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Memorandum No. UCB/ERL M88/41

6 June 1988
(Revised 25 April 1989)

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OPTIMAL DIAGONALIZATION STRATEGIES FOR THE SOLUTION OF A CLASS OF OPTIMAL DESIGN PROBLEMS*

L. He** and E. Polak**

ABSTRACT

The determination by iterative optimization algorithms of both open and closed loop optimal control laws requires discretization of time and/or frequency intervals. Various approaches to discretization are possible. We define a successive approximation algorithm which consists of a sequence of progressively finer stages of discretization, with a prescribed number of iterations of the optimization algorithm carried out in each stage. In the optimization literature, this type of algorithm is often called a *diagonalization* method. We associate with the successive approximation algorithm two optimal discretization problems and propose methods for their solutions. The solutions of these problems are discretization strategies which minimize the time needed to reduce the initial cost-error by a prescribed amount. Since optimal diagonalization strategies depend on a number of problem parameters which are not directly available, we present an implementation scheme based on estimates and show by experiment that it is quite effective.

KEY WORDS

optimal diagonalization, discretization techniques, minimax problems, nondifferentiable optimization.

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* The research reported herein was sponsored in part by the National Science Foundation grant ECS-8713334, the Air Force Office of Scientific Research contract AFOSR-86-0116 and the State of California MICRO Program grant 532410-19900.

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1. INTRODUCTION

The numerical simulation of any dynamical system usually involves some form of discretization. In a design optimization process, as in the design of control systems, seismic resistant structures, electronic circuits, and shapes of structural elements (see [Pol.2, Pol.6, Bha.1, Nye.1, Ben.1] for examples), discretization is used not only in the simulation of the system responses, but also in the finitization of various semi-infinite (continua) constraints. Since many iterations of an optimization algorithm are needed to solve a design problem, each iteration involving several simulations and constraint function evaluations, it is intuitively clear that the selection of a discretization strategy must have a substantial effect on the overall computing time.

To fix our attention on a particular type of optimal design problem that we shall consider in this paper, consider the design of a finite dimensional parametrized controller, as in Fig. 1, which stabilizes a linear feedback-system, suppresses output disturbances and yields satisfactory step responses. First (see [Pol.5]), closed-loop stability is ensured by satisfying the semi-infinite inequality

$$\max_{\omega \in [0,\omega_s]} \operatorname{Re} \left[\chi(x,j\omega)/D(\xi,j\omega) \right] \geq 0, \tag{1.1a}$$

where $x \in \mathbb{R}^n$ is the controller-parameter, $\xi \in \mathbb{R}^m$ is an auxiliary parameter, $\chi(x,s)$ is the closed loop characteristic polynomial, and $D(\xi,s)$ is a stable parameterized polynomial of the same degree as $\chi(x,s)$, and the components ξ^i of ξ are required to remain in certain intervals, $[\underline{\xi}^i,\overline{\xi}^i]$.

Next, output disturbance rejection is ensured by imposing a semi-infinite inequality of the form

$$\max_{\omega \in [\omega_0,\omega_1]} \{ \overline{\sigma}[H_{yd}(x,j\omega)] - b_d(\omega) \} \leq 0 . \tag{1.1b}$$

Finally, the step response $y^j(t,x,r^i)$, in the j–th output channel, resulting from a step input $r^i(t)$, in the i–th reference input channel, can be confined between two bounding functions, $\underline{b}_{ij}(t) < \overline{b}_{ij}(t)$, over the interval $[0,T]$, by requiring that

$$\max_{t \in [0,T]} (y^j(t,x,r^i) - \underline{b}_{ij}(t))(y^j(t,x,r^i) - \overline{b}_{ij}(t)) \leq 0 . \tag{1.1c}$$

In addition, we can expect that the i–th controller parameter must be confined to an interval $[\underline{x}^i,\overline{x}^i]$, $i = 1,2,\ldots,n$. Let $X \triangleq \{x \in \mathbb{R}^n \mid x^i \in [\underline{x}^i,\overline{x}^i],\ i = 1,2,\ldots,n\}$ and let $\Xi \triangleq \{\xi \in \mathbb{R}^m \mid \xi^i \in [\underline{\xi}^i,\overline{\xi}^i]$,

$i = 1,2,...,m\}$. The three inequalities (1.1a-c) can be solved for an $x \in X$ and a $\xi \in \Xi$ by applying a minimax optimization algorithm (such as one of those described in [Pol.3]) to the problem

$$\min_{x \in X, \xi \in \Xi} \max\{ \max_{\omega \in [0,\omega_s]} - \mathrm{Re}\, [\chi(x,j\omega)/D(\xi,j\omega)\,],$$

$$\max_{\omega \in [\omega_0,\omega_1]} \bar{\sigma}[\, H_{yd}(x,j\omega)\,] - b_d(\omega),$$

$$\max_{i,j} \max_{t \in [0,T]} (y^j(t,x,r^j) - \underline{b}_{ij}(t))(y^j(t,x,r^j) - \overline{b}_{ij}(t))\ \}, \tag{1.2}$$

until the value of the minimand becomes negative. When the inequalities (1.1a-c) are consistent and can be satisfied strictly, this is a finite process.

It should be clear that a numerical solution of (1.2) entails discretization of the frequency and time intervals over which the inequalities (1.1a-c) were defined. Although for finite dimensional dynamics the step responses $y^j(t,x,r^j)$ can be computed without discretization of dynamics (see [Wuu.1]), when the dynamics involve PDEs, the use of the finite element method for response evaluation does require discretization of dynamics.

We see that problem (1.2) is a special case of a general design problem in the form

$$\min_{z \in Z} \max_{k \in \mathbf{m}} \max_{\eta^k \in I_k} \phi^k(z,\eta^k)\,. \tag{1.3}$$

There are basically two approaches possible. The first is to select a sufficiently fine (safe) discretization of the intervals I_k and to solve the resulting discretized version of problem (1.3) until a termination test is satisfied. The second approach is to start out with a coarse discretization and to increase discretization progressively, as a solution is approached. This approach is justified by two empirical observations. The first is that when far from a solution, cost reduction is possible even with coarse discretization; the second observation is that the work per iteration for a minimax algorithm is proportional to a polynomial in the number of discretization points. Hence the more discretization points are used, the more expensive the iterations become. The mechanism for increasing the discretization can be either closed-loop or open-loop. Closed-loop techniques (see, e.g., [Pol.1], [Kle.1], [Sch.1]), increase precision whenever the cost-reduction in an iteration drops below a moving floor. Open-loop techniques use preassigned discretization rules to decompose the original semi-infinite minimax problem into an infinite sequence of finite minimax problems whose solutions converge to the solution of the original problem. A minimizing sequence is started for each of the approximating problems and is abandoned progressively closer to a solution, with the last point of one sequence serving as the first point of the next one. The result is a process which can be visualized as a *diagonal progression* along minimizing sequences. Hence they are often referred to as *diagonalization techniques* (see [Tap.1], [Dun.1]). Although there appear to be no optimal discretization strategies to be found in the literature, there is a considerable amount of empirical evidence to indicate that substantial computational savings can be obtained by increasing the number of discretization points slowly, either by open- or by closed-loop techniques.

In this paper we will deal only with diagonalization techniques. In Section 2, we propose a formulation of an optimal diagonalization problem (for design problems of the form (1.3)) and describe an algorithm for its solution. In Section 3 we impose additional structure on the optimal diagonalization problem, proposed in Section 2, to obtain a simplified optimal diagonalization problem, which is much easier to solve. It can be seen from the numerical results in Section 5 that the total work, resulting from the use of the optimal strategies for the simplified diagonalization problem, is quite close to that resulting from the use of the optimal strategies for the original diagonalization problem. In Section 4, we show that when the work function is monomial, the simplified optimal diagonalization problem, proposed in Section 3, has a particularly elegant solution. In Section 5, we propose an implementation of our optimal diagonalization strategies and present numerical results which illustrate the effectiveness of our implemented optimal diagonalization strategies in solving two control design problems.

2. AN OPTIMAL DIAGONALIZATION PROBLEM

To simplify notation, we shall consider optimal design problems in the abstract form

$$\mathbf{P}: \ \min\{\ \psi(x) \mid x \in X\ \}\ , \tag{2.1a}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz continuous function and $X \subset \mathbb{R}^n$ is a compact set defined by

$$X \triangleq \{\ x \in \mathbb{R}^n \mid \underline{x}^i \le x^i \le \bar{x}^i,\ i = 1,2,\ldots n\ \}, \tag{2.1b}$$

Referring to (1.3), we see that the function $\psi(\cdot)$, can have the form

$$\psi(x) \triangleq \max_{k \in \mathbf{m}} \psi^k(x), \tag{2.1c}$$

where $\mathbf{m} \triangleq \{ 1,2,...,m \}$ and

$$\psi^k(x) \triangleq \max_{\eta^k \in I_k} \phi^k(x,\eta^k), \tag{2.1d}$$

the I_k are intervals and the functions $\phi^k(x,\eta^k)$ are locally Lipschitz continuous, with Lipschitz constant, with respect to η^k, L_k on I_k.

Next we introduce a parametrized family of approximating problems, with parameter $q > 0$:

$$\mathbf{P}_q : \ \min \{ \psi_q(x) \mid x \in X \} , \tag{2.2}$$

where $\psi_q : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz continuous function for $q > 0$.

We will assume that the approximations $\psi_q(\cdot)$ are accurate to at least first order in $1/q$, as follows:

Assumption 2.1. There exists a constant $K < \infty$ such that for all $x \in \mathbb{R}^n$ and all $q > 0$

$$| \psi(x) - \psi_q(x) | \leq K/q . \tag{2.3}$$

Suppose that $\psi^k(\cdot)$ is defined as in (2.1d), with $I_k \triangleq [\eta_0^k, \eta_1^k]$. For any integer $p > 0$, we define $\psi_p^k(x) \triangleq \max_{\eta \in I_{p,k}} \phi^k(x,\eta^k)$, where $I_{p,k} \triangleq \{ \eta_0^k , \eta_0^k + l_k/p , \eta_0^k + 2l_k/p,..., \eta_1^k \}$ and $l_k \triangleq \eta_1^k - \eta_0^k$. Suppose that we select a set of positive weights, $\{ \sigma_k \}_{k \in \mathbf{m}}$, which determine the relative fineness of discretization of the various intervals. If for any $q > 0$, we define $p_k(q) \triangleq \lceil \sigma_k q \rceil + 1$ [1], $k \in \mathbf{m}$, then the function $\psi_q(x) \triangleq \max_{k \in \mathbf{m}} \psi^k_{p_k(q)}(x)$ satisfies Assumption 2.1 with $K = \max_{k \in \mathbf{m}} L_k l_k / 2\sigma_k$. In practice, a sensible assignment of weights σ_k would be to make σ_k proportional to $L_k l_k$, the product of a Lipschitz constant and the interval length, as a means for ensuring that the smallest number of discretization points is being used.

