DESIGN OF A FINITE DIMENSIONAL STABILIZING COMPENSATOR FOR A FLEXIBLE BEAM WITH POINT ACTUATORS AND SENSORS

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ABSTRACT

We use the computational stability criterion presented in [Har.1] to design a finite dimensional stabilizing compensator for a controlled flexible beam with point actuators and sensors. To establish the applicability of the computational stability criterion presented in [Har.1], we prove that the controlled flexible beam with point actuators and sensors to be described fits the plant model given in [Har.1]. Numerical results are given.

1. INTRODUCTION

There is considerable interest in the design of control systems for various flexible structures, such as those found in space applications, as well as in robotic manipulators with flexible arms (see e.g., [Bal.1]). The plants of such systems tend to be infinite dimensional and can be modeled by partial differential equations. Robust exponential stability is the most basic requirement of control system design. In [Har.1], we have presented a necessary and sufficient computational stability criterion which is compatible with the use of semi-infinite optimization algorithms for a class of infinite dimensional systems. In this paper, we demonstrate the usefulness of the criterion in [Har.1], by showing that it is applicable to the design of a control system for a particular controlled flexible beam with point actuators and sensors. In Section 2, we will summarize the results in [Har.1]. In Section 3, we will show that our particular plant satisfies the assumptions in [Har.1], and in Section 4 we will present a numerical example.

2. PRELIMINARY RESULTS

Consider the feedback system $S(P,K)$, with n_i inputs and n_o outputs, shown in Fig. 1. We assume that the plant is described by a linear and time-invariant differential equation in a Hilbert space E:

$$\dot{x}_p(t) = A_p x_p(t) + B_p e_2(t); \quad y_2(t) = C_p x_p(t) + D_p e_2(t), \tag{2.1}$$

where $x_p(t) \in E$, $e_2(t) \in \mathbb{R}^{n_i}$, $y_2(t) \in \mathbb{R}^{n_o}$, for $t \geq 0$. The operator A_p from E to E, may be an unbounded operator with domain dense in E, which generates a strongly continuous (C_0) semigroup, $\{e^{A_p t}\}_{t \geq 0}$. The operators $C_p : E \to \mathbb{R}^{n_o}$ and $D_p : \mathbb{R}^{n_i} \to \mathbb{R}^{n_o}$ are assumed to be bounded, while the operator B_p can be unbounded, in the sense that it can be a multiplication operator associated with delta functions which do not belong to the space E. Hence model (2.1) can represent a flexible beam with point actuators and sensors. To define the operator B_p more exactly, we need to extend the state space. For this purpose, we first denote the adjoint operator of A_p by A_p^*, the dual space of E by E^* and the domain and the range of A_p by $D(A_p)$ and $R(A_p)$, respectively. As in [Cur.1], we then let

$$Z^* \triangleq D_{E^*}(A_p^*) \subset E^* \tag{2.2}$$

endowed with the graph norm of A_p^*. Then Z^* is a real, reflexive Banach space and the injection of Z^* into E^* is continuous and dense. Defining the *extended state space* Z to be the dual space of Z^*, we obtain by duality that

$$E \subset Z, \tag{2.3}$$

with a continuous dense injection.

From now on, we will treat the state of the plant , x_p, as an element of the extended state space Z, and we will assume that $B_p : \mathbb{R}^{n_i} \to Z$, is bounded. Because E is dense in Z, C_p can be extended to a bounded operator from Z to $\mathbb{R}^{n_o}$. Referring to [Cur.1, Sal.1], we see that A_p^* can be regarded as a bounded operator from Z^* into E^*. By duality, A_p extends to a bounded operator from E to Z. This extension, regarded as an unbounded operator on Z, is the infinitesimal generator of the extended semigroup $\{e^{A_p t}\} \in L(Z)$. The exponential growth rate of the semigroup $\{e^{A_p t}\}$ is the same on the state spaces E and Z. Furthermore, the spectrum of A_p on the state space E coincides with the spectrum of A_p on Z.

We define the *transfer function* of the plant , $G_p(s)$, to be $C_p(sI - A_p)^{-1}B_p + D_p$, $\forall\ s \in \rho(A_p)$. $G_p(s)$ is analytic on $\rho(A_p)$ [Kat.1, Theorem III 6.7] and $\lim_{\substack{|s| \to \infty \\ \mathrm{Re}\, s > \gamma}} G_p(s) \to D_p$ [Jac.1].

Definition 2.1: For any $\alpha \geq 0$, the semi-group $\{e^{A_p t}\}_{t \geq 0}$ is said to be α*–stable* if there exist $M \in (0,\infty)$ and $\alpha_0 > \alpha$ such that

$$\|e^{A_p t}\|_Z \le M e^{-\alpha_0 t}\ , \quad \forall\ t \ge 0\ . \tag{2.4}$$

For any $\alpha \ge 0$, we define the *stability region* $D_{-\alpha} \triangleq \{s\in \mathbb{C} \mid \mathrm{Re}(s) < -\alpha\}$, with compliment , in $\mathbb{C}$, $U_{-\alpha} = \{s\in \mathbb{C} \mid \mathrm{Re}(s) \ge -\alpha\}$, whose boundary and interior will be denoted by $\partial U_{-\alpha} = \{s\in \mathbb{C} \mid \mathrm{Re}(s) = -\alpha\}$ and $U^o_{-\alpha} = \{s\in \mathbb{C} \mid \mathrm{Re}(s) > -\alpha\}$. Let $\sigma(A_p)$ denote the *spectrum* of A_p and let $\rho(A_p)$ denote the *resolvent set* of A_p.

We assume that the plant in (2.1) is α*–stabilizable* and α*–detectable*, i.e., there exist bounded linear operators $K: Z \to \mathbb{R}^{n_i}$ and $F: \mathbb{R}^{n_o} \to Z$ such that $A_p - B_p K$ and $A_p - FC_p$ are the infinitesimal generators of α–stable C_0–semigroups. It can be shown that the plant is α–stabilizable and α–detectable if and only if there exists a decomposition of $Z = Z_- + Z_+$, with Z_+ finite-dimensional, which induces a decomposition of the plant (2.1), of the form

$$\frac{d}{dt}\begin{bmatrix} x_{p-}(t) \\ x_{p+}(t) \end{bmatrix} = \begin{bmatrix} A_{p-} & 0 \\ 0 & A_{p+} \end{bmatrix}\begin{bmatrix} x_{p-}(t) \\ x_{p+}(t) \end{bmatrix} + \begin{bmatrix} B_{p-} \\ B_{p+} \end{bmatrix} u(t)\ ; \qquad y(t) = [C_{p-} \ \ C_{p+}]\begin{bmatrix} x_{p-}(t) \\ x_{p+}(t) \end{bmatrix} + D_p u(t), \tag{2.5}$$

such that $\sigma(A_{p+}) \subset U_{-\alpha}$, (A_{p+}, B_{p+}) is controllable, (A_{p+}, C_{p+}) is observable, and A_{p-} is the infinitesimal generator of an α–stable C_0–semigroup on Z_- [Nef.1, Jac.1].† We recall that a plant is α–stabilizable and α–detectable if and only if there exists a finite dimensional strictly proper compensator such that the feedback system is α–stable [Jac.1].

