VAN DER POL AND CHAOS

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Memorandum No. UCB/ERL 85/97

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ELECTRONICS RESEARCH LABORATORY

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TITLE PAGE

In a paper entitled "Frequency Demultiplication" [5], Van der Pol described an experiment in which by tuning the capacitor in a neon bulb R-C relaxation oscillator, driven by a sinusoidal voltage source, "currents and voltages appear in the system which are whole submultiples of the driving frequency". He noted that as the capacitance was increased from that value (C_0) for which the natural frequency of the undriven relaxation oscillator equalled that of the sinusoidal source, the system frequency made "discrete jumps from one whole submultiple of the driving frequency to the next"(detected by means of "a telephone coupled loosely in some way to the system"). Van der Pol noted that "often an irregular noise is heard in the telephone receiver before the frequency jumps to the next lower value". Interested primarily in frequency demultiplication, he dismissed the "noise" as "a subsidiary phenomenon". Here, we investigate this noise as an example of the period-adding route to chaos, first verifying and elaborating on the work of Van der Pol, and then modelling the circuit dynamics in order to reproduce the observed phenomena.

2. Experimental Circuit

The circuit we consider is the sinusoidally-driven neon bulb relaxation oscillator described by Van der Pol [5] (Fig. 1). A high voltage d.c. supply E (of approximately 100 V terminal voltage) with large source resistance R (approximately 1 MΩ) is attached to a shunt connection of neon bulb (ALCO type BNE-4R (with current-limiting resistor removed)) (acting as a current-controlled negative resistance) and capacitor C, forming the basic relaxation oscillator. In series with the neon bulb is inserted a sinusoidal voltage source $E_0 \sin\omega t$ (a small current sense resistor R_s was also inserted in series with the neon bulb to detect the current flowing in it).

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ABSTRACT

Experimental confirmation has been made on a driven relaxation oscillator circuit, first presented by Van der Pol, of the period-adding route to chaos. The nonlinear element in the circuit is a neon bulb, modelled by a three-segment piecewise-linear current-controlled resistor. A simple nonlinear circuit model has been used to reproduce in simulations the experimentally-observed period-adding phenomenon.

1. Introduction

Since the pioneering work of Feigenbaum in 1975 on the period-doubling transition to chaos [1], many studies of chaotic phenomena in nonlinear dynamical systems have appeared [2], suggesting that chaos is in some sense a "new" discovery. Indeed, period-adding phenomena and the alternating periodic-chaotic transition sequence [3], were first reported as recently as 1982 [4]. In this paper, we will examine a simple experimental circuit with a negative resistance device, first presented by Van der Pol in 1927 [5], which displays a transition to chaos through an alternating periodic-chaotic sequence, and describe a nonlinear circuit model of the physical circuit which for the first time, to our knowledge, has faithfully reproduced the observed period-adding phenomenon.

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In a paper entitled "Frequency Demultiplication" [5], Van der Pol described an experiment in which by tuning the capacitor in a neon bulb R-C relaxation oscillator, driven by a sinusoidal voltage source, "currents and voltages appear in the system which are whole submultiples of the driving frequency". He noted that as the capacitance was increased from that value (C_0) for which the natural frequency of the undriven relaxation oscillator equalled that of the sinusoidal source, the system frequency made "discrete jumps from one whole submultiple of the driving frequency to the next"(detected by means of "a telephone coupled loosely in some way to the system"). Van der Pol noted that "often an irregular noise is heard in the telephone receiver before the frequency jumps to the next lower value". Interested primarily in frequency demultiplication, he dismissed the "noise" as "a subsidiary phenomenon". Here, we investigate this noise as an example of the period-adding route to chaos, first verifying and elaborating on the work of Van der Pol, and then modelling the circuit dynamics in order to reproduce the observed phenomena.