The above assumption is also satisfied when the functions $\phi^k(\cdot,\cdot)$ are time responses, and the functions $\phi^k_{p_k}(\cdot,\cdot)$ are approximations obtained by means of an integration procedure, provided that the precision parameters in the numerical integration procedure are suitably keyed to the discretization

[1] We denote by $\lceil a \rceil$ the *smallest* integer larger than or equal to a, and by $\lfloor a \rfloor$ the *largest* integer smaller than or equal to a.

parameter q.

Let $\hat{\psi} \triangleq \min\{ \psi(x) \mid x \in X \}$ and let $\hat{\psi}_q \triangleq \min\{ \psi_q(x) \mid x \in X \}$. The effect of Assumption 2.1 is to make the problems $\mathbf{P}_q$ consistent approximations to the problem $\mathbf{P}$ in the following sense.

Lemma 2.1. Suppose that $\hat{x} \in X$ solves $\mathbf{P}$, that $\hat{x}_q \in X$ solves $\mathbf{P}_q$, and that the Assumption 2.1 holds. Then, for all $q > 0$,

$$\hat{\psi} \le \psi(\hat{x}_q) \le \psi_q(\hat{x}_q) + K/q = \hat{\psi}_q + K/q \ , \tag{2.4a}$$

$$\hat{\psi}_q \le \psi_q(\hat{x}) \le \psi(\hat{x}) + K/q = \hat{\psi} + K/q \ . \tag{2.4b}$$

■

We assume that the approximation problems $\mathbf{P}_q$ will be solved by a linearly converging minimax algorithm, defined by an iteration map $A\colon \mathbb{R}^n \times C(\mathbb{R}^n,\mathbb{R}) \times 2^{\mathbb{R}^n} \to \mathbb{R}^n$, which constructs a minimizing sequence $\{x_i\}_{i=0}^{\infty}$ according to the formula $x_{i+1} = A(x_i,\psi_q,X)$. Referring to [Pol.4], we see that the rate constant, θ, of a number of linerly converging minimax algorithms is independent of the number of discretization points used in the intervals I_k, in (2.1d). The fineness of discretization of dynamics also has no effect on the rate constant. Hence we are justified in making the following assumption.

Assumption 2.2. There exists a $\theta \in (0,1)$ such that for all $x \in \mathbb{R}^n$ and all $q > 0$,

$$\psi_q(A(x,\psi_q,X)) - \hat{\psi}_q \le \theta(\psi_q(x) - \hat{\psi}_q) \ , \tag{2.5}$$

i.e., the algorithm converges linearly in cost, uniformly on the family of approximating problems $\mathbf{P}_q$. ■

Assumption 2.3. For any $x \in \mathbb{R}^n$, $q > 0$, let $W(x,\psi_q)$ denote the amount of computational work required to evaluate $A(x,\psi_q,X)$. We assume that it is a polynomial function of the discretization parameter used and that it is independent of the current value of the iterate, i.e., that for all $x \in \mathbb{R}^n$ and all $q > 0$,

$$W(x,\psi_q) = w(q) \triangleq \sum_{i=1}^{p} c_i\, q^{\beta_i} \ , \tag{2.6}$$

where, for $i = 1,2,\ldots,p$, $c_i > 0$ and $\beta_i > 0$ are integers, such that $\beta_i > \beta_{i-1}$. ■

We can now state a diagonalization scheme in the form of a successive approximation algorithm for solving the problem $\mathbf{P}$. The successive approximation algorithm generates an *infinite* sequence of *stages*, each defined by Step 2, below, and indexed by i, in which the discretization parameter q_i is kept constant and in which a *finite number*, n_i, iterations of a linearly converging algorithm are performed on the problem $\mathbf{P}_{q_i}$. The algorithm, below, does not include a rule for determining n_i and q_i. We will provide rules for their determination later.

Successive Approximation Algorithm 2.1.

Data: $x_0 \in \mathbb{R}^n$.

Step 0: Set $n_0 = 0$, $x^0_{n_0} = x_0$ and $i = 1$.

Step 1: Determine the number of iterations n_i and the discretization parameter q_i to be used in the $i\text{–}th$ stage.

Step 2: Set $x^i_0 = x^{i-1}_{n_{i-1}}$. For $j = 1,2,\ldots,n_i$, compute

$$x^i_j = A(\, x^i_{j-1}\, ,\, \psi_{q_i}\, ,\, X\,)\,. \tag{2.7}$$

Step 3: Replace i by $i + 1$, go to Step 1. ■

Before we can formulate the problem of determining the number of stages s and the $i\text{–}th$ stage parameters n_i and q_i, $i = 1,2,\ldots,s$, which minimize the total computational work required to reduce the initial deviation from optimal cost by a specified amount, we must establish what information can be extracted from the assumptions we have introduced. Our first observation is that Assumptions 2.1 and 2.2 do not lead to a necessary condition which must hold whenever a relation of the form

$$\psi(x^s_{n_s}) - \hat{\psi} \le \varepsilon(\psi(x_0) - \hat{\psi}) \tag{2.8}$$

is satisfied after s number of stages of Algorithm 2.1. Hence we now establish a sufficient condition.

Lemma 2.2. Suppose that Assumptions 2.1 and 2.2 hold and that Algorithm 2.1 has generated the sequence $\{\, x^i_j\,\}^{i=\infty\,,\,j=n_i}_{i=1\,,\,j=0}$, from the starting point x_0. Let $e_i \triangleq \psi(x^i_{n_i}) - \hat{\psi}$, $i = 0,1,2,\ldots$. Then

$$e_i \le 4\, K/q_i + \theta^{n_i}\, e_{i-1}\,, \qquad i = 1,2,\ldots \tag{2.9}$$

Proof: Since $x^i_j = A(\, x^i_{j-1}, \psi_{q_i}\, , X\,)$ for $i = 1,2,\ldots$ and $j = 1,2,\ldots,n_i$, it follows from Assumption 2.2 that

$$\psi_{q_i}(x_j^i) - \hat{\psi}_{q_i} \leq \theta\,(\psi_{q_i}(x_{j-1}^i) - \hat{\psi}_{q_i})\,, \quad \forall\, j = 1,2,\ldots,n_i, \forall\, i = 1,2,\ldots \tag{2.10}$$

Hence

$$\psi_{q_i}(x_{n_i}^i) - \hat{\psi}_{q_i} \leq \theta^{n_i}\,(\psi_{q_i}(x_0^i) - \hat{\psi}_{q_i})\,, \quad \forall\, i = 1,2,\ldots \tag{2.11}$$

Since $e_i = \{\, \psi(x_{n_i}^i) - \psi_{q_i}(x_{n_i}^i) \,\} + \{\, \hat{\psi}_{q_i} - \hat{\psi} \,\} + \{\, \psi_{q_i}(x_{n_i}^i) - \hat{\psi}_{q_i} \,\}$, it follows from Assumption 2.1, (2.4b) and (2.11) that

$$\begin{aligned} e_i &\leq 2\,K/q_i + \theta^{n_i}\,(\psi_{q_i}(x_0^i) - \hat{\psi}_{q_i}) \\ &= 2\,K/q_i + \theta^{n_i}\,\{\, (\psi_{q_i}(x_0^i) - \psi(x_0^i)) + (\hat{\psi} - \hat{\psi}_{q_i}) + (\psi(x_0^i) - \hat{\psi}) \,\}\,. \end{aligned} \tag{2.12}$$

Since by construction, $x_0^i = x_{n_{i-1}}^{i-1}$, it follows from Assumption 2.1 and (2.4a) that

$$e_i \leq 2\,K/q_i + \theta^{n_i}\,(2\,K/q_i + e_{i-1})\,. \tag{2.13}$$

Consequently, (2.9) follows from the fact that $\theta^{n_i} < 1$. ■

Corollary 2.1 (Sufficient Condition). Let $r_0 = \psi(x_0) - \hat{\psi}$ and let

$$r_i = 4K/q_i + \theta^{n_i}\, r_{i-1}\,, \qquad i = 1,2,\ldots\,, \tag{2.14a}$$

$$\varepsilon_i = r_i\,/r_{i-1}\,, \qquad i = 1,2,\ldots\,, \tag{2.14b}$$

where n_i and q_i are as in Step 1 of Algorithm 2.1. Then (2.8) is satisfied after s stages if $r_s \leq \varepsilon\, r_0$, or, equivalently, if $\prod_{i=1}^{s} \varepsilon_i \leq \varepsilon$. ■

Definition 2.1. We shall refer to r_i and ε_i in (2.14a-b) as the *estimated cost-error* at the point $x_{n_i}^i$ and *cost-reduction ratio* at the *i–th* stage, respectively. ■

Optimal Diagonalization Problem 2.1. Given $r_0 > 0$ and $\varepsilon \in (0,1)$, find an optimal strategy $\hat{S} = (\hat{s}\,,\, \{\, \hat{n}_i \,\}_{i=1}^{\hat{s}}\,,\, \{\, \hat{q}_i \,\}_{i=1}^{\hat{s}})$, where $\hat{s}$ is the number of stages to be executed, $\hat{n}_i$ and $\hat{q}_i$ are the number of iterations and discretization parameter, respectively, to be used in the *i–th* stage, $i = 1,2,\ldots,\hat{s}$, of Algorithm 2.1, which solves the problem

$$\mathrm{D}(r_0,\varepsilon) : \ \min\{\ \sum_{i=1}^{s} n_i\, w(q_i) \mid r_s \leq \varepsilon\, r_0\,,\ n_i \in \mathbb{N}_+\,,\ q_i > 0\,,\ s \in \mathbb{N}_+ \,\}\,, \tag{2.15}$$

where r_s is determined by means of (2.14a) and $\mathbb{N}_+$ is the set of positive integers. ∎

