ALGEBRAIC THEORY OF LINEAR TIME-INVARIANT FEEDBACK SYSTEMS WITH TWO-INPUT TWO-OUTPUT PLANT AND COMPENSATOR

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ALGEBRAIC THEORY OF LINEAR TIME-INVARIANT FEEDBACK SYSTEMS WITH TWO-INPUT TWO-OUTPUT PLANT AND COMPENSATOR

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Abstract

This paper presents an ***algebraic*** theory for linear time-invariant multi-input multi-output systems with two-input two-output plant and compensator, using the factorization approach. This system configuration considered by Doyle and Nett is most general in that any interconnection of two systems can be represented in terms of this scheme. Other system configurations encountered in compensator design problems are special cases of this system configuration. The analysis and synthesis applies to continuous-time as well as discrete-time systems, lumped as well as distributed systems since the algebraic setting is completely general. The compensator parametrization has four degrees of freedom. The paper is self-contained and is tutorial in the sense that it develops the main results of Nett without introducing the concept of containment.

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INTRODUCTION

The algebraic theory of multi-input multi-output linear time-invariant (l.t-i.) systems rests on a few basic papers [You.1], [Des.1], [Per.1], [Vid.2]. From then on, a large number of papers followed and Vidyasagar's book, [Vid.1], gives a systematic exposition of their results using the factorization approach, and includes a detailed list of references. We list some of these papers and books at the end [Åst.1], [Cal.1], [Chen 1], [Che.1]. [Des.2], [Doy.1], [Hor.1], [Kai.1], [Ros.1], [Sae.1], [Zam.1].

[Vid.1] considers two system configurations: first the unity-gain feedback system $S_1(P, C)$ in which both the plant and the compensator have one vector-input and one vector-output and second, the system $S_2(P, C)$ in which the compensator has two vector-inputs and thus, one added free parameter. [Vid.1] parametrizes the set of all stabilizing compensators and the set of all achievable input-output (I/O) maps for $S_1(P, C)$ and for $S_2(P, C)$. Since [Vid.1] contains a complete list of the related work up to 1985, we will not repeat these references here.

After [Vid.1], two-parameter design was used on a one vector-input two vector-output plant in [Des.4,5], and the I/O map from the exogenous input of the compensator to the output of the plant was diagonalized by adjusting one of the two free parameters. More recently, necessary and sufficient conditions for stabilizing a feedback system with two vector-input two vector-output plant and compensator blocks was presented in [Net.1]. This feedback configuration is a generalization of $S_1(P, C)$ and $S_2(P, C)$ and it will be called $\Sigma(P, C)$ in this paper.

The feedback system $\Sigma(P, C)$ is made up of a two vector-input two vector-output plant and compensator (see figure 1) and is the object of our study. Note that in our feedback configuration, we have a ***negative*** feedback on the left summing node. For a given plant P, $\Sigma(P, C)$ has four degrees-of-freedom and each I/O map that it achieves depends on one of these four "free" parameters. Therefore for this configuration, more design requirements can be satisfied simultaneously. The plant P has two outputs y_o and y_m ; y_m is the measured-output, which is used in feedback to C. The I/O map from u'_o to y_o depends on one of the four parameters whereas the maps from u'_1 , u_o , u_1 to y_o each depend on one other parameter. Here u'_o is the exogenous input and u'_1 , u_o , u_1 model the disturbances at the measured-output, plant input and the compensator output, respectively. The compensator output y'_o is not utilized by P; we view y'_o as an output used for performance monitoring and fault diagnosis.

The objective of this paper is to present systematic and straightforward algebraic development of the feedback system $\Sigma(P, C)$. Our approach is different from that of [Net.1]. First of all, by systematically using the algebraic techniques used for $S_1(P, C)$ and $S_2(P, C)$, we derive the stability conditions in [Net.1] and the set of all achievable I/O maps straightforwardly. Our approach does not require the use and the properties of Nett's concept of containment. The admissibility requirements [Net.1] follow directly from the analysis. Intuitively, the admissibility of P can be viewed as follows: figure 1 shows that only the partial map $P_{22} : e \mapsto y_m$ is in the feedback loop; hence the instabilities of $P_{11} : u_o \mapsto y_o$, $P_{21} : u_o \mapsto y_m$ and $P_{12} : e \mapsto y_o$ must be "contained" in the instabilities of P_{22}: this intuitive concept is made very clear by figure 3 which follows directly from our analysis. Second, we start building our algebraic structure with a *principal* ring h as in [Vid.1], [Des.3,4,5]. Factorizing in this h guarantees the existence of right and left-coprime-fraction representations (c.f.r.), and moreover, of the standard forms (Hermite form in particular) of matrices with elements in h ($m(h)$). Third, the analysis is simplified by using a right-coprime factorization for P and a left-coprime factorization for C.

The reader is assumed to be familiar with some basic algebra; Appendix A, B of [Vid.1] is sufficient background. [Sig.1], [Lang 1], [Bou.1], [Jac.1], [Coh.1], [Mac.1] may also be useful. For the reader's convenience, we use [Vid.1] as a reference. Nevertheless, in order to make the paper self-contained, we have collected the necessary algebraic results in section II and included short proofs where needed.

The paper is organized as follows: Section I presents the algebraic structure and basic results. Section II starts with the system description, analysis and problem formulation. h-stability and Σ-admissibility are defined and the main h-stability theorem (Theorem 2.4) are presented in this section. The set of all stabilizing compensators S and the set of all achievable I/O maps A_{yu} are obtained in section III using the theorems of section II. Conclusions are also given in section III.

The following is a list of symbols and abbreviations.

l.t-i.	linear time-invariant
I/O	input-output
w.l.o.g.	without loss of generality
s.t.	such that
u.t.c.	under these conditions
$a := b$	a is defined as b
e.r.o.'s	elementary row operations
e.c.o.'s	elementary column operations
r.c. (l.c.)	right(left)-coprime
r.f.r. (l.f.r.)	right(left)-fraction representation
c.f.r.	coprime-fraction representation
r.c.f.r. (l.c.f.r.)	right(left)-coprime-fraction representation
det$A$	the determinant of matrix A
$m(h)$	the set of matrices with elements in h

SECTION I

Algebraic Preliminaries

In order to clearly separate algebraic facts from system analysis, in this section we collect relevant definitions, known facts and prove a few lemmas which will be useful in the analysis of $\Sigma(P, C)$.

1.1. Notation [Lang 1, p.71-77], [Vid.1, Appendix A, B]:

h is a principal ring (i.e. an entire commutative ring in which every ideal is principal).

$i \subset h$ is a multiplicative subsystem, $0 \notin i$, $1 \in i$.

$j \subset h$ is the group of units of h.

$g = h / i$ is the ring of fractions of h associated with i.

g_s (Jacobson radical of the ring g) $:= \{x \in g_s : (1 + xy)^{-1} \in g$ for all $y \in g\}$.

1.2. Facts

i) i = the set of units of g which are in h.

ii) Let $A \in m(h)$, $B \in m(g)$. Then a) $A^{-1} \in m(h)$ iff $\det A \in j$ and b) $B^{-1} \in m(g)$ iff $\det B \in i$. (U.t.c. A is called h–unimodular, and B is called g–unimodular.)

iii) Let $Y \in m(g_s)$, $X, Z \in m(g)$. Then $XY, YZ \in m(g_s)$, and $(I+XY)^{-1}$, $(I+YZ)^{-1} \in m(g)$.

1.3. Example : Let $u \supset \mathbb{C}_+$ be a closed subset of $\mathbb{C}$, symmetric about the real axis, and let $\mathbb{C} \setminus u$ be nonempty. Let $\bar{u} := u \cup \{\infty\}$. Let $h = R_u(s) :=$ the ring of proper scalar rational functions which are analytic in u. Let i be the set of elements of R_u s.t. $f \in i$, implies $f(\infty)$ = a nonzero constant. Then g is the ring of proper rational functions in s, j is the set of elements of R_u with no zeros in $\bar{u}$, and g_s is the set of strictly proper rational functions (i.e. goes to 0 as $s \to \infty$).

1.4. Lemma

i) Let $a, b \in h$. Then $ab \in j$ iff a and $b \in j$.

ii) Let $c, d \in h$. Then $cd \in i$ iff c and $d \in i$.

Proof: i) (=>) $ab =: u \in j$ => $u^{-1} \in h$, $(u^{-1}a)b = 1$ and $a(bu^{-1}) = 1$ => $b \in h$ has inverse $u^{-1}a \in h$, and $a \in h$ has inverse $bu^{-1} \in h$, => $b \in j$ and $a \in j$.

(<=) is immediate since j is a group.

ii) (=>) $cd =: v \in i$ => $v^{-1} \in g$, $(v^{-1}c)d = 1$, $c(dv^{-1}) = 1$ => $d \in h$ has inverse $v^{-1}c \in g$, and $c \in h$ has inverse $dv^{-1} \in g$ => $d, c \in i$.

(<=) is immediate since i is a multiplicative system.

