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S. Behtash and S. Sastry**

Department of Electrical Engineering and Electronics Research Laboratory
University of California, Berkeley, CA 94720

ABSTRACT

We consider the problem of local stabilization of nonlinear control systems whose linearizations contain uncontrollable modes on the $j\omega$-axis. A general methodology for designing a stabilizing control is presented. It involves the following steps: 1) Reduction of the stability problem to the stability of the center manifold system. 2) Simplification of the vector field on the center manifold using the theory of normal forms. 3) Finding conditions under which the simplified vector field is asymptotically stable. Following these steps, three cases of degeneracies in the linearized system are treated and necessary and sufficient conditions for the existence of stabilizing controls are given in each case. Finally a theorem is presented regarding the robustness of the above control strategies.

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Department of Electrical Engineering and Electronics Research Laboratory
University of California, Berkeley, CA 94720

1. Introduction

In this paper we discuss the stabilization of nonlinear systems with degenerate linearization, i.e. systems whose linearization is not stabilizable by linear state feedback. We feel that the present work is part of an ongoing effort at developing a dynamical systems viewpoint to the control of nonlinear systems. Such a viewpoint will extend the domain of applicability of techniques for the exact linearization of non-linear systems, by coordinate transformation and state feedback, from completeley linearizable to partially linearizable systems. The present work was inspired by, and is in the spirit of a recent paper of Aeyels [1], and related work by Abed and Fu [4], [5].

2. Formulation

We consider the problem of stabilizing, by state feedback, systems of the form:

$$\dot{\xi}=\varphi(\xi)+bu \tag{2.1}$$

where $\xi\in\mathbb{R}^n$, $u\in\mathbb{R}$, $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ is smooth, $b\in\mathbb{R}^n$, and 0 is an equilibrium point of the undriven system (2.1) ie $\varphi(0)=0$. The extension to systems of the form $\dot{\xi}=\varphi(\xi,u)$ is straightforward and will be discussed in section 6..

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By way of notation, let $A=D_\xi\varphi(0)$, the Jacobian of φ at $\xi=0$. We partition the spectrum of A as:

$$\sigma(A)=\sigma^s \cup \sigma^u \cup \sigma^c$$

where $\sigma^s \subset \mathbb{C}_-$, $\sigma^u \subset \mathbb{C}_+$, and $\sigma^c \subset \{j\omega \mid \omega\in\mathbb{R}\}$. Using basis vectors for the (generalized) eigenspaces of σ^s, σ^u, and σ^c we may transform (2.1) to the form:

$$\begin{bmatrix}\dot\xi_1\\ \dot\xi_2\\ \dot\xi_3\end{bmatrix}=\begin{bmatrix}A_{11} & 0 & 0\\ 0 & A_{22} & 0\\ 0 & 0 & A_{33}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}+\begin{bmatrix}\varphi_1(\xi_1,\xi_2,\xi_3)\\ \varphi_2(\xi_1,\xi_2,\xi_3)\\ \varphi_3(\xi_1,\xi_2,\xi_3)\end{bmatrix}+\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}u \tag{2.2}$$

where $\sigma(A_{11})=\sigma^c$, $\sigma(A_{22})=\sigma^s$, $\sigma(A_{33})=\sigma^u$, $\xi_1\in\mathbb{R}^{n_1}$, $\xi_2\in\mathbb{R}^{n_2}$, and $\xi_3\in\mathbb{R}^{n_3}$.

It is easy to see that (2.2) is locally stabilizable by linear state feedback when (A_{11},b_1), and (A_{33},b_3) are completely controllable. It is also easy to see that if (A_{33},b_3) is not controllable, then no feedback law which is smooth at the origin can stabilize the system (2.2). Consequently we shall be interested in the case when (A_{33},b_3) is controllable, and the critical eigenvalues (those of A_{11}) are completely uncontrollable, ie $b_1=0$. Now our objective is to construct a feedback law $u=F(\xi_1,\xi_2,\xi_3)$ to stabilize the system. From the preceding discussion it is plausible that higher order (quadratic, cubic, etc) terms in ξ_1 are needed to stabilize the system. To make this statement precise we use certain results of center manifold theory as summarized in section 3 below.

3. Mathematical Preliminaries

For our development we need two sets of tools: center manifold theory and normal form theory for differential equations. We review them briefly in the context that we need them here; the center manifold theorems are taken from Carr [2], and the normal form theorems from Guckenheimer and Holmes [3].

3.1. Center Manifold Theory

Consider the following C^k dynamical system in $\mathbb{R}^n$:

$$\dot{x} = f(x) \tag{3.1}$$

A set $S \subset \mathbb{R}^n$ is said to be a *local invariant set* if for all $x_0 \in S$ there exists $T > 0$ such that the solution of the differential equation (3.1) passing through x_0 at $t=0$ remains in S for $|t| < T$. If T can be chosen to be ∞, then S is said to be an *invariant set*.

Now consider the following C^k dynamical system in $\mathbb{R}^n$:

$$\begin{aligned} \dot{x} &= Ax + f(x,y) \qquad && x \in \mathbb{R}^n \\ \dot{y} &= By + g(x,y) \qquad && y \in \mathbb{R}^m \end{aligned} \tag{3.2}$$

where $(x=0, y=0)$ is an equilibrium point, that is

$$f(0,0)=0 \quad ; g(0,0)=0 \tag{3.3}$$

Further f and g comprise only of quadratic and higher order terms , that is

$$D_x f(0,0)=0 \quad ; D_y f(0,0)=0 \quad ; D_x g(0,0)=0 \quad ; D_y g(0,0)=0 \tag{3.4}$$

We also assume that $\sigma(B) \subset \mathbb{C}_-$ and $\sigma(A) \subset \{j\omega \mid \omega \in \mathbb{R}\}$. For this system we have

Definition 3.1 A local invariant manifold M for the system (3.2) is called a *center manifold* if it contains the origin $(x=0, y=0)$ and is tangent to $y=0$ at the origin.

Remarks

1) $\{(x,0) \mid x \in R^n\}$ is the generalized eigenspace of the $j\omega$-axis eigenvalues of the linearization of the system (3.2). Thus a center manifold is a "nonlinear eigenspace" corresponding to the $j\omega$-axis eigenvalues.

2) If M is given locally as the graph of a function $y = h(x)$, then:

$$h(0)=0$$

$$Dh(0)=0$$

It is a basic theorem that center manifolds exist (though elementary examples show that they are not unique) and are locally given as the graph of a function $y=h(x)$.

Theorem 3.1 (Existence of Center Manifolds) If f and g in (3.2) are C^k vector fields for $k \geq 2$, then there exists a center manifold $y=h(x)$, $|x|<\varepsilon$, where h is of class C^k.

The flow on the center manifold is governed by

$$\dot{u}=Au + f(u,h(u)) \tag{3.5}$$

The following theorem connects the stability of the system (3.5) to that of the system (3.2).

Theorem 3.2 If the zero solution of (3.5) is stable (unstable, asymptotically stable), then the zero solution of (3.2) is stable (unstable, asymptotically stable).

Remark

In the instance that the zero solution of (3.5) is stable or asymptotically stable, we can relate the solutions of (3.5) to those of (3.2) for $(x(0),y(0))$ sufficiently small. Let $(x(t),y(t))$ be a solution of (3.2) with $(x(0),y(0))$ small enough. Then there exists a solution $u(t)$ of (3.5) such that

$$x(t)=u(t) + O(e^{-\gamma t})$$

$$y(t)=h(u(t)) + O(e^{-\gamma t}) \qquad \text{as } t \to \infty$$

where the rate of convergence to the center manifold, γ, is related to the eigenvalues of B alone.

Thus we see that the study of stability (instability) of the system (3.2) may be reduced to the study of stability (instability) of (3.5), provided we have an expression for the function h. To solve for $h(x)$, we use the fact that $y=h(x)$ is invariant under the flow of (3.2), thus

$$\dot{y} = \frac{d}{dt}h(x) = Dh(x)[Ax + f(x,h(x))]$$

$$= Bh(x) + g(x,h(x))$$

that is h satisfies the partial differential equation

$$Dh(x)[Ax + f(x,h(x))] = Bh(x) + g(x,h(x)) \tag{3.6}$$

with $h(0)=0$; $Dh(0)=0$

Any solution of the PDE (3.6) is a center manifold for (3.2). Typically, it is difficult to solve the PDE (3.6), consequently the following approximation theorem is of interest.

Theorem 3.3 Let φ be a C^1 mapping from a neighborhood of $\mathbb{R}^n$ into $\mathbb{R}^n$ such that

$$\varphi(0)=0 \quad ; D\varphi(0)=0$$

if φ satisfies the PDE (3.6) modulo terms of $O(|x|^k)$ then as $x \to 0$, we have:

$$|h(x) - \varphi(x)| = O(|x|^k)$$

Remark

In particular, Theorem 3.3 allows us to approximate $h(x)$ by polynomials in x to any desired accuracy.

With Theorem 3.3, we are now ready to study the stability of the center manifold system (3.5). Since the linear part of the vector field on the center manifold has all its eigenvalues on the $j\omega$-axis, we need to study the higher order terms of the vector field. This is done next in a systematic way.

3.2. Normal Forms

To study the behavior of the solutions on the center manifold it is helpful to simplify the vector field but the simplifications should preserve the qualitative behavior of the solutions at least locally around the equilibrium point. In the following discussion a systematic procedure of simplifying the vector fields by means of repeated coordinate transformations is presented. The resulting simplified vector fields are called *normal forms.*

Define H_k to be the real vector space of vector fields whose coefficients are polynomials of degree k. Given a linear vector field $L(x)$ we have the subspace

$$ad\ L\ (H_k) = \{\ ad_L h(x) \mid h(x) \in H_k\ \}$$

and its complement G_k; ie,

$$H_k = ad\ L\ (H_k) \oplus G_k \tag{3.7}$$

Theorem 3.4 Let $\dot{x}=f(x)$ be a C^r dynamical system with $f(0)=0$ and $Df(0)x=L(x)$. Then there exists an analytic change of coordinates in a neighborhood of the origin transforming the system to $\dot{y}=g(y)$ such that

$$g(y)=g^1(y)+g^2(y)+\ldots+g^r(y)+R_r \tag{3.8}$$

where $g^1(y)=L(y)$;$g^k(y)\in G^k$, $2\le k\le r$ and $R_r=o(|y|^r)$.

