_b such that

$$y_{n-1}(t_b) + y_n(t_b) = M_{\bar{n}} + M_{\bar{n}-1}$$

Using the quadratic formula it is straightforward to show that such a t_b exists and is finite and positive. The same argument holds for region II by symmetry.

Now consider an initial condition such that

$$y_{n-1}(t_o) + y_n(t_o) = M_{\bar{n}} + M_{\bar{n}-1}$$

To enter region I, we must have

$$\dot{y}_{n-1}(t_o) > -\dot{y}_n(t_o)$$

since the boundary of region I is a line of slope -1. This implies

$$\begin{aligned} y_n(t_o) &> 2M_{\bar{n}} \\ y_{n-1}(t_o) &< -M_{\bar{n}} + M_{\bar{n}-1} \end{aligned}$$

Assume we enter region I. We show that we return to region III in finite time $t_b > 0$ and that $W(t_b) - W(t_o) < 0$. From the discussion above for trajectories in region I and since

$$y_{n-1}(t_o) + y_n(t_o) = y_{n-1}(t_b) + y_n(t_b)$$

it follows that

$$M_{\bar{n}} \frac{t_b^2}{2} + [2M_{\bar{n}} - y_n(t_o)]t_b = 0$$

This implies

$$t_b = \frac{2}{M_{\bar{n}}}(y_n(t_o) - 2M_{\bar{n}})$$

which is positive because $y_n(t_o) > 2M_{\bar{n}}$. Now consider

$$W(t_b) - W(t_o) = \frac{1}{2}(y_{n-1}^2(t_b) + y_n^2(t_b) - y_{n-1}^2(t_o) - y_n^2(t_o))$$

First consider

$$\frac{1}{2}(y_{n-1}^2(t_b) - y_{n-1}^2(t_o))$$

Observe that in terms of $y_{n-1}(t_o)$

$$t_b = \frac{2}{M_{\bar{n}}}(-y_{n-1}(t_o) + M_{\bar{n}-1} - M_{\bar{n}})$$

Evaluating the closed form solution for y_{n-1} at t_b yields

$$y_{n-1}(t_b) = -y_{n-1}(t_o) + 2(M_{\bar{n}-1} - M_{\bar{n}})$$

A straightforward calculation then shows that

$$y_{n-1}^2(t_b) - y_{n-1}^2(t_o) = 4y_{n-1}(t_o)(M_{\bar{n}} - M_{\bar{n}-1}) + 4(M_{\bar{n}} - M_{\bar{n}-1})^2$$

Since $y_{n-1}(t_o) < -M_{\bar{n}} + M_{\bar{n}-1}$ and $M_{\bar{n}} > M_{\bar{n}-1}$, it follows that $y_{n-1}^2(t_b) - y_{n-1}^2(t_o) < 0$.

Now consider

$$\frac{1}{2}(y_n^2(t_b) - y_n^2(t_o))$$

Evaluating the closed form solution for y_n at t_b yields

$$y_n(t_b) = -y_n(t_o) + 4M_{\bar{n}}$$

A straightforward calculation then shows that

$$y_n^2(t_b) - y_n^2(t_o) = -8y_n(t_o)M_{\bar{n}} + 16M_{\bar{n}}^2$$

Since $y_n(t_o) > 2M_{\bar{n}}$, it follows that $y_n^2(t_b) - y_n^2(t_o) < 0$.

By symmetry, the same analysis holds for trajectories originating on the boundary of region II and entering region II.

Now consider trajectories in region III. We have

$$\begin{aligned} \dot{W} &= y_{n-1}[y_n - \sigma_{\bar{n}}(y_{n-1} + y_n + \sigma_{\bar{n}-1}(y))] \\ &\quad + y_n[-\sigma_{\bar{n}}(y_{n-1} + y_n + \sigma_{\bar{n}-1}(y))] \\ &= (y_{n-1} + y_n)[- \sigma_{\bar{n}}(y_{n-1} + y_n + \sigma_{\bar{n}-1}(y))] + y_{n-1}y_n \\ &= (y_{n-1} + y_n)[y_{n-1} + y_n - \sigma_{\bar{n}}(y_{n-1} + y_n + \sigma_{\bar{n}-1}(y))] \\ &\quad - (y_{n-1} + y_n)^2 + y_{n-1}y_n \\ &= (y_{n-1} + y_n)[y_{n-1} + y_n - \sigma_{\bar{n}}(y_{n-1} + y_n + \sigma_{\bar{n}-1}(y))] \\ &\quad - \frac{1}{2}(y_{n-1} + y_n)^2 - \frac{1}{2}y_{n-1}^2 - \frac{1}{2}y_n^2 \\ &\leq |(y_{n-1} + y_n)|M_{\bar{n}-1} - \frac{1}{2}y_{n-1}^2 - \frac{1}{2}y_n^2 - \frac{1}{2}(y_{n-1} + y_n)^2 \end{aligned}$$

The final inequality follows from definition 2.2, property 2. Consider the level set

$$W = \frac{1}{2}M_{\bar{n}-1}^2$$

On this level set, a circle of radius $M_{\bar{n}-1}$ in the y_{n-1}, y_n plane, we have

$$M_{\bar{n}-1} \leq |y_{n-1} + y_n| \leq \sqrt{2}M_{\bar{n}-1}$$

Consider $|y_{n-1} + y_n| = kM_{\bar{n}-1}$ where $k \in [1, \sqrt{2}]$. Then

$$\begin{aligned} \dot{W} &\leq -\frac{1}{2}M_{\bar{n}-1}^2 - \frac{1}{2}(kM_{\bar{n}-1})^2 + kM_{\bar{n}-1}^2 \\ &= -(\frac{1}{2} - k + \frac{1}{2}k^2)M_{\bar{n}-1}^2 \end{aligned}$$

Since $k \in [1, \sqrt{2}]$, $\dot{W} \leq 0$. Since \dot{W} is bounded by a quadratic negative definite function plus a linear perturbation in region III, $\dot{W} < 0$ outside of the level set $W = \frac{1}{2}M_{\bar{n}-1}^2$ and inside region III. Further, if the trajectory leaves region III, it returns in finite time and at a lower energy level W . Consequently, for any $\epsilon > 0$, the trajectories of y_{n-1}, y_n enter

a circle of radius $M_{\bar{n}-1} + \epsilon$ in finite time and remain in that circle thereafter. If $M_{\bar{n}-1}$ is chosen so that

$$L_{\bar{n}} = \sqrt{2}(M_{\bar{n}-1} + \epsilon) + M_{\bar{n}-1}$$

(i.e. $L_{\bar{n}} > M_{\bar{n}-1}(\sqrt{2} + 1)$), then $\sigma_{\bar{n}}$ operates in its linear region after some finite time. Once $\sigma_{\bar{n}}$ becomes a strictly linear function we have

$$\begin{aligned}\dot{y}_{n-3} &= y_{n-2} - \sigma_{\bar{n}-1}(y_{n-3} + y_{n-2} + \sigma_{\bar{n}-2}(y)) \\ \dot{y}_{n-2} &= -\sigma_{\bar{n}-1}(y_{n-3} + y_{n-2} + \sigma_{\bar{n}-2}(y))\end{aligned}$$

and the same analysis applies to show that y_{n-3}, y_{n-2} eventually enters a sufficiently small neighborhood of the origin. The iterative process continues until it can be shown that, after some finite time, every saturation function is operating in its linear region. After this time, the dynamics of (2.2) are those of an exponentially stable linear system. \square

Remark. The results of theorem 2.2 indicate that it is not possible to further reduce the number of saturation functions by trying to stabilize three states at a time.

After the above results were established, Sontag and Yang were able to make the natural extensions for the general linear setting. For convenience, to prepare for this result, the following class of functions is defined as in [Sontag and Yang, 1991]:

Definition 2.3 Let $\mathcal{F}_n : \mathbb{R}^n \rightarrow \mathbb{R}$ be a class of functions satisfying

1. The zero function $f = 0$ belongs to \mathcal{F} .
2. For any $f \in \mathcal{F}_n$, any vector $H \in \mathbb{R}^n$, and any simple linear saturation σ , the function $\sigma(H^T x + f(x)) \in \mathcal{F}_n$.

Further, let $\mathcal{F}_{n,\epsilon} = \{f \in \mathcal{F}_n : |f| < \epsilon\}$

The following theorem, stated and proved in [Sontag and Yang, 1991], represents the current state of the art for globally stabilizing linear systems with bounded controls:

Theorem 2.5 ([Sontag and Yang, 1991]) Consider the asymptotically null-controllable system

$$\dot{x} = Ax + Bu \tag{2.8}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Then, for each $\epsilon > 0$, $\exists k_i \in \mathcal{F}_{n,\epsilon}$, $i = 1, \dots, m$, such that with the feedback

$$u_i = k_i(x) \quad i = 1, \dots, m$$

the resulting closed-loop system is globally asymptotically stable. Furthermore, the closed-loop system has the converging-input converging state (CICS) property: if $e(t) \in \mathbb{R}^n$ is any (vector-valued) function that satisfies $\lim_{t \rightarrow \infty} e = 0$, then all solutions of

$$\dot{x} = Ax + B[k_1(x), \dots, k_m(x)]^T + e(t) \quad (2.9)$$

converge to zero as $t \rightarrow \infty$.

Remarks.

1. Results for sigmoidal functions more general than *simple linear saturations* are reportedly discussed in [Yang *et al.*, 1992].
2. That there exists a control yielding a closed-loop with the CICS property is significant because it allows for solving this problem with dynamic output feedback for observable systems. The state x will converge to the origin as the estimates of the states converge to the actual states. For more on the CICS property and related topics, see [Sontag, 1989].

2.3 Restricted Tracking

We now discuss how the controls of the previous section can be used to achieve asymptotic tracking for linear null-controllable systems subject to “input saturation”. We begin by considering the tracking problem for the chain of integrators system studied in detail in the previous section. Then we use the general results of [Sontag and Yang, 1991] to state a solution to the multivariable linear regulator problem when the input is subject to saturation. In both cases, we restrict the associated feedforward piece of the reference trajectory to be sufficiently small.

2.3.1 Chain of Integrators

Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_n &= \sigma_{n+1}(u) \\ y &= x_1 \end{aligned} \quad (2.10)$$

Here σ_{n+1} is a linear saturation for (L_{n+1}, M_{n+1}) . The task is to cause y to track a desired reference trajectory y_d given by $y_d, \dot{y}_d, \dots, y_d^{(n)}$.

Corollary 2.1 *If $|y_d^{(n)}(t)| \leq L_{n+1} - \epsilon$ for all $t \geq t_o$ and for some $\epsilon > 0$ then there exist linear functions $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for any set of positive constants $\{(L_i, M_i)\}$ where $M_n \leq \epsilon$, $L_i \leq M_i$ for $i = 1, \dots, n$ and $M_i < \frac{L_{i+1}}{2}$ for $i = 1, \dots, n-1$ and for any set of functions $\{\sigma_i\}$ that are linear saturations for $\{(L_i, M_i)\}$, the feedback*

$$u = y_d^{(n)} - \sigma_n(h_n(\tilde{x}) + \sigma_{n-1}(h_{n-1}(\tilde{x}) + \dots + \sigma_1(h_1(\tilde{x}))) \dots)$$

where \tilde{x} is defined as $\tilde{x}_i = x_i - y_d^{(i-1)}$ for $i = 1, \dots, n$, results in asymptotic tracking for the system (2.10).

Proof. In terms of \tilde{x} , (2.10) becomes

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ &\vdots \\ \dot{\tilde{x}}_n &= -y_d^{(n)} + \sigma_{n+1}(u) \end{aligned}$$

Observe that, with the specified control law, if we chose $M_n \leq \epsilon$, then $\sigma_{n+1}(\cdot)$ is always operating in its linear region so the closed loop system becomes

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ &\vdots \\ \dot{\tilde{x}}_n &= -\sigma_n(h_n(\tilde{x}) + \sigma_{n-1}(h_{n-1}(\tilde{x}) + \dots + \sigma_1(h_1(\tilde{x}))) \dots) \end{aligned}$$

Now if $\{(L_i, M_i)\}$ satisfy $M_i < \frac{L_{i+1}}{2}$ for $i = 1, \dots, n-1$ and $\sigma_i(\cdot)$ satisfies definition 1, then we have the conditions of the stabilization theorem of section 2.2. Consequently, \tilde{x} asymptotically approaches zero. In turn, this implies that $y(t)$ asymptotically approaches $y_d(t)$. \square

For a result with fewer saturation functions we assume, for (2.10), that σ_{n+1} is a linear saturation for $(L_{\bar{n}+1}, M_{\bar{n}+1})$.

Corollary 2.2 *If $|y_d^{(n)}(t)| \leq L_{\bar{n}+1} - \epsilon$ for all $t \geq t_o$ and for some $\epsilon > 0$ then there exist linear functions $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for any set of positive constants $\{(L_i, M_i)\}$ where $M_{\bar{n}} \leq \epsilon$, $L_i \leq M_i$ for $i = 1, \dots, \bar{n}$ and $M_i < \frac{L_{i+1}}{1+\sqrt{2}}$ for $i = 1, \dots, \bar{n}-1$ and for any set of functions $\{\sigma_i\}$ which are simple linear saturations for $\{(L_i, M_i)\}$, the feedback*

$$u = y_d^{(n)} - \sigma_{\bar{n}}(h_{\bar{n}}(\tilde{x}) + \sigma_{\bar{n}-1}(h_{\bar{n}-1}(\tilde{x}) + \dots + \sigma_1(h_1(\tilde{x}))) \dots)$$

where \tilde{x} is defined by $\tilde{x}_i = x_i - y_d^{(i-1)}$ for $i = 1, \dots, n$, results in asymptotic tracking for the system (2.10).

2.3.2 Multivariable Linear Regulation

The results of this chapter can be easily applied to the tracking problem when it is cast in the language of linear regulator theory (see [Francis, 1977]). In this framework, we consider a multivariable linear system with inputs that are subject to saturation together with an exogenous system that generates disturbances and reference trajectories:

$$\begin{aligned}\dot{x} &= Ax + B\sigma(u) + Pw \\ \dot{w} &= Sw \\ e &= Cx + Qw\end{aligned}\tag{2.11}$$

Here, $x \in \mathbb{R}^n$, $w \in \mathbb{R}^s$, $u \in \mathbb{R}^m$, and $e \in \mathbb{R}^p$. The vector e represents the tracking error and $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is given by

$$\sigma([u_1, \dots, u_m]^T) = [\sigma_1(u_1), \dots, \sigma_m(u_m)]^T\tag{2.12}$$

where σ_i is a simple linear saturation for (L_i, M_i) . The two problems to solve are the following:

State Feedback Regulator Problem. Find, if possible, a feedback $u = \alpha(x, w)$ such that

1. the equilibrium $x = 0$ of

$$\dot{x} = Ax + B\sigma(\alpha(x, 0))$$

is globally asymptotically stable and locally exponentially stable.

2. For all $(x(0), w(0)) \in \mathbb{R}^n \times \mathbb{R}^s$, the solution of the closed-loop system satisfies

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

Error Feedback Regulator Problem. Find, if possible, a dynamic error feedback $u = \theta(z)$, $\dot{z} = \eta(z, e)$ where $z \in \mathbb{R}^v$ such that

1. the equilibrium $(x, z) = (0, 0)$ of

$$\begin{aligned}\dot{x} &= Ax + B\sigma(\theta(z)) \\ \dot{z} &= \eta(z, Cx)\end{aligned}$$

is globally asymptotically stable and locally exponentially stable.

2. For all $(x(0), w(0), z(0)) \in \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^v$ the solution of the closed-loop system satisfies

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

In keeping with the natural eigenvalue requirements of this chapter, we make the following assumption:

Assumption 2.1 *The eigenvalues of A have nonpositive real part.*

We now make the following additional assumptions which are standard in linear regulator theory.

Assumption 2.2 *The eigenvalues of S have nonnegative real part.*

Assumption 2.3 *The pair (A, B) is stabilizable.*

Assumption 2.4 *The pair*

$$\begin{bmatrix} C & Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}$$

is detectable.

We then have the following state feedback regulator solution.

Theorem 2.6 *Suppose assumptions 2.1, 2.2, and 2.3 hold. If there exists matrices Π and Γ which solve the linear matrix equations:*

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P \\ C\Pi + Q &= 0 \end{aligned} \tag{2.13}$$

and for $i = 1, \dots, m$, $\exists \epsilon_i, T > 0$ such that $|\Gamma_i w(t)| \leq L_i - \epsilon_i$ for all $t \geq T$ then the state feedback regulator problem is solvable.

Proof. By assumption Π and Γ satisfy (2.13). We make an invertible, triangular coordinate change such that $\xi = x - \Pi w$. We then have

$$\begin{aligned} \dot{\xi} &= \frac{d}{dt}(x - \Pi w) \\ &= Ax + B\sigma(u) + Pw - \Pi Sw \\ &= A(x - \Pi w) + B(\sigma(u) - \Gamma w) + A\Pi w + B\Gamma w + Pw - \Pi Sw \\ &= A\xi + B(\sigma(u) - \Gamma w) \end{aligned} \tag{2.14}$$

Now, given the ϵ_i 's of the theorem we choose

$$u_i = \Gamma_i w + k_i(\xi)$$

where $k_i \in \mathcal{F}_{n, \epsilon_i}$. If we define $k(\xi) = [k_1(\xi), \dots, k_m(\xi)]^T$, then the closed loop is given by

$$\begin{aligned} \dot{\xi} &= A\xi + Bk(\xi) + B[\sigma(\Gamma w + k(\xi)) - \Gamma w - k(\xi)] \\ &= A\xi + Bk(\xi) + \phi(t) \end{aligned} \quad (2.15)$$

Now, since there exists $T > 0$ such that $|\Gamma_i w(t)| \leq L_i - \epsilon_i$ for all $t > T$, it follows that $|\Gamma_i w(t) + k_i(\xi(t))| \leq L_i$ for all $t > T$. Then it follows that

$$\sigma_i(\Gamma_i w(t) + k_i(\xi(t))) = \Gamma_i w(t) + k_i(\xi(t)) \quad \forall t > T$$

From this and the definition of $\phi(t)$ it follows that $\lim_{t \rightarrow \infty} \phi(t) = 0$. Then, from theorem 2.5, since the closed loop has the CICS property, $\lim_{t \rightarrow \infty} \xi(t) = 0$. Finally, consider $e = Cx + Qw$. From the definition of ξ and (2.13),

$$e = C\xi + C\Pi w + Qw = C\xi.$$

Since $\lim_{t \rightarrow \infty} \xi(t) = 0$, it follows that $\lim_{t \rightarrow \infty} e(t) = 0$. \square

Theorem 2.7 *Suppose assumptions 2.1, 2.2, 2.3, and 2.4 hold. If there exists matrices Π and Γ which solve the linear matrix equations (2.13) and for $i = 1, \dots, m$, $\exists \epsilon_i, T > 0$ such that $|\Gamma_i w(t)| \leq L_i - \epsilon_i$ for all $t \geq T$ then the error feedback regulator problem is solvable.*

Proof. The detectability assumption allows us to build a linear observer to asymptotically determine (with exponential convergence) the states x and w from e (see [Francis, 1977]). We construct a feedback exactly as is the proof of theorem 2.6, replacing the states x and w by their estimates. Now since the corresponding closed-loop has the CICS property, the problem is solved. \square

2.4 Performance Issues

The technology of nested saturations to solved the bounded input control problem for linear systems is new. The first results were intended to demonstrate that such a solution exists. Little regard was given to describing the flexibilities of such a design or possible ways to enhance performance. In this section, we do not intend to answer these questions in full.

However, we wish to point out certain performance issues that we have witnessed in our brief experience with these control laws. We anticipate that this section will inspire further research in this area, perhaps leading to the study of optimal stabilizing control laws, given the saturation characteristic and some suitable cost criterion, out of the family $\mathcal{F}_{n,\epsilon}$ of bounded feedbacks.

2.4.1 Pole Placement

We begin by focusing on the flexibility of pole placement in the design of nested saturation control laws. We focus on the chain of integrators system for simplicity. The issue of pole placement naturally affects the performance of the system in a neighborhood of the origin after each saturation function has entered its linear region. The control engineer, in general, would like the assurance of global asymptotic stability without sacrificing the freedom of arbitrary pole placement. However, as one can easily observe, the standard design outlined in section 2.2 places all of the closed-loop poles at $s = -1$. It is worthwhile to point out that there is nothing special about the value $s = -1$. Rather, it is an artifact of our coordinate change. In fact, by constructing our coordinate change $y = Tx$ so that $\dot{y} = Ay + Bu$ where B is as before and A is given by

$$A = \begin{bmatrix} 0 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ 0 & 0 & \lambda_3 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

we are able to achieve arbitrary pole placement subject to the constraint that all eigenvalues have zero imaginary part. The control

$$u = -\sigma_n(\lambda_n y_n + \sigma_{n-1}(\lambda_{n-1} y_{n-1} + \cdots + \sigma_1(\lambda_1 y_1) \cdots))$$

achieves such pole placement.

To prepare for arbitrary pole placement, we wish to consider the stabilization of an integrator chain of length $2n$ where we make a preliminary coordinate change $y = T_\lambda x$

to yield $\dot{y} = Ay + Bu$ where B is as before and A is given by

$$A = \begin{bmatrix} J(\lambda_1) & 1(\lambda_2) & 1(\lambda_3) & \cdots & 1(\lambda_n) \\ 0_2 & J(\lambda_2) & 1(\lambda_3) & \cdots & 1(\lambda_n) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_2 & 0_2 & 0_2 & J(\lambda_{n-1}) & 1(\lambda_n) \\ 0_2 & 0_2 & 0_2 & 0_2 & J(\lambda_n) \end{bmatrix} \quad (2.16)$$

where

$$J(s) = \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix} \quad 1(s) = \begin{bmatrix} s & s \\ s & s \end{bmatrix} \quad 0_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (2.17)$$

To demonstrate that the control

$$u = -\sigma_n(\lambda_n(y_{n-1} + y_n) + \sigma_{n-1}(\lambda_{n-1}(y_{n-3} + y_{n-4}) + \cdots + \sigma_1(\lambda_1(y_1 + y_2)) \cdots)$$

results in global asymptotic stability and with poles placed arbitrarily (in pairs) on the real axis, we must show that

$$\begin{aligned} \dot{y}_{i-1} &= \lambda_i y_i - \sigma_i(\lambda_i(y_{i-1} + y_i) + \sigma_{i-1}) \\ \dot{y}_i &= -\sigma_i(\lambda_i(y_{i-1} + y_i) + \sigma_{i-1}) \end{aligned} \quad (2.18)$$

is such that the trajectories of y_{i-1}, y_i enter in finite time and remain in a region where σ_i is linear, for a sufficiently small M_{i-1} (the bound on the magnitude of σ_{i-1}). If we define $\tilde{y}_{i-1} = \lambda_i y_{i-1}$ and $\tilde{y}_i = \lambda_i y_i$ then (2.18) becomes

$$\begin{aligned} \dot{\tilde{y}}_{i-1} &= \lambda_i \tilde{y}_i - \lambda_i \sigma_i(\tilde{y}_{i-1} + \tilde{y}_i + \sigma_{i-1}) \\ \dot{\tilde{y}}_i &= -\lambda_i \sigma_i(\tilde{y}_{i-1} + \tilde{y}_i + \sigma_{i-1}) \end{aligned} \quad (2.19)$$

Now the result follows from section 2.2 by scaling time by a factor of λ_i .

We are now in a position to achieve arbitrary pole placement. Consider the control

$$u = -\sigma_n(\lambda_n(y_{n-1} + y_n) + \sigma_{n-1}(\lambda_{n-1}(y_{n-3} + y_{n-2}) + \cdots + \sigma_1(\lambda_1(y_1 + y_2) - v) \cdots) \quad (2.20)$$

where $y = T_\lambda x$ is defined (implicitly) above. We have demonstrated that, after some finite time, the closed-loop dynamics are given by

$$\begin{aligned} \dot{y}_1 &= -\lambda_1 y_1 + v \\ \dot{y}_2 &= -\lambda_1 y_1 - \lambda_1 y_2 + v \\ \dot{y}_3 &= -\lambda_1 y_1 - \lambda_1 y_2 - \lambda_2 y_3 + v \\ &\vdots \\ \dot{y}_{2n} &= -\lambda_1 y_1 - \lambda_1 y_2 - \lambda_2 y_3 - \lambda_2 y_4 - \cdots - \lambda_n y_{2n-1} - \lambda_n y_{2n} + v \end{aligned} \quad (2.21)$$

We will now choose v to achieve arbitrary pole placement. We begin by choosing

$$v = -\sigma_{-1}(c_1 y_1 + c_2 y_2 - v') \quad (2.22)$$

where c_1 and c_2 are chosen to place the poles of

$$A_{cl} = \begin{bmatrix} -\lambda_1 & 0 \\ -\lambda_1 & -\lambda_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -c_1 & -c_2 \end{bmatrix} \quad (2.23)$$

First we show that, in general, the system

$$\begin{aligned} \dot{y}_1 &= -\lambda_1 y_1 - b_1 \sigma_{-1}(c_1 y_1 + c_2 y_2) \\ \dot{y}_2 &= -\lambda_1 y_1 - \lambda_1 y_2 - b_2 \sigma_{-1}(c_1 y_1 + c_2 y_2) \end{aligned} \quad (2.24)$$

is globally asymptotically stable as long as both eigenvalues of A_{cl} have real part less than or equal to $-\lambda_1 < 0$ and the pair (A, B) defined by

$$\begin{aligned} A &= \begin{bmatrix} -\lambda_1 & 0 \\ -\lambda_1 & -\lambda_1 \end{bmatrix} \\ B &= \begin{bmatrix} b_1 & b_2 \end{bmatrix}^T \end{aligned} \quad (2.25)$$

is controllable. To do so we will appeal to the Popov criterion (see [Popov, 1973] or [Narendra and Taylor, 1973]) which gives us that (2.24) is globally asymptotically stable if $\exists r \geq 0$ such that

$$P(\omega) \doteq \operatorname{Re}[(1 + j\omega r)\hat{g}(j\omega)] \geq 0 \quad \forall \omega \in \mathbb{R} \quad (2.26)$$

(this is conservative since σ_{-1} lies in the sector $(0, 1]$) where

$$\hat{g}(s) = C^T(sI - A)^{-1}B \quad (2.27)$$

and

$$C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T \quad (2.28)$$

It can be shown that the transfer function $\hat{g}(s)$ is given by

$$\hat{g}(s) = \frac{b_1 c_1 + b_2 c_2}{s + \lambda_1} - \frac{b_1 c_2 \lambda_1}{(s + \lambda_1)^2} \quad (2.29)$$

We now solve for c_1 and c_2 in terms of the coefficients of the Hurwitz polynomial associated with the desired eigenvalues of A_{cl} given by

$$p(s) \doteq \det(sI - A_{cl}) = s^2 + d_1 s + d_2 \quad (2.30)$$

This can always be done since the pair (A, B) is controllable. Solving for c_1, c_2 and substituting into (2.29) we get

$$\hat{g}(s) = \frac{d_1 - 2\lambda_1}{s + \lambda_1} + \frac{\lambda_1^2 - d_1\lambda_1 + d_2}{(s + \lambda_1)^2} \quad (2.31)$$

Now, for Popov's criterion, we choose $r = \frac{1}{\lambda_1}$. Then

$$P(\omega) = \frac{d_1 - 2\lambda_1}{\lambda_1} + \frac{\lambda_1^2 - d_1\lambda_1 + d_2}{\omega^2 + \lambda_1^2} \quad (2.32)$$

Now, $p(s)$ either has two (possibly distinct) real eigenvalues or a pair of complex eigenvalues. Consider first that the roots of $p(s)$ are at $s = -\alpha_1$ and $s = -\alpha_2$ for some $\alpha_1, \alpha_2 > 0$. In this case

$$\begin{aligned} d_1 &= \alpha_1 + \alpha_2 \\ d_2 &= \alpha_1\alpha_2 \end{aligned}$$

We then have

$$P(\omega) = \frac{-2\lambda_1 + \alpha_1 + \alpha_2}{\lambda_1} + \frac{(\lambda_1 - \alpha_1)(\lambda_1 - \alpha_2)}{\omega^2 + \lambda_1^2} \quad (2.33)$$

and, hence, $P(\omega) \geq 0$ if $\alpha_1 \geq \lambda_1$ and $\alpha_2 \geq \lambda_1$.

Consider now that the roots of $p(s)$ are at $s = -\alpha \pm j\omega\beta$ for some $\alpha, \beta > 0$. In this case

$$\begin{aligned} d_1 &= 2\alpha \\ d_2 &= \alpha^2 + \beta^2 \end{aligned}$$

We then have

$$P(\omega) = \frac{-2\lambda_1 + 2\alpha}{\lambda_1} + \frac{(\lambda_1 - \beta)^2}{\omega^2 + \lambda_1^2} \quad (2.34)$$

and, hence, $P(\omega) \geq 0$ if $\alpha \geq \lambda_1$. Finally, since we know that (2.24) is globally asymptotically stable, a converse Lyapunov argument, similar to that used in [Sontag and Yang, 1991]) can be used to demonstrate that for M_{-2} sufficiently small (where $|v'| \leq M_{-2}$) the trajectories of

$$\begin{aligned} \dot{y}_1 &= -\lambda_1 y_1 - \sigma_{-1}(c_1 y_1 + c_2 y_2 - v') \\ \dot{y}_2 &= -\lambda_1 y_1 - \lambda_1 y_2 - \sigma_{-1}(c_1 y_1 + c_2 y_2 - v') \end{aligned} \quad (2.35)$$

enter in finite time and remain in a region where σ_{-1} is linear.

At this point we have the system

$$\begin{aligned} \dot{y}_{1,2} &= A_{cl} y_{1,2} - B_o v' \\ \dot{y}_{3,4} &= A_{cl} y_{1,2} + J(\lambda_2) y_{3,4} - B_o v' \\ &\vdots \\ \dot{y}_{2n-1,2n} &= A_{cl} y_{1,2} + J(\lambda_2) y_{3,4} + J(\lambda_n) y_{2n-1,2n} - B_o v' \end{aligned} \quad (2.36)$$

where

$$B_o = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \quad y_{i-1,i} = \begin{bmatrix} y_{i-1} & y_i \end{bmatrix}^T \quad (2.37)$$

We now focus on the $(y_{1,2}, y_{3,4})$ dynamics. Since the pair (\bar{A}, \bar{B}) given by

$$\bar{A} = \begin{bmatrix} A_{cl} & 0 \\ A_{cl} & J(\lambda_2) \end{bmatrix} \quad \bar{B} = \begin{bmatrix} B_o \\ B_o \end{bmatrix} \quad (2.38)$$

is controllable, there exists a coordinate change transforming (\bar{A}, \bar{B}) into

$$\tilde{A} = \begin{bmatrix} A_{cl} & * \\ 0 & J(\lambda_2) \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} * \\ B \end{bmatrix} \quad (2.39)$$

where the pair $(J(\lambda_2), B)$ is controllable. Now we choose

$$v' = -\sigma_{-2}(\tilde{c}_1 \tilde{y}_3 + \tilde{c}_2 \tilde{y}_4 - v'') \quad (2.40)$$

where

$$\tilde{C} = \begin{bmatrix} \tilde{c}_1 & \tilde{c}_2 \end{bmatrix}$$

is chosen to place the poles of $J(\lambda_2) + B\tilde{C}$ (to the left of $-\lambda_2$). It is now apparent that this process can be continued to arbitrarily place the poles of the closed loop system, since the original λ_i 's were shown to be arbitrary.

2.4.2 Performance

To highlight the performance issues involved in control systems with bounded controls, we focus on the 2-dimensional chain of integrators

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sigma(u) \end{aligned} \quad (2.41)$$

We consider the effects of placing both of the poles of the Jacobian approximation at $s = -\lambda$. Following the procedure of the previous section, this is done by choosing

$$\begin{aligned} y_1 &= \lambda x_1 + x_2 \\ y_2 &= x_2 \end{aligned}$$

This yields the system

$$\begin{aligned} \dot{y}_1 &= \lambda y_2 + \sigma(u) \\ \dot{y}_2 &= \sigma(u) \end{aligned} \quad (2.42)$$

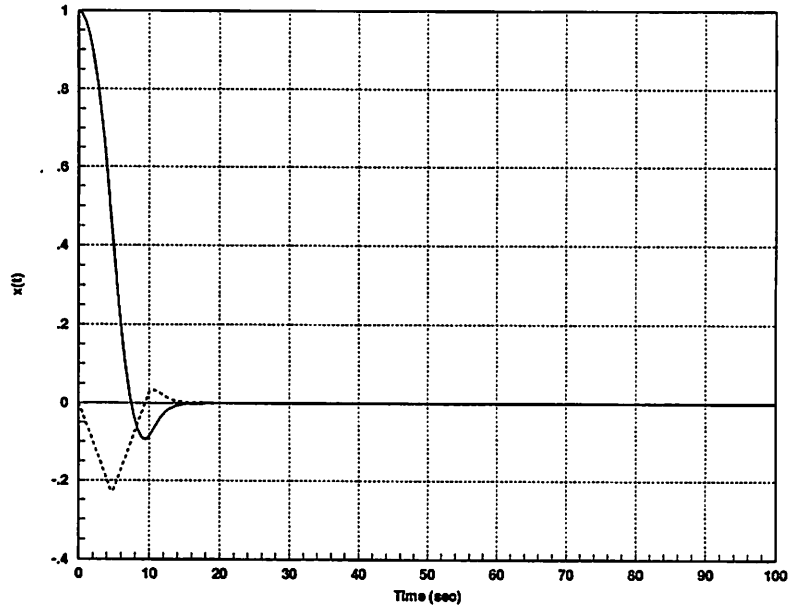


Figure 2.1: Time trajectory for the system (2.41) using the control (2.43) with $\lambda = 1$ and $(x_1(0), x_2(0)) = (1, 0)$.