We assume the compensator to be *finite dimensional, linear, and time-invariant*, with state equations

$$\dot{x}_c(t) = A_c x_c(t) + B_c e_1(t); \quad y_1(t) = C_c x_c(t) + D_c e_1(t)\ , \tag{2.6}$$

where $x_c(t)\in \mathbb{R}^{n_c}$, $e_1(t)\in \mathbb{R}^{n_o}$, $y_1(t)\in \mathbb{R}^{n_i}$ and A_c, B_c, C_c and D_c are matrices of appropriate dimension. The compensator transfer function is $G_c(s) = C_c(sI_{n_c} - A_c)^{-1}B_c + D_c$. The compensator is also assumed to be α–stabilizable and α–detectable. To ensure well-posedness of the feedback system, we assume that $\det(I_{n_i} + D_c D_p) \ne 0$.

We define the inner product space $H = Z \times \mathbb{R}^{n_c}$. Since $e_1 = u_1 - y_2$ and $e_2 = y_1 + u_2$, the state equations for the feedback system are

† In [Nef.1, Jac.1] only 0-stability is considered. Our extension to α–stability is trivial.

$$\begin{bmatrix}\dot{x}_p\\ \dot{x}_c\end{bmatrix} = A\begin{bmatrix}x_p\\ x_c\end{bmatrix} + B\begin{bmatrix}u_1\\ u_2\end{bmatrix}\,; \qquad \begin{bmatrix}e_1\\ e_2\end{bmatrix} = C\begin{bmatrix}x_p\\ x_c\end{bmatrix} + D\begin{bmatrix}u_1\\ u_2\end{bmatrix}\,, \tag{2.7}$$

where

$$A = \begin{bmatrix} A_p - B_pD_c(I_{n_o}+D_pD_c)^{-1}C_p & B_p(I_{n_i}+D_cD_p)^{-1}C_c \\ -B_c(I_{n_o}+D_pD_c)^{-1}C_p & A_c - B_c(I_{n_o}+D_pD_c)^{-1}D_pC_c \end{bmatrix} \tag{2.8a}$$

$$B = \begin{bmatrix} B_pD_c(I_{n_o}+D_pD_c)^{-1} & B_p(I_{n_i}+D_cD_p)^{-1} \\ B_c(I_{n_o}+D_pD_c)^{-1} & -B_c(I_{n_o}+D_pD_c)^{-1}D_p \end{bmatrix}, \tag{2.8b}$$

$$C = \begin{bmatrix} -(I_{n_o}+D_pD_c)^{-1}C_p & -(I_{n_o}+D_pD_c)^{-1}D_pC_c \\ -D_c(I_{n_o}+D_pD_c)^{-1}C_p & (I_{n_i}+D_cD_p)^{-1}C_c \end{bmatrix}; \quad D = \begin{bmatrix} (I_{n_o}+D_pD_c)^{-1} & -(I_{n_o}+D_pD_c)^{-1}D_p \\ D_c(I_{n_o}+D_pD_c)^{-1} & (I_{n_i}+D_cD_p)^{-1} \end{bmatrix}. \tag{2.8c}$$

The domain $D(A) = D(A_p) \times \mathbb{R}^{n_c} \subset H$. It follows from [Paz.1, p. 76], that because, with the exception of A_p, all the operators in the matrix A are bounded, and because A_p generates a C_0-semigroup, the operator A also generates a C_0–semigroup, $\{e^{At}\}_{t\geq 0}$.

Let $x = [x_p, x_c] \in H$. Then the formula $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$ defines a *mild solution* of (2.7) [Paz.1]. We therefore define the *exponential stability* of the feedback system $S(P,K)$ in terms of the semigroup $\{e^{At}\}_{t\geq 0}$.

Definition 2.2: For any $\alpha \geq 0$, the feedback system S(P , K) is α–stable if the semi-group $\{e^{At}\}_{t\geq 0}$ is α–stable. ■

It was shown in [Jac.1] that , under the above assumptions, the feedback system $S(P,K)$ is also α–stabilizable and α–detectable. From the decomposition property in (2.5), for α–stabilizable and α–detectable systems, we can easily deduce the following relationship between α–stability of the feedback system and the spectrum of A, first established in [Jac.1]:

Proposition 2.1: If the above assumptions hold, the feedback system is α–stable if and only if $U_{-\alpha}$ is contained in $\rho(A)$. ■

We define the characteristic function χ: $\mathbb{C} \to \mathbb{C}$, of the feedback system $S(P,K)$, by

$$\chi(s) \triangleq \det(sI_{n_+} - A_{p+})\det(sI_{n_c} - A_c)\det(I_{n_i} + G_c(s)G_p(s))\ , \tag{2.9}$$

where A_{p+} is defined as in (2.5) and n_+ is the dimension of A_{p+}.

Theorem 2.1:[Har.1] The system $S(P,K)$ is α–stable if and only if $Z(\chi(s)) \subset D_{-\alpha}$. ■

Theorem 2.2:[Har.1] Let n_+ and n_c be the dimensions of the matrices A_{p+} in (2.5) and A_c in (2.6), respectively. $Z(\chi(s)) \subset D_{-\alpha}$ if and only if there exists an integer $N_n > 0$, and polynomials $d_0(s)$ and $n_0(s)$, of degree $N_d = N_n + n_s$ and N_n, respectively, with $n_s = n_c + n_+$, such that

$$(i)\ \ Z(d_0(s)) \subset D_{-\alpha}\,, \quad Z(n_0(s)) \subset D_{-\alpha}\,; \qquad (ii)\ \ \mathrm{Re}\left[\frac{\chi(s)n_0(s)}{d_0(s)}\right] > 0 \ \ \forall\ s \in \partial U_{-\alpha}\,. \tag{2.10}$$

■

3. A DESIGN EXAMPLE

Consider the planar bending motion of a flexible beam, shown in Fig. 2. One end of the beam is fixed; a particle with mass M is attached to the other end. The x-axis is the undeformed-beam centroidal line; the y-axis is the cross section principal axis. The associated control system is required to damp out vibrations. Assuming that the beam is of unit length, its bending motion can be described by the partial differential equation

$$m\frac{\partial^2 w(t,x)}{\partial t^2} + cI\frac{\partial^5 w(t,x)}{\partial x^4 \partial t} + EI\frac{\partial^4 w(t,x)}{\partial x^4} = \sum_{j=1}^{n_i} f^j(t)\delta(x - x^j), \ \ t \ge 0,\ \ 0 \le x \le 1\,, \tag{3.1a}$$

with boundary conditions

$$w(t,0) = 0\,,\ \frac{\partial w}{\partial x}(t,0) = 0, \tag{3.1b}$$

$$J\frac{\partial^3 w}{\partial x \partial t^2}(t,1) + cI\frac{\partial^3 w}{\partial x^2 \partial t}(t,1) + EI\frac{\partial^2 w}{\partial x^2}(t,1) = 0\,, \tag{3.1c}$$

$$M\frac{\partial^2 w}{\partial t^2}(t,1) - cI\frac{\partial^4 w}{\partial x^3 \partial t}(t,1) - EI\frac{\partial^3 w}{\partial x^3}(t,1) = 0, \tag{3.1d}$$

where $w(t,x)$ is the vibration along the y-axis, $f^j(t)$ is a control force and $\delta(x - x^j)$ is the Dirac delta function; m is the distributed mass per unit length of the beam, c is the material viscous damping coefficient , E is Young's modulus, M is the end mass, I is the beam sectional moment of inertia with respect to y-axis, J is the inertia of the end mass in the direction of y-axis, and n_i is the number of actuators. The output sensors can be assumed to be modeled by

$$y^i(t) = \int_0^1 \delta(\nu - z^i) w(t,\nu) d\nu \ , \quad t \geq 0 \ , \quad 1 \leq i \leq n_o$$

$$= w(t, z^i), \tag{3.2}$$

where n_o is the number of the sensors, and $\delta(\nu - z^i)$ is the Dirac delta function.

