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Steering Three-Input Chained Form Nonholonomic Systems Using Sinusoids: The Firetruck Example*

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L. Bushnell, D. Tilbury and S.S. Sastry

Intelligent Machines and Robotics Laboratory
University of California
Berkeley, CA 94720
510 642-3248
FAX 510 642-1341
bushnell@robotics.berkeley.edu

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Abstract

In this paper, we steer nonholonomic systems with linear velocity constraints represented mathematically in a special form, called *chained form*. We observe that chained form systems can be steered from an initial configuration to a final configuration with sinusoidal inputs. The controller we use is open loop and no special provisions are made for obstacle avoidance. Sufficient conditions are presented for converting a three-input system with nonholonomic velocity constraints into a "two-chain, single-generator chained form." An algorithm is stated that constructs the sinusoidal control inputs to steer this system from any initial configuration to any desired final point. Our example of a three-input nonholonomic system is a firetruck, or tiller truck. In this three-axle system, the control inputs are the steering velocities of both the front and rear wheels of the truck and the driving velocity of the truck. Simulation results are given for the familiar parallel parking problem and other trajectories.

Keywords: nonholonomic motion planning, Lie brackets.

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1 Introduction

In this paper, we investigate how sinusoids can be used for steering chained form systems and how to convert systems into a special class of chained form systems. The main reason for using sinusoids for steering such systems is stated in [5], motivated by an optimal control problem. The original contributions of this research are the introduction of a three-input system as an example of a nonholonomic system that can be controlled using sinusoids, a steering algorithm for three-input chained form systems and a theorem for converting a nonholonomic system with linear velocity constraints into a special chained form system.

We are interested in steering mechanical systems with nonholonomic, or non-integrable, constraints. Specifically, our interest is in systems with linear velocity constraints

$$\omega_i(x)\dot{x} = 0 \quad i = 1, 2, \cdots, k,$$

where $x \in \mathbf{R}^n$ is the state of the system and $\omega_i(x) \in \mathbf{R}^n$ $i = 1, 2, \cdots, k$ are row vectors. These constraints arise, for example, when a wheel rolls on the road but does not slip in the direction perpendicular to the motion. If these constraints are integrable, they are called *holonomic* constraints and yield level surfaces $h_i(x) = c_i$ for some constant c_i to which the trajectory of the system is restricted. Thus, the holonomic constraints reduce the order of the system. If these constraints are not integrable, i.e., these constraints cannot be written in terms of the configuration variables, then they are called *nonholonomic* constraints.

We assume that the ω_i, $i = 1, 2, \cdots, k$ are linearly independent and smooth. The corresponding co-distribution $\Omega(x) = \text{span}\{\omega_1(x), \omega_2(x), \cdots, \omega_k(x)\}$ has dimension k. Therefore, we can find an $(n-k)$-dimensional distribution $\Delta(x) = \text{span}\{g_1(x), g_2(x), \cdots, g_{n-k}(x)\}$, with $g_i(x) \in \mathbf{R}^n$, $i = 1, 2, \cdots, n-k$ such that $\Delta = \Omega^{\perp}$, i.e., $\omega(x) \cdot g(x) = 0$, $\forall \omega \in \Omega$, $\forall g \in \Delta$. Then a mechanical system with the above nonholonomic linear velocity constraints can be represented as a control system with inputs $u \in \mathbf{R}^{n-k}$:

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 + \cdots + g_{n-k}(x)u_{n-k}\,.$$

It will be shown that under some conditions, it is possible to control such systems after a coordinate transformation and state feedback using sinusoidal inputs.

Our motion planning problem therefore consists of controlling the system

$$\Sigma \;:\; \dot{x}(t) = g_1(x)u_1(t) + \cdots + g_m(x)u_m(t)$$

where $x \in$ open set $U \subset \mathbf{R}^n$, $u \in \mathbf{R}^m$ and the g_i are smooth (C^∞) linearly independent vector fields. All subsequent conditions are assumed to hold on the open set U. Given x^0 and x^f, we wish to find a control law $u(t) = (u_1(t), \ldots, u_m(t))$ to steer $x(0) = x^0$ to $x(T) = x^f$ in $[0, T]$.

Recent work [6, 9] in the area of controlling nonholonomic systems by using sinusoidal inputs has concentrated on systems with two inputs:

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

where $x \in \mathbf{R}^n$, $g_1, g_2 \in \mathbf{R}^n$ linearly independent and smooth and $u_1, u_2 \in \mathbf{R}$. If the system meets certain conditions allowing it to be transformed into what we call a *single-chain, single-generator* chained form

$$\dot{z}_1 = v_1 \quad \begin{aligned} \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ &\vdots \\ \dot{z}_n &= z_{n-1} v_1 \end{aligned}$$

by a change of coordinates and state feedback, then the system is controllable. The main idea for steering such a system is to set the inputs v_1 and v_2 to be sinusoids at integrally related frequencies and systematically steer the state variables. For example, starting at the top of the chain, states z_1 and z_2 are directly controlled with constant (zero-frequency) inputs v_1 and v_2. For the other states in the chain, if the inputs are set as $(1 \leq k \leq n)$

$$\begin{aligned} v_1 &= \alpha \sin \omega t \\ v_2 &= \beta \cos k\omega t \ , \end{aligned}$$

then $\dot{z}_2$ has a frequency component at $k\omega$, which is periodic; $\dot{z}_3$ has a frequency component at $(k-1)\omega$, which is periodic; $\ldots$; and $\dot{z}_k$ has a constant component, which yields net motion in this state. The systematic method continues down the chain with the idea of setting the frequencies of the inputs such that a zero-frequency term emerges after integration over one period, causing a net motion in that variable. Hence, the system is point-to-point controllable.

In this paper, we extend the above idea to three-input systems. The plan is to put a three-input system with nonholonomic velocity constraints into a *two-chain, single-generator* chained form by state feedback and a coordinate transformation. In this special form, it can be controlled using sinusoidal input functions using a similar algorithm to that for the single-chain, single-generator chained form. This idea is also extended to the general case of m inputs. The Appendix states a proposition for converting a m-input system to *(m-1)-chain, single-generator* chained form.

We suggest Chapters 4 and 5 of [4] and Chapter 3 of [8] for references on controllability and Chapter 1 of [4] for reference on distributions and Lie brackets.

The outline of this paper is as follows. In Section 2, a three-input example of a nonholonomic system is introduced. The kinematic equations are derived and represented as a control system. This example, the firetruck, will be used throughout the paper to illustrate the theoretic results. In Section 3, we give sufficient conditions for transforming a three-input nonholonomic system into a two-chain, single-generator chained form and show how the example can be transformed into this special form. In Section 4, the controllability of two-chain, single-generator chained form systems is proven. In Section 5, we present a step-by-step algorithm to control these systems using sinusoids. The example is continued to show how the steps of the algorithm work in driving the system to a desired final configuration. In Section 6, simulation results are given for the firetruck example. Traditional phase plots are given along with clips from a movie of the running simulation. In Section 7, conclusions are drawn.

2 A Nonholonomic System with Three Inputs

In a fire department, firetrucks are used to carry aerial ladders, tools and equipment and have the main purpose of rescue and ventilation. The driver sits up front in the cab, driving the truck

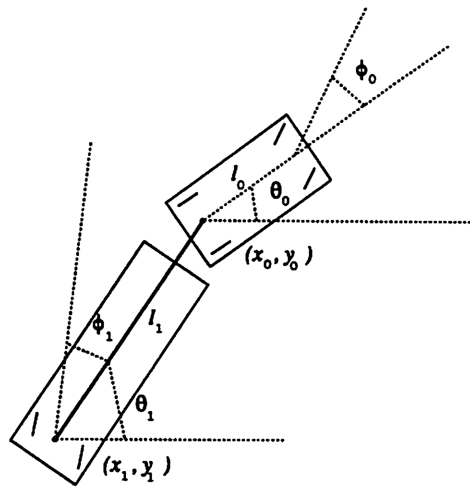


Figure 1: Configuration Space

and steering the front wheels. The tiller person sits in the rear of the truck, steering the rear wheels. The two communicate via an intercom system.

The firetruck is an example of a three-input nonholonomic system. It is mathematically modeled as two planar rigid bodies supported by three axles. The support of the rear body, hereinafter called "trailer", is over the rear axle of the front body, hereinafter called "cab". The two outer axles are allowed to pivot, while the middle axle is rigidly fixed to the cab body. The wheels are assumed to roll but not slip, thus giving linear velocity constraints.

The derivation of the kinematic equations for the firetruck refers to Figure 1, where we emphasize the truck's two rigid bodies. The states of the mathematical model, all functions of time, are chosen as $(x_0, y_0, \phi_0, \theta_0, x_1, y_1, \phi_1, \theta_1)$, where (x_0, y_0) is the Cartesian location of the center of the rear axle of the cab, ϕ_0 is the steering angle of the front wheels with respect to the cab body, and θ_0 is the orientation of the cab body with respect to the horizontal axis of the inertial frame. The states $(x_1, y_1, \phi_1, \theta_1)$ are described similarly for the trailer, except that ϕ_1 is the angle of the rear wheels with respect to the trailer body.

Let the distance between the front and rear axles of the cab be l_0, and the length of the link between trailer and cab, i.e., the distance between the centers of the rear axles of the cab and trailer, be l_1, as shown in Figure 1. This link between the trailer and cab gives the two constraints

$$\begin{aligned} x_1 &= x_0 - l_1 \cos\theta_1 \\ y_1 &= y_0 - l_1 \sin\theta_1 \, , \end{aligned} \tag{1}$$

which are holonomic in the sense that they reduce by two the number of variables needed to specify the state of the system. The six coordinates $x = (x_0, y_0, \phi_0, \theta_0, \phi_1, \theta_1)$ are sufficient to represent the positions of the cab, trailer and wheels.