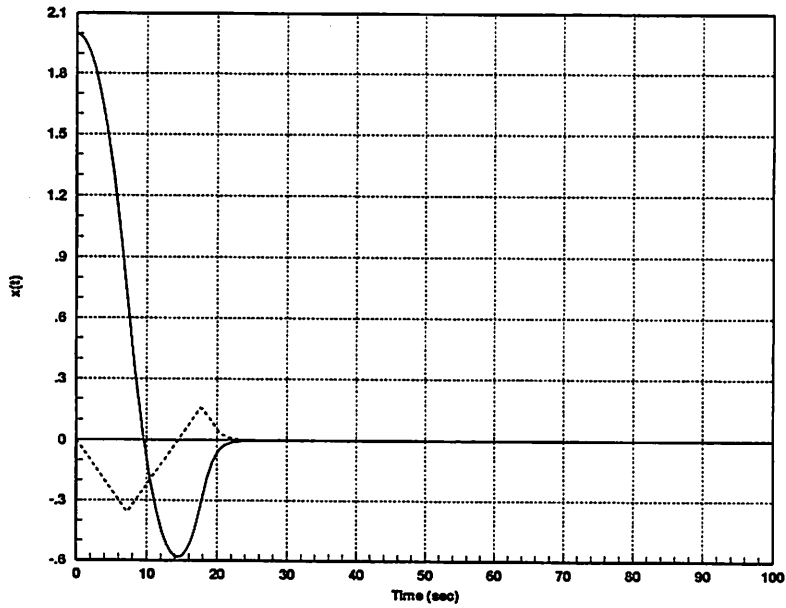


Figure 2.2: Time trajectory for the system (2.41) using the control (2.43) with $\lambda = 1$ and $(x_1(0), x_2(0)) = (2, 0)$.

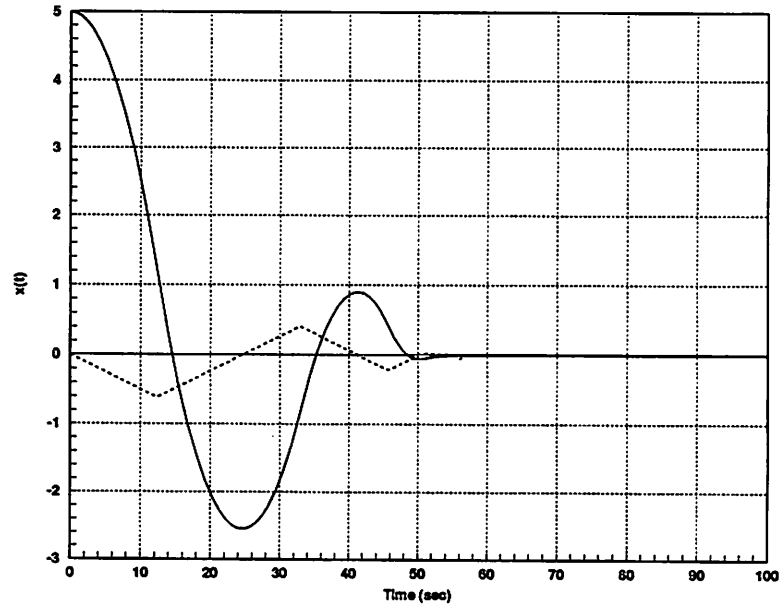


Figure 2.3: Time trajectory for the system (2.41) using the control (2.43) with $\lambda = 1$ and $(x_1(0), x_2(0)) = (5, 0)$.

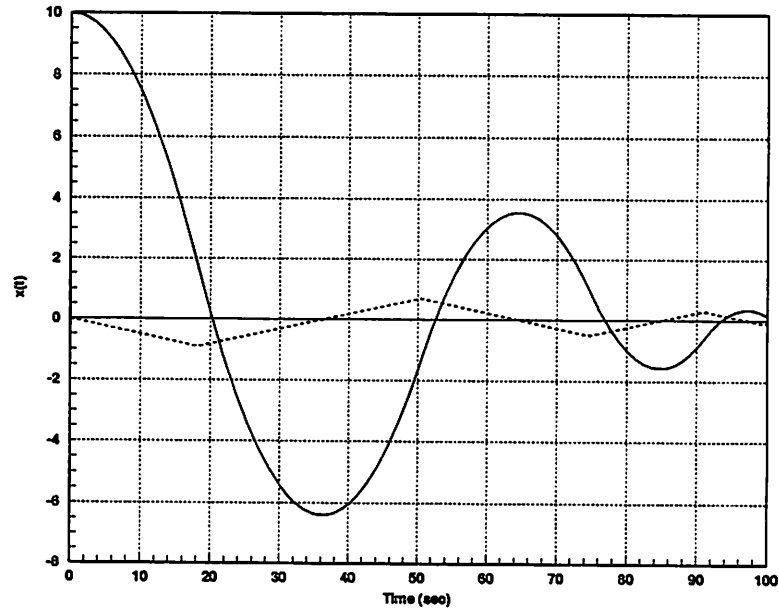


Figure 2.4: Time trajectory for the system (2.41) using the control (2.43) with $\lambda = 1$ and $(x_1(0), x_2(0)) = (10, 0)$.

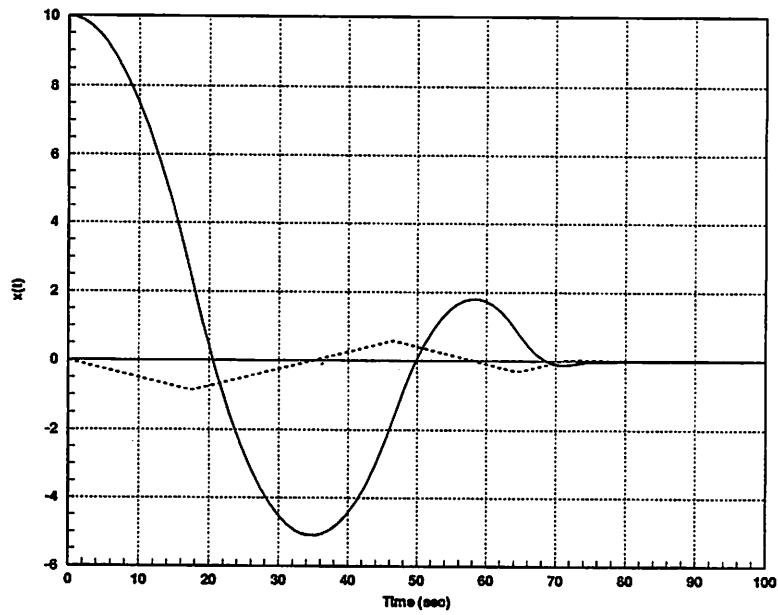


Figure 2.5: Time trajectory for the system (2.41) using the control (2.43) with $\lambda = \sqrt{5}$ and $(x_1(0), x_2(0)) = (10, 0)$.

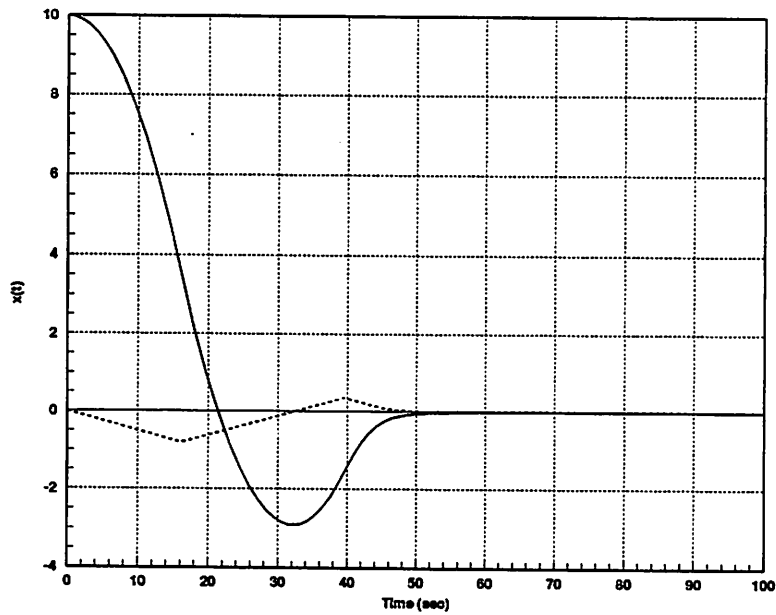


Figure 2.6: Time trajectory for the system (2.41) using the control (2.43) with $\lambda = \sqrt{2}$ and $(x_1(0), x_2(0)) = (10, 0)$.

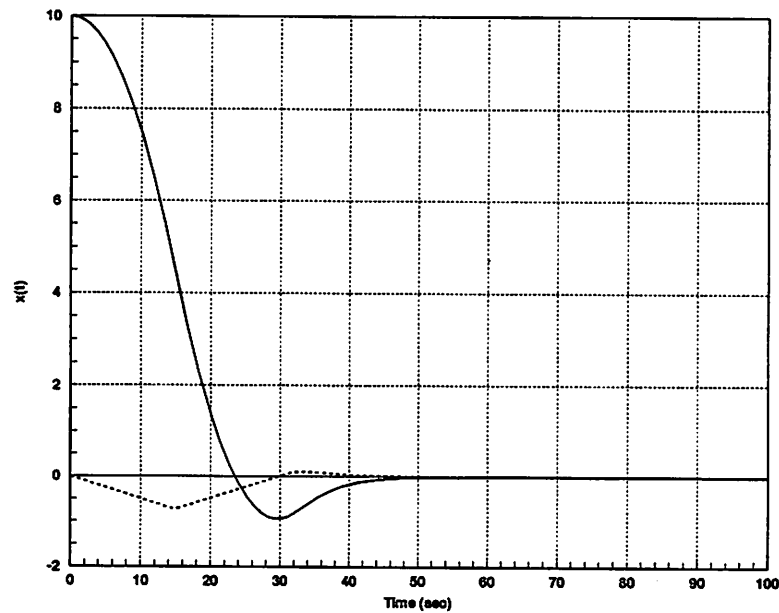


Figure 2.7: Time trajectory for the system (2.41) using the control (2.43) with $\lambda = \sqrt{.1}$ and $(x_1(0), x_2(0)) = (10, 0)$.

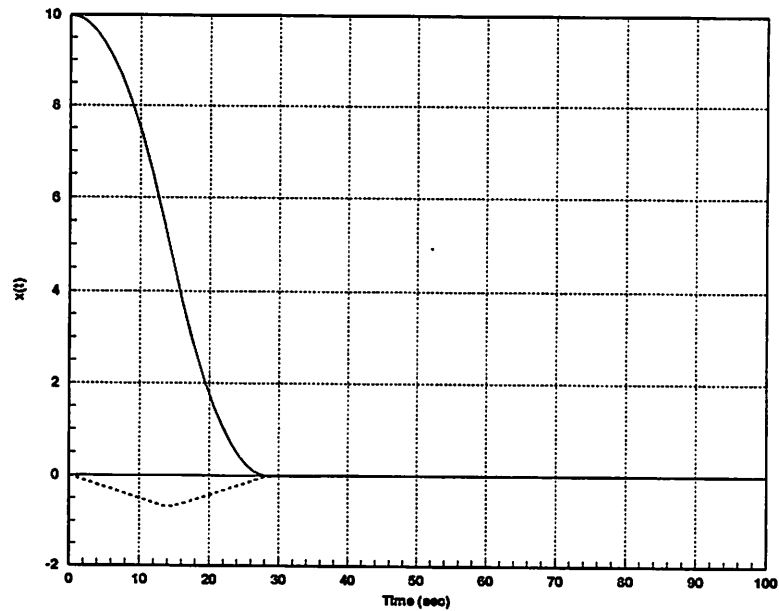


Figure 2.8: Time trajectory for the system (2.41) using the control (2.46) with $\lambda_1 = 0.25$, $\lambda_2 = 1.0$ and $(x_1(0), x_2(0)) = (10, 0)$.

We then choose the control $u = -\lambda(y_1 + y_2)$. In the original coordinates we have

$$u = -\lambda^2 x_1 - 2\lambda x_2. \quad (2.43)$$

If we further define

$$\begin{aligned} \tilde{y}_1 &= \lambda y_1 \\ \tilde{y}_2 &= \lambda y_2 \end{aligned}$$

then we have the system

$$\begin{aligned} \dot{\tilde{y}}_1 &= \lambda \tilde{y}_2 - \lambda \sigma(\tilde{y}_1 + \tilde{y}_2) \\ \dot{\tilde{y}}_2 &= -\lambda \sigma(\tilde{y}_1 + \tilde{y}_2) \end{aligned} \quad (2.44)$$

At this point we can scale time, defining $\tau \doteq \frac{t}{\lambda}$ and denoting $\frac{d}{d\tau}$ by (\prime) , we have

$$\begin{aligned} \tilde{y}_1' &= \tilde{y}_2 - \sigma(\tilde{y}_1 + \tilde{y}_2) \\ \tilde{y}_2' &= -\sigma(\tilde{y}_1 + \tilde{y}_2) \end{aligned} \quad (2.45)$$

So we see that we can achieve the convergence of this canonical system on an arbitrary time scale. One might then be tempted to choose λ arbitrarily large to achieve arbitrarily fast convergence. The reason why this intuition fails, as it must since we are restricted to bounded controls, is that the transient performance is not uniform with respect to the initial conditions. This is significant because as λ grows, the size of the initial conditions of \tilde{y} grows. As alluded to in [Fuller, 1969], the performance of this type of bounded control is much closer to optimal for small initial conditions than it is for large initial conditions. To demonstrate this point we show in figure 2.1 – 2.4, simulations of the two dimensional chain of integrators system (2.41) using the control (2.43) with various size initial conditions. For purposes of comparison, we chose $x_2(0) = 0$ for each simulation. The value of $x_1(0)$ is set at 1, 2, 5 and 10, respectively. The saturating function σ in (2.41) was chosen to be a *simple linear saturation* with $L = M = 0.05$.

We demonstrate the effects of tuning λ by showing in figures 2.4 – 2.7 the trajectories of the system (2.41) using the input (2.43) and choosing λ to be 1, $\sqrt{.5}$, $\sqrt{.2}$ and $\sqrt{.1}$, respectively. Again, σ is a *simple linear saturation* with $L = M = 0.05$. Observe the improved transient performance as λ decreases.

The tradeoff is apparent. From these figures, and the discussion above, we see that, given some a priori knowledge of the size of initial conditions, we can achieve few oscillations and slow convergence in the tail by choosing λ small while we can achieve many oscillations and fast convergence in the tail by choosing λ large. (In fact, given a bound on the initial conditions, picking λ small enough insures that the control never saturates.)

We suggest that we do not have to settle for this tradeoff. The discussion of the previous section on arbitrary pole placement provides the solution. In fact, if we choose

$$u = -\lambda_1^2 x_1 - 2\lambda_2 x_2 - \sigma_2((\lambda_2^2 - \lambda_1^2)x_1 + 2(\lambda_2 - \lambda_1)x_2) \quad (2.46)$$

with $\lambda_2 \geq \lambda_1$ we can use λ_1 to tune the transient performance and λ_2 to tune the performance in the tail. The success of such an approach is determined by comparing figure 2.8 to figure 2.4. Again, the limiting saturation on the control is a *simple linear saturation* with $L = M = 0.05$. The function σ_2 in (2.46) is a *simple linear saturation* with $L = M = 0.02$.

Hence, we have the following design suggestion. If an approximation on the size of initial conditions is known, the λ_i 's of the previous section should be chosen to optimize the transient performance of the system, while the final pole locations should be chosen to achieve the desired convergence in the tail as in figure 2.8.

2.5 Summary

While the nested saturation solution to the bounded control problem is theoretically appealing, its success as a practical design tool is yet to be established. Future work will focus on applying the linear multivariable regulator theory of section 2.3.2 along the the pole placement and performance ideas of sections 2.4.1 and 2.4.2 to the model of the F8 aircraft studied in [Kapasouris *et al.*, 1988].

Chapter 3

Using Saturation in the Semi-global Stabilization of Minimum Phase Systems

In this chapter, we use the bounded controls of the previous chapter to generalize the class of minimum phase nonlinear systems that can be semi-globally stabilized. Previous results can be found in the pioneering work of Byrnes and Isidori ([Byrnes and Isidori, 1991]) and of Sussmann and Kokotovic ([Sussmann and Kokotovic, 1991]). We prove and demonstrate the success of our algorithm and compare it to a competing algorithm [Lin and Saberi, 1992a] that was motivated by our work. We also state a generalization of our work by Lin and Saberi [Lin and Saberi, 1992b] which combines our approach with the general results of Sontag and Yang [Sontag and Yang, 1991] of the previous chapter.

3.1 Introduction

This work is an extension of the semi-global stabilizability results of [Byrnes and Isidori, 1991] and [Sussmann and Kokotovic, 1991] for multi-input minimum phase nonlinear

systems in the normal form:

$$\begin{aligned}
 \dot{\eta} &= f(\eta, \xi, u) \\
 \dot{\xi}_1^i &= \xi_2^i \\
 &\vdots \\
 \dot{\xi}_{r_i}^i &= u_i \\
 y_i &= \xi_1^i \quad \text{for } i = 1, \dots, m
 \end{aligned} \tag{3.1}$$

where the state $x = (\eta, \xi) \in \mathbb{R}^n$ and f is smooth with $f(0, 0, 0) = 0$. By minimum phase it is meant that the equilibrium point $\eta = 0$ of

$$\dot{\eta} = f(\eta, 0, 0)$$

is globally asymptotically stable.

In the works of [Byrnes and Isidori, 1991] and [Sussmann and Kokotovic, 1991], the standard semi-global stabilization problem is to find a family of *linear* feedbacks (of the states ξ only) with tunable gain parameters that allows for local asymptotic stability and regulation to the origin for any initial condition in some (arbitrarily large) a priori bounded set. As described in [Sussmann and Kokotovic, 1991], in general such a family of general feedbacks can fail to exist due to peaking in the linear variables. Loosely speaking, the linear variables can get large before they get small, inducing instability in the nonlinear dynamics. In [Byrnes and Isidori, 1991] the problem is seen as an undesirable reduction of the domain of asymptotic stability of the nonlinear dynamics as a result of redefining new outputs to add *linear* zeros in the left-half plane and by employing high gain output feedback to the new output.

We will be able to achieve our extension by allowing our family of feedbacks to be possibly nonlinear, again as a function of ξ only. Our primary tool will be the bounded controls of chapter 2 (see also [Teel, 1992a]) to eliminate peaking when possible. As motivating examples, we consider two very similar examples in [Byrnes and Isidori, 1991] and [Sussmann and Kokotovic, 1991] that serve as warnings that simple high gain *linear* feedbacks will not always be able to solve the nonlinear semi-global stabilizability problem:

Example 3.1 (Example 8.2 of [Byrnes and Isidori, 1991]) *Consider the system (in*

normal form (3.1))

$$\begin{aligned}\dot{\eta} &= -(1 - \eta\xi_2)\eta \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{3.2}$$

Example 3.2 (Example 1.1 of [Sussmann and Kokotovic, 1991]) *Consider the system (in normal form (3.1))*

$$\begin{aligned}\dot{\eta} &= -0.5(1 + \xi_2)\eta^3 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{3.3}$$

Both examples are globally minimum phase. Our philosophy for semi-globally stabilizing these systems can be summed up in the following heuristic argument that we make more precise in the sequel. In each case, if the state ξ_1 were not a part of the system we would choose a linear high-gain feedback function of ξ_2 alone to drive ξ_2 exponentially to the origin. The necessary rate of decay would be determined by the initial state of η . In both cases, if asymptotic regulation of ξ_2 were not crucial, it would actually be sufficient to drive ξ_2 exponentially to an arbitrarily small neighborhood of the origin. The rate of decay and size of the neighborhood would be chosen based on the initial state of η . But this allows us to reintroduce the state ξ_1 since it can be steered to the origin with an arbitrarily small bounded “control” ξ_2 . In summary, for both examples, we will choose to drive ξ_2 arbitrarily fast to an arbitrarily small control that will (slowly) drive ξ_1 to zero without destabilizing the dynamics of η . The problem with the fully high gain approach of [Byrnes and Isidori, 1991] and [Sussmann and Kokotovic, 1991] is that they drive both ξ_1, ξ_2 rapidly to the origin. To drive ξ_1 fast requires peaking in ξ_2 . The peaking in ξ_2 destabilizes the original zero dynamics. However, in the examples, the rate of convergence of ξ_1 is unimportant.

One interpretation of our approach is that we are adding a (slow) asymptotically stable nonlinear “zero” to the system by reducing the order of the linear subsystem by one. Most importantly, the addition of this nonlinear zero still allows for asymptotic stability of the new composite zero dynamics on arbitrarily large compact sets.

3.2 Problem Statement

We make the following definition to clarify the problem at hand:

Definition 3.1 *The system (3.1) is semi-globally stabilizable by state feedback if for any compact set of initial conditions X there exists a smooth state feedback*

$$u = \alpha(\xi, \eta) \quad (3.4)$$

such that the equilibrium $(0, 0)$ of the closed-loop system (3.1), (3.4) is locally asymptotically stable and X is contained in the domain of attraction of $(0, 0)$.

We will focus on generating feedbacks that depend only on the linear states ξ . (i.e. $u = \alpha(\xi)$.)

We will be able to achieve semi-global stabilization for multi-input systems in the following special normal form:

$$\begin{aligned} \dot{\eta} &= f(\eta, \xi_{j_1}^1, \dots, \xi_{j_m}^m) & j_i &\in \{1, \dots, r_i + 1\} \\ \dot{\xi}_1^i &= \xi_2^i \\ &\dots \\ \dot{\xi}_{r_i}^i &= u \\ y_i &= \xi_1^i & \text{for } i &= 1, \dots, m \end{aligned} \quad (3.5)$$

where $\xi_{r_i+1}^i \equiv u_i$. With respect to the outputs y_i the system (3.5) is said to have *vector relative degree* $\{r_1, \dots, r_m\}$. We define $r = r_1 + \dots + r_m$. We then have $\xi \in \mathbb{R}^r$ and $\eta \in \mathbb{R}^{n-r}$.

We make the following standard assumption:

Assumption 3.1 *The equilibrium point $\eta = 0$ of the dynamics*

$$\dot{\eta} = f(\eta, 0, \dots, 0) \quad (3.6)$$

ie. the zero dynamics of (3.5), are globally asymptotically stable.

The distinguishing feature of the systems in the special normal form of (3.5) is that no more than one state in each of the m chains of integrators appears in the η dynamics. Systems of the form (3.5) are more general than those in [Byrnes and Isidori, 1991] in that the one state is *not* required to be the first state of the chain associated with y_i , namely ξ_1^i . In the terminology of [Sussmann and Kokotovic, 1991], $(0, \dots, 0)$ is not necessarily an achievable sequence of peaking exponents.

3.3 Main Results

Our general approach for nonlinear systems in the form (3.5) is to redefine the nonlinear subsystem to include the dynamics of $\xi_1^i, \dots, \xi_{j_i-1}^i$ for $i = 1, \dots, m$ and redefine the i th output to be $\tilde{y}_i = \xi_{j_i}^i$. We also define the nonnegative constants $\tilde{r}_i = r_i - j_i + 1$. We then have the following nonlinear system:

$$\begin{aligned}
 \dot{\eta} &= f(\eta, \tilde{y}_1, \dots, \tilde{y}_m) \\
 \dot{z}_1^i &= z_2^i \\
 &\vdots \\
 \dot{z}_{j_i-1}^i &= \tilde{y}_i \\
 \dot{\xi}_1^i &= \xi_2^i \\
 &\vdots \\
 \dot{\xi}_{\tilde{r}_i}^i &= u_i \\
 \tilde{y}_i &= \xi_1^i \quad \text{for } i = 1, \dots, m
 \end{aligned} \tag{3.7}$$

The *vector relative degree* with respect to the new outputs \tilde{y}_i is given by $\{\tilde{r}_1, \dots, \tilde{r}_m\}$. Observe that some entries of the vector relative degree may in fact be zero. Define $\tilde{r} = \tilde{r}_1 + \dots + \tilde{r}_m$. We now have $\tilde{\xi} \in \mathbb{R}^{\tilde{r}}$ and $z \in \mathbb{R}^{r-\tilde{r}}$. With respect to the new outputs, it is straightforward to see that the system is not minimum phase. We are now interested in some further output redefinition that makes the system (3.7) minimum phase at least on sets $U = V \times \mathbb{R}^{r-\tilde{r}}$ where $V \subset \mathbb{R}^{n-r}$ is any arbitrarily large compact set. This will be sufficient since we are only interested in semi-global stabilizability.

In preparation for our choice of output redefinition we establish the following result for the system

$$\begin{aligned}
 \dot{\eta} &= f(\eta, \varphi_1(v_1, t), \dots, \varphi_m(v_m, t)) \\
 \dot{z}_1^i &= z_2^i \\
 &\vdots \\
 \dot{z}_{j_i-1}^i &= v_i(z^i) \quad \text{for } i = 1, \dots, m
 \end{aligned} \tag{3.8}$$

Proposition 3.1 *Assume the system (3.8) satisfies assumption 3.1. Then, given the compact set $V \subset \mathbb{R}^{n-r}$, there exists a positive constant ν_0 such that for any set of controls $\{v_i\}_{i=1}^m$ that globally stabilizes z and is such that $|v_i| < \nu_0$ and $v_i(0) = 0$, and for any*

functions $\varphi_i(\cdot, t)$ such that

$$\begin{aligned} \lim_{t \rightarrow \infty} v_i(t) = 0 &\Rightarrow \lim_{t \rightarrow \infty} \varphi_i(v_i, t) = 0 \\ |v_i| < \nu_0 &\Rightarrow |\varphi_i(v_i, t)| < M\nu_0 \end{aligned}$$

for some $M > 0$, the dynamics of (3.8) are asymptotically stable with basin of attraction containing $V \times \mathbb{R}^{r-\bar{r}}$.

Proof. The local asymptotic stability is straightforward (see, for example, [Byrnes and Isidori, 1991, Lemma 4.2].) To determine the basin of attraction note that any state $z \in \mathbb{R}^{r-\bar{r}}$ is driven to the origin by the assumption on $v(z)$. Now consider initial conditions $\eta \in V$. We will demonstrate that $\exists \nu_0$ such that if $|v_i| < \nu_0$ then the trajectories of η remain bounded for all $t \geq 0$. Regulation to the origin then follows from the main theorem of [Sontag, 1989] since $z \rightarrow 0$ as $t \rightarrow \infty$ by assumption, $v_i(\cdot)$ is smooth with $v_i(0) = 0$ and

$$\lim_{t \rightarrow \infty} v_i(t) = 0 \Rightarrow \lim_{t \rightarrow \infty} \varphi(v_i, t) = 0$$

To this end consider a smooth positive definite and proper Lyapunov function

$$W : \mathbb{R}^{n-r} \rightarrow \mathbb{R}$$

such that

$$dW(\eta) \cdot f(\eta, 0) < 0 \tag{3.9}$$

for all nonzero η . The existence of such a Lyapunov function follows from assumption 3.1. It then follows that

$$dW(\eta) \cdot f(\eta, \varphi(v, t)) < 0 \tag{3.10}$$

for all $\|\varphi(v, t)\| < \nu(\|\eta\|)$ for some continuous function ν that is decreasing on $[1, +\infty)$. (See [Sontag, 1990, Lemmas 3.1, 3.2].) Now let c be the largest value of W on the compact set V and let $\|\eta\| \leq R \forall \eta \in \{\eta : W(\eta) \leq c\}$. Such an R exists because W is proper. Then R and the function ν together with the constant M determine a bound ν_0 and an additional constant $L < R$ such that

$$dW(\eta) \cdot f(\eta, \varphi(v, t)) < 0 \tag{3.11}$$

for all $L \leq \|\eta\| \leq R$ and all $\|v\| < \nu_0$. Now, by assumption, $\eta(0) \in V$ and hence $W(0) \leq c$. Finally, since W is decreasing whenever $W = c$ it follows that $W(t) \leq c$ for all $t \geq 0$. This in turn implies $\|\eta(t)\| \leq R$ for all $t \geq 0$. \square

Remark. It is well known that such bounded controls v_i exist since any finite length chain of integrators can be globally stabilized with an arbitrarily small control. (see chapter 2, [Schmitendorf and Barmish, 1980], [Sontag and Sussmann, 1990], or [Teel, 1992a].)

We could now proceed with a standard output definition procedure choosing new outputs as

$$\bar{y}_i = \tilde{y}_i - v_i(z^i)$$

In this case, the zero output dynamics would be given by (3.8) with $\varphi_i(v_i, t) = v_i$. The drawback to this choice is that the procedure to generate the closed loop control involves repeated differentiation of the necessarily complicated (see [Sussmann and Yang, 1991]) bounded controls v_i . Instead we choose a procedure that avoids this repeated differentiation. (This type of procedure has also been used with success in [Teel, 1992b] for certain stabilization problems when the system is not initially minimum phase.) The outputs chosen will depend on the feedback gains used and so will be saved for the last step.

We begin by choosing a high-gain feedback law to stabilize the dynamics of $\tilde{\xi}$ in (3.7). We also include the small bounded control v which will be instrumental in stabilizing the zero dynamics. We choose

$$u_i = -K^{\bar{r}_i} c_{i,\bar{r}_i} \tilde{\xi}_1^i - \dots - K c_{i,1} \tilde{\xi}_{\bar{r}_i}^i + K^{\bar{r}_i} v_i \quad (3.12)$$

where $v_i(\cdot)$ will be specified and $K > 0$.

Next, we make a linear coordinate change to move v_i so that it directly controls the z^i states. To do so, we begin by defining

$$\zeta_k^i = \frac{1}{K^{k-1}} \tilde{\xi}_k^i \quad (3.13)$$

Then the dynamics for ζ^i are

$$\begin{aligned} \dot{\zeta}_1^i &= K \zeta_2^i \\ &\vdots \\ \dot{\zeta}_{\bar{r}}^i &= K(-c_{i,\bar{r}} \zeta_1^i - \dots - c_{i,1} \zeta_{\bar{r}}^i + v) \end{aligned} \quad (3.14)$$

Recalling that

$$\frac{d}{dt} z_{j_i-1}^i = \tilde{\xi}_1^i = \zeta_1^i \quad (3.15)$$

we define

$$\bar{z}_{j_i-1}^i = c_{i,\bar{r}} z_{j_i-1} + \frac{1}{K} (c_{i,\bar{r}-1} \zeta_1^i + \dots + c_{i,1} \zeta_{\bar{r}-1}^i + \zeta_{\bar{r}}^i) \quad (3.16)$$

It can then be shown that

$$\frac{d}{dt} \bar{z}_{j_i-1}^i = v_i \quad (3.17)$$

Likewise, we define \bar{z}_k^i for $k = 1, \dots, j_i - 2$ such that

$$\frac{d}{dt} \bar{z}_k^i = \bar{z}_{k+1}^i \quad (3.18)$$

It is straightforward to show that this can be done in a way such that the transformation between (ζ^i, z^i) and $(\bar{\zeta}^i, \bar{z}^i)$ is invertible.