We now proceed to show that the system (3.1a-d), (3.2) can be transcribed into the first order form (2.1), with the assumptions stated. For simplicity and without loss of generality, we assume that there is only one actuator and one sensor. First we rewrite (3.1a-d) in the form

$$\ddot{\overline{w}}(t) + D_0\dot{\overline{w}}(t) + A_0\overline{w}(t) = B_0 f(t) \quad , t \geq 0 \ , \ 0 \leq x \leq 1 \ , \tag{3.3a}$$

$$y(t) = C_0 \overline{w}, \tag{3.3b}$$

where

$$\overline{w} = (w(\cdot),\ w_1,\ w_2)^T \in D(A_0) = D(D_0) = \left\{ w \in H^4([0,1]), w(0) = w'(0) = 0,\ w_1 = w(1),\ w_2 = w'(1) \right\} \tag{3.3c}$$

$$\subset H_0 \triangleq L^2([0,1]) \times \mathbb{R}^2 \ ,$$

$$D_0\overline{w} = \left(\frac{cI}{m}\frac{d^4w(x)}{dx^4},\ \frac{-cI}{M}\frac{d^3w}{dx^3}(1),\ \frac{cI}{J}\frac{d^2w}{dx^2}(1)\right)^T , \tag{3.3d}$$

$$A_0\overline{w} = \left(\frac{EI}{m}\frac{d^4w(x)}{dx^4},\ \frac{-EI}{M}\frac{d^3w}{dx^3}(1),\ \frac{EI}{J}\frac{d^2w}{dx^2}(1)\right)^T , \tag{3.3e}$$

$$B_0 = (\delta(x,x_i),\ 0,\ 0)^T \ , \quad C_0\overline{w} = w(x_o) \ , \tag{3.3f}$$

$H^4([0,1])$ denotes the set of functions whose fourth derivative belongs to $L^2([0,1])$ and w' denotes the derivative of w with respect to the spatial variable x. Note that $D_0 = \frac{E}{c}A_0$ in the above example. Let $\overline{u} = (u(\cdot),\ u_1,\ u_2)^T$ and $\overline{v} = (v(\cdot),\ v_1,\ v_2)^T$. We define the inner product in H_0 as follows:

$$\langle \overline{u},\ \overline{v} \rangle_{H_0} = \langle u,\ v \rangle_{L^2([0,1])} + \frac{M}{m} u_1 v_1 + \frac{J}{m} u_2 v_2 \ . \tag{3.4}$$

We have the following nice property for the operator A_0.

Proposition 3.1: The linear stiffness operator A_0 is a positive definite and self-adjoint operator from

$D(A_0)$, which is dense in H_0, onto H_0, with compact inverse. In fact, A_0 is coercive, i.e., there exists $\rho > 0$ such that

$$\langle A_0\bar{v}, \bar{v}\rangle_{H_0} \geq \rho^2 \|\bar{v}\|^2_{H_0}, \quad \forall\ \bar{v} \in D(A_0)\ . \tag{3.5}$$

Proof: The following proof is similar to that given in [Sch.1]. We first prove that $D(A_0)$ is dense in H_0. Let $\bar{v} = [v(\cdot), v_1, v_2]^T \in H_0$. Define

$$z_n(x) = \begin{cases} 0, & x \in [0,\ 1/n] \\ v(x), & x \in [1/n,\ 1 - 1/n] \\ v_1 + v_2 \cdot (x-1), & x \in [1 - 1/n,\ 1 + 1/n] \end{cases} \tag{3.6}$$

Let $\phi_\varepsilon(\cdot)$ be a positive function in C^∞ (the space of infinitely differentiable real valued functions on $(-\infty, \infty)$) such that

$$\begin{aligned} &\phi_\varepsilon(-x) = \phi_\varepsilon(x)\ , \\ &\int_{-\infty}^{\infty} \phi_\varepsilon(x)dx = 1\ , \\ &\phi_\varepsilon(x) = 0 \ \text{ for } x \notin (-\varepsilon, \varepsilon)\ . \end{aligned} \tag{3.7}$$

We define

$$u_n(x) \triangleq \int_{-\infty}^{\infty} z_n(x-y)\phi_{\frac{1}{4n}}(y)dy = \int_{-\infty}^{\infty} z_n(y)\phi_{\frac{1}{4n}}(x-y)dy\ , \qquad 0 \leq x \leq 1\ . \tag{3.8}$$

Then $u_n(\cdot) \in C^\infty([0,\ 1])$, $u_n(0) = u_n'(0) = 0$, $u_n(1) = v_1$, $u_n'(1) = v_2$ and u_n converges to v in $L^2([0,1])$. Therefore $[u_n(\cdot), v_1, v_2]^T \in D(A_0)$ and it converges to $\bar{v}$ in H_0.

Now we prove that A_0 is invertible. For any $\bar{v} = [v(\cdot), v_1, v_2]^T \in H_0$, we define

$$\begin{aligned} u(x) &= \frac{m}{EI}\int_0^x d\varepsilon_1 \int_0^{\varepsilon_1} d\varepsilon_2 \left\{ \int_1^{\varepsilon_2} d\varepsilon_3 \left[\int_1^{\varepsilon_3} v(\varepsilon_4)d\varepsilon_4 - \frac{M}{m}v_1 \right] + \frac{J}{m}v_2 \right\} \\ &= \frac{m}{EI}\int_0^x d\varepsilon_1 \int_0^{\varepsilon_1} d\varepsilon_2 \int_1^{\varepsilon_2} d\varepsilon_3 \int_1^{\varepsilon_3} v(\varepsilon_4)d\varepsilon_4 + \frac{J}{EI}\frac{v_2}{2}x^2 - \frac{M}{EI}\frac{v_1}{6}x^2(x-3)\ . \end{aligned} \tag{3.9}$$

Then $\bar{u} = [u(\cdot), u(1), u'(1)]^T \in D(A_0)$ and $A_0\bar{u} = \bar{v}$. Since A_0^{-1} is an integral operator, it is compact, and therefore bounded.