For mechanical systems with wheels rolling and turning on a surface, the non-slipping constraint states that the velocity of a body in the direction perpendicular to each wheel must be zero. This can be stated in terms of coordinates as follows: for a wheel centered at location (x, y) and at an angle φ with respect to the horizontal axis of the fixed frame,

$$0 = v_x \sin\varphi - v_y \cos\varphi .$$

In order to simplify our model of the firetruck, each pair of wheels is modeled as a single wheel centered at the midpoint of the axle. In other words, we will assume that the pairs of wheels all have the same angle[1]. Assuming that none of the wheels slip, the linear velocity constraints are

$$\begin{aligned} 0 &= \frac{d}{dt}(x_0 + l_0\cos\theta_0)\sin(\theta_0+\phi_0) - \frac{d}{dt}(y_0 + l_0\sin\theta_0)\cos(\theta_0+\phi_0) \\ 0 &= \dot{x}_0\sin\theta_0 - \dot{y}_0\cos\theta_0 \\ 0 &= \dot{x}_1\sin(\theta_1+\phi_1) - \dot{y}_1\cos(\theta_1+\phi_1) \end{aligned}$$

which may be expressed using the holonomic constraints (1) as

$$\begin{aligned} 0 &= \dot{x}_0\sin(\theta_0+\phi_0) - \dot{y}_0\cos(\theta_0+\phi_0) - l_0\dot{\theta}_0\cos\phi_0 \\ 0 &= \dot{x}_0\sin\theta_0 - \dot{y}_0\cos\theta_0 \\ 0 &= \dot{x}_0\sin(\theta_1+\phi_1) - \dot{y}_0\cos(\theta_1+\phi_1) - l_1\dot{\theta}_1\cos\phi_1 . \end{aligned}$$

These constraints are nonintegrable, or nonholonomic, and will not further reduce the reachable configuration space. They can be expressed more compactly as $\omega_i(x)\cdot\dot{x} = 0$, where we represent the entire state as $x = (x_0, y_0, \phi_0, \theta_0, \phi_1, \theta_1)$ and the 1-forms $\omega_i(x)$ are row vector fields in $\mathbb{R}^6$:

$$\begin{aligned} \omega_1(x) &= [\begin{array}{cccccc} \sin(\theta_0+\phi_0) & -\cos(\theta_0+\phi_0) & 0 & -l_0\cos\phi_0 & 0 & 0 \end{array}] \\ \omega_2(x) &= [\begin{array}{cccccc} \sin\theta_0 & -\cos\theta_0 & 0 & 0 & 0 & 0 \end{array}] \\ \omega_3(x) &= [\begin{array}{cccccc} \sin(\theta_1+\phi_1) & -\cos(\theta_1+\phi_1) & 0 & 0 & 0 & l_1\cos\phi_1 \end{array}] . \end{aligned}$$

The corresponding co-distribution is $\Omega(x) = \mathrm{span}\{\omega_1(x), \omega_2(x), \omega_3(x)\}$. Since Ω has dimension three and the state space is of dimension six, we can find a three-dimensional distribution $\Delta(x) = \mathrm{span}\{g_1(x), g_2(x), g_3(x)\}$, such that $\Delta = \Omega^\perp$, i.e.,

$$\omega(x)\cdot g(x) = 0, \quad \forall\omega\in\Omega, \forall g\in\Delta .$$

A simple calculation will show that the following vector fields g_1, g_2, g_3 form a basis for Δ.

$$g_1 = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ 0 \\ \frac{\tan\phi_0}{l_0} \\ 0 \\ \frac{-\sin(\phi_1-\theta_0+\theta_1)}{l_1\cos\phi_1} \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad g_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The nonholonomic constraints $\omega(x)\cdot\dot{x} = 0$, $\forall\omega\in\Omega$ are equivalent to $\dot{x}\in\Delta = \mathrm{span}\{g_1, g_2, g_3\}$, i.e., $\dot{x}$ is a linear combination of vector fields in Δ. Therefore the kinematic equations of the firetruck as a control system with three inputs can be written as

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3 ,$$

[1]This is a simplification since it may be shown (see Alexander and Maddocks [1]) that the two wheels in fact have different angles and their normals all intersect at a point.

where the state is $x = (x_0, y_0, \phi_0, \theta_0, \phi_1, \theta_1)$ and input u_1 corresponds to the forward driving velocity of the truck, u_2 corresponds to the steering velocity of the front wheel of the cab and u_3 corresponds to the steering velocity of the rear wheel of the trailer.

It will be necessary later on to consider an equivalent representation obtained by dividing g_1 by $\cos\theta_0$, which is the same as an input transformation $\tilde{u}_1 = u_1 \cos\theta_0$. The state remains the same, and the the system equations become

$$\begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{\phi}_0 \\ \dot{\theta}_0 \\ \dot{\phi}_1 \\ \dot{\theta}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \tan\theta_0 \\ 0 \\ \frac{\tan\phi_0}{l_0\cos\theta_0} \\ 0 \\ \frac{-\sin(\phi_1-\theta_0+\theta_1)}{l_1\cos\phi_1\cos\theta_0} \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_3 \, .$$

In this representation three of the states are controlled directly, so their velocities are the inputs.

3 Converting Systems to Two-Chain, Single-Generator Chained Form

The above kinematic equation for the firetruck becomes very complicated when we investigate how find control inputs $\{u_1(t), u_2(t), u_3(t)\}$ that will steer the state $x \in \mathbf{R}^6$ from an initial point to a desired point. For this reason, we seek a simpler form of the system. Chained form systems are constructed in such a way that they are easily steered with sinusoidal inputs. Transforming the kinematic equation to a chained form equation uses a similar method as in transforming a nonlinear system to a linear system.

Recall that a Lie bracket is defined as

$$\begin{aligned} [f,g](x) &= \frac{\partial g}{\partial x}f(x) - \frac{\partial f}{\partial x}g(x) \\ \mathrm{ad}_f^0 g(x) &= g(x) \\ \mathrm{ad}_f^k g(x) &= [f(x), \mathrm{ad}_f^{k-1} g(x)] \end{aligned}$$

where f, g are vector fields. A distribution, Δ, is said to be *involutive* if

$$f, g \in \Delta \Rightarrow [f,g] \in \Delta \, .$$

Using similar conditions to those in Isidori [4] for full state linearization of nonlinear control systems, it has been shown [6, 9] that a two-input nonholonomic system can be put into a chained canonical form structure for which a simple steering algorithm has been developed. For example, [6] gives sufficient conditions for a two-input system

$$\dot{x}(t) = g_1(x(t))u_1(t) + g_2(x(t))u_2(t)$$

with $x \in U \subset \mathbf{R}^n$, g_1, g_2 linearly independent smooth vector fields and $u_1, u_2 \in \mathbf{R}$ to be put in the single-chain, single-generator chained form

$$\dot{z}_1 = v_1$$

$$
\begin{aligned}
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2 v_1 \\
&\vdots \\
\dot{z}_n &= z_{n-1} v_1
\end{aligned}
$$

by a change of coordinates and state feedback. The sufficient conditions are that the distributions

$$
\begin{aligned}
\Delta_0 &:= \operatorname{span}\{g_1, g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{n-2} g_2\} \\
\Delta_1 &:= \operatorname{span}\{g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{n-2} g_2\} \\
\Delta_2 &:= \operatorname{span}\{g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{n-3} g_2\}
\end{aligned}
$$

are constant rank and that (1) $\Delta_0(x) = \mathbf{R}^n$ and (2) Δ_1 and Δ_2 are involutive.

Extending this idea to three-input nonholonomic systems is more complicated; the sufficient conditions for transformation are similar, but the distributions are constructed differently. A set of sufficient conditions for transforming a three-input nonholonomic system into a two-chain, single-generator chained form is stated in the following proposition.

The proposition will use the fact that linearly independent vector fields g_1, g_2, g_3 can locally be put into the form

$$
\begin{aligned}
g_1 &= \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} g_1^i(x) \frac{\partial}{\partial x_i} \\
g_2 &= \sum_{i=2}^{n} g_2^i(x) \frac{\partial}{\partial x_i} \\
g_3 &= \sum_{i=2}^{n} g_3^i(x) \frac{\partial}{\partial x_i}
\end{aligned}
\tag{2}
$$

by a non-unique change of input.

Proposition 1 (Converting Three-input Systems to Two-Chain, Single-Generator Chained Form)
Consider a three-input, drift-free, nonholonomic system

$$
\dot{x} = g_1(x) u_1 + g_2(x) u_2 + g_3(x) u_3
$$

with smooth, linearly independent input vector fields g_1, g_2, g_3 in the form of equation (2). Define the distributions

$$
\begin{aligned}
\Delta_0 &= \operatorname{span}\{g_1, g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{j} g_2,\ g_3, \mathrm{ad}_{g_1} g_3, \ldots, \mathrm{ad}_{g_1}^{k} g_3\} \\
\Delta_1 &= \operatorname{span}\{g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{j} g_2,\ g_3, \mathrm{ad}_{g_1} g_3, \ldots, \mathrm{ad}_{g_1}^{k} g_3\} \\
\Delta_2 &= \operatorname{span}\{g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{j-1} g_2,\ g_3, \mathrm{ad}_{g_1} g_3, \ldots, \mathrm{ad}_{g_1}^{k} g_3\} \\
\Delta_3 &= \operatorname{span}\{g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{j-1} g_2,\ g_3, \mathrm{ad}_{g_1} g_3, \ldots, \mathrm{ad}_{g_1}^{k-1} g_3\}
\end{aligned}
$$

where $j+k+3=n$. If for some open set $U \subset \mathbb{R}^n$, $\Delta_0(x)=\mathbb{R}^n\ \forall x \in U$ and Δ_1, Δ_2 and Δ_3 are involutive on U, then there exists a local feedback transformation on U

$$\begin{aligned}(\xi,\zeta,\eta) &= \Phi(x)\\ u &= \beta(x)v\end{aligned}$$

such that the transformed system is in two-chain, single-generator chained form

$$\begin{array}{lll}\dot{\xi}_0 = & v_1 & \\ \dot{\zeta}_0 = & v_2 & \dot{\eta}_0 = v_3\\ \dot{\zeta}_1 = & \zeta_0 v_1 & \dot{\eta}_1 = \eta_0 v_1\\ \vdots & & \vdots\\ \vdots & & \dot{\eta}_k = \eta_{k-1} v_1\\ \dot{\zeta}_j = & \zeta_{j-1} v_1 & \end{array} \tag{3}$$

Proof. We will need the fact (see [4] section 1.4) that for $\Delta = \text{span}\{f_1, f_2, \ldots, f_n\}$ having rank n on an open set U and $\Delta' = \text{span}\{f_2, f_3, \ldots, f_n\}$ involutive, there exists a smooth function $h : U \to \mathbb{R}$ such that $dh \cdot f = 0\ \forall f \in \Delta'$ (or more compactly, $dh \cdot \Delta' = 0$) and $dh \cdot f_1 = a(x) \neq 0$.