We are now ready to define the appropriate outputs. To do so, we denote by A_i the controllable canonical form matrix associated with the Hurwitz polynomial

$$s^{\bar{r}_i} + c_{i,1} s^{\bar{r}_i-1} + \dots + c_{i,\bar{r}_i} \quad (3.19)$$

We also let $C_i \in \mathbb{R}^{1 \times \bar{r}_i}$ and $B_i \in \mathbb{R}^{\bar{r}_i \times 1}$ be such that

$$\begin{aligned} C_i &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \\ B_i &= \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T \end{aligned} \quad (3.20)$$

Then we define the i th output to be

$$\bar{y}_i = C_i e^{K A_i t} \zeta^i(0) \quad (3.21)$$

Observe that

$$\begin{aligned} \bar{y}_i(t) = \bar{\xi}_1^i &= \zeta_1^i \\ &= \bar{y}_i(t) + \int_0^t C_i e^{K A_i (t-\tau)} K B_i v_i(\tau) d\tau \\ &= \bar{y}_i(t) + \varphi_i(v_i, t) \end{aligned} \quad (3.22)$$

Finally, we have the nonlinear system

$$\begin{aligned} \dot{\eta} &= f(\eta, \bar{y}_i + \varphi_i(v_i, t)) \\ \dot{\bar{z}}_1^i &= \bar{z}_2^i \\ &\vdots \\ \dot{\bar{z}}_{j_i-1}^i &= v_i \\ \dot{\zeta}_1^i &= K \zeta_2^i \\ &\vdots \\ \dot{\zeta}_{\bar{r}}^i &= K(-c_{i,\bar{r}} \zeta_1^i - \dots - c_{i,1} \zeta_{\bar{r}}^i + v_i) \\ \bar{y}_i &= C_i e^{K A_i t} \zeta^i(0) \end{aligned} \quad \text{for } i = 1, \dots, m \quad (3.23)$$

To check the minimum phase property we must examine the system

$$\begin{aligned}
 \dot{\eta} &= f(\eta, \varphi_i(v_i, t)) \\
 \dot{\bar{z}}_1^i &= \bar{z}_2^i \\
 &\vdots \\
 \dot{\bar{z}}_{j_i-1}^i &= v_i
 \end{aligned} \tag{3.24}$$

It is easy to show from (3.22) that the functions φ_i satisfy the requirements of proposition 3.1 with the constant M independent of the choice of K . Further, it is important to note that K can be chosen to drive the outputs \bar{y}_i to zero exponentially with an arbitrarily fast rate of decay without exhibiting peaking. We then have the following results.

Theorem 3.1 *Assume the system (3.5) satisfies assumption 3.1. Then the system (3.5) is semi-globally stabilized by the family of feedbacks (3.12). That is, (3.12) locally asymptotically stabilizes (3.5) and for any compact set X of the state space (η, ξ) there exists a $K_X > 0$ and $\nu_X > 0$ such that, for all $K > K_X$ and all globally asymptotically stabilizing $v(\bar{z})$ such that $\|v(\bar{z})\| < \nu_X$, the basin of attraction for the closed-loop system (3.5), (3.12) contains X .*

Proof. The proof of this theorem follows from the proof of [Byrnes and Isidori, 1991, Theorem 7.2] together with proposition 3.1. Following the proof of Theorem 7.2 in [Byrnes and Isidori, 1991], we can show that it is possible to choose K in (3.23) large enough such that the trajectories of η, \bar{z} with exponentially decaying inputs converge to trajectories of the undriven η, \bar{z} dynamics that take initial conditions in some compact set \bar{X} determined by X . Then applying proposition 3.1, given \bar{X} , there exists ν_0 sufficiently small such that if v is chosen with $\|v(z)\| < \nu_0$, all trajectories of η, \bar{z} that originate in the compact set X are driven to zero. Finally, the states ζ converge to zero since they are the states of a linear system driven by bounded inputs that converge to zero. \square

It is possible to slightly weaken the compact set restriction since the dynamics of \bar{z} are autonomous and globally asymptotically stable.

Corollary 3.1 *Assume the system (3.5) satisfies assumption 3.1. Then the feedbacks (3.12) locally asymptotically stabilizes the origin of (3.5) and, for all initial conditions in the set $Y = X_\eta \times \mathbb{R}^{r-\bar{r}} \times X_\xi$ where $X_\eta \subset \mathbb{R}^{n-r}$ is compact and $X_\xi \subset \mathbb{R}^{\bar{r}}$ is compact, there exists $K_Y > 0$ and $\nu_Y > 0$ such that, for all $K > K_Y$ and all $v(z)$ such that $\|v(z)\| < \nu_Y$, the basin of attraction for the closed-loop system (3.5), (3.12) contains Y .*

3.4 Examples

We return now to examples 3.1 and 3.2. Both examples have essentially the same structure and are solved by the same family of feedbacks. To describe this class of feedbacks we define the smooth function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} s\sigma(s) &> 0 \quad \text{for } s \neq 0 \\ |\sigma(s)| &\leq \nu \end{aligned} \tag{3.25}$$

The semi-global stabilization problem for examples 3.1 and 3.2 are then solved by the family of feedbacks

$$u = -K[\xi_2 + \sigma(\xi_1 + \frac{1}{K}\xi_2)] \tag{3.26}$$

parametrized by $K, \nu > 0$.

We give one further example to demonstrate the methods when $j = r + 1$.

Example 3.3 *Consider the single-input system (in normal form (3.1))*

$$\begin{aligned} \dot{\eta} &= -\eta + \eta^2 u \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{3.27}$$

Given $\eta \in V$ where $V \subset \mathbb{R}$ compact, the family of feedbacks is specified by

$$u = -\sigma(c_1\xi_1 + c_2\xi_2) \tag{3.28}$$

where c_1 and c_2 are chosen such that the dynamics of ξ_1, ξ_2 are globally asymptotically stable and where ν is chosen such that

$$\eta > \eta^2 \nu \tag{3.29}$$

for all $\eta \in V$.

3.5 Mention of Subsequent Results

An alternative approach to solving the semi-global stabilization problem for systems in the special normal form of (3.5) has subsequently been proposed in [Lin and Saberi, 1992a]. In this work the authors proposed a fully linear solution by implementing very low gain feedback to replace the bounded controls. The big selling point is that it is a linear

solution that requires only one tuning parameter. (The time constant of the slow dynamics is the inverse of the time constant of the fast dynamics.) The most obvious drawback to this approach is that the convergence rate of the slow dynamics disappears as the size of the compact set grows. This is contrasted with the nonlinear, bounded control approach where arbitrary convergence in the tail can be achieved. In fact, we can use the methodology of chapter 2 to tune the transient performance as well as the convergence in the tail.

On the other hand, in [Lin and Saberi, 1992b] these same authors recognized the equation (3.7) to be a special case of a cascade of an asymptotically stable nonlinear system with a right-invertible linear system with invariant zeros in the closed left-half plane. For (3.7) the zeros are all located at $s = 0$. They then applied the general bounded control results of [Sontag and Yang, 1991], which were mentioned in chapter 2, to achieve semi-global stabilization for all such cascaded systems.

One further way to view all of these developments is to return to the original normal form [Byrnes and Isidori, 1991] (where the nonlinear dynamics were driven exclusively by the states at the top of the integrator chains.) By this we mean that we would now consider the invariant zeros of the linear system in the cascade to be part of the nonlinear system. With the results of this chapter and the ideas of [Lin and Saberi, 1992b] we see that the minimum phase assumption of [Byrnes and Isidori, 1991] can be weakened. In fact, all that is required of the η dynamics is that they can be asymptotically stabilized with an arbitrarily small control. Unfortunately, this is a difficult condition to check except for the cases considered in this chapter and in [Lin and Saberi, 1992b]. Further, under these circumstances, semi-global stabilization will require explicit knowledge of the normal form and (bounded) feedback of (some) of the η states.

Chapter 4

Using Saturation to Stabilize a Class of Single-Input Non-minimum Phase Nonlinear Systems

In this chapter, we use the nested saturation technology of chapter 1 to globally and semi-globally stabilize nonlinear systems that are not feedback linearizable and not minimum phase. We solve the global and semi-global stabilization problem for systems that apparently had no previous solution.

4.1 Introduction

We will consider partially linear single-input composite systems of the form

$$\begin{aligned}\dot{\eta} &= f(\eta, z, u, t) \\ \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_n &= u\end{aligned}\tag{4.1}$$

where $\eta \in \mathbb{R}^p$ and f is smooth with $f(0, 0, 0, t) = 0$ for all $t \geq t_0$.

Interest in such systems has been driven by input-output linearization theory [Isidori, 1989] which allows partial linearization for systems that cannot be full-state lin-

earized. There have been many recent global stabilization results for such composite systems ([Byrnes *et al.*, 1991], [Kokotovic and Sussmann, 1989], [Marino, 1988], [Praly *et al.*, 1991], [Saber *et al.*, 1990], [Sastry and Isidori, 1989], [Sontag, 1989], [Sussmann and Kokotovic, 1991]). In general these results either assume that the nonlinear subsystem is zero-input asymptotically stable or that f depends only on η and z_1 . In the latter case it is also assumed that a smooth “input” z_1 is known which globally stabilizes the nonlinear subsystem.

The approach presented in this chapter aims at globally (semi-globally) stabilizing a subclass of systems described by (4.1) where the nonlinear subsystem is *not* zero-input asymptotically stable and where f can depend on the complete state vector z as well as the input u . We will rely heavily on the “converging input bounded state” property of [Sontag, 1989] and incorporate the recent result for stabilizing a (linear) chain of integrators with bounded controls described in chapter 2 (see also [Teel, 1992a]) to achieve nonlinear stabilization. Interestingly, our design will provide intuition for determining coordinates and a feedback that yield a composite system of the form (4.1) where the nonlinear subsystem *is* zero-input globally asymptotically stable. More importantly, the approach outlined here depends only on the general properties of the nonlinear terms and not on their explicit form. Consequently, our approach is robust in the presence of a class of unmodeled nonlinear terms and in the presence of unknown (possibly time-varying) bounded parameters.

The assumptions we impose are not generic, but do allow us to handle systems that do not satisfy the conditions of existing methods. In this sense, our method presents a specialized tool intended to complement other existing methods in the nonlinear stabilization toolbox.

Section 4.2 begins this chapter by describing our algorithm for a special subclass of systems known as *feedforward* systems. We defer to an appendix discussion about geometric conditions for generating this form. In section 4.3 we describe the general class of systems for which our algorithm is applicable. Section 4.3.1 will define the general concepts used to expand the result and will review the work of [Sontag, 1989] as it applies to our problem. In section 4.3.2 we state our main results for global stabilization. The proof is also deferred to an appendix. Section 4.3.3 contains our main results for semi-global stabilizability. Finally, in section 4.4 we provide examples for both global and semi-global stabilization. In the global case, we show that our algorithm provides a solution to a previously unsolved benchmark problem [Kokotovic *et al.*, 1991]. In the semi-global case, we show that our algorithm provides a solution to the popular “ball and beam” example [Hauser *et al.*, 1992].

To our knowledge, the only existing stabilizing solutions to this problem were local in nature.

4.2 Feedforward Systems

To prepare for our general result, we begin by considering a special class of systems for which the proof of our result is straightforward. We consider systems of the form:

$$\begin{aligned}\dot{x}_1 &= x_2 + f_1(x_2, x_3, \dots, x_n, u, t) \\ \dot{x}_2 &= x_3 + f_2(x_3, \dots, x_n, u, t) \\ &\vdots \\ \dot{x}_n &= u + f_n(u, t)\end{aligned}\tag{4.2}$$

where we require the functions f_i 's to be continuous and

$$f_i(x_{i+1}, \dots, x_n, u, t) = O(x, u)^2 \quad i = 1, \dots, n$$

The notation $O(x, u)^2$ is used to refer to functions that contain terms which are quadratic and higher in (x, u) . Because of this condition, together with the structure of (4.2), which is in direct contrast to the *feedback* systems of [Kanellakopoulos *et al.*, 1991], we call these systems *higher order feedforward systems*. These systems, in general, are not feedback linearizable. For a discussion of geometric conditions for generating this normal form, see appendix section 4.6.1.

Our nonlinear global stabilizability results rely on a recent linear result for stabilizing a chain of integrators using nested saturation functions with linear arguments [Teel, 1992a]. To that end, we repeat definition 2.1:

Definition 4.1 *Given two constants δ and ϵ satisfying $0 < \delta \leq \epsilon$, a function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a linear saturation if it is a continuous, nondecreasing function satisfying*

1. $\sigma(s) = s$ when $|s| \leq \delta$
2. $|\sigma(s)| \leq \epsilon$ for all $s \in \mathbb{R}$.

The proof of the following theorem will be constructive, yielding a globally stabilizing control law.

Theorem 4.1 (Feedforward Stabilizability) *There exist linear functions $T_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, \dots, n$ and a set of linear saturations $\{\sigma_i\}_{i=1}^n$ such that the control*

$$u = -\sigma_n(T_n(x) + \sigma_{n-1}(T_{n-1}(x) + \dots + \sigma_1(T_1(x))) \dots)$$

globally asymptotically stabilizes the origin of (4.2).

Proof. The proof is very much like the proof of theorem 2.3 in chapter 2. We begin by considering the linear coordinate transformation $y = Tx$ which transforms the Jacobian linearization of (4.2) into $\dot{y} = Ay + Bu$ where A and B are given by

$$A = \begin{bmatrix} 0 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad (4.3)$$

The matrix T is upper triangular and hence T^{-1} is upper triangular. The i th row of the matrix T defines the linear function T_i and $y_i = T_i x$. In transformed coordinates the system (4.2) is given by

$$\begin{aligned} \dot{y}_i &= y_{i+1} + \dots + y_n + u + \sum_{j=i}^n T_{ij} f_j \\ &= y_{i+1} + \dots + y_n + u + \varphi_i(y_{i+1}, \dots, y_n, u, t) \end{aligned} \quad (4.4)$$

where $\varphi_i(y_{i+1}, \dots, y_n, u, t) = O(y, u)^2$ and is continuous. The functional dependence of φ_i follows from the triangular structure of T and the functional dependence of f .

We now show that a set of linear saturations can be chosen such that the control

$$u = -\sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \dots + \sigma_1(y_1)) \dots) \quad (4.5)$$

globally asymptotically stabilizes the origin of (4.2). We begin by considering the evolution of the state y_n determined by

$$\dot{y}_n = u + \varphi_n(u, t) \quad (4.6)$$

Consider the Lyapunov function $V_n = y_n^2$ where the derivative of V_n is given by

$$\dot{V}_n = -2y_n[\sigma_n(y_n + \sigma_{n-1}(\cdot)) - \varphi_n(u, t)] \quad (4.7)$$

Since $\varphi_n = O(u)^2$, we have, for ϵ_n sufficiently small,

$$|u| \leq \epsilon_n \Rightarrow |\varphi_n(u, t)| \leq C_n \epsilon_n^2$$

for some positive constant C_n which does not depend on ϵ_n . We then choose ϵ_n (and hence δ_n) sufficiently small such that

$$\frac{\delta_n}{2} - C_n \epsilon_n^2 > 0$$

and then, given δ_n , we require that ϵ_{n-1} be chosen such that

$$\epsilon_{n-1} \leq \frac{\delta_n}{4}$$

With these bounds and using (4.7), we can show that $\dot{V}_n < 0$ for all $y_n \notin Q_n = \{y_n : |y_n| \leq \frac{3\delta_n}{4}\}$. In fact, \dot{V} is negative and bounded away from zero since δ_n and ϵ_{n-1} are constants. Consequently, y_n enters Q_n in finite time and remains in Q_n thereafter. Furthermore, after y_n has entered Q_n , the argument of σ_n is bounded by

$$|y_n + \sigma_{n-1}(\cdot)| \leq \frac{3\delta_n}{4} + \epsilon_{n-1} \leq \delta_n$$

Consequently, after y_n enters Q_n , σ_n operates in its linear region. At this point, for δ_n (and hence $|y_n|$) sufficiently small, the dynamics of y_n are of an exponentially stable nonlinear system perturbed by an input of magnitude bounded by ϵ_{n-1} . A converse Lyapunov argument can then be used to show that, for ϵ_{n-1} sufficiently small, there exists some finite time τ_n such that $|y_n(t)| \leq a_n \epsilon_{n-1}$ and $|u(t)| \leq a_n \epsilon_{n-1}$ for some $a_n > 0$ and for all $t > \tau_n$.

Now consider the evolution of the state y_{n-1} . First, since φ_{n-1} is continuous and y_n and u are bounded for all time, y_{n-1} remains bounded for any finite time. After the finite time τ_n , since σ_n is now linear, the evolution of y_{n-1} is given by

$$\begin{aligned} \dot{y}_{n-1} &= y_n + u + \varphi_{n-1}(y_n, u, t) \\ &= -\sigma_{n-1}(y_{n-1} + \sigma_{n-2}(\cdot)) + \varphi_{n-1}(y_n, u, t) \end{aligned} \tag{4.8}$$

Now the same argument as for y_n can be used to show that, for ϵ_{n-1} and ϵ_{n-2} sufficiently small, there exists some finite time τ_{n-1} such that

$$\begin{aligned} |y_{n-1}(t)| &\leq a_{n-1} \epsilon_{n-2} \\ |y_n(t)| &\leq a_{n-1} \epsilon_{n-2} \\ |u(t)| &\leq a_{n-1} \epsilon_{n-2} \end{aligned}$$

for some $a_{n-1} > 0$ and for all $t > \tau_{n-1}$. In fact, this procedure can be continued until, after some finite time τ_1 , we are left with the system

$$\begin{aligned} \dot{y}_1 &= -y_1 + \varphi_1(y_2, \dots, y_n, u, t) \\ \dot{y}_2 &= -y_1 - y_2 + \varphi_2(y_3, \dots, y_n, u, t) \\ &\vdots \\ \dot{y}_n &= -y_1 - \dots - y_n + \varphi_n(u, t) \end{aligned} \tag{4.9}$$

Since $\varphi_i = O(y, u)^2$ is higher order, and the Jacobian linearization is exponentially stable, for ϵ_1 (and hence $\|y(\tau_1)\|$) sufficiently small, we have exponential convergence to the origin. \square

4.3 General Results

4.3.1 Preliminaries

The results of the previous section can be extended for two reasons. First, concerning the φ_i 's, the important property was that $\varphi_i = O(y_{i+1}, \dots, y_n, u)^2$. If this property can be retained while allowing φ_i to depend on other states, the results will still hold. Consequently, we make the following definition:

Definition 4.2 *A function $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ denoted $g(v, w)$ is said to be higher order in w uniformly in v if \exists positive constants ϵ_o, C such that $\forall \epsilon < \epsilon_o$,*

$$\|w\| < \epsilon \Rightarrow |g(v, w)| < C\epsilon^2 \quad \forall v \in \mathbb{R}^n$$

Secondly, regarding the class of saturation functions we are using, we could use the more restrictive class of saturations given in definition 2.2 and take advantage of the fact that outside of some neighborhood of the origin σ is constant. For example, this property would cause the extremal value of σ_{j-1} to serve as a temporary, attractive set point for the state y_j that is held by keeping the states y_{j+1}, \dots, y_n and the input u identically zero. We can exploit this property to ensure that y_{j-1} remains bounded for finite time while allowing even more complicated functional dependence in φ_{j-1} . To prepare for this we repeat definition 2.2:

Definition 4.3 *Given two positive constants δ and ϵ satisfying $0 < \delta \leq \epsilon$ a function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a simple linear saturation if it is a continuous, nondecreasing function satisfying*

1. $\sigma(s) = s$ when $|s| \leq \delta$
2. $s[\sigma(s) - s] \geq 0$ when $|s| \leq \epsilon$
3. $|\sigma(s)| = \epsilon$ when $|s| \geq \epsilon$.

Next recall the “converging input bounded state” results of [Sontag, 1989] extended to allow for certain time-varying dynamics.

Consider a finite-dimensional composite nonlinear system

$$\dot{\eta} = f(\eta, x, t) \quad (4.10)$$

$$\dot{x} = g(x, t) \quad (4.11)$$

where f and g are smooth and $f(0, 0, t) = 0$ and $g(0, t) = 0$ for all $t \geq t_0$. Assume the composite system has the following properties:

Property 4.1 *The equilibrium point $\eta = 0$ of*

$$\dot{\eta} = f(\eta, 0, t)$$

is uniformly globally asymptotically stable.

Property 4.2 *The equilibrium $x = 0$ of (4.11) is uniformly globally asymptotically stable and locally exponentially stable.*

Property 4.3 *For each bounded “control” $x(\cdot)$ on $[t_0, \infty)$ with an exponentially decaying tail (i.e. $\exists \tau \geq t_0, \alpha > 0$ such that $\|x(t)\| \leq e^{-\alpha(t-\tau)}$ for all $t > \tau$) and for each initial state η_0 , the solution of (4.10) with $\eta(t_0) = \eta_0$ exists for all $t \geq t_0$ and is bounded uniformly in t_0 .*

Under these conditions there is the following result:

Theorem 4.2 ([Sontag, 1989]) *If properties 4.1, 4.2 and 4.3 are satisfied then the equilibrium $(0, 0)$ of (4.10), (4.11) is globally asymptotically stable.*

4.3.2 Global Results

We now apply the approach of section 4.2 to single-input non-minimum phase nonlinear systems of the form

$$\begin{aligned} \dot{\bar{\eta}} &= \bar{f}(\bar{\eta}, z, u, t) \\ \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_n &= u \end{aligned} \quad (4.12)$$

where $\bar{\eta} \in \mathbb{R}^p$ and \bar{f} is continuous with $\bar{f}(0, 0, 0, t) = 0$ for all $t \geq t_0$.

The systems of this form that we globally stabilize are specified by two assumptions. First we decompose the state vector $\bar{\eta}$:

$$\bar{\eta} = \begin{bmatrix} \eta \\ x \end{bmatrix}$$

where $\eta \in \mathbb{R}^k$ and $x \in \mathbb{R}^m$. We write

$$\begin{aligned} \dot{\eta} &= f(\eta, x, z, u, t) \\ \dot{x} &= g(\eta, x, z, u, t) \end{aligned} \tag{4.13}$$

The η dynamics will correspond to the typical, zero-input asymptotically stable nonlinear subsystem. The x dynamics will correspond to a system very similar to the *higher order feedforward systems* of the previous section. It is these dynamics that make the composite system non-minimum phase. To be more precise, we make the following assumptions:

Assumption 4.1 *The dynamics of*

$$\dot{\eta} = f(\eta, x, z, u, t) \tag{4.14}$$

with $x(\cdot), z(\cdot), u(\cdot)$ considered as “controls” satisfy property 4.1 and property 4.3.

Assumption 4.2 *The dynamics of x have the form*

$$\dot{x}_i = g_i(\eta, x, z, u, t) = x_{i+1} + h_i(\eta, x, z, u, t) \tag{4.15}$$

for $i = 1, \dots, m$ ($x_{m+1} := z_1$) where

1. *h_i is higher order in x_i, \dots, x_m, z, u uniformly in $\eta, x_1, \dots, x_{i-1}, t$.*

2. *$h_i(\eta, x_1, \dots, x_{i+1}, 0, \dots, 0, t) = h_i^a + h_i^b$ where*

(a) *h_i^a is higher order in x_{i+1} uniformly in η, x_1, \dots, x_i, t*

(b) *$x_i h_i^b \leq 0$ for all $\eta, x_1, \dots, x_{i+1}, t$.*

3. *$\exists \epsilon_0 > 0$ such that*

$$h_i(\eta, x_1, \dots, x_{i-1}, x_i, 0, \dots, 0, t) = 0$$

for all x_i satisfying $|x_i| < \epsilon_0$.

4. For each finite $c > 0$, $\exists \epsilon_0 > 0$ such that for all $\epsilon_i < \epsilon_0$ the dynamics of x_i satisfy property 4.3 with $[c\sigma_i(x_i) + x_{i+1}], x_{i+2}, \dots, x_m, z, u$ as “controls” uniformly in η, x_1, \dots, x_i, t . The function σ_i is a simple saturation with positive constants δ_i, ϵ_i .

Remarks.

1. Feedforward systems are a special class of systems that satisfy this assumption. For other examples, see section 4.4.
2. The most difficult requirement to check in the above assumption is point 4. Sufficient conditions to guarantee point 4 is satisfied are either
 - (a) The dynamics of x_i have the “bounded input bounded state” property uniformly in $\eta, x_1, \dots, x_{i-1}, t$ or
 - (b) \bar{h}_i defined by

$$\bar{h}_i := h_i(\eta, x, z, u, t) - h_i(\eta, x_1, \dots, x_i, -c\sigma_i, 0, \dots, 0, t)$$

can be bounded as

$$|\bar{h}_i| \leq \kappa_1(|\zeta|) + \kappa_2(|\zeta|)|x_i|$$

where $\zeta = (c\sigma_i + x_{i+1}, x_{i+2}, \dots, x_m, z, u)^T$ and $\kappa_i(\cdot)$ are strictly increasing functions such that $\kappa_i(0) = 0$ and for some $k, \epsilon_0 > 0$ $\kappa_i(\epsilon) \leq k\epsilon$ for $0 \leq \epsilon \leq \epsilon_0$.

The latter requirement follows from a simple application of the Bellman-Gronwall lemma.

Theorem 4.3 *If assumptions 4.1 and 4.2 are satisfied, then there exists a control of the form*

$$u = Kz - \sigma_m(T_m(x, z) + \sigma_{m-1}(T_{m-1}(x, z) + \dots + \sigma_1(T_1(x, z)))) \dots$$

which globally asymptotically stabilizes the origin of (4.12) where σ_i satisfies definition 4.2, T_i is a linear function and the gains K are the coefficients of a Hurwitz polynomial.

Proof. See appendix section 4.6.2.

Remarks.

1. T_i is the i th row of invertible matrix that transforms the coordinates (x, z) into (y, z) where, in the transformed coordinates and subject to the control $u = Kz + v$, the new dynamics are

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} B \\ B_c \end{bmatrix} v$$

where A and B are given by (4.3), A_K is the canonical Hurwitz matrix associated with the gains K and B_c is in controllable canonical form.

2. An easy consequence of this theorem is that the system (4.12) can be globally stabilized using a bounded control. This follows by simply redefining the dynamics of x to include the dynamics of z .

4.3.3 Semi-global Results

For the global results of section 4.3.2 we have ruled out unbounded dependence on x_1, \dots, x_{i-1} in the \dot{x}_i equation. However, if we employ the second bounded control strategy of chapter 2 which stabilizes two states at a time rather than just one we can slightly weaken this requirement. In doing so, we replace global stabilizability by semi-global stabilizability. By semi-global stabilizability we mean that, given initial conditions in a compact set X , we can find a control that renders the origin of (4.12) locally asymptotically stable and with basin of attraction that contains X . For a precise definition, see definition 3.1.

We replace assumption 4.2 by the following (recursive) assumption:

Assumption 4.3 *Let $i = m$ and consider the dynamics of x_i :*

1. *if assumption 4.2 holds then let $i = i - 1$.*
2. *otherwise, the dynamics of x_{i-1}, x_i have the form*

$$\begin{aligned} \dot{x}_{i-1} &= x_i + h_{i-1}(\eta, x, z, u, t) \\ \dot{x}_i &= x_{i+1} + h_i(\eta, x, z, u, t) \end{aligned} \tag{4.16}$$

where for $j = i - 1, i$

$$(a) |h_j - \bar{h}_j(\eta, x_1, \dots, x_{i+1}, 0, \dots, 0, t)| \leq (|x_{i-1}| + |x_i| + 1)|\bar{h}_j|$$

where \bar{h}_j is continuous and higher order in $x_{i+2}, \dots, x_m, z, u$ for bounded x_{i+1} uniformly in η, x_1, \dots, x_i, t .

- (b) i. $|h_{i-1}(\eta, x_1, \dots, x_{i+1}, 0, \dots, 0, t)| \leq (|x_i| + 1)|\hat{h}_{i-1}|$
 where \hat{h}_{i-1} is continuous and higher order in x_{i+1} uniformly in η, x_1, \dots, x_i, t .
 ii. $h_i(\eta, x_1, \dots, x_{i+1}, 0, \dots, 0, t)$ depends only on x_{i+1} and is higher order.
- (c) $\exists \epsilon_o > 0$ such that

$$h_j(\eta, x_1, \dots, x_{i-1}, 0, \dots, 0, t) = 0$$

for all x_{i-1} satisfying $|x_{i-1}| < \epsilon_o$.

Let $i = i - 2$.

Theorem 4.4 *If assumptions 4.1 and 4.3 are satisfied, then there exists a family of control laws of the form*

$$u = Kz - \sigma_m(T_m(x, z) + \sigma_{m-j}(T_{m-j}(x, z) + \dots + \sigma_1(T_1(x, z)))) \dots$$

which semi-globally stabilizes the origin of (4.12) where σ_i satisfies definition 4.2, T_i is a linear function and the gains K are the coefficients of a Hurwitz polynomial.

Proof. See appendix section 4.6.3.

Remarks.

1. In the control, $j = 1$ if point 1 of assumption 4.3 holds for $i = m$. Otherwise $j = 2$.
2. The family of semi-globally stabilizing control laws is parameterized by the absolute bounds ϵ_i on the simple linear saturations σ_i .
3. Again, the system (4.12) can be semi-globally stabilized with a bounded control by redefining the dynamics of x to include the dynamics of z as well.

4.4 Examples

4.4.1 Global stabilizability

Example 4.1 Our first example is the system

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta(t)x_2^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \tag{4.17}$$

This system can be globally stabilized, using the methods of [Kokotovic and Sussmann, 1989], [Prally *et al.*, 1991], [Sontag, 1989] for example, in the case where the constant parameter θ is known. In the case where the parameter θ is fixed but unknown, this system can be *locally* stabilized using the adaptive method of [Kanellakopoulos *et al.*, 1991]. On the other hand, our method is able to yield global asymptotic stability in the presence of an unknown parameter θ which can be time-varying as long as a bound on $|\theta(t)|$ is known. Accordingly, assume $|\theta(t)| \leq K$. In the notation of section 4.3.2, we have

$$h_1 = \theta(t)x_2^2$$

and, hence, assumption 4.2 is satisfied. We choose

$$u = -x_2 - x_3 + v$$

where v will be specified to stabilize x_1 . We form the coordinate transformation

$$\begin{aligned} y_1 &= x_1 + x_2 + x_3 \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned}$$

and we let $v = -\sigma(y_1)$ where σ is a *linear saturation* for some ϵ, δ . This yields the closed loop dynamics

$$\begin{aligned} \dot{y}_1 &= -\sigma(y_1) + \theta(t)y_2^2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= -y_2 - y_3 - \sigma(y_1) \end{aligned}$$

We see that the states y_2, y_3 have a bound proportional to ϵ after some finite time τ since σ is bounded by ϵ . Further we see that y_1 is bounded for all finite time since its derivative is bounded for all time. Next, we see that we can pick ϵ small enough such that σ dominates $\theta(t)y_2^2$ for all $t > \tau$. Hence, eventually σ enters and remains in its linear region. Finally, if ϵ is small enough, all the states are close enough to the origin so that the exponential stability of the linear approximation dominates the higher order terms and we have exponential stability.

Example 4.2 This example has been mentioned in recent work as an unsolved problem, both in the adaptive and known parameter context (see [Kokotovic *et al.*, 1991].)

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta(t)x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \tag{4.18}$$

Again we allow θ to be time dependent but we will restrict it such that $|\theta(t)| \leq K$. In the notation of section 4.3.2, we have

$$h_1 = \theta(t)x_3^2$$

and assumption 4.2 is satisfied. The control is constructed in the same manner as in the previous example. We choose

$$u = -x_2 - x_3 + v$$

where v will be specified to stabilize x_1 . We form the coordinate transformation

$$y_1 = x_1 + x_2 + x_3$$

$$y_2 = x_2$$

$$y_3 = x_3$$

Then we let $v = -\sigma(y_1)$ where σ is a *linear saturation* for some ϵ, δ .

Remarks.

1. With θ constant, the above example fails the well-known involutivity condition that is required for the system to be full-state linearizable. However, it is interesting to note that with the output

$$h(x) = x_3 + x_2 + \sigma(x_1 + x_2)$$

the system is relative degree one with zero dynamics given by

$$\dot{x}_1 = x_2 + \theta[x_2 + \sigma(x_1 + x_2)]^2$$

$$\dot{x}_2 = -x_2 - \sigma(x_1 + x_2)$$

In the coordinates $y_1 = x_1 + x_2, y_2 = x_2$ these dynamics are given by

$$\dot{y}_1 = -\sigma(y_1) + \theta[y_2 + \sigma(y_1)]^2$$

$$\dot{y}_2 = -y_2 - \sigma(y_1)$$

But we have shown that this system is globally asymptotically stable if σ is a simple saturation with sufficiently small δ, ϵ .

2. It should also be noted that this system can be globally stabilized with the bounded control

$$u = -\sigma_3(x_3 + \sigma_2(x_2 + x_3 + \sigma_1(x_1 + 2x_2 + x_3)))$$

with each σ_i a simple saturation and ϵ_i, δ_i chosen appropriately.

Example 4.3 We add to the complexity of the previous example by adding nonlinear terms and an extra dimension. This is done to illustrate the kind of nonlinearities that are allowed by assumption 4.2.

$$\begin{aligned}
\dot{x}_1 &= \sin(x_2) - x_1 x_2^2 + x_1 x_3 \cos(u) \\
\dot{x}_2 &= x_3 + \theta(t)x_4^2 + \sin(x_1 t)x_3^2 e^u + u^2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u
\end{aligned} \tag{4.19}$$

In the notation of section 4.3.2, we have

$$\begin{aligned}
h_1 &= (\sin(x_2) - x_2) - x_1 x_2^2 + x_1 x_3 \cos(u) \\
h_2 &= \theta(t)x_4^2 + \sin(x_1 t)x_3^2 e^u + u^2
\end{aligned}$$

Since $|\theta(t)| \leq K$ and $|\sin(x_1 t)| \leq 1$, it is obvious that h_2 is higher order in x_2, x_3, x_4, u uniformly in x_1, t . Likewise h_1 is higher order in x_1, x_2, x_3, x_4, u uniformly in t .