2. Experimental Circuit

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3. Experimental Results

With the signal source zeroed, the natural frequency of the undriven oscillator was set to 1 kHz, by tuning capacitance C to C_0. Then a sinusoidal signal of peak amplitude E_0 (7.7 V) and frequency 1 kHz was applied as shown. The resulting frequency of the voltage pulses measured across R_s was recorded as the value of the bifurcation parameter C was brought gradually from C_0 to a much larger value. As the capacitance is increased, the system at first continues to oscillate at 1 kHz (where we define the system frequency as the repetition rate of the neon bulb current pulse pattern) over a wide range of C, until the frequency "suddenly" drops to 1000/2 Hz, to maintain that value over a further range of capacitance. If C is increased still more, the frequency drops to 1000/3 Hz, then 1000/4 Hz, 1000/5 Hz, and so on up to 1000/20 Hz, exactly as reported by Van der Pol in 1927 [5].

It is a tribute to the remarkable experimental skill of Van der Pol that the above delicate observations were originally made using no more than a telephone !

Figure 2 is a plot of measured system period as a function of the setting of capacitor C. The shaded areas are those "noisy" regions which warrant further investigation.

Between each two submultiples of the oscillator driving frequency, a further rich structure of submultiples is found. At the macroscopic level (the coarse structure examined by Van der Pol) increasing the value of C causes the system period to step from T (1 ms) to 2T, 3T, 4T,..., where the range of C for which the period is fixed is much greater than that over which the transitions occur (Fig. 2).

Examining the "step" transitions more closely, one finds that between any two "macroscopic" regions where the period is fixed at (n-1)T and nT (n>1) respectively, there lies a narrower region over which the system oscillates with stable period (2n-1)T. Further, between (n-1)T and (2n-1)T, one finds a region of C for which the period is (3n-2)T, and between (2n-1)T and nT, a region with period (3n-1)T. Indeed, between any two stable regions with periods (n-1)T and nT respectively, we expect to find a region with period (2n-1)T. Figure 3 shows an enlargement of the C axis in the region of the T to 2T macro-transition, showing the finer period-adding structure. Between T and 2T is a region with stable period 3T. Figure 4 shows neon bulb current waveforms for orbits of periods T, 2T and between them, an orbit of period 3T (note the reference direction for I_{neon}). Between this and 2T, regions of periods 5T, 7T, 9T,... up to 25T were detected. A region of period 4T lies between T and 3T, with steps 7T, 10T, 13T,... up to 25T between that and 3T.

In practice, it becomes difficult to observe higher periods since their windows of existence become narrower, so the behaviour becomes more unstable (stochastic noise in the experimental circuit can throw the solution out of the narrow window of existence of a high period orbit) and consequently more difficult to observe.

Transition from one periodic state to another is characterised by a cascade of period-doubling bifurcations to chaos, followed by recovery to the next periodic state. Figure 5 shows the frequency spectrum of a period-doubling transition to chaos. [At lower signal amplitudes, where $2E_0 < (V_{on} - V_{off})$ (the potential difference between the neon bulb's "on" and "off" states), transition between locked states occurs without period-doubling bifurcations.]

4. Computer Simulation

Figure 6 is the circuit of the computer simulation. E is a d.c. voltage source of 100 V terminal voltage. R has value 1 MΩ. The sinusoidal voltage source has peak amplitude 7.7 V at frequency 1 kHz. The neon bulb has been modelled by a current-controlled resistor with three-segment piecewise linear $i-v$ characteristic, defined by

$$v = f(i) = 29.9925 + 5.0005i - 6\,|\,i - 7.5\,| + 1.0005\,|\,i - 15\,|$$

where v is in Volts, and i in μA (Fig. 7).

A parasitic inductance L_s has been included in series with the current-controlled resistor in the neon bulb model to account for the element's dynamic behaviour. (The circuit is very sensitive to the value of L_s (chosen to be 1 pH); too high a value of parasitic inductance reduces the widths of the transition regions between periods (n-1)T and nT). This inductance is an essential part of the neon bulb dynamic model since its $i-v$ characteristic is non-monotonic and current-controlled. Note that the state equation of the circuit does not exist if L_p is not included [6].