Suppose there exists an open set U, such that $\Delta_0(x) = \mathbb{R}^n\ \forall x \in U$ and Δ_1, Δ_2 and Δ_3 are involutive on U. Noting that by definition $\Delta_3 \subset \Delta_2 \subset \Delta_1 \subset \Delta_0$, there exist smooth functions h_1, h_2 and $h_3 : U \to \mathbb{R}$ such that

$$\begin{array}{ll} dh_1 \cdot \Delta_1(x) = 0 & dh_1 \cdot g_1(x) = 1\\ dh_2 \cdot \Delta_2(x) = 0 & dh_2 \cdot \text{ad}_{g_1}^j g_2(x) = a_2(x) \neq 0\\ dh_3 \cdot \Delta_3(x) = 0 & dh_3 \cdot \text{ad}_{g_1}^k g_3(x) = a_3(x) \neq 0. \end{array} \tag{4}$$

Here we used the fact that h_1 can be chosen as x_1 since the g_i's are in the special form in equation (2). Therefore consider the local coordinate transformation $\Phi : x \mapsto (\xi, \zeta, \eta)$ (see [4] section 5.1)

$$\begin{aligned}\xi_0 &= h_1 = x_0\\ \zeta_0 &= L_{g_1}^j h_2\\ &\ \vdots\\ \zeta_{j-1} &= L_{g_1} h_2\\ \zeta_j &= h_2\\ \eta_0 &= L_{g_1}^k h_3\\ &\ \vdots\\ \eta_{k-1} &= L_{g_1} h_3\\ \eta_k &= h_3\,.\end{aligned}$$

To verify that the above coordinate transformation is valid, we show that it is a local diffeomorphism. First calculate the derivative of the coordinate transformation with respect to x.

$$\frac{\partial \Phi}{\partial x} = \begin{bmatrix} dh_1 \\ dL_{g_1}^j h_2 \\ \vdots \\ dL_{g_1} h_2 \\ dh_2 \\ dL_{g_1}^k h_3 \\ \vdots \\ dL_{g_1} h_3 \\ dh_3 \end{bmatrix}$$

We now multiply on the right by the nonsingular matrix whose columns are the n independent vector fields in the definition of Δ_0. Without loss of generality, assume that $j > k$ (the calculations for the other case are similar, or indeed can be reduced to this case by a renumbering of the vector fields).

$$\frac{\partial \Phi}{\partial x} \cdot [\Delta_0] = \begin{bmatrix} dh_1 \\ dL_{g_1}^j h_2 \\ \vdots \\ dL_{g_1} h_2 \\ dh_2 \\ dL_{g_1}^k h_3 \\ \vdots \\ dL_{g_1} h_3 \\ dh_3 \end{bmatrix} \begin{bmatrix} g_1 & g_2 & \mathrm{ad}_{g_1} g_2 & \cdots & \mathrm{ad}_{g_1}^j g_2 & g_3 & \mathrm{ad}_{g_1} g_3 & \cdots & \mathrm{ad}_{g_1}^k g_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \pm a_2(x) & * & \cdots\cdots & * & * & \cdots & \cdots & * \\ * & 0 & \pm a_2(x) & & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & \ddots & * & \vdots & & & * \\ * & 0 & 0 & \cdots\ 0 & a_2(x) & 0 & 0 & \cdots & 0 \\ * & 0 & * & \cdots\cdots & * & \pm a_3(x) & * & \cdots & * \\ \vdots & 0 & & \ddots & * & 0 & \pm a_3(x) & & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & * \\ * & 0 & 0 & \cdots\ 0 & * & 0 & \cdots & 0 & a_3(x) \end{bmatrix}$$

Here, the functions $a_2(x)$ and $a_3(x)$ are nonzero by definition.

That the resulting matrix has rank n for all x can be seen most easily by considering row operations. Using the first row, one can eliminate all the terms in the first column without altering the other columns. Then using row $j+1$ (corresponding to the dh_2 terms), all the $*$'s in column $j+1$ can be similarly eliminated. Now the only nonzero term in the n^{th} row is $a_3(x)$, and this can be used to reduce column n, and so forth. The matrix $\frac{\partial\Phi}{\partial x}\cdot[\Delta_0]$ is therefore equivalent under row operations to a nonsingular diagonal matrix with $1, a_2(x), \ldots, a_2(x), a_3(x), \ldots, a_3(x)$ on the diagonal.

Since $[\Delta_0]$ is of full rank, the Jacobian matrix $\frac{\partial\Phi}{\partial x}$ must also be nonsingular locally and thus $(\xi, \zeta, \eta) = \Phi(x)$ is a local diffeomorphism and a valid coordinate transformation on the open set U (see [4] Proposition 1.2.3).

The input transformation needed to put the system into the two-chain, single-generator chained form can be easily computed by taking derivatives of transformed coordinates and cancelling terms by using the zero entries of the above matrix $\frac{\partial\Phi}{\partial x}\cdot[\Delta_0]$.

$$
\begin{aligned}
\dot{\xi}_0 &= u_1 \\
\dot{\zeta}_0 &= L_{g_1}^{j+1}h_2u_1 + L_{g_2}L_{g_1}^{j}h_2u_2 + L_{g_3}L_{g_1}^{j}h_2u_3 \\
&\vdots \\
\dot{\zeta}_{j-1} &= L_{g_1}^{2}h_2u_1 = \zeta_{j-2}u_1 \\
\dot{\zeta}_{j} &= L_{g_1}h_2u_1 = \zeta_{j-1}u_1 \\
\dot{\eta}_0 &= L_{g_1}^{k+1}h_3u_1 + L_{g_3}L_{g_1}^{k}h_3u_3 \\
&\vdots \\
\dot{\eta}_{k-1} &= L_{g_1}^{2}h_3u_1 = \eta_{k-2}u_1 \\
\dot{\eta}_{k} &= L_{g_1}h_3u_1 = \eta_{k-1}u_1
\end{aligned}
$$

The input transformation

$$
\begin{aligned}
v_1 &= u_1 \\
v_2 &= L_{g_1}^{j+1}h_2u_1 + L_{g_2}L_{g_1}^{j}h_2u_2 + L_{g_3}L_{g_1}^{j}h_2u_3 \\
v_3 &= L_{g_1}^{k+1}h_3u_1 + L_{g_3}L_{g_1}^{k}h_3u_3
\end{aligned}
$$

will result in the two-chain, single-generator chained form (3). □

Example. We will now apply Proposition 1 to the firetruck system described in Section 2. We use the system equations in the following form.

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3$$

$$
\begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{\phi}_0 \\ \dot{\theta}_0 \\ \dot{\phi}_1 \\ \dot{\theta}_1 \end{pmatrix}
=
\begin{pmatrix} 1 \\ \tan\theta_0 \\ 0 \\ \frac{\tan\phi_0}{l_0\cos\theta_0} \\ 0 \\ \frac{-\sin(\phi_1-\theta_0+\theta_1)}{l_1\cos\phi_1\cos\theta_0} \end{pmatrix} u_1
+
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_2
+
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_3
$$

It can be seen that the choice of $(j,k) = (2,1)$ will resulting in the following distributions. For notation's sake, let $g_4 = \mathrm{ad}_{g_1} g_2$, $g_5 = \mathrm{ad}_{g_1} g_3$ and $g_6 = \mathrm{ad}^2_{g_1} g_2$.

$$
\begin{aligned}
\Delta_0 &= \mathrm{span}\{g_1, g_2, \mathrm{ad}_{g_1} g_2, \mathrm{ad}^2_{g_1} g_2, g_3, \mathrm{ad}_{g_1} g_3\} = \mathrm{span}\{g_1, g_2, g_4, g_6, g_3, g_5\} \\
&= \mathrm{span}\left\{ \begin{pmatrix} 1 \\ \tan\theta_0 \\ 0 \\ \frac{\tan\phi_0}{l_0\cos\theta_0} \\ 0 \\ \frac{-\sin(\phi_1-\theta_0+\theta_1)}{l_1\cos\phi_1\cos\theta_0} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-1}{l_0\cos^2\phi_0\cos\theta_0} \\ 0 \\ 0 \end{pmatrix}, \right. \\
&\qquad \left. \begin{pmatrix} 0 \\ \frac{1}{l_0\cos^2\phi_0\cos^3\theta_0} \\ 0 \\ 0 \\ 0 \\ \frac{\cos(\phi_1+\theta_1)}{l_0 l_1\cos^2\phi_0\cos\phi_1\cos^3\theta_0} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\cos(\theta_1-\theta_0)}{l_1\cos^2\phi_1\cos\theta_0} \end{pmatrix} \right\} \\
\Delta_1 &= \mathrm{span}\{g_2, \mathrm{ad}_{g_1} g_2, \mathrm{ad}^2_{g_1} g_2, g_3, \mathrm{ad}_{g_1} g_3\} \\
\Delta_2 &= \mathrm{span}\{g_2, \mathrm{ad}_{g_1} g_2, g_3, \mathrm{ad}_{g_1} g_3\} \\
\Delta_3 &= \mathrm{span}\{g_2, \mathrm{ad}_{g_1} g_2, g_3\}
\end{aligned}
$$

The distribution Δ_0 is of full rank[2] on $U = \{(x_0, y_0, \phi_0, \theta_0, \phi_1, \theta_1) : \theta_0 - \theta_1, \phi_0, \phi_1, \theta_0 \neq \frac{\pi}{2}, \} \subset \mathbf{R}^6$.