For point 2 of assumption 4.2,

$$\begin{aligned}
h_2(x_1, x_2, x_3, 0, 0, t) &= \sin(x_1 t)x_3^2 \\
h_1(x_1, x_2, 0, 0, 0, t) &= (\sin(x_2) - x_2) - x_1 x_2^2
\end{aligned}$$

For h_2 , $h_2^b \equiv 0$ and h_2^a is higher order in x_3 uniformly in x_1, x_2, t . For h_1 , $h_1^b = -x_1 x_2^2$ and hence $x_1 h_1^b \leq 0$ for all x_1, t . Also h_1^a is higher order in x_2 uniformly in x_1, t .

For point 3 of assumption 4.2,

$$\begin{aligned}
h_2(x_1, x_2, 0, 0, 0, t) &= 0 \\
h_1(x_1, 0, 0, 0, 0, t) &= 0
\end{aligned}$$

And finally, for point 4 of assumption 4.2, both h_1 and h_2 satisfy point (b) of the second remark after assumption 4.2.

We choose

$$u = -x_3 - x_4 + v$$

We form the coordinate transformation

$$\begin{aligned}
y_1 &= x_1 + 2x_2 + 2x_3 + x_4 \\
y_2 &= x_2 + x_3 + x_4 \\
y_3 &= x_3 \\
y_4 &= x_4
\end{aligned}$$

Then $v = -\sigma_2(y_2 + \sigma_1(y_1))$ where σ_i is a *simple linear saturation* for some ϵ_i, δ_i sufficiently small.

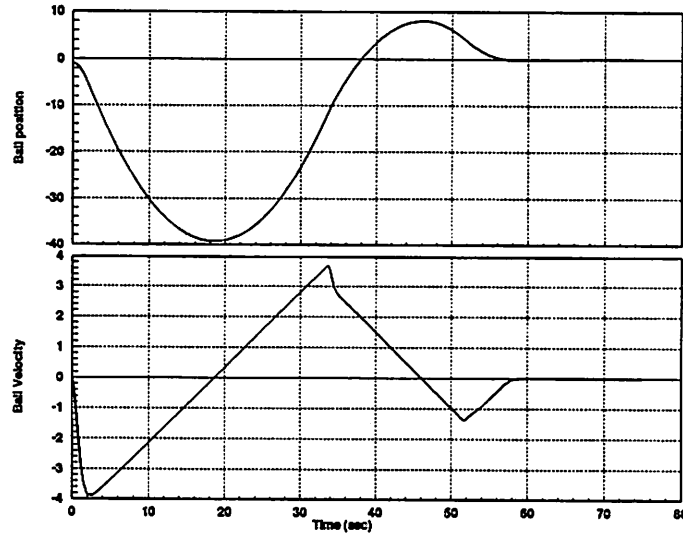


Figure 4.1: Ball position and velocity

4.4.2 Semi-global stabilizability: the “ball and beam” example

Finally, we present a physical example to demonstrate the semi-global result.

Example 4.4 (“ball and beam”) The dynamics for the “ball and beam” were derived in [Hauser *et al.*, 1992]. After a globally invertible nonlinear transformation between torque and angular acceleration we have

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= -G \sin(x_3) + x_1 x_4^2 \\
 \dot{x}_3 &= x_4 \\
 \dot{x}_4 &= u
 \end{aligned} \tag{4.20}$$

where x_1 is the ball position, x_2 is the ball velocity, x_3 is the beam angle, and x_4 is the beam angular velocity. In the notation of section 4.3.3, we have

$$\begin{aligned}
 h_1 &= 0 \\
 h_2 &= G(x_3 - \sin(x_3)) + x_1 x_4^2
 \end{aligned}$$

Assumption 4.3.2 is satisfied for h_1, h_2 . We choose

$$u = -4x_3 - 4x_4 + v$$

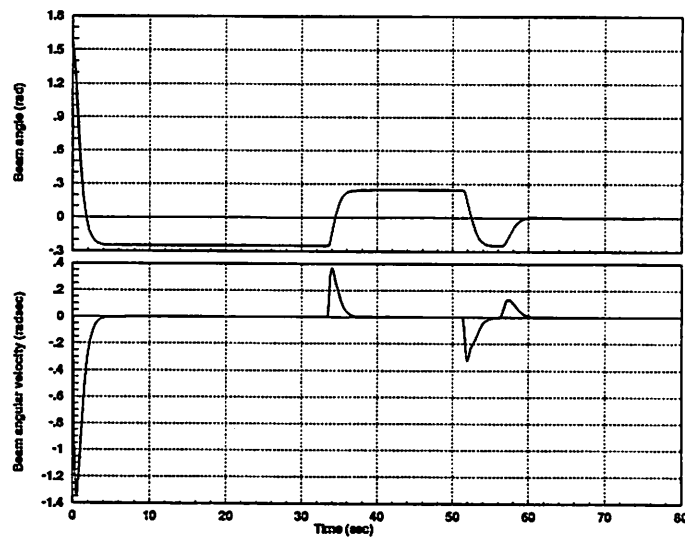


Figure 4.2: Beam angle and angular velocity

We form the coordinate transformation

$$\begin{aligned} y_1 &= -\frac{4}{G}x_1 - \frac{8}{G}x_2 + 5x_3 + x_4 \\ y_2 &= -\frac{4}{G}x_2 + 4x_3 + x_4 \\ y_3 &= x_3 \\ y_4 &= x_4 \end{aligned}$$

Then $v = -\sigma(y_1 + y_2)$ where σ is a *simple linear saturation* for some ϵ, δ . The value of ϵ , which parametrizes the family of semi-globally stabilizing control laws, will be inversely proportional to the bound on the set of initial conditions. To demonstrate the capability of such a control law we present, in figures 4.1 and 4.2, simulation results starting the beam at a 90° angle and the ball at a position below the pivot of the beam. The function σ was chosen to be C^∞ with $\delta = \epsilon = 1$.

4.5 Conclusion

We have proposed a globally (semi-globally) stabilizing control approach for a class of single-input nonlinear systems that is especially useful for systems that cannot be globally full-state linearized. We employ saturation functions to systematically drive the state to the origin. In certain instances our control approach can be used to globally (semi-globally)

stabilize a nonlinear system using a bounded control. An important feature of our approach is that it is robust to unknown (possibly time-varying) parameters as well as unmodeled nonlinear perturbations that satisfy certain general properties.

4.6 Appendix

4.6.1 Geometric Conditions for feedforward forms

We will use this section to discuss geometric conditions for transforming a general single-input, nonlinear system into a feedforward system. Typically, the first thing that is done when transforming a system into a chain of integrators with perturbations is to decompose the nonlinear system into a piece that is feedback linearizable and a perturbation piece. (For example, see references on *pure feedback* systems; [Kanellakopoulos *et al.*, 1991], [Akrif and Blankenship, 1988], and [Marino and Tomei, 1991].) We will proceed along these lines. (Finding condition that are not decomposition dependent is, as far as we know, an open problem.)

We will use the following example to show that the required decomposition is not always the naive decomposition. Also, we will show that the procedure is more delicate than a transformation to a *pure feedback* system.

Example 4.5 *Consider the system*

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1 + (x_1 + x_2 + x_3)^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u\end{aligned}\tag{4.21}$$

It is easy to check that this system is not feedback linearizable. Can this system be transformed into a feedforward system? We propose the following feedback and coordinate transformation:

$$\begin{aligned}z_1 &= x_1 \\ z_2 &= x_2 + x_1 \\ z_3 &= x_3 + x_2 + x_1 \\ z_4 &= x_4 + x_3 + x_2 + x_1 + (x_1 + x_2 + x_3)^2\end{aligned}\tag{4.22}$$

and

$$u = -z_4 - 2z_3z_4 + v\tag{4.23}$$

In the new coordinates we have

$$\begin{aligned}\dot{z}_1 &= z_2 + z_3^2 \\ \dot{z}_2 &= z_3 + z_3^2 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v\end{aligned}\tag{4.24}$$

Hence, we have successfully transformed the system into feedforward form. We have done so by first decomposing the original nonlinear system as

$$f = \begin{bmatrix} x_2 + x_1 \\ x_3 \\ x_4 + (x_1 + x_2 + x_3)^2 \\ 0 \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\tag{4.25}$$

and

$$\Delta f = \begin{bmatrix} (x_1 + x_2 + x_3)^2 \\ 0 \\ -(x_1 + x_2 + x_3)^2 \\ 0 \end{bmatrix} \quad \Delta g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\tag{4.26}$$

The second step was to pick an output function that leads to a linearizing transformation for the unperturbed system (f, g) . We chose $h(x) = x_1$. It needs to be pointed out that this choice is a very delicate one. In fact, any function $h(x_1)$ with $dh \neq 0$ can be used to exactly linearized the unperturbed system. In general, however, these output function choices will not lead to coordinates in which the system is a feedforward system. This problem is not encountered in transforming to *pure feedback* systems because of the triangular feedback structure.

As this discussion indicates, it is not trivial to find a useful decomposition of the vector fields or the right output function on which to base a coordinate transformation. We will not address these limitations here. Instead we will assume that a decomposition and output function have been chosen, and we will give conditions to test whether the system can be transformed into *feedforward* form.

Consider a single-input nonlinear system

$$\dot{x} = F(x, u, t)\tag{4.27}$$

where $x \in \mathbb{R}^n$. We assume the system has been decomposed as

$$\dot{x} = f(x) + g(x)u + \Delta f(x, u, t)\tag{4.28}$$

where the vector fields f, g satisfy conditions for exact feedback linearization. Further we assume that an output function $h(x)$ has been chosen which has relative degree n . We assume that the linearizing feedback defined by

$$u = \alpha(x) + \beta(x)v \quad (4.29)$$

where

$$\begin{aligned} \alpha(x) &= -\frac{1}{L_g L_f^{n-1} h(x)} L_f^n h(x) \\ \beta(x) &= \frac{1}{L_g L_f^{n-1} h(x)} \end{aligned} \quad (4.30)$$

is applied to (4.28). This yields the closed-loop system

$$\dot{x} = \tilde{f}(x) + \tilde{g}(x)v + \Delta \tilde{f}(x, v, t) \quad (4.31)$$

where

$$\begin{aligned} \tilde{f} &= f + g\alpha \\ \tilde{g} &= g\beta \\ \Delta \tilde{f}(x, v, t) &= \Delta f(x, \alpha(x) + \beta(x)v, t) \end{aligned} \quad (4.32)$$

We now define the following codistributions:

$$\Omega^i = \text{span}\{dL_f^{n-1}h, \dots, dL_f^{n-i}h\}$$

for $i = 1, \dots, n$. Further, we define the following distributions

$$G^i = \{v \in \mathbb{R}^n : \langle w^*, v \rangle = 0, \forall w^* \in \Omega^i\} \quad (4.33)$$

(ie., the annihilator of Ω^i .) We then have the following result.

Theorem 4.5 *If*

$$[\Delta \tilde{f}, X] \in G^{i+1} \quad \forall X \in G^i \quad (4.34)$$

then in the coordinates

$$z_1 = h(x), \quad z_2 = L_f h(x), \quad \dots, \quad z_n = L_f^{n-1} h(x) \quad (4.35)$$

the system (4.28) has the feedforward form of (4.2).

Proof. In coordinates

$$\Omega^i = \text{span}\{dz_n, \dots, dz_{n-i}\} \quad (4.36)$$

Hence, in coordinates

$$G^i = \text{span}\left\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{n-i-1}}\right\} \quad (4.37)$$

Therefore, condition (4.34) is equivalent to the condition

$$[\Delta \tilde{f}, \frac{\partial}{\partial z_i}] \in \text{span}\left\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{i-1}}\right\} \quad (4.38)$$

for $i = 1, \dots, n$. This implies that

$$\Delta \tilde{f}(\phi^{-1}(z)) = \begin{pmatrix} f_1(z_2, \dots, z_n, v, t) \\ f_2(z_3, \dots, z_n, v, t) \\ \vdots \\ f_{n-1}(z_n, v, t) \\ f_n(v, t) \end{pmatrix} \quad (4.39)$$

□

4.6.2 Proof of theorem 4.3

The proof is constructive and divides into three major parts. First we develop a convenient linear coordinate change that will simplify our analysis. Then we develop how the conditions of assumption 4.2 translate in the new coordinates. Finally, we show how these conditions allow for a globally stabilizing control law.

Coordinate change

Our first step in developing our coordinate change is to choose the input as $u = Kz + v$ where the gains K are the coefficients of a Hurwitz polynomial. We then have

$$\begin{aligned} \dot{\eta} &= f(\eta, x, z, u, t) \\ \dot{x} &= g(\eta, x, z, u, t) \\ \dot{z} &= Az + Bv \end{aligned} \quad (4.40)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ k_1 & \dots & \dots & \dots & k_n \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (4.41)$$

The additional control v will be bounded and chosen to stabilize the x states. We proceed to make a linear coordinate change to achieve a convenient form for our approach. We choose

$$\begin{aligned}\tilde{\eta} &= \eta \\ \tilde{y} &= T_1 x + T_2 z \\ \tilde{z} &= z\end{aligned}\tag{4.42}$$

where T_1 and T_2 are constructed below. For purposes of compact notation, we employ the following selection operators:

$$\begin{aligned}S_i &: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n \\ S_i(w) &= [w_i, \dots, w_{i+n-1}]^T\end{aligned}$$

and

$$\begin{aligned}P_i &: \mathbb{R}^{m+n} \rightarrow \mathbb{R} \\ P_i(w) &= w_i\end{aligned}$$

S_i is defined for $i = 1, \dots, m$ and P_i is defined for $i = 1, \dots, m+n$. Both operate on the concatenation of x and z :

$$w = [x^T, z^T]^T$$

We choose \tilde{y} to have the following recursive construction:

$$\begin{aligned}\tilde{y}_m &= -K S_m(w) + P_{m+n}(w) \\ \tilde{y}_{m-1} &= \tilde{y}_m - K S_{m-1}(w) + P_{m+n-1}(w) \\ &\vdots \\ \tilde{y}_1 &= \tilde{y}_2 - K S_1(w) + P_{n+1}(w)\end{aligned}$$

It is apparent from this construction that T_1 has the form

$$T_1 = \begin{bmatrix} -k_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & -k_1 \end{bmatrix} \quad T_1^{-1} = \begin{bmatrix} -1/k_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & -1/k_1 \end{bmatrix}\tag{4.43}$$

(T_1 is invertible because $k_1 < 0$ for A to be Hurwitz.) In the new coordinates, the dynamics of (4.40) are given by

$$\begin{aligned}\dot{\tilde{\eta}} &= \bar{f}(\tilde{\eta}, T_1^{-1}(\tilde{y} - T_2 \tilde{z}), \tilde{z}, u, t) \\ \dot{\tilde{y}} &= \bar{g}(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \\ \dot{\tilde{z}} &= A \tilde{z} + B v\end{aligned}\tag{4.44}$$

It is obvious that the dynamics of $\tilde{\eta}$ satisfy assumption 4.1 with $\tilde{y}(\cdot), \tilde{z}(\cdot), u(\cdot)$ as “controls”.

For the dynamics of \tilde{y} we have

$$\begin{aligned}\dot{\tilde{y}}_i &= \tilde{g}_i(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \\ &= \tilde{y}_{i+1} + \dots + \tilde{y}_m + v + \sum_{j=i}^m T_{1,i,j} h_j(\tilde{\eta}, T_1^{-1}(\tilde{y} - T_2 \tilde{z}), \tilde{z}, u, t) \\ &= \tilde{y}_{i+1} + \dots + \tilde{y}_m + v + \tilde{h}_i(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t)\end{aligned}\quad (4.45)$$

We proceed to determine the relevant properties of \tilde{h}_i .

Properties of Perturbation Terms

Define $\tilde{y}_{m+1} \equiv -k_1 \tilde{z}_1$. The following properties of \tilde{h}_i follow from assumption 4.2:

1. \tilde{h}_i is higher order in $\tilde{y}_i, \dots, \tilde{y}_m, \tilde{z}, u$ uniformly in $\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i-1}, t$
2. $\tilde{h}_i(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0, t) = \tilde{h}_i^a + \tilde{h}_i^b$ where
 - (a) \tilde{h}_i^a is higher order in \tilde{y}_{i+1} uniformly in $\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_i, t$
 - (b) for some $\epsilon_0, d > 0$ and $\forall \epsilon < \epsilon_0$, $\tilde{y}_i \tilde{h}_i^b \leq 0$ for all $\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i-1}$ and $|\tilde{y}_{i+1}| < \epsilon$ and $|\tilde{y}_i| > d\epsilon$.
3. for some $\epsilon_0 > 0$, $\tilde{h}_i(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i-1}, \tilde{y}_i, 0, \dots, 0, t) = 0$ for all \tilde{y}_i such that $|\tilde{y}_i| < \epsilon_0$.
4. $\exists \epsilon_0 > 0$ such that for all $\epsilon_i < \epsilon_0$ the dynamics of \tilde{y}_i satisfy property 4.3 with $[\sigma_i(\tilde{y}_m) + \tilde{y}_{i+1}], \tilde{y}_{i+2}, \dots, \tilde{y}_m, \tilde{z}, u$ as “controls” uniformly in $\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_i, t$. The function σ_i is a *simple linear saturation* with positive constants δ_i, ϵ_i .

Consider point 1. For some $\epsilon_0 > 0$ and any $\epsilon < \epsilon_0$ assume that $|\tilde{y}_j| < \epsilon$ for $j = i, \dots, m$ and $||\tilde{z}|| < \epsilon, |u| < \epsilon$. From T_1^{-1} this implies, for some constant D , $|x_j| < D\epsilon$ for $j = i, \dots, m$. Further $||z|| < \epsilon$. By assumption 4.2.1 this implies, for some constants C_j , $|h_j| < C_j \epsilon^2, j = i, \dots, m$. Finally, from (4.45), for some constant \tilde{C} , $|\tilde{h}_i| < \tilde{C} \epsilon^2$.

Consider point 2. Decompose $\tilde{h}_i(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0, t)$ as $\tilde{h}_i = \tilde{h}_i^a + \tilde{h}_i^b$ where

$$\begin{aligned}\tilde{h}_i^a &= T_{1,i,i} h_i^a + \sum_{j=i+1}^m T_{1,i,j} h_j \\ \tilde{h}_i^b &= T_{1,i,i} h_i^b\end{aligned}$$

where h_i^a and h_i^b are defined by assumption 4.2.2. Consider point 2a above. Assume that $|\tilde{y}_{i+1}| < \epsilon$ and $\tilde{y}_j = 0$ for $j = i+2, \dots, m$ and $\tilde{z} = 0, u = 0$. From T_1^{-1} this implies $|x_{i+1}| < D\epsilon, x_j = 0$ for $j = i+2, \dots, m$ and $z = 0$. By assumption 4.2.2a this implies

$|h_i^a| < C\epsilon^2$. Further, assumption 4.2.3 implies $h_j = 0$ for $j = i + 1, \dots, m$. Hence, for some constant \tilde{C} , $|\tilde{h}_i^a(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0, t)| < \tilde{C}\epsilon^2$. Consider point 2b above. Again assume $|\tilde{y}_{i+1}| < \epsilon$ and $\tilde{y}_j = 0$ for $j = i + 2, \dots, m$ and $\tilde{z} = 0, u = 0$. It follows that

$$\tilde{y}_i \tilde{h}_i^b(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0, t) = (T_{1,ii}x_i + T_{1,i,i+1}x_{i+1})T_{1,ii}h_i^b$$

From assumption 4.2.2.b it follows that $\tilde{y}_i \tilde{h}_i^b \leq 0$ for $|x_i| > |\frac{T_{1,i,i+1}}{T_{1,ii}}x_{i+1}|$ and all $\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i-1}$.

Consider point 3. For some $\epsilon_0 > 0$ assume that $|\tilde{y}_i| < \epsilon_0$. Further, assume $\tilde{y}_j = 0$ for $j = i + 1, \dots, m$ and $\tilde{z} = 0, u = 0$. From T_1^{-1} this implies, for some constant D , $|x_i| < D\epsilon_0$, $y_j = 0$ for $j = i + 1, \dots, m$ and $z = 0$. By assumption 4.2.3, for ϵ_0 small enough, $h_j = 0$ for $j = i, \dots, m$. Finally, from (4.45), $\tilde{h}_i(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i-1}, \tilde{y}_i, 0, \dots, 0, t) = 0$.

Consider point 4. Let $[\sigma_i(\tilde{y}_i) + \tilde{y}_{i+1}], \tilde{y}_{i+2}, \dots, \tilde{y}_m, \tilde{z}, u$ converge to zero with an exponential tail. From T_1^{-1} , we have $[-\frac{1}{k_1}\sigma(\tilde{y}_i) + x_{i+1}], x_{i+2}, \dots, x_m, z, u$ converge to zero with an exponential tail. Note that, for any bounded x_{i+1}, \dots, x_m, z and sufficiently large x_i , $\sigma(\tilde{y}_i) = \sigma(x_i)$. Since we are trying to establish the boundedness of x_i we can, without loss of generality, assume $|x_i|$ is sufficiently large. Then we have that $[-\frac{1}{k_1}\sigma(x_i) + x_{i+1}], x_{i+2}, \dots, x_m, z, u$ converge to zero with an exponential tail. Hence, from assumption 4.2.4, x_i is bounded. Hence, by T_1 , \tilde{y}_i is bounded.

Stability Analysis

Throughout our analysis we will rely on lemmas taken from [Hahn, 1967] which apply to the finite-dimensional unperturbed differential equation

$$\dot{x} = f(x, t) \tag{4.46}$$

with f satisfying certain smoothness assumptions and such that $f(0, t) = 0$ for $t \geq t_0$, and the perturbed differential equation

$$\dot{x} = f(x, t) + g(x, t) \tag{4.47}$$

Lemma 4.1 *If the equilibrium of (4.46) is exponentially stable and if $g(x, t)$ satisfies an estimate $g(x, t) = o(\|x\|)$ then the equilibrium of (4.47) is also exponentially stable, in fact with the same exponent.*

Lemma 4.2 *Let the equilibrium of (4.46) be (locally) exponentially stable. Then (for sufficiently small $\|x\|$) there exists a Lyapunov function $V(x, t)$ which satisfies estimates of the*

form

$$\begin{aligned} a_1 \|x\|^2 &\leq V(x, t) \leq a_2 \|x\|^2 \\ \dot{V} &\leq -a_3 \|x\|^2 \\ \left\| \frac{\partial V}{\partial x} \right\| &\leq a_4 \|x\| \end{aligned} \tag{4.48}$$

for certain positive constants a_1, a_2, a_3, a_4 .

Lemma 4.3 *If the equilibrium of (4.46) is exponentially stable and if $g(x, t)$ satisfies an estimate $\|g(x, t)\| \leq \epsilon$ for ϵ sufficiently small then for sufficiently small $\|x(t_0)\|$, $\|x(t)\|$ satisfies an estimate of the form $\|x(t)\| \leq a\epsilon$ for all $t > T$ for some $T \geq t_0$ and for some positive constant a which depends on a_1, a_2, a_3, a_4 .*

We now propose the following for the remaining control v :

$$v = -\sigma_m(\tilde{y}_m + \sigma_{m-1}(\tilde{y}_{m-1} + \cdots + \sigma_1(\tilde{y}_1))) \cdots \tag{4.49}$$

where σ_i is a *simple linear saturation* for δ_i, ϵ_i and we show that the values δ_i, ϵ_i can be chosen to yield global asymptotic stability.

The first thing to observe is that, with all of the saturating limits removed, the dynamics of (\tilde{y}, \tilde{z}) are of an asymptotically stable linear system perturbed by higher order terms. Hence, from lemma 4.1, for the system with the saturating limits removed, there is an open neighborhood $U \subset \mathbb{R}^{m+n}$ of 0 such that if $(\tilde{y}_0, \tilde{z}_0) \in U$ then the equilibrium $(\tilde{y}, \tilde{z}) = (0, 0)$ is exponentially stable. It follows that, if we can show from any initial condition the states (\tilde{y}, \tilde{z}) enter and remain in a small neighborhood $V \subset U$ in which the functions σ_i for $i = 1, \dots, m$ operate in their linear region, lemma 4.1 allows us to conclude global asymptotic stability and local exponential stability for the dynamics of \tilde{y}, \tilde{z} with the saturating limits included. Finally, by assumption 4.1 and theorem 4.2, the complete composite system has $(0, 0, 0)$ as a G.A.S. equilibrium.

We set out to establish that all of the states (\tilde{y}, \tilde{z}) can be steered to the set V in finite time by judicious choice of δ_i, ϵ_i .

Observe that the dynamics of \tilde{z} are given by an asymptotically stable linear system perturbed by a small disturbance (with maximum absolute amplitude of ϵ_m). Here the estimates of lemma 4.2 apply globally. Hence lemma 4.3 applies for any initial condition $\tilde{z}(0)$. This leads to a bound $|\tilde{z}(t)| \leq a\epsilon_m$ for all $t > T_{m+1}$ for some $T_{m+1} > t_0$. Observe that $|u(t)| \leq a_K \epsilon_m$ for all $t > T_{m+1}$ where a_K depends on a and the feedback gains K . We define $a_m = \max\{a, a_K, k_1 a\}$.

With this bound on \tilde{z} we define $\tilde{y}_{m+1} = -k_1 \tilde{z}_1$ and proceed by induction showing that given ϵ_{i-1} sufficiently small, $\exists \epsilon_i$ sufficiently small such that if

$$\begin{aligned} |\tilde{y}_{j+1}(t)| &\leq a_i \epsilon_i \quad j = i, \dots, m \\ ||\tilde{z}(t)|| &\leq a_i \epsilon_i \\ |u(t)| &\leq a_i \epsilon_i \end{aligned}$$

for all $t > T_{i+1}$, then

$$\begin{aligned} |\tilde{y}_j(t)| &\leq a_{i-1} \epsilon_{i-1} \quad j = i, \dots, m \\ ||\tilde{z}(t)|| &\leq a_{i-1} \epsilon_{i-1} \\ |u(t)| &\leq a_{i-1} \epsilon_{i-1} \end{aligned}$$

for all $t > T_i > T_{i+1}$.

Assume that ϵ_i is chosen sufficiently small such that σ_{i+1} operates in its linear region for all $t > T_{i+1}$. (σ_{m+1} can be considered a globally linear function.) Consider the dynamics for $\tilde{y}_i, \dots, \tilde{y}_m, \tilde{z}$ after time T_{i+1} :

$$\begin{aligned} \dot{\tilde{y}}_i &= -\sigma_i + \tilde{h}_i(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \\ \dot{\tilde{y}}_{i+1} &= -\tilde{y}_{i+1} - \sigma_i + \tilde{h}_i(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \\ \dot{\tilde{y}}_{i+2} &= -\tilde{y}_{i+2} - \tilde{y}_{i+1} - \sigma_i + \tilde{h}_i(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \\ &\vdots \\ \dot{\tilde{z}} &= A\tilde{z} - B(\tilde{y}_m + \dots + \tilde{y}_{i+1} + \sigma_i) \end{aligned} \tag{4.50}$$

We show that, for ϵ_i sufficiently small \tilde{y}_i becomes small and after some finite time $T_i > T_{i+1}$ remains in a region such that σ_i is linear. Consider \tilde{y}_i such that $|\tilde{y}_i| > \epsilon_i + \epsilon_{i-1}$ and make the coordinate change

$$\begin{aligned} \bar{y}_{i+1} &= \tilde{y}_{i+1} + \sigma_i \\ \bar{y}_j &= \tilde{y}_j \quad j = i+2, \dots, m \\ \bar{z} &= \tilde{z} \end{aligned}$$

Then the dynamics of $\bar{y}_i, \bar{y}_{i+1}, \dots, \bar{y}_m, \bar{z}$ are

$$\begin{aligned} \dot{\bar{y}}_i &= -\sigma_m(\bar{y}_i) + \tilde{h}_i(\tilde{\eta}, \bar{y}, \bar{z}, u, t) \\ \dot{\bar{y}}_{i+1} &= -\bar{y}_{i+1} + \tilde{h}_{i+1}(\tilde{\eta}, \bar{y}, \bar{z}, u, t) \\ \dot{\bar{y}}_{i+2} &= -\bar{y}_{i+2} - \bar{y}_{i+1} + \tilde{h}_{i+2}(\tilde{\eta}, \bar{y}, \bar{z}, u, t) \\ &\vdots \\ \dot{\bar{z}} &= A\bar{z} - B(\bar{y}_m + \dots + \bar{y}_{i+1}) \end{aligned} \tag{4.51}$$

since, when $|\tilde{y}_i| > \epsilon_i + \epsilon_{i-1}$,

$$\begin{aligned}\dot{\sigma}_i &= 0 \\ \sigma_m(\tilde{y}_i) &= \sigma_i(\tilde{y}_i + \sigma_{i-1}(\cdot))\end{aligned}$$

Observe that, for the dynamics of $\tilde{y}_{i+1}, \dots, \tilde{y}_m, \bar{z}$, lemma 4.1 applies so that, for ϵ_i sufficiently small, $\tilde{y}_{i+1}, \dots, \tilde{y}_m, \bar{z}$ converge exponentially toward zero. Point 3 above is crucial for the perturbations \tilde{h}_j for $j = i+1, \dots, m$ to remain higher order in the $\tilde{y}_{i+1}, \dots, \tilde{y}_m, \bar{z}$ coordinates. Note that, since the control is $u = K\bar{z} - \tilde{y}_m - \dots - \tilde{y}_{i+1}$, u also converges exponentially toward zero. We assert that, for small enough ϵ_i , and with these “controls” set to zero, the set $M = \{\tilde{y}_i : |\tilde{y}_i| \leq \epsilon_i + \epsilon_{i-1}\}$ is attractive. Further, since the dynamics of \tilde{y}_i satisfy property 2, by theorem 4.2, at some finite time $T > T_{i+1}$, \tilde{y}_i will enter M .

Consider the dynamics of \tilde{y}_i with the “controls” $\bar{z}, \tilde{y}_m, \dots, \tilde{y}_{i+1}, u$ set to zero:

$$\begin{aligned}\dot{\tilde{y}}_i &= -\sigma_i(\tilde{y}_i) + \tilde{h}_i(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_i, -\sigma_i, 0, \dots, 0, t) \\ &= -\sigma_i(\tilde{y}_i) + \tilde{h}_i^a + \tilde{h}_i^b\end{aligned}\tag{4.52}$$

With regard to point 2 above we have $|\tilde{y}_{i+1}| = \epsilon_i$. Consider the time derivative of the Lyapunov function $V_i = \tilde{y}_i^2$ along the trajectories of (4.52):

$$\begin{aligned}\dot{V}_i &= 2\tilde{y}_i[-\sigma_i(\tilde{y}_i) + \tilde{h}_i^a + \tilde{h}_i^b] \\ &\leq 2|\tilde{y}_i|[-\epsilon_i + C_1\epsilon_i^2 + C_2\epsilon_i^2]\end{aligned}$$

(Note that the term $\tilde{y}_i\tilde{h}_i^b \leq 0$ for $|\tilde{y}_m| > d\epsilon_m$ from point 2, and is uniformly higher order for $|\tilde{y}_m| \leq d\epsilon_i$ from point 1.) It follows that we must choose ϵ_i such that

$$\epsilon_i - (C_1 + C_2)\epsilon_i^2 < 0$$

to insure that the set M is attractive with the controls $\tilde{y}_{i+1}, \dots, \tilde{y}_m, \bar{z}, u$ set to zero.

We show now that for ϵ_{i-1} sufficiently small, \tilde{y}_i enters and stays in a region where $\sigma_i(\cdot)$ is linear. (Note that $\epsilon_0 \equiv 0$.) Again consider the dynamics of \tilde{y}_i beginning at the time when \tilde{y}_i enters M :

$$\dot{\tilde{y}}_i = -\sigma_m(\tilde{y}_i + \sigma_{i-1}) + \tilde{h}_i(\tilde{\eta}, \tilde{y}, \bar{z}, u, t)\tag{4.53}$$

We take the derivative of the Lyapunov function $V_i = \tilde{y}_i^2$ along the trajectories of (4.53) and employ point 1 from above:

$$\begin{aligned}\dot{V}_i &= 2\tilde{y}_i[-\sigma_i(\tilde{y}_i + \sigma_{i-1}) + \sigma_i(\tilde{y}_i) - \sigma_i(\tilde{y}_i) + \tilde{h}_i] \\ &\leq 2|\tilde{y}_i|[-|\sigma_i(\tilde{y}_i)| + \epsilon_{i-1} + \tilde{C}\epsilon_i^2]\end{aligned}$$

First note that if $\epsilon_i - \epsilon_{i-1} - C_3\epsilon_i^2 > 0$ then the set M is invariant. Second, observe that given δ_i, ϵ_i , if ϵ_{i-1} satisfies

$$\epsilon_{i-1} < \frac{\delta_i - C_3\epsilon_i^2}{2}$$

then \tilde{y}_i will enter the set $Q_i = \{\tilde{y}_i : |\tilde{y}_i| \leq \frac{\delta_i + C_3\epsilon_i^2}{2}\}$ in finite time and remain in Q_i thereafter. With $\tilde{y}_i \in Q_i$ the argument of σ_i is bounded by

$$\begin{aligned} |\tilde{y}_i + \sigma_{i-1}| &\leq |\tilde{y}_i| + |\sigma_{i-1}| \\ &\leq \frac{\delta_i + C_3\epsilon_i^2}{2} + \frac{\delta_i - C_3\epsilon_i^2}{2} \\ &\leq \delta_i \end{aligned}$$

Hence $\sigma_i(\cdot)$ enters in finite time and thereafter remains in its linear region.

