KHARITONOV'S THEOREM AND A GRAPHICAL STABILITY TEST FOR LINEAR TIME-INVARIANT SYSTEMS

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0. SUMMARY

This paper presents a simple rigorous development of two major results in robust stability theory: Kharitonov's Theorem for interval polynomials (a previously known result); and a graphical U-stability Nyquist type test for a broad class of parameterized linear time-invariant systems. An example is included for a time delay system with non-linear parameter dependence.

1. INTRODUCTION

This paper presents a simple rigorous development of two major results in robust stability theory. Section 2 presents simplified analytical methods for proving Kharitonov's stability theorem and several extensions, all previously known results. Section 3 presents a generalization of these analytical methods to develop a graphical technique--related to the classical Nyquist criterion--for a much broader class of robust stability problems.

Kharitonov's Theorem states that an *interval class* of polynomials, defined by letting each coefficient vary *independently* in arbitrarily defined intervals, is Hurwitz if and only if *four* special, well-defined polynomials in the class are Hurwitz. The original proof of this theorem was rather complex, but a series of simplifications in the exposition have led up to the simple proof included here; we refer to [Barm. 1], [Yeu. 1], [Das. 1] and [Min. 1] in particular, with additional references included in these. Indeed, the cornerstone of our exposition is the observation that the *image* of the interval class of polynomials, evaluated at any point on the *imaginary axis*, is a level rectangle in $\mathbb{C}$, with corners specified by the four Kharitonov polynomials (Lemma 2.2); this observation is due to Dasgupta [Das. 1]. We consider the motion of this rectangle, using very elementary facts about Hurwitz polynomials, to prove the result without reference to the Hermite-Biehler Theorem.

In Section 3 we apply the analytical methods developed in Section 2 to a much broader class of problems. Instead of the closed right half-plane, we consider arbitrary closed sets U of forbidden zero locations; and instead of an interval class of polynomials--a parallelepiped with edges parallel to the coordinate axes in coefficient space--we consider more general sets: arbitrary polyhedra, convex sets, compact connected sets and even completely arbitrary sets. The resulting test is a graphical technique--first proposed in [Ana. 1]--based on a result sometimes referred to as the 'zero-exclusion principle,' an old idea discernible in Bode's notion of gain and phase margin. Section 3 includes a proof of this result appropriate for the broad class of problems under consideration. Of course, the polyhedral case, with its linear constraints, will have the most numerically efficient solutions. Finally, instead of considering only polynomial functions, we consider a broader class of functions on $\mathbb{C}$; specifically, functions analytic in the

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region U. This allows us to analyze systems with a variety of infinite-dimensional components: PDE's and time delays being the most important. An example demonstrating the latter is included in Section 3.4.

Related work has been developed by Barmish [Barm. 2,3]; similar graphical tests are proposed, although not exactly of the 'Nyquist' variety used here. The Nyquist type tests, actually plotting the 'nearest point' in the image set, seem to be the most efficient, and the resulting graphical information can be utilized with a traditional Nyquist plot interpretation for the whole family of systems. For the polyhedral case mentioned above (either polynomials or analytic functions), an alternative approach to the Nyquist type tests described here is the root-locus type tests of Bartlett, Hollot and Lin ([Bart. 1], the 'Edge Theorem'). There are even finite algorithms for implementing the Edge Theorem (see [Fu 1] for the special case where U is the right half-plane; [Kra. 1] for the unit circle with low order polynomials; and [Ana. 2] for arbitrary closed U with parameterizable boundary). Even considering these finite implementations, we feel the Nyquist type algorithms will, in general, prove most efficient (see our discussion in [Ana. 2]).

2. STREAMLINED PROOF OF KHARITONOV'S THEOREM

2.1 Statement of the Theorem

We consider a family of *real* polynomials of degree n

$$p(s,a)=\sum_{0}^{n} a_k s^k \qquad 0 < \underline{a}_k \le a_k \le \bar{a}_k \quad k=0,1,\ldots,n \tag{2.1}$$

where the real numbers $\underline{a}_k$ and $\bar{a}_k$ are given and $a:=(a_0, a_1, \ldots, a_n)$. Define

$$A:=\{a\in\mathbb{R}^{n+1} : \underline{a}_k \le a_k \le \bar{a}_k, \quad k=0,1,\ldots,n\}. \tag{2.2}$$

A is a parallelepiped in $\mathbb{R}^{n+1}$ with 2^{n+1} vertices. There is an obvious bijection between the points of A and the polynomials $p(\cdot,a)$, so we consider A as a *family of polynomials*. Using standard set notation we write, for any fixed $s\in\mathbb{C}$,

$$p(s,A)=\{p(s,a) \mid a\in A\} \tag{2.3}$$

Recall that a *polynomial* $q(\cdot)$ (with either real or complex coefficients) is called *Hurwitz* iff all its zeros have *negative* real parts. We say that the *family* A is *Hurwitz* iff all members of A are Hurwitz. We define the four Kharitonov polynomials with respect to A as follows:

$$k_{11}(s)=\underline{a}_0+\underline{a}_1 s+\bar{a}_2 s^2+\bar{a}_3 s^3+\underline{a}_4 s^4+\underline{a}_5 s^5+\ldots \tag{2.4}$$

$$k_{12}(s)=\underline{a}_0+\bar{a}_1 s+\bar{a}_2 s^2+\underline{a}_3 s^3+\underline{a}_4 s^4+\bar{a}_5 s^5+\ldots \tag{2.5}$$

$$k_{21}(s)=\bar{a}_0+\underline{a}_1 s+\underline{a}_2 s^2+\bar{a}_3 s^3+\bar{a}_4 s^4+\underline{a}_5 s^5+\ldots \tag{2.6}$$

$$k_{22}(s)=\bar{a}_0+\bar{a}_1 s+\underline{a}_2 s^2+\underline{a}_3 s^3+\bar{a}_4 s^4+\bar{a}_5 s^5+\ldots \tag{2.7}$$

Theorem 2.1 *Kharitonov's Theorem (Real Coefficients)* [Kha. 1]. Consider A, the family of polynomials defined in (2.1) and (2.2). The family A is Hurwitz if and only if the four Kharitonov polynomials $k_{11}(\cdot)$, $k_{12}(\cdot)$, $k_{21}(\cdot)$, $k_{22}(\cdot)$ are Hurwitz.

2.2 Two lemmas

We start with a useful characterization of Hurwitz polynomials. The following analytical tools have been available to the engineering community for a long time; the earliest reference we have found to similar concepts is [Whi. 1, Sec. 6.31, Ex. 2 (1902)]. They are proven here for completeness.

Lemma 2.2 *Characterization of Hurwitz polynomials.* Let

$$q(s) = \sum_{0}^{n} q_k s^k \quad \text{with } q_k \in \mathbb{R}, \forall k, \quad \text{and } q_n > 0. \tag{2.8}$$

i) The polynomial q is Hurwitz if and only if

a) $\arg q(j\omega)$ is well defined $\forall \omega \in \mathbb{R}$, and (2.9)

b) $$\lim_{\omega\to\infty} \arg[q(j\omega)] - \arg q(0) = \frac{n\pi}{2}. \tag{2.10}$$

ii) If q is Hurwitz then the map $\omega \to \arg q(j\omega)$ is strictly increasing on $\mathbb{R}$.

iii) If q is Hurwitz with $q_n > 0$, then all of its coefficients are positive.

Comment. Geometrically, statements i) and iii) imply that an nth degree real polynomial q with positive coefficients is Hurwitz if and only if the curve traced in $\mathbb{C}$ by $q(j\omega)$, as ω increases from 0 to ∞, starts on the *positive* real axis and rotates around the origin *counterclockwise* by a net change of angle of $n\pi/2$. Statement ii) implies that this rotation is *monotonic.*

Remark. If the coefficients of $q(\cdot)$ in (2.8) are complex with $|q_n| \neq 0$ then statements i) and ii) of Lemma 2.2 remain valid provided (2.10) is replaced by

$$\lim_{\omega\to\infty} [\arg q(+j\omega) - \arg q(-j\omega)] = n\pi. \tag{2.10a}$$

In the proof below we would add $\arg(q_n)$ to the RHS of (2.11) for the complex coefficient case.

Proof. i) $\Rightarrow$. Statement (2.9) holds since q has no zeros on the $j\omega$-axis. Now since $q_n > 0$, denoting the zeros as $z_i = \sigma_i + j\omega_i$, we have

$$\arg[q(j\omega)] = \sum_{1}^{n} \arg(j\omega - z_i). \tag{2.11}$$

Since $\forall i$, $\sigma_i < 0$, for each *real* zero, $\arg(j\omega - z_i)$ increases by $\pi/2$ as ω increases from 0 to ∞; for each pair of complex conjugate zeros, $\arg(j\omega - z_i) + \arg(j\omega - \bar{z}_i)$ increases by π. Hence (2.10) follows.