Proof It suffices to show that for a given $k\ge 2$ the components of $ad\ L\ (H_k)$ can be locally removed from the vector field by an analytic change of coordinates. Performing this for $k=2,..,r$ we obtain the desired coordinate transformation as the composition of the transformations for each k. Thus we let

$$x = y + P(y)$$

where $P(y)$ is a polynomial of degree k. We point out that $D_y x(0)=I$ so that we have a local diffeomorphism (preserving the local behavior of the flow of the vector field around the origin). Using this transformation, we get

$$\dot{y}=(I+DP(y))^{-1}[f(y)+Df(y)P(y)+o(|y|^r)] \tag{3.9}$$

Now note that

$$(I+DP(y))^{-1}=I-DP(y)+o(|y|^r) \tag{3.10}$$

and

$$Df(y)P(y)=Df(0)P(y)+o(|y|^r)$$

$$=DL\ P(y)+o(|y|^r) \tag{3.11}$$

Using (3.10) and (3.11) in (3.9) we have:

$$\dot{y}=f(y)+DL\ P-DP\ L+o(|y|^r) \tag{3.12}$$

If we set

$$f(y)=f_{k-1}(y)+f^k(y)+o(|y|^r)$$

with $f_{k-1}(y)=\sum_{j=1}^{k-1} f^j(y)$ and $f^j(y)\in H_j$ for $j=1,..,k$ we get:

$$g(y)=g_{k-1}(y)+g^k(y)+o(|y|^r) \tag{3.13}$$

where

$$g_{k-1}(y)=f_{k-1}(y) \tag{3.14}$$

$$g^k(y)=f^k(y)+ad\ L\ (P(y)) \tag{3.14}$$

By choosing the coefficients of $P(y)$, components of $ad\ L\ (H_k)$ can be removed from $f^k(y)$ while all lower order terms remain unchanged. Thus $g^k(y)\in G_k$.

Although the transformations for each k leave all lower order terms unchanged, they do alter the higher order terms. We have the following corollary to this effect.

Corollary 3.1 Let $\dot{x}=f(x)$ be a C^r dynamical system, such that $f^j(x)\in G_j$ for $j=2,..,k-1$, $k<r$. Let $\dot{y}=g(y)$ represent the transformed system after the removal of $O(k)$ terms in the span of $ad\ L\ (H_k)$. Then we have:

$$g^{j+k}(y)=f^{j+k}(y)+ad\ f^{j+1}(y)(P(y)) \qquad j=0,..,k-2 \tag{3.16}$$

$$g^{2k-1}(y)=f^{2k-1}(y)+ad\ f^{k}(y)(P(y))-DP(y)[ad\ L(P(y))] \tag{3.17}$$

where (3.16) and (3.17) make sense provided $j+k\le r$ and $2k-1\le r$ respectively.

Proof Using the change of coordinates $x=y+P(y)$ we have:

$$\dot{y}=(I+DP(y))^{-1}f(y+P(y)) \tag{3.18}$$

We note

$$(I+DP(y))^{-1}=I-DP(y)+(DP(y))^2+O(2k-2) \tag{3.19}$$

$$f(y+P(y))=f(y)+Df(y)P(y)+O(2k) \tag{3.20}$$

Now using (3.19) and (3.20) in (3.18) we get:

$$\dot{y}=f(y)+DfP-DPf-DP[DfP-DPf]+O(2k) \tag{3.21}$$

Collecting the $O(l)$ terms for various values of l we obtain (3.16) and (3.17) for $g(y)$.

4. Stabilizing Control Laws

Consider the system (2.2), with $b_1=0$

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\end{bmatrix}=\begin{bmatrix}A_{11} & 0 & 0\\ 0 & A_{22} & 0\\ 0 & 0 & A_{33}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}+\begin{bmatrix}\varphi_1(\xi_1,\xi_2,\xi_3)\\ \varphi_2(\xi_1,\xi_2,\xi_3)\\ \varphi_3(\xi_1,\xi_2,\xi_3)\end{bmatrix}+\begin{bmatrix}0\\ b_2\\ b_3\end{bmatrix}u \tag{2.2}$$

with $\sigma(A_{11})\subset\{j\omega \mid \omega\in R\}$, $\sigma(A_{22})\subset \mathbb{C}_-$, $\sigma(A_{33})\subset \mathbb{C}_+$, and (A_{33},b_3) controllable. By choosing u of the form $u=v+K_3\xi_3$ and $\sigma(A_{33}+b_3K_3)\subset \mathbb{C}_-$, we may consider the problem of stabilizing

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\end{bmatrix}=\begin{bmatrix}A_{11} & 0 & 0\\ 0 & A_{22} & 0\\ 0 & 0 & A'_{33}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}+\begin{bmatrix}\varphi_1(\xi_1,\xi_2,\xi_3)\\ \varphi_2(\xi_1,\xi_2,\xi_3)\\ \varphi_3(\xi_1,\xi_2,\xi_3)\end{bmatrix}+\begin{bmatrix}0\\ b_2\\ b_3\end{bmatrix}u \tag{4.1}$$

with $\sigma(A'_{33})\subset \mathbb{C}_-$. We will drop the prime superscript on A'_{33} in the sequel and assume that $\sigma(A_{33}) \subset \mathbb{C}_-$. Now our objective is to find $v = F(\xi_1, \xi_2, \xi_3)$, an analytic feedback law, such that the equilibrium point of (4.1) is asymptotically stable.

It follows that when we set $v = F(\xi_1, \xi_2, \xi_3)$ with A_{33} stable, that the center manifold is tangent to $\{(\xi_1, 0, 0) \mid \xi_1 \in \mathbb{R}^{n_1}\}$ and is given locally by

$$\begin{bmatrix} \xi_2 \\ \xi_3 \end{bmatrix} = h(\xi_1) = \begin{bmatrix} h_2(\xi_1) \\ h_3(\xi_1) \end{bmatrix}$$

Further from (3.6) it follows that h satisfies the following PDE:

$$Dh(\xi_1)[A_{11}\xi_1 + \varphi_1(\xi_1, h_2(\xi_1), h_3(\xi_1))] =$$

$$\begin{bmatrix} A_{22}h_2(\xi_1) + \varphi_2(\xi_1, h_2(\xi_1), h_3(\xi_1)) + b_2 F(\xi_1, h_2(\xi_1), h_3(\xi_1)) \\ A_{33}h_3(\xi_1) + \varphi_3(\xi_1, h_2(\xi_1), h_3(\xi_1)) + b_3 F(\xi_1, h_2(\xi_1), h_3(\xi_1)) \end{bmatrix} \tag{4.2}$$

and the flow on the center manifold is governed by

$$\dot{\xi}_1 = A_{11}\xi_1 + \varphi_1(\xi_1, h_2(\xi_1), h_3(\xi_1)) \tag{4.3}$$

Thus, we need to choose $F(\xi_1, \xi_2, \xi_3)$ in such a way that the resulting $h(\xi_1)$ produces an asymptotically stable equilibrium point on the center manifold. While a general solution is not available to this problem we consider several cases for the matrix A_{11}. The case where $A_{11} \in \mathbb{R}^{2\times 2}$ and has a pair of imaginary eigenvalues was solved by D. Aeyels [1]. In [4], Abed and Fu treat the same case using bifurcation formulae derived from the projection method. The same technique is also employed in [5] where, the case of a single critical mode is treated. The cases covered here, which have not been treated by Aeyels or Abed and Fu, are the following

(i) Double zero eigenvalues.

(ii) Pair of imaginary and a simple zero eigenvalue.

(iii) Two pairs of imaginary eigenvalues.

In the remainder of this section we assume, for simplicity, that $n_2 = 0$ and $n_3 = 1$. We will show in section 6, by way of an example, that there is no loss of generality in this assumption.

4.1. Case of Double Zero Eigenvalues

We consider here the case where $A_{11} \in \mathbb{R}^{2\times 2}$ and has the form:

$$A_{11}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We let $\begin{bmatrix} x \\ y \end{bmatrix}$ represent ξ_1 and we will drop the subscript from ξ_3 and represent it by ξ. We further let

$$\varphi_1(\xi_1,\xi)=\begin{bmatrix} f(x,y,\xi) \\ g(x,y,\xi) \end{bmatrix}$$

Now rewriting (4.1) with the above notation we get:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} f(x,y,\xi) \\ g(x,y,\xi) \end{bmatrix} \tag{4.4}$$

$$\dot{\xi}=-k\xi+\varphi_3(x,y,\xi)+v$$

where $k>0$. Since we will choose v to be of the form $F(x,y,\xi)$, we can assume that $\varphi_3(x,y,\xi)=0$. The center manifold is given locally by $\xi=h(x,y)$ with h satisfying

$$Dh(x,y)\begin{bmatrix} y+f(x,y,h(x,y)) \\ g(x,y,h(x,y)) \end{bmatrix}=-kh(x,y)+F(x,y,h(x,y)) \tag{4.5}$$

$$h(0)=0\,;\; Dh(0)=0$$

We now use Theorem 3.3 to approximate the center manifold upto terms of $O(3)$, ie,

$$h(x,y)=ax^2+bxy+cy^2+O(3) \tag{4.6}$$

Note that the choice of h in (4.6) automatically gives $h(0,0)=0$; $Dh(0,0)=0$. Next we choose F to be of the form:

$$F(x,y,\xi)=\alpha x^2+\beta xy+\gamma y^2 \tag{4.7}$$

Using (4.6) and (4.7) in (4.5) we get:

$$(2ax+by)(y+f(x,y,h))+(2cy+bx)g(x,y,h)=$$
$$-kax^2-kbxy-kcy^2+\alpha x^2+\beta xy+\gamma y^2+O(3) \tag{4.8}$$

Recalling that f and g are both of $O(2)$, we may equate terms of $O(2)$ in (4.8) to get:

$$\begin{bmatrix} k & 0 & 0 \\ 2 & k & 0 \\ 0 & 1 & k \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \tag{4.9}$$

For $k\neq 0$, we see from (4.9) that (a,b,c) can be arbitrarily assigned by choice of (α,β,γ) in the control law (4.7). In other words, the control law determines a center manifold upto terms of $O(3)$. The remaining problem is to determine what choice of the parameters (a,b,c) in (4.6) stabilizes the flow on the center manifold, which is given by:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} f(x,y,h(x,y)) \\ g(x,y,h(x,y)) \end{bmatrix} \tag{4.10}$$

This program is continued using the normal form theory of section 3.2 in the following theorem, where by a slight abuse of notation we let $f(x,y)=f(x,y,h(x,y))$ and $g(x,y)=g(x,y,h(x,y))$.