Note that after this finite time the dynamics of $(\tilde{y}_i, \dots, \tilde{y}_m, \tilde{z})$ are of an asymptotically stable linear system perturbed by higher order terms as well as a perturbation of maximum amplitude ϵ_{i-1} . Combining lemma 4.1 and lemma 4.3, if ϵ_i is sufficiently small (to start in a small neighborhood of the origin) then we can establish bounds $|\tilde{y}_j| < a_{i-1}\epsilon_{i-1}$ for $j = i, \dots, m$ and $|\tilde{z}| < a_{i-1}\epsilon_{i-1}$ and $|u| < a_{i-1}\epsilon_{i-1}$ for all $t > T_i > T_{i+1}$. \square

4.6.3 Proof of theorem 4.4

The proof is again constructive. We employ the same convenient coordinate change as the the case of global stabilization. We will develop how the conditions of assumption 4.3 translate in these new coordinates. Most of the work then lies in showing how these conditions allow for a semi-globally stabilizing class of control laws.

Coordinate change

As in the case for global stabilization we begin by choosing the input as $u = Kz + v$ where the gains K are the coefficients of a Hurwitz polynomial. In addition, we add the condition that the gains K are such that $\text{Re } \sigma(A) \leq -1$ where A is defined in (4.41). The coordinate change then proceeds in the same way as in the global case (see section 4.6.2.) Once again we have

$$\begin{aligned} \dot{\tilde{y}}_i &= \tilde{g}_i(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \\ &= \tilde{y}_{i+1} + \dots + \tilde{y}_m + v + \sum_{j=i}^m T_{1,j} h_j(\tilde{\eta}, T_1^{-1}(\tilde{y} - T_2 \tilde{z}), \tilde{z}, u, t) \\ &= \tilde{y}_{i+1} + \dots + \tilde{y}_m + v + \tilde{h}_i(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \end{aligned} \tag{4.54}$$

We proceed to determine the relevant properties of \tilde{h}_i .

Properties of Perturbation Terms

Define $\tilde{y}_{m+1} \equiv -k_1 \tilde{z}_1$. Next we establish the properties of \tilde{h}_i that follow from assumption 4.3. First observe that if assumption 4.3.1 applies to \tilde{h}_i then the four points established in section 4.6.2 apply. Otherwise we establish the following properties for \tilde{h}_i and \tilde{h}_{i-1} that follow from assumption 4.3.2:

for $j = i - 1, i$

1.

$$|\tilde{h}_j - \tilde{h}_j(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0, t)| \leq (|\tilde{y}_{i-1}| + |\tilde{y}_i| + 1)|\tilde{h}_j|$$

where the function \tilde{h}_j is bounded for bounded $\tilde{y}_{i+1}, \dots, \tilde{y}_m, z, u$ and higher order in $\tilde{y}_{i+2}, \dots, \tilde{y}_m, z, u$ for bounded \tilde{y}_{i+1} uniformly in $\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_i, t$.

2. (a)

$$|\tilde{h}_{i-1}(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0, t)| \leq (|\tilde{y}_i| + 1)|\tilde{h}_{i-1}|$$

where \tilde{h}_{i-1} is higher order in \tilde{y}_{i+1} uniformly in $\tilde{y}_1, \dots, \tilde{y}_i, t$ and bounded for bounded \tilde{y}_{i+1} .

(b) $\tilde{h}_i(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0, t)$ depends only on \tilde{y}_{i+1} . Further it is higher order in \tilde{y}_{i+1} and is bounded for bounded \tilde{y}_{i+1} .

3. For some $\epsilon_0 > 0$, $\tilde{h}_j(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i-1}, 0, \dots, 0, t) = 0$ for $|\tilde{y}_{i-1}| < \epsilon_0$.

Point 1 follows from (4.54) by apply assumptions 4.3.2.a and 4.2.1 to the appropriate terms in the summation that defines \tilde{h}_j and then using T_1 to return to the \tilde{y} coordinates.

Consider point 2a. Assume $\tilde{y}_k = 0$ for $k = i + 2, \dots, m$ and $\tilde{z} = 0$ and $u = 0$. From T_1^{-1} this implies $x_k = 0$ for $k = i + 2, \dots, m$ and $z = 0$. By assumption 4.3.2.b this implies $|\tilde{h}_{i-1}| < (|\tilde{x}_i| + 1)|\hat{h}_{i-1}|$ where \hat{h}_{i-1} is higher order in x_{i+1} uniformly in η, x_1, \dots, x_i, t and is bounded for bounded x_{i+1} . Also \tilde{h}_i is higher order in x_{i+1} uniformly in η, x_1, \dots, x_i, t and bounded for bounded x_{i+1} . Further, assumptions 4.3.2.c and 4.2.3 imply $\tilde{h}_k = 0$ for $k = i + 1, \dots, m$. Hence, from (4.54) and T_1 , $|\tilde{h}_{i-1}(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0, t)| < (|\tilde{y}_i| + 1)|\tilde{h}_{i-1}|$ where \tilde{h}_{i-1} is higher order in \tilde{y}_{i+1} uniformly in $\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_i, t$. Consider point 2b. Since $\tilde{h}_k = 0$ for $k = i + 1, \dots, m$, $\tilde{h}_i = T_{1,i} \tilde{h}_i$. Now \tilde{h}_i depends only on x_{i+1} and is higher order in x_{i+1} and bounded for bounded x_{i+1} . Finally, since $\tilde{y}_j = 0$ for $j = i + 2, \dots, m$, it follows that $\tilde{y}_{i+1} = T_{1,i} x_{i+1}$. Hence, $\tilde{h}_i(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0, t)$ depends only on \tilde{y}_{i+1} and is higher order in \tilde{y}_{i+1} and bounded for bounded \tilde{y}_{i+1} .

Consider point 3. For some $\tilde{\epsilon}_0 > 0$, assume that $|\tilde{y}_{i-1}| < \tilde{\epsilon}_0$ for some $\tilde{\epsilon}_0 > 0$. Further, assume $\tilde{y}_j = 0$ for $j = i, \dots, m$ and $\tilde{z} = 0$ and $u = 0$. From T_1^1 this implies $|x_{i-1}| < \tilde{\epsilon}_0/c$ for some constant $c > 0$ and $y_j = 0$ for $j = i, \dots, m$ and $z = 0$. Define $\tilde{\epsilon}_0 \equiv c\epsilon_0$. By assumptions 4.3.2.c and 4.2.3 this implies $h_j = 0$ for $j = i-1, \dots, m$. Finally, from (4.54), $\tilde{h}_j(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i-2}, \tilde{y}_{i-1}, 0, \dots, 0, t) = 0$ for $j = i-1, i$.

Stability Analysis

Again we will rely on lemmas taken from [Hahn, 1967] which are stated in section 4.6.2. In addition we will use the following lemma in our proof.

Lemma 4.4 *Consider the n -dimensional nonlinear system*

$$\dot{x} = f(x, t)$$

where $|f_i(x, t)| \leq q_i(t) + \sum_{j=1}^n a_{ij}|x_j|$ for all $t \geq t_0$. Define the constant matrix \bar{A} by $\bar{A}_{ij} = a_{ij}$. Consider the vectors

$$\begin{aligned}\bar{x}(t) &= [|x_1(t)|, \dots, |x_n(t)|]^T \\ \bar{q}(t) &= [|q_1(t)|, \dots, |q_n(t)|]^T.\end{aligned}$$

Then $\bar{x}(t)$ is bounded as

$$\bar{x}(t) \leq e^{\bar{A}(t-t_0)} \bar{x}(t_0) + \int_{t_0}^t e^{\bar{A}(t-\tau)} \bar{q}(\tau) d\tau$$

We begin by formulating a bounded control v to stabilize \tilde{y} using the following algorithm:

1. let $k = m$ and let $v = -\sigma_m$ where σ_m is a *simple linear saturation* for ϵ_m, δ_m to be specified.
2. if assumption 4.3.1 applies to \tilde{h}_k then
 - (a) let the argument of σ_k be $\tilde{y}_k + \sigma_{k-1}(\cdot)$ where σ_{k-1} is a *simple linear saturation* for $\epsilon_{k-1}, \delta_{k-1}$ to be specified.
 - (b) let $k=k-1$
 - (c) return to step 2.
3. if assumption 4.3.2 applies to \tilde{h}_k (and hence \tilde{h}_{k-1}) then

- (a) let the argument of σ_k be $\tilde{y}_k + \tilde{y}_{k-1} + \sigma_{k-2}(\cdot)$ where σ_{k-2} is a *simple linear saturation* for $\epsilon_{k-2}, \delta_{k-2}$ to be specified.
- (b) let $k=k-2$
- (c) return to step 2.

We show that, given initial conditions in some bounded set X , the values ϵ_i, δ_i can be chosen to yield asymptotic stability.

The proof proceeds in the same manner as the proof for global stabilizability. Again we define $\tilde{y}_{m+1} \equiv -k_1 \tilde{z}_1$. In addition, we define k to be the largest index such that assumption 4.3.2 applies to \tilde{h}_i rather than assumption 4.3.1. It follows then from the proof of the global result that, $\exists \epsilon_j$ for $j = k, \dots, m$ sufficiently small such that σ_j for $j = k+1, \dots, m$ operate in their linear region for all $t > T_{k+1}$ (σ_{m+1} can be considered as a globally linear function.)

We now show that $\exists \epsilon_k$ sufficiently small such that for ϵ_{k-2} sufficiently small σ_k operates in its linear region for all $t > T_k > T_{k+1}$. (For $k = 2$, observe that $\epsilon_{k-2} \equiv 0$.)

First, we know that u, \tilde{z} and \tilde{y}_i for $i = k+1, \dots, m$ are bounded for all $t > 0$. Then since $\tilde{h}_{k-1}, \tilde{h}_k$ are globally Lipschitz in $\tilde{y}_{k-1}, \tilde{y}_k$ for bounded $u, \tilde{z}, \tilde{y}_i$ for $i = k+1, \dots, m$, $\exists R$ which depends on the initial conditions of $\tilde{y}_i(t_0)$ for $i = k-1, \dots, m$ and $\tilde{z}(t_0)$ and on ϵ_i for $i = k+1, \dots, m$ such that for $j = k-1, k$

$$|\tilde{y}_j(T_{k+1})| \leq R.$$

Consider the dynamics for $\tilde{y}_{k-1}, \dots, \tilde{y}_m, \tilde{z}$ for $t > T_{k+1}$:

$$\begin{aligned}
 \dot{\tilde{y}}_{k-1} &= \tilde{y}_k - \sigma_k(\tilde{y}_{k-1} + \tilde{y}_k + \sigma_{k-2}) + \tilde{h}_{k-1}(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \\
 \dot{\tilde{y}}_k &= -\sigma_k(\tilde{y}_{k-1} + \tilde{y}_k + \sigma_{k-2}) + \tilde{h}_k(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \\
 \dot{\tilde{y}}_{k+1} &= -\tilde{y}_{k+1} - \sigma_k + \tilde{h}_{k+1}(\tilde{\eta}, \tilde{y}, \tilde{z}, u, t) \\
 &\vdots \\
 \dot{\tilde{z}} &= A\tilde{z} - B(\tilde{y}_m + \dots + \tilde{y}_{k+1} + \sigma_k)
 \end{aligned} \tag{4.55}$$

Again from the proof of global stabilizability we know that, for all $t > T_{k+1}$,

$$\begin{aligned}
 |\tilde{y}_i(t)| &\leq a_{k+1}\epsilon_{k+1} \quad i = k+1, \dots, m \\
 \|\tilde{z}(t)\| &\leq a_{k+1}\epsilon_{k+1} \\
 |u(t)| &\leq a_{k+1}\epsilon_{k+1}.
 \end{aligned}$$

Then, since $\tilde{h}_{k-1}, \tilde{h}_k$ satisfy points 1 and 2 of section 4.6.3, the dynamics of $\tilde{y}_{k-1}, \tilde{y}_k$ are of a 2-dimensional nonlinear system satisfying the conditions of lemma 4.4 with

$$\bar{A} = \begin{bmatrix} C\epsilon_{k+1}^2 & 1 + C\epsilon_{k+1}^2 \\ C\epsilon_{k+1}^2 & C\epsilon_{k+1}^2 \end{bmatrix}$$

and

$$\bar{q}(t) = \begin{bmatrix} \epsilon_{k+1} + C\epsilon_{k+1}^2 \\ \epsilon_{k+1} + C\epsilon_{k+1}^2 \end{bmatrix}$$

(since we will choose $\epsilon_k < \epsilon_{k+1}$.) A simple calculation using lemma 4.4 shows that for some \bar{K} depending on ϵ_{k+1} , for $j = k-1, k$ and $\forall t > T_{k+1}$

$$|\tilde{y}_j(t)| \leq R\bar{K}e^{\alpha(t-T_{k+1})} \quad (4.56)$$

where

$$\alpha = C\epsilon_{k+1}^2 + \sqrt{C\epsilon_{k+1}^2(1 + C\epsilon_{k+1}^2)}$$

For convenience, we choose ϵ_{k+1} such that $\alpha < 0.5$.

Now since the linear approximation at the origin of the dynamics of $\tilde{y}_{k+1}, \dots, \tilde{y}_m, \tilde{z}$ has eigenvalues with real part less than or equal to -1 , we can conclude from the lemmas 4.1 and 4.3 that

$$\begin{aligned} |\tilde{y}_{i+1}(t)| &\leq a_k\epsilon_k + a_{k+1}\epsilon_{k+1}e^{-(t-T_{k+1})} \quad i = k, \dots, m \\ ||\tilde{z}(t)|| &\leq a_k\epsilon_k + a_{k+1}\epsilon_{k+1}e^{-(t-T_{k+1})} \\ |u(t)| &\leq a_k\epsilon_k + a_{k+1}\epsilon_{k+1}e^{-(t-T_{k+1})} \end{aligned}$$

for all $t > T_{k+1}$.

We solve for the time t_ϵ such that, for all $t \geq t_\epsilon$

$$\begin{aligned} |\tilde{y}_{i+1}(t)| &\leq 2a_k\epsilon_k \quad i = k, \dots, m \\ ||\tilde{z}(t)|| &\leq 2a_k\epsilon_k \\ |u(t)| &\leq 2a_k\epsilon_k \end{aligned}$$

We find

$$t_\epsilon = T_{k+1} - \ln \frac{a_k\epsilon_k}{a_{k+1}\epsilon_{k+1}}.$$

Further, from (4.56), we determine a bound on $\tilde{y}_j(t)$ for $T_{k+1} \leq t \leq t_\epsilon$ for $j = k-1, k$ to be

$$|\tilde{y}_j| \leq R\bar{K} \left(\frac{a_{k+1}\epsilon_{k+1}}{a_k\epsilon_k} \right)^\alpha \equiv R_{\epsilon_k} \quad (4.57)$$

So then it remains to determine whether ϵ_k can be chosen sufficiently small such that a small neighborhood of the origin is attractive for the dynamics

$$\begin{aligned}\dot{\tilde{y}}_{k-1} &= \tilde{y}_k - \sigma_k(\tilde{y}_{k-1} + \tilde{y}_k + \sigma_{k-2}) + \tilde{h}_{k-1}(t) \\ \dot{\tilde{y}}_k &= -\sigma_k(\tilde{y}_{k-1} + \tilde{y}_k + \sigma_{k-2}) + \tilde{h}_k(t)\end{aligned}\tag{4.58}$$

from initial conditions such that

$$|\tilde{y}_j| \leq R_{\epsilon_k}$$

for $j = k-1, k$ and where \tilde{h}_j satisfy the properties of section 4.6.3. To show that this is possible we begin with the coordinate change

$$\begin{aligned}x_1 &= \tilde{y}_{k-1} + \tilde{y}_k \\ x_2 &= \tilde{y}_k\end{aligned}$$

yielding the dynamics

$$\begin{aligned}\dot{x}_1 &= x_2 - 2\sigma_k(x_1 + \sigma_{k-2}) + f_1(t) \\ \dot{x}_2 &= -\sigma_k(x_1 + \sigma_{k-2}) + f_2(t)\end{aligned}\tag{4.59}$$

It can be shown that for $j = 1, 2$

$$|f_j - f_j(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{i+1}, 0, \dots, 0)| \leq (|x_1| + |x_2| + 1)\bar{f}_j\tag{4.60}$$

where \bar{f}_j is higher order in $\tilde{y}_{k+2}, \dots, \tilde{y}_m, \tilde{z}, u$ uniformly in $\tilde{y}_1, \dots, \tilde{y}_{k+1}, t$ (since \tilde{y}_{k+1} is bounded.) Further, it can be shown that

$$|f_1(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{k+1}, 0, \dots, 0)| \leq (|x_2| + 1)D\epsilon_k^2\tag{4.61}$$

and $f_2(\tilde{\eta}, \tilde{y}_1, \dots, \tilde{y}_{k+1}, 0, \dots, 0)$ depends only on \tilde{y}_{k+1} and is higher order.

Observe that if $|x_1(t_b)| \geq \epsilon_k + \epsilon_{k-2}$, then for all $t \geq t_b$ such that $|x_1(t)| \geq \epsilon_k + \epsilon_{k-2}$ we have

$$\begin{aligned}|\tilde{y}_{i+1}(t)| &\leq \epsilon_k + 2a_k\epsilon_k e^{-(t-t_b)} \\ |\tilde{y}_{i+2}(t)| &\leq 2a_k\epsilon_k e^{-(t-t_b)} \quad i = k, \dots, m \\ |\tilde{z}(t)| &\leq 2a_k\epsilon_k e^{-(t-t_b)} \\ |u(t)| &\leq 2a_k\epsilon_k e^{-(t-t_b)}\end{aligned}\tag{4.62}$$

This follows from point 3 of section 4.6.2. Hence, for $t \geq t_b$ and such that $|x_1(t)| \geq \epsilon_k + \epsilon_{k-2}$,

$$\begin{aligned}|f_1| &\leq (|x_2| + 1)D\epsilon_k^2 + (|x_1| + |x_2| + 1)D\epsilon_k^2 e^{-(t-t_b)} \\ f_2 &= C + \hat{f}_2\end{aligned}$$

where C is a constant and

$$\begin{aligned} |\hat{f}_2| &\leq (|x_1| + |x_2| + 1)D\epsilon_k^2 e^{-(t-t_b)} \\ |C| &\leq D\epsilon_k^2 \end{aligned}$$

Then from (4.56) and (4.57), we have the bound

$$\begin{aligned} |f_1| &\leq |x_2|D\epsilon_k^2 + D\epsilon^{2-\alpha} \\ |\hat{f}_2| &\leq D\epsilon_k^{2-\alpha} e^{-(1-\alpha)(t-t_b)} \end{aligned} \tag{4.63}$$

These bounds on the nonlinear terms when $|x_1| \geq \epsilon_k + \epsilon_{k-2}$ will play a crucial part in our analysis.

The remainder of the proof consists of three points. Consider the set

$$Q = \{(x_1, x_2) : |x_1| \leq \epsilon_k + \epsilon_{k-2}\}$$

Point 1 will be to show that if $x_1(t_c) \notin Q$ then $\exists t_c > t_c$ which is finite such that $x_1(t_c) \in Q$ and we establish a worst case value for $|x_2(t_c)|$. Point 2 will be to consider the “Lyapunov-like” function

$$W = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}x_2^2 \tag{4.64}$$

which we will demonstrate is uniformly decreasing when $x_1(t) \in Q \cap U^c$. Here U is a neighborhood of the origin depending on ϵ_{k-2} and such that $\sigma_k(x_1 + \sigma_{k-2}) = x_1 + \sigma_{k-2}$ for all $x_1 \in U$. (U^c is the complement of U .) Point 3 will be to show that whenever the trajectory leaves Q it returns to Q and when it does it returns at a lower energy level for W .

For point 1, define the set

$$Q_r = \{(x_1, x_2) : x_1 > \epsilon_k + \epsilon_{k-2}\}$$

and without loss of generality assume $x_1(t_c) \in Q_r$. We demonstrate that for ϵ_k sufficiently small and for

$$2\epsilon_k + D\epsilon^{2-\alpha} < x_2(t_0) \leq R_{\epsilon_k}$$

$\exists t_d > t_c$ such that $x_1(t_d) = x_1(t_c)$ and further,

$$|x_2(t_d)| \leq x_2(t_0)$$

In the set Q_r the dynamics of (x_1, x_2) are given by

$$\begin{aligned} \dot{x}_1 &= x_2 - 2\epsilon_k + f_1(t) \\ \dot{x}_2 &= -\epsilon + C + \hat{f}_2(t) \end{aligned} \tag{4.65}$$

From the bounds on C and $\hat{f}_2(t)$, x_2 is monotonically decreasing for sufficiently small ϵ_k . We now consider the forms of f_1, \hat{f}_2 which will maximize $|x_2(t_d)|$. We do this by considering the instantaneous slope of the trajectory in the (x_1, x_2) plane. The instantaneous slope is given by

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{-\epsilon_k + C + \hat{f}_2(t)}{x_2 - 2\epsilon_k + f_1(t)} \quad (4.66)$$

Since x_2 is monotonically decreasing, the actual trajectory will be bounded by two curves. The outer curve is produced by flowing along the vector field that minimizes the magnitude of the negative instantaneous slope (x_1 increasing) and maximizes the positive instantaneous slope (x_1 decreasing.) The inner curve is produced by flowing along the vector field that maximizes the magnitude of the negative instantaneous slopes and minimizes positive instantaneous slopes. It is straightforward to see that, as long as $|x_2| \leq R_{c_k}$, the outer curve is generated by setting $f_1(t) = D\epsilon_k^{2-\alpha}$ and

$$\hat{f}_2(t) = \begin{cases} D\epsilon_k^{2-\alpha}e^{-(t-t_b)} & x_2(t) \geq 2\epsilon_k - D\epsilon_{2-\alpha} \\ -D\epsilon_k^{2-\alpha}e^{-(t-t_b)} & x_2(t) < 2\epsilon_k - D\epsilon_{2-\alpha} \end{cases}$$

Likewise, the inner curve is generated by setting $f_1(t) = -D\epsilon_k^{2-\alpha}$ and

$$\hat{f}_2(t) = \begin{cases} -D\epsilon_k^{2-\alpha}e^{-(t-t_b)} & x_2(t) \geq 2\epsilon_k + D\epsilon_{2-\alpha} \\ D\epsilon_k^{2-\alpha}e^{-(t-t_b)} & x_2(t) < \epsilon_k + D\epsilon_{2-\alpha} \end{cases}$$

(The value of C is fixed as a function of ϵ_k .) The outer curve gives us a least upper bound on $|x_2(t_d)|$. To compute this bound, we first calculate the time t_i on the outer curve such that $x_2(t_i) = 2\epsilon_k - D\epsilon_k^{2-\alpha}$. The value of $x_2(t)$ for $t_e \leq t \leq t_i$ along the outer curve is given by

$$x_2(t) = x_2(t_e) - (\epsilon_k + C)t - \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}[e^{-(1-\alpha)t} - 1] \quad (4.67)$$

(We have temporarily reinitialized $t_e = 0$ for convenience.) Thus, we have (implicitly)

$$t_i = \frac{1}{\epsilon_k - C}x_2(t_e) - 2\epsilon_k - \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}[e^{-(1-\alpha)t_i} - 1] \quad (4.68)$$

The value of $x_1(t)$ for $t_e \leq t \leq t_i$ along the outer curve is given by

$$\begin{aligned} x_1(t) = & x_1(t_e) - \frac{1}{2}(\epsilon_k - C)t^2 + [x_2(0) - 2\epsilon_k + \frac{D\epsilon_k^{2-\alpha}}{1-\alpha} + D\epsilon_k^{2-\alpha}]t \\ & + \frac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}[e^{-(1-\alpha)t} - 1] \end{aligned} \quad (4.69)$$

Thus, we have

$$\begin{aligned} x_1(t_i) = & x_1(t_\epsilon) + \frac{1}{2(\epsilon_k - C)}(x_2(t_\epsilon) - 2\epsilon_k + D\epsilon_k^{2-\alpha} + \frac{D\epsilon_k^{2-\alpha}}{(1-\alpha)})^2 \\ & - \frac{1}{2(\epsilon_k - C)}[\frac{D\epsilon_k^{2-\alpha}}{1-\alpha}e^{-(1-\alpha)t_i}]^2 - \frac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}[1 - e^{-(1-\alpha)t_i}] \end{aligned} \quad (4.70)$$

We now continue the flow beginning at the point $(x_1(t_i), x_2(t_i))$. Then for $t_i < t \leq t_d$ the flow along the outer curve is given by

$$\begin{aligned} x_1(t) = & x_1(t_i) + [x_2(t_i) - 2\epsilon_k + D\epsilon_k^{2-\alpha} - \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}e^{-(1-\alpha)t_i}]t - \frac{1}{2}(\epsilon_k - C)t^2 \\ & - \frac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}e^{-(1-\alpha)t_i}[e^{-(1-\alpha)t} - 1] \\ x_2(t) = & x_2(t_i) - (\epsilon_k - C)t + \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}e^{-(1-\alpha)t_i}[e^{-(1-\alpha)t} - 1] \end{aligned} \quad (4.71)$$

(We have reinitialized $t_i = 0$ for convenience.) We are now interested in determining $x_2(t_d)$ where t_d is such that $x_1(t_d) = x_1(t_0)$. Since we are interested in a worst case bound for $|x_2(t_d)|$ and x_2 is monotonically decreasing in the region we are considering, it suffices to determine a least upper bound for t_d . We find that

$$t_d \leq t_i + \frac{1}{\epsilon_k - C} \sqrt{[x_2(t_\epsilon) - 2\epsilon_k + D\epsilon_k^{2-\alpha} + \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}]^2 + 2(\epsilon_k - C)\frac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}} \quad (4.72)$$

Consequently, we can conclude that

$$\begin{aligned} x_2(t_d) \geq & 2\epsilon_k - D\epsilon_k^{2-\alpha} - (x_2(t_0) - 2\epsilon_k + D\epsilon_k^{2-\alpha} + \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}) \\ & - \sqrt{2(\epsilon_k - C)\frac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}} \end{aligned} \quad (4.73)$$

It is readily apparent that ϵ_k can be chosen sufficiently small so that $|x_2(t_d)| \leq |x_2(t_\epsilon)|$ since it was assumed that $x_2(t_\epsilon)$ is positive. In fact, for later purposes it is important to note that ϵ_k can be chosen sufficiently small so that $x_2(t_d) \geq -x_2(t_0) + 3\epsilon_k$.

We continue now with point 1 and, without loss of generality, assume that the trajectory of (x_1, x_2) begins at the point $(x_1(t_d), x_2(t_d)) = (R_{\epsilon_k}, -R_{\epsilon_k})$. Again note that x_2 is monotonically decreasing. In (4.63) we will assume a bound on $|x_2|$ to be $|x_2| \leq aR_{\epsilon_k}$ (a constant and independent of ϵ_k) and hence $|f_1| \leq aD\epsilon_k^{2-\alpha}$. Then, since x_2 is monotonically decreasing from $-R_{\epsilon_k}$, if we can show that $|x_2(t_c)| \leq aR_{\epsilon_k}$ (where $x_1(t_c) = \epsilon_k + \epsilon_{k-2}$) then this is a worst case bound on $|x_2(t_c)|$. To maximize $|x_2(t_c)|$ we again flow along the outer curve described previously. The flow is given by

$$\begin{aligned} x_1(t) = & x_1(t_d) + [x_2(t_d) - 2\epsilon_k + aD\epsilon_k^{2-\alpha} - \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}]t - \frac{1}{2}(\epsilon_k - C)t^2 \\ & - \frac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}[e^{-(1-\alpha)t} - 1] \\ x_2(t) = & x_2(t_d) - (\epsilon_k - C)t + \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}[e^{-(1-\alpha)t} - 1] \end{aligned} \quad (4.74)$$

In this instance, a worst case bound on t_c is given by

$$t_c \leq \frac{1}{\epsilon_k - C} [b + \sqrt{b^2 + 2(\epsilon_k - C)c}] \quad (4.75)$$

where

$$\begin{aligned} b &= R_{\epsilon_k} - 2\epsilon_k + aD\epsilon_k^{2-\alpha} - \frac{D\epsilon_k^{2-\alpha}}{1-\alpha} \\ c &= R_{\epsilon_k} + \frac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2} - \epsilon_k - \epsilon_{k-2} \end{aligned} \quad (4.76)$$

Then $|x_2(t_c)|$ is bounded by

$$|x_2(t_c)| \leq R_{\epsilon_k} + \frac{D\epsilon_k^{2-\alpha}}{1-\alpha} b + \sqrt{b^2 + 2(\epsilon_k - C)c} \quad (4.77)$$

It is straightforward to see that, for ϵ_k sufficiently small, a worst case bound on $|x_2(t_c)|$ is given by

$$|x_2(t_c)| \leq aR_{\epsilon_k} \quad (4.78)$$

for $a \geq 3$.

We are now ready to move to point 2. Here we show that we can choose ϵ_k and ϵ_{k-2} sufficiently small, such that for W defined by (4.64), $\dot{W} \leq 0$ for all $x \in Q$. Consider \dot{W} along the trajectories of (4.65):

$$\begin{aligned} \dot{W} &= (x_1 - x_2)[\dot{x}_1 - \dot{x}_2] + x_2\dot{x}_2 \\ &= (x_1 - x_2)[x_2 - 2\sigma_k(x_1 + \sigma_{k-2}) + f_1(t) + \sigma_k(x_1 + \sigma_{k-2}) - f_2(t)] \\ &\quad + x_2[-\sigma_k(x_1 + \sigma_{k-2}) + f_2(t)] \end{aligned} \quad (4.79)$$

Recall that, in Q , we have the bounds (for $j = 1, 2$):

$$|f_j| \leq (|x_1| + |x_2| + 1)D\epsilon_k^2$$

Hence,

$$\begin{aligned} \dot{W} &\leq -x_1\sigma_k(x_1 + \sigma_{k-2}) + x_1x_2 - x_2^2 \\ &\quad + (|x_1| + |x_2|)(|x_1| + |x_2| + 1)D\epsilon_k^2 \\ &\leq -0.5x_1^2 - 0.5x_2^2 - 0.5(x_1 - x_2)^2 + x_1(x_1 - \sigma_k(x_1 + \sigma_{k-2})) \\ &\quad + (|x_1| + |x_2|)(|x_1| + |x_2| + 1)D\epsilon_k^2 \end{aligned}$$

Consider the level set

$$W = \frac{1}{2}(\beta\epsilon_{k-2})^2$$

and define U to be the interior of this level set. On this level set, (a circle of radius $\beta\epsilon_k$ in the original y_{k-1}, y_k coordinates), it can be shown that

$$\beta\epsilon_{k-2} \leq |x_i| \leq \sqrt{2}\beta\epsilon_{k-2}$$

for $i = 1, 2$. Also notice that for $x_1 \in Q$,

$$|x_1 - \sigma_k(x_1 + \sigma_{k-2})| \leq \epsilon_{k-2}$$

Consequently, we have on this level set

$$\dot{W} \leq -0.5(\beta\epsilon_{k-2})^2 - 0.5(k\beta\epsilon_{k-2})^2 + k\beta\epsilon_{k-2}^2 + (2k\beta\epsilon_{k-2})(2k\beta\epsilon + 1)D\epsilon_k^2$$

where $k \in [1, \sqrt{2}]$. As a function of k we have

$$\dot{W} \leq -[.5\beta^2 - 4\beta^2 D\epsilon_k^2]k^2 - (\beta + 2\beta D\epsilon_k^2)k + .5\beta^2]\epsilon_{k-2}^2$$

Then since $k \in [1, \sqrt{2}]$, we can choose β ($\beta > 2$ is sufficient) such that ϵ_k can be chosen sufficiently small such that $\dot{W} < 0$ on this level set. Since, for ϵ_k small enough, \dot{W} is bounded by a quadratic negative definite function plus a linear perturbation in Q , $\dot{W} < 0$ in $Q \cap U^c$. Notice also, for ϵ_{k-2} small enough, σ_k operates in its linear region for all $x \in U$.