Problem $\mathbf{D}(r_0,\varepsilon)$ is a mixed integer programming problem which can be solved using a combination of branch-and-bound and embedding methods. Alternatively, making use of (2.14a) to replace r_s in (2.15), we can rewrite $\mathbf{D}(r_0,\varepsilon)$ in the form

$$\mathbf{D}(r_0,\varepsilon) : \min_{s \in \mathbb{N}_+} \min\{ \sum_{i=1}^{s} n_i\, w(q_i) \mid \sum_{i=1}^{s} \frac{4K}{q_i}\theta^{k_i} + r_0\theta^{k_0} \leq \varepsilon\, r_0 ,\ k_i = \sum_{j=i+1}^{s} n_j ,\ n_i \in \mathbb{N}_+ ,\ q_i > 0 \} ,\tag{2.16}$$

where $k_s \triangleq 0$. When the number of iterations used in each stage is likely to be at least 10, a very good approximate solution to $\mathbf{D}(r_0,\varepsilon)$ can be obtained by using a nonlinear programming algorithm on the inner problem in (2.16), with n_i relaxed to be a real number, for increasing values of s, until the cost starts to increase, and then rounding upwards the final values of the n_i. Although we are not able to prove it analytically, our experimental results indicate that the optimal value of the inner problem in (2.16) is unimodal in s and hence the enumeration approach, incorporated in the algorithm below, is a practical, but costly tool for solving $\mathbf{D}(r_0,\varepsilon)$.

Algorithm 2.2.

Data: $\varepsilon \in (0,1)$, r_0.

Step 0: Set $\bar{w}_0 = +\infty$ and $s = 1$.

Step 1: Compute the strategy $\bar{S}_s = (s\ ,\ \{\ \bar{n}_i^s\ \}_{i=1}^{s}\ ,\ \{\ \bar{q}_i^s\ \}_{i=1}^{s})$ and value $\bar{w}_s = \sum_{i=1}^{s} \bar{n}_i^s\, w(\bar{q}_i^s)$, by solving

$$\min\{ \sum_{i=1}^{s} n_i\, w(q_i) \mid \sum_{i=1}^{s} \frac{4K}{q_i}\theta^{k_i} + r_0\theta^{k_0} \leq \varepsilon\, r_0 ,\ k_i = \sum_{j=i+1}^{s} n_j ,\ n_i > 0 ,\ q_i > 0 \} .\tag{2.17}$$

Step 2: If $\bar{w}_s < \bar{w}_{s-1}$, replace s by $s+1$ and go to Step 1. Else, set $\hat{s} = s-1$, $\hat{n}_i = \lceil \bar{n}_i^{s-1} \rceil$ and $\hat{q}_i = \bar{q}_i^{s-1}$ for $i = 1,2,\ldots,\hat{s}$, and $\hat{S} = (\hat{s}\ ,\ \{\ \hat{n}_i\ \}_{i=1}^{\hat{s}}\ ,\ \{\ \hat{q}_i\ \}_{i=1}^{\hat{s}})$, to be an approximate solution to $\mathbf{D}(r_0,\varepsilon)$, and stop. ∎

3. A SIMPLIFIED OPTIMAL DIAGONALIZATION PROBLEM

We will now show that if we impose the additional requirement on $D(r_0,\varepsilon)$ that the fractional cost-error reductions in each stage be equal, then we obtain a simplified optimal diagonalization problem which decomposes into an easily solvable sequence of one stage problems. As we will see in Section 5, the resulting optimal strategies are much cheaper to compute and almost as effective as those of $D(r_0,\varepsilon)$. The simplified optimal diagonalization problem can be stated as follows:

Simplified Optimal Diagonalization Problem 3.1. Given $r_0 > 0$ and $\varepsilon \in (0,1)$, find an optimal strategy $\hat{S} = (\hat{s}\,,\hat{\alpha}\,,\{\hat{n}_i\}_{i=1}^{\hat{s}}\,,\{\hat{q}_i\}_{i=1}^{\hat{s}})$, which solves the problem

$$SD(r_0,\varepsilon) : \min\{ \sum_{i=1}^{s} n_i\, w(q_i) \mid 4K/q_i + \theta^{n_i} r_0\, \alpha^{i-1} = \alpha\, r_0\, \alpha^{i-1} ,$$

$$n_i \in \mathbb{N}_+,\ q_i > 0\ ,\alpha^s \le \varepsilon,\ s \in \mathbb{N}_+ \ \}. \tag{3.1}$$

■

The theorem below shows that problem $SD(r_0,\varepsilon)$ decomposes into a sequence of one-stage problems of the form:

One-Stage Optimal Diagonalization Problem 3.2. Given $r > 0$ and $\varepsilon \in (0,1)$, find the optimal solution $\hat{S}_1 = (\hat{n},\hat{q})$ to the problem

$$1\text{–}D(r,\varepsilon) : \min\{ n\, w(q) \mid 4K/q + \theta^n r \le \varepsilon r,\ n \in \mathbb{N}_+,\ q > 0 \} . \tag{3.2}$$

■

Theorem 3.1. Suppose that $\hat{S} = (\hat{s},\hat{\alpha},\{\hat{n}_i\}_{i=1}^{\hat{s}},\{\hat{q}_i\}_{i=1}^{\hat{s}})$ is an optimal strategy of $SD(r_0,\varepsilon)$, then

(i) $\hat{\alpha}^{\hat{s}} = \varepsilon$.

(ii) $(\hat{n}_i,\hat{q}_i)$ is a solution of $1\text{–}D(r_0\hat{\alpha}^{i-1},\hat{\alpha})$.

Proof: (i) Suppose that $\hat{\alpha}^{\hat{s}} < \varepsilon$, then, from $\hat{S}$, we can construct another strategy $\bar{S} = (\hat{s}\,,\bar{\alpha}\,,\{\hat{n}_i\}_{i=1}^{\hat{s}}\,,\{\bar{q}_i\}_{i=1}^{\hat{s}})$, where $\bar{\alpha} = \varepsilon^{1/\hat{s}}$ and $\bar{q}_i = 4K/r_0\bar{\alpha}^{i-1}(\bar{\alpha} - \theta^{\hat{n}_i})$ for $i = 1,2,\ldots,\hat{s}$. Making use of the fact that $\hat{q}_i = 4K/r_0\hat{\alpha}^{i-1}(\hat{\alpha} - \theta^{\hat{n}_i})$ and $\bar{\alpha} > \hat{\alpha}$, we obtain that $\bar{q}_i < \hat{q}_i$ for $i = 1,2,\ldots,\hat{s}$. Since $w(\cdot)$ is monotone increasing, $w(\bar{q}_i) < w(\hat{q}_i)$ for $i = 1,2,\ldots,\hat{s}$. Therefore, the cost associated with the strategy $\bar{S}$ is smaller than the cost associated with $\hat{S}$, which contradicts the optimality of $\hat{S}$.

(ii) First, because the cost in (3.1) is a separable sum, and the constraints are decoupled, we note that $SD(r_0,\varepsilon)$ can be rewritten in the decoupled form

$$SD(r_0,\varepsilon): \min_{\substack{s \in \mathbb{N}_+ \\ \alpha^s \le \varepsilon}} \left\{ \sum_{i=1}^{s} \min\{ \ n_i \, w(q_i) \mid 4K/q_i + \theta^{n_i} r_0 \, \alpha^{i-1} = \alpha \, r_0 \, \alpha^{i-1}, \ n_i \in \mathbb{N}_+, \ q_i > 0 \ \} \right\}. \tag{3.3}$$

Next, it is obvious that if $(\hat{n},\hat{q})$ is a solution to $1\text{–}D(r,\varepsilon)$, then $4K/\hat{q} + \theta^{\hat{n}} r = \varepsilon r$ must hold. Hence (ii) follows from (3.3). ∎

We will now show that a method for finding a solution of the problem $1\text{–}D(r,\varepsilon)$ can be obtained from an examination of the following relaxed one-stage problem in which n is a real number.

One-Stage Relaxed Optimal Diagonalization Problem 3.3. Given $r > 0$ and $\varepsilon \in (0,1)$, find the optimal solution $\hat{S}_1 = (\hat{n},\hat{q})$ to the problem

$$1\text{–}RD(r,\varepsilon): \ \min\{ \ n \, w(q) \mid 4K/q + \theta^n \, r \le \varepsilon r, \ n > 0, \ q > 0 \ \} . \tag{3.4}$$

∎

Lemma 3.1. Problem $1\text{–}RD(r,\varepsilon)$ has a unique solution $(\hat{n},\hat{q})$ which is given by

$$\hat{n} = \ln z(r,\varepsilon)/\ln \theta , \qquad \hat{q} = 4K/r(\varepsilon - z(r,\varepsilon)) , \tag{3.5a}$$

where $z(r,\varepsilon) \in (0,\varepsilon)$ is the unique minimizer of the strictly convex function $\gamma(z)$ defined by

$$\gamma(z) \triangleq \sum_{j=1}^{p} [c_j \ln z/\ln \theta] \, [4K/r(\varepsilon - z)]^{\beta_j} , \qquad z \in (0,\varepsilon) . \tag{3.5b}$$

Furthermore, we have that $4K/\hat{q} + \theta^{\hat{n}} \, r = \varepsilon r$ holds.