Next we prove that A_0 is self-adjoint. Consider $\bar{u} = [u(\cdot), u(1), u'(1)]^T$ and $\bar{v} = [v(\cdot), v(1), v'(1)]^T \in D(A_0)$. Then

$$\langle \overline{u}, A_0\overline{v} \rangle_{H_0} = \langle (u, u(1), u'(1))^T, (\frac{EI}{m}v^{(iv)}(x), -\frac{EI}{m}v'''(1), \frac{EI}{J}v''(1))^T \rangle_{H_0}$$
$$= \frac{EI}{m}\int_0^1 u(\tau)v^{(iv)}(\tau)d\tau - \frac{EI}{m}u(1)v'''(1) + \frac{EI}{m}u'(1)v''(1) \,. \tag{3.10}$$

Integrating by parts, we obtain

$$\langle \overline{u}, A_0\overline{v} \rangle_E = \frac{EI}{m}\int_0^1 u''(\tau)v''(\tau)d\tau = \frac{EI}{m}\langle u''(\cdot), v''(\cdot) \rangle_{L^2([0,1])} \,. \tag{3.11a}$$

Similarly, we have that

$$\langle A_0\overline{u}, \overline{v} \rangle = \frac{EI}{m}\langle u''(\cdot), v''(\cdot) \rangle_{L^2([0,1])} = \langle \overline{u}, A_0\overline{v} \rangle_E \,. \tag{3.11b}$$

Hence we have shown that A_0 is symmetric, and therefore $A_0 \subset A_0^*$. To prove that $A_0 = A_0^*$, we have to show that $D(A_0^*) = D(A_0)$. Suppose $\overline{y} \in D(A_0^*)$ and $A_0^*\overline{y} = \overline{z}$. From the definition of A_0^*, we have that

$$\langle \overline{y}, A_0\overline{u} \rangle = \langle \overline{z}, \overline{u} \rangle \quad \forall \ \overline{u} \in D(A_0) \,. \tag{3.12}$$

Since $\overline{v} \in H_0$ and $R(A_0) = H_0$, there exists $\overline{v} \in D(A_0)$ such that $A_0\overline{v} = \overline{z}$. Hence

$$\langle \overline{y}, A_0\overline{u} \rangle = \langle A_0\overline{v}, \overline{u} \rangle = \langle \overline{v}, A_0\overline{u} \rangle, \quad \forall \ \overline{u} \in D(A_0) \,. \tag{3.13}$$

Therefore $\overline{y} = \overline{v} \in D(A_0)$ because $R(A_0) = H_0$. So we have shown that A_0 is self-adjoint. (Therefore, it is closed.)

Now we prove that A_0 is coercive. Consider $\overline{v} = (v(\cdot), v(1), v'(1))^T \in D(A_0)$. From (3.11a-b), we have that

$$\langle \overline{v}, A_0\overline{v} \rangle_{H_0} = \frac{EI}{m}\|v''\|^2_{L^2([0,1])} \,. \tag{3.14}$$

Since $v(x) = \int_0^x v'(\tau)d\tau$, it follows from the Schwartz Inequality that

$$|v(x)| \le \int_0^x |v'(\tau)| \, d\tau \le \int_0^1 |v'(\tau)| \, d\tau \le \left(\int_0^1 |v'(\tau)|^2 d\tau\right)^{1/2} = \|v'\|_{L^2([0,1])} \,, \tag{3.15a}$$

which implies that

$$\|v\|_{L^2([0,1])} = \left(\int_0^1 |v(x)|^2 dx\right)^{1/2} \le \|v'\|_{L^2([0,1])} \,. \tag{3.15b}$$

Similarly,

$$|v'(x)| \le \|v''\|_{L^2([0,1])} \tag{3.16a}$$

and

$$\|v'\|_{L^2([0,1])} \le \|v''\|_{L^2([0,1])} . \tag{3.16b}$$

Note that

$$\|\overline{v}\|^2_{H_0} = \|v\|^2_{L^2([0,1])} + \frac{M}{m} v(1)^2 + \frac{J}{m} v'(1)^2 . \tag{3.17}$$

From (3.15a),(3.16a-b), we have that

$$\frac{M}{m} v(1)^2 \le \frac{M}{m} \|v''\|^2_{L^2([0,1])}, \tag{3.18a}$$

$$\frac{J}{m} v'(1)^2 \le \frac{J}{m} \|v''\|^2_{L^2([0,1])}, \tag{3.18b}$$

and

$$\begin{aligned} \|\overline{v}\|^2_{H_0} &\le \left(1 + \frac{M}{m} + \frac{J}{m}\right)\|v''\|^2_{L^2([0,1])} \\ &= \left(1 + \frac{M}{m} + \frac{J}{m}\right)\frac{m}{EI}\langle \overline{v}, A_0\overline{v}\rangle_{L^2([0,1])} = \left(\frac{m}{EI} + \frac{M}{EI} + \frac{J}{EI}\right)\langle \overline{v}, A_0\overline{v}\rangle_{L^2([0,1])} . \end{aligned} \tag{3.19}$$

and the proof is completed. ■

Remark 3.1: Referring to [Kat.1, p. 187], we find that the spectrum of A_0 is an infinitely increasing sequence of positive real eigenvalues ω_n^2, each of finite multiplicity, and that the corresponding mutually orthogonal eigenvectors η_n comprise a complete basis in H_0. The ω_n's and η_n's are, respectively, the natural frequencies and mode shapes of free, undamped oscillations. ■

We define the *energy space* $E = V \times H_0$, where $V = D(A_0^{1/2})$ is a Hilbert space with inner product

$$\langle \overline{v}_1, \overline{v}_2 \rangle_V = \langle A_0^{1/2}\overline{v}_1, A_0^{1/2}\overline{v}_2 \rangle_{H_0}, \quad \overline{v}_1, \overline{v}_2 \in V . \tag{3.20}$$

The space E has the energy inner product

$$\langle (\overline{v}_1, \overline{h}_1)^T, (\overline{v}_2, \overline{h}_2)^T \rangle_E = \langle \overline{v}_1, \overline{v}_2 \rangle_V + \langle \overline{h}_1, \overline{h}_2 \rangle_{H_0} , \quad \overline{v}_1, \overline{v}_2 \in V , \ \overline{h}_1, \overline{h}_2 \in H_0 \tag{3.21}$$

Remark 3.2: (i) The eigenvectors of A_0 are also mutually orthogonal and complete in V, and the pairs $(\eta_n, 0)$ and $(0, \eta_n)$ are thus mutually orthogonal and complete in E. (ii) $V = D(A_0^{1/2})$ is the closure of $D(A_0)$ with respect to the norm defined by (3.14). For the above example of the flexible beam, it can be easily seen from (3.14),(3.15a) and (3.16a) that $V = \{\bar{v} = (v(\cdot), v_1, v_2)^T \mid v \in H^2([0,1]),$ $v(0) = v'(0) = 0,\ v_1 = v(1),\ v_2 = v'(1)\}$ and $\|\bar{v}\|_V = \sqrt{\frac{EI}{m}}\|v''\|_{L^2([0,1])}$ [Sch.1]. ■

Let $x_p(t) = (w(t), \dot{w}(t))^T$. Then (3.3) can be rewritten in the following first order form:

$$\dot{x_p}(t) = \begin{bmatrix} 0 & 1 \\ -A_0 & -D_0 \end{bmatrix} x_p(t) + \begin{bmatrix} 0 \\ B_0 \end{bmatrix} f(t)$$
$$\triangleq A_p x_p(t) + B_p f(t) \tag{3.22a}$$

$$y(t) = (C_0,\ 0)x_p(t)$$
$$\triangleq C_p x_p(t)\ . \tag{3.22b}$$

Proposition 3.2: $C_p : E \to \mathbb{R}$ is a bounded operator.