1.5. Definition

i) The pair $(N, D) \in m(h)$ is called **right-coprime (r.c.)** iff there exist $U, V \in m(h)$ s.t.

$$UN + VD = I \tag{1.1}$$

ii) The pair $(N, D) \in m(h)$ is called a **right-fraction representation (r.f.r.)** of $P \in m(g)$ iff

$$D \text{ is square}, \ \det D \in i \ \text{ and } \ P = ND^{-1} \tag{1.2}$$

iii) The pair $(N, D) \in m(h)$ is called a **right-coprime-fraction representation (r.c.f.r.)** of $P \in m(g)$ iff (N, D) is a r.f.r. of P and (N, D) is r.c.

1.6. Definition

i) The pair $(\tilde{D}, \tilde{N}) \in m(h)$ is called **left-coprime (l.c.)** iff there exist $\tilde{U}, \tilde{V} \in m(h)$ s.t.

$$\tilde{N}\tilde{U} + \tilde{D}\tilde{V} = I \tag{1.3}$$

ii) The pair $(\tilde{D}, \tilde{N}) \in m(h)$ is called a **left-fraction representation (l.f.r.)** of $P \in m(g)$ iff

$$\tilde{D} \text{ is square}, \ \det\tilde{D} \in i \ \text{ and } \ P = \tilde{D}^{-1}\tilde{N} \tag{1.4}$$

iii) The pair $(\tilde{D}, \tilde{N}) \in m(h)$ is called a **left-coprime-fraction representation (l.c.f.r.)** iff $(\tilde{D}, \tilde{N})$ is a l.f.r. of P and $(\tilde{D}, \tilde{N})$ is l.c.

1.7. Facts [Vid.1, chap. 4]

i) Every $P \in \mathcal{M}(\mathcal{G})$ has a r.c.f.r. $(N, D) \in \mathcal{M}(\mathcal{H})$ and a l.c.f.r. $(\tilde{D}, \tilde{N}) \in \mathcal{M}(\mathcal{H})$ because $\mathcal{H}$ is a principal ring.

ii) Let (N, D) be a r.c.f.r. of $P \in \mathcal{M}(\mathcal{G})$. Then (X, Y) is a r.f.r. (r.c.f.r.) of P iff $(X, Y) = (NR, DR)$ for some $\mathcal{G}$-unimodular ($\mathcal{H}$-unimodular) $R \in \mathcal{M}(\mathcal{H})$.

iii) Let $(\tilde{D}, \tilde{N})$ be a l.c.f.r. of $P \in \mathcal{M}(\mathcal{G})$. Then $(\tilde{X}, \tilde{Y})$ is a l.f.r. (l.c.f.r.) of P iff $(\tilde{X}, \tilde{Y}) = (L\tilde{D}, L\tilde{N})$ for some $\mathcal{G}$-unimodular ($\mathcal{H}$-unimodular) $L \in \mathcal{M}(\mathcal{H})$.

iv) Let (N, D) be a r.c.f.r. and $(\tilde{D}, \tilde{N})$ be a l.c.f.r. of P. Then there exist $U, V, \tilde{U}, \tilde{V} \in \mathcal{M}(\mathcal{H})$ s.t.

$$\begin{bmatrix} U & \vdots & V \\ \cdots & & \cdots \\ \tilde{D} & \vdots & -\tilde{N} \end{bmatrix} \begin{bmatrix} N & \vdots & \tilde{V} \\ \cdots & & \cdots \\ D & \vdots & -\tilde{U} \end{bmatrix} = \begin{bmatrix} I & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I \end{bmatrix} \tag{1.5}$$

Equation (1.5) is called the ***generalized Bezout Identity*** for the coprime-fraction representations (c.f.r.) of P.

1.8. Lemma [Vid.1, Net.1]

Let (N, D) and $(\tilde{D}, \tilde{N})$ be a r.c.f.r. and a l.c.f.r. of $P \in \mathcal{M}(\mathcal{G})$, respectively. Then there exists $m_1 \in \mathcal{J}$ s.t.

$$\det\tilde{D} = m_1 \det D \tag{1.6}$$

Proof: By Fact 1.7.iv, using obvious notation, we write equation (1.5) as

$$M_1 M_2 = I \quad ; \quad M_1, M_2 \in \mathcal{M}(\mathcal{H})$$

Then $\det M_1$ and $\det M_2 \in \mathcal{H}$; moreover, $\det M_1 \det M_2 = 1 \in \mathcal{H}$. By Lemma 1.4.i, $\det M_1 =: m_1 \in \mathcal{J}$. Then $\det M_1 = \det \begin{bmatrix} I & 0 \\ 0 & \tilde{D} \end{bmatrix} \det \begin{bmatrix} U & V \\ I & -P \end{bmatrix} = \det\tilde{D} \begin{bmatrix} U & VD \\ I & -N \end{bmatrix} \det \begin{bmatrix} I & 0 \\ 0 & D^{-1} \end{bmatrix}$. By e.r.o.'s in $\mathcal{H}$, $\det M_1 = \det\tilde{D} \det(UN+VD)(\det D)^{-1} = \det\tilde{D} \det D^{-1} = m_1 \in \mathcal{J}$, and equation (1.6) follows.

■

Comment : We say that $\det\tilde{D}$ is equivalent to $\det D$, denoted $\det\tilde{D} \approx \det D$, if and only if there exists $m_1 \in \mathcal{J}$ s.t. equation (1.6) holds. Clearly, with $\det\tilde{D}$, $\det D \in \mathcal{H}$, " $\approx$ " is an equivalence relation on $\mathcal{h}$. Similarly, with $a \in \mathcal{h}$, $a \approx 1$ is equivalent to $a \in \mathcal{j}$.

The following lemma is useful in studying the equations describing the system in section II.

1.9. Lemma : Let $\begin{bmatrix} A \\ \cdots \\ B \end{bmatrix} = E \begin{bmatrix} N \\ \cdots \\ D \end{bmatrix}$ and let $\begin{bmatrix} \tilde{A} & \vdots & \tilde{D} \end{bmatrix} = \begin{bmatrix} \tilde{N} & \vdots & \tilde{D} \end{bmatrix} F$, where E , $F \in \mathcal{m}(\mathcal{h})$ are of appropriate dimension. Then

i) for all $\mathcal{h}$–unimodular E , the pair (N, D) is r.c. iff the pair (A, B) is r.c.

ii) for all $\mathcal{h}$–unimodular F , the pair $(\tilde{D}, \tilde{N})$ is l.c. iff the pair $(\tilde{B}, \tilde{A})$ is l.c.

Proof: i) (N, D) is r.c. <=> there exist U , $V \in \mathcal{m}(\mathcal{h})$ s.t.

$$\begin{bmatrix} U & \vdots & V \end{bmatrix} \begin{bmatrix} N \\ \cdots \\ D \end{bmatrix} = I = (\begin{bmatrix} U & \vdots & V \end{bmatrix} E^{-1}) \begin{bmatrix} A \\ \cdots \\ B \end{bmatrix} \iff (A, B) \text{ is r.c.}$$

■

ii) The proof is similar to part (i).

Because of our interest in the system $\Sigma(P, C)$ shown in figure 1, we partition P and exhibit a special c.f.r. of it. Note that every $P \in \mathcal{m}(\mathcal{g})$ has both a r.c.f.r. and a l.c.f.r. by Fact 1.7.i. The following lemma states that these c.f.r.'s can be put into a special form.

1.10. Lemma : Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{m}(\mathcal{g})$. Let

(N_{22}, D_{22}) be a r.c.f.r. and $(\tilde{D}_{22}, \tilde{N}_{22})$ be a l.c.f.r. of P_{22}. (1.7)

Then there exist $N_{11}, N_{12}, N_{21}, D_{11}, D_{21}, R_{22} \in \mathcal{m}(\mathcal{h})$; $\tilde{D}_{11}, \tilde{D}_{12}, \tilde{N}_{11}, \tilde{N}_{12}, \tilde{N}_{21}, \tilde{L}_{22} \in \mathcal{m}(\mathcal{h})$ s.t.

$$\text{i)} \quad (N, D) =: (\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22}R_{22} \end{bmatrix}, \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22}R_{22} \end{bmatrix}) \text{ is a r.c.f.r. of } P. \tag{1.8}$$

$$\text{ii)} \quad (\tilde{D}, \tilde{N}) =: (\begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & \tilde{L}_{22}\tilde{D}_{22} \end{bmatrix}, \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{L}_{22}\tilde{N}_{22} \end{bmatrix}) \text{ is a l.c.f.r. of } P. \tag{1.9}$$

Moreover,

$$\text{iii)} \quad \frac{\det D}{\det D_{22}} \in \mathcal{i} \text{ and } \frac{\det \tilde{D}}{\det \tilde{D}_{22}} \in \mathcal{i} \tag{1.10}$$

1.11. Remark (Generalized Bezout Identity for the c.f.r.'s of P_{22})

By Fact 1.7.iv there exist U_{22} , V_{22} , $\tilde{U}_{22}$, $\tilde{V}_{22} \in \mathcal{m}(\mathcal{h})$ s.t.

$$\begin{bmatrix} U_{22} & \vdots & V_{22} \\ \cdots & \vdots & \cdots \\ \tilde{D}_{22} & \vdots & -\tilde{N}_{22} \end{bmatrix} \begin{bmatrix} N_{22} & \vdots & \tilde{V}_{22} \\ \cdots & \vdots & \cdots \\ D_{22} & \vdots & -\tilde{U}_{22} \end{bmatrix} = I \tag{1.11}$$

Proof of Lemma 1.10 : By Fact 1.7.iv every $P \in \mathcal{m}(\mathcal{g})$ has a r.c.f.r. (call it (X, Y)) and a l.c.f.r. (call it $(\tilde{Y}, \tilde{X})$).

i) By the existence of the Hermite Column Form [Vid. 1, Appendix B] there is an $\mathcal{h}$-unimodular R

s.t. $YR =: \begin{bmatrix} D_{11} & 0 \\ D_{21} & \bar{D}_{22} \end{bmatrix} \in \mathcal{m}(\mathcal{h})$ and $XR =: \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & \bar{N}_{22} \end{bmatrix} \in \mathcal{m}(\mathcal{h})$.