Proof: Let $z = \theta^n$. Then $n = \ln z/\ln \theta$. Assuming that q satisfies the constraints in (3.4), we must have that $q \ge 4K/r(\varepsilon - z)$. Hence, for any feasible pair (n,q), the objective function in (3.4) satisfies

$$n \, w(q) \ge [\ln z/\ln \theta] \sum_{j=1}^{p} c_j \, [4K/r(\varepsilon - z)]^{\beta_j} = \sum_{j=1}^{p} [c_j \ln z/\ln \theta] \, [4K/r(\varepsilon - z)]^{\beta_j} . \tag{3.6a}$$

Since $\ln \theta < 0$ and $z \in (0,\varepsilon)$, we conclude from Lemma A.2 that each term in the right hand side of (3.6a) is strictly convex in z. Making use of the fact that the sum of strictly convex functions is still a strictly convex function, we claim that the right hand side of (3.6a) is strictly convex in z. Since it goes to $+\infty$ as z goes to 0 or ε, the right hand side of (3.6a) has a unique minimizer $z(r,\varepsilon) \in (0,\varepsilon)$. Thus, for any feasible pair (n,q),

$$n\, w(q) \geq \sum_{j=1}^{p} [c_j \ln z(r,\varepsilon)/\ln \theta]\, [4K/r(\varepsilon - z(r,\varepsilon))]^{\beta_j} . \tag{3.6b}$$

In addition we conclude that for any feasible pair (n,q), equality holds in (3.6b) if and only if $z = z(r,\varepsilon)$ and $q = 4K/r(\varepsilon - z)$, which is equivalent to equality holding in (3.6b) if and only if $n = \ln z/\ln\theta$ and $q = 4K/r(\varepsilon - z)$ for $z = z(r,\varepsilon)$. Therefore the solution to the problem 1–**RD**(r,ε) is unique and it has the form of (3.5a-b). ■

The following theorem shows that the one-stage optimal diagonalization problem 1–**D**(r,ε) can be solved by scanning the positive integers until a decrease in cost is followed by an increase.

Theorem 3.2. (i) Problem 1–**D**(r,ε) is equivalent to the problem:

$$\min\{\, n\, w(4K/r(\varepsilon - \theta^n)) \mid n > \ln \varepsilon/\ln \theta \ ,\ n \in \mathbb{N}_+ \} . \tag{3.7}$$

Furthermore, if $\hat{n}$ is a solution of (3.7) and $\hat{q} = 4K/r(\varepsilon - \theta^{\hat{n}})$, then $(\hat{n},\hat{q})$ is a solution of 1–**D**(r,ε).

(ii) The objective function $nw(4K/r(\varepsilon - \theta^n))$, in (3.7), is unimodal in n.

Proof: (i) The equivalence follows from the constraints that $q \geq 4K/r(\varepsilon - \theta^n)$ and and that $q > 0$ in (3.2).

(ii) Let $z = \theta^n$, then

$$n\, w(4K/r(\varepsilon - \theta^n)) = \sum_{j=1}^{p} [c_j \ln z/\ln \theta]\, [4K/r(\varepsilon - z)]^{\beta_j} . \tag{3.8}$$

Since the right hand side of (3.8) is unimodal in z and since n decreases as z increases, we conclude that the objective function in (3.7) is unimodal in n. ■

In view of the above, it is clear that the following algorithm provides an efficient means for obtaining a solution of the problem SD(r_0,ε), assuming that the total work function is unimodal in the number of stages s. Our computational results indicate that such an assumption is valid. If any doubt exists, a larger range of integer values of s should be scanned than by the algorithm below.

Algorithm 3.1.

Data: $\varepsilon \in (0,1)$, r_0.

Step 0: Set $\bar{w}_0 = +\infty$ and $s = 1$.

Step 1: Set $\alpha = \varepsilon^{1/s}$. For $i = 1,2,\ldots,s$, compute the solution $(\bar{n}_i^s, \bar{q}_i^s)$ to the problem $1\text{–D}(r_0\, \alpha^{i-1}, \alpha)$ and value $\bar{w}_s = \sum_{i=1}^{s} \bar{n}_i^s\, w(\bar{q}_i^s)$.

Step 2: If $\bar{w}_s < \bar{w}_{s-1}$, then replace s by $s+1$ and go to Step 1. Else, set $\hat{s} = s - 1$, $\hat{\alpha} = \varepsilon^{1/\hat{s}}$, $\hat{n}_i = \bar{n}_i^{s-1}$ and $\hat{q}_i = \bar{q}_i^{s-1}$ for $i = 1,2,\ldots,\hat{s}$, and $\hat{S} = (\hat{s}\,,\, \hat{\alpha}\,,\, \{\,\hat{n}_i\,\}_{i=1}^{\hat{s}}\,,\, \{\,\hat{q}_i\,\}_{i=1}^{\hat{s}})$, to be an optimal strategy for $\mathrm{SD}(r_0,\varepsilon)$, and stop. ■

4. A SPECIAL CASE

In this section we will consider a special case of the simplified optimal diagonalization problem $\mathrm{SD}(r_0,\varepsilon)$, which we will denote by $\mathrm{SDM}(r_0,\varepsilon)$, obtained by assuming that the work function in (3.1) is monomial, i.e., $w(q) = cq^{\beta}$, where $c > 0$ and $\beta > 0$. We will show that a very good approximation to a solution of problem $\mathrm{SDM}(r_0,\varepsilon)$ can be obtained without scanning the positive integers for the optimal number of stages. There are two reasons for considering this special case. The first is that for large values of q, the work function is often quoted as a monomial (in the form $O(q^{\beta})$); the second is that the approximations to optimal strategies for $\mathrm{SDM}(r_0,\varepsilon)$, that we will construct in this section, lead to an efficient scheme for obtaining very good approximations to the optimal strategies for the general case of $\mathrm{SD}(r_0,\varepsilon)$.

To obtain a good approximation to the solution of problem $\mathrm{SDM}(r_0,\varepsilon)$, we relax the requirement that the n_i be integers, and thus embed it into the following relaxed problem:

$$\mathrm{SDM}'(r_0,\varepsilon)\ :\ \min\{\ \sum_{i=1}^{s} n_i\, w(q_i)\ \mid\ 4K/q_i + \theta^{n_i} r_0\, \alpha^{i-1} = \alpha\, r_0\, \alpha^{i-1}\ ,$$

$$n_i > 0\ ,\ q_i > 0\ ,\ \alpha^s \le \varepsilon\ ,\ s \in \mathbb{N}_+\ \}\ , \tag{4.1}$$

where $r_0 > 0$ and $\varepsilon \in (0,1)$ are given, $w(q) = cq^{\beta}$ and $c > 0$, $\beta > 0$.

First we note that the following result is a direct consequence of Lemma 3.1 and Lemma A.2.

Lemma 4.1. If $w(q) = cq^{\beta}$, with c, $\beta > 0$, then

(i) $\gamma(z) = [c \ln z/\ln \theta]\, [4K/r(\varepsilon - z)]^{\beta}$ and $z(r,\varepsilon) = \arg\min_{z \in (0,\varepsilon)} \gamma(z) = z_{\beta}(\varepsilon)$, the unique solution of

$$\varepsilon - z + \beta z \ln z = 0 \,, \qquad z \in (0,\varepsilon) \,. \tag{4.2a}$$

(ii) The unique solution $(\hat{n},\hat{q})$ of 1–**RD**(r,ε) (defined in (3.4)) and the associated cost, are given by

$$\hat{n} = \ln z_{\beta}(\varepsilon)/\ln \theta \,, \tag{4.2b}$$

$$\hat{q} = 4K/r(\varepsilon - z_{\beta}(\varepsilon)) = -4K/\beta\, r\, z_{\beta}(\varepsilon) \ln z_{\beta}(\varepsilon) \,, \tag{4.2c}$$

$$\hat{n}\, w(\hat{q}) = [c \ln z_{\beta}(\varepsilon)/\ln \theta]\, [-4K/\beta\, r\, z_{\beta}(\varepsilon) \ln z_{\beta}(\varepsilon)]^{\beta} \,. \tag{4.2d}$$

■

Remark 4.1. Since $z_{\beta}(\varepsilon)$ does not depend on c, K, θ and r, it follows that if $w(q) = cq^{\beta}$ and $(\hat{n},\hat{q})$ is optimal for 1–**RD**(r,ε), then $\hat{n}$ does not depend on c, K and r, and $\hat{q}$ does not depend on c and θ. ■

The following theorem can be established by following the reasoning used to prove Theorem 3.1.

Theorem 4.1. Suppose that $\hat{S} = (\hat{s}\,,\, \hat{\alpha}\,,\, \{\hat{n}_i\}_{i=1}^{\hat{s}}\,,\, \{\hat{q}_i\}_{i=1}^{\hat{s}})$ is an optimal strategy for **SDM**$'(r_0,\varepsilon)$, then

(i) $\hat{\alpha}^{\hat{s}} = \varepsilon$.

(ii) $(\hat{n}_i,\hat{q}_i)$ is the unique solution of 1–**RD**$(r_0\hat{\alpha}^{i-1},\hat{\alpha})$, and has the following form:

$$\hat{n}_i = \ln z_{\beta}(\varepsilon^{1/\hat{s}})/\ln \theta \,, \qquad \hat{q}_i = (-4K/\beta r_0 z_{\beta}(\varepsilon^{1/\hat{s}}) \ln z_{\beta}(\varepsilon^{1/\hat{s}}))\, (1/\varepsilon^{1/\hat{s}})^{i-1} \,. \tag{4.3}$$

■

It is clear from Theorem 4.1 that the optimal number of stages $\hat{s}$ uniquely determines all the other quantities, $\hat{\alpha}$, $\hat{n}_i$, and $\hat{q}_i$, $i = 1,2,3,\ldots,\hat{s}$. Hence the optimal cost has the form

$$F(\hat{s}) \triangleq \sum_{i=1}^{\hat{s}} (\ln z_{\beta}(\varepsilon^{1/\hat{s}})/\ln \theta)\; c\; [(-4K/\beta r_0 z_{\beta}(\varepsilon^{1/\hat{s}}) \ln z_{\beta}(\varepsilon^{1/\hat{s}}))(1/\varepsilon^{1/\hat{s}})^{i-1}]^{\beta}$$

$$= [c \ln z_{\beta}(\varepsilon^{1/\hat{s}})/\ln \theta]\, [-4K/\beta r_0 z_{\beta}(\varepsilon^{1/\hat{s}}) \ln z_{\beta}(\varepsilon^{1/\hat{s}})]^{\beta}\, [\varepsilon^{\beta/\hat{s}}(1 - \varepsilon^{\beta})/\varepsilon^{\beta}(1 - \varepsilon^{\beta/\hat{s}})] \,. \tag{4.4}$$

Note that (4.4) defines a function $F(\cdot)$ on $(0,\infty)$. We will now prove that $F(\cdot)$ is continuous and unimodal on $(0,\infty)$.