Theorem 4.1 The zero solution of the center manifold system (4.10) is not stabilizable unless

$$D_x^2 g=0 \tag{4.11a}$$

$$D_{xy}^2 g+D_x^2 f=0 \tag{4.11b}$$

Furthermore if (4.11) is satisfied, then the zero solution of (4.10) is locally asymptotically stable provided that

$$\frac{1}{3}D_x^3 g+(D_x^2 f)^2<0 \tag{4.12a}$$

$$D^3_{xxy} g + D^3_x f - D^2_x f \, (D^2_{xy} f + D^2_y g) < 0 \tag{4.12b}$$

where all the derivatives above are evaluated at the origin.

Proof For the vector fields in $\mathbb{R}^2$,

$$H_2 = span\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix} \begin{pmatrix} y^2 \\ 0 \end{pmatrix} \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix} \begin{pmatrix} 0 \\ y^2 \end{pmatrix} \begin{pmatrix} 0 \\ xy \end{pmatrix} \right\}$$

Further for the system in (4.10),

$$L(x,y) = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

Then,

$$ad\ L\ (H_2) = span\left\{ \begin{pmatrix} -xy \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} y^2 \\ 0 \end{pmatrix} \begin{pmatrix} x^2 \\ -2xy \end{pmatrix} \begin{pmatrix} y^2 \\ 0 \end{pmatrix} \begin{pmatrix} xy \\ -y^2 \end{pmatrix} \right\}$$

Thus a complement to $ad\ L\ (H_2)$ is given by,

$$G_2 = span\left\{ \begin{pmatrix} 0 \\ x^2 \end{pmatrix} \begin{pmatrix} 0 \\ xy \end{pmatrix} \right\}$$

Thus , the normal form of (4.10) can be written as,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \delta x^2 + \varepsilon xy \end{aligned} \quad + O(3) \tag{4.13}$$

where $\delta = \frac{1}{2} D^2_x g$ and $\varepsilon = D^2_{xy} g + D^2_x f$. It is easy to see that the zero solution of (4.13) is unstable for all nonzero values of δ and ε. Thus a necessary condition for stabilization is (4.11). Further with (4.11) holding, we may consider the $O(3)$ terms in the expansion of the vector field in (4.10). We have,

$$H_3 = span\left\{ \begin{pmatrix} x^3 \\ 0 \end{pmatrix} \begin{pmatrix} x^2 y \\ 0 \end{pmatrix} \begin{pmatrix} xy^2 \\ 0 \end{pmatrix} \begin{pmatrix} y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3 \end{pmatrix} \begin{pmatrix} 0 \\ x^2 y \end{pmatrix} \begin{pmatrix} 0 \\ xy^2 \end{pmatrix} \begin{pmatrix} 0 \\ y^3 \end{pmatrix} \right\}$$

Then,

$$ad\ L\ (H_3)=span\left\{\begin{bmatrix}3x^2y\\0\end{bmatrix}\begin{bmatrix}2xy^2\\0\end{bmatrix}\begin{bmatrix}y^3\\0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}x^3\\-3x^2y\end{bmatrix}\begin{bmatrix}x^2y\\2xy^2\end{bmatrix}\begin{bmatrix}xy^2\\-y^3\end{bmatrix}\begin{bmatrix}y^3\\0\end{bmatrix}\right\}$$

Therefore a complement to $ad\ L\ (H_3)$ is given by,

$$G_3=span\left\{\begin{bmatrix}0\\x^3\end{bmatrix}\begin{bmatrix}0\\x^2y\end{bmatrix}\right\}$$

Thus we see that the normal form of (4.10) upto terms of $O(4)$ may be written as,

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=\lambda x^3+\mu x^2y+\ O(4)\end{aligned}\tag{4.14}$$

where, $\lambda=\frac{1}{6}D_x^3g'$, and $\mu=\frac{1}{2}(D_{xxy}^3g'+D_x^3f')$. Here $\begin{bmatrix}f'\\g'\end{bmatrix}$ is the $O(3)$ vector field obtained from $\begin{bmatrix}f\\g\end{bmatrix}$ by removal of the $O(2)$ terms. Now using Corollary 3.1 to relate $\begin{bmatrix}f'\\g'\end{bmatrix}$ and $\begin{bmatrix}f\\g\end{bmatrix}$ we find for λ and μ,

$$\lambda=\frac{1}{6}D_x^3g+\frac{1}{2}(D_x^2f)^2$$

$$\mu=\frac{1}{2}(D_{xxy}^3g+D_x^3f)-\frac{1}{2}D_x^2f\,(D_{xy}^2+D_y^2g)$$

Next using the Lyapunov function candidate

$$V=-\frac{1}{4}\lambda x^4+\frac{1}{2}y^2$$

we have for $\dot{V}$

$$\dot{V}=-\lambda x^3y+\lambda x^3y+\mu x^2y^2$$

Thus for $\lambda<0$ and $\mu<0$, the equilibrium point of (4.14) is locally asymptotically stable.

Corollary 4.1 The zero solution of (4.4) is stabilizable by the control $v=\alpha x^2+\beta xy+\gamma y^2$, provided (4.11) is satisfied and $D^2_{x\xi}g\neq 0$.

Proof By inspection, if $D^2_{x\xi}g\neq 0$, then the values of $D^3_x g$ and $D^3_{xxy}g$ can be assigned arbitrarily by a proper choice of (a,b,c) (and thus by (α,β,γ)). Thus through the control v the parameters λ and μ can be made negative.

4.2. Case of a Pair of Imaginary and a Simple Zero Eigenvalues

We consider here the case where $A_{11}\in\mathbb{R}^{3\times 3}$ and is of the form

$$A_{11}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

In this case (4.1) may be written as

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}+\begin{bmatrix}f(x,y,z,\xi)\\ g(x,y,z,\xi)\\ p(x,y,z,\xi)\end{bmatrix} \tag{4.15}$$

$$\dot{\xi}=-k\xi+v \qquad\qquad k>0$$

The center manifold is represented by $\xi=h(x,y,z)$. Letting $v=F(x,y,z,\xi)$, we get that h satisfies the following

$$Dh(x,y,z)\begin{bmatrix}-y+f(x,y,z,h(x,y,z))\\ x+g(x,y,z,h(x,y,z))\\ p(x,y,z,h(x,y,z))\end{bmatrix}=-kh(x,y,z)+F(x,y,z,h(x,y,z)) \tag{4.16}$$

$$h(0)=0\,;\ Dh(0)=0$$

As before, using Theorem 3.3, we approximate the center manifold upto terms of $O(3)$

$$h(x,y,z)=ax^2+by^2+cz^2+dxy+exz+lyz+O(3) \tag{4.17}$$

Next we choose the following form for the feedback law F

$$F(x,y,z,\xi)=\alpha x^2+\beta y^2+\gamma z^2+\sigma xy+\eta xz+\mu yz \tag{4.18}$$

Using (4.17) and (4.18) in (4.16) and equating the coefficients of the $O(2)$ terms we get

$$\begin{bmatrix} k & 0 & 0 & 1 & 0 & 0 \\ 0 & k & 0 & -1 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & 0 \\ -2 & 2 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & k & 1 \\ 0 & 0 & 0 & 0 & -1 & k \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ l \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \sigma \\ \eta \\ \mu \end{bmatrix} \tag{4.19}$$

For $k \neq 0$, (4.19) implies that $(a, b, \ldots)$ can be arbitrarily assigned by a choice of the control prameters $(\alpha, \beta, \ldots)$. Thus the control law determines a center manifold upto $O(3)$ terms. Next we wish to determine what choice of $(a, b, \ldots)$ renders an asymptotically stable equilibrium point for the center manifold system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} f(x,y,z,h(x,y,z)) \\ g(x,y,z,h(x,y,z)) \\ p(x,y,z,h(x,y,z)) \end{bmatrix} \tag{4.20}$$

This is done in the next theorem, where as before $(f', g', p')^T$ denotes the vector field obtained from $(f, g, p)^T$ after removal of the $O(2)$ terms.

Theorem 4.2 The zero solution of (4.20) is not stabilizable unless

$$D^2_{xz} f + D^2_{yz} g = 0 \tag{4.21a}$$

$$D^2_{x} p + D^2_{y} p = 0 \tag{4.21b}$$

$$D^2_{z} p = 0 \tag{4.21c}$$

Furtheremore if (4.21) is satisfied, then the zero solution of (4.21) is asymptotically stable provided that

$$D^3_x f' + D^3_y g' + D^3_{xyy} f' + D^3_{xxy} g' < 0 \tag{4.22a}$$

$$D^3_z p' < 0 \tag{4.22b}$$

$$D^3_{xzz} f' + D^3_{yzz} g' < 0 \tag{4.22c}$$

$$D^3_{xxz} p' + D^3_{yyz} p' < 0 \tag{4.22d}$$

Conditions (4.22c) and (4.22d) may be replaced by the single condition

$$sgn\,(D^3_{xzz}f\,'+D^3_{yzz}g\,')=-sgn\,(D^3_{xxz}p\,'+D^3_{yyz}p\,') \tag{4.22e}$$

Remark

We may use Corollary 3.1 to express (4.22) in terms of the vector field $(f\,,g\,,p\,)^T$. The resulting expressions, although extremely involved, would consist of the terms appearing in (4.22) plus additional terms involving various second order partial derivatives of $(f\,,g\,,p\,)^T$. The stabilizability conditions on $(f\,,g\,,p\,)^T$, however, can be determined from (4.22) alone, since the control can only affect the $O(3)$ terms in $(f\,,g\,,p\,)^T$.