$\Leftarrow$. Assumption (2.9) implies that q has no $j\omega$-axis zeros. To prove that q is Hurwitz use contraposition. Suppose q has a real zero in the open right-half plane, say $z_1 > 0$. Then $\arg(j\omega - z_1)$ *decreases* by $\pi/2$ as ω increases from 0 to ∞. Since q has *precisely n* zeros, the equality in (2.10) cannot be satisfied. Hence (2.10) rules out open right half plane zeros of q.

ii) If q is Hurwitz then its n zeros are in the open left half plane: hence $\operatorname{Re} z_i < 0$, $\forall i$. Now $\arg(j\omega - z_i)$ is a strictly increasing function of ω for all $\omega \in \mathbb{R}$, hence by (2.11) the same holds for $\arg q(j\omega)$.

iii) Since q is a *real* Hurwitz polynomial, its zeros are real or occur in complex conjugate pairs: so $q(s)$ is a product of q_n, monomials $(s - \sigma_i)$ with $\sigma_i < 0$ and binomials $s^2 - 2\sigma_k s + (\sigma_k^2 + \omega_k^2)$ with $\sigma_k < 0$. Since each factor has positive coefficients, their product, $q(\cdot)$, will have positive coefficients. □

We now prove a key property due to Dasgupta.

Lemma 2.3 [Das. 1]. For each fixed $\omega \in \mathbb{R}_+$, $p(j\omega, A)$ is a *rectangle* with edges parallel to the coordinate axes and with vertices determined by the four Kharitonov polynomials $k_{11}(j\omega)$, $k_{12}(j\omega)$, $k_{22}(j\omega)$ and $k_{21}(j\omega)$.

Comment. By (2.1), for any fixed $\omega \in \mathbb{R}$, the map $a \to p(j\omega, a)$ is a *linear* map from $A \subset \mathbb{R}^{n+1}$ into $\mathbb{C}$: since A is a parallelepiped in $\mathbb{R}^{n+1}$, $p(j\omega, A)$ is a *convex polygon* in $\mathbb{C}$. The thrust of Lemma 2.3 is that this polygon is the rectangle with corners $k_{11}(j\omega)$, $k_{12}(j\omega)$, $k_{21}(j\omega)$, and $k_{22}(j\omega)$. (Fig. 2.1)

Proof. From (2.1) we have, $\forall \omega \in \mathbb{R}$,

$$\underline{a}_0 - \bar{a}_2\omega^2 + \underline{a}_4\omega^4 - \ldots \quad \leq \ \mathrm{Re}\{p(j\omega, a)\} \ \leq \ \bar{a}_0 - \underline{a}_2\omega^2 + \bar{a}_4\omega^4 - \ldots \tag{2.12}$$

and, $\forall \omega \in \mathbb{R}_+$,

$$\omega(\underline{a}_1 - \bar{a}_3\omega^2 + \underline{a}_5\omega^4 - \ldots) \ \leq \ \mathrm{Im}\{p(j\omega, a)\} \ \leq \ \omega(\bar{a}_1 - \underline{a}_3\omega^2 + \bar{a}_5\omega^4 - \ldots). \tag{2.13}$$

Let $g_1(j\omega)$ and $g_2(j\omega)$ be the LHS and RHS of (2.12), and $h_1(j\omega)$ and $h_2(j\omega)$ be j times the LHS and RHS of (2.13). Note that the four polynomials $g_1(s)$, $g_2(s)$, $h_1(s)$ and $h_2(s)$ have real coefficients and that the four Kharitonov polynomials, defined in (2.4)–(2.7), are given by

$$k_{lm}(s) = g_l(s) + h_m(s) \qquad l, m = 1, 2. \tag{2.14}$$

By (2.12) and (2.13), we see that $p(j\omega, A)$ is the convex hull of the images of the four Kharitonov polynomials:

$$p(j\omega, A) = \mathrm{co}\{k_{11}(j\omega), k_{21}(j\omega), k_{22}(j\omega), k_{12}(j\omega)\} \tag{2.15}$$

Examination of (2.12-2.15) shows that the sides of $p(j\omega, A)$ are parallel to the coordinate axes. □

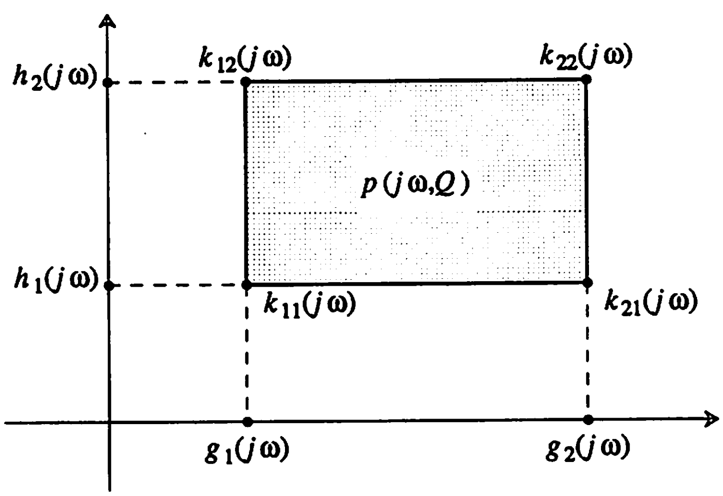

Figure 2.1:
Rectangular image $p(j\omega, Q)$ (for $\omega > 0$).

2.3 Proof of Kharitonov's Theorem

Proof. $\Rightarrow$. If all polynomials of the family A are Hurwitz, then, in particular, k_{11}, k_{12}, k_{21} and k_{22} are Hurwitz.

$\Leftarrow$. By assumption k_{11}, k_{12}, k_{21} and k_{22} are Hurwitz polynomials.

Step 1. Consider the motion of the rectangle $p(j\omega, A)$ as ω increases from 0 to ∞. Clearly, $p(0, A) = [\underline{a}_0, \bar{a}_0]$, a segment of the *positive* real axis since $\underline{a}_0 > 0$. Since the $k_{lm}(j\omega)$ are polynomials in ω, the rectangle $p(j\omega, A)$ will move continuously in $\mathbb{C}$ as ω increases, keeping at all times its sides parallel to the coordinate axes (Lemma 2.3).

Now k_{21} is Hurwitz and $k_{21}(j\omega)$ is the lower right hand corner of $p(j\omega, A)$; since arg $k_{21}(j\omega)$ is strictly increasing, $k_{21}(j\omega)$ will push $p(j\omega, A)$ off the real axis into the open first quadrant and then, if $n > 1$, will push the *whole rectangle* $p(j\omega, A)$ into the open second quadrant. Now, with $p(j\omega, A)$ in the (open) second quadrant, consider its upper right hand corner $k_{22}(j\omega)$: since arg $k_{22}(j\omega)$ is strictly increasing, if $n > 2$, $k_{22}(j\omega)$ will push the whole rectangle $p(j\omega, A)$ into the open third quadrant. Now consider the upper left hand corner $k_{12}(j\omega)$: since arg $k_{12}(j\omega)$ is strictly increasing, if $n > 3$, $k_{12}(j\omega)$ will push the whole rectangle into the open fourth quadrant. Now consider the lower left hand corner of $p(j\omega, A)$, namely $k_{11}(j\omega)$: since arg $k_{11}(j\omega)$ is strictly increasing, if $n > 4$, $k_{11}(j\omega)$ will push the whole rectangle into the open first quadrant. At this point, we consider a second time $k_{21}(j\omega)$, the lower right hand corner of the rectangle: if $n > 5$, $k_{21}(j\omega)$ pushes the rectangle from the open first quadrant into the open second quadrant, as before. This motion--the whole rectangle moving continuously from one open quadrant into the next, never passing through the origin--will go on until the net argument of each corner asymptotically approaches $n\pi/2$, since each corner polynomial is Hurwitz of degree n and (2.10) holds. Thus the whole rectangle $p(j\omega, A)$ starts from the positive real axis and travels continuously through a net angle of $n\pi/2$ as ω increase from zero to infinity.

Step 2. Let $p(\cdot, a)$ be an arbitrary polynomial in A. By (2.12) and (2.13), $\forall \omega \in \mathbb{R}_+$,

$$p(j\omega, a) \in p(j\omega, A) \tag{2.16}$$

By step 1, $\forall \omega \in \mathbb{R}_+$, $p(j\omega, a) \neq 0$, so its argument is a well defined continuous function of ω; furthermore, by step 1 and condition (2.16), condition (2.10) of Lemma 2.2 holds. Hence, by Lemma 2.2, $p(\cdot, a)$ is Hurwitz. □

2.4 Special Cases

Corollary 2.4 ($n = 3, 4$ and 5) [And. 1]. Let the notation of (2.1), (2.2) and (2.4)–(2.7) hold.

For $n = 3$, the family A is Hurwitz if and only if k_{21} is Hurwitz.

For $n = 4$, the family A is Hurwitz if and only if k_{21} and k_{22} are Hurwitz.

For $n = 5$, the family A is Hurwitz if and only if k_{21}, k_{22} and k_{12} are Hurwitz.