Lemma 4.2. Function $F(\cdot)$ defined in (4.4) is a continuous, unimodal function in $(0,\infty)$ and has a unique minimizer $\hat{s}^*$:

$$\hat{s}^* = \ln \varepsilon / \ln [(\hat{y}_\beta + 1) \exp(-\hat{y}_\beta/\beta)] , \tag{4.5a}$$

where $\hat{y}_\beta$ is the unique solution of the equation

$$1 + y - \exp(y/(\beta+1)) = 0, \qquad y \in (0,\infty) . \tag{4.5b}$$

Proof: From Lemma A.5 and the facts that $\varepsilon^{1/s}$ is strictly increasing and transforms s from $(0,\infty)$ to $(0,1)$ and that $-\beta \ln(\cdot)$ is strictly decreasing and maps $(0,z_\beta^*)$ onto (y_β^*,∞) (where z_β^* is defined in Lemma A.5 and $y_\beta^* \triangleq -\beta \ln z_\beta^*$), we conclude that $-\beta \ln (z_\beta(\varepsilon^{1/s}))$ is strictly decreasing as a function of s and maps $(0,\infty)$ onto (y_β^*,∞) . Now, let $y = -\beta \ln (z_\beta(\varepsilon^{1/s}))$. Then $z_\beta(\varepsilon^{1/s}) = \exp(-y/\beta)$ and $\varepsilon^{1/s} = (y+1) \exp(-y/\beta)$. Hence

$$F(s) = [-c\,(1-\varepsilon^\beta)/\beta\, \varepsilon^\beta \ln \theta]\, [4K/r_0]^\beta \left[\frac{(y+1)^\beta}{y^{\beta-1}\,(1-(y+1)^\beta \exp(-y))} \right] . \tag{4.6}$$

Since $y \in (y_\beta^*,\infty)$ and $\ln \theta < 0$, we conclude from Lemma A.8 that the right hand side of (4.6) is a continuous, unimodal function of y and has unique minimizer $\hat{y}_\beta$, which is the unique solution of (4.5b). Making use of the fact that the variable transformation $y = -\beta \ln (z_\beta(\varepsilon^{1/s}))$ is strictly decreasing, we claim that $F(\cdot)$ is unimodal function in $(0,\infty)$ and has a unique minimizer satisfying (4.5a). ■

The unimodality of $F(\cdot)$ and Theorem 4.1 lead to the following theorem.

Theorem 4.2. Suppose that $\hat{s}^*$ is defined as in (4.5a). Let $\hat{s} = \mathrm{argmin}\{ F(s) \mid s \in \{ \lfloor \hat{s}^* \rfloor, \lceil \hat{s}^* \rceil \} \}$, let $\hat{\alpha} = \varepsilon^{1/\hat{s}}$, and let $\hat{n}_i,\ \hat{q}_i,\ i = 1,2,\ldots,\hat{s}$ be defined by (4.3). Then $\hat{S} = (\hat{s}\ ,\ \hat{\alpha}\ ,\ \{\hat{n}_i\}_{i=1}^{\hat{s}}\ ,\ \{\hat{q}_i\}_{i=1}^{\hat{s}})$ is an optimal strategy for $\mathrm{SDM}'(r_0,\varepsilon)$. ■

Remark 4.2. Note that if $\hat{s}^*$, defined by (4.5a), is an integer, then $\hat{s}^*$ is the optimal number of stages for $\mathrm{SDM}'(r_0,\varepsilon)$. In this case, it follows from (4.5a) and the fact that $\hat{\alpha} = \varepsilon^{1/\hat{s}}$, that $\hat{\alpha} = \hat{\alpha}_\beta \triangleq (\hat{y}_\beta + 1) \exp(-\hat{y}_\beta/\beta)$, which depends only on β.

■

Theorem 4.2 leads to the following fast algorithm for the solution of problem SDM(r_0,ε).

Algorithm 4.1.

Data: r_0, $\varepsilon \in (0,1)$, $\beta > 0$.

Step 1: Find the solution $\hat{y}_\beta$ of (4.5b) and set $\hat{\alpha}_\beta = (\hat{y}_\beta + 1) \exp(-\hat{y}_\beta/\beta)$.

Step 2: If $F(\lfloor \ln \varepsilon / \ln \hat{\alpha}_\beta \rfloor) \le F(\lceil \ln \varepsilon / \ln \hat{\alpha}_\beta \rceil)$, set $\hat{s} = \lfloor \ln \varepsilon / \ln \hat{\alpha}_\beta \rfloor$. Else set $\hat{s} = \lceil \ln \varepsilon / \ln \hat{\alpha}_\beta \rceil$.

Step 3: Set $\hat{\alpha} = \varepsilon^{1/\hat{s}}$. For $i = 1,2,\ldots,\hat{s}$, compute the solution $(\hat{n}_i,\hat{q}_i)$ to the problem 1–D$(r_0\hat{\alpha}^{i-1},\hat{\alpha})$. Set $\hat{S} = (\hat{s}, \hat{\alpha}, \{\hat{n}_i\}_{i=1}^{\hat{s}}, \{\hat{q}_i\}_{i=1}^{\hat{s}})$, to be an approximate solution to SDM(r_0,ε), and stop. ■

One can also modify Algorithm 4.1 to obtain an algorithm which yields an approximate solution to the problem SD(r_0,ε). This modification, stated below, in stage i, approximates the work function $w(q)$, by a monomial of the form $c_i q^{\beta_i}$, and then calls Algorithm 4.1, as a subprocedure. The algorithm, below, requires an initial discretization parameter $\hat{q}_0$.

Algorithm 4.2.

Data: $\varepsilon \in (0,1)$, r_0, $\hat{q}_0$.

Step 0: Set $\hat{r}_0 = r_0$ and $i = 1$.

Step 1: Set $\beta_i = \hat{q}_{i-1}\, w'(\hat{q}_{i-1})/w(\hat{q}_{i-1})$ and $c_i = w(\hat{q}_{i-1})/\hat{q}_{i-1}^{\beta_i}$, which yields the monomial $c_i q^{\beta_i}$ matching the value and derivative of $w(\cdot)$ at $\hat{q}_{i-1}$.

Step 2: Find the solution y_i of (4.5b), for $\beta = \beta_i$, and set $\alpha_i = (y_i + 1)\exp(-y_i/\beta_i)$.

Step 3: Set s_i to be the nearest integer to $\ln(\varepsilon \hat{r}_0/\hat{r}_{i-1})/\ln \alpha_i$ and $\hat{\varepsilon}_i = (\varepsilon \hat{r}_0/\hat{r}_{i-1})^{1/s_i}$. Compute the solution $(\hat{n}_i,\hat{q}_i)$ to the problem 1–D$(\hat{r}_{i-1},\hat{\varepsilon}_i)$ where $w(q) = c_i \cdot q^{\beta_i}$.

Step 4: Set $\hat{r}_i = \hat{\varepsilon}_i \hat{r}_{i-1}$. If $\hat{r}_i \le \varepsilon \hat{r}_0$, set $\hat{s} = i$, and $\hat{S} = (\hat{s}, \{\hat{n}_i\}_{i=1}^{\hat{s}}, \{\hat{q}_i\}_{i=1}^{\hat{s}})$, to be an approximate solution to SD(r_0,ε), and stop. Else, replace i by $i + 1$ and go to step 1. ■

5. IMPLEMENTATION OF DIAGONALIZATION ALGORITHMS

In the first part of this section we will report on computational experiments aimed at comparing the solutions to the problems D(r_0,ε) and SD(r_0,ε), as well as on the performance of Algorithms 2.2, 3.1 and 4.2, and draw conclusions as to their relative merits. In the second part of this section we will present an implementation of the Successive Approximation Algorithm 2.1, and report on its use in solving two control design problems. The implementation supplements the Successive Approximation Algorithm 2.1 with procedures for estimating the constant K in (2.3), the convergence constant θ in (2.5), the work function $w(q)$ and the cost-errors $\hat{r}_i$.

In our computational experiments, Algorithm 2.2 was implemented using a penalty method in conjunction with the global, multi-start optimization routine ZXMWD, in the IMSL library (see [IMS.1]), for solving (2.17). Our computational experiments indicate that Algorithms 3.1 and 4.2 yield almost as good diagonalization strategies as Algorithm 2.2, but require much less cpu time to compute. In particular, our computational experiments indicate that the assumption, that the estimated cost-reduction ratios are the same in each stage, results in very little degradation of the resulting diagonalization strategy.

Table 5.1, below, presents a comparison of the results yielded by Algorithm 2.2, Algorithm 3.1 and Algorithm 4.2, on a typical diagonalization problem, where $K = 10.0$, $\theta = 0.8$, $\varepsilon = 0.001$, $r_0 = 100.0$, $\hat{q}_0 = 16.0K/r_0$ and $w(q) = 40q + q^2$. In Table 5.1, $\hat{w} \triangleq \sum_{i=1}^{\hat{s}} \hat{n}_i\, w(\hat{q}_i)$. The algorithms were executed on a VAX-11/780 computer.

Algorithm	$\hat{w}$	$\hat{s}$	$\hat{r}_{\hat{s}}$	cpu time (secs.)
Algorithm 2.2	3.631724e+06	5	0.1022	389.67
Algorithm 3.1	3.710678e+06	7	0.1000	0.32
Algorithm 4.2	3.754011e+06	6	0.1000	0.14

Table 5.1. Results for a Typical Diagonalization Problem

As we have already indicated, one may have to estimate some or all of the data, i.e., K, θ, $w(\cdot)$ and $\hat{r}_i$, needed for computing an optimal diagonalization strategy. Since in Algorithm 4.2 we obtain the number of iterations $\hat{n}_i$ and the discretization parameter $\hat{q}_i$ one stage at a time, we can estimate the required data as we go along. This fact is incorporated in the following algorithm which is an implementation of the Successive Approximation Algorithm 2.1, for solving problem (2.1a) where $\psi(x) = \max_{k \in \mathbf{m}} \max_{\eta^k \in [\eta_0^k, \eta_1^k]} \phi^k(x, \eta^k)$. For a given set of positive weights $\{ \sigma_k \}_{k \in \mathbf{m}}$, the approximating functions $\psi_q(x)$ are defined as follows:

$$\psi_q(x) = \max_{k \in \mathbf{m}} \max_{t_k = 0,1,\ldots,\lceil \sigma_k q \rceil} \phi^k(x, \eta_0^k + t_k l_k / \lceil \sigma_k q \rceil) , \tag{5.1}$$

where $l_k = \eta_1^k - \eta_0^k$.