Proof: Consider $x_p = (\bar{v}, \bar{u})^T \in E = V \times H_0$, where $\bar{v} = (v(\cdot), v_1, v_2)^T \in V$ and $\bar{u} = (u(\cdot), u_1, u_2)^T \in H_0$. Then (see (3.3b), (3.3f))

$$C_p x_p = (C_0,\ 0)\begin{bmatrix} \bar{v} \\ \bar{u} \end{bmatrix} = C_0 \bar{v} = v(x_i)\ . \tag{3.23}$$

From (3.16b), $|v(x_i)| \le \|v''\|_{L^2([0,1])}$. Note that

$$\|x_p\|_E = \|\bar{v}\|_V + \|\bar{u}\|_{H_0} = \sqrt{\frac{EI}{m}}\|v''\|_{L^2([0,1])} + \|\bar{u}\|_{H_0}\ . \tag{3.24}$$

Therefore

$$\frac{|C_p x_p|}{\|x_p\|_E} = \frac{|v(x_i)|}{\|x_p\|_E} = \frac{|v(x_i)|}{\sqrt{\frac{EI}{m}}\|v''\|_{L^2([0,1])} + \|\bar{u}\|_{H_0}}$$
$$\le \frac{\|v''\|_{L^2([0,1])}}{\sqrt{\frac{EI}{m}}\|v''\|_{L^2([0,1])} + \|\bar{u}\|_{H_0}} \le \sqrt{\frac{m}{EI}}\ . \tag{3.25}$$

Hence we have shown that C_p: $E \to \mathbb{R}$ is a bounded operator. ■

Let

$$D(A_p) = R\left\{\begin{bmatrix} -\widetilde{A_0^{-1}D_0} & -A_0^{-1} \\ I & 0 \end{bmatrix}\right\} \subset E, \tag{3.26}$$

where $\widetilde{A_0^{-1}D_0}$ is the bounded extension of $A_0^{-1}D_0$ to V. Then A_p generates the C_0-semigroup $\{e^{A_pt}\}_{t\geq 0}$ and $\|e^{A_pt}\|_E \leq 1$, $\forall\ t \geq 0$ [Gib.1]. For the above example,

$$D(A_p) = \{v_p = (\bar{v}_1,\bar{v}_2)^T \mid \bar{v}_1 = \bar{u}_0 + \bar{u}_1,\ \bar{v}_2 = -\bar{u}_0 + \bar{u}_2,\ \bar{u}_1\&\bar{u}_2 \in D(A_0),\ \bar{u}_0 \in V\ D(A_0)\}\ , \tag{3.27a}$$

and

$$A_pv_p = \begin{bmatrix} 0 & I \\ -A_0 & -D_0 \end{bmatrix}\begin{bmatrix} \bar{u}_0 + \bar{u}_1 \\ -\bar{u}_0 + \bar{u}_2 \end{bmatrix} \triangleq \begin{bmatrix} -\bar{u}_0 + \bar{u}_2 \\ -A_0\bar{u}_1 - D_0\bar{u}_2 \end{bmatrix}. \tag{3.27b}$$

Now we can find A_p^*, the adjoint of A_p. Its domain is given by [Gib.1]

$$D(A_p^*) = R\left(\begin{bmatrix} -\widetilde{A_0^{-1}D_0} & A_0^{-1} \\ -I & 0 \end{bmatrix}\right) \subset E^* = E\ . \tag{3.28}$$

Note that $E^* = E$ because E is a Hilbert space. For our example,

$$D(A_p^*) = \{v_p^* = (\bar{v}_1,\bar{v}_2)^T \mid \bar{v}_1 = -\bar{u}_0 + \bar{u}_1,\ \bar{v}_2 = -\bar{u}_0 + \bar{u}_2,\ \bar{u}_1\&\bar{u}_2 \in D(A_0),\ \bar{u}_0 \in V\ D(A_0)\}\ , \tag{3.29a}$$

and

$$A_p^*v_p^* = \begin{bmatrix} 0 & -I \\ A_0 & -D_0 \end{bmatrix}\begin{bmatrix} -\bar{u}_0 + \bar{u}_1 \\ -\bar{u}_0 + \bar{u}_2 \end{bmatrix} \triangleq \begin{bmatrix} \bar{u}_0 - \bar{u}_2 \\ A_0\bar{u}_1 - D_0\bar{u}_2 \end{bmatrix}. \tag{3.29b}$$

For our example, $D(A_p^*)$ in (3.29a) is the Z^* in (2.2a) and the Z in (2.2b) is the adjoint space of $D(A_p^*)$. Now we are ready to present the following result:

Proposition 3.3: $B_p \in Z$.

Proof: Recall that $B_p = [0\ B_0^T]^T$ and $B_0 = (\delta(x - x_i),0,0)^T$. To show $B_p \in Z$, we have to prove that $g: v \to \langle v, B_p\rangle_E$ is a linear functional from Z^* to $\mathbb{R}$. Clearly, $g(\cdot)$ is linear. Now we will prove that it is bounded. Consider any $v^* = (\bar{v}_1,\bar{v}_2)^T \in Z^*$.

$$g(v^*) = \langle v^*, B_p \rangle_E = \langle \overline{v}_1, 0 \rangle_V + \langle \overline{v}_2, B_0 \rangle_{H_0}$$
$$= \langle \overline{v}_2, B_0 \rangle_{H_0}, \tag{3.30}$$

where $\overline{v}_2 = (v_2(\cdot),\, v_2(1),\, v_2'(1))^T \in V$. Therefore

$$g(v^*) = \int_0^1 v_2(\tau)\delta(\tau - x_i)\, d\tau = v_2(x_i)\,. \tag{3.31}$$

From (3.16b),

$$|g(v^*)| = |v_2(x_i)| \le \|v_2''\|_{L^2([0,1])}\,. \tag{3.32}$$

Since Z^* is equipped with graph norm, for any $v \in Z^*$,

$$\|v^*\|_{Z^*} = \|v^*\|_E + \|A_p^* v^*\|_E$$
$$= \|(\overline{v}_1,\, \overline{v}_2)^T\|_E + \|(\overline{v}_2,\, A_0\overline{v}_1 - D_0\overline{v}_2)^T\|_E \tag{3.33}$$
$$= \|\overline{v}_1\|_V + \|\overline{v}_2\|_{H_0} + \|\overline{v}_2\|_V + \|A_0\overline{v}_1 - D_0\overline{v}_2\|_{H_0}\,,$$

where $A_0\overline{v}_1 - D_0\overline{v}_2$ is defined in the sense of (3.29b). Since $\|\overline{v}_2\|_V^2 = \dfrac{EI}{m}\|v_2''\|^2_{L^2([0,1])}$,

$$\frac{|g(v^*)|}{\|v^*\|_{Z^*}} \le \frac{\|v_2''\|_{L^2([0,1])}}{\sqrt{\dfrac{EI}{m}}\|v_2''\|_{L^2([0,1])}} = \sqrt{\frac{m}{EI}}\,. \tag{3.34}$$

Therefore $g(\cdot)$ is bounded and $B_p \in Z$. ■

Remark 3.3:

(i) It can be shown that the use of ideal point moment actuators, instead of (or in addition to) point force actuators, which result in the replacement of $\delta(x - x^i)$ by (or the addition of) $-\delta'(x - x^i)$ in (3.1a) is compatible with the assumptions stated for model (2.1).

(ii) The infinitesimal generator A_p of the flexible beam defined in (3.27a-b) generates an analytic semigroup [Hua.1]. Hence it is easy to show that a decomposition of the form (2.5) holds for our example [Gib.1]. Therefore the computational stability criterion presented in [Har.1] can be applied.