Resistance R_s (current sensing resistor) has been included in the model in order to reproduce, as closely as possible, the conditions of the experiment; it has value 50 Ω, and may be omitted with little effect on the behaviour of the circuit. C_0 is approximately 1 nF.

State equations for the circuit are of the form:

$$\dot{v}_C = -\frac{1}{RC}v_C - \frac{1}{C}i_L + \frac{E}{RC}$$

$$\dot{i}_L = \frac{1}{L_p}v_C - \frac{R_s}{L_p}i_L - \frac{f(i_L)}{L_p} - \frac{E_0 \sin(\omega t)}{L_p}$$

The circuit has been analysed using a third order Backward Difference integration routine [7], with initial conditions $v_C = 0$ and $i_L = 0$.

Figure 8 shows the period-capacitance plot for our computer simulation of Van der Pol's neon bulb circuit in which the staircase pattern of transition between periods T (1 ms) and 2T has been reproduced. Stable oscillations with the indicated periods have been found in the regions marked with solid lines.

Figure 9 shows the output waveforms from simulations for various values of the bifurcation parameter C, which are qualitatively identical to those of the experimental circuit. Figure 10 shows a complex waveform for C = 1.26 nF, in the chaotic regime. Here, the bold vertical lines are the current surges flowing at intervals through the neon bulb (Some current spikes appear to be of greater duration than others. An enlargement of the time axis shows that this is not the case, but because of limited resolution graphics output, the spikes as plotted must be either one or two pixels wide). The system appears to oscillate with period T, with intermittent interruptions, returning to quasi-period T oscillation within a few cycles each time.

Chaotic states contain component waveforms belonging to nearby periodic states, but of course they appear in an apparently random manner. For example, between cycles with rotation numbers $\frac{2}{3}$ and $\frac{3}{5}$, where we define the rotation number as *the ratio of the number of current pulses to the number of periods of the driving source per system cycle*, we obtain the chaotic transition shown (Fig. 11). In each case, the diagram on the left is an enlargement of the corresponding time segment (20 - 28 ms) of the neon current waveform on the right. Note that waveform (b) contains component oscillations of periods 3 and 5, which appear at random throughout the waveform.

Thus, in our simulation, we have accurately modelled the dynamics of the system under investigation and have consequently succeeded in reproducing those period-adding phenomena present in the original circuit.

5. Discussion

The alternating periodic-chaotic transition sequence, and the forward period-adding phenomenon observed in Van der Pol's neon bulb circuit also appear in several other systems [3]. The rotation numbers of the successive period-adding states are related by a definite law. Kaneko [8] noted that in a period-adding sequence, any locking with rotation number $\frac{\alpha q+\beta s}{\alpha p+\beta r}$ appears in principle between lockings $\frac{q}{p}$ and $\frac{s}{r}$, where q and p, and s and r are relatively prime, and α and β are strictly positive integers. The sequence of locking states (devil's staircase) [9] is constructed as follows (for integers n, k, and m):

$$\frac{1}{1}\to\frac{1}{2}\to\frac{1}{3}\to\cdots\to\frac{1}{n}\to\cdots\to\frac{1}{\infty}\quad(n>1)$$

(this is the macro-level transition sequence investigated by Van der Pol and corresponds to the T, 2T, 3T,... steps).

$$\frac{1}{n-1}\to\frac{2}{2n-1}\to\cdots\to\frac{k}{kn-1}\to\cdots\to\frac{1}{n}\quad(k>1)$$

between $\frac{1}{n-1}$ and $\frac{1}{n}$. Here, we can identify $q=1$, $p=n-1$, $s=1$, $r=n$, $\alpha=1$, $\beta=1,2,3,\ldots,(k-1),\ldots$.

$$\frac{k-1}{(k-1)n-1}\to\frac{2k-1}{(kn-1)+(k-1)n-1}\to\cdots\to\frac{mk-1}{(m-1)(kn-1)+(k-1)n-1}\to\cdots\to\frac{k}{(kn-1)}\quad(m>1)$$

between $\frac{k-1}{(k-1)n-1}$ and $\frac{k}{kn-1}$, and so on for each pair of orbits. In this case, we identify $q=k-1, p=(k-1)n-1, s=k, r=kn-1, \alpha=1$, and $\beta=1,2,3,\ldots,(m-1),\ldots$.