Now $\det Y \in \mathcal{i}$ => $\det D_{11} \det \bar{D}_{22} \in \mathcal{i}$ => $\det \bar{D}_{22} \in \mathcal{i}$ by Lemma 1.4.ii. Then $(\bar{N}_{22}, \bar{D}_{22})$ is a r.f.r. of P_{22}. By Fact 1.7.ii, $(\bar{N}_{22}, \bar{D}_{22}) = (N_{22}R_{22}, D_{22}R_{22})$ for some $\mathcal{g}$-unimodular $R_{22} \in \mathcal{m}(\mathcal{h})$ and the result follows.

ii) Same as the proof of (i), except the Hermite Row Form and Fact 1.7.iii are used.

iii) From equation (1.8), D_{11} , D_{22} , $R_{22} \in \mathcal{m}(\mathcal{h})$, $\det D \in \mathcal{i}$ and $\det D = \det D_{11} \det D_{22} \det R_{22}$. By Lemma 1.4.ii, each factor is in $\mathcal{i}$ and hence, $\det D \in \mathcal{i}$ <=> $\det D_{11} \det R_{22} = \dfrac{\det D}{\det D_{22}} \in \mathcal{i}$. The second equality in equation (1.10) is proved similarly.

■

1.12. Lemma : Let $P_{22} \in \mathcal{m}(\mathcal{g}_{s})$. and let statement (1.7) hold. Consider the equation

$$\tilde{D}'_{22} D_{22} + \tilde{N}'_{22} N_{22} = I \tag{1.12}$$

Then $(\tilde{D}'_{22}, \tilde{N}'_{22}) \in \mathcal{m}(\mathcal{h})$ is a solution of equation (1.12) if and only if there exists $Q'_{22} \in \mathcal{m}(\mathcal{h})$ s.t.

$$\begin{bmatrix} \tilde{N}'_{22} & \vdots & \tilde{D}'_{22} \end{bmatrix} = \begin{bmatrix} I & \vdots & Q'_{22} \end{bmatrix} \begin{bmatrix} U_{22} & \vdots & V_{22} \\ \cdots & \vdots & \cdots \\ \tilde{D}_{22} & \vdots & -\tilde{N}_{22} \end{bmatrix} \tag{1.13}$$

Proof : (<=) Suppose equation (1.13) holds. Then $\tilde{N}'_{22} N_{22} + \tilde{D}'_{22} D_{22} = \begin{bmatrix} \tilde{N}'_{22} & \tilde{D}'_{22} \end{bmatrix} \begin{bmatrix} N_{22} \\ D_{22} \end{bmatrix}$

$= \begin{bmatrix} I & Q'_{22} \end{bmatrix} \begin{bmatrix} U_{22} & V_{22} \\ \tilde{D}_{22} & -\tilde{N}_{22} \end{bmatrix} \begin{bmatrix} N_{22} \\ D_{22} \end{bmatrix} = \begin{bmatrix} I & Q'_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = I$, where the third equality follows from equation (1.11).

(=>) Since equation (1.12) holds, we have

$$\begin{bmatrix} \tilde{D}'_{22} & \tilde{N}'_{22} \end{bmatrix} \begin{bmatrix} N_{22} & \tilde{V}_{22} \\ D_{22} & -\tilde{U}_{22} \end{bmatrix} = \begin{bmatrix} I & Q'_{22} \end{bmatrix} \tag{1.14}$$

where $Q'_{22} := \tilde{D}'_{22}\tilde{V}_{22} - \tilde{N}'_{22}\tilde{U}_{22} \in \mathcal{M}(\mathcal{H})$.

Multiplying both sides by $\begin{bmatrix} U_{22} & V_{22} \\ \tilde{D}_{22} & -\tilde{N}_{22} \end{bmatrix}$, and using equation (1.11), we obtain equation (1.13).

1.13. Lemma : Let $P_{22} \in \mathcal{M}(\mathcal{G})$, and let equation (1.7) hold. Consider the equation

$$\tilde{D}_{22}D'_{22} + \tilde{N}_{22}N'_{22} = I \tag{1.15}$$

Then (N'_{22}, D'_{22}) is a solution of equation (1.15) if and only if there exists $\hat{Q}'_{22} \in \mathcal{M}(\mathcal{H})$ s.t.

$$\begin{bmatrix} D'_{22} \\ -N'_{22} \end{bmatrix} = \begin{bmatrix} N_{22} & \tilde{V}_{22} \\ D_{22} & -\tilde{U}_{22} \end{bmatrix} \begin{bmatrix} -\hat{Q}'_{22} \\ I \end{bmatrix} \tag{1.16}$$

Proof: Entirely analogous to the proof of Lemma 1.12.

■

1.14. Lemma : Let $P_{22} \in \mathcal{M}(\mathcal{G}_s)$. Consider equation (1.12). Then for all Q'_{22}, $\hat{Q}'_{22} \in \mathcal{M}(\mathcal{H})$, the following properties hold.

i) $\det(V_{22} - Q'_{22}N_{22}) \in \mathcal{J}$ (1.17)

ii) $\det(\tilde{V}_{22} - N_{22}\hat{Q}'_{22}) \in \mathcal{J}$ (1.18)

Proof : Using the Bezout identity for (N_{22}, D_{22}) from equation (1.11), we obtain

$$\begin{aligned}(V_{22} - Q'_{22}\tilde{N}_{22})^{-1} &= [(V_{22}D_{22} - Q'_{22}\tilde{N}_{22}D_{22})D_{22}^{-1}]^{-1} \\ &= D_{22}[I - U_{22}N_{22} - Q'_{22}\tilde{N}_{22}D_{22}]^{-1}\end{aligned} \tag{1.19}$$

Similarly, using the Bezout identity for $(\tilde{D}_{22}, \tilde{N}_{22})$,

$$(\tilde{V}_{22} - N_{22}\hat{Q}'_{22})^{-1} = (I - \tilde{N}_{22}\tilde{U}_{22} - \tilde{D}_{22}N_{22}\hat{Q}'_{22})^{-1}\tilde{D}_{22} \tag{1.20}$$

Now $P \in \mathcal{M}(\mathcal{G}_s)$ implies that both $N_{22} = P_{22}D_{22}$ and $\tilde{N}_{22} = \tilde{D}_{22}P_{22} \in \mathcal{M}(\mathcal{G}_s)$. Consequently, the product of either N_{22} or $\tilde{N}_{22}$ with any matrix in $\mathcal{M}(\mathcal{H})$ is also in $\mathcal{M}(\mathcal{G}_s)$. Since $\mathcal{M}(\mathcal{G}_s)$ is closed under addition, the inverses in equations (1.19) and (1.20) are of the form $(I + T)^{-1}$ with $T \in \mathcal{M}(\mathcal{G}_s)$; hence $(I + T) \in \mathcal{M}(\mathcal{G})$ by Fact 1.2.iii. Therefore, $\det(I + T) \in \mathcal{J}$ by Fact 1.2.ii and equations (1.17), (1.18) follow.

■

SECTION II

Problem and System Description

Consider the system $\Sigma(P, C)$ in figure 1. In the case that $h = R_u$ as in example 1.3, assume that P and C have no "unstable hidden dynamics" so that an I/O representation is valid.

We impose the following assumptions on $\Sigma(P, C)$. Note that by Lemma 1.10, *any* $P \in \mathcal{M}(\mathcal{G})$ and *any* $C \in \mathcal{M}(\mathcal{G})$ has the following c.f.r.'s.