Proof In $\mathbb{R}^3$ we have:

$$H_2=span\left\{\begin{pmatrix}x^2\\0\\0\end{pmatrix}\begin{pmatrix}y^2\\0\\0\end{pmatrix}\begin{pmatrix}z^2\\0\\0\end{pmatrix}\begin{pmatrix}xy\\0\\0\end{pmatrix}\begin{pmatrix}xz\\0\\0\end{pmatrix}\begin{pmatrix}yz\\0\\0\end{pmatrix},\right.$$

$$\begin{pmatrix}0\\x^2\\0\end{pmatrix}\begin{pmatrix}0\\y^2\\0\end{pmatrix}\begin{pmatrix}0\\z^2\\0\end{pmatrix}\begin{pmatrix}0\\xy\\0\end{pmatrix}\begin{pmatrix}0\\xz\\0\end{pmatrix}\begin{pmatrix}0\\yz\\0\end{pmatrix},$$

$$\left.\begin{pmatrix}0\\0\\x^2\end{pmatrix}\begin{pmatrix}0\\0\\y^2\end{pmatrix}\begin{pmatrix}0\\0\\z^2\end{pmatrix}\begin{pmatrix}0\\0\\xy\end{pmatrix}\begin{pmatrix}0\\0\\xz\end{pmatrix}\begin{pmatrix}0\\0\\yz\end{pmatrix}\right\}$$

And for the system in (4.20),

$$L(x,y,z)=\begin{pmatrix}-y\\x\\0\end{pmatrix}$$

Thus,

$$ad\ L\ (H_2)=span\left\{\begin{pmatrix}2xy\\x^2\\0\end{pmatrix}\begin{pmatrix}-2xy\\y^2\\0\end{pmatrix}\begin{pmatrix}0\\z^2\\0\end{pmatrix}\begin{pmatrix}y^2-x^2\\xy\\0\end{pmatrix}\begin{pmatrix}yz\\xz\\0\end{pmatrix}\begin{pmatrix}-xz\\yz\\0\end{pmatrix},\right.$$

$$\begin{pmatrix}-x^2\\2xy\\0\end{pmatrix}\begin{pmatrix}-y^2\\-2xy\\0\end{pmatrix}\begin{pmatrix}-z^2\\0\\0\end{pmatrix}\begin{pmatrix}-xy\\y^2-x^2\\0\end{pmatrix}\begin{pmatrix}-xz\\yz\\0\end{pmatrix}\begin{pmatrix}-yz\\-xz\\0\end{pmatrix},$$

$$\begin{pmatrix}0\\0\\2xy\end{pmatrix}\begin{pmatrix}0\\0\\-2xy\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\y^2-x^2\end{pmatrix}\begin{pmatrix}0\\0\\yz\end{pmatrix}\begin{pmatrix}0\\0\\-zx\end{pmatrix}\Bigg\}$$

It is easy to show that a complement to $ad\ L\ (H_2)$ is given by,

$$G_2=span\left\{\begin{pmatrix}xz\\yz\\0\end{pmatrix}\begin{pmatrix}yz\\-xz\\0\end{pmatrix}\begin{pmatrix}0\\0\\x^2+y^2\end{pmatrix}\begin{pmatrix}0\\0\\z^2\end{pmatrix}\right\}$$

Therefore the $O(2)$ normal form associated with (4.20) is of the form,

$$\begin{aligned}\dot{x}&=-y+\delta xz+\varepsilon yz\\ \dot{y}&=x+\delta yz-\varepsilon xz \quad +O(3)\\ \dot{z}&=\lambda(x^2+y^2)+\rho z^2\end{aligned}\tag{4.23}$$

where, $\delta=\frac{1}{2}(D^2_{xz}f+D^2_{yz}g)$, $\varepsilon=\frac{1}{2}(D^2_{yz}f-D^2_{xz}g)$, $\lambda=\frac{1}{4}(D^2_xp+D^2_yp)$, and $\rho=\frac{1}{2}D^2_zp$.

Now transforming the normal form in (4.23) to cylindrical coordinates we get,

$$\begin{aligned}\dot{r}&=\delta rz \qquad +O(|r,z|^3)\\ \dot{z}&=\lambda r^2+\rho z^2+O(|r,z|^3)\\ \dot{\vartheta}&=1 \qquad +O(|r,z|^2)\end{aligned}\tag{4.24}$$

It is easy to show that the zero solution of (4.24) is not asymptotically stable for any nonzero values of δ,λ, and ρ. Therefore conditions (4.22) are necessary for stabilization. Note also that ε does not appear in (4.24).

Next,assuming (4.22) is satisfied, we consider the $O(3)$ terms in the expansion of the vector field in (4.20). We have,

$$H_3=span\left\{\begin{pmatrix}x^3\\0\\0\end{pmatrix}\begin{pmatrix}y^3\\0\\0\end{pmatrix}\begin{pmatrix}z^3\\0\\0\end{pmatrix}\begin{pmatrix}x^2y\\0\\0\end{pmatrix}\begin{pmatrix}x^2z\\0\\0\end{pmatrix}\begin{pmatrix}y^2x\\0\\0\end{pmatrix}\begin{pmatrix}y^2z\\0\\0\end{pmatrix}\begin{pmatrix}z^2x\\0\\0\end{pmatrix}\begin{pmatrix}z^2y\\0\\0\end{pmatrix}\begin{pmatrix}xyz\\0\\0\end{pmatrix}\right.,$$

$$\begin{pmatrix}0\\x^3\\0\end{pmatrix}\begin{pmatrix}0\\y^3\\0\end{pmatrix}\begin{pmatrix}0\\z^3\\0\end{pmatrix}\begin{pmatrix}0\\x^2y\\0\end{pmatrix}\begin{pmatrix}0\\x^2z\\0\end{pmatrix}\begin{pmatrix}0\\y^2x\\0\end{pmatrix}\begin{pmatrix}0\\y^2z\\0\end{pmatrix}\begin{pmatrix}0\\z^2x\\0\end{pmatrix}\begin{pmatrix}0\\z^2y\\0\end{pmatrix}\begin{pmatrix}0\\xyz\\0\end{pmatrix},$$

$$\left.\begin{pmatrix}0\\0\\x^3\end{pmatrix}\begin{pmatrix}0\\0\\y^3\end{pmatrix}\begin{pmatrix}0\\0\\z^3\end{pmatrix}\begin{pmatrix}0\\0\\x^2y\end{pmatrix}\begin{pmatrix}0\\0\\x^2z\end{pmatrix}\begin{pmatrix}0\\0\\y^2x\end{pmatrix}\begin{pmatrix}0\\0\\y^2z\end{pmatrix}\begin{pmatrix}0\\0\\z^2x\end{pmatrix}\begin{pmatrix}0\\0\\z^2y\end{pmatrix}\begin{pmatrix}0\\0\\xyz\end{pmatrix}\right\}$$

Then we get,

$$ad\ L\ (H_3)=span\left\{\begin{pmatrix}3x^2y\\ x^3\\ 0\end{pmatrix}\begin{pmatrix}-3xy^2\\ y^3\\ 0\end{pmatrix}\begin{pmatrix}0\\ z^3\\ 0\end{pmatrix}\begin{pmatrix}2xy^2-x^3\\ x^2y\\ 0\end{pmatrix}\begin{pmatrix}2xyz\\ x^2z\\ 0\end{pmatrix}\right.$$

$$\begin{pmatrix}y^3-2x^2y\\ y^2x\\ 0\end{pmatrix}\begin{pmatrix}-2xyz\\ y^2z\\ 0\end{pmatrix}\begin{pmatrix}z^2y\\ z^2x\\ 0\end{pmatrix}\begin{pmatrix}-z^2x\\ z^2y\\ 0\end{pmatrix}\begin{pmatrix}y^2z-x^2z\\ xyz\\ 0\end{pmatrix},$$

$$\begin{pmatrix}-x^3\\ 3x^2y\\ 0\end{pmatrix}\begin{pmatrix}-y^3\\ -3xy^2\\ 0\end{pmatrix}\begin{pmatrix}-z^3\\ 0\\ 0\end{pmatrix}\begin{pmatrix}-x^2y\\ 2xy^2-x^3\\ 0\end{pmatrix}\begin{pmatrix}-x^2z\\ 2xyz\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-y^2x\\ y^3-2x^2y\\ 0\end{pmatrix}\begin{pmatrix}-y^2z\\ -2xyz\\ 0\end{pmatrix}\begin{pmatrix}-z^2x\\ z^2y\\ 0\end{pmatrix}\begin{pmatrix}-z^2y\\ -z^2x\\ 0\end{pmatrix}\begin{pmatrix}-xyz\\ y^2z-x^2z\\ 0\end{pmatrix},$$

$$\begin{pmatrix}0\\ 0\\ 3x^2y\end{pmatrix}\begin{pmatrix}0\\ 0\\ -3xy^2\end{pmatrix}\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\begin{pmatrix}0\\ 0\\ 2xy^2-x^3\end{pmatrix}\begin{pmatrix}0\\ 0\\ 2xyz\end{pmatrix}$$

$$\left.\begin{pmatrix}0\\ 0\\ y^3-2x^2y\end{pmatrix}\begin{pmatrix}0\\ 0\\ -2xyz\end{pmatrix}\begin{pmatrix}0\\ 0\\ z^2y\end{pmatrix}\begin{pmatrix}0\\ 0\\ -z^2x\end{pmatrix}\begin{pmatrix}0\\ 0\\ y^2z-x^2z\end{pmatrix}\right\}$$

It can be shown that a complement to $ad\ L\ (H_3)$ is given by,

$$G_3=span\left\{\begin{pmatrix}x^3+xy^2\\ y^3+x^2y\\ 0\end{pmatrix}\begin{pmatrix}y^3+x^2y\\ -x^3-xy^2\\ 0\end{pmatrix}\begin{pmatrix}xz^2\\ yz^2\\ 0\end{pmatrix}\begin{pmatrix}-yz^2\\ xz^2\\ 0\end{pmatrix}\begin{pmatrix}0\\ 0\\ x^2z+y^2z\end{pmatrix}\begin{pmatrix}0\\ 0\\ z^3\end{pmatrix}\right\}$$