Between any two states, we find a self-similar (reproduced qualitatively at all scales of the bifurcation parameter) staircase of locking states.

Using the definition of rotation numbers given above, Van der Pol's neon bulb circuit is seen to follow this law exactly.

Also, our results confirm the observation of Kaneko [4] (from numerical simulations of a discrete map displaying period-adding phenomena) that the most stable orbit (having the widest window of existence for variations in C) between the periodic cycles with rotation numbers $\frac{q}{p}$ and $\frac{s}{r}$ has rotation number $\frac{q+s}{p+r}$; i.e. α and β both equal to unity. Further, the uniform period-adding sequence most easily observed between cycles with rotation numbers $\frac{1}{n-1}$ and $\frac{1}{n}$ is precisely that sequence with rotation numbers $\frac{k}{kn-1}$, k = 2,3,4,...; namely, the next level of the self-similar staircase structure.

Whether or not it is feasible to observe a particular period depends on the region of the bifurcation parameter where the cycle is stable and the stability of the cycle. Cycles of shorter period are easily observed; those with longer periods are susceptible to corruption by experimental noise, and are consequently more difficult to detect in practice. For chaotic states, the waveform contains periodic oscillations from adjacent periodic states randomly distributed through the waveform.

The phenomena described above have been observed in Van der Pol's neon bulb circuit using a variety of bulb types, sinusoidal source amplitudes, d.c. voltages and source resistors, nearly sixty years (and half a century of development in neon bulb technology) after they were first reported; it follows that the behaviour is robust.

The circuit model described above is characterised by a second order *stiff* differential equation (the time constant for the current spike is orders of magnitude less than the system period). In order to reproduce the results of our computer simulations, therefore, one should use a reliable stiff differential equation solver such as [7]; otherwise, it is unlikely that it will be possible to duplicate the period-adding phenomena outlined.

It should be noted that a neon bulb is a very complicated device, yet we have been able to duplicate its dynamic behaviour over the frequency range of interest (below 1 kHz) using the simplest possible model [10]. This confirms our belief that a neon bulb may be realistically modelled at low frequencies by a series connection of inductance (which we have indicated is an essential component of the dynamic model) and current-controlled nonlinear resistor.

A third order non-autonomous circuit (with a negative resistance device synthesised using bipolar transistors) which exhibits period-adding phenomena has already been reported [3]. Van der Pol's Neon Bulb Circuit is important because it is described by a *second* order non-autonomous ordinary differential equation, and is consequently *the system of lowest order capable of exhibiting the period-adding transition to chaos.*

6. Conclusion

We have investigated the period-adding route to chaos in the simplest non-autonomous system, a driven relaxation oscillator circuit with two dynamic elements (described by a second order nonlinear non-autonomous ordinary differential equation), and have presented a piecewise-linear model of the circuit for which the experimentally-observed period-adding phenomena have been reproduced in computer simulations.

Our future goal is to simplify the computer model in order to make a more detailed analytical analysis of the circuit dynamics, and ultimately to develop a theory which explains the observed phenomena.

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Figure Captions

Fig. 1.

Sinusoidally-driven Neon Bulb Relaxation Oscillator.

Fig. 2.

Pulse Pattern Repetition Rate vs. C [Experimental], showing coarse staircase structure.

Fig. 3.

Pulse Pattern Repetition Rate vs. C [Experimental], showing fine period-adding structure.

Fig. 4.

Neon Bulb Current Waveforms (I_{neon}) [Experimental]: (a) C = 2.0 nF (Period T); (b) C = 2.7 nF (Period 3T); (c) C = 3.4 nF (Period 2T).

Fig. 5.