Proof. We use the ideas of the proof of Sec. 2.3 to prove sufficiency. For $n = 3$, the Hurwitzness of k_{21} will move the rectangle $p(j\omega, A)$ from the real axis into the open first quadrant and then into the open second quadrant. As ω increases further, arg $k_{21}(j\omega)$ will increase to become asymptotic to $3\pi/2$. Thus the lower horizontal edge of $p(j\omega, A)$ will enter and remain in the open third quadrant. The top edge will do the same because, for $n = 3$,

$$k_{22}(j\omega) = -j\omega^3 \underline{a}_3 - \underline{a}_2\omega^2 + \bar{a}_1 j\omega + \bar{a}_0 \tag{2.17}$$

hence, for ω large, $\mathrm{Re}[k_{22}(j\omega)]$ and $\mathrm{Im}[k_{22}(j\omega)]$ will eventually become and remain *negative* with arg $k_{22}(j\omega) \to 3\pi/2$. A similar calculation applied to $k_{12}(j\omega)$ leads to the same conclusion. Thus we see that $p(j\omega, A)$ will travel counterclockwise around the origin through a total angle of $3\pi/2$, never intersecting the origin. By step 2 of the proof above, $\forall a \in A$, $p(j\omega, a)$ will do the same, hence the family A is Hurwitz.

The proof for $n = 4$ and 5 follows from a similar argument. □

2.5 Complex Coefficients Case

Let A^* be the family of polynomials defined as follows:

$$p(s,a) = (\alpha_n + j\beta_n)s^n + (\alpha_{n-1} + j\beta_{n-1})s^{n-1} + \ldots + (\alpha_0 + j\beta_0) \qquad (2.18)$$

where $a_k = \alpha_k + \beta_k$, and $\underline{\alpha}_k \le \alpha_k \le \bar{\alpha}_k$ and $\underline{\beta}_k \le \beta_k \le \bar{\beta}_k$, for $k = 0, 1, \ldots, n$; the $\underline{\alpha}_k$'s, $\bar{\alpha}_k$'s, $\underline{\beta}_k$'s and $\bar{\beta}_k$'s are given, and the rectangle $[\underline{\alpha}_n, \bar{\alpha}_n] \times [\underline{\beta}_n \times \bar{\beta}_n]$ is bounded away from zero. Again we visualize A^* as a parallelepiped in $\mathbb{R}^{2n+2}$. Since, $\forall s \in \mathbb{C}$, $a \rightarrow p(s,a)$ is a linear map, it maps the convex set A^* into a convex polygon in $\mathbb{C}$.

Since the arguments for the complex case are entirely analogous to those of the real case, we sketch them briefly. There are two important differences: first we must consider two cases $\omega \ge 0$ and $\omega \le 0$; second, $p(0, A^*)$ is now a rectangle in $\mathbb{C}$, namely $[\underline{\alpha}_0, \bar{\alpha}_0] \times [\underline{\beta}_0, \bar{\beta}_0]$.

As in Lemma 2.3, we observe that $\forall a \in A^*$, $\forall \omega \ge 0$

$$g_1^+(j\omega) \le \mathrm{Re}[p(j\omega, a)] \le g_2^+(j\omega) \qquad (2.19)$$

and

$$\mathrm{Im}[h_1^+(j\omega)] \le \mathrm{Im}[p(j\omega, a)] \le \mathrm{Im}[h_2^+(j\omega)] \qquad (2.20)$$

where

$$g_1^+(s) := \underline{\alpha}_0 + j\bar{\beta}_1 s + \bar{\alpha}_2 s^2 + j\underline{\beta}_3 s^3 + \underline{\alpha}_4 s^4 + \cdots \qquad (2.21)$$

$$g_2^+(s) := \bar{\alpha}_0 + j\underline{\beta}_1 s + \underline{\alpha}_2 s^2 + j\bar{\beta}_3 s^3 + \bar{\alpha}_4 s^4 + \cdots \qquad (2.22)$$

$$h_1^+(s) := j\underline{\beta}_0 + \underline{\alpha}_1 s + j\bar{\beta}_2 s^2 + \bar{\alpha}_3 s^3 + j\underline{\beta}_4 s^4 + \cdots \qquad (2.23)$$

$$h_2^+(s) := j\bar{\beta}_0 + \bar{\alpha}_1 s + j\underline{\beta}_2 s^2 + \underline{\alpha}_3 s^3 + j\bar{\beta}_4 s^4 + \cdots \qquad (2.24)$$

Clearly the four complex polynomials $k_{lm}^+ := g_l^+ + h_m^+$, $l, m = 1, 2$, are in the family A^*. By (2.19) and (2.20) we see that $\forall \omega \ge 0$

$$p(j\omega, A^*) = \mathrm{co}\{k_{11}^+(j\omega), k_{21}^+(j\omega), k_{22}^+(j\omega), k_{12}^+(j\omega)\}. \qquad (2.25)$$

Again, $\forall \omega \ge 0$, $p(j\omega, A^*)$ is a rectangle with edges parallel to the coordinate axes.

For $\omega \le 0$, $\forall a \in A^*$, we obtain again Eqns. (2.19)–(2.24) with g_1^+, g_2^+, h_1^+ and h_2^+ replaced by g_1^-, g_2^-, h_1^-, h_2^-, where g_1^-, g_2^-, h_1^-, h_2^- are defined as in (2.21)–(2.24) except that all the odd power coefficients are interchanged ($\bar{\beta}_1$ is replaced by $\underline{\beta}_1$, etc.). With $k_{lm}^- := g_l^- + h_m^-$, $l, m = 1, 2$ we have $\forall \omega \le 0$, $\forall a \in A^*$

$$p(j\omega, A^*) = \mathrm{co}\{k_{11}^-(j\omega), k_{12}^-(j\omega), k_{21}^-(j\omega), k_{22}^-(j\omega)\}. \qquad (2.26)$$

Thus we have the generalization of Lemma 2.3 for the complex case. Using Lemma 2.2 and Eqns. (2.10a), (2.25) and (2.26), we easily prove the following using arguments analogous to those used for Theorem 2.1 (for the complete proof, see [Min. 1]).

Theorem 2.5 *Kharitonov's Theorem (Complex Coefficients)* [Kha. 2, Bos. 1]. Let A^* be the family of *complex* polynomials defined in (2.18). The family A^* is Hurwitz if and only if the eight complex Kharitonov polynomials $k_{11}^+, k_{11}^-, k_{12}^+, k_{12}^-, k_{21}^+, k_{21}^-, k_{22}^+, k_{22}^-$ are Hurwitz.

3. ROBUST STABILITY FOR LINEAR TIME-INVARIANT SYSTEMS

We propose to generalize the results of Section 2 in two directions: allow for a more flexible definition of stability and allow for less restrictive parameter dependence of the characteristic polynomial coefficients. In Section 2, each of the $n+1$ coefficients of $p(s,a)$ were allowed to vary *independently* inside prescribed intervals. Examples of control systems and circuits show that physical parameters—such as mass, inertia tensor, spring constants, damping factors, resistances, transconductances, inductances, etc.—appear as variables in polynomials that specify the coefficients; usually a parameter appears as a variable in several coefficients, thus the coefficients are no longer independent. (See example in Section 3.4.)

3.1 U-stability

Roughly speaking, we let U be the *closed* set containing all those values of s that are viewed as "undesirable" from a stability point of view. Typically, U includes the closed right-half plane, is symmetric with respect to the real axis, and has a boundary, ∂U, consisting of C^1 curves (hence parameterizable curves). A simple example is

$$U = R_{\sigma_0} := \{s \in \mathbb{C} \mid \mathrm{Re}\{s\} \geq \sigma_0\} \quad \text{for some fixed } \sigma_0 < 0. \tag{3.1}$$

We consider a family of polynomial functions of $s \in \mathbb{C}$ parameterized by $q \in Q \subset \mathbb{R}^m$, $p(s,q)$. Using standard set and functional notation, this family of polynomials is denoted $p(\cdot,Q)$. Note that $p(\cdot,q)$ denotes a particular polynomial in $p(\cdot,Q)$ if $q \in Q$, and $p(s,q) \in \mathbb{C}$ denotes the *value* of that polynomial evaluated as a point $s \in \mathbb{C}$. Finally, $p(s,Q)$ denotes the set of values of all the polynomials $p(\cdot,Q)$ evaluated at s; e.g. $p(s,Q) = \{p(s,q) : q \in Q\}$. A polynomial $p(\cdot,q)$ is said to be *U–stable* iff $p(s,q) \neq 0$ $\forall s \in U$. A family of polynomials is *U–stable* iff each polynomial $p(\cdot,q)$ is U-stable; i.e. iff $0 \notin p(s,Q)$ $\forall s \in U$.

3.2 Lumped linear time-invariant systems

We consider lumped systems with characteristic polynomials of the form

$$p(s,q) = \sum_{0}^{n} a_k(q)\, s^k, \quad q \in Q \subset \mathbb{R}^m \tag{3.2}$$

where the parameterization $p(\cdot,q)$ has been expressed as a parameterization of the polynomial coefficients.

Assumptions:

(A1) The parameter vector $q \in \mathbb{R}^m$ is in a given non-empty set Q, and Q is *connected* and *compact*.

(A2) For $k = 0, 1, \ldots, n$, the coefficient parameterizations $a_k : Q \to \mathbb{R}$ are *continuous*.