(iii) Generally speaking, the above results can be applied to systems which can be formulated in the form of (3.3) where $\overline{w}(t)$ is in a real Hilbert space H and $u(t)=(u_1(t),\, u_2(t),\, \cdots,u_{n_i}(t))^T \in \mathbb{R}^{n_i}$; A_0

satisfies the assumptions given in Proposition 3.1; C_0 is a nonnegative, symmetric linear operator with its domain containing $D(A_0)$ and there exists $\gamma \geq 0$ such that $\|C_0x\|_H \leq \gamma^2\|A_0x\|_H$,$\forall$ $x \in D(A_0)$ [Gib.1]; $B_0u(t) = \sum_{j=1}^{n_i} b^j u^j(t)$ where each b^j, $1 \leq j \leq n_i$, defines a linear functional $g^j : V \to \mathbb{R}$, where $V = D(A_0^{1/2})$, defined by $g^j(\cdot) = \langle b^j, \cdot \rangle_H$. ■

4. A NUMERICAL EXAMPLE FOR THE CONTROL SYSTEM DESIGN OF THE FLEXIBLE BEAM

In practice, the test (2.10) can only be used as a sufficient condition, because one must choose in advance the degree N_d of the polynomial $d_0(s)$. We shall now sketch out some of the numerical aspects of using the test (2.10) in the design of a stabilizing controller. First , the order n_c of the controller (2.3) must be selected and the elements of the matrices in (2.3) must be made continuously differentiable in the design parameter vector p_c. Second, the polynomials $d_0(s)$ and $n_0(s)$ must be parametrized. In [Pol.1] we find a computationally efficient parametrization for $d_0(s)$ and $n_0(s)$ which is based on the following observation. When $a,b \in \mathbb{R}$, $Z[(s+\alpha) + a)] \subset D_{-\alpha}$ if and only if $a > 0$, and $Z[(s+\alpha)^2 + a(s + \alpha) + b] \subset D_{-\alpha}$ if and only if $a > 0$, $b > 0$. Hence, when the degree of $d_0(s)$ is odd, we set $d_0(s,q_d) \triangleq ((s + \alpha) + a_0)\prod_{i=1}^{m}((s + \alpha)^2 + a_i(s + \alpha) + b_i)$, where $q_d \triangleq (a_0,a_1,a_2, \cdots ,a_m,b_1,b_2, \cdots ,b_m)^T \in \mathbb{R}^{2m+1}$ and $N_d = 2m+1$. When N_d is even, the linear term is omitted. The polynomial $n_0(s)$, which is of degree $N_n = N_d - n_s$ can be parametrized similarly, with corresponding parameter vector q_n. As a result , the system of inequalities (2.10) becomes

$$q_d^i - \varepsilon \geq 0, \quad \text{for } i = 1,2, \cdots ,N_d\,; \quad q_n^i - \varepsilon \geq 0, \quad \text{for } i = 1,2, \cdots ,N_n, \tag{4.1a}$$

$$\mathrm{Re}\left(\frac{\chi(-\alpha + j\omega,p_c)n_0(-\alpha + j\omega,q_n)}{d_0(-\alpha + j\omega,q_d)}\right) - \varepsilon \geq 0, \quad \forall\ \omega \in [0,\infty)\ , \tag{4.1b}$$

where q_d^i, q_n^i are the components of q_d, q_n, and ε is a small positive number.

We shall now describe our numerical experience in designing a fourth order compensator for a single-input single-output feedback system with the plant described by (3.1a-d), (3.2). We refer the reader to [Har.1] about the discussion of the numerical implementation of the stability criterion given in Theorem 2.2.. We assumed that $m = 2$, $cI = 0.01$, $EI = 1$, $M = 5$, $J = 0.5$, that the required stability margin $\alpha = 0.2$, and that the colocated point actuator and sensor are put at $x = 1$.

To obtain an initial compensator design and to provide a testbed for the study of truncation effects, we carried out a modal expansion of the plant dynamics to obtain the first eight modes: $-0.0023 \pm 0.6716i$, $-0.0447 \pm 2.9890i$, $-1.3718 \pm 16.5069i$, $-9.7845 \pm 43.1411i$. In the corresponding truncated state space plant model, the matrix A_p has the form $A_p = diag(A_{11}, A_{22}, A_{33}, A_{44})$, where

$$A_{11} = \begin{bmatrix} 0 & 1 \\ -0.451053 & -0.004511 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 0 & 1 \\ -8.936154 & -0.089362 \end{bmatrix},$$
$$A_{33} = \begin{bmatrix} 0 & 1 \\ -274.359603 & -2.743596 \end{bmatrix} \qquad A_{44} = \begin{bmatrix} 0 & 1 \\ -1956.894214 & -19.568942 \end{bmatrix}. \tag{4.2}$$

$B_p = (0, -0.272993, 0, -0.112681, 0, 0.073277, 0, -0.047885)^T$, $C_p =$ (-0.545986, 0, -0.225362, 0 0.146553 , 0 , -0.095770 , 0) , and $D_p = 0$. We chose to design the compensator in transfer function form: $G_c(p_c, s) = c_0(c_1 s^2 + c_2 s + 1)(c_3 s^2 + c_4 s + 1)/(d_1 s^2 + d_2 s + 1)(d_3 s^2 + d_4 s + 1)$, which results in $p_c = (c_o, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4)^T$. We set $n_0(s) = 1$ and $d_0(s, q_d) = \prod_{i=1}^{4} ((s+\alpha) + a_i(s+\alpha) + b_i)$, so that $q_d \triangleq (a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)^T$. We set $\varepsilon = 0$ in (4.1a,b).

Using pole assignment on the fourth order truncated model, we obtained the initial compensator transfer function: $G_c(p_c, s) = \frac{21044.8s^3 + 96356.5s^2 + 88286.1s + 858018}{s^4 + 2.94613s^3 + 177.301s^2 - 3333.83s - 7930.13}$, which stabilizes the truncated model. However, it fails to stabilize the truncated plant of order 6 and 8, as well as the full precision model.

Using this compensator as the starting point for our semi-infinite optimization algorithm, we obtained in two iterations of a semi-infinite minimax algorithm the following transfer function of the stabilizing compensator for our controlled flexible structure: $G_c(p_c, s) = \frac{-12.5806s^4 + 20658.8s^3 + 94255.7s^2 + 87402.1s + 841483}{s^4 + 2.12762s^3 + 171.79s^2 - 3262.91s - 7774.42}$. The critical frequency interval for the evaluation of $\chi(p_c, s)$ was [0.1 , 200] and the number of sampling points used was 50; 500 points were used to produce the plots in Figures 3 and 4. The plot corresponding to (4.1b) for the initial value of the compensator is shown in Fig. 3 and for the final value in Fig. 4.

It is interesting to observe that the closed-loop system poles which result from the use of this stabilizing compensator and the truncated plant of order 4 are $0.695414 \pm i9.82352$, $-1.4397 \pm i7.04732$,

$-0.128045 \pm i4.91775$, $-0.238414 \pm i2.99904$. As we can see, there are two unstable poles. However, when the plant model is truncated to order 6 and 8, respectively, the closed-loop system is stable and has poles at $-0.521081 \pm i16.3213$, $-1.02523 \pm i9.92591$, $-0.459227 \pm i7.0698$, $-0.23843 \pm i4.9936$, $-0.238574 \pm i2.99953$; and $-9.75924 \pm i43.1321$, $-0.51818 \pm i16.3271$, $-1.09369 \pm i9.94782$, $-0.411156 \pm i7.04619$, $-0.246175 \pm i4.99733$, $-0.238581 \pm i2.99956$, respectively.