Algorithm 5.1.

Data: $x_0 \in \mathbb{R}^n$, $\varepsilon \in (0,1)$, $\hat{q}_0 > 0$, $\theta_0 \in (0,1)$, ψ_0 and $\beta_0 > 0$.

Step 0: Set $x_0^0 = x_0$, $\hat{n}_0 = 0$. Compute K_0 and r_0 according to (5.8a) and (5.8c), respectively, with $i = 0$. Then, set $i = 1$.

Step 1: Find the unique solution y_i of following equation:

$$1 + y - \exp (y/(\beta_{i-1} + 1)) = 0, \qquad y \in (0,\infty) . \tag{5.2}$$

Set $\alpha_i = (y_i + 1) \exp(-y_i / \beta_{i-1})$.

Step 2: Set s_i to be the nearest integer to $\ln (\varepsilon\, r_0 / r_{i-1}) / \ln \alpha_i$ and $\hat{\varepsilon}_i = (\varepsilon\, r_0 / r_{i-1})^{1/s_i}$. Compute [2]

$$(\hat{n}_i, \hat{q}_i) = \arg \min_{\substack{n \in \mathbb{N}_+ \\ q > 0}} \{ \, n\, q^{\beta_{i-1}} \mid 2K_{i-1}/q + \theta_{i-1}^n\, r_{i-1} \le \hat{\varepsilon}_i r_{i-1} \} . \tag{5.3}$$

Step 3: Set $x_0^i = x_{\hat{n}_{i-1}}^{i-1}$. For $j = 1,2,\ldots,\hat{n}_i$, compute

$$x_j^i = A(x_{j-1}^i , \psi_{\hat{q}_i} , X) , \tag{5.4}$$

and store w_j^i, the cpu time needed to compute x_j^i.

[2] Since $\psi_q(x) \le \psi(x)$ for all x and q, the inequality (2.9) can be tightened by replacing $4K$ by $2K$. Hence we use $2K_i$ instead of $4K_i$ in (5.3).

Step 4: Set the average computational work per iteration, for $q = \hat{q}_i$, to be

$$w_i = \Big(\sum_{j=1}^{\hat{n}_i} w_j^i \Big) / \hat{n}_i \, . \tag{5.5}$$

Step 5: For $i = 1$, set $\beta_i = \beta_0$. For $i > 1$, solve the linear least squares minimization problem, below, to obtain

$$(\beta_i, c_i) = \arg \min_{\beta, c} \sum_{j=1}^{i} [\ln (w_j) - \beta \ln (\hat{q}_j) - \ln(c)]^2 \, , \tag{5.6}$$

Step 6: Compute the estimate θ_i of θ as follows:

(a) set $a_0 = \theta_{i-1}$.

(b) For $t = 1,2,\ldots,$ until $| a_t - a_{t-1} | < 0.001$, solve the linear least squares minimization problem

$$(a_t, b_t) = \arg \min_{a,b} \sum_{j=0}^{\hat{n}_i} [\ln (\psi_{\hat{q}_i}(x_j^i) - d_t) - j \ln(a) - b]^2 \, , \tag{5.7a}$$

where

$$d_t \triangleq \frac{1}{\hat{n}_i} \sum_{j=0}^{\hat{n}_i - 1} [\psi_{\hat{q}_i}(x_{\hat{n}_i}^i) - a_{t-1}^{\hat{n}_i - j} \psi_{\hat{q}_i}(x_j^i)] / [1 - a_{t-1}^{\hat{n}_i - j}] \, . \tag{5.7b}$$

(c) Set $\theta_i = a_t$.

Step 7: Compute the estimates K_i, ψ_i, r_0 and r_i of K, $\hat{\psi}$, $\hat{r}_0$, and $\hat{r}_i$, respectively, as follows:

$$K_i = \max_{k \in \mathbf{m}} \max_{\substack{j = 0,\ldots,\hat{n}_i \\ t_k = 1,\ldots,\lceil \sigma_k \hat{q}_i \rceil}} \frac{| \phi^k(x_j^i, \eta_0^k + (t_k - 1) l_k / \lceil \sigma_k \hat{q}_i \rceil) - \phi^k(x_j^i, \eta_0^k + t_k l_k / \lceil \sigma_k \hat{q}_i \rceil) | \lceil \sigma_k \hat{q}_i \rceil}{2\sigma_k} \, , \tag{5.8a}$$

$$\psi_i = \min_{j = 0,\ldots,\hat{n}_i - 1} [\psi_{\hat{q}_i}(x_{\hat{n}_i}^i) - \theta_i^{\hat{n}_i - j} \psi_{\hat{q}_i}(x_j^i)] / [1 - \theta_i^{\hat{n}_i - j}] - K_i / \hat{q}_i \, , \tag{5.8b}$$

$$r_0 = \psi_{\hat{q}_i}(x_0) + K_i / \hat{q}_i - \psi_i \, , \tag{5.8c}$$

$$r_i = \psi_{\hat{q}_i}(x^i_{\hat{n}_i}) + K_i/\hat{q}_i - \psi_i \,. \tag{5.8d}$$

Step 8: If $r_i \le \varepsilon r_0$, stop. Otherwise, replace i by $i+1$ and go to step 1. ■

In Steps 1 and 2 of Algorithm 5.1, Algorithm 4.2 is used to compute $\hat{n}_i$ and $\hat{q}_i$ in terms of K_{i-1}, θ_{i-1}, β_{i-1}, r_0 and r_{i-1}, which are the successively improved estimates of the quantities K, θ, β, $\hat{r}_0$ and $\hat{r}_{i-1}$. In Steps 4-7 of Algorithm 5.1, all the quantities are estimated in terms of the previous estimates and the stored function values. Note that the procedure for obtaining θ_i in Step 6 is very robust and gives the exact rate of convergence θ provided that the sequence $\{\ \psi_{\hat{q}_i}(x^i_j)\ \}_{j=0}^{\hat{n}_i}$ is exactly linear.

In our numerical experiments, the solution y_i of algebraic equation (5.2) was found by bisection; on the basis of Theorem 3.2, the solution of (5.3) was computed by scanning the positive integers. The iteration map A in (5.4) was defined as one iteration of the Pshenichnyi minimax algorithm (see [Psh.1, Pol.3 (Algorithm 5.2 modification)]). The left hand sides in (5.6) and (5.7a) were calculated by solving linear least squares problems in two unknown variables, $(\beta, \ln c)$ and $(\ln a, b)$, respectively. The left hand side of (5.8a) was obtained in the process of computing $\psi_{\hat{q}_i}(x^i_j)$ for $j = 0,1,\ldots,\hat{n}_i$ in the Step 3. Therefore the computational cost incurred in Steps 1-2 and Steps 4-7 is quite small, compared with the computational cost incurred in Step 3, where the design parameters are iterated.

To evaluate the effectiveness of Algorithm 5.1, we have compared it with a fixed discretization scheme, in which the precision parameter q was set to be the smallest value compatible with the precision required, as well as with the well-tried adaptive precision method in [Kle.1], below.

Adaptive Precision Algorithm 5.2.

Data: $x_0 \in \mathbb{R}^n$, $\hat{q}_0 > 0$, $\delta_0 > 0$, $\gamma_1 \in (0,1)$, $\gamma_2 > 1$.

Step 0: Set $n_0 = 0$, $x^0_{n_0} = x_0$, $i = 1$, $\delta_i = \delta_0$ and $\hat{q}_i = \hat{q}_0$.

Step 1: Set $x^i_0 = x^{i-1}_{n_{i-1}}$ and $j = 0$.

Step 2: Compute $x^i_{j+1} = A(\, x^i_j\,,\, \psi_{\hat{q}_i}\,,\, X\,)$.

Step 3: If $\psi_{\hat{q}_i}(x^i_{j+1}) - \psi_{\hat{q}_i}(x^i_j) > -\delta_i$, set $n_i = j + 1$ and go to Step 5. Else, go to step 4.

Step 4: Set $j = j + 1$, go to Step 2.

Step 5: Set $i = i + 1$, $\delta_i = \gamma_1 \delta_{i-1}$ and $\hat{q}_i = \gamma_2 \hat{q}_{i-1}$. Go to Step 1. ∎

Tables 5.2 and 5.3 below, compare results obtained using Algorithms 5.1, 5.2 and the Pshenichnyi minimax algorithm (see [Psh.1]), with a fixed discretization, in two designs of a stabilizing controller which minimizes frequency domain tracking error, for the 2-input, 2-output feedback system in Fig.1. In the first design the plant was defined by (5.9a), while in the second one it was defined by (5.9b), below:

$$P_1(s) = \frac{1}{(s+2)^2(s+3)} \begin{bmatrix} s^2 + 8s + 10 & 3s^2 + 7s + 4 \\ 2s + 2 & 3s^2 + 9s + 8 \end{bmatrix}, \tag{5.9a}$$

$$P_2(s) = \frac{1}{(s+2)(s+3)} \begin{bmatrix} s+3 & s+3 \\ s-2 & -s-5 \end{bmatrix}. \tag{5.9b}$$

Using Q-parametrization (see [You.1]), with

$$Q(x,s) = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} + \frac{1}{(s+6)} \begin{bmatrix} x_5 & x_6 \\ x_7 & x_8 \end{bmatrix}, \tag{5.9c}$$

the two design problems were transcribed into the form (2.1a), with $\psi(x) = \max_{\omega \in [0,2]} \phi_1(x,\omega)$ and $\psi(x) = \max_{\omega \in [0,2]} \phi_2(x,\omega)$, respectively, where

$$\phi_1(x,\omega) \triangleq \sigma\,[I - P_1(j\omega)\, Q(x,j\omega)], \tag{5.9d}$$

$$\phi_2(x,\omega) \triangleq \sigma\,[I - P_2(j\omega)\, Q(x,j\omega)], \tag{5.9e}$$

$\sigma[\,A\,]$ denotes the largest singular value of the matrix A, and I is the 2x2 identity matrix.