Then the $O(3)$ normal form of (4.20) can be written in the form,

$$\begin{aligned}\dot{x} &= -y+\varepsilon yz+\alpha'(x^3+xy^2)+\beta'(y^3+x^2y)+\gamma' xz^2-\eta' yz^2\\ \dot{y} &= x-\varepsilon xz+\alpha'(y^3+x^2y)-\beta'(x^3+xy^2)+\gamma' yz^2+\eta' xz^2 \quad + \quad O(4)\\ \dot{z} &= \sigma'(x^2+y^2)z+\mu' z^3\end{aligned} \tag{4.25}$$

where with some algebra we may find,

$$\alpha'=\frac{1}{16}(D_x^3 f'+D_y^3 g'+D_{xyy}^3 f'+D_{xxy}^3 g')$$

$$\beta' = \frac{1}{16}(-D_x^3 g' + D_y^3 f' - D_{xyy}^3 g' + D_{xxy}^3 f')$$

$$\gamma' = \frac{1}{4}(D_{xzz}^3 f' + D_{yzz}^3 g')$$

$$\eta' = \frac{1}{4}(D_{xzz}^3 g' - D_{yzz}^3 f')$$

$$\sigma' = \frac{1}{4}(D_{xxz}^3 p' + D_{yyz}^3 p')$$

$$\mu' = \frac{1}{6} D_z^3 p'$$

Next transforming (4.25) into cylindrical coordinates we get,

$$\begin{aligned} \dot{r} &= \alpha' r^3 + \gamma' r z^2 + O(|r,z|^4) \\ \dot{z} &= \delta' r^2 z + \mu' z^3 + O(|r,z|^4) \\ \dot{\vartheta} &= 1 \qquad\qquad + O(|r,z|^2) \end{aligned} \tag{4.26}$$

Now using with the Lyapunov function candidate,

$$V = \frac{1}{2} R r^2 + \frac{1}{2} S z^2$$

we have,

$$\dot{V} = R\alpha' r^4 + R\gamma' r^2 z^2 + S\sigma' r^2 z^2 + S\mu' z^4$$

Therefore for $\alpha', \gamma', \sigma'$, and μ' all negative, or for α', μ' negative and $sgn(\gamma') = -sgn(\sigma')$, the equilibrium point of (4.26) is asymptotically stable.

Corollary 4.2 The zero solution of (4.15) is stabilizable by the control $v = \alpha x^2 + \beta y^2 + \gamma z^2 + \sigma xy + \eta xz + \mu yz$ provided (4.21) is satisfied ,$D_{z\xi}^2 p \neq 0$, and either $D_{x\xi}^2 f \neq 0$, $D_{z\xi}^2 g \neq 0$, or $D_{z\xi}^2 f \neq 0$, $D_{y\xi}^2 g \neq 0$.

Proof In view of the Remark following Theorem 4.2, we see that with $D_{z\xi}^2 p \neq 0$ (4.22b) and (4.22d) can be satisfied by making $D_z^3 p$, and $D_{xxz}^3 p$ or $D_{yyz}^3 p$ arbitrarily negative with a proper choice of parameters c , and a or b of the center manifold. Furthermore with $D_{x\xi}^2 f \neq 0$, $D_{z\xi}^2 g \neq 0$ (4.22a) and (4.22c) may be satisfied through the parameters a and l by making $D_x^3 f$ and $D_{yzz}^3 g$ arbitrarily negative. (4.22a) , (4.22c) may be satisfied with $D^2_{z\xi} f \neq 0$, $D^2_{y\xi} g \neq 0$ in a similar fashion.

4.3. Case of Two Pairs of Imaginary Eigenvalues

Here we consider the case where $A_{11} \in \mathbb{R}^{4\times 4}$ and has the following form:

$$A_{11} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta \\ 0 & 0 & \delta & 0 \end{bmatrix}$$

where we assume $\delta \notin \{\pm\frac{1}{2}, \pm 1, \pm 2, \pm 3\}$. Rewritting (4.1) for this form of A_{11} we get,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta \\ 0 & 0 & \delta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} + \begin{bmatrix} f(x,y,z,w,\xi) \\ g(x,y,z,w,\xi) \\ p(x,y,z,w,\xi) \\ q(x,y,z,w,\xi) \end{bmatrix} \tag{4.27}$$

$$\dot{\xi} = -k\xi + v$$

Letting $v = F(x,y,z,w)$ and representing the center manifold by $\xi = h(x,y,z,w)$ we get,

$$Dh(x,y,z,w) \begin{bmatrix} -y + f(x,y,z,w,h(x,y,z,w)) \\ x + g(x,y,z,w,h(x,y,z,w)) \\ -\delta w + p(x,y,z,w,h(x,y,z,w)) \\ -\delta z + q(x,y,z,w,h(x,y,z,w)) \end{bmatrix} = -kh(x,y,z,w) + F(x,y,z,w) \tag{4.28}$$

$$h(0)=0\,;\ Dh(0)=0$$

Choosing the following form for the control law F,

$$F(x,y,z,w) = \alpha x^2 + \beta y^2 + \gamma z^2 + \delta w^2 + \eta xy + \mu xz + \rho xw +$$

$$\lambda yz + \nu yw + \zeta zw \tag{4.29}$$

and approximating the center manifold upto terms of $O(3)$ as

$$h(x,y,z,w) = ax^2 + by^2 + cz^2 + dw^2 + exy + lxz + mxw +$$

$$nyz+syw+tzw+O(3) \tag{4.30}$$

Using (4.29) and (4.30) in (4.28) and equating the $O(2)$ terms we find that for nonzero values of k there is a 1–1 correspondence between the control parameters $(\alpha,\beta,...)$ and the center manifold paramters $(a,b,...)$ regardless of the value of δ. Thus again the control law determines a center manifold completely upto terms of $O(3)$. Now in relation to the stability of the center manifold system,

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \dot{w}\end{bmatrix}=\begin{bmatrix}0&-1&0&0\\ 1&0&0&0\\ 0&0&0&-\delta\\ 0&0&\delta&0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}+\begin{bmatrix}f(x,y,z,w,h(x,y,z,w))\\ g(x,y,z,w,h(x,y,z,w))\\ p(x,y,z,w,h(x,y,z,w))\\ q(x,y,z,w,h(x,y,z,w))\end{bmatrix} \tag{4.31}$$

we have the following theorem,

Theorem 4.3 The zero solution of (4.31) is asymptotically stable if

$$D_x^3 f'+D_y^3 g'+D_{xyy}^3 f'+D_{xxy}^3 g'<0 \tag{4.32a}$$

$$D_z^3 p'+D_w^3 q'+D_{zww}^3 p'+D_{zzw}^3 q'<0 \tag{4.32b}$$

$$D_{xzz}^3 f'+D_{xww}^3 f'+D_{yzz}^3 g'+D_{yww}^3 g'<0 \tag{4.32c}$$

$$D_{xxz}^3 p'+D_{yyz}^3 p'+D_{xxw}^3 q'+D_{yyw}^3 q'<0 \tag{4.32d}$$

Conditions (4.32c) and (4.32d) may be replaced by the single condition

$$\begin{aligned} &sgn(D_{xzz}^3 f'+D_{xww}^3 f'+D_{yzz}^3 g'+D_{yww}^3 g')=\\ &\qquad -sgn(D_{xxz}^3 p'+D_{yyz}^3 p'+D_{xxw}^3 q'+D_{yyw}^3 q') \end{aligned} \tag{4.32e}$$

Proof We give a short sketch of the proof since the details, although quite similar to the proof of Theorem 4.2, are extremely lengthy and tedious. Calculating $ad\ L\ (H_2)$ and $ad\ L\ (H_3)$ for the system (4.31) we can show that all $O(2)$ terms of the vector field may be removed and that the $O(3)$ normal form can be written as,

$$\left.\begin{aligned} \dot{x}&=-y+(\alpha'x+\beta'y)(x^2+y^2)+(\gamma'x+\sigma'y)(x^2+y^2)\\ \dot{y}&=x+(\alpha'y-\beta'x)(x^2+y^2)+(\gamma'y-\sigma'x)(x^2+y^2)\\ \dot{z}&=-\delta w+(\eta'z+\mu'w)(z^2+w^2)+(\rho'z+\nu'w)(x^2+y^2)\\ \dot{w}&=\delta z+(\eta'w-\mu'z)(z^2+w^2)+(\rho'w-\nu'z)(x^2+y^2) \end{aligned}\right. +O(4) \tag{4.33}$$

where,

$$\alpha' = \frac{1}{16}(D_x^3 f' + D_y^3 g' + D_{xyy}^3 f' + D_{xxy}^3 g')$$

$$\beta' = \frac{1}{16}(D_y^3 f' - D_x^3 g' + D_{xxy}^3 f' - D_{xyy}^3 g')$$

$$\gamma' = \frac{1}{8}(D_{xzz}^3 f' + D_{xww}^3 f' + D_{yzz}^3 g' + D_{yww}^3 g')$$

$$\sigma' = \frac{1}{8}(D_{yzz}^3 f' + D_{yww}^3 f' - D_{xzz}^3 g' - D_{xww}^3 g')$$

$$\eta' = \frac{1}{16}(D_z^3 p' + D_w^3 q' + D_{zww}^3 p' + D_{zzw}^3 q')$$

$$\mu' = \frac{1}{16}(D_w^3 p' - D_z^3 q' + D_{zzw}^3 p' - D_{zww}^3 q')$$

$$\rho' = \frac{1}{8}(D_{xxz}^3 p' + D_{yyz}^3 p' + D_{xxw}^3 q' + D_{yyw}^3 q')$$

$$\nu' = \frac{1}{8}(D_{xxw}^3 p' + D_{yyw}^3 p' - D_{xxz}^3 q' - D_{yyz}^3 q')$$

Now transforming (4.33) into cylindrical coordinates we obtain,

$$\begin{aligned} \dot{r}_1 &= \alpha' r_1^3 + \gamma' r_1 r_2^2 + O(|r_1, r_2|^4) \\ \dot{r}_2 &= \rho' r_1^2 r_2 + \eta' r_2^3 + O(|r_1, r_2|^4) \\ \dot{\vartheta}_1 &= 1 \qquad\qquad + O(|r_1, r_2|^2) \\ \dot{\vartheta}_2 &= \delta \qquad\qquad + O(|r_1, r_2|^2) \end{aligned} \tag{4.34}$$

Conditions (4.32) are now clear by considering a Lyapunov function candidate of the form $V = Rr_1^2 + Sr_2^2$.