Frequency Spectrum of a Period-Doubling transition to Chaos: (a) C = 2.165 nF; (b) C = 2.170 nF; (c) C = 2.180 nF.

Fig. 6.

Van der Pol's Neon Bulb Circuit - Computer Model.

Fig. 7.

Neon Bulb Characteristics: (a) Measured; (b) Simulated.

Fig. 8.

Pulse Pattern Repetition Rate vs. C [Simulation], replicating the period-adding structure of the real circuit.

Fig. 9.

Neon Bulb Current Waveforms (I_{neon}) [Simulated]: (a) C = 1.00 nF (Rotation Number $\frac{1}{1}$); (b) C = 1.54 nF (Rotation Number $\frac{2}{3}$); (c) C = 2.00 nF (Rotation Number $\frac{1}{2}$).

Fig. 10.

Neon Bulb Current Waveform in the Chaotic Regime (C = 1.26 nF).

Fig. 11.

Neon Bulb Current Waveforms; (a) C = 1.50 nF (Rotation Number $\frac{2}{3}$); (b) C = 1.57 nF; (c) C = 1.60 nF (Rotation Number $\frac{3}{5}$).

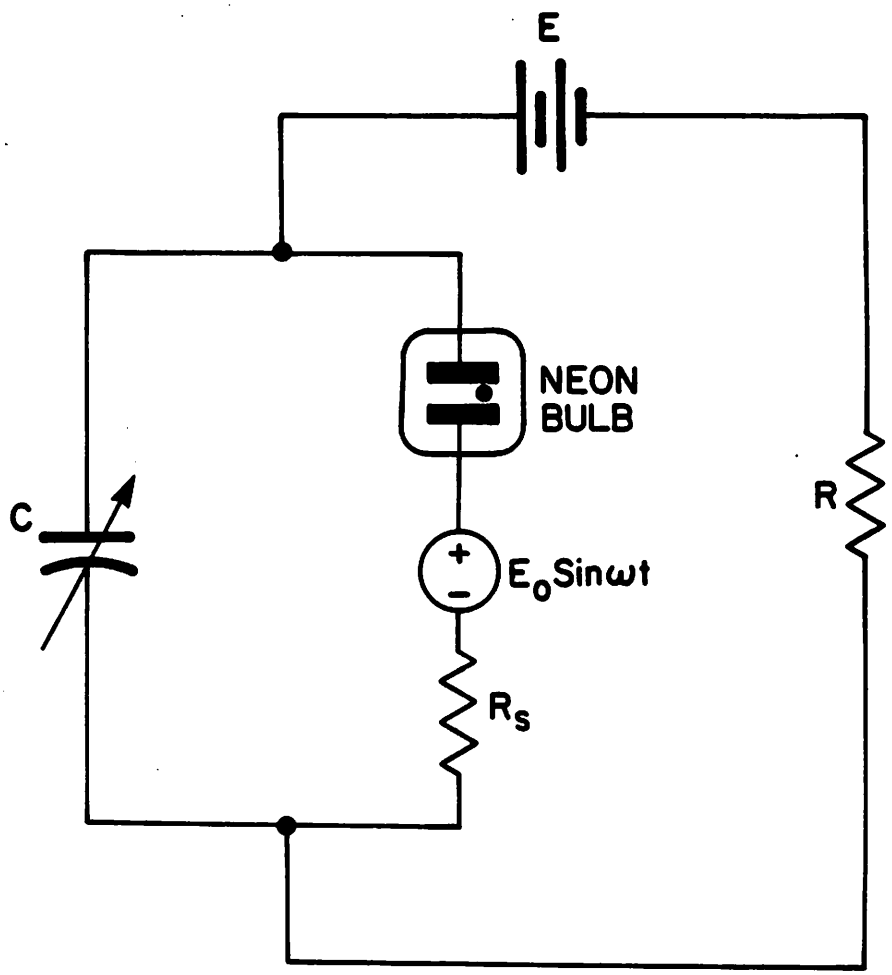

Fig. 1 SINUSOIDALLY DRIVEN NEON BULB RELAXATION OSCILLATOR.