For both designs, we required that the initial cost-error r_0 be reduced by $\varepsilon = 0.0001$. The initial point for the first design problem was (30,−20,30,30,50,−20,30,40), while the initial point for the second design problem was (10,−10,10,10,10,−10,10,−10). All three algorithms tested used the same iteration map A defined by the Pshenichnyi minimax algorithm (see [Psh.1]) and the same approximating functions $\psi_q(\cdot)$ defined by (5.1) with $\sigma_1 = 1$. The initial cost-error r_0, the required final cost value, as well as the constant K, were all estimated in the first series of runs, using Algorithm 5.1, and then were used to define a stopping rule and minimum discretization level for the other two methods.

(a) We initialized Algorithm 5.1 with $\varepsilon = 0.0001$, $\hat{q}_0 = 3.0$, $\theta_0 = 0.8$, $\psi_0 = 0.0$, and $\beta_0 = 2.0$ for both design problems. The required final number of discretization points was found to be 346 points for the first problem and 256 for the second one.

(b) We carried out two runs using the Adaptive Precision Algorithm 5.2 for each problem. In the first run we set $\hat{q}_0 = 3$, $\delta_0 = 0.1$, $\gamma_1 = 0.5$, $\gamma_2 = 1.5$, while in the second run we kept the same values of $\hat{q}_0$, δ_0 and γ_1, and set $\gamma_2 = 1.25$. The computation was stopped when the number of discretization points used was at least 346 and 256, for the first and second design problems, respectively, and the cost value was approximately equal to that obtained by Algorithm 5.1.

(c) For the fixed discretization design we used 346 points in the first design problem and 256 points in the second one. The computation was stopped when the cost value was approximately equal to that obtained by Algorithm 5.1.

The results, produced on a Sun 3/140 computer, are shown in Table 5.2 and 5.3, where the final cost value is denoted by $\bar{\psi}$, while the final number of discretization points used is denoted by $\bar{p}$. Note that of the 4577.34 cpu secs. used by Algorithm 5.1 for the first design problem, 4391.42 cpu secs. were used in Step 3, computing the successive iterates x_j^i while of the 6487.38 cpu secs. used by Algorithm 5.1 for the second design problem, 6370.44 cpu secs. were used in Step 3, which shows that the overhead in computing an approximation to an optimal diagonalization strategy is quite small relative to the benefits which it yields.

Algorithm	$\bar{\psi}$	$\bar{p}$	cpu time (secs.)
Algorithm 5.1	0.296241	346	4577.34
Algorithm 5.2, first run	0.296960	1232	146146.88
Algorithm 5.2, second run	0.296048	362	28877.18
Fixed discretization scheme	0.296397	346	79905.00

Table 5.2. Results for the First Design Problem

Algorithm	$\bar{\psi}$	$\bar{p}$	cpu time (secs.)
Algorithm 5.1	0.172502	256	6487.38
Algorithm 5.2, first run	0.172597	1849	171641.38
Algorithm 5.2, second run	0.172530	289	25572.06
Fixed discretization scheme	0.172588	256	88631.20

Table 5.3. Results for the Second Design Problem

6. CONCLUSION

We have shown that it is possible to obtain optimal diagonalization strategies for the discretization of semi-infinite minimax optimal design problems. We propose both exact and approximate methods for the computation of these optimal diagonalization strategies. The algorithms for computing approximate diagonalization strategies yield very good approximations in much less computing time than needed to compute an optimal diagonalization strategy exactly. Our optimal diagonalization strategies can be implemented by using estimation schemes to obtain approximations to the various quantities which determine an optimal strategy. Our experimental results, involving the solution of optimal loop-shaping problems for multivariable linear feedback systems, show that the the use of these implementable strategies leads to considerable savings in computing time over alternative approaches.

7. APPENDIX: TECHNICAL RESULTS

We now present a number of technical results which were used in our proofs. All these results depend on a parameter $\beta > 0$.

Lemma A.1. Let $\beta > 0$. Then, for any $\varepsilon \in (0,1)$, the equation

$$\varepsilon - z + \beta z \ln z = 0 , \qquad z \in (0,\varepsilon) , \tag{A.1}$$

has a unique solution.

Proof: Let $z \in (0,\varepsilon)$ and let

$$g_\varepsilon(z) \triangleq \varepsilon - z + \beta z \ln z . \tag{A.2}$$

Then $g_\varepsilon''(z) = \beta/z > 0$ for all $z \in (0,\varepsilon)$. Hence $g_\varepsilon(\cdot)$ is strictly convex on $(0,\varepsilon)$. Since

$g_\varepsilon(0+) = \varepsilon > 0$ and $g_\varepsilon(\varepsilon) = \beta\, \varepsilon \ln \varepsilon < 0$, it follows that there exists a unique zero of $g_\varepsilon(\cdot)$ in $(0,\varepsilon)$ ∎

Definition A.1. Let $\beta > 0$ be given. For every $\varepsilon \in (0,1)$, we shall denote by $z_\beta(\varepsilon)$ the unique solution of the equation (A.1). ∎

Lemma A.2. Let $\beta > 0$ and $\varepsilon \in (0,1)$. Then (i)

$$\max_{0<z<\varepsilon} \ln z/(\varepsilon - z)^\beta = \ln z_\beta(\varepsilon)/[\, -\beta\, z_\beta(\varepsilon) \ln z_\beta(\varepsilon)\,]^\beta , \tag{A.3}$$

and $z_\beta(\varepsilon)$ is the unique solution of the above maximization problem. (ii) If $\beta \geq 1$, then $(\ln z/(\varepsilon - z)^\beta)$ is strictly concave on $(0,\varepsilon)$.

Proof: (i) For $z \in (0,\varepsilon)$, let

$$f_\varepsilon(z) \triangleq \ln z/(\varepsilon - z)^\beta . \tag{A.4}$$

and let $g_\varepsilon(z)$ be defined as in (A.2). Then

$$f_\varepsilon'(z) = g_\varepsilon(z)/z\,(\varepsilon - z)^{\beta+1} . \tag{A.5}$$

It now follows from (A.5), Lemma A.1 and the fact that (i) $g_\varepsilon(z) > 0$ for all $z \in (0, z_\beta(\varepsilon))$, and (ii) $g_\varepsilon(z) < 0$ for all $z \in (z_\beta(\varepsilon), \varepsilon)$, that $z_\beta(\varepsilon)$ is the unique solution of the maximization problem. Since $\varepsilon - z_\beta(\varepsilon) = -\beta\, z_\beta(\varepsilon) \ln z_\beta(\varepsilon)$, we obtain (A.3).

(ii) We claim that f_ε is concave if $\beta \geq 1$. Now

$$f_\varepsilon(z) = \ln (z/\varepsilon)/[\varepsilon^\beta\,(1 - z/\varepsilon)^\beta] + \ln \varepsilon/[\varepsilon^\beta\,(1 - z/\varepsilon)^\beta] . \tag{A.6}$$

Since $1/(1 - z/\varepsilon)^\beta$ is strictly convex in $(0,\varepsilon)$ and $\ln \varepsilon < 0$, it follows that the second term on the right hand side of (A.6) is strictly concave. Hence it suffices to show that $\ln z/(1 - z)^\beta$ is concave on (0,1). Let $f(z) = \ln z/(1 - z)^\beta$. Then

$$f'' = h(z)/z^2\,(1 - z)^{\beta+2} , \tag{A.7a}$$

where

$$h(z) = (1 - z)\,((2\beta + 1)z - 1) + \beta\,(\beta + 1) z^2 \ln z . \tag{A.7b}$$

Let $\hat{z} = \arg\max_{z \in [0,1]} h(z)$. We will show that $h(\hat{z}) \leq 0$ which ensures that $f(\cdot)$ is concave on (0,1). If $\hat{z} = 0$ or 1, then $h(\hat{z}) \leq 0$. If $\hat{z} \in (0,1)$, then $h'(\hat{z}) = 0$, i.e.,

$$(2\beta + 2) - 2(2\beta + 1)\hat{z} + \beta(\beta + 1)\hat{z} + 2\beta(\beta + 1)\hat{z}\ln\hat{z} = 0\,. \tag{A.8}$$

Multiplying (A.8) by $\hat{z}$ and rearranging , we obtain

$$\beta(\beta + 1)\hat{z}^2\ln\hat{z} = -(\beta + 1)\hat{z} + (2\beta + 1)\hat{z}^2 - [\beta(\beta + 1)/2]\hat{z}^2\,. \tag{A.9}$$

Hence

$$\begin{aligned} h(\hat{z}) &= (1 - \hat{z})((2\beta + 1)\hat{z} - 1) + \beta(\beta + 1)\hat{z}^2\ln\hat{z} \\ &= (1 - \hat{z})((2\beta + 1)\hat{z} - 1) - (\beta + 1)\hat{z} + (2\beta + 1)\hat{z}^2 - [\beta(\beta + 1)/2]\hat{z}^2 \\ &= -[\beta(\beta + 1)/2]\,(\hat{z} - 1/\beta)^2 - (\beta - 1)/2\beta \;\leq\; 0\,. \end{aligned} \tag{A.10}$$

Therefore $h(\hat{z}) \leq 0$ irrespective of whether $\hat{z} = 0$, or $\hat{z} = 1$, or $\hat{z} \in (0,1)$. Hence f is concave on (0,1). ■

The following result is obvious:

Lemma A.3. For all $\beta \in (1,\infty)$,

$$1 - (1/\beta) + \ln(1/\beta) < 0. \tag{A.11}$$ ■

Lemma A.4. Let $\beta > 0$ be given and let z_β^* be the smallest zero of the equation

$$1 - z + \beta z \ln z = 0\,, \qquad z \in (0,+\infty)\,. \tag{A.12}$$

Then

(i) $1 - z + \beta z \ln z$ is strictly decreasing in $(0,z_\beta^*)$;

(ii) $$z_\beta^* \begin{cases} = 1 & \text{if } \beta \in (0,1] \\ < 1/\beta & \text{if } \beta \in (1,\infty) \end{cases}\;; \tag{A.13a}$$