Finally, for point 3, we demonstrate that, for ϵ_k small enough, whenever the trajectory leaves Q it returns to Q at a lower energy level of W . We simply need to show that this is true for $|x_2| \leq aR_{\epsilon_k}$ where a comes from (4.78). We demonstrate that this follows from the first part of point 1. We can consider trajectories that enter Q_r from Q at some time t_0 without loss of generality. Consequently $x_1(t_0) = \epsilon_k + \epsilon_{k-2}$. From the first part of point 1, by incorporating the constant a into the constant D perhaps further decreasing ϵ_k , it follows that for each $x_1(t_0) \in Q_r$ and each $aR_{\epsilon_k} \geq x_2(t_0) \geq 2\epsilon_k + D\epsilon_k^{2-\alpha}$ there exists a t_d such that $x_1(t_d) = x_1(0)$ and

$$x_2(t_d) \geq -x_2(t_0) + 3\epsilon_k. \quad (4.80)$$

Consider

$$W(t_d) - W(t_0) = \frac{1}{2}[(x_1(t_d) - x_2(t_d))^2 + x_2(t_d)^2] - \frac{1}{2}[(x_1(t_0) - x_2(t_0))^2 + x_2(t_0)^2]$$

From (4.80) and the lower bound of $x_2(t_0)$ we can conclude that

$$\frac{1}{2}[x_2(t_d)^2 - x_2(t_0)^2] < 0.$$

Also from (4.80) the remaining terms are bounded as

$$\begin{aligned} [x_1(t_d) - x_2(t_d)]^2 - [x_1(t_0) - x_2(t_0)]^2 &\leq [x_1(t_0) + x_2(t_0) - 3\epsilon_k]^2 - [x_1(t_0) - x_2(t_0)]^2 \\ &\leq [x_2(t_0 - x_1(t_0) - \epsilon_k + 2\epsilon_{k-2})]^2 - [x_2(t_0) - x_1(t_0)]^2 \end{aligned}$$

If $\epsilon_{k-2} < \frac{1}{2}\epsilon_k$ then this quantity is also less than zero since $x_2(t_0) > 2\epsilon_k + D\epsilon_k^{2-\alpha}$ and $x_1(t_0) = \epsilon_k + \epsilon_{k-2}$.

The above three points demonstrate that x_1, x_2 eventually enter U where σ_k is linear. So it follows that $\tilde{y}_{k-1}, \tilde{y}_k$ eventually enter a neighborhood of the origin where σ_k is linear. The size of this neighborhood is determined by ϵ_{k-2} . The remainder of the proof follows by induction using either the global or semi-global result when appropriate. (Point 3 of section 4.6.3 is used to conclude (4.62) in the subsequent step of the induction.) \square

Chapter 5

Beyond Linear Feedback for the Nonlinear Regulator

In this chapter, we address the tracking problem for nonlinear systems. We demonstrate that, for small reference signals, solving the stabilization problem goes a long way toward solving the tracking problem. The method for augmenting the stabilizing control to achieve tracking comes directly from the nonlinear regulator theory developed by Byrnes and Isidori ([Byrnes and Isidori, 1990]). We use the stabilizing control laws developed in the previous chapters to achieve small signal tracking with large domains of attraction for the associated class of systems. In point of fact, the systems of chapter 4 typically do not have a well-defined relative degree, and hence, the results of [Grizzle *et al.*, 1991] indicate that exact tracking for an open set of trajectories is not possible. Nevertheless, combining the regulator theory of [Byrnes and Isidori, 1990] with the control laws of the previous chapter, we are able to achieve approximate tracking results that compare quite favorably to the approximate linearization results of [Hauser *et al.*, 1992] when comparing domains of attraction and ability to achieve arbitrarily small tracking error.

5.1 Introduction

As in [Teel, 1991], we seek to expand the region of attraction of the zero-error manifold of nonlinear regulator theory developed in [Byrnes and Isidori, 1990]. In [Teel, 1991], we approached this problem by deforming the manifold so that the initial state of the system started close to the deformed manifold and then allowed the deformed manifold to

decay slowly to the zero-error manifold. We then used standard linear feedback to regulate to this deformed manifold. For some systems this approach yielded dramatic improvements. Nevertheless, the result was still inherently local. A further drawback to this approach was that (approximate) knowledge of the initial state of the system was needed. Also, dynamic states equal to the number of states of the system were added to the compensator.

In this paper, we seek to expand the region of attraction without deforming the manifold. We propose replacing the standard linear feedback used to regulate to the manifold with nonlinear feedback based on global or semi-globally stabilizing control laws for unperturbed systems. We show that this approach yields theoretical reasons for an increased domain of attraction. We demonstrate an application of this approach using the frequently studied “ball and beam” example presented in [Hauser *et al.*, 1992]. For this example, we choose the semi-globally stabilizing control law developed in the previous chapter.

5.2 Problem Statement

The task at hand is to achieve (perhaps approximate) tracking for the system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u + p(x)w \\ y &= h(x)\end{aligned}\tag{5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w \in W \subset \mathbb{R}^s$ is a disturbance. As usual, f and the columns of g and p are assumed to be smooth vector fields and $h(x)$ is a smooth mapping on \mathbb{R}^n . We assume that the desired trajectory and the disturbance are generated by an autonomous, Poisson stable exosystem

$$\begin{aligned}\dot{w} &= s(w) \\ y_d &= -q(w)\end{aligned}\tag{5.2}$$

where s is a smooth vector field and $q(w)$ is a smooth mapping defined on W . The Poisson stability of the exosystem implies that the eigenvalues of the linear approximation of the exosystem lie on the imaginary axis. For simplicity we assume that $f(0) = 0$, $s(0) = 0$, $h(0) = 0$ and $q(0) = 0$ so that, for $u = 0$ the composite system (5.1), (5.2) has an equilibrium state $(x, w) = (0, 0)$ which yields zero tracking error.

We will focus on finding a state feedback $u = \alpha(x, w)$ that yields (perhaps approximate) tracking.

5.3 The solution

As is standard in nonlinear regulator theory (see [Byrnes and Isidori, 1990], [Huang and Rugh, 1990b]) our starting point will be to assume that we can solve the following partial differential and algebraic equations (at least approximately) for $\pi(w)$ and $c(w)$ which characterize the zero-error manifold and the feedforward that renders the manifold invariant, respectively:

$$\begin{aligned}\frac{\partial \pi}{\partial w} s(w) &= f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w \\ h(\pi(w)) + q(w) &= 0\end{aligned}\tag{5.3}$$

The standard nonlinear regulator solution is then to choose the feedback

$$u = c(w) + K[x - \pi(w)]\tag{5.4}$$

where K is a linear gain matrix that stabilizes the Jacobian linear approximation of (5.1). This, of course, assumes that the linear approximation of (5.1) is stabilizable. For the nonlinear regulator problem, the feedback (5.4) solves the tracking problem for sufficiently small $(x(0), w(0))$. We will retain the requirement that $w(0)$ is sufficiently small, but we will allow $x(0)$ to be large.

Consider the system (5.1) disconnected from the exosystem:

$$\dot{x} = f(x) + g(x)u\tag{5.5}$$

Let $u = \varphi(x)$, with $\varphi(0) = 0$, be a smooth control that renders the equilibrium $x = 0$ of (5.5) globally asymptotically stable and locally *exponentially* stable. We then have the following result.

Theorem 5.1 *$\exists \epsilon_0$ such that for any $\epsilon < \epsilon_0$, if $|w(t)| < \epsilon$ for all $t \geq 0$, then the control $u = c(w) + \varphi(x - \pi(w))$ solves the nonlinear regulator problem with basin of attraction containing the ball $|x(0)| \leq \kappa(\frac{1}{\epsilon})$ for some class- K function $\kappa(\cdot)$.*

Proof. The proof uses the total stability result of Sontag [Sontag, 1990]. Define

$$F(x, w) := f(x) + g(x)[c(w) + \varphi(x - \pi(w))] + p(x)w\tag{5.6}$$

Since $c(0) = 0$ and $\pi(0) = 0$, we have that $\dot{x} = F(x, 0)$ is globally asymptotically stable. Therefore, there exists a smooth, positive definite and proper Lyapunov function

$$V : \mathbb{R}^n \rightarrow \mathbb{R}$$

such that

$$dV(x) \cdot F(x, 0) < 0$$

for all nonzero x . It then follows that

$$dV(x) \cdot F(x, w) < 0 \quad (5.7)$$

for all $|w| < \theta(|x|)$ for some continuous function $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\theta(0) = 0$ and that is decreasing on $[1, \infty)$. (See [Sontag, 1990, Lemmas 3.1, 3.2].) Then, for some ϵ_0 sufficiently small and any $\epsilon < \epsilon_0$, we can deduce from the function $\theta(\cdot)$ two class-K functions κ_1 and κ_2 such that

$$dV(x) \cdot F(x, w) < 0 \quad (5.8)$$

for all $x \in \mathbb{R}^n$ satisfying

$$\kappa_1(\epsilon) \leq |x| \leq \kappa_2\left(\frac{1}{\epsilon}\right) \quad (5.9)$$

Since V is proper we can deduce a class-K function κ such that every initial condition satisfying $|x(0)| \leq \kappa(\frac{1}{\epsilon})$ leads to a trajectory that is driven to some small neighborhood of the origin. If ϵ_0 sufficiently small then for all $\epsilon < \epsilon_0$ we have returned to the local nonlinear regulator problem. Since, $u = \varphi(x)$ is smooth and locally *exponentially* stabilizes the origin of (5.5) the linear approximation of the composite closed loop is in the form for which center manifold theory applies. Since $\varphi(0) = 0$, $u = c(w) + \varphi(x - \pi(w))$, and $c(w)$ and $\pi(w)$ satisfy (5.3) and since φ is a locally exponential stabilizer, $x = \pi(w)$ is an attractive, invariant manifold for the closed loop. Finally, also from (5.3), the tracking error approaches zero asymptotically. \square

Remark. Although we will not show it here, the results of the theorem extend readily to the approximate regulator problem (where the manifold equation (5.3) is solved up to some arbitrary order), and to the use of semi-globally stabilizing controls ($u = \varphi(x, p)$ where the basin of attraction of the system (5.5) can be made arbitrarily large by choice of p .)

5.4 Example: the “ball and beam”

We demonstrate the capabilities of this approach on the “ball and beam” example which has been studied with regard to approximate tracking in [Hauser *et al.*, 1992],

[Castillo, 1990], [Huang and Rugh, 1990a] and [Teel, 1991]. The dynamics of this system can be modeled as

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= x_1 x_4^2 - G \sin(x_3) \\
 \dot{x}_3 &= x_4 \\
 \dot{x}_4 &= u \\
 y &= x_1
 \end{aligned} \tag{5.10}$$

where x_1 is ball position, x_2 is ball velocity, x_3 is the angle of the beam, and x_4 is the beam's angular velocity. (For a derivation of these equations, see [Hauser *et al.*, 1992].) For simplicity, we have normalized the acceleration due to gravity to be $G = 1$ in our simulations. In chapter 4 it was shown that the control law

$$u = \varphi(x) = -4x_3 - 4x_4 - \sigma(y_1 + y_2) \tag{5.11}$$

where

$$\begin{aligned}
 y_1 &= -\frac{4}{G}x_1 - \frac{8}{G}x_2 + 5x_3 + x_4 \\
 y_2 &= -\frac{4}{G}x_2 + 4x_3 + x_4
 \end{aligned} \tag{5.12}$$

and $\sigma(\cdot)$ satisfies

1. $\sigma(s) = s$ for all $|s| \leq \delta$
2. $|\sigma(s)| = \delta$ for all $|s| > \delta$

for some $\delta > 0$ is an example of a semi-globally stabilizing control law for (5.10). The basin of attraction for $x = 0$ can be made arbitrarily large by making δ arbitrarily small.

The task at hand is to cause the ball position x_1 to (at least almost) track a sinusoid generated by the exosystem

$$\begin{aligned}
 \dot{w}_1 &= -\lambda w_2 \\
 \dot{w}_2 &= \lambda w_1 \\
 q(w) &= -w_1
 \end{aligned} \tag{5.13}$$

As seen in [Castillo, 1990] and [Huang and Rugh, 1990a] approximating the manifold to either first or third order yields nice approximate tracking results. A first order approximation

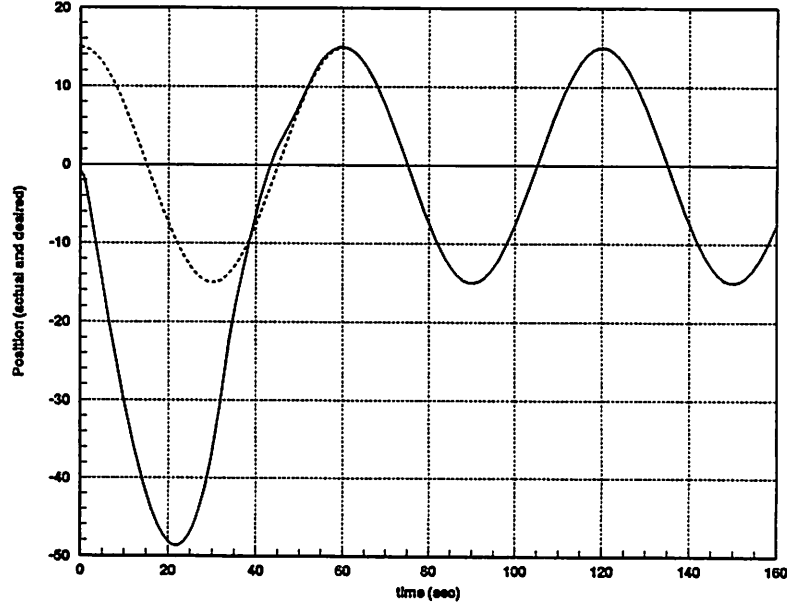


Figure 5.1: Tracking Results for the “ball and beam”

to the mappings $x = \pi(w)$ and $u = c(w)$ are given by

$$\begin{aligned}
 \pi_1(w) &= w_1 \\
 \pi_2(w) &= -\lambda w_2 \\
 \pi_3(w) &= \frac{1}{G} \lambda^2 w_1 \\
 \pi_4(w) &= -\frac{1}{G} \lambda^3 w_2 \\
 c(w) &= -\frac{1}{G} \lambda^4 w_1
 \end{aligned} \tag{5.14}$$

For simulation purposes, for the exosystem (5.13), we chose $\lambda = \frac{\pi}{30}$, $w_1(0) = 15$ and $w_2(0) = 0$. Consequently, the task is for the ball position, x_1 , to track $15 \cdot \cos(\frac{\pi}{30}t)$. We choose the control

$$u = c(w) + \varphi(x - \pi(w)) \tag{5.15}$$

with $c(w)$ and $\pi(w)$ specified in (5.14) and φ specified in (5.11). To demonstrate regulation from a difficult initial condition, we choose the initial angle of the beam to be 90° and the ball to be a position slightly below the pivot of the beam at $x_1 = -1$. We give the ball zero initial velocity and the beam zero initial angular velocity. The results of the simulation are demonstrated in figures 5.1, 5.2 and 5.3.

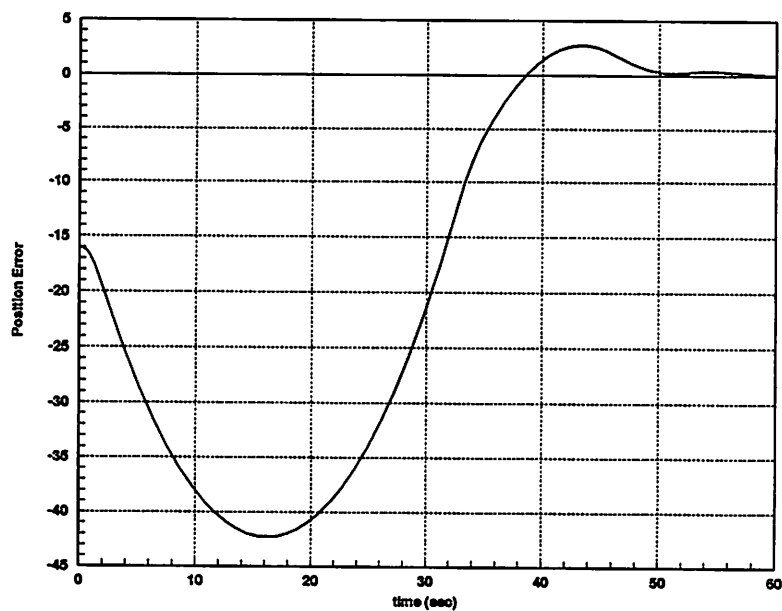


Figure 5.2: Transient performance

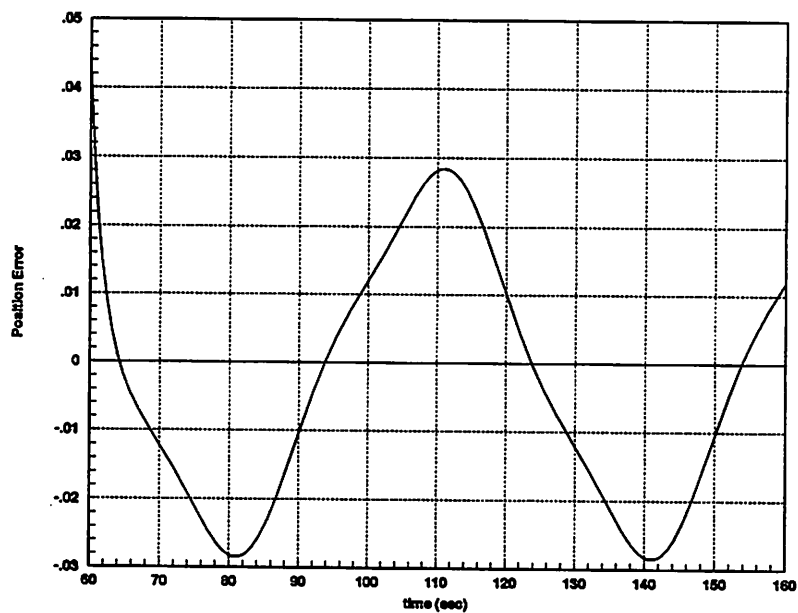


Figure 5.3: Steady-state performance

5.5 Conclusion

We have demonstrated that the use of nonlinear feedback in place of linear feedback in the nonlinear regulator problem expands the domains of attraction when the nonlinear feedback is known to be a global or semi-global stabilizer. This was done to show the usefulness of studying stabilization problems independent of tracking problems.

Chapter 6

Nonholonomic Control Systems: From Steering to Stabilization with Sinusoids

In this chapter, we investigate the stabilizability of nonholonomic control systems. Much of the work in this chapter is joint work with Richard Murray and Greg Walsh (see [Teel *et al.*, 1992]). After reviewing the general setting and discussing previous results, we propose a new family of stabilizing control laws for a class of nonholonomic control systems. We do so by combining previous open loop steering with sinusoids results in the literature [Murray and Sastry, 1991a] with feedback.

6.1 Introduction

This paper focuses on the problem of point stabilization for a control system of the form

$$\dot{x} = \sum_{i=1}^m g_i(x) u_i \quad x \in \mathbb{R}^n, \quad (6.1)$$

where each g_i is a smooth vector field on \mathbb{R}^n and the g_i 's are linearly independent for all $x \in \mathbb{R}^n$. Systems of this form arise in the study of mechanical systems with velocity constraints and have received renewed attention as an example of strongly nonlinear systems. For such systems, control methods based on linearization cannot be applied and nonlinear techniques must be utilized. We are particularly interested in the case where the nonlinear system (6.1)

is completely controllable, corresponding to a set of maximally nonholonomic constraints which do not restrict the state of the system to a submanifold of the state space. See [Murray and Sastry, 1990] for a more detailed derivation and motivation. We refer to a system with these properties as a *nonholonomic control system*.

A fundamental problem in the study of nonholonomic control systems is the generation of open-loop trajectories connecting two states. That is, given an initial state x_0 and a final state x_1 , find an input $u(t)$, $t \in [0, 1]$ such that $x(0) = x_0$ and $x(1) = x_1$. Such an input induces a feasible state trajectory which automatically satisfies the constraints on the system. The condition for the existence of a path between two configurations is given by Chow's theorem. We let $[f, g]$ be the Lie bracket between two vector fields,

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g,$$

and define the involutive closure of a distribution Δ as the closure of Δ under Lie bracketing. Briefly, Chow's theorem states that if the involutive closure of the distribution associated with equation (6.1) spans \mathbb{R}^n at each configuration, the system can be steered between any two configurations. Initial work in constructing paths between configurations includes [Jacobs *et al.*, 1990, Laumond and Siméon, 1989], [Li and Canny, 1990], and [Lafferriere and Sussmann, 1991, Sussmann and Liu, 1991], as well as [Murray and Sastry, 1990, Murray and Sastry, 1991b]. In this paper we concentrate on a different problem: stabilization to a point.

A control law $u = k(x, t)$ stabilizes a point $x_0 \in \mathbb{R}^n$ if $x(t) \rightarrow x_0$ as $t \rightarrow \infty$ for all initial conditions of the system. For a nonholonomic control system, the dependence of a stabilizing control law on time is essential since the system (6.1) does not satisfy Brockett's necessary condition for smooth stabilization [Brockett, 1983]. Hence there does not exist a smooth static state feedback law which stabilizes the system to a point. Recent work by Coron has shown that it is possible to stabilize a nonholonomic system using time-varying feedback [Coron, 1991]. Constructive approaches have been presented by Samson [Samson and Ait-Abderrahim, 1991] and Pomet [Pomet, 1992]. In this paper we present some new control laws for a specific class of systems, namely those in so-called chained form [Murray and Sastry, 1991b]. These control laws are based on earlier work using sinusoids for open-loop planning and have connections with the recent work in [Sussmann and Liu, 1991].

Chained systems. We restrict attention to a special class of nonholonomic systems, called chained systems [Murray and Sastry, 1991b]. A two-input system with a single chain has the form:

$$\begin{aligned}\dot{\xi}_1 &= v_1 \\ \dot{\xi}_2 &= v_2 \\ \dot{\xi}_3 &= \xi_2 v_1 \\ \dot{\xi}_4 &= \xi_3 v_1 \\ &\vdots \\ \dot{\xi}_n &= \xi_{n-1} v_1.\end{aligned}\tag{6.2}$$

This system is controllable using the input vector fields and Lie brackets of the form $\text{ad}_{g_1}^k g_2$, where $\text{ad}_f g$ is the iterated Lie bracket $[f, [f, \dots, [f, g] \dots]]$ (k copies of f).

Under some conditions, it is possible to convert a two-input nonholonomic system into a system with the form of equation (6.2) using feedback transformations. Sufficient conditions for doing this are presented in [Murray and Sastry, 1991b]. In particular, it can be shown that under certain regularity conditions all two-input nonholonomic systems in \mathbb{R}^3 can be put into this form. More complicated examples of nonholonomic systems which are locally feedback equivalent to a chained form include kinematic models of an automobile and an automobile towing a trailer.

Chained systems can be steered between two arbitrary configurations using the following algorithm.

Algorithm 1

1. Steer ξ_1 and ξ_2 to their desired values.
2. For each ξ_{k+2} , $k \geq 1$, steer ξ_{k+2} to its final value using $v_1 = a \sin t$, $v_2 = b \cos kt$, where a and b satisfy

$$\xi_{k+2}(2\pi) - \xi_{k+2}(0) = \frac{(a/2)^k b}{k!} \cdot 2\pi.$$

This algorithm uses n path segments to steer the system. It is also possible to steer the system using a linear combination of sinusoidal terms at different frequencies by solving a polynomial equation for the coefficients of the sinusoids.

Power form. Related to chained form is a second canonical form which we refer to as “power form”:

$$\begin{aligned}
 \dot{x}_1 &= u_1 \\
 \dot{x}_2 &= u_2 \\
 \dot{x}_3 &= x_1 u_2 \\
 \dot{x}_4 &= \frac{1}{2} x_1^2 u_2 \\
 &\vdots \\
 \dot{x}_n &= \frac{1}{(n-2)!} x_1^{n-2} u_2.
 \end{aligned} \tag{6.3}$$

Like chained form, the control Lie algebra for this system is spanned by the input vector fields and Lie products of the form $\text{ad}_{g_1}^k g_2$. The power form is related to the chained form through a global coordinate transformation:

$$\begin{aligned}
 x_1 &= \xi_1 \\
 x_2 &= \xi_2 \\
 x_3 &= -\xi_3 + \xi_1 \xi_2 \\
 x_4 &= \xi_4 - \xi_1 \xi_3 + \frac{1}{2} \xi_1^2 \xi_2 \\
 &\vdots \\
 x_n &= (-1)^n \xi_n + \sum_{i=2}^{n-1} (-1)^i \frac{1}{(n-i)!} \xi_1^{n-i} \xi_i
 \end{aligned} \tag{6.4}$$

The advantage of using power form over chained form is that given u_1 and u_2 , we can quickly solve for the motion of any of the state variables using only the trajectory of x_1 and the function u_2 . This canonical form also arises in the work of Grayson and Grossman in the context of generating systems of vector fields which realize a nilpotent control Lie algebra of a given order [Grayson and Grossman, 1987]. It is also worthwhile to note that this form satisfies some of the simplifying assumptions used by Pomet to generate controllers for more general nonholonomic control systems [Pomet, 1992].

In the sequel, we will restrict our results to those that apply to systems in chained form or, equivalently, power form. There are several reasons for taking this action. Systems which are in chained form characterize the fundamental difficulties of nonholonomic systems in a very simple and useful form. By understanding the geometry of controllers applied to chained form, we hope to understand the geometry of controllers applied to more general nonholonomic systems. This point of view has been used very successfully by Sussmann, who has shown how results applied to a “symbolic” representation of the control system can be used to understand systems with a compatible control Lie algebra [Lafferriere and

Sussmann, 1991]. Chained systems can be regarded as a realization of a class of “symbolic” control systems with a particular Lie algebraic structure.

The goal of this paper is to present a class of control laws with strong geometric intuition which asymptotically stabilize an arbitrary chained system with two inputs and a single chain. We are optimistic that the stabilizing controllers presented here can be extended to the more general case and that by understanding their action on a canonical system we can understand their extension to systems with a similar Lie algebraic structure.

6.2 Local Stabilization

In this section we propose a class of locally stabilizing inputs for (6.3). To motivate our approach, we consider first the simplest such system:

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2\end{aligned}\tag{6.5}$$

From the discussion of chained systems above, we know that motion in the x_3 direction can be achieved using sinusoidal inputs $u_1 = a \sin t$ and $u_2 = b \cos t$. Integrating the differential equations over one period, the resulting motion is a closed curve in x_1 and x_2 and a net motion of $-(ab)\pi$ in x_3 . This suggests that the following control law

$$\begin{aligned}u_1 &= -x_1 - x_3^2 \sin t \\ u_2 &= -x_2 - x_3 \cos t\end{aligned}\tag{6.6}$$

might be used to stabilize the system. The intuition is that if x_3 is slowly varying then the average motion (over one period) in the x_3 coordinate can be approximated by setting $a = -x_3^2$, $b = -x_3$ which would give a net motion in x_3 of $-x_3^3\pi$, i.e., x_3 would converge to zero.

To prove stability in a more rigorous fashion we make use of center manifold theory and averaging. For the purposes of the proof, we realize the time-varying feedback law by augmenting the controller with an exosystem

$$\begin{aligned}\dot{w}_1 &= w_2 & w_1(0) &= 0 \\ \dot{w}_2 &= -w_1 & w_2(0) &= 1,\end{aligned}$$

and write the control law as

$$\begin{aligned} u_1 &= -x_1 - x_3^2 w_1 \\ u_2 &= -x_2 - x_3 w_2. \end{aligned}$$

The closed loop system (including exosystem) has a local center manifold given by

$$\begin{aligned} x_1 &= \pi_1(x_3, w_1, w_2) \\ x_2 &= \pi_2(x_3, w_1, w_2), \end{aligned}$$

which is approximately given by

$$\begin{aligned} \pi_1 &= -\frac{1}{2}x_3^2(w_1 - w_2) \\ \pi_2 &= -\frac{1}{2}x_3(w_1 + w_2). \end{aligned}$$

The dynamics of the system evaluated on the center manifold are (approximately) given by

$$\dot{x}_3 = -\frac{1}{4}x_3^3(w_1 - w_2)^2.$$

An averaging-like coordinate change can then be made to show that the complete system is locally, asymptotically stable to the origin. For x_3 small, the higher order nature of x_3^3 plays the role of the small parameter ϵ usually found in averaging results.

We now consider the stabilization of an arbitrary system in power form. We begin with a local result and extend the controller to provide global convergence in the next section.

Theorem 6.1 *Every pair of inputs*

$$\begin{aligned} u_1 &= -x_1 - (\sum_{j=1}^{n-2} x_{j+2}^2)(\sin(t) - \cos(t)) \\ u_2 &= -x_2 - \sum_{j=1}^{n-2} c_j x_{j+2} \cos(jt) \end{aligned} \tag{6.7}$$

with $c_j > 0$ locally asymptotically stabilizes the origin of (6.3).

Remark. The control law given in theorem 6.1 is a generalization of the simple controller presented earlier. We have added a cosine term to u_1 to make the proof tractable. It can be seen that, for the simple example, this extra term adds a term on the manifold of zero average. Sinusoids at integrally related frequencies are used to generate motion in the different bracket directions in such a way as to stabilize the system to the origin. We note that the control law requires neither the use of high-frequency sinusoids, such as those used by Sussmann and Liu for open loop steering [Sussmann and Liu, 1991] (see also [Tilbury *et al.*, 1992]), nor does it require the use of a leading ϵ coefficient as typically used when

applying averaging techniques. Likewise, compared to the work of [Gurvits and Li, 1992], even though we employ an averaging like analysis, we do not require high-frequency sinusoids and we do not settle for stabilization to an arbitrarily small set. Furthermore, the weights c_j can be adjusted to control the rate of convergence in the different coordinate directions in a straightforward manner.

Proof of theorem 6.1. The proof of theorem 6.1 will require applications of center manifold theory (see [Carr, 1981]), techniques used in averaging theory (see [Guckenheimer and Holmes, 1983] or [Hale, 1969]) and a case specific Lyapunov result. Center manifold theory does not apply directly to (6.3), (6.7) because the time-varying terms in (6.7) are $O(1)$. Nevertheless, we can demonstrate the following lemma regarding a class of systems to which (6.3), (6.7) can be transformed. We use the notation of [Carr, 1981] so that $f'(0,0,w)$ refers to the partial derivative of f with respect to all variables and evaluated at $(y,z,w) = (0,0,w)$.

Lemma 6.1 ("Time-varying" Center Manifold) *Consider the system*

$$\begin{aligned}\dot{y} &= By + g(y,z,w) \\ \dot{z} &= Az + f(y,z,w) \\ \dot{w} &= Sw\end{aligned}\tag{6.8}$$

with $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $w \in \mathbb{R}^p$ and where the eigenvalues of B have negative real part and the eigenvalues of A and S have zero real part. The functions f, g and h are C^2 with $f(0,0,w) = 0$, $f'(0,0,w) = 0$, $g(0,0,w) = 0$, and $g'(0,0,w) = 0$. Then, given $M > 0$, there exists a center manifold for (6.8), $y = h(z,w)$ for $|w| < M$, $|z| < \delta(M)$, for some $\delta > 0$ and dependent on M , where h is C^2 and $h(0,w) = 0$, $h'(0,w) = 0$.

Proof. See appendix.