It may be verified that Δ_1, Δ_2 and Δ_3 are involutive on U by computing the Lie brackets of their elements (g_i, $g_j \in \Delta_k$ for $1 \leq i,j \leq 3$ implies $[g_i, g_j] \in \Delta_k$ for $k = 1,2,3$).

A comment should be made here on why $(j,k) = (2,1)$ was chosen when there are actually four possible combinations (j,k) such that $j + k + 3 = 6$. For the two cases $(j,k) = (1,2)$ and $(j,k) = (0,3)$, Δ_0 is not of full rank. For the case $(j,k) = (3,0)$, Δ_0 is of full rank, but $\Delta_2 = \mathrm{span}\{g_2, g_4, g_6, g_3\}$ is not involutive. Therefore $(j,k) = (2,1)$ is the only combination that satisfies all conditions of Proposition 1.

The functions $h_1 = x_0$, $h_2 = y_0$ and $h_3 = \theta_1$ satisfy equation (4) for $(j,k) = (2,1)$. Note that there is a lack of uniqueness in the h's; other choices may yield a more intuitive numerical interpretation. Using these h's, the coordinate transformation $(\xi, \zeta, \eta) = \Phi(x)$ from equation (5) is computed.

$$
\begin{aligned}
\xi_0 = \quad & h_1 & = x_0 \\
\zeta_0 = \quad & L^2_{g_1} h_2 & = \frac{\tan\phi_0}{l_0\cos^3\theta_0}
\end{aligned}
$$

[2]This can be easily checked by showing that the six vector fields that define Δ_0 are linearly independent, i.e.,

$$
\det[g_1\ g_2\ g_4\ g_6\ g_3\ g_5] = \frac{-\cos(\theta_0 - \theta_1)}{l_0^2 l_1 \cos^4\phi_0\cos^2\phi_1\cos^5\theta_0}.
$$

$$
\begin{aligned}
\zeta_1 = \quad & L_{g_1} h_2 & = \tan\theta_0 \\
\zeta_2 = \quad & h_2 & = y_0 \\
\eta_0 = \quad & L_{g_1} h_3 & = \frac{-\sin(\phi_1 - \theta_0 + \theta_1)}{l_1 \cos\phi_1 \cos\theta_0} \\
\eta_1 = \quad & h_3 & = \theta_1
\end{aligned}
$$

This is a valid coordinate transformation since the matrix

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x} \cdot [\Delta_0] \quad &= \begin{bmatrix} dh_1 \\ dL_{g_1}^2 h_2 \\ dL_{g_1} h_2 \\ dh_2 \\ dL_{g_1} h_3 \\ dh_3 \end{bmatrix} \begin{bmatrix} g_1 & g_2 & \mathrm{ad}_{g_1} g_2 & \mathrm{ad}_{g_1}^2 g_2 & g_3 & \mathrm{ad}_{g_1} g_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ * & a_2(x) & * & * & * & * \\ * & 0 & -a_2(x) & * & 0 & * \\ * & 0 & 0 & a_2(x) & 0 & 0 \\ * & 0 & * & * & -a_3(x) & * \\ * & 0 & 0 & * & 0 & a_3(x) \end{bmatrix}
\end{aligned}
$$

has determinant $a_2^3(x) a_3^2(x) \neq 0$; therefore it is of full rank.

Next, we take the derivatives with respect to time

$$
\begin{aligned}
\dot{\xi}_0 &= \dot{x}_0 \\
\dot{\zeta}_0 &= \frac{\partial \zeta_0}{\partial \theta_0} \dot{\theta}_0 + \frac{\partial \zeta_0}{\partial \phi_0} \dot{\phi}_0 \\
\dot{\zeta}_1 &= \frac{\partial \zeta_1}{\partial \theta_0} \dot{\theta}_0 \\
\dot{\zeta}_2 &= \dot{y}_0 \\
\dot{\eta}_0 &= \frac{\partial \eta_0}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial \eta_0}{\partial \theta_0} \dot{\theta}_0 + \frac{\partial \eta_0}{\partial \phi_0} \dot{\phi}_0 \\
\dot{\eta}_1 &= \dot{\theta}_1 \, ,
\end{aligned}
$$

which give the following equations showing the required state feedback to put the system equations into the two-chain, single-generator chained form

$$
\begin{aligned}
\dot{\xi}_0 &= u_1 \\
&= v_1 \\
\dot{\zeta}_0 &= \frac{3 \, \tan^2\phi_0 \tan\theta_0}{l_0^2 \cos^4\theta_0} u_1 + \frac{1}{l_0 \cos^2\phi_0 \cos^3\theta_0} u_2 \\
&= v_2 \\
\dot{\zeta}_1 &= \frac{\tan\phi_0}{l_0 \cos^3\theta_0} u_1
\end{aligned}
$$

$$
\begin{aligned}
&= \zeta_0 v_1 \\
\dot{\zeta}_2 &= \tan\theta_0 \, u_1 \\
&= \zeta_1 v_1 \\
\dot{\eta}_0 &= \left(\frac{\cos(\phi_1+\theta_1)\sin\phi_0}{l_0 l_1 \cos\phi_0 \cos\phi_1 \cos^3\theta_0} + \frac{\cos(\phi_1-\theta_0+\theta_1)\sin(\phi_1-\theta_0+\theta_1)}{l_1^2\cos^2\phi_1\cos^2\theta_0} \right) u_1 \\
&\quad + \frac{-\cos(\theta_1-\theta_0)}{l_1\cos^2\phi_1\cos\theta_0}\, u_3 \\
&= v_3 \\
\dot{\eta}_1 &= \frac{-\sin(\phi_1-\theta_0+\theta_1)}{l_1\cos\phi_1\cos\theta_0}\, u_1 \\
&= \eta_0 v_1 \,.
\end{aligned}
$$

□

4 Controllability of Two-Chain, Single-Generator Chained Form Systems

In this section we show that a system in two-chain, single-generator chained form is completely controllable. Since controllability is unaffected by state feedback and coordinate transformation, it follows that the original system is also completely controllable.

Proposition 2 (Controllability of Systems in Two-Chain Single-Generator Chained Form)
The three-input, two-chain, single-generator chained form system in equation (3), where $(\xi, \zeta, \eta) \in \mathbb{R}^n$ *and* $n = j + k + 3$, *is completely controllable.*

Proof. The system is of the form

$$
\begin{pmatrix} \dot{\xi} \\ \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = f_1(\xi,\zeta,\eta)v_1 + f_2(\xi,\zeta,\eta)v_2 + f_3(\xi,\zeta,\eta)v_3\,,
$$

where the input vector fields are

$$
\begin{aligned}
f_1 &= \frac{\partial}{\partial \xi_0} + \sum_{i=1}^{j} \zeta_{i-1}\frac{\partial}{\partial \zeta_i} + \sum_{i=1}^{k} \eta_{i-1}\frac{\partial}{\partial \eta_i} \\
f_2 &= \frac{\partial}{\partial \zeta_0} \\
f_3 &= \frac{\partial}{\partial \eta_0}\,.
\end{aligned}
$$

Recall that a system of the form

$$
\dot{x} = \sum_{i=1}^{m} f_i(x) v_i
$$

is completely controllable if the involutive closure of the distribution $\Delta = \text{span}\{f_1, \ldots, f_m\}$ at each configuration is equal to the entire state space $\mathbb{R}^n$ (see Chow's Theorem [3]). Define $\Delta = \text{span}\{f_1, f_2, f_3\}$. The existence of n independent vector fields in the involutive closure, $\bar{\Delta}$, will imply complete controllability.

Consider the n-dimensional distribution, a subset of $\bar{\Delta}$, resulting from taking successive Lie brackets with f_1.

$$
\begin{aligned}
G &= \text{span}\{f_1, f_2, \text{ad}_{f_1} f_2, \ldots, \text{ad}_{f_1}^j f_2, f_3, \text{ad}_{f_1} f_3, \ldots, \text{ad}_{f_1}^k f_3\} \\
&= \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & & & & & & 0 \\
\zeta_0 & 0 & 1 & & & & & & 0 \\
\vdots & & & \ddots & & & & & \vdots \\
\zeta_{j-1} & & & & 1 & & & & \vdots \\
0 & & & & & 1 & & & \vdots \\
\eta_0 & & & & & & 1 & & \vdots \\
\vdots & & & & & & & \ddots & 0 \\
\eta_{k-1} & 0 & 0 & \cdots & 0 & 0 & 0 & & 1
\end{bmatrix}
\end{aligned}
$$

The columns are linearly independent vector fields in for each $x \in U$. Therefore the system is completely controllable. □

5 Steering Two-Chain, Single-Generator Chained Form Systems

An algorithm to steer the two-chain, single-generator chained form system in equation (3) from a given initial configuration to a desired final configuration will be stated for the case $j \geq k$, since through a renaming of the vector fields this can always be achieved. The algorithm systematically steers the states, starting at the top (those states with zero subscript) and working down the chains.