2.1. Assumptions :

i) $$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{M}(\mathcal{G}) \tag{2.1}$$

$$(N_{22}, D_{22}) \text{ is a r.c.f.r. of } P_{22}, \ (\tilde{D}_{22}, \tilde{N}_{22}) \text{ is a l.c.f.r. of } P_{22} \tag{2.2}$$

Then by Lemma 1.10

$$(N, D) = (\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22}R_{22} \end{bmatrix}, \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22}R_{22} \end{bmatrix}) \text{ is a r.c.f.r. of } P \tag{2.3}$$

$$(\tilde{D}, \tilde{N}) = (\begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & \tilde{L}_{22}\tilde{D}_{22} \end{bmatrix}, \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{L}_{22}\tilde{N}_{22} \end{bmatrix}) \text{ is a l.c.f.r. of } P \tag{2.4}$$

By Fact 1.7.iv, the generalized Bezout identity equation (1.11) holds for the c.f.r.'s of P_{22}.

ii) $$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \in \mathcal{M}(\mathcal{G}) \tag{2.5}$$

$$(N'_{22}, D'_{22}) \text{ is a r.c.f.r. and } (\tilde{D}'_{22}, \tilde{N}'_{22}) \text{ is a l.c.f.r. of } C_{22} \tag{2.6}$$

Then by Lemma 1.10 applied to C,

$$(\tilde{D}', \tilde{N}') = (\begin{bmatrix} \tilde{D}'_{11} & \tilde{D}'_{12} \\ 0 & \tilde{L}'_{22}\tilde{D}'_{22} \end{bmatrix}, \begin{bmatrix} \tilde{N}'_{22} & \tilde{N}'_{12} \\ \tilde{N}'_{21} & \tilde{L}'_{22}\tilde{D}'_{22} \end{bmatrix}) \text{ is a l.c.f.r. of } C \tag{2.7}$$

$$(N', D') = (\begin{bmatrix} N'_{11} & N'_{12} \\ N'_{21} & N'_{22}R'_{22} \end{bmatrix}, \begin{bmatrix} D'_{11} & 0 \\ D'_{21} & D'_{22}R'_{22} \end{bmatrix}) \text{ is a r.c.f.r. of } C \tag{2.8}$$

By Fact 1.7.iv, the generalized Bezout identity for C_{22} reads as follows: There exist U'_{22}, V'_{22}, $\tilde{U}'_{22}$, $\tilde{V}'_{22}$ s.t.

$$\begin{bmatrix} U'_{22} & \vdots & V'_{22} \\ \cdots & \vdots & \cdots \\ \tilde{D}'_{22} & \vdots & -\tilde{N}'_{22} \end{bmatrix} \begin{bmatrix} N'_{22} & \vdots & \tilde{V}'_{22} \\ \cdots & \vdots & \cdots \\ D'_{22} & \vdots & -\tilde{U}'_{22} \end{bmatrix} = I \tag{2.9}$$

2.2. Comment : By Fact 1.7.ii, any other r.c.f.r. (X, Y) of P is given by $(X, Y) = (NR, DR)$, where (N, D) is the r.c.f.r. in equation (2.3) and $R \in \mathcal{M}(\mathcal{H})$ is $\mathcal{H}$-unimodular. Similarly, by Fact 1.7.iii, any other l.c.f.r. $(\tilde{Y}, \tilde{X})$ of P is given by $(\tilde{Y}, \tilde{X}) = (L\tilde{D}, L\tilde{N})$ where $(\tilde{D}, \tilde{N})$ is the l.c.f.r. in equation (2.4) and $L \in \mathcal{M}(\mathcal{H})$ is $\mathcal{H}$-unimodular. Since the same holds for any other c.f.r.'s of C, there is no loss of generality in taking the c.f.r.'s always as in equations (2.3)-(2.4) and (2.7)-(2.8) respectively.

Analysis of $\Sigma(P, C)$

Using the representations of P and C as in Assumptions (2.1)-(2.9) we redraw the system $\Sigma(P, C)$ as in figure 2. Let $y := \begin{bmatrix} y_o \\ y_m \\ y'_o \\ y'_m \end{bmatrix}$, $u := \begin{bmatrix} u_o \\ u_1 \\ u'_o \\ u'_1 \end{bmatrix}$, $\xi := \begin{bmatrix} \xi_p \\ y'_o \\ y'_m \end{bmatrix}$. Then $\Sigma(P, C)$ is described by:

$$\left[\begin{array}{c:c} D & \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \\ \cdots & \cdots \\ \tilde{N}'\begin{bmatrix} 0 & 0 \\ N_{21} & N_{22}R_{22} \end{bmatrix} & \tilde{D}' \end{array}\right] \begin{bmatrix} \xi_p \\ \cdots \\ y'_o \\ y'_m \end{bmatrix} = \left[\begin{array}{c:c} I & 0 \\ \cdots & \cdots \\ 0 & \tilde{N}' \end{array}\right] \begin{bmatrix} u_o \\ u_1 \\ \cdots \\ u'_o \\ u'_1 \end{bmatrix} \tag{2.10a}$$

$$\left[\begin{array}{c:c} N & 0 \\ \cdots & \cdots \\ 0 & I \end{array}\right] \begin{bmatrix} \xi_p \\ \cdots \\ y'_o \\ y'_m \end{bmatrix} = \begin{bmatrix} y_o \\ y_m \\ \cdots \\ y'_o \\ y'_m \end{bmatrix} \tag{2.10b}$$

Using obvious notations we write equations (2.10a)-(2.10b) in the form

$$D_H \xi = N_L u \tag{2.11}$$

$$N_R \xi = y \tag{2.12}$$

By e.r.o.'s in $\mathcal{H}$, $E \begin{bmatrix} D_H \\ \cdots \\ N_R \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \\ \cdots & \cdots \\ N & 0 \\ 0 & I \end{bmatrix} =: \begin{bmatrix} B \\ \cdots \\ A \end{bmatrix}$ where, by inspection, (A, B) is r.c.

Hence, by Lemma 1.9, (N_R, D_H) is r.c. Similarly, by e.c.o.'s in $\mathcal{H}$,

$$\left[D_H \ \vdots \ N_L \right] F = \left[\begin{array}{cc:cc} 0 & 0 & I & 0 \\ 0 & \tilde{D}' & 0 & \tilde{N}' \end{array}\right] =: \left[\tilde{B} \ \vdots \ \tilde{A} \right]$$ where, by inspection, $(\tilde{B}, \tilde{A})$ is l.c.

By Lemma 1.9, (D_H, N_L) is l.c.

If $\det D_H \in \boldsymbol{i}$, then from equations (2.10)-(2.12) we obtain H_{yu} : $\begin{bmatrix} u_o \\ u_1 \\ u'_o \\ u'_1 \end{bmatrix} \mapsto \begin{bmatrix} y_o \\ y_m \\ y'_o \\ y'_m \end{bmatrix} \in \boldsymbol{m}(\boldsymbol{g})$

since
$$H_{yu} = N_R D_H^{-1} N_L \tag{2.13}$$
In terms of P and C, H_{yu} is given by

$$H_{yu} = \begin{bmatrix} P_{11}-P_{12}(T)^{-1}C_{22}P_{21} & P_{12}(T)^{-1} & P_{12}(T)^{-1}C_{21} & P_{12}(T)^{-1}C_{22} \\ (I-P_{22}(T)^{-1}C_{22})P_{21} & P_{22}(T)^{-1} & P_{22}(T)^{-1}C_{21} & P_{22}(T)^{-1}C_{22} \\ -C_{12}(I-P_{22}(T)^{-1}C_{22})P_{21} & -C_{12}P_{22}(T)^{-1} & C_{11}-C_{12}P_{22}(T)^{-1}C_{21} & C_{12}(I-P_{22}(T)^{-1}C_{22}) \\ -(T)^{-1}C_{22}P_{21} & (T)^{-1}-I & (T)^{-1}C_{21} & (T)^{-1}C_{22} \end{bmatrix} \tag{2.14}$$

where $T := I + C_{22}P_{22}$.

2.3. Definition : $\Sigma(P, C)$ is called $\boldsymbol{h}$*-stable* if and only if $H_{yu} \in \boldsymbol{m}(\boldsymbol{h})$.

The following theorem is essentially contained in [Net. 1].

2.4. Theorem ($\boldsymbol{h}$-stability of $\Sigma(P, C)$)

Consider $\Sigma(P, C)$ shown in figure 2. Let Assumptions (2.1)-(2.9) hold. Then the following statements are equivalent:

i) $\Sigma(P, C)$ is $\boldsymbol{h}$-stable

ii) $\det D_H \asymp 1$

iii) $\det D \asymp \det D_{22}$ and (2.15)

$\det \tilde{D}' \asymp \det \tilde{D}'_{22}$ and (2.16)

$\det(\tilde{D}'_{22} D_{22} + \tilde{N}'_{22} N_{22}) \asymp 1$ (2.17)

2.5. Remarks 1) Following Comment 2.2, equations (2.15)-(2.17) hold for *any* r.c.f.r. of P and *any* l.c.f.r. of C as well as those in equations (2.3) and (2.7) respectively.