(A3) For all $q \in Q$, $a_n(q) > 0$.

For example, Q could be a convex polyhedron in $\mathbb{R}^m$ or a closed ball. (A1) and (A2) imply that $a_k(Q)$ is a compact interval, $k=0, \ldots, n$; then (A3) implies that the interval $a_n(Q)$ is *bounded away* from 0.

We begin the derivation of the main result of this section (Theorem 3.3) by stating two facts. Fact 3.1 is a corollary of [Die. 1, Thm. 9.17.4]; it is a rigorous version of the statement that the zeroes of a polynomial vary *continuously* with respect to its coefficients.[1] Fact 3.2 states that the zeroes of the set of

[1] We mention an alternative--more formal--approach to characterizing 'continuity' of polynomial zeroes with respect to coefficients. Consider the set of zeroes of a polynomial as an element of the *finite power set* of $\mathbb{C}$, $P_f(\mathbb{C})$, the set of all nonempty finite subsets of $\mathbb{C}$, equipped with the Hausdorff metric [Mun. 1, Ex. 7, p. 279]. In this metric, two subsets of $\mathbb{C}$ are ε-close if and only if every element of either set is ε-close to some

polynomials defined above can be uniformly bounded.

Fact 3.1 Given any *bounded* set $V \subset \mathbb{C}$ and any $q_0 \in Q$, if $p(s, q_0) \neq 0$, $\forall s \in \partial V$, then the number of zeros of $p(s, q)$ in V, *counting multiplicities*, is a locally constant function of $q \in Q$ at q_0.

Fact 3.2 Under assumptions (A1)-(A3), there is an $\alpha \in (0, \infty)$ such that for all $q \in Q$, the zeros of $p(s, q)$ belong to the disc $D(0, \alpha) \subset \mathbb{C}$, centered on 0 with radius α.

Proof. Define $\underline{a}_k$, $\bar{a}_k$ by $a_k(Q) = [\underline{a}_k, \bar{a}_k]$. Given any $q \in Q$, suppose z is a zero of $p(\cdot, q)$; i.e. $p(z, q) = 0$. Of course, if $z = 0$ then $z \in D(0, \alpha)$; so we may assume $z \neq 0$. Then

$$\underline{a}_n |z| \;\le\; |a_n(q) z| \;\le\; \Big| \sum_0^{n-1} a_k(q) z^{k+1-n} \Big| \;\le\; \sum_0^{n-1} \bar{a}_k \, |z|^{k+1-n} . \tag{3.3}$$

If $|z| \ge 1$, we have

$$|z| \;\le\; \underline{a}_n^{-1} \sum_0^{n-1} \bar{a}_k \;=:\; \alpha_1 \tag{3.4}$$

so we choose $\alpha = \max\{1, \alpha_1\}$. □

Theorem 3.3 *U-stability of p(s,Q).* Let the set of polynomials $p(\cdot, Q)$ as defined in (3.2) satisfy assumptions (A1)-(A3), and recall that $U \subset \mathbb{C}$ is closed. Then

1) the set of polynomials $p(\cdot, Q)$ is U-stable (3.5)

if and only if

2) (a) for some $q_0 \in Q$, $p(\cdot, q_0)$ is U-stable, and (3.6)

(b) $\forall s \in \partial U$, $0 \notin p(s, Q)$. (3.7)

Remarks. a) Assumption (A1) can be modified to read:

(A1') The parameter vector $q \in \mathbb{R}^m$ is in a given non-empty set Q, and Q is *pathwise connected.*

So Q is required to be neither closed nor bounded. The simplest way to prove this is to observe that *any* point q' in Q can be connected to the point q_0 of condition 2(a) by a path C in Q; this path is compact. Since $p(s, C) \subset p(s, Q)$, condition 2(b) implies that $0 \notin p(s, C)$ $\forall s \in \partial U$, and Theorem 3.3 (with the original assumption (A1)) implies that $q' \in C$ is U-stable.

b) Theorem 3.3 can be further extended to arbitrary connected parameter sets Q satisfying (A3) using a different method of proof (see footnote 1); of course, the distinction between connected and pathwise connected parameter sets is not a real engineering concern. We may even consider completely arbitrary sets $Q \subset \mathbb{R}^m$ if we require condition 2(a) to hold for some q_0 in *each connected component* of Q.

---

element of the other set. So set ordering and multiplicity are not issues, and two sets can be arbitrarily close to each other even if they don't have the same number of elements. It is not difficult to show that the map from the space of polynomials to the set of polynomial zeroes in $\mathbf{P}_f(\mathbb{C})$ is continuous; indeed, this is implied by Fact 3.1. The proof of Theorem 3.3 then proceeds in a straightforward manner.

This approach to the problem is *mathematically* more powerful, as it produces the result (Theorem 3.3) quite naturally for arbitrary *connected* parameter sets Q. Theorem 3.3 is stated for *compact connected* Q, and in the remark which follows the theorem, it is extended to non-compact *pathwise connected* Q. However, from the engineering perspective of plausible parameter sets, the distinction between connected parameter sets and pathwise connected parameter sets is insignificant.

The approach taken in the text is used because the topological and geometric arguments in the complex plane have a direct conceptual connection to the resulting graphical test; the connection to the Hausdorff topology of $\mathbf{P}_f(\mathbb{C})$ is considerably more abstract.

Comments. a) The choice of U allows for great designer freedom.

b) Note that Q is not required to be convex. The freedom in choosing Q and U allows the engineer to evaluate trade-offs: higher degree of stability versus greater parameter variations.

c) The theorem is a labor saving device: "the set $p(\cdot, Q)$ is U-stable" is equivalent to $0 \notin p(U, Q)$; once condition 2(a) holds, we need only check $0 \notin p(\partial U, Q)$. Condition 2(b) is to be tested on a work station: hence any possibility for obtaining $p(s, Q)$ efficiently should be exploited (see special case below). Note that, $\forall s \in \partial U$, it is not required to actually determine the whole set $p(s, Q)$, we need only check that $0 \notin p(s, Q)$. In case $p(s, Q)$ is convex, we need only a line separating $p(s, Q)$ from the origin, $\forall s \in \partial U$.

Proof. $\underline{1 \Rightarrow 2}$: Suppose $p(\cdot, Q)$ is U-stable. Clearly condition 2(a) is satisfied. We show 2(b) by contradiction. If $s \in \partial U$ and $0 \in p(s, Q)$, then there is some parameter $q^* \in Q$ with $p(s, q^*)=0$. Thus $p(\cdot, q^*)$ has a zero in $\partial U \subset U$ and is not U-stable, which contradicts the assertion that $p(\cdot, Q)$ is U-stable.

$\underline{2 \Rightarrow 1}$: From Fact 2.1 the zeroes of $p(\cdot, q)$ are uniformly bounded for all q in Q, say $|s| < \alpha$ whenever $p(s, q)=0$ $\forall q \in Q$. Then let $V = U \cap \{s : |s| \le \alpha\}$. Clearly $p(\cdot, q)$ is V-Hurwitz $\Leftrightarrow$ $p(\cdot, q)$ is U-Hurwitz, for all q in Q. Condition 2(b) implies that $p(s, q) \neq 0$ $\forall s \in \partial V$ $\forall q \in Q$, since $s \in \partial V$ implies $s \in \partial U$ or $|s| = \alpha$. Now Fact 3.1 implies that the number of zeroes that $p(\cdot, q)$ has in V is a locally constant function of q on all of Q. Since Q is connected, the number of zeroes that $p(\cdot, q)$ has in V is globally constant on Q [Dug. 1, p. 108], and condition 2(a) guarantees that that number is zero. So $p(\cdot, Q)$ is V-stable, and thus U-stable. $\square$

Special Case: Q is a Convex Polyhedron.

Consider the polynomial $p(s, q)$ given by (3.2), but now replace assumptions (A1)-(A3) by

(A1*) Q is a *convex polyhedron* in $\mathbb{R}^m$ with vertices $\{v_1, v_2, \ldots, v_l\}$.

(A2*) For $k = 0, 1, \ldots, n$, the coefficient parameterization $a_k(\cdot)$ is *affine*; i.e.

$$a_k = \alpha_k + \beta_k^T q \tag{3.8}$$

where $\alpha_k \in \mathbb{R}$ and $\beta_k \in \mathbb{R}^m$ are given and $q \in Q \subset \mathbb{R}^m$.

(A3) For all $q \in Q$, $a_n(q) > 0$.

Assumptions (A1*) and (A2*) imply assumptions (A1) and (A2), so Theorem 3.3 applies. Assumption (A2*) implies that, for any fixed $s \in \mathbb{C}$ the map $q \to p(s, q)$ is *affine*. Hence by (A1*), $p(s, Q)$ is a *convex polygon* in $\mathbb{C}$. In fact

$$p(s, Q) = \mathrm{co}\{p(s, v_1), \ldots, p(s, v_l)\} \tag{3.9}$$

where $\mathrm{co}\{p_1, \ldots, p_m\}$ denotes the *convex hull* of $\{p_1, \ldots, p_m\}$. Usually only a proper subset of the points $p(s, v_i)$ are vertices of the polygon $p(s, Q)$; note that as s varies, that subset may change.