To transform (6.3), (6.7) into a system for which lemma 6.1 applies, we begin by defining $n - 2$ linear oscillators which will generate the time-varying terms of (6.7). Let

$$\dot{w}_j = \begin{bmatrix} \dot{w}_{1j} \\ \dot{w}_{2j} \end{bmatrix} = \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix} \begin{bmatrix} w_{1j} \\ w_{2j} \end{bmatrix} = S_{jj} w_j\tag{6.9}$$

We choose $w_{1j}(0) = 0$, $w_{2j}(0) = 1$ so that $w_{1j} = \sin(jt)$ and $w_{2j} = \cos jt$. If we define the vector

$$w = \begin{bmatrix} w_1 & \dots & w_{n-2} \end{bmatrix}^T$$

we have

$$\dot{w} = Sw \quad (6.10)$$

where S is a block diagonal matrix with the j th block given by S_{jj} . Next, partition the original state space as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ z \end{bmatrix} \quad (6.11)$$

so that $y \in \mathbb{R}^2$ and $z \in \mathbb{R}^m$ with $m \equiv n - 2$. For the closed loop system we have

$$\begin{aligned} \dot{y}_1 &= -y_1 - w^T D z^T z \\ \dot{y}_2 &= -y_2 - w^T C z \\ \dot{z} &= f(y, z, w) \\ \dot{w} &= Sw \end{aligned} \quad (6.12)$$

where f is C^2 with $f(0, 0, w) = 0$ and $f'(0, 0, w) = 0$. The matrix $C \in \mathbb{R}^{2m \times m}$ is block diagonal with the j th block given by the column vector

$$C_{jj} = \begin{bmatrix} 0 \\ c_j \end{bmatrix} \quad (6.13)$$

and $D \in \mathbb{R}^{2m}$ is given by

$$D = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \end{bmatrix}^T \quad (6.14)$$

We then make a coordinate change in y_2 to eliminate the linear time-varying dependence of z in the \dot{y}_2 equation. We choose $\tilde{y}_2 = y_2 - z^T \Pi_2 w$ where Π_2 solves the matrix equation

$$\Pi_2 S = -I \Pi_2 - C^T \quad (6.15)$$

(The solution to this matrix equation always exists because the spectrum of S is disjoint from the spectrum of I .) We then have

$$\begin{aligned} \dot{\tilde{y}}_2 &= \dot{y}_2 - z^T \Pi_2 \dot{w} - \dot{z}^T \Pi_2 w \\ &= -y_2 - z^T C^T w - z^T \Pi_2 S w - f^T(y, z, w) \Pi_2 w \\ &= -y_2 + z^T \Pi_2 w - f^T(y, z, w) \Pi_2 w \\ &= -\tilde{y}_2 + g_2(\tilde{y}, z, w) \end{aligned} \quad (6.16)$$

where $g_2(0, 0, w) = 0$ and $g_2'(0, 0, w) = 0$. We make the same kind of coordinate change for y_1 . We choose $\tilde{y}_1 = y_1 - z^T z \Pi_1 w$ where Π_1 solves the matrix equation

$$\Pi_1 S = -I \Pi_1 - D^T \quad (6.17)$$

We then have

$$\begin{aligned} \dot{\tilde{y}}_1 &= \dot{y}_1 - z^T z \Pi_1 \dot{w} - 2z^T \dot{z} \Pi_1 w \\ &= -y_1 - z^T z D^T w - z^T z \Pi_1 S w - 2z^T f(y, z, w) \Pi_1 w \\ &= -y_1 + z^T z \Pi_1 w - 2z^T f(y, z, w) \Pi_1 w \\ &= -\tilde{y}_1 + g_1(\tilde{y}, z, w) \end{aligned} \quad (6.18)$$

where $g_1(0, 0, w) = 0$, $g_1'(0, 0, w) = 0$ and $g_1''(0, 0, w) = 0$.

Now, from lemma 6.1 there is a center manifold $\tilde{y} = h(z, w)$, $|z| < \delta$, $|w| < M$ for

$$\begin{aligned} \dot{\tilde{y}} &= -I \tilde{y} + g(\tilde{y}, z, w) \\ \dot{z} &= \tilde{f}(\tilde{y}, z, w) \\ \dot{w} &= S w \end{aligned} \quad (6.19)$$

where $h(0, w) = 0$ and $h'(0, w) = 0$. In fact, since $g_1''(0, 0, w) = 0$, one can use an approximation theorem [Carr, 1981, theorem 3] or calculate to show that $h_1''(0, w) = 0$ (where $h = [h_1, h_2]^T$). Now it is sufficient to analyze the dynamics of the reduced system

$$\begin{aligned} \dot{z} &= \tilde{f}(h(z, w), z, w) \\ \dot{w} &= S w \end{aligned} \quad (6.20)$$

Further, since $h(0, w) = 0$ and the dynamics of w are autonomous with $|w(t)| < M$ for all $t \geq 0$ for some $M > 0$, it is sufficient to check the stability of $z = 0$ for the following “time-varying” nonlinear differential equation:

$$\begin{aligned} \dot{z}_1 &= (h_1 + z^T z \Pi_1 w)(-h_2 + w^T S^T \Pi_2^T z) \\ &\vdots \\ \dot{z}_m &= \frac{1}{m!} (h_1 + z^T z \Pi_1 w)^m (-h_2 + w^T S^T \Pi_2^T z) \end{aligned} \quad (6.21)$$

First, because $h(0, w) = 0$, $h'(0, w) = 0$ and $h_1''(0, w) = 0$ we can write the dynamics of z as

$$\begin{aligned} \dot{z}_1 &= (z^T z \Pi_1 w) z^T \Pi_2 S w + O(z)^4 \\ \dot{z}_2 &= \frac{1}{2} (z^T z \Pi_1 w)^2 (z^T \Pi_2 S w) + O(z)^6 \\ &\vdots \\ \dot{z}_m &= \frac{1}{m!} (z^T z \Pi_1 w)^m (z^T \Pi_2 S w) + O(z)^{2(m+1)} \end{aligned} \quad (6.22)$$

Now we determine expressions for Π_1 and Π_2 to examine the explicit time dependence of (6.22). From the block structure of S and C it follows that Π_2 also has a block structure where the j th block satisfies the matrix equation

$$\Pi_{2jj} S_{jj} = -I \Pi_{2j} - C_{jj}^T$$

It can be shown that

$$\Pi_{2jj} = \begin{bmatrix} -\frac{j}{1+j^2} c_j & -\frac{1}{1+j^2} c_j \end{bmatrix}$$

and, hence,

$$\Pi_{2jj} S_{jj} = \begin{bmatrix} \frac{j}{1+j^2} c_j & -\frac{j^2}{1+j^2} c_j \end{bmatrix}$$

Thus we have

$$w^T S^T \Pi_2^T z = \sum_{j=1}^m c_j \left[\frac{j}{1+j^2} \sin(jt) - \frac{j^2}{1+j^2} \cos(jt) \right] z_j \quad (6.23)$$

Now from (6.17) it can be shown that

$$\Pi_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix}$$

so that $z^T z \Pi_1 w = z^T z \cos(t)$. We now consider the product

$$\frac{1}{i!} (z^T z \Pi_1 w)^i (w^T S^T \Pi_2^T z)$$

given by

$$\frac{1}{i!} (z^T z \cos(t))^i \left(\sum_{j=1}^m c_j \left[\frac{j}{1+j^2} \sin(jt) - \frac{j^2}{1+j^2} \cos(jt) \right] z_j \right) \quad (6.24)$$

Using the identity

$$\cos(t) \cos(kt) = \frac{1}{2} [\cos((k-1)t) + \cos((k+1)t)]$$

it can be shown that

$$\cos^i(t) = \sum_{k=1}^{\ell} \alpha_{ik} \cos([i - 2(k-1)]t) \quad (6.25)$$

where $\alpha_{ik} > 0$ and $\ell = \frac{i}{2} + 1$ if i is even and $\frac{i+1}{2}$ if i is odd.

At this point, we would like to apply averaging to the terms in (6.24) to conclude asymptotic stability. However, since we are not using high frequency sinusoids and we do not have exponential stability for the averaged system, general averaging results do not apply. Nevertheless, a very specific averaging result which covers the class of systems we

have can be asserted. We describe this result in the next two lemmas. The uniformly higher order characteristic of our equations eliminates the need for a small parameter (or alternatively, very high frequencies). We are able to find a case specific Lyapunov function that demonstrates asymptotic stability in the presence of small time-varying disturbances without requiring exponential stability.

Lemma 6.2 (“Averaging” transformation) *Consider the time-varying nonlinear system*

$$\dot{x} = f(x, t) \quad (6.26)$$

where f is of period T in t and is C^r and the i th entry of the vector f satisfies $f_i = O(x)^{2i+1}$. Then there exists a C^r local change of coordinates $x = y + \Psi(y, t)$ under which (6.26) becomes

$$\dot{y} = \bar{f}(y) + \hat{f}(y, t) \quad (6.27)$$

where \bar{f} is the time average of f and $\hat{f}_i(y, t) = O(y)^{2i+2}$ and of period T in t .

Proof. See appendix.

Lemma 6.3 (Case Specific Lyapunov result) *Consider the system*

$$\dot{y} = \bar{f}(y) + \tilde{f}(y, t) \quad (6.28)$$

where $y \in \mathbb{R}^n$. If

$$|\tilde{f}_i(y, t)| \leq \beta_i \|y\|^{2(1+i)} \quad (6.29)$$

for all y in some open neighborhood of the origin and

$$\bar{f}(y) = A\psi(y) \quad (6.30)$$

where A is a square lower triangular matrix with $a_{ii} < 0$ for $i = 1, \dots, n$ and

$$\psi_i(y) = y_i \|y\|^{2i} \quad (6.31)$$

then the origin of (6.28) is locally asymptotically stable.

Proof. See appendix.

Now we make the coordinate transformation of lemma 6.2 to pull out the lowest order terms on each line of equation (6.22) with nonzero average. Using (6.24) and (6.25)

we can show that this transformation yields a system possessing the (triangular) structure of the system in lemma 6.3. In fact, the a_{jj} 's of lemma 6.3 are given by

$$a_{jj} = -\frac{1}{2j!} \frac{j^2}{1+j^2} \alpha_{j1} c_j$$

Since $\alpha_{j1}, c_j > 0$, the local asymptotic stability of the origin of (6.3), (6.7) then follows from lemma 6.3. \square

6.3 Global Stabilization

In this section we propose a class of smooth, time-varying, *globally* stabilizing inputs for (6.3). Near the origin these control laws will exactly match the locally stabilizing control laws proposed in section 6.2. We introduce saturation functions in these control laws to eliminate destabilizing effects that take place away from the origin.

Theorem 6.2 *Given any pair of inputs*

$$\begin{aligned} u_1 &= -x_1 - \sigma((\sum_{j=1}^{n-2} x_{j+2}^2)^{\frac{1}{2}})^2 (\sin(t) - \cos(t)) \\ u_2 &= -x_2 - \sum_{j=1}^{n-2} c_j \sigma(x_{j+2}) \cos(jt) \end{aligned} \quad (6.32)$$

with $c_j > 0$ and with $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ a nondecreasing C^3 function satisfying

1. $\sigma(s) = s$ when $|s| \leq \delta$
2. $|\sigma(s)| \leq \epsilon$ for all $s \in \mathbb{R}$

for some $0 < \delta < \epsilon$, $\exists \epsilon_0$ such that if $\epsilon < \epsilon_0$ then the origin of (6.3) is globally asymptotically stable.

Proof of theorem 6.2. The proof of theorem 6.2 is very much in the spirit of the proof of theorem 6.1. We begin by defining the same oscillators as in (6.9) and we make the same partition of the state space as in (6.11). For (6.3), (6.32) we have

$$\begin{aligned} \dot{y}_1 &= -y_1 - w^T D \sigma(\|z\|)^2 \\ \dot{y}_2 &= -y_2 - w^T C \bar{\sigma}(z) \\ \dot{z} &= f(y, z, w) \\ \dot{w} &= Sw \end{aligned} \quad (6.33)$$

where

$$\bar{\sigma}(z) = \begin{bmatrix} \sigma(z_1) & \cdots & \sigma(z_m) \end{bmatrix}^T$$

The matrices C and D are as defined in (6.13) and (6.14) respectively.

We make the coordinate change

$$\begin{aligned} \tilde{y}_1 &= y_1 - \sigma(\|z\|)^2 \Pi_1 w \\ \tilde{y}_2 &= y_2 - \bar{\sigma}^T(z) \Pi_2 w \end{aligned}$$

where Π_1 and Π_2 satisfy (6.17) and (6.15) respectively.

We then have

$$\begin{aligned} \dot{\tilde{y}}_1 &= -\tilde{y}_1 - 2\sigma(\|z\|) \frac{\partial \sigma}{\partial \|z\|} \|z\|^{-1} z^T f(y, z, w) \Pi_1 w \\ &= -\tilde{y}_1 + g_1(\tilde{y}, z, w) \\ \dot{\tilde{y}}_2 &= -\tilde{y}_2 - f^T(y, z, w) \frac{\partial \bar{\sigma}^T}{\partial z} \Pi_2 w \\ &= -\tilde{y}_2 + g_2(\tilde{y}, z, w) \end{aligned} \tag{6.34}$$

We now wish to show that given ϵ sufficiently small, there is a center manifold $\tilde{y} = h(z, w)$, $z \in \mathbb{R}^m$, $|w| < M$ for

$$\begin{aligned} \dot{\tilde{y}} &= -I\tilde{y} + g(\tilde{y}, z, w) \\ \dot{z} &= \tilde{f}(\tilde{y}, z, w) \\ \dot{w} &= Sw \end{aligned} \tag{6.35}$$

where $h(0, w) = 0$ and $h'(0, w) = 0$. To do so, following the proof of [Carr, 1981, theorem 1], we must show that given $M > 0$ and for ϵ sufficiently small, there exists a continuous function $\kappa(\epsilon)$ with $\kappa(0) = 0$ such that

$$\begin{aligned} |\tilde{f}(\tilde{y}, z, w)| + |g(\tilde{y}, z, w)| &\leq \epsilon \kappa(\epsilon) \\ |\tilde{f}(\tilde{y}, z, w) - \tilde{f}(\tilde{y}', z', w')| &\leq \kappa(\epsilon) (|\tilde{y} - \tilde{y}'| + |z - z'| + |w - w'|) \\ |g(\tilde{y}, z, w) - g(\tilde{y}', z', w')| &\leq \kappa(\epsilon) (|\tilde{y} - \tilde{y}'| + |z - z'| + |w - w'|) \end{aligned} \tag{6.36}$$

for all $z, z' \in \mathbb{R}^m$, and all $w, w' \in \mathbb{R}^p$ with $|w|, |w'| < M$ and all $y, y' \in \mathbb{R}^n$ with $|y|, |y'| < \epsilon$. It can be shown that \tilde{f} satisfies this relationship, since every dependence on z in \tilde{f} is as the argument of a saturation function bounded by ϵ . Then, since \tilde{f} satisfies these relationships, it follows from (6.34) that g also satisfies these relationships by noting that σ is C^3 and hence its partials are bounded and $|\frac{z}{\|z\|}| \leq b$ for some positive constant b .

Next we show that, for ϵ sufficiently small, the manifold $h(z, w)$ is globally attractive. First, observe that the dynamics of y are of an exponentially stable linear system

perturbed by bounded disturbances of magnitude proportional to ϵ . Consequently, after some finite time y is contained in a ball of radius proportional to ϵ . Then, by the nature of the coordinate change from y to \tilde{y} , \tilde{y} is also contained in a ball of radius proportional to ϵ . Now we know that the manifold is locally attractive, so for ϵ sufficiently small the ϵ ball is contained in the basin of attraction for $h(z, w)$. Hence, the manifold $h(z, w)$ is globally attractive.

We will eventually establish that the dynamics

$$\dot{z} = \tilde{f}(h(z, w) + (\tilde{y} - h(z, w)), z, w) \quad (6.37)$$

have the “converging input bounded state” property of [Sontag, 1989] with $e \equiv \tilde{y} - h(z, w)$ as input. Then, since $h(0, w) = 0$ it is sufficient to consider the dynamics of

$$\dot{z} = \tilde{f}(h(z, w), z, w) \quad (6.38)$$

For now, we simply consider the global stability property of (6.38). To do so, we begin by establishing a bound on $h(z, w)$. We follow the approximation of center manifolds in [Carr, 1981]. As in [Carr, 1981], for functions $\phi : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^2$ which are C^1 in a neighborhood of the origin we define the operator N to be

$$(N\phi)(z, w) = \frac{\partial \phi}{\partial z} \tilde{f}(\phi(z, w), z, w) + \frac{\partial \phi}{\partial w} Sw + I\phi(z, w) - g(\phi(z, w), z, w)$$

where g is defined in (6.34). We choose to approximate $h(z, w)$ by the function $\phi(z, w) \equiv 0$. We then have

$$(N\phi)(z, w) = -g(0, z, w)$$

It follows from (6.34) and \tilde{f} that $(N\phi)(z, w) = O(\sigma(\|z\|)^3)$ for all $z \in \mathbb{R}^m$ and all $w \in \mathbb{R}^p$ with $|w| < M$. We can then mimic the proof of [Carr, 1981, theorem 3] to establish that

$$|h(z, w) - \phi(z, w)| = |h(z, w)| = O(\sigma(\|z\|)^3) \quad (6.39)$$

for all $z \in \mathbb{R}^m$ and all $w \in \mathbb{R}^p$ with $|w| < M$.

We are now ready to establish lemmas similar to lemmas 6.1 and 6.2 that apply to the global stability problem.

Lemma 6.4 (Global “Averaging” transformation)
time-varying system

Consider the nonlinear

$$\dot{x} = f(x, t) \quad (6.40)$$

where f is of period T in t and is C^r and where the i th entry of the vector f satisfies $f_i = O(\sigma(\|x\|)^{2i+1})$. If the ϵ associated with the saturation function σ is sufficiently small, then there exists a C^r global change of coordinates $x = y + \Psi(y, t)$ under which (6.40) becomes

$$\dot{y} = \bar{f}(y) + \hat{f}(y, t) \quad (6.41)$$

where \bar{f} is the time average of f and \hat{f} is of period T in t with $\hat{f}_i(y, t) = O(\sigma(\|y\|)^{2i+2})$.

Proof. See appendix.

Lemma 6.5 (Global Case Specific Lyapunov result) *Consider the system*

$$\dot{y} = \bar{f}(y) + \tilde{f}(y, t) \quad (6.42)$$

where $y \in \mathbb{R}^n$. If

$$|\tilde{f}_i(y, t)| \leq \beta_i \sigma(\|y\|)^{2(1+i)} \quad (6.43)$$

for all $y \in \mathbb{R}^n$ and

$$\bar{f}(y) = A\psi(y) \quad (6.44)$$

where A is a square lower triangular matrix with $a_{ii} < 0$ for $i = 1, \dots, n$ and

$$\psi_i(y) = \sigma(y_i) \sigma(\|y\|)^{2i} \quad (6.45)$$

then, for ϵ sufficiently small, the origin of (6.42) is globally asymptotically stable.

Proof. See appendix.

Now using the expression for Π_1 and Π_2 from the proof of theorem 6.1 we can show that these lemmas apply and thus the reduced dynamics are globally asymptotically stable. It remains to verify that the z dynamics have the “converging input bounded state” property of [Sontag, 1989]. Since \tilde{f} is bounded for bounded e , and hence z is bounded for all finite time, it is sufficient to prove the following result:

Lemma 6.6 (Converging input bounded state) *Under the conditions of lemma 6.5, if the perturbation in the equation*

$$\dot{y} = \bar{f}(y) + \tilde{f}(y, t) + p(t) \quad (6.46)$$

satisfies $|p(t)| \leq \nu$, then, for ν sufficiently small, y satisfies $|y(t)| \leq G$ for all $t \geq 0$ for some $G > 0$.

Proof. See appendix.

Now the main theorem of [Sontag, 1989] provides global asymptotic stability for the system (6.3), (6.32). \square

It is also possible to deduce a *locally* stabilizing control law for (6.2) without using the transformation to power form given in (6.4).

Corollary 6.1 *Every pair of inputs*

$$\begin{aligned} v_1 &= -\xi_1 - (\sum_{j=1}^{n-2} \xi_{j+2}^2)(\sin(t) - \cos(t)) \\ v_2 &= -\xi_2 - \sum_{j=1}^{n-2} (-1)^j c_j \xi_{j+2} \cos(jt) \end{aligned} \quad (6.47)$$

with $c_j > 0$ locally asymptotically stabilizes the origin of (6.2).

Proof of corollary 6.1. Let the transformation (6.4) that takes us from chain form to power form be written as $x = \Phi(\xi) = T\xi + \bar{\Phi}(\xi)$ where $\bar{\Phi}(\xi)$ is higher order. Let $v_{chain}(\cdot)$ denote the controls given by (6.47) and let $u_{power}(\cdot)$ denote the controls given by 6.7. Then we have $v_{chain}(\xi) = u_{power}(T^{-1}x)$. For (6.2), (6.47) if we make the transformation $x = \Phi(\xi)$, we have a power form system (6.3) with controls given by (6.7) plus higher order terms. Now the proof is exactly equivalent to the proof of theorem 6.1 since the higher order terms would simply contribute higher order terms on the manifold which were shown to be unimportant. \square

6.4 Example: an automobile

Our example system will be a simple kinematic model of an automobile as shown in figure 1. This system is controllable using two levels of Lie Brackets. A derivation of the kinematic equations may be found in [Murray and Sastry, 1990]. A sketch of the car is found in Figure 6.1

$$\begin{aligned} \dot{x} &= \cos(\theta)u_1 \\ \dot{y} &= \sin(\theta)u_1 \\ \dot{\phi} &= u_2 \\ \dot{\theta} &= \frac{1}{L} \tan(\phi)u_1 \end{aligned} \quad (6.48)$$

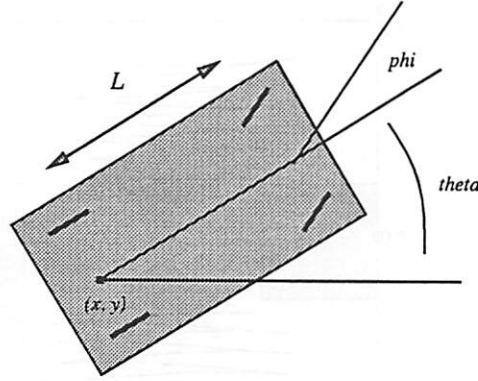


Figure 6.1: Kinematic model of the car

where (x, y) is the position of the car in the plane, ϕ is the angle of the front wheels with respect to the car (or the steering wheel angle), θ is the orientation of the car with respect to some reference frame, and the constant L is the length of the wheel base. For simplicity, we choose $L = 1$.

The following change of coordinates will put the car into power form coordinates, locally:

$$\begin{aligned} x_1 &= x \\ x_2 &= \sec^3(\theta) \tan(\phi) \\ x_3 &= x \sec^3(\theta) \tan(\phi) - \tan(\theta) \\ x_4 &= y + \frac{1}{2} x^2 \sec^3(\theta) \tan(\phi) - x \tan(\theta) \end{aligned}$$

with the following input transformation:

$$\begin{aligned} u_1 &= v_1 \sec(\theta) \\ u_2 &= -3v_1 \sec(\theta) \sin^2(\phi) \tan(\theta) + v_2 \cos^3(\theta) \cos^2(\phi) \end{aligned}$$

The control law used for the simulation was:

$$\begin{aligned} v_1 &= -x_1 - \sigma^2 \left(\sqrt{x_3^2 + x_4^2} \right) (\sin(t) - \cos(t)) \\ v_2 &= -x_2 - k\sigma(x_3) \cos(t) - k\sigma(x_4) \cos(2t) \end{aligned}$$

The gain k was chosen to be 2, and the ϵ of the saturating function $\sigma(\cdot)$ was chosen to be $\epsilon = 0.5$. The initial conditions chosen for these two simulations were $(0, \pm 1, 0, 0)$. The plot

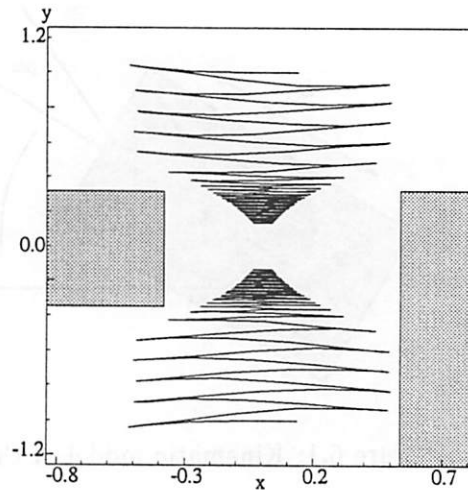


Figure 6.2: Phase plane plot, x versus y , of the two simulations. Note the effects of the saturation function on the limits of travel in the x direction.

demonstrates the effect using a saturation function. At first the error is large enough to cause the saturation functions to limit the magnitude of the input sinusoids, hence limiting the x and ϕ travel of the car. After the error drops sufficiently, the controls are no longer saturated and the range of travel drops.

6.5 Summary and Discussion

We have presented a control law which globally asymptotically stabilizes a system in power form. This control law uses sinusoids at integrally related frequencies to achieve motion in bracketing directions and saturation functions to achieve globally convergence. Convergence in the coordinate directions can be adjusted by setting the appropriate weights in the control law. By making use of a feedback transformation to convert a nonholonomic system into power form, we have applied this control law to a kinematic model of an automobile.

The primary limitation of the control law presented here is that it can only be applied to systems which are feedback equivalent to a system in power form. However, there is strong evidence to suggest that control laws of this form can be extended to more general nonholonomic systems by using an “extended system” such as that used by Sussmann and co-workers [Lafferriere and Sussmann, 1991, Sussmann and Liu, 1991]. The generalization

of the results presented here would be to systems which are controllable through the input vector fields and Lie products of the form $\text{ad}_{g_1}^k g_2$. Controllers for this same basic class of systems can be found in the recent work of Pomet [Pomet, 1992]. The extension of the ideas presented here to this more general situation is the subject of current research.

6.6 Appendix

6.6.1 Proof of lemma 6.1

The proof of lemma 6.1 mimics the proof of [Carr, 1981, theorem 1, pages 16-19]. Accordingly, let $\psi : \mathbb{R}^+ \rightarrow [0, 1]$ be a C^∞ function with $\psi(s) = 1$ when $s \leq 1$ and $\psi(s) = 0$ when $s \geq 2$. Then for $\epsilon, M > 0$ define F and G by

$$\begin{aligned} F(y, z, w) &= f(y, z\psi(\frac{|z|}{\epsilon}), w\psi(\frac{|w|}{M})) \\ G(y, z, w) &= g(y, z\psi(\frac{|z|}{\epsilon}), w\psi(\frac{|w|}{M})) \end{aligned}$$

We prove that, given $M > 0$, the system

$$\begin{aligned} \dot{y} &= By + G(y, z, w) \\ \dot{z} &= Az + F(y, z, w) \\ \dot{w} &= Sw \end{aligned} \tag{6.49}$$

has a center manifold $y = h(z, w)$, $z \in \mathbb{R}^m$, $w \in \mathbb{R}^p$ for ϵ sufficiently small. Then since F and G agree with f and g for all $|z| < \epsilon$ and for all $|w| < M$, this proves the existence of a local center manifold for (6.8). The existence of the global center manifold for (6.49) can be demonstrated using the same contraction mapping calculations as in the proof of [Carr, 1981, theorem 1] since we can show, as was needed in [Carr, 1981], that there is a continuous function $\kappa(\epsilon)$ with $\kappa(0) = 0$ such that

$$\begin{aligned} |F(y, z, w)| + |G(y, z, w)| &\leq \epsilon \kappa(\epsilon) \\ |F(y, z, w) - F(y', z', w')| &\leq \kappa(\epsilon) (|y - y'| + |z - z'| + |w - w'|) \\ |G(y, z, w) - G(y', z', w')| &\leq \kappa(\epsilon) (|y - y'| + |z - z'| + |w - w'|) \end{aligned}$$

for all $z, z' \in \mathbb{R}^m$, and all $w, w' \in \mathbb{R}^p$ and all $y, y' \in \mathbb{R}^n$ with $|y|, |y'| < \epsilon$. Following [Carr, 1981], these inequalities yield a center manifold $y = h(z, w)$ with $h(0, w) = 0$ and $h'(0, w) = 0$. \square

6.6.2 Proof of lemma 6.2

The proof of this lemma follows closely the exposition of [Guckenheimer and Holmes, 1983, pages 168-169]. We split $f(x, t)$ as

$$f(x, t) = \bar{f}(x) + \tilde{f}(x, t)$$

where \bar{f} is the mean of f and \tilde{f} is its oscillating part. Now we make the coordinate change

$$x = y + \Psi(y, t) \quad (6.50)$$

where Ψ will be specified. (We will show Ψ to be strictly higher order so that this is a valid coordinate change locally.) Differentiating we have

$$[I + D_y \Psi] \dot{y} + \frac{\partial \Psi}{\partial t} = \dot{x} = \bar{f}(y + \Psi) + \tilde{f}(y + \Psi, t) \quad (6.51)$$

Reorganizing we get

$$\dot{y} = [I + D_y \Psi]^{-1} [\bar{f}(y + \Psi) + \tilde{f}(y + \Psi, t) - \frac{\partial \Psi}{\partial t}] \quad (6.52)$$

We now choose Ψ such that

$$\frac{\partial \Psi}{\partial t} = \tilde{f}(y, t)$$

(Since \tilde{f} has zero mean, Ψ is bounded as a function of time.) This choice produces

$$\dot{y} = [I + D_y \Psi]^{-1} [\bar{f}(y) + f(y + \Psi, t) - f(y, t)] \quad (6.53)$$

Expanding with respect to Ψ we have

$$\begin{aligned} \dot{y} &= [I - D_y \Psi + O(\|D_y \Psi\|^2)] [\bar{f} + D_y f \Psi + O(\|\Psi\|^2)] \\ &\equiv \bar{f}(y) + \hat{f}(y, t) \end{aligned} \quad (6.54)$$

Now we check the order of \hat{f}_i . The first term we consider is the term

$$I[f(y + \Psi, t) - f(y, t)]$$

It suffices to check the order of the i th entry of

$$D_y f \cdot \Psi$$

Accordingly, the entries of the i th row of $D_y f$ are of order $2i$. Further, since

$$\frac{\partial \Psi_i}{\partial t} = \tilde{f}_i(y, t)$$

it follows that Ψ_i is of order $1 + 2i$ in y . Hence, the lowest order in w is 3 ($i=1$) and the product yields terms of order $2i + 3$.

The final terms we need to consider are given by $D_y \Psi N(y, t)$ where

$$N_i(y, t) \equiv \bar{f}_i(y) + f_i(y + \Psi, t) - f_i(y, t)$$

By assumption, we know that $N_i(y, t)$ is of order $1 + 2i$ in y . Since Ψ_i is of order $1 + 2i$ it follows that the entries of the i th row of $D_y \Psi$ are of order $2i$. The lowest order in $N(y, t)$ is 3 ($i = 1$) and so the i th entry of $D_y \Psi N(y, t)$ is of order $2i + 3$. \square

6.6.3 Proof of lemma 6.3

Consider the Lyapunov function

$$V = \sum_{i=1}^n \frac{\alpha_i}{2(n+1-i)} y_i^{2(n+1-i)} \quad (6.55)$$

where the α_i 's will be specified later. The derivative of V along the trajectories of (6.28) is given by

$$\begin{aligned} \dot{V} &= \sum_{i=1}^n \alpha_i y_i^{2(n-i)+1} [\bar{f}_i(y) + \tilde{f}_i(y, t)] \\ &\leq [\sum_{i=1}^n \alpha_i y_i^{2(n-i)+1} A_i \psi(y)] + \gamma \|y\|^{2(n+1)+1} \end{aligned} \quad (6.56)$$

where A_i is the i th row of the matrix A and γ is a constant that depends on α_i, β_i for $i = 1, \dots, n$. We claim that the α_i 's can be chosen such that

$$S(y) \equiv \sum_{i=1}^n \alpha_i y_i^{2(n-i)+1} A_i \psi(y) \leq -\|y\|^{2(n+1)} \quad (6.57)$$

This will give

$$\dot{V} \leq -(1 - \gamma \|y\|) \|y\|^{2(n+1)} \quad (6.58)$$

and hence asymptotic stability of the origin for $\|y\|$ sufficiently small. The proof of this claim will involve an iterative process of completing squares, bookkeeping coefficients and judiciously choosing the α_i 's.

We begin by multiplying the i th term ($i = 1, \dots, n$) in the summation $S(y)$ by

$$\left(\frac{\|y\|}{\|y\|} \right)^{2(n-i+1)}$$

for $\|y\| \neq 0$. This yields

$$S(y) = \|y\|^{2(n+1)} \sum_{i=1}^n \left[\alpha_i \frac{y_i^{2(n-i)+1}}{\|y\|^{2(n-i+1)}} \sum_{j=1}^i a_{ij} y_j \right] \equiv \|y\|^{2(n+1)} T(y) \quad (6.59)$$

Now we begin to complete squares by first considering the quadratic terms (i.e. those terms generated by $i = n$ in the summation). Doing so, we have

$$T(y) \leq \left[\sum_{i=1}^{n-1} \alpha_i \frac{y_i^{2(n-i)+1}}{\|y\|^{2(n-i+1)}} \sum_{j=1}^i a_{ij} y_j \right] + \alpha_n \frac{a_{nn}}{2} \left(\frac{y_n}{\|y\|} \right)^2 + \alpha_n \sum_{j=1}^{n-1} \tilde{a}_{nj} \left(\frac{y_j}{\|y\|} \right)^2 \quad (6.60)$$

Here \tilde{a}_{nj} are positive constants that depend on a_{nn}, a_{nj} , and n . Now, by the definition of $\|y\|$, we have

$$y_n^2 = \|y\|^2 - y_1^2 - \dots - y_{n-1}^2 \quad (6.61)$$

and choosing

$$\alpha_n = -\frac{4}{a_{nn}} \quad (6.62)$$

we have

$$T(y) \leq \left[\sum_{i=1}^{n-1} \alpha_i \frac{y_i^{2(n-i)+1}}{\|y\|^{2(n-i+1)}} \sum_{j=1}^i a_{ij} y_j \right] - 2 + \alpha_n \sum_{j=1}^{n-1} \tilde{a}_{nj} \left(\frac{y_j}{\|y\|} \right)^2 \quad (6.63)$$

with the \tilde{a}_{nj} 's appropriately redefined positive constants.