Corollary 4.3 The zero solution of (4.27) is stabilizable by the control law in (4.29) if $D_{x\xi}^2 f$, $D_{y\xi}^2 g$ are not both zero, and $D_{z\xi}^2 p$, $D_{w\xi}^2 q$ are not both zero.

The proof is very reminiscent of that of Corollary 4.2 and will, therefore, be omitted here.

5. Robustness Considerations

In this section we study the effects of perturbations of the vector field on the stability properties of the system. We need the following definitions to characterize the notion of closeness of two vector fields.

Definition 5.1 Let $F(\xi)$ be a C^k vector field, then it may be expressed as

$$F(\xi)=f^1(\xi)+f^2(\xi)+\cdots+f^k(\xi)+R(\xi)$$

where $f^j \in H_j$, $j=1,..,k$, and $R=o(|\xi|^k)$. We define the C^k norm of F as

$$\|F\|_k=max\{\|f^1\|,\|f^2\|,\cdots\|f^k\|\} \tag{5.1}$$

where

$$\|f^j\|=\sup_{|\xi|\le 1}\frac{|f^j(\xi)|}{|\xi|^j} \tag{5.2}$$

Definition 5.2 A vector field $G(\xi)$ is a C^k ε- perturbation of F if $\|F-G\|_k\le\varepsilon$.

Following the assumptions of section 4, we let the unperturbed system $\dot{\xi}=\varphi(\xi)+bu$ be of the form

$$\begin{aligned}\dot{\xi}_1&=A_{11}\xi_1+\varphi_1(\xi_1,\xi_2)\\ \dot{\xi}_2&=A_{22}\xi_2+\varphi_2(\xi_1,\xi_2)+b_2u(\xi_1,\xi_2)\end{aligned} \tag{5.3}$$

where $\xi_1\in\mathbb{R}^{n_1}$, $\xi_2\in\mathbb{R}^{n_2}$, $\sigma(A_{11})\subset\{j\omega|\omega\in\mathbb{R}\}$, $\sigma(A_{22})\subset\mathbb{C}_-$, φ_1,φ_2 are at least C^k with k the order of the smallest order jet of the center manifold system associated with (5.3), on which the stabilizing control law, $u(\xi_1,\xi_2)$, is based.

Let the perturbed system $\dot{\eta}=\bar{\varphi}(\eta)+\bar{b}u$ be such that

$$\|\varphi(.)-\bar{\varphi}(.)\|_k\le\varepsilon_1 \tag{5.4}$$

$$|b-\bar{b}|\le\varepsilon_2 \tag{5.5}$$

Defining $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\|u(.)\|_k\}$, we may express the perturbed system as

$$\dot{\eta}=\varphi(\eta)+\varepsilon\hat{\varphi}(\eta)+bu(\eta)+\varepsilon\hat{b}u(\eta) \tag{5.6}$$

for some $\hat{\varphi}(.)$ and $\hat{\delta}$ such that

$$\|\hat{\varphi}(.)\|_k \leq 1 \tag{5.7}$$

$$\|\hat{\delta}u(.)\|_k \leq 1 \tag{5.8}$$

Rewriting (5.6) in the form of (5.3) we have

$$\begin{aligned}\dot{\eta}_1 &= A_{11}\eta_1+\varepsilon\hat{A}_{11}\eta_1+\varepsilon\hat{A}_{12}\eta_2+\varphi_1(\eta_1,\eta_2)+\varepsilon\tilde{\varphi}_1(\eta_1,\eta_2)\\ \dot{\eta}_2 &= A_{22}\eta_2+\varepsilon\hat{A}_{21}\eta_1+\varepsilon\hat{A}_{22}\eta_2+\varphi_2(\eta_1,\eta_2)+\varepsilon\tilde{\varphi}_2(\eta_1,\eta_2)+b_2u(\eta_1,\eta_2)\end{aligned} \tag{5.9}$$

where $\tilde{\varphi}(\eta)=\hat{\varphi}(\eta)-\hat{A}\eta+\hat{\delta}u(\eta)$.

Now since the stability properties of (5.3) were derived from the study of the flow on the center manifold, we wish to find a local invariant manifold for the system (5.9) such that the stability properties of the vector field restricted to this invariant coincide with those of (5.9). This is done in the following theorem.

Theorem 5.1 The system (5.9) possesses a C^k local invariant manifold given locally by the graph of a function $\hat{h}:\mathbb{R}_+\times\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$, such that

$$\hat{h}(0,0)=0 \ ; \ D\hat{h}(0,0)=0$$

Moreover this invarint manifold is C^k close to a center manifold of (5.3), and its stability properties coincide with those of the overall system (5.9).

Proof The technique used is called the suspension technique. We rewrite (5.9) as follows

$$\begin{aligned}\dot{\eta}_1 &= A_{11}\eta_1+\varepsilon\hat{A}_{11}\eta_1+\varepsilon\hat{A}_{12}\eta_2+\varphi_1(\eta_1,\eta_2)+\varepsilon\tilde{\varphi}_1(\eta_1,\eta_2)\\ \dot{\eta}_2 &= A_{22}\eta_2+\varepsilon\hat{A}_{21}\eta_1+\varepsilon\hat{A}_{22}\eta_2+\varphi_2(\eta_1,\eta_2)+\varepsilon\tilde{\varphi}_2(\eta_1,\eta_2)+b_2u(\eta_1,\eta_2)\\ \dot{\varepsilon} &= 0\end{aligned} \tag{5.10}$$

Since ε is a state variable in this system, we see that $\varepsilon\hat{A}_{11}\eta_1, \varepsilon\hat{A}_{12}\eta_2, \varepsilon\hat{A}_{21}\eta_1$, and $\varepsilon\hat{A}_{22}\eta_2$ are nonlinear terms in the new system. Therefore the linear part of the vector field in (5.10) has eigenvalues associated with η_1 and ε on the $j\omega$-axis and those associated with η_2 in the left half plane. Thus from theorem 3.1, there exists a C^k center manifold for (5.10), $\eta_2=\hat{h}(\eta_1,\varepsilon,\ |\eta_1|<\delta_1,|\varepsilon|<\delta_2$. Furthermore since $\hat{h}(\eta_1,\varepsilon)$ is C^k in η_1 and ε, there exists $\delta:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\|\hat{h}(\eta_1,\varepsilon)-\hat{h}(\eta_1,0)\|_k\le\delta(\varepsilon) \quad \text{and} \quad \delta(\varepsilon)\to 0 \ \ as \ \ \varepsilon\to 0$$

Now clearly $\hat{h}(\eta_1,0)$ is a center manifold for (5.3) and this completes the proof.

The flow on this invariant manifold is governed by

$$\dot{\zeta}=A_{11}\zeta+\varepsilon\hat{A}_{11}\zeta+\varepsilon\hat{A}_{12}\hat{h}(\zeta,\varepsilon)+\varphi_1(\zeta,\hat{h}(\zeta,\varepsilon))+\varepsilon\tilde{\varphi}_1(\zeta,\hat{h}(\zeta,\varepsilon)) \tag{5.11}$$

Note that for $\varepsilon=0$, (5.11) represents the dynamical system on the center manifold of (5.3). Therefore by assumption the zero solution of (5.11) is locally asymptotically stable for $\varepsilon=0$. We have the following theorem for the case when $\varepsilon\neq 0$.

Theorem 5.2 For the system (5.11) there exists ε^* such that for all $\varepsilon<\varepsilon^*$, there exists a ball $B_{r(\varepsilon)}$ around the origin which is locally asymptotically stable in the sense that locally outside $B_{r(\varepsilon)}$ the vector field is directed inward. Furthermore $r(\varepsilon)\to 0$ *as* $\varepsilon\to 0$.

Proof Since all the terms on the right hand side of (5.11) are C^k we may rewrite (5.11) as

$$\dot{\zeta}=A_{11}\zeta+\varphi_1(\zeta,\hat{h}(\zeta,0))+\sigma_1(\varepsilon)f^1(\zeta)+\ldots+\sigma_k(\varepsilon)f^k(\zeta)+R(\zeta,\varepsilon) \tag{5.12}$$

where $\sigma_j(\varepsilon)$ are C^k in ε and $\sigma_j(\varepsilon)\to 0$ as $\varepsilon\to 0$; $f^i\in H_i$ with $\|f^i\|\le 1$;$R(\zeta,\varepsilon)=o\,(|\zeta|^k,\varepsilon^k)$.

By assumption the zero solution of (5.11) is locally asymptotically stable for $\varepsilon=0$. Therefore by a converse theorem on asymptotic stability [6], there exists a locally positive definite Lyapunov function $V(\zeta)$ with continuous partial derivatives such that

$$-\dot{V}|_{\varepsilon=0}=-\frac{\partial V}{\partial\zeta}[A_{11}\zeta+\varphi_1(\zeta,\hat{h}(\zeta,0))] \tag{5.13}$$

is locally positive definite. Now computing $-\dot{V}|_{\varepsilon\neq 0}$ along the flow of (5.12) we obtain

$$-\dot{V}|_{\varepsilon\neq 0}=-\dot{V}|_{\varepsilon=0}-\frac{\partial V}{\partial \zeta}[\sigma_1 f^1+\dots+\sigma_k f^k]-\frac{\partial V}{\partial \zeta}R(\zeta,\varepsilon) \tag{5.14}$$

For any fixed value of ε, $R(\zeta,\varepsilon)$ is $o(|\zeta|^k)$; therefore there exists $\mu(\varepsilon)>0$ and a monotone increasing function $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$-\dot{V}|_{\varepsilon=0}-\frac{\partial V}{\partial \zeta}R(\zeta,\varepsilon)\geq\alpha(|\zeta|) \quad \text{for} \quad |\zeta|<\mu(\varepsilon) \tag{5.15}$$

and $\alpha(r)>0$ for $r>0$, $\alpha(0)=0$, μ is a continuous function of ε with $\mu(0)>0$. Furthermore by continuity of the partials of V we may define

$$L(\varepsilon)=\sup_{|\zeta|\leq\mu(\varepsilon)}\left|\frac{\partial V}{\partial \zeta}\right| \tag{5.16}$$

Using (5.15) and (5.16) in (5.14) we have

$$-\dot{V}|_{\varepsilon\neq 0}\geq\alpha(|\zeta|)-L(\varepsilon)\sigma(\varepsilon)[|\zeta|+|\zeta|^2+\dots+|\zeta|^k] \tag{5.17}$$

$$\geq\alpha(|\zeta|)-kL(\varepsilon)\sigma(\varepsilon)|\zeta| \quad \text{for} \quad |\zeta|\leq\min\{1,\mu(\varepsilon)\} \tag{5.18}$$

where $\sigma(\varepsilon)=max[\sigma_1(\varepsilon),\dots,\sigma_k(\varepsilon)]$ and we have used the fact that $\|f^j\|\leq 1$.