The algorithm exploits the decoupling of the two chains, allowing for simultaneous steering. The main idea, considering for a moment only the ζ chain, is that if $v_1 = \alpha \sin \omega t$ and $v_2 = \beta \cos \ell \omega t$, then $\dot{\zeta}_0$ will have a frequency component at $\ell\omega$, $\dot{\zeta}_1$ will have a frequency component at $(\ell - 1)\omega$, $\cdots$ and $\dot{\zeta}_\ell$ will have a frequency component at zero. By simple integration over one period, this yields net movement in ζ_ℓ while $\zeta_0, \cdots, \zeta_{\ell-1}$ return to their previous values.

This algorithm is an extension of the algorithm for two-input systems in Murray and Sastry [7].

Algorithm 1 (Step-by-step Steering with Sinusoids)

STEP 0. *Set the inputs to be constant over the time interval* $[0, T]$.

$$v_1 = \tfrac{1}{T}(\xi_0^f - \xi_0^o)$$
$$v_2 = \tfrac{1}{T}(\zeta_0^f - \zeta_0^o)$$
$$v_3 = \tfrac{1}{T}(\eta_0^f - \eta_0^o)$$

This will drive ξ_0, ζ_0 *and* η_0 *to their final positions.*

STEP I. *For $\ell = 2, \ldots, k$. Over the time interval $[\ell T, (\ell+1)T]$, set the inputs to be*

$$v_1 = \alpha \sin \omega t$$
$$v_2 = \beta \cos \ell \omega t$$
$$v_3 = \gamma \cos \ell \omega t \ ,$$

where $\omega = \frac{2\pi}{T}$ and α, β and γ are chosen such that

$$\zeta_\ell^f - \zeta_\ell(kT) = \frac{\alpha^\ell \beta}{(2\omega)^\ell \ell!} T$$
$$\eta_\ell^f - \eta_\ell(kT) = \frac{\alpha^\ell \gamma}{(2\omega)^\ell \ell!} T,$$

causing ζ_ℓ and η_ℓ to reach their final values. After step ℓ, the first ℓ coordinates in each chain, namely $\xi_0, \zeta_0, \ldots, \zeta_{\ell-1}, \eta_0, \ldots, \eta_{\ell-1}$ have returned to their final values.

STEP II. *For $\ell = k+1, \ldots, j$. Over the time interval $[\ell T, (\ell+1)T]$, set the inputs to be*

$$v_1 = \alpha \sin \omega t$$
$$v_2 = \beta \cos \ell \omega t$$
$$v_3 = 0,$$

where $\omega = \frac{2\pi}{T}$ and α and β are chosen such that

$$\zeta_\ell^f - \zeta_\ell(kT) = \frac{\alpha^\ell \beta}{(2\omega)^\ell \ell!} T,$$

causing ζ_ℓ to reach its final value. After step ℓ, the first ℓ coordinates in each chain, namely $\xi_0, \zeta_0, \ldots, \zeta_{\ell-1}, \eta_0, \ldots, \eta_k$ have returned to their final values.

Example (Continued). The details of the algorithm are shown below for controlling the firetruck, a system in two-chain, single-generator chained form with $(j, k) = (2, 1)$.

$$\begin{aligned}
\dot{\xi}_0 &= v_1 & & \\
\dot{\zeta}_0 &= v_2 & \dot{\eta}_0 &= v_3 \\
\dot{\zeta}_1 &= \zeta_0 v_1 & \dot{\eta}_1 &= \eta_0 v_1 \\
\dot{\zeta}_2 &= \zeta_1 v_1 & &
\end{aligned} \tag{5}$$

Let the initial state be (ξ^o, ζ^o, η^o) and the desired final state be (ξ^f, ζ^f, η^f).

STEP 0: The tops of the chains are steered first since they are directly controllable by the inputs. This is accomplished by setting the input functions to be constants in the time interval $[0, T]$

$$\begin{aligned}
v_1 &= \frac{1}{T}(\xi_0^f - \xi_0^o) \\
v_2 &= \frac{1}{T}(\zeta_0^f - \zeta_0^o) \\
v_3 &= \frac{1}{T}(\eta_0^f - \eta_0^o) \, ,
\end{aligned}$$

which yield, after integration,

$$\begin{aligned}
\xi_0(T) &= \xi_0^f \\
\zeta_0(T) &= \zeta_0^f \\
\eta_0(T) &= \eta_0^f \, .
\end{aligned}$$

The constant inputs affect the other three states, but these states will be steered in the subsequent steps.

STEP 1: The second coordinates in the chains, ζ_1 and η_1, are steered as follows. In the time interval $[T, 2T]$ the inputs are

$$\begin{aligned} v_1 &= \alpha \sin \omega t \\ v_2 &= \beta \cos \omega t \\ v_3 &= \gamma \cos \omega t \, , \end{aligned}$$

where $\omega = \frac{2\pi}{T}$. Therefore, by directly integrating the state equations from T to t, the states in the interval $[T, 2T]$ are

$$\begin{aligned} \xi_0(t) &= \xi_0(T) - \frac{\alpha}{\omega}(\cos \omega t - 1) \\ \zeta_0(t) &= \zeta_0(T) + \frac{\beta}{\omega} \sin \omega t \\ \eta_0(t) &= \eta_0(T) + \frac{\gamma}{\omega} \sin \omega t \\ \zeta_1(t) &= \zeta_1(T) + \int_T^t \left[\zeta_0(T) + \frac{\beta}{\omega} \sin \omega \tau \right] \alpha \sin \omega \tau d\tau \\ &= \zeta_1(T) - \frac{\alpha \zeta_0(T)}{\omega}(\cos \omega t - 1) + \frac{\alpha\beta}{2\omega}(t - T) + \frac{\alpha\beta}{4\omega^2} \sin 2\omega t \\ \eta_1(t) &= \eta_1(T) + \int_T^t \left[\eta_0(T) + \frac{\gamma}{\omega} \sin \omega \tau \right] \alpha \sin \omega \tau d\tau \\ &= \eta_1(T) - \frac{\alpha \eta_0(T)}{\omega}(\cos \omega t - 1) + \frac{\alpha\gamma}{2\omega}(t - T) + \frac{\alpha\gamma}{4\omega^2} \sin 2\omega t \\ \zeta_2(t) &= \zeta_2(T) + \int_T^t \zeta_1(\tau) v_1(\tau) d\tau \, . \end{aligned}$$

Evaluating the states at $2T$,

$$\begin{aligned} \xi_0(2T) &= \xi_0(T) = \xi_0^f \\ \zeta_0(2T) &= \zeta_0(T) = \zeta_0^f \\ \eta_0(2T) &= \eta_0(T) = \eta_0^f \\ \zeta_1(2T) &= \zeta_1(T) + \frac{\alpha\beta}{2\omega} T \\ \eta_1(2T) &= \eta_1(T) + \frac{\alpha\gamma}{2\omega} T \\ \zeta_2(2T) &= \zeta_2(T) + \int_T^{2T} \zeta_1(\tau) v_1(\tau) d\tau \, , \end{aligned}$$

the tops of the chains have returned to their final desired values while the second-level coordinates moved a net amount. Notice how both chains are steered simultaneously due to the decoupling. Here again, the third-level coordinate ζ_2 was affected, but it will be steered in the next step. If the parameters α, β and γ are chosen such that

$$\zeta_1^f - \zeta_1(T) = \frac{\alpha\beta}{2\omega} T$$

$$\eta_1^f - \eta_1(T) = \frac{\alpha\gamma}{2\omega}T$$

then

$$\begin{aligned} \zeta_1(2T) &= \zeta_1^f \\ \eta_1(2T) &= \eta_1^f . \end{aligned}$$

STEP 2: Since $j = 2$, this is the last step of the algorithm, in which state ζ_2 is controlled. In the time interval $[2T, 3T]$, set the input functions to be

$$\begin{aligned} v_1 &= \alpha \sin \omega t \\ v_2 &= \beta \cos 2\omega t \\ v_3 &= 0 \end{aligned}$$

with $\omega = \frac{2\pi}{T}$ as above. By integrating the state equations from $2T$ to t the states in the interval $[2T, 3T]$ are

$$\begin{aligned} \xi_0(t) &= \xi_0(2T) - \frac{\alpha}{\omega}(\cos\omega t - 1) \\ \zeta_0(t) &= \zeta_0(2T) + \frac{\beta}{2\omega}\sin 2\omega t \\ \eta_0(t) &= \eta_0(2T) \\ \zeta_1(t) &= \zeta_1(2T) + \int_{2T}^{t}\left[\zeta_0(2T) + \frac{\beta}{2\omega}\sin 2\omega\tau\right]\alpha\sin\omega\tau d\tau \\ &= \zeta_1(2T) - \frac{\alpha\zeta_0(2T)}{\omega}(\cos\omega t - 1) + \frac{\alpha\beta}{4\omega^2}\sin\omega t - \frac{\alpha\beta}{12\omega^2}\sin 3\omega t \\ \eta_1(t) &= \eta_1(2T) + \int_{2T}^{t}\eta_0(2T)\alpha\sin\omega\tau d\tau \\ &= \eta_1(2T) - \frac{\alpha\eta_0(2T)}{\omega}(\cos\omega t - 1) \\ \zeta_2(t) &= \zeta_2(2T) + \int_{2T}^{t}\left[\zeta_1(2T) - \frac{\alpha\zeta_0(2T)}{\omega}(\cos\omega\tau - 1) + \frac{\alpha\beta}{4\omega^2}\sin\omega\tau\right. \\ &\quad \left. -\frac{\alpha\beta}{12\omega^2}\sin 3\omega\tau\right]\alpha\sin\omega\tau d\tau \\ &= \zeta_2(2T) - \frac{\zeta_1(2T)\alpha}{\omega}(\cos\omega t - 1) + \frac{\alpha^2\zeta_0(2T)}{2\omega^2}(\cos^2\omega t - 1) \\ &\quad + \frac{\alpha^2\beta}{4\omega^2}\left[\frac{1}{2}(t - 2T) + \frac{1}{2\omega}\sin 2\omega t\right] - \frac{\alpha^2\beta}{12\omega^2}\left[\frac{1}{4\omega}\sin 2\omega t - \frac{1}{8\omega}\sin 4\omega t\right] . \end{aligned}$$