2) From equation (2.10a), calculate $\det D_H$:

$$\det D_H = \det \begin{bmatrix} D & 0 \\ 0 & \tilde{D}' \end{bmatrix} \det \begin{bmatrix} I & \begin{bmatrix} 0 & 0 \\ 0 & -R_{22}^{-1} D_{22}^{-1} \end{bmatrix} \\ C \begin{bmatrix} 0 & 0 \\ N_{21} & N_{22}R_{22} \end{bmatrix} & I \end{bmatrix}$$

By e.c.o.'s in h ,

$$\det D_H = \det D \det \tilde{D}' \det(I + C_{22}P_{22}) \tag{2.18}$$

Since $\det(I + C_{22}P_{22}) = \det(I + P_{22}C_{22})$,

$$\det D_H = \det D \det \tilde{D}' \det(I + P_{22}C_{22}) \tag{2.19}$$

Substituting $P_{22} = N_{22}D_{22}^{-1}$ and $C_{22} = \tilde{D}'_{22}{}^{-1}\tilde{N}'_{22}$ in equation (2.18), we get

$$\det D_H = \det D\,(\det D_{22})^{-1}\det\tilde{D}'(\det\tilde{D}'_{22})^{-1}\det(\tilde{D}'_{22}D_{22}+\tilde{N}'_{22}N_{22}) \tag{2.20}$$

and substituting $P_{22} = \tilde{D}_{22}^{-1}\tilde{N}_{22}$ and $C_{22} = N'_{22}D'^{-1}_{22}$ in equation (2.19), we get

$$\det D_H = \det D\,(\det\tilde{D}_{22})^{-1}\det\tilde{D}'(\det D'_{22})^{-1}\det(\tilde{D}_{22}D'_{22}+\tilde{N}_{22}N'_{22}) \tag{2.21}$$

Equations (2.17)-(2.21) are important for compensator design, and have important interpretations: First, the form of equations (2.18) and (2.19) are reminiscent of standard form for $\det D_H$ when P and C both have one vector-input and one vector-output as in the system $S_1(P, C)$; the difference is that, instead of $\det(I+CP)$, in (2.18) we have $\det(I+C_{22}P_{22})$. This modification is natural since the feedback affects only the second inputs and outputs of P and C. (Similar comment holds for (2.19)). Second, equation (2.17) is simply the requirement that the feedback-loop be h-stable . Third, consider equation (2.20) together with (2.15)-(2.16). Conditions (2.15)-(2.16) express the necessary and sufficient condition for P to be stabilizable by the scheme $\Sigma(P, C)$ of figure 1: indeed, if either one fails, it is easy to see that for all compensators C satisfying (2.5)-(2.6), $\Sigma(P, C)$ will *not* be h-stable. These restrictions on the class of plants is a consequence of the fact that the feedback only includes the plant input e and the plant output y_m.

3) Using equations (2.20) and (2.21), Theorem 2.4 states that $\Sigma(P, C)$ is h-stable if and only if each of the factors in $\det D_H$ is in $\mathcal{j}$. Therefore, w.l.o.g. if $\det D_H = 1$ then, by normalization,

$$\tilde{D}'_{22}D_{22} + \tilde{N}'_{22}N_{22} = I \tag{2.22}$$

$$\tilde{D}_{22}D'_{22} + \tilde{N}_{22}N'_{22} = I \tag{2.23}$$

Since $P_{22} = N_{22}D_{22}^{-1} = \tilde{D}_{22}^{-1}\tilde{N}_{22}$, and $C_{22} = \tilde{D}'^{-1}_{22}\tilde{N}'_{22} = N'_{22}D'^{-1}_{21}$, we have the following important conclusion :

If $\Sigma(P, C)$ is h-stable, then, with the above normalizations,

$$\begin{bmatrix} \tilde{N}'_{22} & \vdots & \tilde{D}'_{22} \\ \cdots & \vdots & \cdots \\ \tilde{D}_{22} & \vdots & -\tilde{N}_{22} \end{bmatrix} \begin{bmatrix} N_{22} & \vdots & D'_{22} \\ \cdots & \vdots & \cdots \\ D_{22} & \vdots & -N'_{22} \end{bmatrix} = \begin{bmatrix} I & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & I \end{bmatrix} \tag{2.24}$$

4) Note that

$$\frac{\det D}{\det D_{22}} = (\det D_{11} \det R_{22}) \approx 1 \tag{2.26}$$

$$\frac{\det \tilde{D}'}{\det \tilde{D}'_{22}} = (\det \tilde{D}'_{11} \det \tilde{L}'_{22}) \approx 1 \tag{2.27}$$

Equations (2.26) and (2.27) imply that R_{22} and $\tilde{L}'_{22}$, D_{11} and $\tilde{D}'_{11}$ are h –unimodular; that is, P is stabilizable by C if and only if the only source of instability in P is D_{22} (or $\tilde{D}_{22}$) (see figure 3).

5) Note that the equivalence of the conditions in Theorem 2.4 does ***not*** require the special assumption that $\Sigma(P, C)$ be well-posed [Vid.1]. Any of the equivalences guarantees well-posedness as well as h –stability.

■

For convenience, following [Net.1] we define

2.6. Definition : P (C) is called ***Σ–admissible*** if and only if $\det D \approx \det D_{22}$ ($\det \tilde{D}' \approx \det \tilde{D}'_{22}$ respectively).

From equations (2.3), (2.7) and (2.26), (2.27), and by Lemma 1.4.i, P (C) is Σ–admissible if and only if $\det D_{11} \approx 1$ and $\det R_{22} \approx 1$ ($\det \tilde{D}'_{11} \approx 1$ and $\det \tilde{L}'_{22} \approx 1$ respectively). W.l.o.g. by suitable normalizations, P is Σ–admissible if and only if

$$D_{11} = I \quad \text{and} \quad R_{22} = I \tag{2.28}$$

and C is Σ–admissible if and only if

$$\tilde{D}'_{11} = I \quad \text{and} \quad \tilde{L}'_{22} = I. \tag{2.29}$$

From Remark 2.5 and Definition 2.6 we reformulate Theorem 2.4 as [Net.1]

2.7. Corollary : Let assumptions (2.1)-(2.9) hold. Then $\Sigma(P, C)$ is h –stable if and only if

P is Σ–admissible and (2.30)

C is Σ–admissible and (2.31)

$\tilde{D}'_{22} D_{22} + \tilde{N}'_{22} N_{22} = I$ (2.32)

2.8. Comments : 1) $P \in m(h)$ if and only if P is Σ–admissible and $P_{22} \in m(h)$.

2) Consider the system $S_1(P, C)$ in which both P and C each have only one vector-input and one vector-output (see for example [Des.3], [Vid.1]). Then P and C are automatically Σ–admissible , and hence h–stability of $S_1(P, C)$ reduces to the well known equation (2.32).

■

Proof of Theorem 2.4 : (i) => (ii) The map $H_{y'_m u_1} : u_1 \mapsto y'_m$ is given by $H_{y'_m u_1} = (I+C_{22}P_{22})^{-1} - I$. $\Sigma(P, C)$ is h–stable $\Rightarrow H_{y'_m u_1} :\ u_1 \mapsto y'_m \in m(h)$, => $(I+C_{22}P_{22})^{-1} \in m(h)$ and hence,

$$\det(I + C_{22}P_{22})^{-1} \in h \tag{2.33}$$

From equation (2.18),

$$(\det D_H)^{-1} = (\det D)^{-1}(\det \tilde{D}')^{-1} \det(I + C_{22}P_{22})^{-1} \tag{2.34}$$

By assumptions (2.1)-(2.9), $\det D \in i$ and $\det \tilde{D}' \in i$, and by equation (2.10a), $D_H \in m(h)$. Thus $\det D_H \in h$ and using equation (2.33) and (2.34), $(\det D_H)^{-1} \in g$. Therefore, $\det D_H \in i$. Since (N_R, D_H) is r.c., and (D_H, N_L) is l.c., $H_{yu} = N_R D_H^{-1} N_L \in m(h)$ implies that $D_H^{-1} \in m(h)$ [Vid.1]. By Lemma 1.2.ii, $\det D_H \approx 1$.

(ii) => (i) By Lemma 1.2.ii, $\det D_H \in j \Rightarrow H_{yu} = N_R D_H^{-1} N_L \in m(h)$.

(ii) => (iii) Consider equation (2.20). By Lemma 1.10.iii, and Lemma 1.4.i, and since $\det(\tilde{D}'_{22} D_{22} + \tilde{N}'_{22} N_{22}) \in h$,

$$\det D_H = \frac{\det D}{\det D_{22}} \frac{\det \tilde{D}'}{\det \tilde{D}'_{22}} \det(\tilde{D}'_{22} D_{22} + \tilde{N}'_{22} N_{22}) \in j$$

if and only if each of the factors $\dfrac{\det D}{\det D_{22}} \in j$, $\dfrac{\det \tilde{D}'}{\det \tilde{D}'_{22}} \in j$ and $\det(\tilde{D}'_{22} D_{22} + \tilde{N}'_{22} N_{22}) \in j$ and hence, (ii) <=> (iii).

■

With Definition 2.6 in mind, we now parametrize the class of all Σ–admissible plants P and the class of all Σ–admissible compensators C.