Let $\bar{\sigma}(\varepsilon)=kL(\varepsilon)\sigma(\varepsilon)$. Since $-\dot{V}|_{\varepsilon=0}$ is at least $O(k+1)$, $\alpha(.)$ is higher order than linear. Therefore there exists $r(\varepsilon)>0$ such that

$$-\dot{V}|_{\varepsilon\neq 0}\geq\alpha(|\zeta|)-\bar{\sigma}(\varepsilon)|\zeta|>0 \quad \text{for} \quad |\zeta|\geq r(\varepsilon) \tag{5.19}$$

Moreover since $\bar{\sigma}(\varepsilon)\to 0$ as $\varepsilon\to 0$, we see that $r(\varepsilon)\to 0$ as $\varepsilon\to 0$. Thus for ε small enough $r(\varepsilon)<\mu(\varepsilon)$ and we may write

$$-\dot{V}|_{\varepsilon\neq 0}>0 \quad \text{for} \quad r(\varepsilon)<|\zeta|<\mu(\varepsilon) \tag{5.20}$$

Together with the positive definiteness of $\alpha(.)$, this shows that the ball of radius $r(\varepsilon)$, $B_{r(\varepsilon)}$, around the origin is locally asymptotically stable. The size of the perturbation ε is limited by $\mu(\varepsilon)$, the domain of attraction of the unperturbed system, and the parameter δ_2 from theorem 5.1.

Theorems 5.2 and 3.2 together imply the following result.

Theorem 5.3 Let the zero solution of (5.3) be locally asymptotically stable for some control law constructed based on the k-jet of the center manifold system. Then there exists ε^* such that for all C^k ε-perturbations of the vector field with $\varepsilon<\varepsilon^*$, there exists a ball $B_{r(\varepsilon)}$ around the origin which is locally asymptotically stable. Further $r(\varepsilon)\to 0$ as $\varepsilon\to 0$.

6. Discussion and Examples

The previous sections were based on several seemingly restrictive assumptions. Here we will show the extension of the previous results to more general cases. In section 2 we considered systems of the form $\dot{\xi}=\varphi(\xi)+bu$. Let us now consider the general case of the form

$$\dot{\xi}=\hat{\varphi}(\xi,u) \tag{6.1}$$

where $\hat{\varphi}(0,0)=0$. Since we are interested in local stability properties of the origin we may expand the vector field as

$$\dot{\xi}=\hat{\varphi}(\xi,0)+D_u\hat{\varphi}(\xi,0)u+D_u^2\hat{\varphi}(\xi,0)u^2+O(u^3) \tag{6.2}$$

Letting $D_u\hat{\varphi}(0,0)=b$, we may define $\psi(\xi)$ by

$$D_u\hat{\varphi}(\xi,0)=b+\psi(\xi) \tag{6.3}$$

We further define

$$\varphi(\xi)=\hat{\varphi}(\xi,0) \tag{6.4}$$

$$\Gamma(\xi)=D_u^2\hat{\varphi}(\xi,0) \tag{6.5}$$

Rewriting (6.1) using the above notation we have

$$\dot{\xi}=\varphi(\xi)+bu+\psi(\xi)u+\Gamma(\xi)u^2+O(u^3) \tag{6.6}$$

where $\varphi(0)=0$. Now letting $A=D_\xi\varphi(0)$ and transforming the system as in section 2 we have

$$\begin{bmatrix}\dot\xi_1\\ \dot\xi_2\\ \dot\xi_3\end{bmatrix}=\begin{bmatrix}A_{11} & 0 & 0\\ 0 & A_{22} & 0\\ 0 & 0 & A_{33}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}+\begin{bmatrix}\varphi_1(\xi)\\ \varphi_2(\xi)\\ \varphi_3(\xi)\end{bmatrix}+\begin{bmatrix}0\\ b_2\\ b_3\end{bmatrix}u+\begin{bmatrix}\psi_1(\xi)\\ \psi_2(\xi)\\ \psi_3(\xi)\end{bmatrix}u+\begin{bmatrix}\Gamma_1(\xi)\\ \Gamma_2(\xi)\\ \Gamma_3(\xi)\end{bmatrix}u^2+O(u^3) \tag{6.7}$$

Representing the center manifold by $(\xi_2{}^T,\xi_3{}^T)=(h_2(\xi_1)^T,h_3(\xi_1)^T)$ we get

$$Dh(\xi_1)[A_{11}\xi_1+\varphi_1(\xi_1,h_2,h_3)+\psi_1(\xi_1,h_2,h_3)u+\Gamma_1(\xi_1,h_2,h_3)u^2+O(u^3)]=$$

$$\begin{bmatrix}A_{22}h_2+\varphi_2(\xi_1,h_2,h_3)+b_2u+\psi_2(\xi_1,h_2,h_3)u+\Gamma_2(\xi_1,h_2,h_3)u^2\\ A_{33}h_3+\varphi_3(\xi_1,h_2,h_3)+b_3u+\psi_3(\xi_1,h_2,h_3)u+\Gamma_3(\xi_1,h_2,h_3)u^2\end{bmatrix}+O(u^3) \tag{6.8}$$

Assuming the control u is a smooth function of the ξ's , it is clear that the terms $\psi_j u$ and $\Gamma_j u^2$ for $j=1,2,3$ are at least of $O(2)$ and will, therefore, have no effect on the $O(2)$ expansion of the center manifold. The flow on the center manifold, on the other hand, is now determined by

$$\dot\xi_1=A_{11}\xi_1+\varphi_1(\xi_1,h_2,h_3)+\psi_1(\xi_1,h_2,h_3)u+\Gamma_1(\xi_1,h_2,h_3)u^2+O(u^3) \tag{6.9}$$

Since the stability of the zero solution of (6.9) is determined by the quadratic and higher order terms, the presence of $\psi_1 u$ and $\Gamma_1 u_2$ will only relax the stabilizability conditions by adding more flexibility in satisfying the conditions of Theorems 4.1, 4.2, and 4.3 . In other words the special class of systems $\dot\xi=\varphi(\xi)+bu$ represents a least controllable situation.

We next present two illustrative examples. The first example demonstrates the control design procedure and the effects of perturbations on the stabilized system. The second example involves the case where the controllable part of the system is not a scalar.

Example 1 We consider the system

$$\begin{aligned}\dot x &= y-x^3+xy^2-2y\xi\\ \dot y &= x^3+x\xi\\ \dot\xi &= -5\xi+u\end{aligned} \tag{6.10}$$

With $u \equiv 0$, a center manifold for (6.10) is given by $\xi \equiv 0$. The flow on this center manifold is governed by

$$\begin{aligned} \dot{x} &= y - x^3 + xy^2 \\ \dot{y} &= x^3 \end{aligned} \tag{6.11}$$

The origin of (6.11) is unstable as shown in Fig. 1 by the phase portrait of the system.

To stabilize the system we choose the control as in (4.7) and represent the center manifold by (4.6). Then the flow on the center manifold is governed by

$$\left.\begin{aligned} \dot{x} &= y - x^3 + xy^2 - 2y(ax^2 + bxy + cy^2) \\ \dot{y} &= x^3 + x(ax^2 + bxy + cy^2) \end{aligned}\right. + O(4) \tag{6.12}$$

From Theorem 4.1 we see that a choice of parameters of the center manifold which stabilizes the origin of (6.12) is given by: $a = -2$, $b = 0$, and $c = 0$. Using (4.9), the corresponding control parameters are given by: $\alpha = -10$, $\beta = -4$, and $\gamma = 0$. Thus a stabilizing control law is given by

$$u = -10x^2 - 4xy \tag{6.13}$$

Fig. 2 shows the phase portrait of the stabilized system (6.12).

Next we introduce a linear perturbation in the original system. The perturbed system is given by

$$\begin{aligned} \dot{x} &= \varepsilon x + y - x^3 + xy^2 - 2y\xi \\ \dot{y} &= \varepsilon y + x^3 + x\xi \qquad\qquad \varepsilon > 0 \\ \dot{\xi} &= -5\xi + u \end{aligned} \tag{6.14}$$

Clearly the origin of (6.14) is unstable irrespective of the control u. From Theorem 5.3, however, we expect that for ε small, using the control (6.13) the trajectories of the system converge to a small ball around the origin. To demonstrate this we compute the center manifold of the suspended system obtained from (6.14). This is given by

$$h(x,y,\varepsilon)=-2x^2+0.8\varepsilon x^2-0.32\varepsilon xy+0.064\varepsilon y^2+O(4) \tag{6.15}$$

Then the flow on this invariant manifold is determined by

$$\begin{aligned}\dot{x}&=\varepsilon x+y-x^3+xy^2-2yh(x,y,\varepsilon)\\ \dot{y}&=\varepsilon y+x^3+xh(x,y,\varepsilon)\end{aligned} \tag{6.16}$$

The phase portrait of (6.16) is shown in Fig. 3 for $\varepsilon=0.5$. Comparison of figures 2 and 3 shows that as ε changes from zero, the stable equilibrium point at the origin bifurcates into a stable periodic orbit around the origin and an unstable equilibrium point at the origin.