Evaluating the states at $3T$, we note that those states that have a zero frequency term do not return to their previous values. The non-zero frequency terms vanish when integrated over one period.

$$\begin{aligned} \xi_0(3T) &= \xi_0(2T) = \xi_0^f \\ \zeta_0(3T) &= \zeta_0(2T) = \zeta_0^f \end{aligned}$$

$$
\begin{aligned}
\eta_0(3T) &= \eta_0(2T) = \eta_0^f \\
\zeta_1(3T) &= \zeta_1(2T) = \zeta_1^f \\
\eta_1(3T) &= \eta_1(2T) = \eta_1^f \\
\zeta_2(3T) &= \zeta_2(2T) + \frac{\alpha^2\beta}{8\omega^2}T
\end{aligned}
$$

All of the states except ζ_2 have returned to their final values. Selecting the parameters α, β and γ such that

$$
\zeta_2^f - \zeta_2(2T) = \frac{\alpha^2\beta}{8\omega^2}T\,,
$$

yields $\zeta_2(3T) = \zeta_2^f$. Thus, the states are all driven to their desired values by Algorithm 1.

6 Simulation Results

The simulation of the firetruck system was performed on the transformed system in equation (5). The transformed states (ξ, ζ, η) were steered from an initial point to a final point by using sinusoids for the the transformed inputs v_1, v_2 and v_3 as in Algorithm 1. Then the inverse coordinate transformation

$$
\begin{aligned}
x_0 &= \xi_0 \\
y_0 &= \zeta_2 \\
\phi_0 &= \tan^{-1}(l_0\,\zeta_0 \cos^3(\tan^{-1}\zeta_1)) \\
\theta_0 &= \tan^{-1}\zeta_1 \\
\phi_1 &= \tan^{-1}\left\{\frac{\sin(\eta_1 - \tan^{-1}\zeta_1) + l_1\,\eta_0 \cos(\tan^{-1}\zeta_1)}{\cos(\eta_1 - \tan^{-1}\zeta_1)}\right\} \\
\theta_1 &= \eta_1
\end{aligned}
\tag{6}
$$

was performed on the simulation data to simulate the trajectory of the firetruck in the original coordinates. The results are presented for the familiar parallel-parking maneuver and for an arbitrary trajectory. Snap-shots from a movie animation of the simulation results are presented along with plots showing the steps of the algorithm.

Figure 2 shows nine frames from a movie of the simulation results for the parallel parking maneuver. In the original coordinates, this corresponds to steering the firetruck from $(x_0, y_0, \phi_0, \theta_0, \phi_1, \theta_1) = (0,3,0,0,0,0)$ to $(0,0,0,0,0,0)$. This maneuver is carried out by Step 2 of Algorithm 1, since the only state that must be changed is $y_0 = \zeta_2$, which is the third coordinate of the "ζ" chain. A trace of the trajectory is shown in Figure 3.

Figure 4 shows frames from the movie of the simulation results for steering the firetruck from an arbitrary initial configuration $(x_0, y_0, \phi_0, \theta_0, \phi_1, \theta_1) = (-2, 2, 0.099, 0.197, 0.544, 0.4)$ to $(0,0,0,0,0,0,0)$. The last two frames correspond to the parallel parking trajectory shown in more detail in Figure 2.

The steps of Algorithm 1 can be seen clearly in Figure 5, which shows sample trajectories with the arbitrary initial configuration above. The following discussion assumes small angles. The first

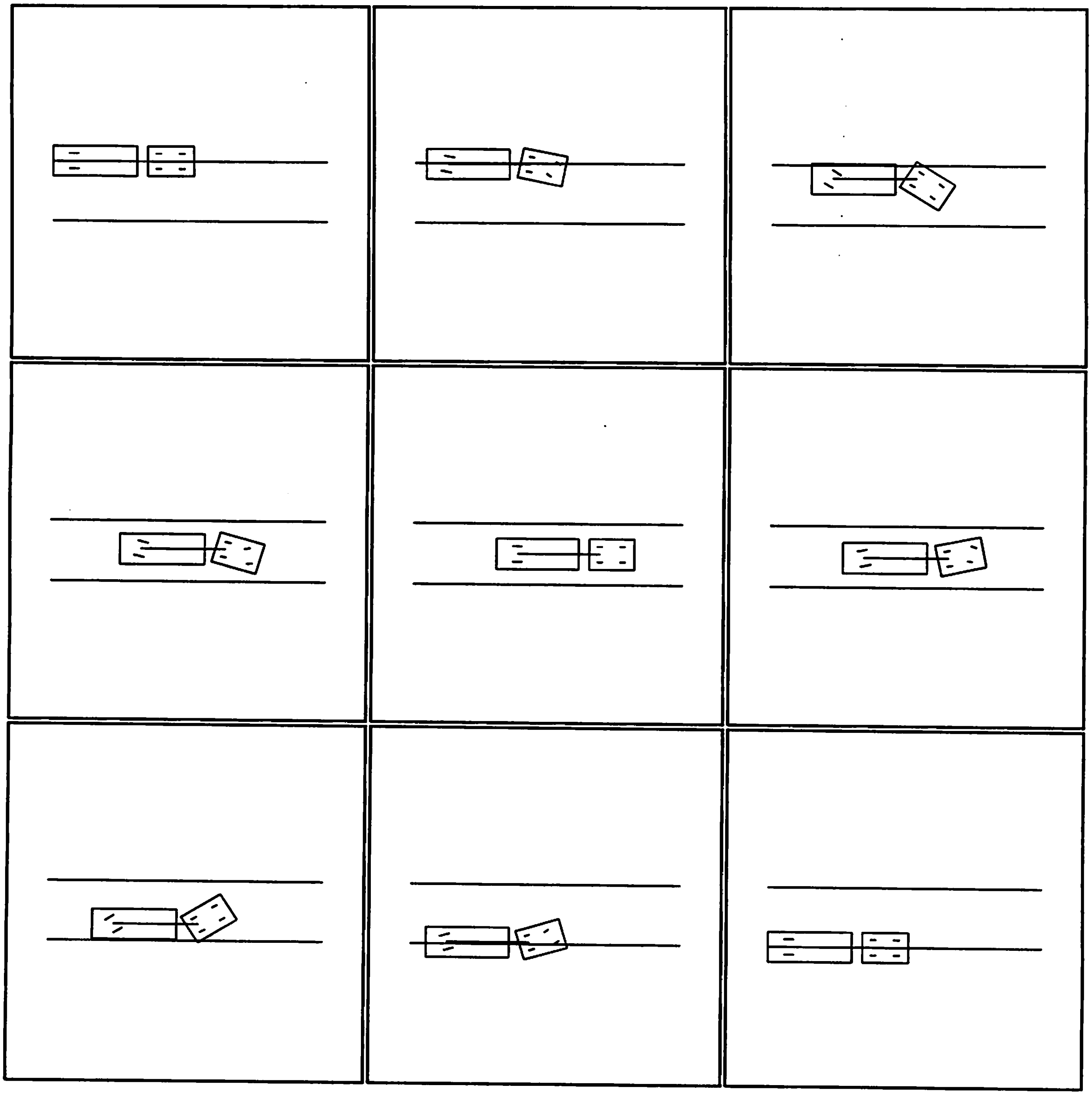


Figure 2: Movie of Parallel Parking Trajectory

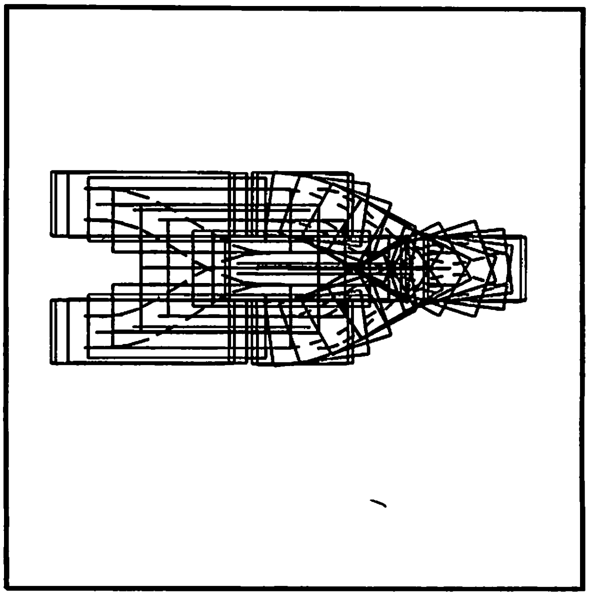


Figure 3: Trace of Parallel Parking Trajectory

part of the path, labeled A, corresponds to Step 0 and uses constant input to steer the transformed coordinates ξ_0, ζ_0, η_0 (tops of the chains) to their final values. In the original coordinates, this means that x_0 and ϕ_0 are steered to their desired position while the other four states drift. The second part, labeled B, corresponds to Step 1 and uses sinusoidal input to steer ζ_1 and η_1 to their final positions. Referring to equation (6) with small angles, this means that the body orientations θ_0 and θ_1 are steered to their desired positions. The wheel orientation ϕ_1 is now at its final value, since it involves both first and second coordinates of the chains. Step 1 also brings states x_0, ϕ_0 back to their correct values. The last part, labeled C, drives y_0 to its desired value and returns the other states to their final values.