By Theorem 3.3, once it has been verified that for some q_0, $p(\cdot, q_0)$ is U-stable, it remains to check that the *convex polygon* $p(s, Q)$ does not contain the origin $\forall s \in \partial U$. We define the nearest point function, $\mathrm{Nr}(\cdot)$, on closed convex subsets $S \subset \mathbb{C}$ by

$$\mathrm{Nr}(S) := \arg\min_{s \in S} \{ |s| \}. \tag{3.10}$$

Checking that $p(s, Q)$ does not contain the origin is equivalent to checking that $\mathrm{Nr}(p(s, Q))$ does not equal zero. This can easily be done using Wolfe's nearest point algorithm [Wol. 1, Hau. 1], which computes

$$\mathrm{Nr}(\,\mathrm{co}\{k_1, \ldots, k_l\}\,) \tag{3.11}$$

for any finite set $\{k_1, \ldots, k_l\} \subset \mathbb{C}$ in an efficient and finite manner. In our case we wish to compute

$$\mathrm{Nr}(p(s, Q)) = \mathrm{Nr}(\mathrm{co}\{p(s, v_1), p(s, v_2), \ldots, p(s, v_l)\}) \tag{3.12}$$

for various values of $s \in \partial U$. It is easily verified that (3.12) is a *continuous* function from $s \in \partial U$ to $\mathrm{Nr}(p(s,Q)) \in \mathbb{C}$. So, for the typical case where ∂U is composed of a finite number of C^1 curves, we have the following procedure to implement Theorem 3.3:

a) check *any* polynomial in $p(\cdot, Q)$; if it is Hurwitz, condition 2(a) of Theorem 3.3 is satisfied; otherwise $p(\cdot, Q)$ is not U-stable;

b) parameterize the curve(s) ∂U;

c) as s travels along the components of ∂U, plot the point $\mathrm{Nr}(p(s, Q))$;

d) if the resulting curve is bounded away from zero, then condition 2(b) of Theorem 3.3 is satisfied; otherwise, 2(b) is violated and $p(\cdot, Q)$ is not U-stable.

Comments. a) One obviously cannot plot the locus in step c) for *every* value of $s \in \partial U$. Indeed, one would normally partition ∂U, plot those points, and then fill in points as necessary (either manually or 'automatically') until the locus becomes sufficiently smooth to yield a reliable conclusion. The 'filling in' problem is identical to the 'filling in' problem for the conventional Nyquist test.

b) The nearest point calculation in step c) is finite (using the Wolfe algorithm [Wol. 1]). Indeed, in our experience, by using the resulting edge of the polygon containing the nearest point as the starting point for the search at the next value of $s \in \partial U$, we usually required only one or two iterations (mostly one) of the numerical procedure defined in [Wol. 1]. Of course, this cannot be guaranteed; in general, we may need up to l iterations.

Remark. We now consider modifying (A1*) once again to read

(A1**) $Q \subset \mathbb{R}^m$ is *closed* and *convex*

leaving (A2*) and (A3) intact (e.g. Q might be a closed convex set in coefficient space itself, with $a_k(q)$ being simply coordinate projections). The algorithm above still works (that is, the nearest point function is still well-defined), although the nearest point calculation is no longer finite. For this convex case, however, efficient convergent algorithms do exist [Hau. 1]. If we specify *a priori* some acceptable precision for approximating the nearest point, we obtain a finite algorithm for finding $\mathrm{Nr}(p(s,Q))$.

3.3 Linear distributed time-invariant systems

Theorem 3.3 is easily generalized to linear time-invariant *distributed* systems. Consider a control system made up of subsystems whose matrix transfer functions have elements in the algebra $\hat{B}(\sigma_0)$, where σ_0 is typically negative [Cal. 1-3, Des. 1, Nett 1; for connections to the semi-group literature, see Cal. 4 and the references therein]. Let R_σ denote the closed right half-plane $\{s \in \mathbb{C} : \mathrm{Re}\{s\} \geq \sigma\}$ and fix $\sigma_0 < 0$. We say that a function $\hat{f}(s) \in \hat{B}(\sigma_0)$ if, for some $\sigma < \sigma_0$:

1. $\hat{f}$ has a finite number of poles in R_σ, and

2. the inverse transform of $\hat{f}$ includes--in addition to the exponentials due to the poles in R_σ--

$$f_a(t) + \sum_{k=0}^{\infty} f_k\, \delta(t - t_k) \tag{3.18}$$

where

$$\int_0^\infty |f_a(t)| e^{-\sigma t} dt + \sum_{k=0}^\infty |f_k| e^{-\sigma t_k} < \infty \tag{3.19}$$

with $t_0=0$ and $t_k>0 \;\; \forall k$.

So, except for its poles in R_σ, $\hat{f}(s)$ is *analytic* in R_{σ_0} and, except for arbitrarily small neighborhoods of its poles, $\hat{f}(s)$ is *bounded* in R_{σ_0}. In particular, $\hat{f}(s)$ is bounded in R_{σ_0} as $|s|$ goes to infinity. Note that $\hat{f}(s)$ is not necessarily defined on all of $\mathbb{C}$; indeed, there are functions in $\hat{B}(\sigma_0)$ which have no analytic continuation beyond a half-plane R_σ for some $\sigma<\sigma_0$.

The framework above guarantees that the control system matrix transfer functions have coprime matrix factorizations with factors which are *analytic* in s for $s \in R_{\sigma_0}$. In [Cal. 3, Des. 1] it is shown that such a control control system is σ_0-stable[2] if and only if

$$\inf_{\mathrm{Re}\{s\}\ge\sigma_0} |\chi(s)| > 0 \tag{3.20}$$

where the characteristic function $\chi(s)$ is a sum of products of the elements of these matrix factors. Thus we see that the zeroes of a characteristic function of a distributed system have dynamical interpretations similar to zeroes of a characteristic polynomial of a finite dimensional system.

Now suppose that each element of these factorizations depends *continuously* on a parameter $q \in Q \subset \mathbb{R}^m$ with Q *compact* and *connected.* Then the characteristic function becomes $\chi(s,q)$, continuous in q for $q \in Q$ and analytic in s for $s \in R_{\sigma_0}$. We consider 'undesirable' closed sets U of the complex plane which satisfy the assumption:

(A4) $\quad U \subset R_{\sigma_0}$.

Of course, assumption (A4) guarantees that $\chi(s,q)$ will be analytic in s for all $s \in U$. Note that there was no analogous assumption required in the polynomial case, since polynomials are automatically analytic on all of $\mathbb{C}$. We also impose the *well-posedness* assumption:

(A5) $\quad |\chi(s,q)|$ is bounded away from zero as $|s| \to \infty$ in R_{σ_0} for all $q \in Q$.

Assumption (A5) is analogous to assumption (A3) for the polynomial case; indeed, it is equivalent to Fact 3.2, whose proof required (A3). From [Die. 1, Thm. 9.17.4], the zeroes of $\chi(s,q)$ are 'continuous' with respect to q; more precisely, Fact 3.1 holds for arbitrary analytic functions, not just for polynomials. Hence the reasoning for Theorem 3.3 applies and we conclude immediately:

Theorem 3.4 For all $q \in Q$, the distributed control system satisfying all the assumptions of the previous paragraph has a characteristic function $\chi(s,q)$ with no zeroes in the closed set $U \subset \mathbb{C}$ if and only if

(a) for some $q_0 \in Q$, $\chi(s,q_0) \neq 0 \;\; \forall s \in U$, and

(b) $0 \notin \chi(s,Q)$ for all $s \in \partial U$.

So the graphical algorithms discussed in Section 3.2 apply to distributed systems. In particular, if $\chi(s,q)$ depends *affinely* on $q \in Q$ and $Q \subset \mathbb{R}^m$ is compact and *convex*, then $\chi(s,Q) \subset \mathbb{C}$ is convex for all s in $\mathbb{C}$

[2] Here, we say that a control system is σ_0-stable (for $\sigma_0<0$) if it is L_p-stable $\forall p \in [1,\infty]$, and, for any input with compact support, the responses are $o(e^{\sigma_0 t})$ as $t \to \infty$.

and $\mathrm{Nr}(\chi(s,Q))$, $s \in \partial U$, can be computed using the method of Hauser [Hau. 1], as was the case for polynomials. Further, if Q is a *convex polyhedron,* we can apply Wolfe's finite 'nearest point' algorithm as described in Section 3.2.

Remarks. a) The discussion of $\hat{B}(\sigma_0)$ provides an *a priori* guarantee that factorizations of the open-loop transfer functions exist which are analytic in R_{σ_0}; this implies that the resulting characteristic function is analytic in R_{σ_0}. Combined with assumption (A5), this implies that closed-loop transfer functions are well-defined as elements of $\hat{B}(\sigma_0)$. Thus the discussion of $\hat{B}(\sigma_0)$ and assumption (A5) are to be thought of as system theoretic considerations.

b) In many specific examples, the characteristic function is easily obtained and analyticity over certain regions--or all of $\mathbb{C}$--can be verified by inspection (see, e.g., the example of Section 3.4); here we *implicitly* assume that the characteristic function was derived using some appropriate algebraic structure. In these cases, we may consider *arbitrary* closed sets $U \subset \mathbb{C}$ if we modify assumptions (A4) and (A5) to

(A4*) $\chi(s,q)$ is analytic in s for all $s \in U$ and $q \in Q$.