(iii) $$1 - z + \beta z \ln z \begin{cases} > 0 & \text{if } z \in (0,z_\beta^*) \\ < 0 & \text{if } z \in (z_\beta^*,1)\,. \end{cases} \tag{A.13b}$$

Proof: Let

$$g_1(z) \triangleq 1 - z + \beta z \ln z\,, \qquad z \in (0,+\infty). \tag{A.14}$$

Since $g_1''(z) = \beta/z > 0$ for all $z \in (0,\infty)$, $g_1(\cdot)$ is strictly convex on $(0,\infty)$. If $\beta \in (0,1]$, then the results follow from the facts that $g_1(\cdot)$ is strictly convex on $(0,\infty)$, that $g_1(+0) = 1 > 0$, that

$g_1(1) = 0$ and that $g_1'(1) = -1 + \beta \le 0$. If $\beta \in (1,+\infty)$, it follows from Lemma A.3 that $g_1(1/\beta) = 1 - (1/\beta) + \ln(1/\beta) < 0$. Making use of the facts that $g_1(\cdot)$ is strictly convex on $(0,\infty)$, that $g_1(+0) = 1 > 0$, that z_β^* is the smallest zero of $g_1(\cdot)$ in $(0,\infty)$, that $g_1(1/\beta) < 0$ and that $g_1(1) = 0$, we again obtain (i)-(iii). ■

Lemma A.5. Let $\beta > 0$ be given and let $z_\beta^* \in (0,+\infty)$ be the smallest zero of equation (A.12). Then the function $z_\beta(\cdot)$ defined by Definition A.1 is continuous, strictly increasing, and it maps $(0,1)$ onto $(0,z_\beta^*)$.

Proof: Let

$$p(z) \triangleq z - \beta z \ln z, \qquad \forall\, z \in (0,z_\beta^*). \tag{A.15}$$

Then it follows from Lemma A.4(i) that $p(\cdot)$ is strictly increasing on $(0,z_\beta^*)$. Making use of the fact that $p(0+) = 0$ and $p(z_\beta^*) = 1$, we conclude that $p(\cdot)$ is one-to-one from $(0,z_\beta^*)$ to $(0,1)$. Now, for $\varepsilon \in (0,1)$, it follows from the Definition A.1 that

$$1 - z_\beta(\varepsilon) + \beta z_\beta(\varepsilon) \ln z_\beta(\varepsilon) = 1 - \varepsilon > 0, \tag{A.16}$$

and that $z_\beta(\varepsilon) \in (0,\varepsilon)$. Making use of Lemma A.4 (iii), we conclude that $z_\beta(\varepsilon) \in (0,z_\beta^*)$ for all $\varepsilon \in (0,1)$. Since

$$p(z_\beta(\varepsilon)) = z_\beta(\varepsilon) - \beta z_\beta(\varepsilon) \ln z_\beta(\varepsilon) = \varepsilon, \tag{A.17}$$

and since $p(\cdot)$ is continuous, strictly increasing and one-to-one, we conclude that $p(\cdot)$ is the inverse of $z_\beta(\cdot)$. Hence, $z_\beta(\cdot)$ is continuous and strictly increasing, and it maps $(0,1)$ onto $(0,z_\beta^*)$. ■

Lemma A.6. Let $\beta > 0$ be given, let $z_\beta^* \in (0,+\infty)$ be the smallest zero of equation (A.12), and let $y_\beta^* \triangleq -\beta \ln z_\beta^*$. Then

(i) $\exp(y_\beta^*/\beta) - y_\beta^* - 1 = 0$; (A.18a)

(ii) $y_\beta^* \begin{cases} = 0 & \text{if } \beta \in (0,1] \\ > \beta - 1 & \text{if } \beta \in (1,\infty) \end{cases}$; (A.18b)

(iii) $1 - (y+1)\exp(-y/\beta) \begin{cases} < 0 & \text{if } y \in (0,y_\beta^*) \\ > 0 & \text{if } y \in (y_\beta^*,\infty) \end{cases}$. (A.18c)

Proof: (i) Since $z_\beta^* = \exp(-y_\beta^*/\beta)$ and z_β^* is the solution of the equation (A.12), we get

(A.18a).

(ii) Making use of Lemma A.4 (ii) and Lemma A.3, we obtain (A.18b).

(iii) Making use of Lemma A.4 (iii) and the fact that the one-to-one function $y(z) \triangleq -\beta \ln z$ maps $(0,z_\beta^*)$ and $(z_\beta^*,1)$ into (y_β^*,∞) and $(0,y_\beta^*)$, respectively, we obtain (A.18c). ■

Lemma A.7. Let $\beta > 0$ be given. Then the equation

$$1-(1+y)^{\beta+1}\exp(-y) = 0\,, \qquad y \in (0,\infty)\,, \tag{A.19a}$$

has a unique solution which will be denoted by $\hat{y}_\beta$. Furthermore, the following hold, with y_β^* defined as in Lemma A.6:

$$\text{(i)} \quad 1-(1+y)^{\beta+1}\exp(-y) \begin{cases} <0 & \text{if } y \in (0,\hat{y}_\beta) \\ >0 & \text{if } y \in (\hat{y}_\beta,\infty)\,; \end{cases} \tag{A.19b}$$

$$\text{(ii)} \quad \hat{y}_\beta > y_\beta^*\,. \tag{A.19c}$$

Proof: Observe that (A.19a) is equivalent to

$$\exp\,(y/(\beta+1)) - y - 1 = 0\,, \qquad y \in (0,\infty)\,. \tag{A.20}$$

Let $h(y) \triangleq \exp\,(y/(\beta+1)) - y - 1$ for all $y \in (-\infty,\infty)$. Since $h(0) = 0$, $h'(0) = -\beta/(\beta+1) < 0$ and since $h(\infty) = \infty$, we conclude that there exists a $\hat{y}_\beta \in (0,\infty)$ such that $h(\hat{y}_\beta) = 0$. Since $h''(y) = \exp(y/(\beta+1))/(\beta+1)^2 > 0$, $h(\cdot)$ is strictly convex on $(-\infty,+\infty)$. Hence it has at most two zeroes. But $h(0) = 0$ and $h(\hat{y}_\beta) = 0$, which leads to the conclusion that $\hat{y}_\beta$ is the only zero of $h(\cdot)$ in $(0,\infty)$. Furthermore, $h(y) < 0$ for all $y \in (0,\hat{y}_\beta)$ and $h(y) > 0$ for all $y \in (\hat{y}_\beta,\infty)$. Hence, (A.19b) is true.

Next we will establish (A.19c). If $\beta \in (0,1]$, then, since $y_\beta^* = 0$ by Lemma A.6 (ii), (A.19c) is obvious. If $\beta \in (1,\infty)$, then, since $y_\beta^* > 0$, we have

$$h(y_\beta^*) = \exp\,(y_\beta^*/(\beta+1)) - y_\beta^* - 1 < \exp\,(y_\beta^*/\beta) - y_\beta^* - 1\,. \tag{A.21}$$

Since y_β^* satisfies (A.18a), $h(y_\beta^*) < 0$. Thus, (A.19c) follows from the facts that $h(y) < 0$ for all $y \in (0,\hat{y}_\beta)$ and that $h(y) > 0$ for $y \in (\hat{y}_\beta,\infty)$.

■

Lemma A.8. Let $\beta > 0$ be given and let $\hat{y}_\beta$ be the unique solution of (A.19a). Then $\hat{y}_\beta$ is the unique solution of the minimization problem

$$\min \left\{ \frac{(y+1)^\beta}{y^{\beta-1}\,[\,1-(y+1)^\beta \exp(-y)\,]} \;\middle|\; y \in (y_\beta^*, \infty) \right\} . \tag{A.22}$$

and the objective function in (A.22) is unimodal.

Proof: For $y \in (y_\beta^*, \infty)$, let

$$f(y) \triangleq \frac{(y+1)^\beta}{y^{\beta-1}\,[\,1-(y+1)^\beta \exp(-y)\,]} . \tag{A.23}$$

By Lemma A.6 (iii), the denominator of $f(\cdot)$ is positive in (y_β^*, ∞) . Hence $f(\cdot)$ is differentiable in (y_β^*, ∞) . After lengthy calculation we obtain

$$f'(y) = \frac{[\,1-(y+1)^{\beta+1}\exp(-y)\,]\,(y-\beta+1)\,(y+1)^{\beta-1}}{y^\beta\,[\,1-(y+1)^\beta \exp(-y)\,]^2} . \tag{A.24}$$

It follows from Lemma A.6 (ii) and (iii), that all the right hand side terms, except the first term of the numerator, are positive for all $y \in (y_\beta^*, \infty)$. Hence, by Lemma A.7, $f'(\cdot)$ has only one zero $\hat{y}_\beta$ in (y_β^*, ∞) . Furthermore, $f'(y) < 0$ for all $y \in (0, \hat{y}_\beta)$ and $f'(y) > 0$ for all $y \in (\hat{y}_\beta, \infty)$. Therefore $\hat{y}_\beta$ is the unique solution of the minimization problem (A.22) and $f(\cdot)$ is unimodal. ■

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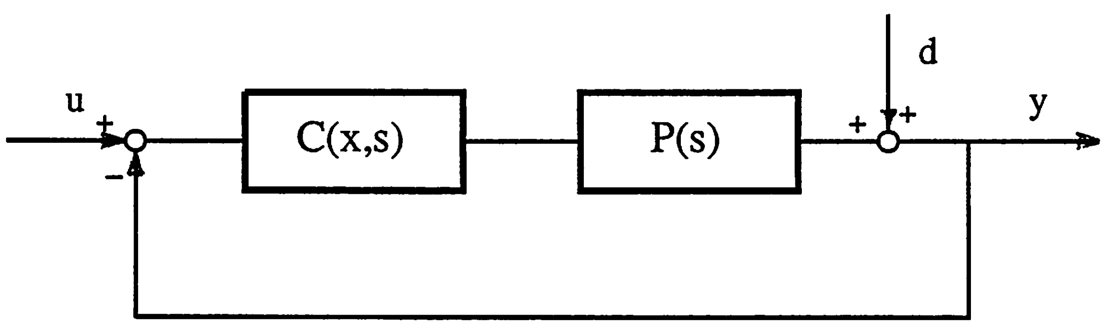

Fig.1 : The feedback system