The Lissajous figures arise because of the model's structure. Refer to the portion of the trajectory labeled C in Figure 5 and consider only the first three plots (one chain). The ϕ_0 vs. x_0 phase plot has a Lissajous figure with two loops, the θ_0 vs. x_0 plot has one loop and the y_0 vs. x_0 plot has net motion in the y_0 variable. The state x_0 is controlled directly by the input u_1 and therefore is moved in the direction of vector field g_1. Similarly, ϕ_0 is moved in the g_2 direction. Taking first-order Lie brackets shows that θ_0 is moved in the direction $ad_{g_1}g_2$. The state y_0 is driven in the direction of the second-order Lie bracket $ad^2_{g_1}g_2$. The number of loops is determined by the order of the Lie bracket needed to get net motion in a desired direction. For example, the figures show that in order to get net motion in the y_0 direction, ϕ_0 vs. x_0 goes through two loops and θ_0 vs. x_0 goes through one loop. We are essentially rectifying the harmonics of the sinusoidal input functions [2] using the Lie bracket vector field directions.

In Figure 6 the three parts of the transformed and original input functions needed to control the above example are shown. The transformed inputs are the open-loop control laws for the system in two-chain, single-generator chained form. The original inputs, however, depend on the original

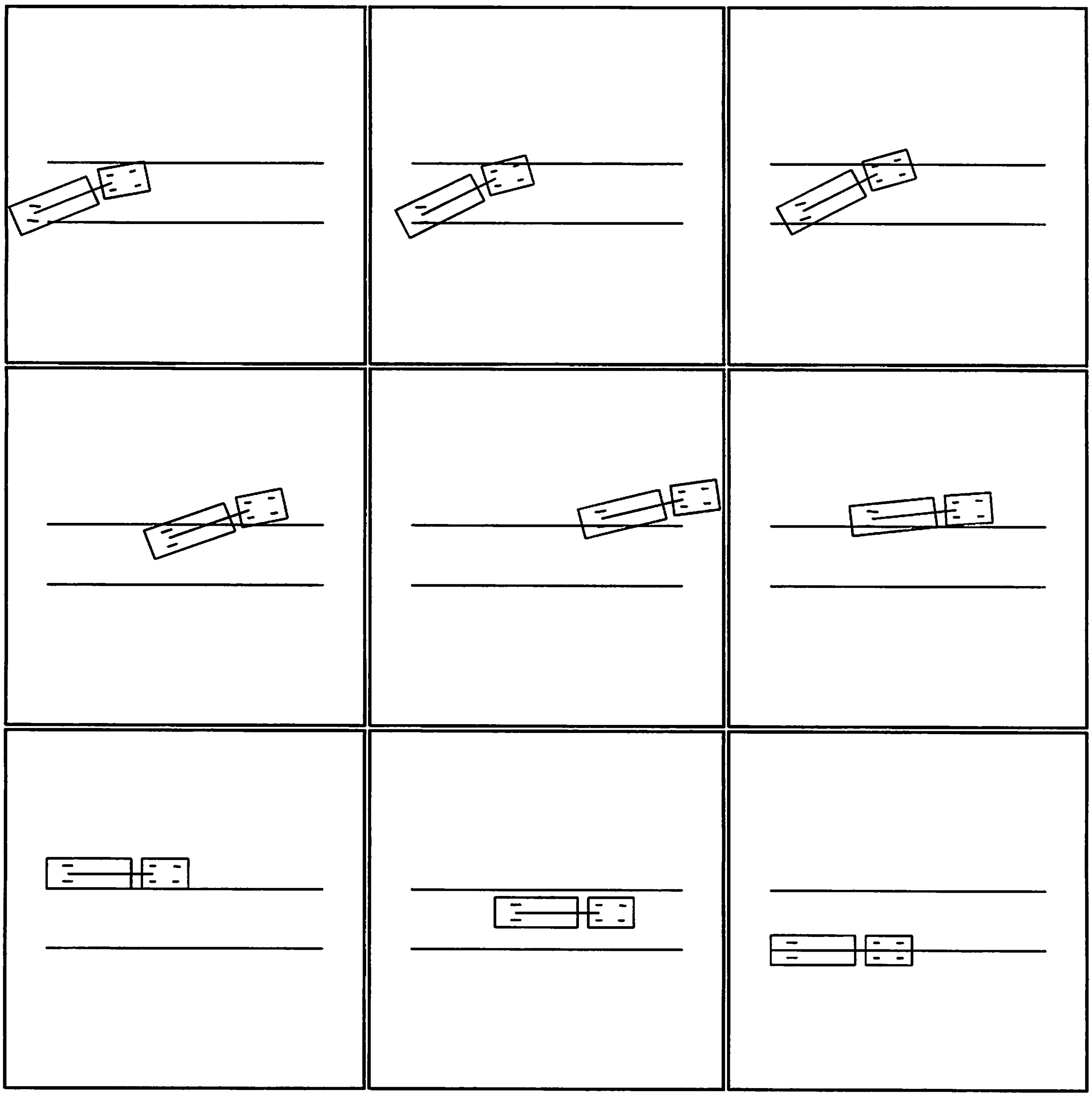
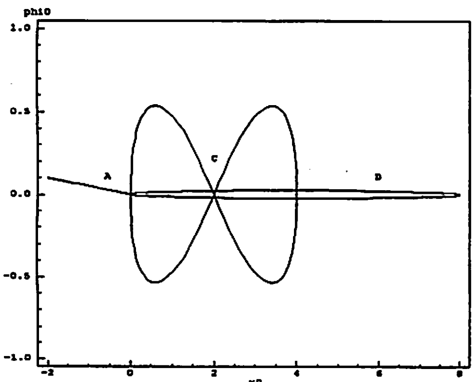
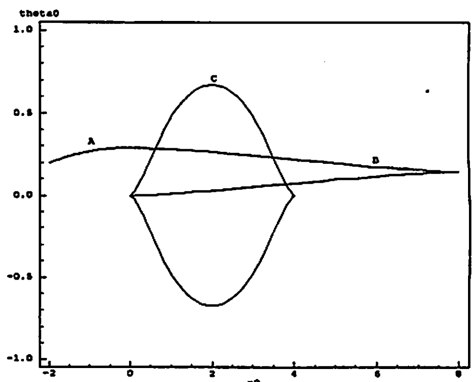


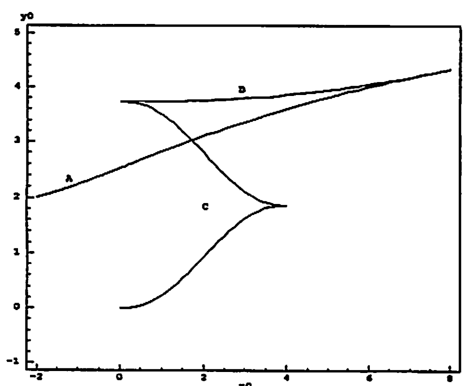
Figure 4: Movie of Trajectory with Arbitrary Initial Configuration