2.9. Theorem (The Class of Σ–admissible Plants and Compensators)

i) Let assumptions (2.1)-(2.4) hold ; then P is Σ–admissible if and only if P has a r.c.f.r. as in equa-

$$(N, D) = \left(\begin{bmatrix} \hat{N}_{11} & \vdots & N_{12} \\ \cdots & \vdots & \cdots \\ \tilde{V}_{22}\tilde{N}_{21} & \vdots & N_{22} \end{bmatrix}, \begin{bmatrix} I & \vdots & 0 \\ \cdots & \vdots & \cdots \\ -\tilde{U}_{22}\tilde{N}_{21} & \vdots & D_{22} \end{bmatrix} \right) \tag{2.35}$$

$$(\tilde{D}, \tilde{N}) = \left(\begin{bmatrix} I & \vdots & -N_{12}U_{22} \\ \cdots & \vdots & \cdots \\ 0 & \vdots & \tilde{D}_{22} \end{bmatrix}, \begin{bmatrix} \hat{N}_{11} & \vdots & N_{12}V_{22} \\ \cdots & \vdots & \cdots \\ \tilde{N}_{21} & \vdots & \tilde{N}_{22} \end{bmatrix} \right) \tag{2.36}$$

ii) Let C satisfy assumptions (2.5)-(2.9). Then C is Σ–admissible if and only if it has a l.c.f.r. as in equation (2.38) an equivalently, a r.c.f.r. as in equation (2.39) below.

$$(\tilde{D}', \tilde{N}') = \left(\begin{bmatrix} I & \vdots & -Q'_{12}U'_{22} \\ \cdots & \vdots & \cdots \\ 0 & \vdots & \tilde{D}'_{22} \end{bmatrix}, \begin{bmatrix} Q'_{11} & \vdots & Q'_{12}V'_{22} \\ \cdots & \vdots & \cdots \\ Q'_{21} & \vdots & \tilde{N}'_{22} \end{bmatrix} \right) \tag{2.38}$$

$$(N', D') = \left(\begin{bmatrix} Q_{11} & \vdots & Q'_{12} \\ \cdots & \vdots & \cdots \\ \tilde{V}'_{22}Q'_{21} & \vdots & N'_{22} \end{bmatrix}, \begin{bmatrix} I & \vdots & 0 \\ \cdots & \vdots & \cdots \\ -\tilde{U}'_{22}Q'_{21} & \vdots & D'_{22} \end{bmatrix} \right) \tag{2.39}$$

2.10. Comments : 1) Suppose we are given 1) P as in equation (2.1), 2) P_{22} factorized as in equation (2.2), and 3) the Bezout identity for (N_{22}, D_{22}) from equation (1.11); then the general expression for ***Σ–admissible*** plants, (2.36) shows that P is Σ–admissible if and only if a) $P_{11} - P_{12}D_{22}U_{22}P_{21} \in \mathcal{M}(\mathcal{H})$, b) $P_{12}D_{22} \in \mathcal{M}(\mathcal{H})$, c) $\tilde{D}_{22}P_{21} \in \mathcal{M}(\mathcal{H})$ (cf. [Net.1, Lemma (31)]).

2) Here we have chosen to call the three compensator parameters $Q'_{11}, Q'_{12}, Q'_{21} \in \mathcal{M}(\mathcal{H})$ instead of $\hat{N}'_{11}, N'_{12}, \tilde{N}'_{21}$; in Theorem 3.5 below, we will see that $Q'_{11}, Q'_{12}, Q'_{21}$ are three of the four "free" parameters used in compensator design. Using Q'_{ij} instead of N'_{ij} should remind us that these parameters, unlike the given plant parameters $\hat{N}_{11}, N_{12}, \tilde{N}_{21}$, can be chosen arbitrarily to meet other design specifications. As far as Σ–admissibility is concerned, (N'_{22}, D'_{22}), or $(\tilde{D}'_{22}, \tilde{N}'_{22})$, are also free; with the stabilization requirement (2.22) ((2.23), respectively), there is an additional constraint on this pair of parameters resulting in four free parameters.

3) From equation (2.28), P is Σ–admissible if and only if $D_{11} = I$ and $R_{22} = I$. Consequently, the r.c.f.r. of P in equation (2.3) and the l.c.f.r. of P in equation (2.4) are each left with four "parameters": $N_{11}, N_{12}, N_{21}, D_{21}$ and $\tilde{N}_{11}, \tilde{N}_{12}, \tilde{N}_{21}$, $\tilde{D}_{12}$ respectively. Theorem 2.9 claims that there are in fact only ***three*** independent "parameters", namely $\hat{N}_{11}, N_{12}, \tilde{N}_{21}$.

4) From equation (2.35),

$$D^{-1} = \begin{bmatrix} I & 0 \\ D_{22}^{-1}\tilde{U}_{22}\tilde{N}_{21} & D_{22}^{-1} \end{bmatrix} \tag{2.40}$$

Let $\mathcal{h} = R_u$ as in Example 1.3. Then Σ-admissibility of P implies that every $\mathcal{U}$-pole of P_{11} , P_{12} and P_{21} is a pole of P_{22} with ***at most*** the same McMillan degree. This conclusion is obvious from figure 3, where the block diagrams for Σ–admissible P and Σ–admissible C are obtained from equations (2.35) and (2.38), respectively. Note the duality between these two block diagrams.

5) Consider figure 3 which shows $\Sigma(P, C)$ with P and C Σ–admissible . Then clearly, $\Sigma(P, C)$ is $\mathcal{h}$–stable if and only if the "loop" $S_1(P_{22}, C_{22})$ is $\mathcal{h}$–stabilized ; equivalently, equation (2.32) holds. With $u'_o = 0$, figure 3 reduces to the system considered in [Des.5], which considers a Σ–admissible plant with one vector-input and two vector-outputs..

Proof of Theorem 2.9 : By Lemma 1.8, $\det D \approx \det\tilde{D}$ and $\det D_{22} \approx \det\tilde{D}_{22}$. Consequently, from equations (2.3) and (2.4), $\det D_{11}\det R_{22} \approx \det\tilde{D}_{11}\det\tilde{L}_{22}$. Therefore from equation (2.28) and by normalization, P is Σ–admissible iff

$$D_{11} = I \ , R_{22} = I \ , \tilde{D}_{11} = I \ , \tilde{L}_{22} = I \tag{2.41}$$

Using equations (2.41) in $\tilde{N}D = \tilde{D}N$ we obtain

$$\tilde{N}_{12}D_{22} + (-\tilde{D}_{12})N_{22} = N_{12} \tag{2.42}$$

$$\tilde{D}_{22}N_{21} + \tilde{N}_{22}(-D_{21}) = \tilde{N}_{21} \tag{2.43}$$

Using Lemma 1.12, it is easy to show that $(\tilde{D}_{12}, \tilde{N}_{12})$ is a solution of equation (2.42) if and only if there exists $Q_{22} \in \mathcal{m}(\mathcal{h})$ s.t.

$$\begin{bmatrix} -\tilde{D}_{12} & \vdots & \tilde{N}_{12} \end{bmatrix} = \begin{bmatrix} N_{12} & \vdots & Q_{22} \end{bmatrix} \begin{bmatrix} U_{22} & \vdots & V_{22} \\ \cdots & \vdots & \cdots \\ \tilde{D}_{22} & \vdots & -\tilde{N}_{22} \end{bmatrix} \tag{2.44}$$

and using Lemma 1.13, (N_{21}, D_{21}) is a solution of equation (2.43) if and only if there exists

$\hat{Q}_{22} \in \boldsymbol{m}(\boldsymbol{h})$ s.t.

$$\begin{bmatrix} N_{21} \\ \cdots \\ D_{21} \end{bmatrix} = \begin{bmatrix} N_{22} & \vdots & \tilde{V}_{22} \\ \cdots & \vdots & \cdots \\ D_{22} & \vdots & -\tilde{U}_{22} \end{bmatrix} \begin{bmatrix} -\hat{Q}_{22} \\ \cdots \\ \tilde{N}_{21} \end{bmatrix} \tag{2.45}$$

Using equations (2.41) and (2.45) in equation (2.3) we obtain

$$P = \begin{bmatrix} N_{11} & N_{12} \\ \tilde{V}_{22}\tilde{N}_{21} - N_{22}\hat{Q}_{22} & N_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\tilde{U}_{22}\tilde{N}_{21} - D_{22}\hat{Q}_{22} & D_{22} \end{bmatrix}^{-1} \tag{2.46}$$

and by Fact 1.7.ii, performing e.c.o.'s on each of the matrices in equation (2.46) we get

$$P = ND^{-1} = \begin{bmatrix} N_{11} + N_{12}\hat{Q}_{22} & N_{12} \\ \tilde{V}_{22}\tilde{N}_{21} & N_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\tilde{U}_{22}\tilde{N}_{21} & D_{22} \end{bmatrix}^{-1} \tag{2.47}$$

Similarly, using equations (2.41), (2.44) in equation (2.4) and by Fact 1.7.ii and e.r.o.'s

$$P = \tilde{D}^{-1}\tilde{N} = \begin{bmatrix} I & -N_{12}U_{22} \\ 0 & \tilde{D}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_{11} + Q_{22}\tilde{N}_{21} & N_{12}V_{22} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix} \tag{2.48}$$

Using once again $\tilde{N}D = \tilde{D}N$, and equations (1.11), (2.47), (2.48), we obtain

$$N_{11} + N_{12}\hat{Q}_{22} = \tilde{N}_{11} + Q_{22}\tilde{N}_{21} =: \hat{N}_{11} \tag{2.49}$$

and hence, equations (2.35)-(2.36) follow.