5. CONCLUSION

We have shown that a necessary and sufficient computational stability criterion presented in [Har.1] can be used in the design of stabilizing controllers for flexible structures with point actuators and sensors. The stability criterion is suitable for optimization-based computer-aided-design by using semi-infinite optimization algorithms. Our approach can avoid common "spill-over" effects which result from modal truncation of partial differential equation models. There remains a certain amount of numerical analysis type work to be done in developing efficient techniques for the repeated evaluation of frequency responses of distributed parameter systems, and for the computation of their unstable poles. Furthermore, because of local minima effects, we remind the reader that the successful use of our stability criterion in conjunction with semi-infinite optimization algorithms may be predicated on a good initial design of a stabilizing controller.

6. ACKNOWLEDGEMENT

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7. REFERENCES

[Bal.1] M. J. Balas, "Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams", *IEEE Trans. on Automatic Control* Vol. AC-27, No. 3, pp. 522-535, 1982.

[Cur.1] R. F. Curtain and D. Salamon, "Finite Dimensional Compensators for Infinite Dimensional Systems with Unbounded Input Operators," *SIAM J. Control and Optimization*, Vol. 24, No. 4, pp. 797-816, July 1986.

[Gib.1] J. S. Gibson, "An analysis of optimal modal regulation: Convergence and stability," *SIAM J. Control and Optimization*, Vol. 19, No. 5, pp. 686-707, Sept. 1981.

[Har.1] Y. P. Harn and E. Polak, "On the Design of Finite Dimensional Stabilizing Compensators for Infinite Dimensional Feedback-Systems Via Semi-Infinite Optimization," *IEEE Trans. Automat. Contr*, in press.

[Hua.1] F. Huang, "On the Mathematical Model for Linear Elastic Systems with Analytic Damping," SIAM J. Control and Optimization, Vol. 26, No. 3, pp. 714-724, May 1988.

[Jac.1] C. A. Jacobson and C. N. Nett , "Linear state-space systems in infinite-dimensional space: The role and characterization of joint stabilizability/detectability," *IEEE Trans. Automat. Contr.*, Vol. 33, No. 6, pp. 541-549, June 1988.

[Kat.1] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.

[Nef.1] S. A. Nefedov and F. A. Sholokhovich, "A criterion for the stabilizability of dynamical systems with finite-dimensional input, " *Differential Equations*, pp. 163-166, 1986.

[Paz.1] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

[Pol.1] E. Polak and T. L. Wuu, "On the Design of Stabilizing Compensators Via Semi-infinite Optimization", *IEEE Trans. Automat. Contr*, Vol. 34, No. 2, pp. 196,200, Feb. 1989.

[Sal.1] D. Salamon, *Control and Observation of Neutral Systems*, Pitman Advanced Publishing Program, 1984.

[Sch.1] R. A. Schulz, "Control of a Large Space Structure Using a Distributed Parameter Model," Ph.D. Thesis, UCLA, 1986.

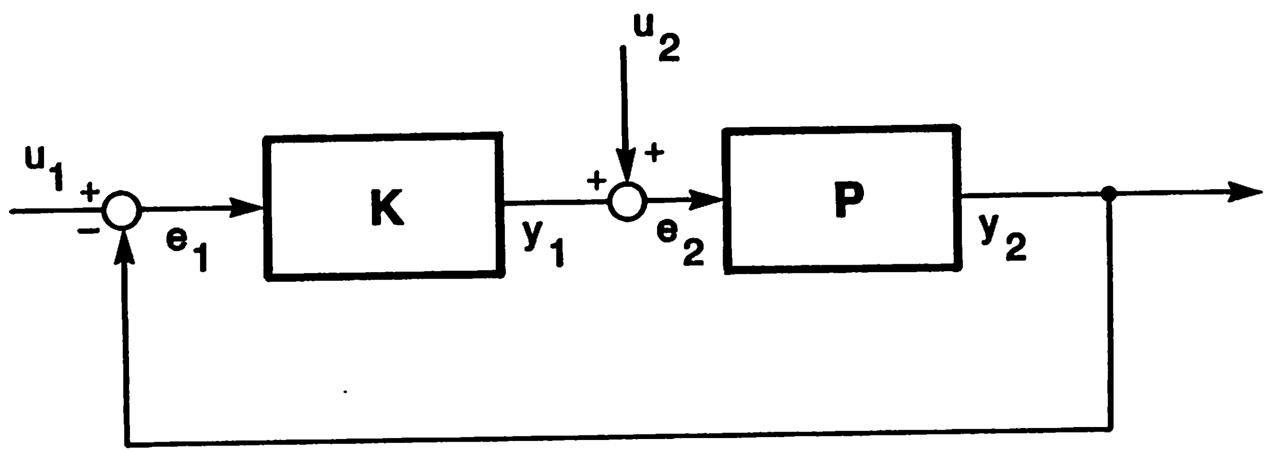

Figure 1: The feedback system S(P,K).

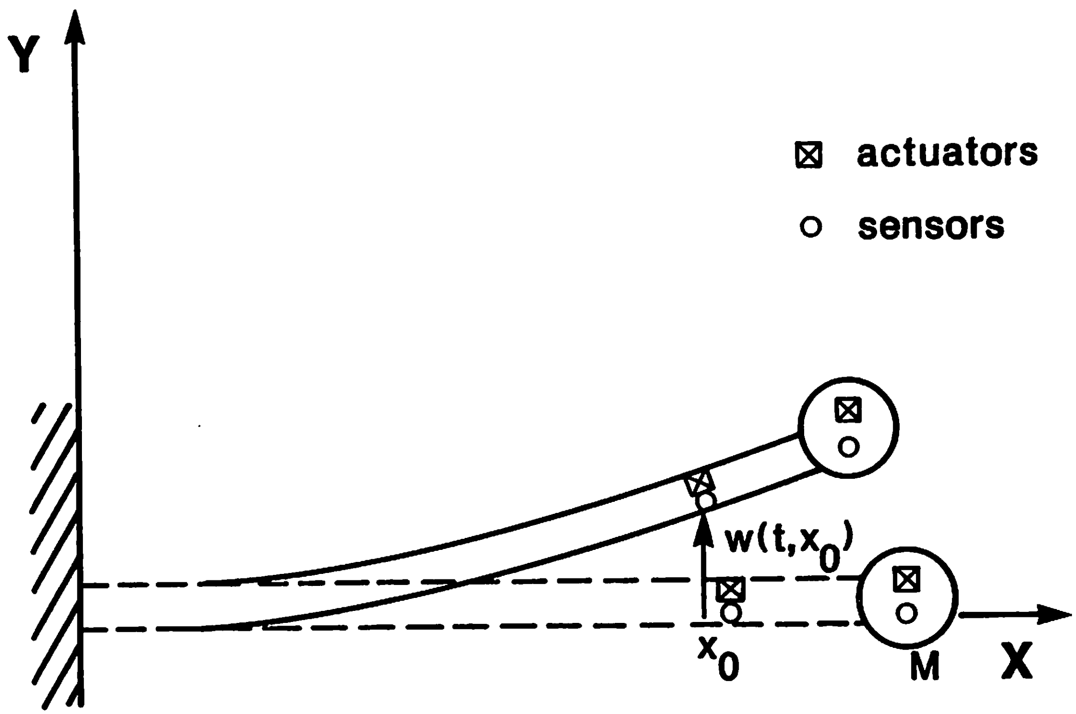

Figure 2: Planar bending motion of a flexible beam.