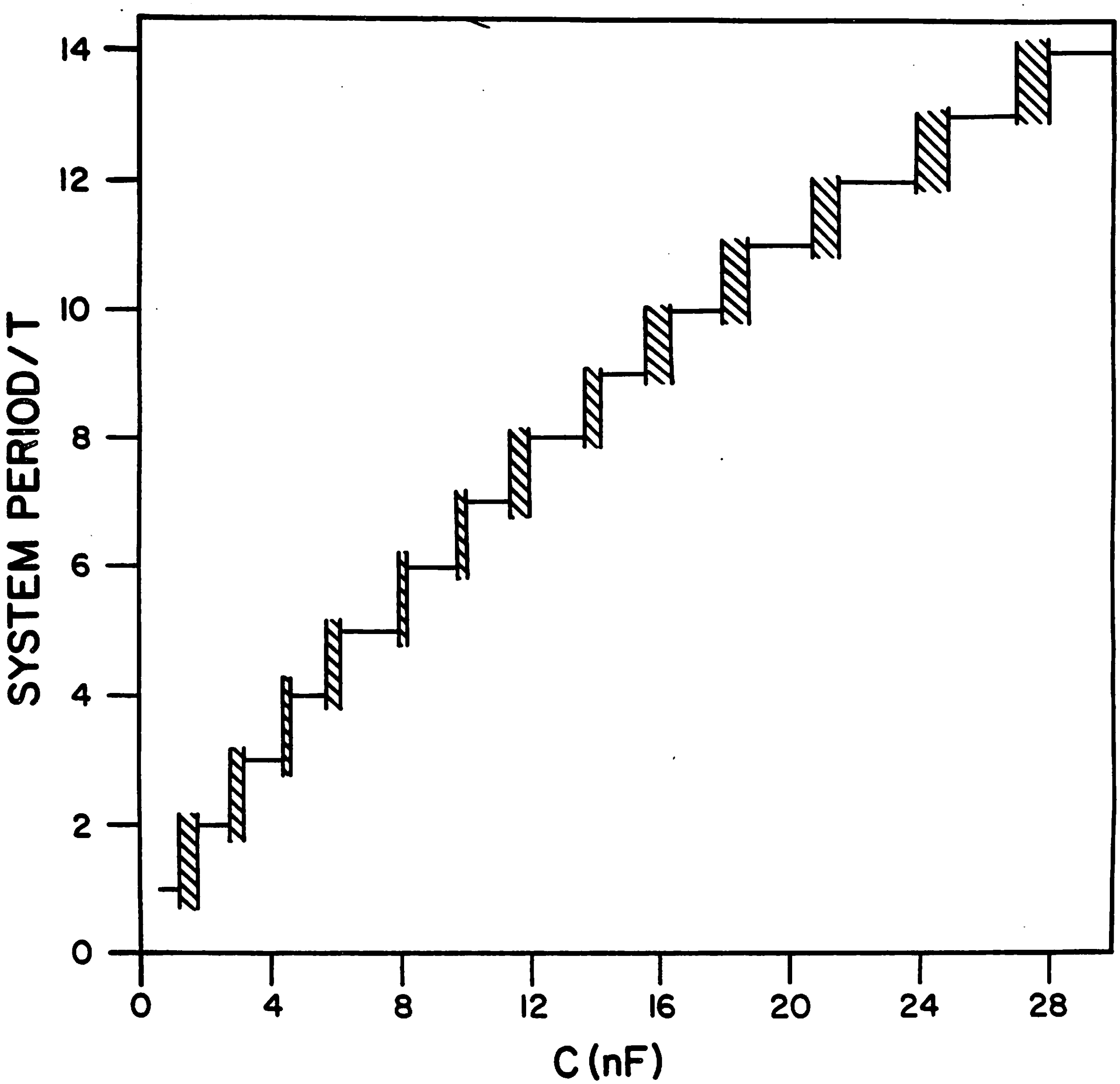

Fig. 2