■

SECTION III

Compensator Synthesis

In this section we describe the set of all compensators C such that, for a given Σ–admissible P, the system $\Sigma(P, C)$ is h–stable.

By Corollary 2.7, if the given P is not Σ–admissible , then $\Sigma(P, C)$ cannot be made h–stable by any C. Therefore we make the following assumption.

3.1. Assumption : Let assumptions (2.1)-(2.9) hold and let P be Σ–admissible . Hence, by Theorem 2.9, P is described by equations (2.35) and (2.36).

Assumption 3.1 holds throughout this section.

3.2. Definition : C is called an ***h–stabilizing compensator*** for P (equivalently, C h–stabilizes P) iff C is Σ–admissible and $\Sigma(P, C)$ is h–stable.

3.3. Definition

$$S := \{ C : C \ \ h - \text{stabilizes } P \} \tag{3.1}$$

is called the set of all ***h–stabilizing compensators*** (for given P in the configuration $\Sigma(P, C)$).

3.4. Definition

$$A_{yu} := \{ H_{yu} \ : C \in S \} \tag{3.2}$$

is called the set of all ***achievable I/O maps of*** $\Sigma(P, C)$.

3.5. Theorem

Let $P \in m(g)$ be given and let Assumption 3.1 hold. Assume that $P_{22} \in m(g_s)$. Then the set of all stabilizing compensators S is given by equation (3.3) or, equivalently, equation (3.4) below.

$$S = \{ \begin{bmatrix} I & -Q'_{12}\tilde{N}_{22} \\ 0 & V_{22}-Q'_{22}\tilde{N}_{22} \end{bmatrix}^{-1} \begin{bmatrix} Q'_{11} & Q'_{12}\tilde{D}_{22} \\ Q'_{21} & U_{22}+Q'_{22}\tilde{D}_{22} \end{bmatrix}$$

$$: Q'_{11}, Q'_{12}, Q'_{21}, Q'_{22} \in m(h) \ \} \tag{3.3}$$

$$S = \left\{ \begin{bmatrix} \hat{Q}'_{11} & \hat{Q}'_{12} \\ D_{22}\hat{Q}'_{21} & \tilde{U}_{22}+D_{22}\hat{Q}'_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -N_{22}\hat{Q}'_{21} & \tilde{V}_{22}-N_{22}\hat{Q}'_{22} \end{bmatrix}^{-1} \right.$$

$$\left. : \hat{Q}'_{11}, \hat{Q}'_{12}, \hat{Q}'_{21}, \hat{Q}'_{22} \in \mathcal{M}(\mathcal{H}) \right\} \tag{3.4}$$

where the matrices Q'_{ij} and $\hat{Q}'_{ij}$ are of suitable dimensions.

3.6. Comments : 1) Theorem 3.5 shows that, given a Σ-admissible P with $P_{22} \in \mathcal{M}(\mathcal{G}_s)$, the class of all $\mathcal{H}$-stabilizing compensators C is parametrized by *four* parameters : $Q'_{11}, Q'_{12}, Q'_{21}, Q'_{22} \in \mathcal{M}(\mathcal{H})$: indeed, the theorem shows that the map $(Q'_{11}, Q'_{12}, Q'_{21}, Q'_{22}) \mapsto C$ is surjective, and Lemma 3.7 below shows that this map is injective.

2) If $P_{22} \in \mathcal{M}(\mathcal{G})$ instead of $\mathcal{M}(\mathcal{G}_s)$, then in equations (3.3) and (3.4) we take those Q'_{22} and $\hat{Q}'_{22} \in \mathcal{M}(\mathcal{H})$ s.t. $\det(V_{22}-Q'_{22}\tilde{N}_{22}) \in \mathcal{I}$ and $\det(\tilde{V}_{22}-N_{22}\hat{Q}'_{22}) \in \mathcal{I}$. Lemma 1.14 guarantees that if $P_{22} \in \mathcal{M}(\mathcal{G}_s)$ these determinants are $\in \mathcal{I}$ for *all* $Q'_{22}, \hat{Q}'_{22} \in \mathcal{M}(\mathcal{H})$.

3.7. Lemma : The map $(Q'_{11}, Q'_{12}, Q'_{21}, Q'_{22}) \mapsto C$ defined in equation (3.3) or (3.4) is injective.

Proof : Consider

$$C = \tilde{D}'^{-1}\tilde{N}' = \begin{bmatrix} I & -Q'_{12}\tilde{N}_{22} \\ 0 & V_{22}-Q'_{22}\tilde{N}_{22} \end{bmatrix}^{-1} \begin{bmatrix} Q'_{11} & Q'_{12}\tilde{D}_{22} \\ Q'_{21} & U_{22}+Q'_{22}\tilde{D}_{22} \end{bmatrix} \tag{3.5}$$

and

$$\hat{C} = \hat{N}'\hat{D}'^{-1} = \begin{bmatrix} \hat{Q}'_{11} & \hat{Q}'_{12} \\ D_{22}\hat{Q}'_{21} & \tilde{U}_{22}+D_{22}\hat{Q}'_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -N_{22}\hat{Q}'_{21} & \tilde{V}_{22}-N_{22}\hat{Q}'_{22} \end{bmatrix}^{-1} \tag{3.6}$$

Then $C = \hat{C}$ iff

$$\tilde{D}'\hat{N}' = \tilde{N}'\hat{D}' \tag{3.7}$$

Using the generalized Bezout Identity equation (1.11) in equation (3.7), and substituting for $\tilde{D}', \tilde{N}', \hat{N}', \hat{D}'$ from equations (3.5)-(3.6) it is easy to verify that

$$C = \hat{C} \text{ if and only if } Q'_{11} = \hat{Q}'_{11},\ Q'_{12} = \hat{Q}'_{12},\ Q'_{21} = \hat{Q}'_{21} \text{ and } Q'_{22} = \hat{Q}'_{22} \tag{3.8}$$

Let $\hat{C} = \hat{N}'\hat{D}'^{-1} = \tilde{\hat{D}}'^{-1}\tilde{\hat{N}}'$ and consider $\bar{C} = \bar{N}'\bar{D}'^{-1}$ with $\bar{Q}'_{11}, \bar{Q}'_{12}, \bar{Q}'_{21}, \bar{Q}'_{22}$ replacing the $\hat{Q}'_{ij}$'s in equation (3.6). Then by equations (3.7), (3.8), (1.11), $\hat{C} = \bar{C}$ <=> $\hat{N}'\hat{D}'^{-1} = \bar{N}'\bar{D}'^{-1}$ <=> $\tilde{\hat{D}}'^{-1}\tilde{\hat{N}}' = \bar{N}'\bar{D}'^{-1}$ <=> $\hat{Q}'_{11} = \bar{Q}'_{11}, \hat{Q}'_{12} = \bar{Q}'_{12}, \hat{Q}'_{21} = \bar{Q}'_{21}, \hat{Q}'_{22} = \bar{Q}'_{22}$.

This shows that there is a one-to-one correspondence between the "free" parameters $Q'_{11}, Q'_{12}, Q'_{21}, Q'_{22} \in \mathcal{M}(\mathcal{H})$ and the compensator C. Equivalently, suppose P is a given Σ–admissible plant and we have chosen a ***particular*** r.c.f.r. (N, D), l.c.f.r. $(\tilde{D}, \tilde{N})$ as well as particular matrices $U_{22}, \tilde{U}_{22}, V_{22}, \tilde{V}_{22}$ s.t. equation (1.11) holds. Then corresponding to each $C \in \mathcal{S}$, there is unique $Q'_{11}, Q'_{12}, Q'_{21}, Q'_{22}$ s.t. C is given by equation (3.5).

■

Proof of Theorem 3.5 : We prove only equation (3.3) since the proof of equation (3.4) is similar. Since by assumption, P is Σ–admissible , C $\mathcal{H}$–stabilizes P if and only if C is Σ–admissible and equation (2.32) holds. Then, by equation (2.38) and Lemma 1.12, C $\mathcal{H}$–stabilizes P iff for some $Q'_{11}, Q'_{12}, Q'_{21}, Q_{22} \in \mathcal{M}(\mathcal{H})$ and U'_{22}, V'_{22} satisfying the generalized Bezout equation (2.9),

$$C = \begin{bmatrix} I & -Q'_{12}U'_{22} \\ 0 & V_{22}-Q'_{22}\tilde{N}_{22} \end{bmatrix}^{-1} \begin{bmatrix} Q'_{11} & Q'_{12}V'_{22} \\ Q'_{21} & U_{22}+Q'_{22}\tilde{D}_{22} \end{bmatrix} \tag{3.9}$$

In equation (3.9) we used the fact that, by Lemma 1.14, $\det(V_{22}-Q'_{22}\tilde{N}_{22}) \in \mathcal{I}$ for all $Q'_{22} \in \mathcal{M}(\mathcal{H})$ and hence by Lemma 1.12, $(V_{22}-Q'_{22}\tilde{N}_{22})^{-1}\,(U_{22}+Q'_{22}\tilde{D}_{22})$ is a legitimate left-coprime factorization of $C_{22} = \tilde{D}'^{-1}_{22}\tilde{N}'_{22}$. From equation (2.24),

$$V'_{22} = \tilde{D}_{22} \ , \ U'_{22} = \tilde{N}_{22} \tag{3.10}$$

satisfies equation (2.9). Therefore using equation (3.10) in equation (3.9) , we use the l.c.f.r. of C which is given in equation (3.3).