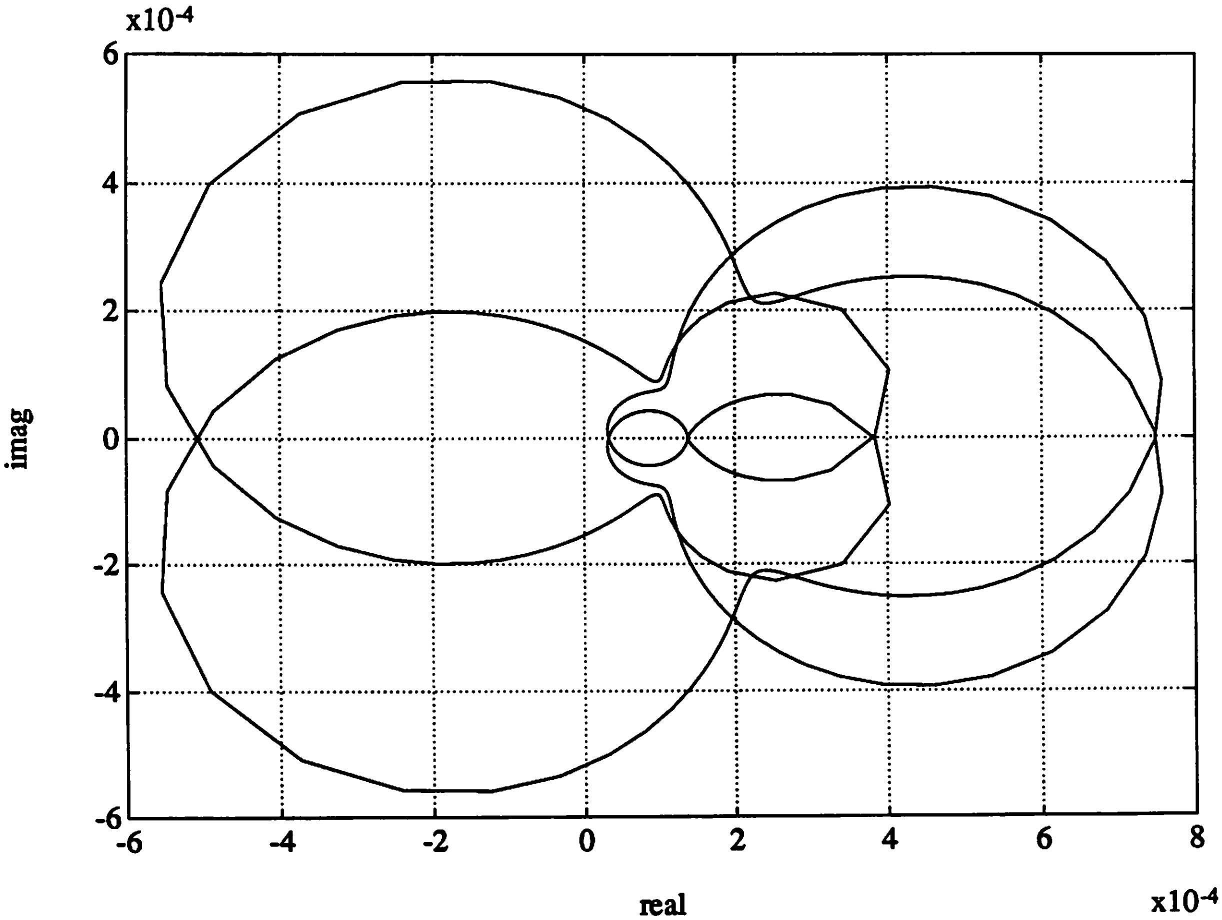

Figure 3: Modified Nyquist diagram (initial design).

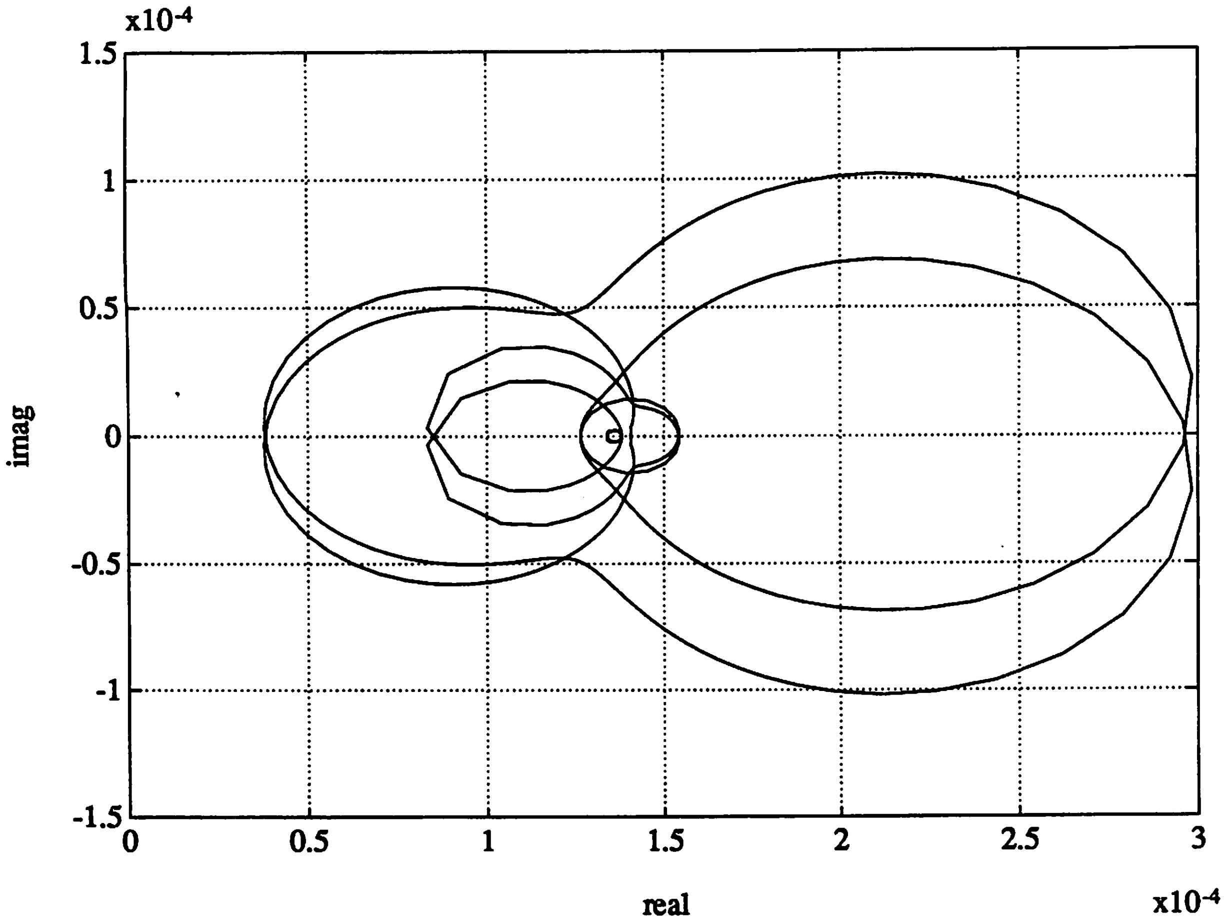

Figure 4: Modified Nyquist diagram for the stabilized system.