Now we consider the quartic terms generated by $i = n - 1$ in the summation. Again completing squares, and using the fact that

$$\left(\frac{y_{n-1}}{\|y\|} \right)^2 \left(\frac{y_k}{\|y\|} \right)^2 \leq \left(\frac{y_k}{\|y\|} \right)^2$$

we have

$$\begin{aligned} T(y) \leq & -2 + \left[\sum_{i=1}^{n-2} \alpha_i \frac{y_i^{2(n-i)+1}}{\|y\|^{2(n-i+1)}} \sum_{j=1}^i a_{ij} y_j \right] + \alpha_n \left[\sum_{j=1}^{n-1} \tilde{a}_{nj} \left(\frac{y_j}{\|y\|} \right)^2 \right] \\ & + \alpha_{n-1} \frac{a_{n-1,n-1}}{2} \left(\frac{y_{n-1}}{\|y\|} \right)^4 + \alpha_{n-1} \sum_{j=1}^{n-2} \tilde{a}_{n-1,j} \left(\frac{y_j}{\|y\|} \right)^2 \end{aligned} \quad (6.64)$$

We now choose α_{n-1} sufficiently large so that

$$\alpha_{n-1} \frac{a_{n-1,n-1}}{2} \left(\frac{y_{n-1}}{\|y\|} \right)^4 + \alpha_n \tilde{a}_{n,n-1} \left(\frac{y_{n-1}}{\|y\|} \right)^2 \leq \frac{1}{n-1} \quad (6.65)$$

In fact, we continue this process of completing squares and choosing α_i large enough such that all the terms involving y_i are bounded by $\frac{1}{n-1}$. This can always be done because of the triangular structure. Finally we have that

$$T(y) \leq -2 + \sum_{i=1}^{n-1} \frac{1}{n-1} \leq -1 \quad (6.66)$$

From this we conclude that

$$\mathcal{S}(y) \leq -\|y\|^{2(n+1)} \quad (6.67)$$

for $\|y\| \neq 0$ and our claim is established. \square

6.6.4 Proof of lemma 6.4

The proof of this lemma is a virtual duplication of the proof of lemma 6.2. We split f as before and make a similar coordinate change

$$x = y + \Psi(y, t) \quad (6.68)$$

This time we will establish that for ϵ sufficiently small, this is a globally valid coordinate transformation. In fact, we again pick

$$\frac{\partial \Psi}{\partial t} = \tilde{f}(y, t) \quad (6.69)$$

Since $\tilde{f}_i(y, t) = O(\sigma(\|y\|)^{2i+1})$ and σ is C^3 it follows that $\Psi = O(\sigma(\|y\|)^{2i+1})$ and $D_y \Psi = O(\sigma(\|y\|)^{2i})$. We can now use the same kind of bookkeeping as in the proof of lemma 6.2 to establish the result. \square

6.6.5 Proof of lemma 6.5

The proof of this lemma is a virtual duplication of the proof of lemma 6.3. This time we start with the Lyapunov function

$$V = \sum_{i=1}^n \alpha_i \int_0^{y_i} \sigma^{2(n-i)+1}(s) ds \quad (6.70)$$

where the α_i 's will be specified. The derivative along the trajectories of (6.42) is given by

$$\begin{aligned} \dot{V} &= \sum_{i=1}^n \alpha_i \sigma^{2(n-i)+1}(y_i) [\tilde{f}_i(y) + \tilde{f}_i(y, t)] \\ &\leq [\sum_{i=1}^n \alpha_i \sigma^{2(n-i)+1}(x_i) A_i \psi(y)] + \gamma \sigma(\|y\|)^{2(n+1)+1} \end{aligned} \quad (6.71)$$

where A_i is the i th row of the matrix A and γ is a constant that depends on α_i, β_i for $i = 1, \dots, n$. We claim that the α_i 's can be chosen such that

$$S(y) \equiv \sum_{i=1}^n \alpha_i \sigma^{2(n-i)+1}(y_i) A_i \psi(y) \leq -\sigma(\|y\|)^{2(n+1)} \quad (6.72)$$

This will give

$$\dot{V} \leq -(1 - \gamma \sigma(\|y\|)) \sigma(\|y\|)^{2(n+1)} \quad (6.73)$$

and hence global asymptotic stability of the origin for ϵ sufficiently small. To prove this claim we now follow the proof of lemma 6.3, everywhere replacing $\|y\|^k$ by $\sigma(\|y\|)^k$ and

y_i^k by $\sigma(y_i)^k$. The only difficulty we have is that the equality (6.61) does not carry over. However, it is sufficient to have the inequality

$$\sigma(y_n)^2 \geq \sigma(\|y\|)^2 - \sigma(y_1)^2 - \dots - \sigma(y_{n-1})^2 \quad (6.74)$$

Completely squares and judiciously choosing the α_i 's again produces the result. \square

6.6.6 Proof of lemma 6.6

The proof of this lemma follows from the proof of lemma 6.5. We use the same Lyapunov function V as in (6.70). From lemma 6.5 and from the nature of the partial derivative of V with respect to y , we have, for ϵ sufficiently small, that the derivative of V along the trajectories of the perturbed system satisfies

$$\dot{V} \leq -[1 - \gamma\sigma(\|y\|) - \gamma \frac{\sigma(\|y\|)}{\sigma(\|y\|)^{2(n+1)}}\nu]\sigma(\|y\|)^{2(n+1)} \quad (6.75)$$

Since we are simply trying to establish that y is bounded we can assume without loss of generality that $\delta^{2(n+1)} < \sigma(\|y\|)^{2(n+1)} \leq \epsilon^{2(n+1)}$. Therefore we see that if

$$1 - \gamma\epsilon - \gamma \frac{\epsilon}{\delta^{2(n+1)}}\nu > 0 \quad (6.76)$$

then $\dot{V} < 0$ for $|y|$ sufficiently large. Since V is proper, this implies that $|y|$ is bounded. We see that, given ϵ such that $1 - \gamma\epsilon > 0$, (6.76) is satisfied for all ν satisfying

$$\nu < (1 - \gamma\epsilon) \frac{\delta^{2(n+1)}}{\gamma\epsilon} \quad (6.77)$$

\square

Chapter 7

Recent Adaptive Control Algorithms

In this chapter, we recast a recently developed adaptive stabilization algorithm for pure-feedback form nonlinear systems into an error-based algorithm. This enlarges the subset of pure-feedback form nonlinear systems that can be stabilized globally (with respect to the state of the system).

7.1 Introduction

Several recent nonlinear adaptive control algorithms have focused on stabilization and tracking for systems that can be described in pure-feedback form. The development of these algorithms were initiated in [Kanellakopoulos *et al.*, 1991] and have been refined in [Krstic *et al.*, 1991]. These schemes fall into the category of *direct* adaptive control in that the parameter estimates are driven by the mismatch between the plant states and the control objective (stabilization or tracking) for these states. These algorithms have not been cast into an error-based or *indirect* framework. By indirect adaptive control we mean that the parameter estimates are driven by the mismatch between the plant states and a dynamic estimate of the plant states. For recent examples of this approach, see [Campion and Bastin, 1990], [Pomet and Praly, 1989] and [Teel *et al.*, 1991]. An appealing feature of the indirect approach is that parameter estimates that begin close to the actual parameter values remain close to the actual parameter values. This feature can play an important role in the feasibility of the adaptive control algorithm. For instance, consider the following

academic example:

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta x_2^3 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u\end{aligned}\tag{7.1}$$

This system is in pure-feedback form. For this system, the feasibility region of [Kanellakopoulos *et al.*, 1991] is expressed as a set $\mathcal{F} = B_x \times B_\theta$ where B_x is an open set in \mathbb{R}^3 and B_θ is an open set in \mathbb{R} such that

$$|1 + \theta x_2^2| > 0 \quad \forall x \in B_x \quad \forall \theta \in B_\theta$$

We see that one possible feasibility region is given by $B_x = \mathbb{R}^3$ and $B_\theta = \mathbb{R}_+$ so that the global stabilization problem is possible. However, the direct algorithms of [Kanellakopoulos *et al.*, 1991] and [Krstic *et al.*, 1991] cannot guarantee that θ remains in B_θ unless the initial state $x(0)$ is sufficiently small. Reformulating the algorithm of [Kanellakopoulos *et al.*, 1991] as an error-based algorithm will eliminate the restriction on the size of the initial state.

7.2 The Class of Systems and Feasibility Regions

For simplicity, we will consider single-input systems of the form

$$\begin{aligned}\dot{x}_1 &= \theta^T f_1(x_1, x_2) \\ \dot{x}_2 &= \theta^T f_2(x_1, x_2, x_3) \\ &\vdots \\ \dot{x}_{n-1} &= \theta^T f_{n-1}(x_1, \dots, x_n) \\ \dot{x}_n &= \theta^T [f_n(x) + g_n(x)u]\end{aligned}\tag{7.2}$$

Here $\theta \in \mathbb{R}^p \times \{1\}$ is the vector of unknown parameters augmented to allow for terms that are independent of the unknown parameters $\theta^* \in \mathbb{R}^p$. i.e.

$$\theta = \begin{bmatrix} \theta^* \\ 1 \end{bmatrix}$$

The vector $g_n \in \mathbb{R}^{p+1}$ is smooth and the smooth vectors $f_i \in \mathbb{R}^{p+1}$ are such that $f_i(0) = 0$. Geometric conditions for transforming a general single-input nonlinear system into this form (locally) generalize easily from the conditions in [Akhrif and Blankenship, 1988] and [Kanellakopoulos *et al.*, 1991].

We demonstrate our algorithm by solving the adaptive stabilization problem. (As in [Kanellakopoulos *et al.*, 1991], the algorithm presented here naturally extends to the tracking and multi-input problems.) Our algorithm is most powerful when the feasibility region is global in the state x but (possibly) not global in the parameter θ . Consequently, following [Kanellakopoulos *et al.*, 1991], we make the following definition:

Definition 7.1 *A feasibility region for the system (7.2) is any connected set $\mathcal{F} \subset \mathbb{R}^p \times \{1\}$ such that*

$$\begin{aligned} |\theta^T \frac{\partial f_i}{\partial x_{i+1}}| &> 0 \quad \text{for } i = 1, \dots, n-1 \\ |\theta^T g_n(x)| &> 0 \end{aligned}$$

for all $x \in \mathbb{R}^n$ and for all $\theta \in \mathcal{F}$.

Remarks.

1. As noted in [Kanellakopoulos *et al.*, 1991], the sets \mathcal{F} are connected sets where the system is full-state linearizable.
2. It is important to note that feasibility regions are connected. For example, in the case that $p = 1$ it may be true that the conditions of definition 1 are satisfied for all $\theta \neq 0 \times \{1\}$. However $\mathcal{F} = (0 \times \{1\})^c$ is not a feasibility region.

We now restrict the augmented parameter vector $\theta \in \mathbb{R}^p \times \{1\}$ so that our algorithm remains feasible. To do so, let $\{S_\theta^i\}$ be the collection of sets known to contain θ and define $S_\theta = \cap S_\theta^i$. Further, let $\{\mathcal{F}^j\}$ be the collection of feasibility regions such that $S_\theta \subset \mathcal{F}^j$ and define $\mathcal{F} = \cup \mathcal{F}^j$. (\mathcal{F} is connected since $\theta \in \mathcal{F}^j$.)

Assumption 7.1 *If $\mathcal{F} \neq \mathbb{R}^p \times \{1\}$ then we assume:*

1. *If $p = 1$ and \mathcal{F} is unbounded, then $\text{clos}(S_\theta) \subset \mathcal{F}$*
2. *otherwise, $S_\theta \subset B_{(r,\theta')} \times \{1\} \subset B_{(2r,\theta')} \times \{1\} \subset \mathcal{F}$ where $B_{(r,\theta')} \subset \mathbb{R}^p$ is a ball of radius r centered at some $\theta' \in \mathbb{R}^p$.*

Remark. We see that when the entire space $\mathbb{R}^p \times \{1\}$ is not a feasibility region, we restrict the possible values of the unknown parameter vector θ . In the case of one unknown parameter, we do not necessarily restrict θ to lie in a bounded set. For example, if $\mathcal{F} = \mathbb{R}^+ \times \{1\}$ then it is sufficient to know that $\theta \in \mathbb{R}_+ \times \{1\}$. If $\mathcal{F} = (0, +\infty) \times \{1\}$ then it is sufficient to know that $\theta \in [\epsilon, +\infty) \times \{1\}$ for some $\epsilon > 0$.

If $p = 1$ and \mathcal{F} is bounded or if $p > 1$, we restrict θ to lie in a bounded set. For example, if $\mathcal{F} = \mathbb{R}_+ \times \mathbb{R}_+ \times \{1\}$, then we require θ to lie in some ball such that a ball of twice the radius and centered at the same point is contained in \mathcal{F} . The reason for this will become clear in the stability proof.

7.3 The Stabilization Algorithm

We recast the basic algorithm of [Kanellakopoulos *et al.*, 1991] into an error-based algorithm.

Step 0. Define $z_1 = x_1$.

Step 1. The previous step gives

$$\dot{z}_1 = \theta^T f_1(x_1, x_2) \quad (7.3)$$

Now define

$$z_2 = \hat{\theta}_1^T f_1(x_1, x_2) \quad (7.4)$$

where $\hat{\theta}_1$ is an estimate of θ . Substituting (7.4) into (7.3) yields

$$\begin{aligned} \dot{z}_1 &= z_2 + [\theta - \hat{\theta}_1]^T f_1(x_1, x_2) \\ &= z_2 + [\theta - \hat{\theta}_1]^T w_1(z_1, z_2, \hat{\theta}_1) \end{aligned} \quad (7.5)$$

(We will demonstrate in the stability proof that assumption 1 ensures this algorithm is feasible and hence the inverse relation between z_2 and x_2 is well-defined. We write w_1 as a function of z_1, z_2 and $\hat{\theta}_1$ for completeness. When implementing this algorithm, it will be easier to employ this function expressed in the original coordinates x_1, x_2 .)

We choose the update law for $\hat{\theta}_1$ to be driven by the mismatch between the state z_1 and a dynamic estimate of this state \hat{z}_1 :

$$\begin{aligned} \dot{\hat{z}}_1 &= -\alpha_1(\hat{z}_1 - z_1) + z_2 \\ \dot{\hat{\theta}}_1 &= (z_1 - \hat{z}_1)w_1(z_1, z_2, \hat{\theta}_1) \end{aligned} \quad (7.6)$$

where $\alpha_1 > 0$.

Step 2. The previous step gives

$$\begin{aligned} \dot{z}_2 &= \hat{\theta}_1^T \frac{\partial f_1}{\partial x_2} \theta^T f_2(x_1, x_2, x_3) + \hat{\theta}_1^T \frac{\partial f_1}{\partial x_1} \theta^T f_1(x_1, x_2) \\ &\quad + (z_1 - \hat{z}_1)w_1^T(z_1, z_2, \hat{\theta}_1)f_1(x_1, x_2) \\ &= \hat{\theta}_1^T \frac{\partial f_1}{\partial x_2} \theta^T f_2(x_1, x_2, x_3) + \theta^T \psi_1(z_1, z_2, \hat{\theta}_1) + \chi_1(z_1, z_2, \hat{z}_1, \hat{\theta}_1) \end{aligned} \quad (7.7)$$

Now define

$$z_3 = \hat{\theta}_1^T \frac{\partial f_1}{\partial x_2} \hat{\theta}_2^T f_2(x_1, x_2, x_3) + \hat{\theta}_2^T \psi_1(z_1, z_2, \hat{\theta}_1) + \chi_1(z_1, z_2, \hat{z}_1, \hat{\theta}_1) \quad (7.8)$$

where $\hat{\theta}_2$ is an (independent) estimate of θ . Substituting (7.8) into (7.7) yields

$$\dot{z}_2 = z_3 + [\theta - \hat{\theta}_2]^T w_2(z_1, z_2, z_3, \hat{z}_1, \hat{\theta}_1, \hat{\theta}_2) \quad (7.9)$$

We choose the update law for $\hat{\theta}_2$ to be driven by the mismatch between the actual state z_2 and a dynamic estimate of this state \hat{z}_2 :

$$\begin{aligned} \dot{\hat{z}}_2 &= -\alpha_2(\hat{z}_2 - z_2) + z_3 \\ \dot{\hat{\theta}}_2 &= (z_2 - \hat{z}_2)w_2(z_1, z_2, z_3, \hat{z}_1, \hat{\theta}_1, \hat{\theta}_2) \end{aligned} \quad (7.10)$$

where $\alpha_2 > 0$.

Step i: i=3,...,n-1. The previous step gives

$$\begin{aligned} \dot{z}_i &= \hat{\theta}_1^T \frac{\partial f_1}{\partial x_2} \dots \hat{\theta}_{i-1}^T \frac{\partial f_{i-1}}{\partial x_i} \theta^T f_i(x_1, \dots, x_{i+1}) \\ &\quad + \theta^T \psi_i(z_1, \dots, z_i, \hat{z}_1, \dots, \hat{z}_{i-2}, \hat{\theta}_1, \dots, \hat{\theta}_{i-1}) \\ &\quad + \chi_i(z_1, \dots, z_i, \hat{z}_1, \dots, \hat{z}_{i-1}, \hat{\theta}_1, \dots, \hat{\theta}_{i-1}) \end{aligned} \quad (7.11)$$

Now define

$$\begin{aligned} z_{i+1} &= \hat{\theta}_1^T \frac{\partial f_1}{\partial x_2} \dots \hat{\theta}_{i-1}^T \frac{\partial f_{i-1}}{\partial x_i} \hat{\theta}_i^T f_i(x_1, \dots, x_{i+1}) \\ &\quad + \hat{\theta}_i^T \psi_i(z_1, \dots, z_i, \hat{z}_1, \dots, \hat{z}_{i-2}, \hat{\theta}_1, \dots, \hat{\theta}_{i-1}) \\ &\quad + \chi_i(z_1, \dots, z_i, \hat{z}_1, \dots, \hat{z}_{i-1}, \hat{\theta}_1, \dots, \hat{\theta}_{i-1}) \end{aligned} \quad (7.12)$$

where $\hat{\theta}_i$ is an (independent) estimate of θ . Substituting (7.12) into (7.11) yields

$$\dot{z}_i = z_{i+1} + [\theta - \hat{\theta}_i]^T w_i(z_1, \dots, z_{i+1}, \hat{z}_1, \dots, \hat{z}_{i-1}, \hat{\theta}_1, \dots, \hat{\theta}_i) \quad (7.13)$$

We choose the update law for $\hat{\theta}_i$ to be driven by the mismatch between the state z_i and a dynamic estimate of this state \hat{z}_i :

$$\begin{aligned} \dot{\hat{z}}_i &= -\alpha_i(\hat{z}_i - z_i) + z_{i+1} \\ \dot{\hat{\theta}}_i &= (z_i - \hat{z}_i)w_i(z_1, \dots, z_{i+1}, \hat{z}_1, \dots, \hat{z}_{i-1}, \hat{\theta}_1, \dots, \hat{\theta}_i) \end{aligned} \quad (7.14)$$

where $\alpha_i > 0$.

Step n. The previous step gives

$$\begin{aligned} \dot{z}_n &= \hat{\theta}_1^T \frac{\partial f_1}{\partial x_2} \dots \hat{\theta}_{n-1}^T \frac{\partial f_{n-1}}{\partial x_n} \theta^T g_n(x)u \\ &\quad + \theta^T \psi_n(z_1, \dots, z_n, \hat{z}_1, \dots, \hat{z}_{n-2}, \hat{\theta}_1, \dots, \hat{\theta}_{n-1}) \\ &\quad + \chi_n(z_1, \dots, z_n, \hat{z}_1, \dots, \hat{z}_{n-1}, \hat{\theta}_1, \dots, \hat{\theta}_{n-1}) \end{aligned} \quad (7.15)$$

We choose the input

$$u = \Delta^{-1}[-\hat{\theta}_n^T \psi_n - \chi_n - k_1 z_1 - \dots - k_n z_n] \quad (7.16)$$

where k_i are the coefficients of a Hurwitz polynomial and

$$\Delta = \hat{\theta}_1^T \frac{\partial f_1}{\partial x_2} \dots \hat{\theta}_{n-1}^T \frac{\partial f_{n-1}}{\partial x_n} \hat{\theta}_n^T g_n(x) \quad (7.17)$$

and $\hat{\theta}_n$ is an (independent) estimate of θ . (We will demonstrate in the stability proof that assumption 1 ensures the algorithm remains feasible and, hence, Δ^{-1} is well-defined.)

Substituting (7.16) into (7.15) yields

$$\dot{z}_n = -k_1 z_1 - \dots - k_n z_n + [\theta - \hat{\theta}_n]^T w_n(z_1, \dots, z_n, \hat{z}_1, \dots, \hat{z}_{n-1}, \hat{\theta}_1, \dots, \hat{\theta}_n) \quad (7.18)$$

We choose the update law for $\hat{\theta}_n$ to be driven by the mismatch between the state z_n and a dynamic estimate of this state \hat{z}_n :

$$\begin{aligned} \dot{\hat{z}}_n &= -\alpha_n(\hat{z}_n - z_n) - k_1 z_1 - \dots - k_n z_n \\ \dot{\hat{\theta}}_n &= (z_n - \hat{z}_n) w_n(z_1, \dots, z_n, \hat{z}_1, \dots, \hat{z}_{n-1}, \hat{\theta}_1, \dots, \hat{\theta}_n) \end{aligned} \quad (7.19)$$

where $\alpha_n > 0$.

Step n+1. Consider the set S_θ and \mathcal{F} of assumption 1. If $\mathcal{F} = \mathbb{R}^p \times \{1\}$, then $\hat{\theta}_i(0)$ can be chosen anywhere in $\mathbb{R}^p \times \{1\}$. Otherwise, if $p = 1$ and \mathcal{F} is unbounded then the projection of \mathcal{F} onto \mathbb{R} has either a well-defined least upper bound or greatest lower bound, but not both. Denote whichever is well-defined by β . Finally, let $\hat{\theta}_i(0)$ be that point in the closure of S_θ with the shortest distance to $(\beta, 1) \in \mathbb{R} \times \{1\}$. If \mathcal{F} is bounded or $p > 1$ then consider the ball $B_{(r, \theta')}$ associated with S_θ as defined in assumption 1. Choose the initial state of the parameter estimates as

$$\hat{\theta}_i(0) = \begin{bmatrix} \theta' \\ 1 \end{bmatrix} \quad (7.20)$$

This, together with $x(0)$ completely defines $z(0)$. Now choose the initial state of the state estimates such that $\hat{z}(0) = z(0)$.

Remarks.

1. It is clear that $\hat{\theta}_{i,p+1}(0) = \theta_{p+1} = 1$. Consequently, updating $\hat{\theta}_{i,p+1}$ is not necessary.

2. It follows from the algorithm and the above remark that the dimension of the dynamic adaptive compensator is $np + n$. The n additional states are due to estimating the states dynamically to construct an error-based identifier. These additional states are not found in the algorithm of [Kanellakopoulos *et al.*, 1991].
3. Because of the error-based scheme we are able to place the poles of the certainty equivalence z dynamics arbitrarily with the Hurwitz polynomial coefficients k_i .
4. Let $f(x, \theta)$ denote the drift vector field and $g(x, \theta)$ denote the input vector field both associated with (7.2) and let $h(x) = x_1$. It follows from the algorithm that if

$$\hat{\theta}_i \equiv \theta, \quad \hat{z}_i \equiv z_i \quad \text{for } i = 1, \dots, n \quad (7.21)$$

then

$$z_i = L_{f(x, \theta)}^{i-1} h(x)$$

and

$$u = (L_{g(x, \theta)} L_{f(x, \theta)}^{n-1} h(x))^{-1} [-L_{f(x, \theta)}^n h(x) - k_1 h(x) - \dots - k_n L_{f(x, \theta)}^{n-1} h(x)]$$

The condition (7.21) is an equilibrium point of the identifier, independent of the value of z . Consequently, if (7.21) is satisfied at $t = 0$ then the the control implemented for $t \geq 0$ is an exact linearizing control.

5. As seen in step $n+1$, the selection of the initial value of $\hat{\theta}$ is not arbitrary. It is selected to ensure that the algorithm remains feasible.

7.4 Closed-loop Stability

In this section we prove the following theorem:

Theorem 7.1 (Adaptive Regulation) *Under assumption 1, if the algorithm of section 7.3 is applied to the system (7.2), the resulting closed loop system is such that*

$$\lim_{t \rightarrow \infty} x = 0 \quad (7.22)$$

for all $x(0) \in \mathbb{R}^n$.

Proof. The algorithm of section 7.3 yields the following closed loop system:

$$\begin{aligned}
 \dot{z}_1 &= z_2 + (\theta - \hat{\theta}_1)^T w_1 \\
 &\vdots \\
 \dot{z}_{n-1} &= z_n + (\theta - \hat{\theta}_{n-1})^T w_{n-1} \\
 \dot{z}_n &= -k_1 z_1 - \dots - k_n z_n + (\theta - \hat{\theta}_n)^T w_n \\
 \dot{\hat{z}}_1 &= -\alpha_1 (\hat{z}_1 - z_1) + z_2 \\
 &\vdots \\
 \dot{\hat{z}}_{n-1} &= -\alpha_{n-1} (\hat{z}_{n-1} - z_{n-1}) + z_n \\
 \dot{\hat{z}}_n &= -\alpha_n (\hat{z}_n - z_n) - k_1 z_1 - \dots - k_n z_n \\
 \dot{\hat{\theta}}_1 &= (z_1 - \hat{z}_1) w_1 \\
 &\vdots \\
 \dot{\hat{\theta}}_n &= (z_n - \hat{z}_n) w_n \\
 \dot{\hat{\theta}} &= 0
 \end{aligned} \tag{7.23}$$

We make the following linear coordinate change:

$$\begin{bmatrix} e \\ \hat{z} \\ \phi_i \\ \theta \end{bmatrix} = \begin{bmatrix} I & -I & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} z \\ \hat{z} \\ \hat{\theta}_i \\ \theta \end{bmatrix} \tag{7.24}$$

The dynamics of (7.23) in the new coordinates become:

$$\begin{aligned}
 \dot{e}_1 &= -\alpha_1 e_1 - \phi_1^T w_1 \\
 &\vdots \\
 \dot{e}_n &= -\alpha_n e_n - \phi_n^T w_n \\
 \dot{\phi}_1 &= e_1 w_1 \\
 &\vdots \\
 \dot{\phi}_n &= e_n w_n \\
 \dot{\hat{z}}_1 &= \hat{z}_2 + \alpha_1 e_1 + e_2 \\
 &\vdots \\
 \dot{\hat{z}}_{n-1} &= \hat{z}_n + \alpha_{n-1} e_{n-1} + e_n \\
 \dot{\hat{z}}_n &= -k_1 \hat{z}_1 - \dots - k_n \hat{z}_n + \alpha_n e_n - k_1 e_1 - \dots - k_n e_n \\
 \dot{\hat{\theta}} &= 0
 \end{aligned} \tag{7.25}$$

We denote by A_k the controllable canonical form matrix corresponding to the Hurwitz polynomial

$$s^n + k_n s^{n-1} + \dots + k_2 s + k_1$$

We then choose $P > 0$ to satisfy

$$A_k^T P + P A_k = -I \quad (7.26)$$

To prove stability, we choose the following Lyapunov function candidate:

$$V = \mu \left[\frac{1}{2} (e^T e + \sum_{i=1}^n \phi_i^T \phi_i) \right] + \frac{1}{2} \hat{z}^T P \hat{z} + \theta^T \theta \quad (7.27)$$

The derivative of V along the trajectories of (7.25) is given by

$$\dot{V} = \mu \left(\sum_{i=1}^n -\alpha_i e_i^2 \right) - \hat{z}^T \hat{z} + \hat{z}^T M e \quad (7.28)$$

where M is a constant matrix independent of μ . It is obvious that $\exists \mu > 0$ such that $\dot{V} \leq 0$ for all e, \hat{z}, ϕ, θ . This establishes the stability (i.s.L.) of the closed loop system.

We now focus on the dynamics of the identifier itself to verify that the proposed algorithm is indeed feasible. The n identifier systems are given by

$$\begin{aligned} \dot{e}_1 &= -\alpha_1 e_1 - \phi_1^T w_1 \\ \dot{\phi}_1 &= e_1 w_1 \\ &\vdots \\ \dot{e}_n &= -\alpha_n e_n - \phi_n^T w_n \\ \dot{\phi}_n &= e_n w_n \end{aligned} \quad (7.29)$$

Consider the Lyapunov function candidate for the i th system of (7.29):

$$V_i = \frac{1}{2} (e_i^2 + \phi_i^T \phi_i) \quad (7.30)$$

The derivative for V_i along the trajectories of the i th system of (7.29) is given by

$$\dot{V}_i = -\alpha_i e_i^2 \quad (7.31)$$

Since $\dot{V}_i \leq 0$ for all e_i, ϕ_i we can conclude that

$$V_i(t) \leq V_i(0) \quad (7.32)$$

Since we have chosen $\hat{z}_i(0)$ such that $e_i(0) = 0$ we can then conclude that

$$\|\phi_i(t)\| \leq \|\phi_i(0)\| \quad (7.33)$$

We only need to consider the case when $\mathcal{F} \neq \mathbb{R}^p \times \{1\}$. If $p = 1$ and \mathcal{F} is unbounded, we have chosen $\hat{\theta}_i(0) = (s, 1) \in \text{clos}(S_\theta)$ for some $s \in \mathbb{R}$. Define $E_l = (-\infty, s] \times \{1\}$ and $E_r = [s, +\infty) \times \{1\}$ and let E denote the one set, E_l or E_r , that is contained in \mathcal{F} . (One and only one will satisfy this condition since \mathcal{F} is unbounded but not $\mathbb{R} \times \{1\}$.) We then have $\theta \in S_\theta \subset E \subset \mathcal{F}$. The choice of $\hat{\theta}_i(0)$, the definition of E , the fact that $\theta \in E$ and (7.33) imply $\hat{\theta}_i(t) \in E$ for all $t \geq 0$. Since $E \subset \mathcal{F}$ it follows that the proposed algorithm is feasible. For $p > 1$ or \mathcal{F} bounded, we have chosen $\hat{\theta}_i(0)$ such that $\|\phi_i(0)\| \leq r$. Since we know that $\theta \in S_\theta \subset B_{(r, \theta')} \times \{1\}$ it follows from (7.33) that $\hat{\theta}_i(t) \in B_{(2r, \theta')} \times \{1\}$ for all $t \geq 0$. Finally, since $B_{(2r, \theta')} \times \{1\} \subset \mathcal{F}$ it follows that the proposed algorithm is feasible.

We now demonstrate asymptotic stability of the state x . First, from (7.31) it follows that

$$\int_0^\infty \sum_{i=1}^n \alpha_i e_i^2 < \infty \quad (7.34)$$

Next, from the stability of the overall system (see (7.28)) it follows that \dot{e}_i is bounded. With this we are able to conclude that

$$\lim_{t \rightarrow \infty} e_i = 0 \quad (7.35)$$

Then a simple application of the Bellman-Gronwall lemma to the dynamics of \hat{z} shows that

$$\lim_{t \rightarrow \infty} \hat{z} = 0 \quad (7.36)$$

From (7.35), (7.36) and (7.24) we conclude that

$$\lim_{t \rightarrow \infty} z = 0 \quad (7.37)$$

Finally, from the algorithm of section 7.3, since $f_i(0) = 0$ and from the definition of a feasibility region, z is a global diffeomorphism of x without translation. Hence,

$$\lim_{t \rightarrow \infty} x = 0 \quad (7.38)$$

□.

7.5 Conclusion

We have modified the nonlinear adaptive algorithm of [Kanellakopoulos *et al.*, 1991] to produce an error-based algorithm. This allows global stabilizability for a larger subset of pure-feedback nonlinear systems. The algorithm was demonstrated on the single-input stabilization problem but easily extends to the multi-input and tracking problems.

Chapter 8

Conclusion

In this dissertation, we have investigated the stabilization and tracking problems for several different classes of systems. We have focused on systems that fail to satisfy differential geometric conditions for input-to-state linearizability under state feedback and change of coordinates. We have attempted to emphasize that certain underlying structural properties can still be exploited to yield systematic stabilizing (and tracking) algorithms. In doing so, we have added some specialized, but potentially very useful, tools to the nonlinear control toolbox. Foremost, we have demonstrated the power of introducing saturation in a control law to overcome certain nonlinearities or actuator limitations. This notion led to a solution to the global stabilization and small signal tracking problem for linear systems subject to actuator constraints. Further, introducing saturation provided new global and semi-global stabilizing and approximate tracking solutions for nonlinear systems in special normal forms. This included the discussion of *higher order feedforward forms* and related systems like “the ball and beam”.

We hope that the developments of this dissertation will not remain simply of theoretical interest. There is work ahead to test the usefulness of our algorithms, beyond systems like the “ball and beam”. It might also be possible to refine our algorithms to achieve design specifications beyond the qualitative stability or tracking properties. Further, we are inclined to believe that as we continue to study physical examples, other nonlinear normal forms, ready to produce new design algorithms, will manifest themselves. The problems of interest will be to identify useful underlying structures, to generate conditions for recognizing these structures, to produce systematic control algorithms, and to convince the control community of their usefulness.

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