■

3.8. Corollary : Let the assumptions of Theorem 3.5 hold. Then the set of all achievable I/O maps $\mathcal{A}_{yu}$ is given by equation (3.11) below.

$$\mathcal{A}_{yu} := \left\{ \begin{bmatrix} \hat{N}_{11}+N_{12}Q'_{22}\tilde{N}_{21} & \vdots & N_{12}(V_{22}-Q'_{22}\tilde{N}_{22}) & \vdots & N_{12}Q'_{21} & \vdots & N_{12}(U_{22}+Q'_{22}\tilde{D}_{22}) \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ (\tilde{V}_{22}-N_{22}Q'_{22})\tilde{N}_{21} & \vdots & N_{22}(V_{22}-Q'_{22}\tilde{N}_{22}) & \vdots & N_{22}Q'_{21} & \vdots & N_{22}(U_{22}+Q'_{22}\tilde{D}_{22}) \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ -Q'_{12}\tilde{N}_{21} & \vdots & -Q'_{12}\tilde{N}_{22} & \vdots & Q'_{11} & \vdots & Q'_{12}\tilde{D}_{22} \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ -(\tilde{U}_{22}+D_{22}Q'_{22})\tilde{N}_{21} & \vdots & (\tilde{U}_{22}+D_{22}Q'_{22})\tilde{N}_{22} & \vdots & D_{22}Q'_{21} & \vdots & D_{22}(U_{22}+Q'_{22}\tilde{D}_{22}) \end{bmatrix} \right.$$

$$: Q'_{12}, Q'_{21}, Q'_{22} \in \mathcal{m}(\mathcal{h}) \Big\} \qquad (3.11)$$

Proof : Substitute equations (3.3)-(3.4) into equation (2.14).

■

3.9. Comments : 1) Note that the set of achievable I/O maps $\mathcal{A}_{yu}$ in equation (3.11) uses the r.c.f.r. in equation (2.35) for P. Similarly, we could use a l.c.f.r. of P in equation (2.36) to obtain the set of all achievable I/O maps.

2) The parametrization of all $\mathcal{h}$-stabilizing compensators has *four* degrees of freedom as seen from equations (3.3)-(3.4). Each of the closed-loop maps depends on only one of the four parameters Q'_{11}, Q'_{12}, Q'_{21}, Q'_{22}. Consider for example $H_{y_o u'_o} = N_{12}Q'_{21}$ which has N_{12} as a left factor. In the case that $\mathcal{h} = R_{\mathcal{u}}$ as in Example 1.3, this implies that the $\bar{\mathcal{U}}$-zeros of N_{12} are also $\bar{\mathcal{U}}$-zeros of $H_{y_o u'_o}$. In the case where N_{12} is square, if we wish to diagonalize the map $H_{y_o u'_o} : u'_o \mapsto y'_o$, the "free" compensator parameter Q'_{21} should be chosen appropriately (see [Des.4]).

■

Conclusions

In this paper the analysis of linear time-invariant control systems was extended to the system configuration in figure 1. Although the results are essentially contained in [Net.1], the techniques used are simpler and do not require the introduction of new concepts. The present derivation follows in spirit the one used in design with the previous system configurations $S_1(P, C)$ and $S_2(P, C)$. These systems are in fact special cases of the system $\Sigma(P, C)$ considered here.

The concept of Σ–admissibility is of key importance here because the system $\Sigma(P, C)$ can be h–stabilized only if all the "instabilities" of the plant P are "included in " P_{22}. Note that P_{22} is the only partial map of the plant P in the feedback loop. Similarly, the stabilizing compensator has to be also Σ–admissible . The parametrization of these compensators has four degrees-of-freedom. Each of the I/O maps achieved by the system $\Sigma(P, C)$ depends on one and only one of the four free parameters Q'_{11} , Q'_{12} , Q'_{21} , and Q'_{22}. Therefore, the map $H_{y_o u'_o} : u'_o \mapsto y_o$, which depends on Q'_{21} can be chosen independently of the map from u_o to y_o, which depends on Q'_{22} . In [Des.4], the map $H_{y_o u'_o}$ was diagonalized in the case that P_{12} is square. The asymptotic tracking at y_o of a class of input signals going into u'_o was discussed in [Des.5]. This parametrization may further be used in optimal design problems and in fault diagnosis via the compensator output y'_o.

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Figure Captions:

Fig. 1 The system $\Sigma(P, C)$

Fig. 2 The system $\Sigma(P, C)$ after factorization

Fig. 3 The system $\Sigma(P, C)$ with Σ–admissible P and Σ–admissible C. Note that all the instabilities of P_{11}, P_{12}, P_{21} are a subset of those of P_{22}, i.e., of those generated by D_{22}^{-1}. Note the duality between the block diagrams of P and C.

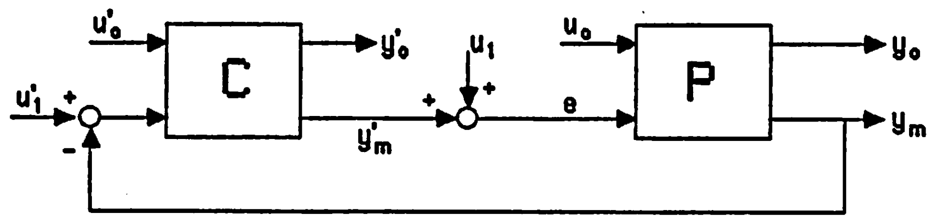


Fig. 1

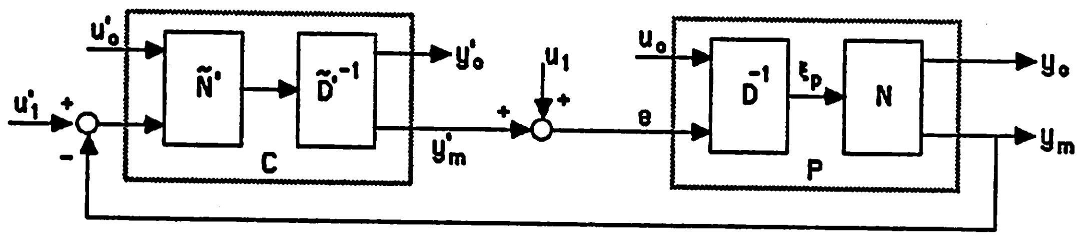
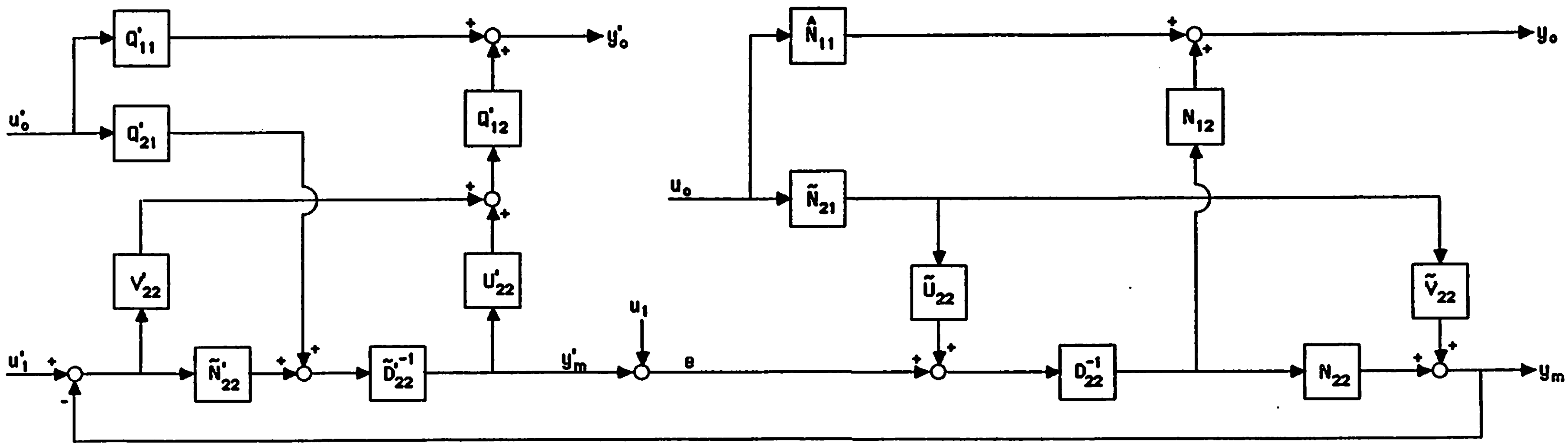


Fig. 2



Q'11
Q'21
u'o
+
y'o
Q'12
V22
U'22
u'1
Ñ'22
D̃'22 -1
y'm
u1
e
N̂11
yo
N12
uo
Ñ21
Ũ22
Ṽ22
D22 -1
N22
ym

Fig. 3