Example 2 In this example we consider a system whose hyperbolic portion is not a scalar. Since the approach to all higher dimensional problems is identical, we consider a two dimensional example. Thus consider the system

$$\begin{aligned}\dot{\xi}_1&=\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}x^3+xy^2-x\xi_2\\ y\xi_2+x\xi_3+x^2y\end{bmatrix}\\ \dot{\xi}_2&=-\xi_2+u\\ \dot{\xi}_3&=-2\xi_3+u\end{aligned} \tag{6.17}$$

Representing the center manifold as

$$\begin{bmatrix}\xi_2\\ \xi_3\end{bmatrix}=\begin{bmatrix}h_2(x,y)\\ h_3(x,y)\end{bmatrix}=\begin{bmatrix}a_1x^2+b_1xy+c_1y^2\\ a_2x^2+b_2xy+c_2y^2\end{bmatrix}+O(3) \tag{6.18}$$

we have that with the control $u=\alpha x^2+\beta xy+\gamma y^2$, $h(x,y)$ satisfies

$$Dh(x,y)\begin{bmatrix}y+x^3+xy^2-xh_2(x,y)\\ x^2y+yh_2(x,y)+xh_3(x,y)\end{bmatrix}=$$

$$\begin{bmatrix}-a_1x^2-b_1xy-c_1y^2+\alpha x^2+\beta xy+\gamma y^2\\ -2a_2x^2-2b_2xy-2c_2y^2+\alpha x^2+\beta xy+\gamma y^2\end{bmatrix}+O(3) \tag{6.19}$$

Now equating the $O(2)$ terms we get

$$\begin{bmatrix}1&0&0\\2&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}=\begin{bmatrix}2&0&0\\2&2&0\\0&1&2\end{bmatrix}\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}=\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} \tag{6.20}$$

The flow on the center manifold is given by

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}y+(1-a_1)x^3+(1-c_1)xy^2-b_1x^2y\\ a_2x^3+(a_1+b_2+1)x^2y+(b_1+c_2)xy^2+c_1y^3\end{bmatrix}+O(4) \tag{6.21}$$

Then from Theorem 4.1 the zero solution of (6.21) is asymptotically stable if $a_2<0$ and $3(1-a_1)+a_1+b_2+1<0$. Thus for example choosing $(a_1,b_1,c_1)=(-2,-14,2)$ and $(a_2,b_2,c_2)=(-1,-8,-2)$ satisfies (6.20) and the above inequalities.Then we get that the control law

$$u=-2x^2-18xy-12y^2 \tag{6.22}$$

stabilizes the zero solution of (6.21) and thus that of (6.17).

It is clear from (6.20) that although the control law does not determine the center manifold completely, it does give us the same number of degrees freedom in choosing the center manifold as was available in the case with a scalar hyperbolic state.

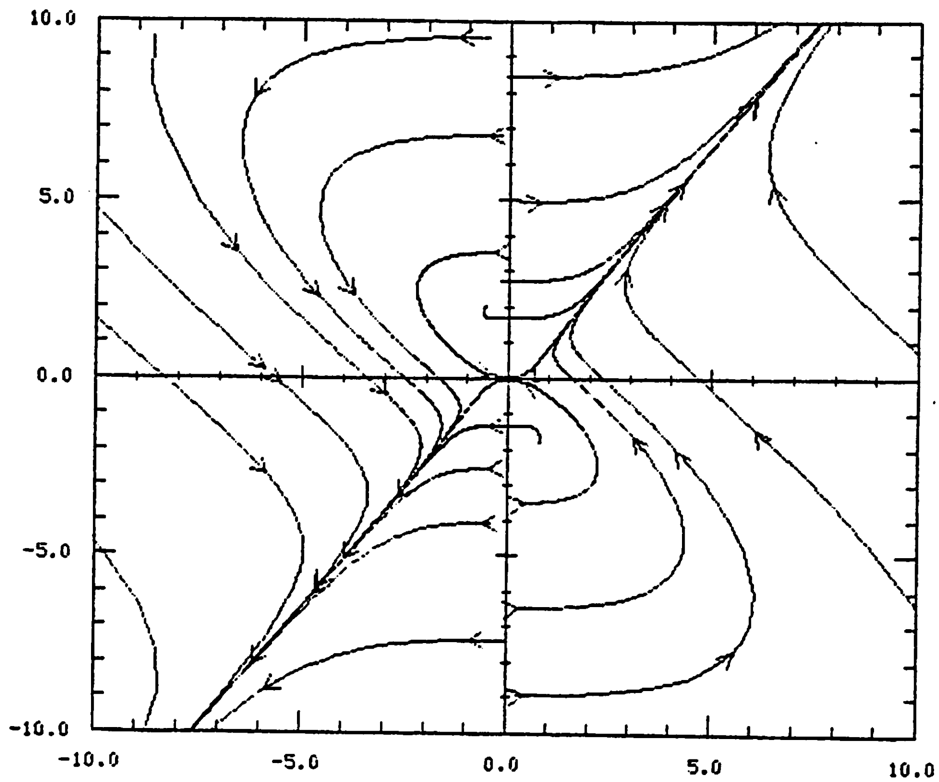

Figure 1. Phase portrait of center manifold system of the unstable system.

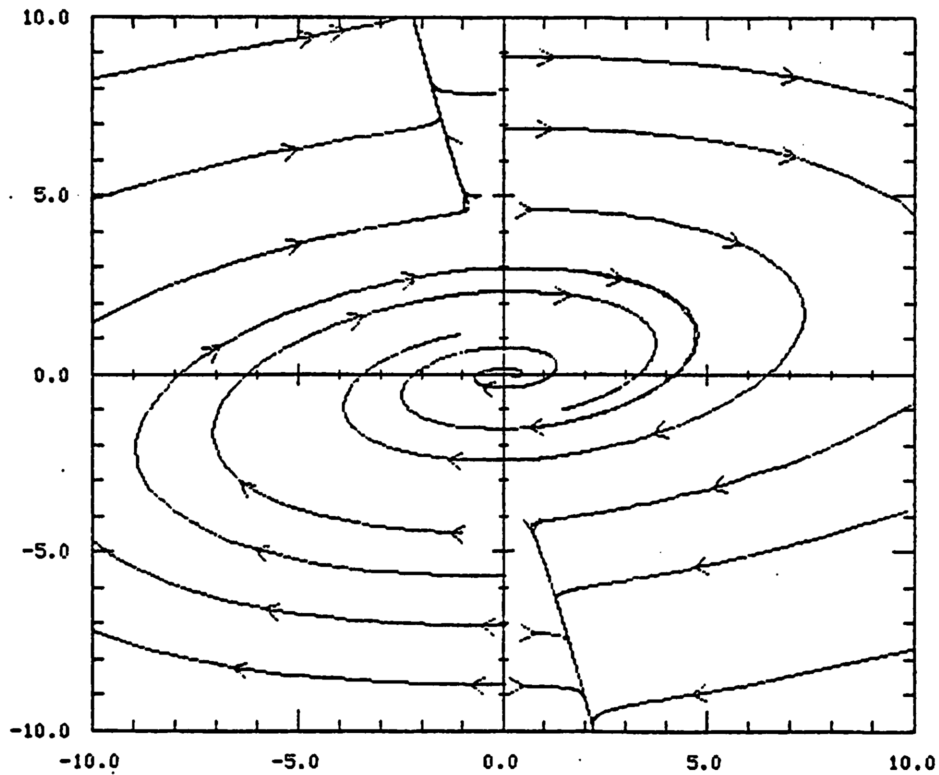

Figure 2. Phase portrait of center manifold system of the stabilized system.

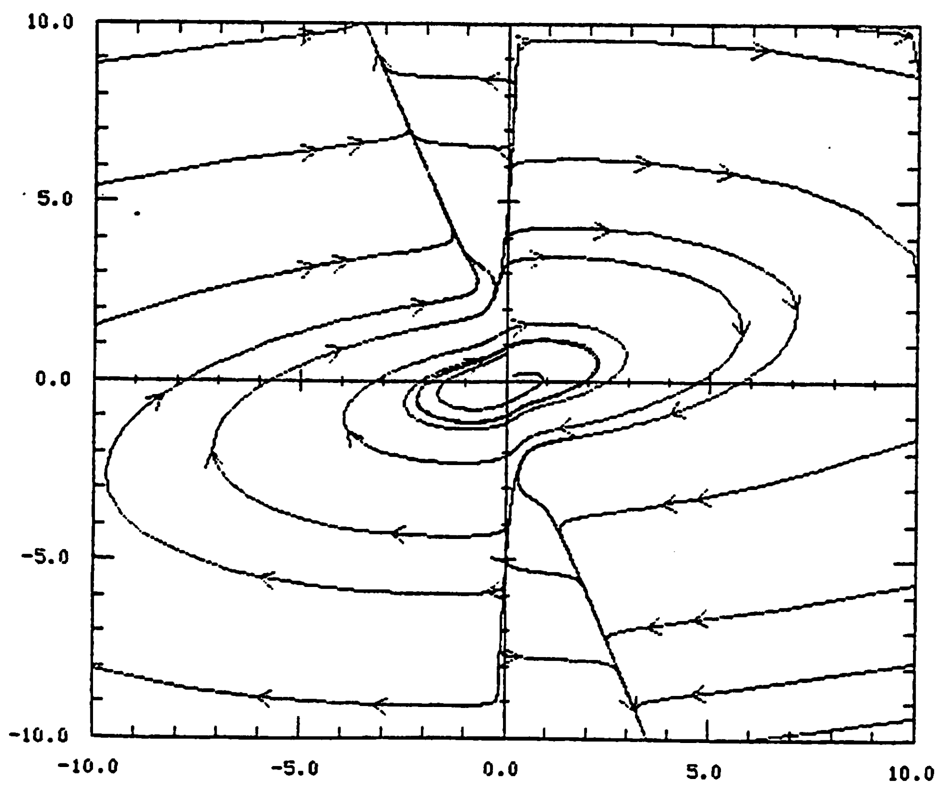

Figure 3. Phase portrait on the invariant manifold of the perturbed system.

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