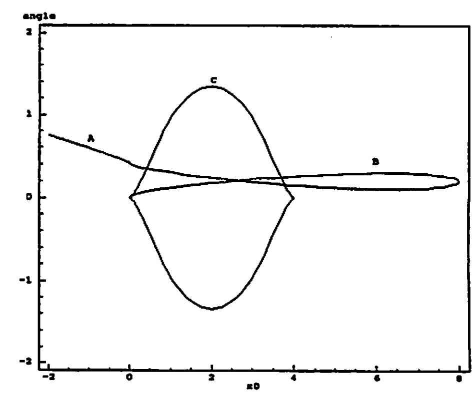
ϕ_0 vs. x_0



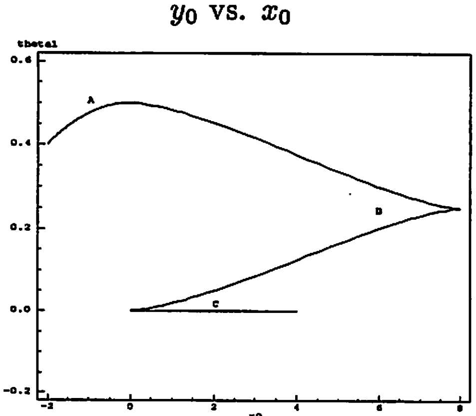
θ_0 vs. x_0



y_0 vs. x_0

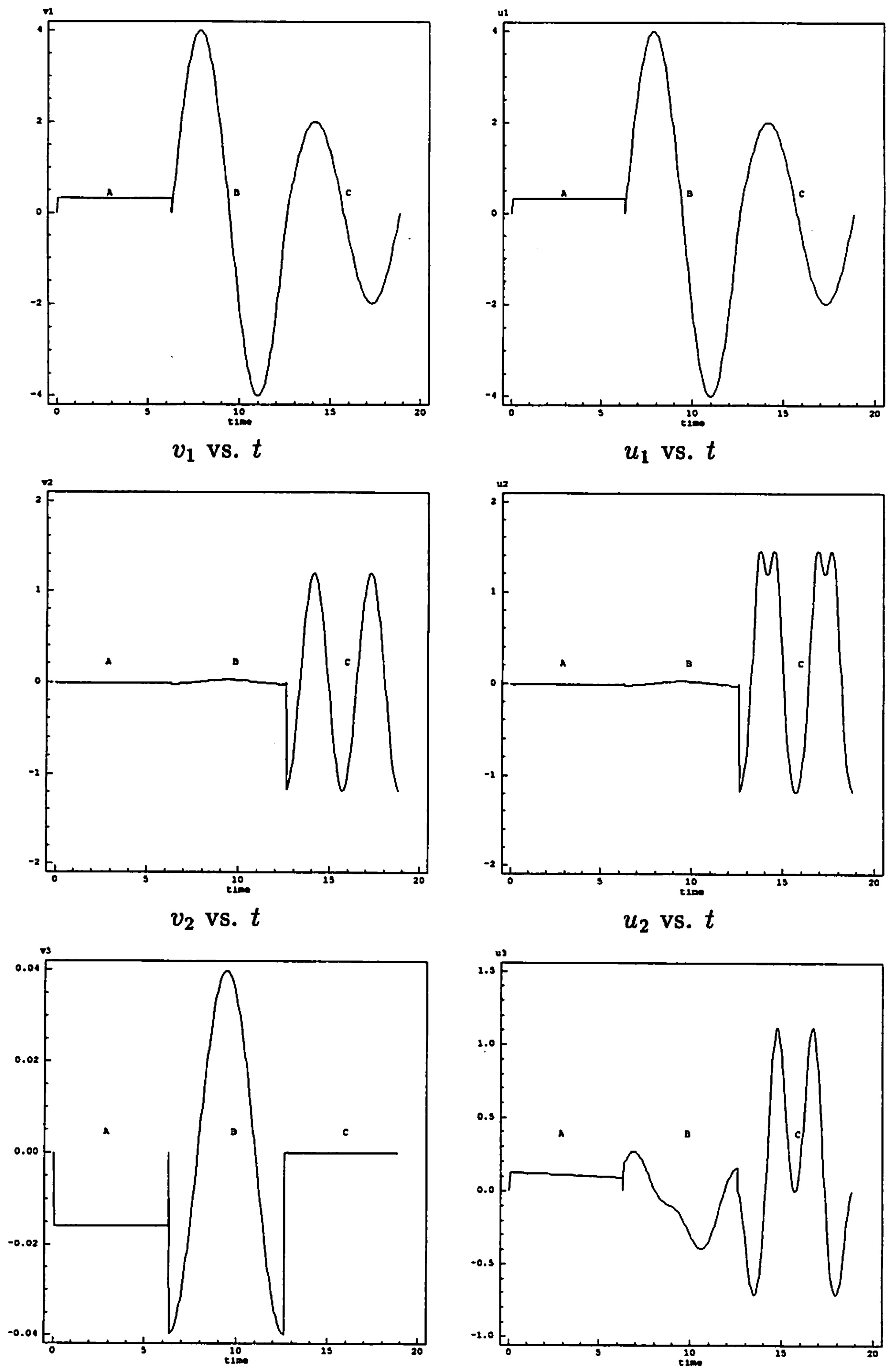


$\phi_1 - \theta_0 + \theta_1$ vs. x_0



θ_1 vs. x_0

Figure 5: Sample Trajectories for Arbitrary Initial Configuration



v_1 vs. t u_1 vs. t

v_2 vs. t u_2 vs. t

v_3 vs. t u_3 vs. t

Figure 6: Inputs for Arbitrary Initial Configuration

states of the system, as can be seen in the following equations.

$$
\begin{aligned}
u_1 &= v_1 \\
u_2 &= l_0 \cos^2 \phi_0 \cos^3 \theta_0 \left(v_2 - u_1 \, \frac{3 \, \tan^2 \phi_0 \tan \theta_0}{l_0^2 \cos^4 \theta_0} \right) \\
u_3 &= \frac{-l_1 \cos^2 \phi_1 \cos \theta_0}{\cos(\theta_1 - \theta_0)} \left(v_3 - u_1 \left(\frac{\cos(\phi_1 + \theta_1) \sin \phi_0}{l_0 l_1 \cos \phi_0 \cos \phi_1 \cos^3 \theta_0} \right.\right. \\
&+ \left.\left. \frac{\cos(\phi_1 - \theta_0 + \theta_1) \sin(\phi_1 - \theta_0 + \theta_1)}{l_1^2 \cos^2 \phi_1 \cos^2 \theta_0} \right) \right)
\end{aligned}
$$

7 Conclusions

Sufficient conditions have been given for transforming a three-input drift-free nonholonomic system into a two-chain, single-generator chained form by a coordinate transformation and state feedback. In this special form, the system was shown to be completely controllable and easily steerable by using sinusoidal input functions with integrally related frequencies. The steering algorithm provided a step-by-step method for open-loop, point-to-point control.

The main example used to illustrate the ideas of this paper was a firetruck, our example of a three-input nonholonomic system. The kinematic equations were derived using nonholonomic linear velocity constraints for the non-slipping conditions of the wheels rolling on the road. The system was transformed into a two-chain, single-generator chained form and steered using sinusoids at integrally related frequencies. Simulation results show the effectiveness of this algorithm.

We would like to mathematically investigate how the ability to steer the firetruck's back wheels allows for sharper turning. Future work also includes extending the theory to m-input nonholonomic systems, which may be associated with the example of automating trucks with multiple trailers, where each trailer has the ability to steer one set of wheels. Such m-input systems would be converted to a m(m-1)-chain, m-generator form. The Appendix gives sufficient conditions for the special case of an (m-1)-chain, single-generator chained form.

Other future work on the firetruck example includes decentralized control, deriving a collision avoidance controller and using the Alexander-Maddocks condition [1] for the simulations, which would restrict the wheel angles of the firetruck to be more realistic (wheel differentials).

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Appendix.
We extend the theory in this paper to general chained form systems with only one generator. For a m-input system, the m-input, m(m-1)-chain, m-generator chained form is stated in [5] as follows.

$$\begin{aligned} \dot{z}_j^0 &= v_j \quad 1 \le j \le m \\ \dot{z}_{ij}^1 &= z_i^0 v_j \quad i > j \;\; and \;\; z_{ji}^1 := z_j^0 z_i^0 - z_{ij}^1 \\ \dot{z}_{ij}^k &= z_{ij}^{k-1} v_j \quad 1 \le i,j \le m;\;\; i \ne j;\;\; k > 1 \end{aligned}$$

For example, when m = 3 , the chained form system has six chains:

$$\begin{array}{cccccc} \dot{z}_1^0 = v_1 & & \dot{z}_2^0 = v_2 & & \dot{z}_3^0 = v_3 & \\ \dot{z}_{21}^1 = z_2^0 v_1 & \dot{z}_{31}^1 = z_3^0 v_1 & & \dot{z}_{32}^1 = z_3^0 v_2 & & \\ \vdots & \vdots & \dot{z}_{12}^2 = z_{12}^1 v_2 & \vdots & \dot{z}_{13}^2 = z_{13}^1 v_3 & \dot{z}_{23}^2 = z_{23}^1 v_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dot{z}_{21}^k = z_{21}^{k-1} v_1 & \dot{z}_{31}^k = z_{31}^{k-1} v_1 & \dot{z}_{12}^k = z_{12}^{k-1} v_2 & \dot{z}_{32}^k = z_{32}^{k-1} v_2 & \dot{z}_{13}^k = z_{13}^{k-1} v_3 & \dot{z}_{23}^k = z_{23}^{k-1} v_3 \end{array}$$

where $k > 1$, $z_{12}^1 := z_1^0 z_2^0 - z_{21}^1$, $z_{13}^1 := z_1^0 z_3^0 - z_{31}^1$ and $z_{23}^1 := z_2^0 z_3^0 - z_{32}^1$.

Here we give conditions for transforming an m-input system into a (m-1)-chain, single-generator chained form.

Proposition 3 Converting Systems to m-Input, (m-1)-Chain, Single-Generator Chained Form
Given the system

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m \,,$$

with $x \in U \subset \mathbf{R}^n$, $u_i \in \mathbf{R}$ and the g_i smooth linearly independent vector fields. Define the distributions

$$\begin{aligned} \Delta_0 &:= \operatorname{span}\{g_1, \operatorname{ad}_{g_1}^{k_j} g_j;\; k_j = 0,\ldots,n_j;\; j = 2,\ldots,m\} \\ \Delta_1 &:= \operatorname{span}\{\operatorname{ad}_{g_1}^{k_j} g_j;\; k_j = 0,\ldots,n_j;\; j = 2,\ldots,m\} \\ \Delta_2 &:= \operatorname{span}\{\operatorname{ad}_{g_1}^{k_j} g_j;\; k_2 = 0,\ldots,n_2 - 1;\; k_j = 0,\ldots,n_j \; j \ne 2;\; j = 2,\ldots,m\} \\ &\;\;\vdots \\ \Delta_m &:= \operatorname{span}\{\operatorname{ad}_{g_1}^{k_j} g_j;\; k_j = 0,\ldots,n_j - 1;\; j = 2,\ldots,m\} \end{aligned}$$

where $\sum_{j=2}^m n_j + m + 1 = n$.

If (1) $\Delta_0(x) = \mathbf{R}^n$ and (2) Δ_j for $1 \le j \le m$ are involutive, then $\exists$ local feedback transformation on U

$$\begin{aligned} z &= \Phi(x) \\ u &= \beta(x) v \end{aligned}$$

such that system is in m-input, (m-1)-chain, 1-generator chained form:

$$\begin{array}{llll}
\dot{z}_1^0 = v_1 & & & \\
\dot{z}_2^0 = v_2 & \dot{z}_3^0 = v_3 & \cdots & \dot{z}_m^0 = v_m \\
\dot{z}_{21}^1 = z_2^0 v_1 & \dot{z}_{31}^1 = z_3^0 v_1 & \cdots & \dot{z}_{m1}^1 = z_m^0 v_1 \\
\vdots & \vdots & \vdots & \vdots \\
\dot{z}_{21}^{n_2} = z_{21}^{n_2-1} v_1 & \dot{z}_{31}^{n_3} = z_{31}^{n_3-1} v_1 & \cdots & \dot{z}_{m1}^{n_m} = z_{m1}^{n_m-1} v_1 \ .
\end{array}$$

The proof follows the same method as in the proof of Proposition 1.