(A5*) $|\chi(s,q)|$ is bounded away from zero as $|s| \to \infty$ in R_{σ_0} for all $q \in Q$.

Now (A5*) is not to be interpreted as a well-posedness condition, nor even as a system theoretic consideration, but simply as an analytic requirement (compare with (A3) and Fact 3.2 in the polynomial case). *Very roughly speaking,* if U and U^c are both unbounded, we may think of the boundary of U as consisting of $\partial U \subset \mathbb{C}$ and a 'point' (or a 'set') at infinity; (A5*) extends condition (b) of Theorem 3.4 to this extra boundary at infinity, so that zeroes cannot enter U neither through ∂U nor through 'infinity.'

3.4 Example: LTI System with Delay

In this section we illustrate the use of Theorem 3.4 by analyzing an LTI plant with a delay. The plant is a regulated motor-inertia system modelled as shown in Figure 3.1. The transfer function is given by

$$\hat{\omega}(s) = \frac{K_m e^{-Ts}}{LJs^2 + (JR + Lb)s + (Rb + K_m^2)} \hat{V}_o(s) \tag{3.21}$$

where ω is the shaft angular velocity and V_0 is the applied voltage to the motor. The *nominal* design parameters are given as

$K_m = 1$ N m/amp
$L = 1$ H
$R = .01\ \Omega$
$J = 1$ kg m^2
$b = 0$ kg m^2/sec
$T = 0$ sec

Due to uncertainty, we would like the system to meet specification for parameters in the ranges: $J \in [0.8,1]$, $b \in [0,0.2]$, and $T \in [0,\tau]$. We have not specified τ ahead of time, but instead, we would like to estimate the maximum τ which will work (if any).

The design objective is steady-state regulation of the motor output ω with "good" disturbance rejection. Steady-state regulation is obtained with integral control, while disturbance rejection will be specified in terms of closed-loop pole location according to the region U shown in Figure 3.2 ($\zeta > 0.707$ and $\sigma > 1\ \mathrm{sec}^{-1}$). Specifications were met for the nominal system by using a root-locus design to obtain the following compensator:

$$C_{nom}(s) = \frac{100(s^2+2s+2)}{s(s+20)}. \tag{3.22}$$

The four closed-loop poles are at $-1.75 \pm j1.47$, -2.77 and -13.73. A Nyquist plot of the compensated system shows the nominal system has a 50° phase margin, infinite upper gain margin, and a lower gain margin of -40 dB.

Now we want to find out if the specifications are met for all possible plant perturbations. Using Theorem 3.4, condition (a) is met by the nominal plant, and condition (b) amounts to checking if the characteristic function

$$\chi(s;J,b,T) = 100e^{-Ts}(s^2+2s+2) + [Js^2+(b+0.01J)s+(1+0.01b)](s+20)s \tag{3.23}$$

satisfies $\chi(s_0;J_0,b_0,T_0)=0$ for some $s_0 \in \partial U$, $J_0 \in [0.8,1]$, $b_0 \in [0,0.2]$ and $T_0 \in [0,0.1]$.

First we discuss the restricted problem of considering variations of $J \in [0.8,1]$ and $b \in [0,0.2]$, fixing $T=0$. In this case, $\chi(s;J,b,0)$ is a polynomial in s whose coefficients are affine functions of J and b. Thus $\chi(s;[0.8,1],[0,0.2],0)$ is a parallelogram in $\mathbb{C}$ for each fixed s. So Theorem 3.3 applies, and, furthermore, the system is one of the 'special cases' discussed after Theorem 3.3. Applying the nearest point algorithm to this parallelogram, we have plotted

$$\mathrm{Nr}\,\{\chi(s;[0.8,1],[0,0.2],0)\} \tag{3.24}$$

in Fig. 3.3 for $s \in \partial U$. As s traverses ∂U (see Figure 3.2), the locus circles around the origin without ever intersecting the origin. We conclude that this *restricted* class of perturbed systems ($T \equiv 0$) does robustly meet specifications.

Now we want to consider the whole class $J \in [0.8,1]$, $b \in [0,0.2]$, $T \in [0,\tau]$ using Theorem 3.4 and the extension (for U *not* contained in a right half-plane) in remark (b) following Theorem 3.4. We know by inspection of (3.23), however, that as $T \to 0$ (for J and b fixed), there are infinitely many zeroes of $\chi(s;J,b,T)$ whose real and imaginary parts both go to infinity (in the left half-plane), while the imaginary parts increase in magnitude more rapidly than the real parts (i.e. the ratio is unbounded). Thus 'every' such zero enters the region U of Figure 3.2 for T sufficiently small (but not equal to 0); and for any fixed $T \neq 0$, there are infinitely many zeroes in U, and their magnitudes are not bounded. So we see that Theorem 3.4 does not apply, since the assumption (A5*) is not satisfied: $\chi(s;J,b,T)$ obviously cannot be uniformly (with respect to T) bounded away from zero as $s \to \infty$ in U. (In case one were to apply the test of the 'extended' Theorem 3.3 'blindly' to this problem--i.e. without satisfying assumption (A5*)--there do exist proper finite-dimensional linear controllers which would *seem* to 'guarantee' U-stability for the whole class of systems, a conclusion which is clearly erroneous. This demonstrates the importance of satisfying assumption (A5*)--or (A5) in the case $U \subset R_{\sigma_0}$.)

We repeat the following fact for emphasis: although the zeroes of $\chi(s;J,b,T)$ enter U as $T \to 0$, they also have real parts which tend to $-\infty$. This suggests the following modification of the undesirable region U. We intersect the original region U with the half-plane $\{s : \mathrm{Re}\{s\} \geq -7.5\}$. The modified region U is shown in Figure 3.4. In terms of modal settling times and damping ratios, the modified specifications on the characteristic function zeroes is $\sigma > 1\ \mathrm{sec}^{-1}$, and $\zeta > 0.707$ *for any zero with* $\sigma \leq 7.5\ \mathrm{sec}^{-1}$; zeroes with a corresponding $\sigma > 7.5\ \mathrm{sec}^{-1}$ have no damping margin requirement.

With the modified region U, it is clear that assumptions (A4*) and (A5*) are satisfied. In fact, we can give bounds on the zero locations in U if we restrict τ to some arbitrary interval, and we choose $\tau \in [0,.1]$. (If this turns out to be too restrictive--i.e. the whole class meets specification even for $\tau = 0.1$--we can just increase the interval size, or just choose some maximum τ of interest to begin with.) A simple calculation based on Eq. 3.23 shows that $|\chi(s;J,b,T)| > 15$ for all $J \in [0.8,1]$, $b \in [0,0.2]$, $T \in [0,.1]$ and for all s satisfying $\mathrm{Re}\{s\} > -7.5$ and $|s| > 50$. So the zeroes of $|\chi(s;J,b,T)|$ which are in the modified region U must satisfy $|s| < 50$.

Now we apply the test of Theorem 3.4. $\chi(s;[0.8,1],[0,0.2],[0,\tau])$ is *not* a polygon in $\mathbb{C}$, since $\chi(s;J,b,T)$ is not affine in T. We therefore cannot apply Wolfe's nearest point algorithm directly. However, for each *fixed* $T \in [0,\tau]$, we can apply Wolfe's algorithm to find $\mathrm{Nr}\{\chi(s;[0.8,1],[0,0.2],T)\}$. Performing a line search over $T \in [0,\tau]$, we determine $\mathrm{Nr}\{\chi(s;[0.8,1],[0,0.2],[0,\tau]\}$. In Figures 3.5 and 3.6 we have plotted $\mathrm{Nr}\{\chi(s;[0.8,1],[0,0.2],[0,\tau]\}$ for $s \in \partial U$, for $\tau = 0.03$ and $\tau = 0.04$. From the discussion in the previous paragraph, we do not need to plot the image of the whole unbounded boundary ∂U; we need only plot the image of the compact intersection of ∂U with the ball $\{s : |s| \le 50\}$. Figure 3.5 shows that, for $\tau = 0.03$, the locus remains bounded away from the origin. The shape of the locus is similar to Figure 3.3 at low frequencies, drawing slightly closer to the origin during the 'first pass' around it (probably indicating that the delay does push some of the original poles closer to the boundary of U). Figure 3.6 shows that, for $\tau = 0.04$, the class of characteristic functions (3.23) is *not* U-stable. The low-frequency locus is similar to Figure 3.5, still coming near to, yet remaining bounded away from, the origin as it circles around the origin, and then travels away from the origin. At higher frequencies, indeed, near $s = -5.3 \pm j5.3$, the locus comes back in and *does* intersect the origin. Roughly speaking, we conclude that it is not one of the 'original' zeroes that crosses into the region U, but, instead, one of the (infinitely many) zeroes introduced by the delay.

Figure 3.7 shows a close up near the origin of the locus for $\tau = .025, .030, .035, .040$. Note that once the locus hits the origin for $\tau = .035$, it must intersect the origin for $\tau > .035$. Using additional analysis, we found the limit to be just slightly less than 0.035.

<u>Summary</u>

This example has demonstrated the following:

1) Theorem 3.4 provides a workable test for distributed parameter systems, even when the characteristic function depends nonlinearly on one or more of the unknown parameters. However, the inclusion of nonlinear dependence can severely intensify the numerical calculations. In this example, the nonlinear dependence on T lead to a line search with respect to $T \in [0,\tau]$ in the nearest point calculation. Including *several* nonlinear dependencies would mandate more sophisticated minimization algorithms.

2) The role and importance of assumption (A5*). Note that a finite-dimensional system will automatically satisfy assumption (A5*) (or (A5)) if we bound the leading coefficient of the characteristic polynomial away from zero and consider a bounded set of characteristic polynomials (regardless of how the set U is defined). Also note that, in the case where we start with a finite-dimensional system satisfying assumption (A5), if the region U is contained in some right half-plane (assumption (A4)), assumption (A5) will automatically accommodate the addition of unknown time delays. In our example, we had to modify U to fit in a right half-plane when we included the delay.

3) The theory allows assumptions (A4) and (A5) to be replaced by (A4*) and (A5*) (see remark (b) following Theorem 3.4). However, in many common examples of distributed systems, (A5*) will be very difficult to satisfy for regions U which are not bounded on the 'left,' unless the region is very carefully crafted. For our example system (and for any time delay system), a simple positive damping coefficient bound cannot generate a region U for which (A5*) will be satisfied.

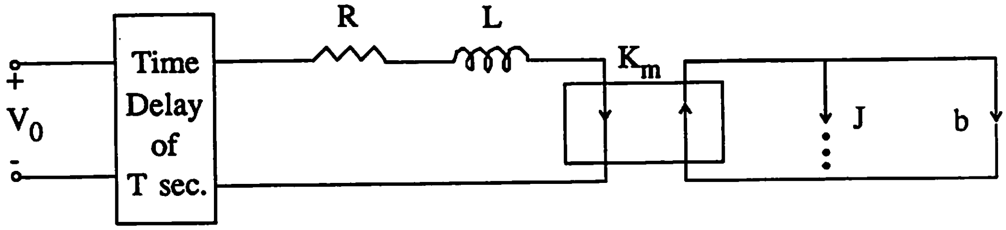

Figure 3.1 - Motor-inertia dynamical system with delay

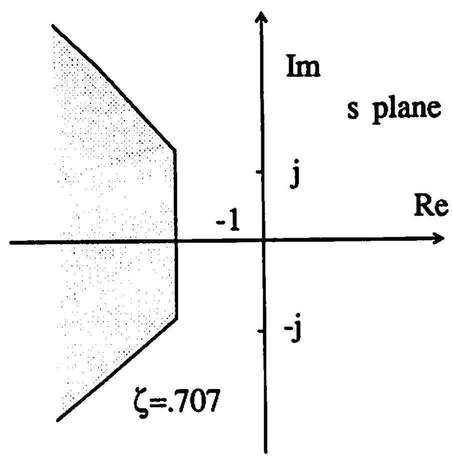

Figure 3.2 - Desired region for closed loop poles

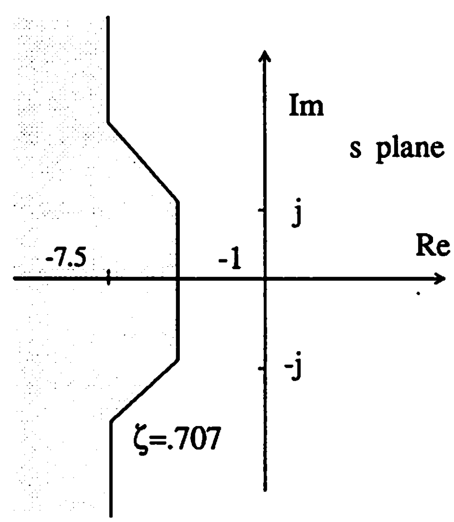

Figure 3.4 - Desired region for closed loop poles

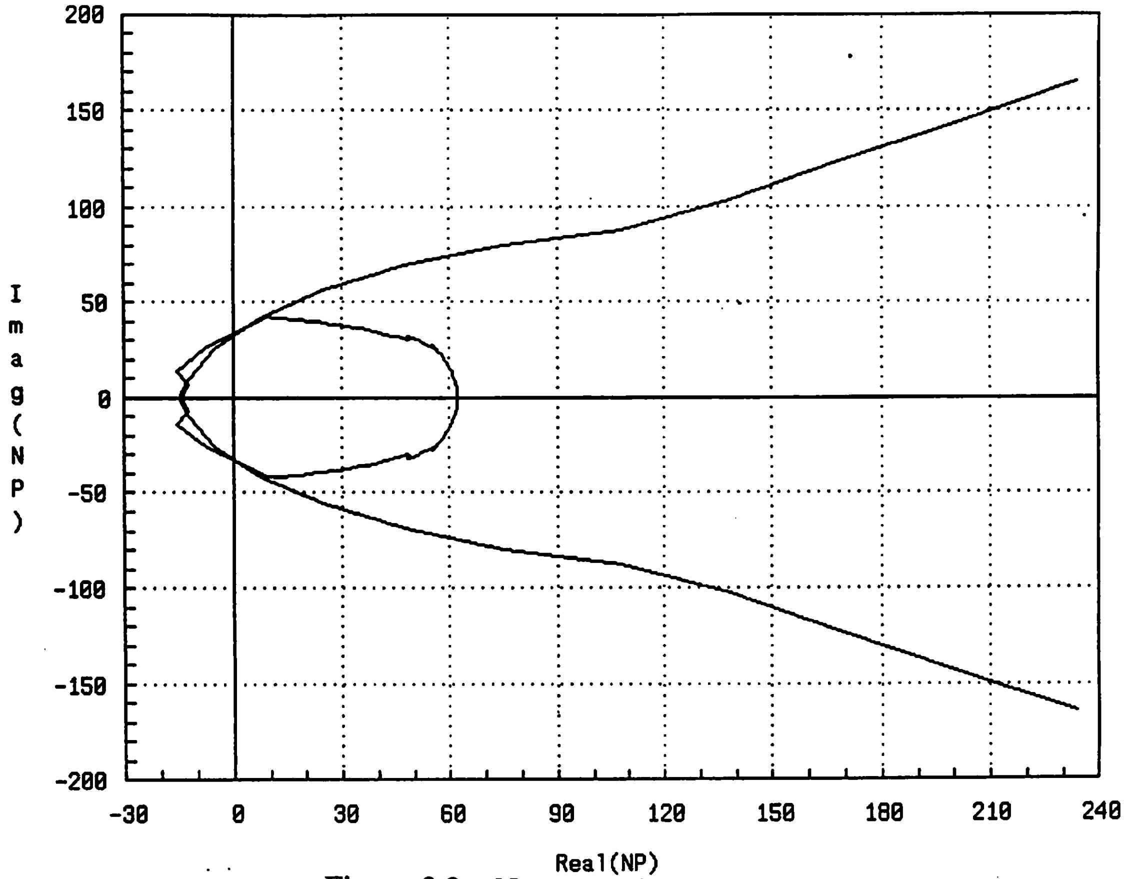

Figure 3.3 - Nearest Point Locus with T=0

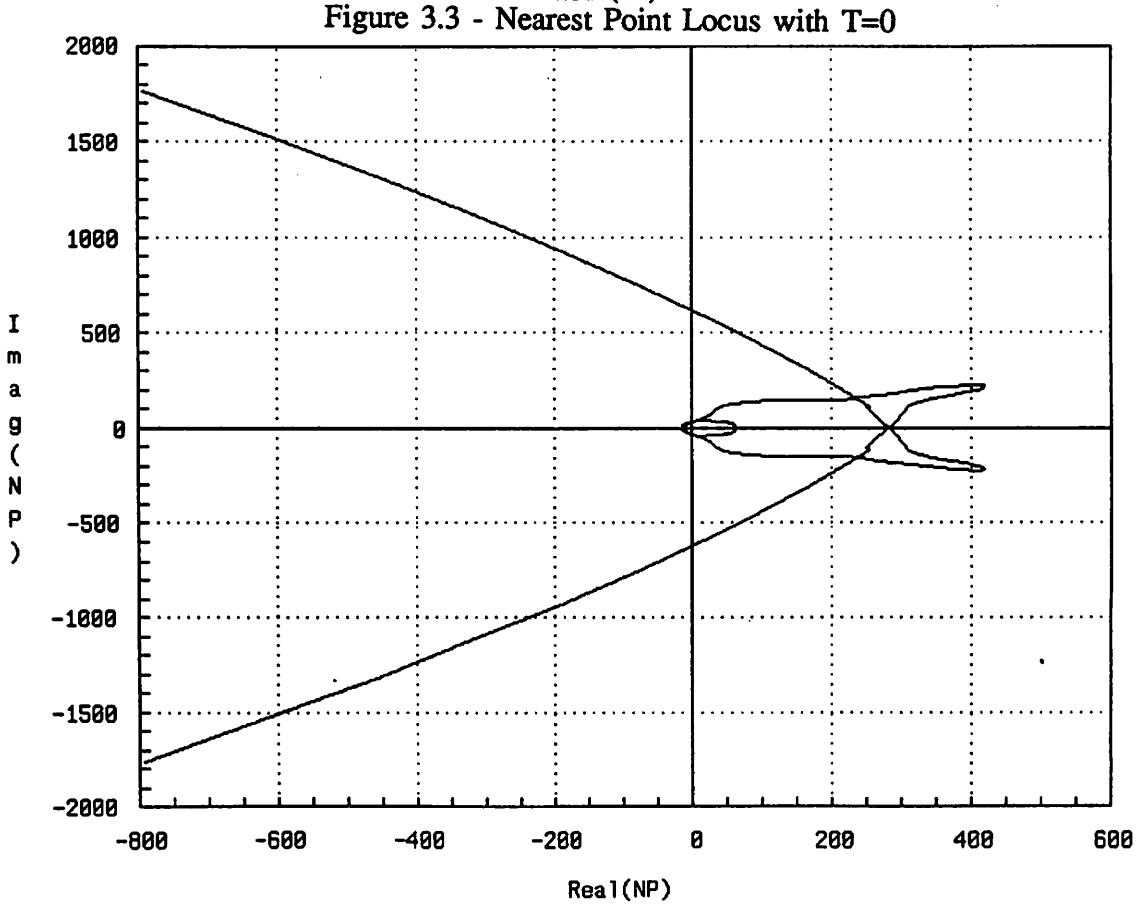

Figure 3.5 - Nearest Point Locus for $\tau = .030$

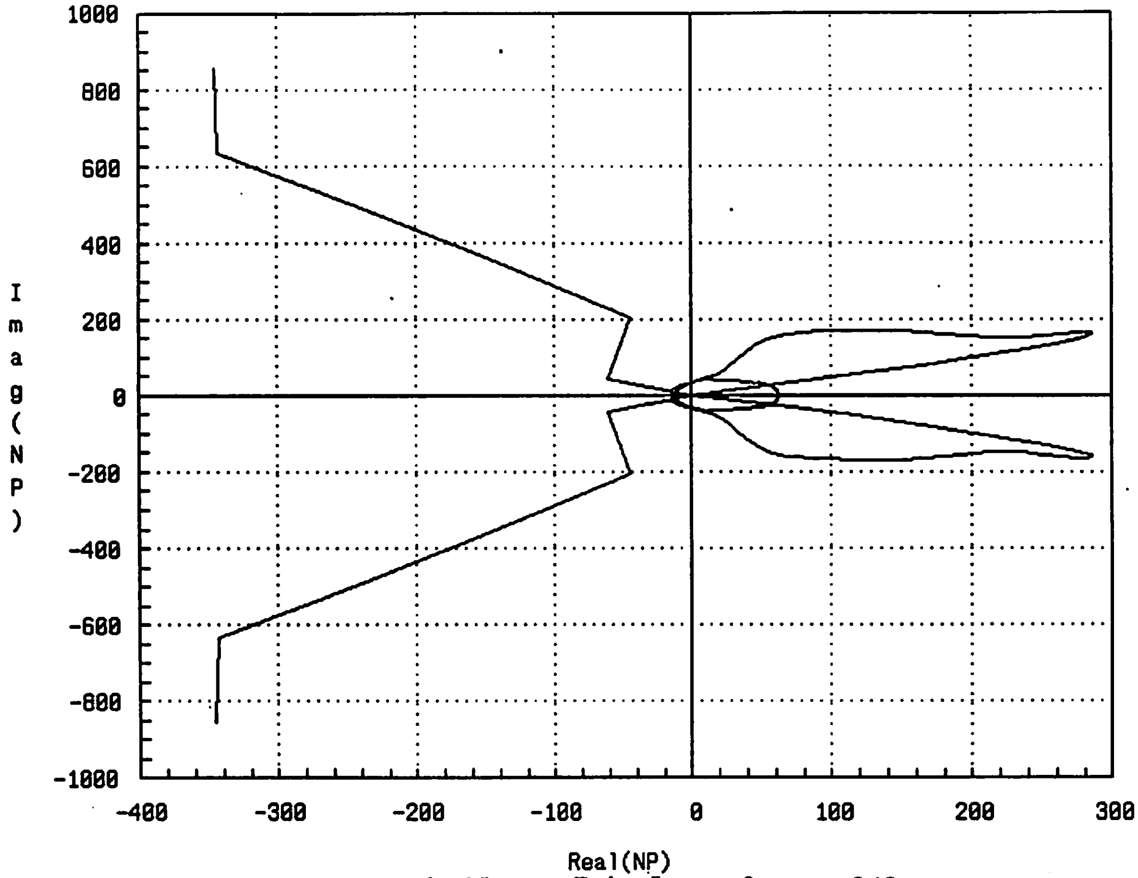

Figure 3.6 - Nearest Point Locus for $\tau = .040$

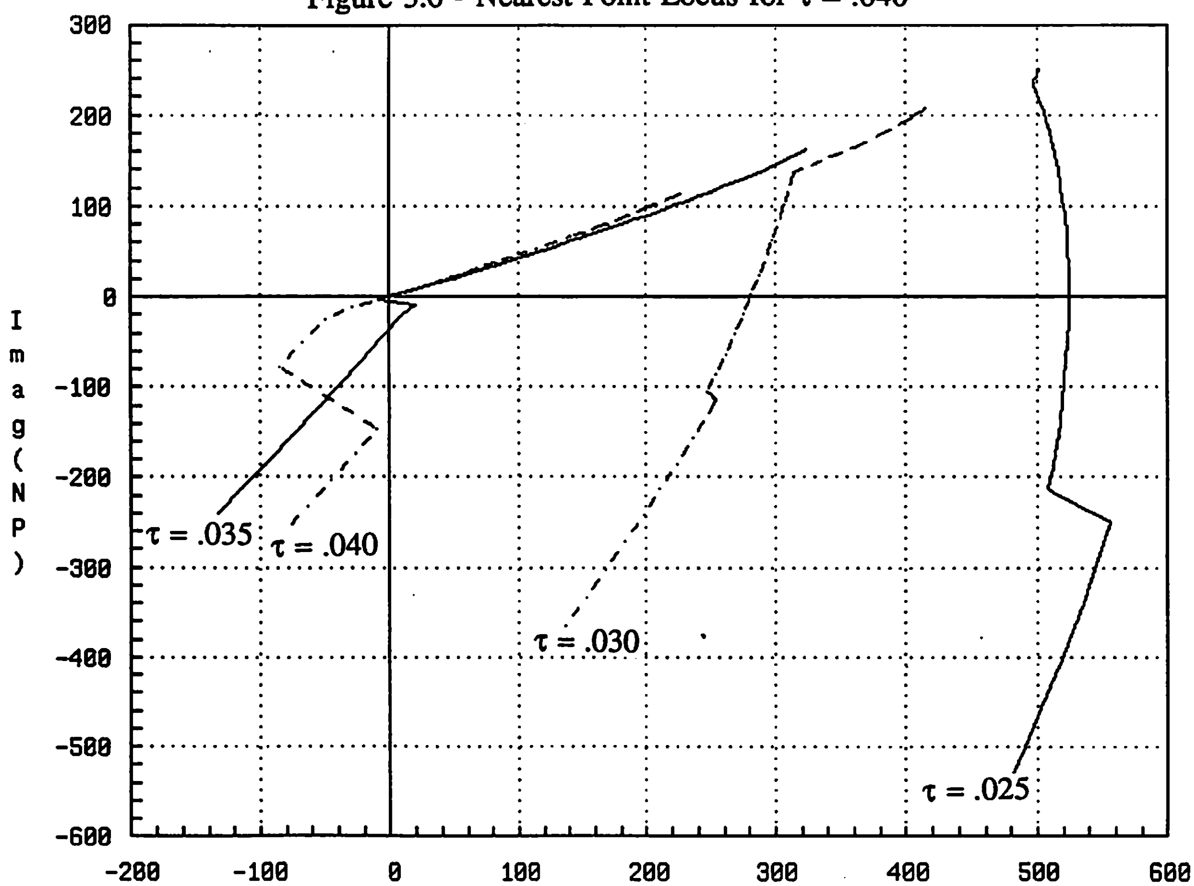

Figure 3.7 - Nearest Point Locus for $\tau = .025, .030, .035, .040$

4. CONCLUSION

The contributions of this paper should be viewed from the following perspective. Workstations have revolutionized engineering design by their computing power, sophisticated software and graphics. The computing power and the software allow designers to consider much more complicated dynamics as well as more complicated constraints on performance. The combination of computing power, software and graphics allows the study of design trade-offs. Kharitonov's Theorem streamlines the study of the tradeoff between the degree of stability of the nominal system and the size of *coefficient* perturbations that will not destroy stability. Section 3 of this paper develops tools for looking at trade-offs between the degree of stability (choice of region U) for a parameterized class of systems, and the size of that class (choice of the set Q).

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