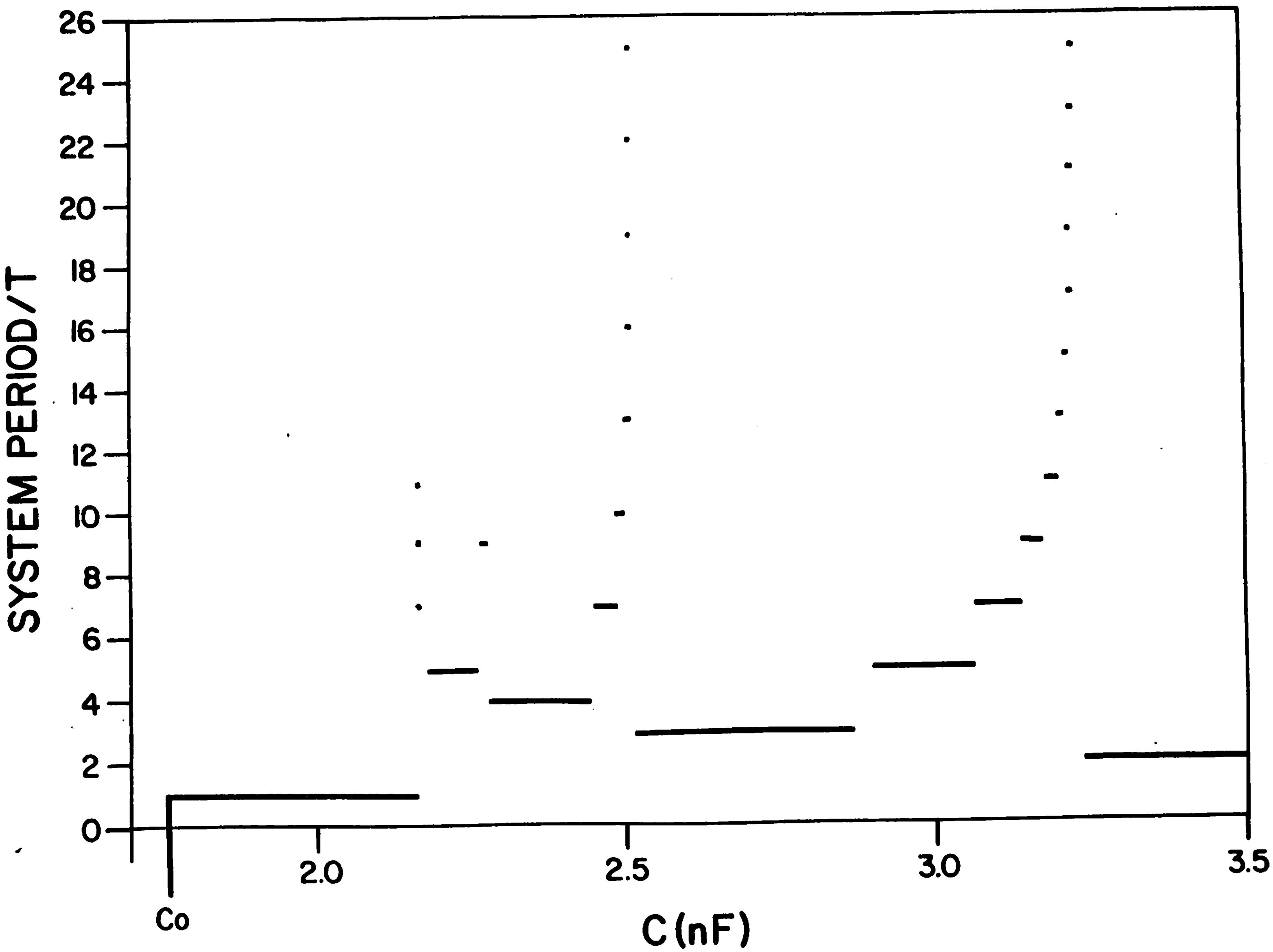

Fig. 3

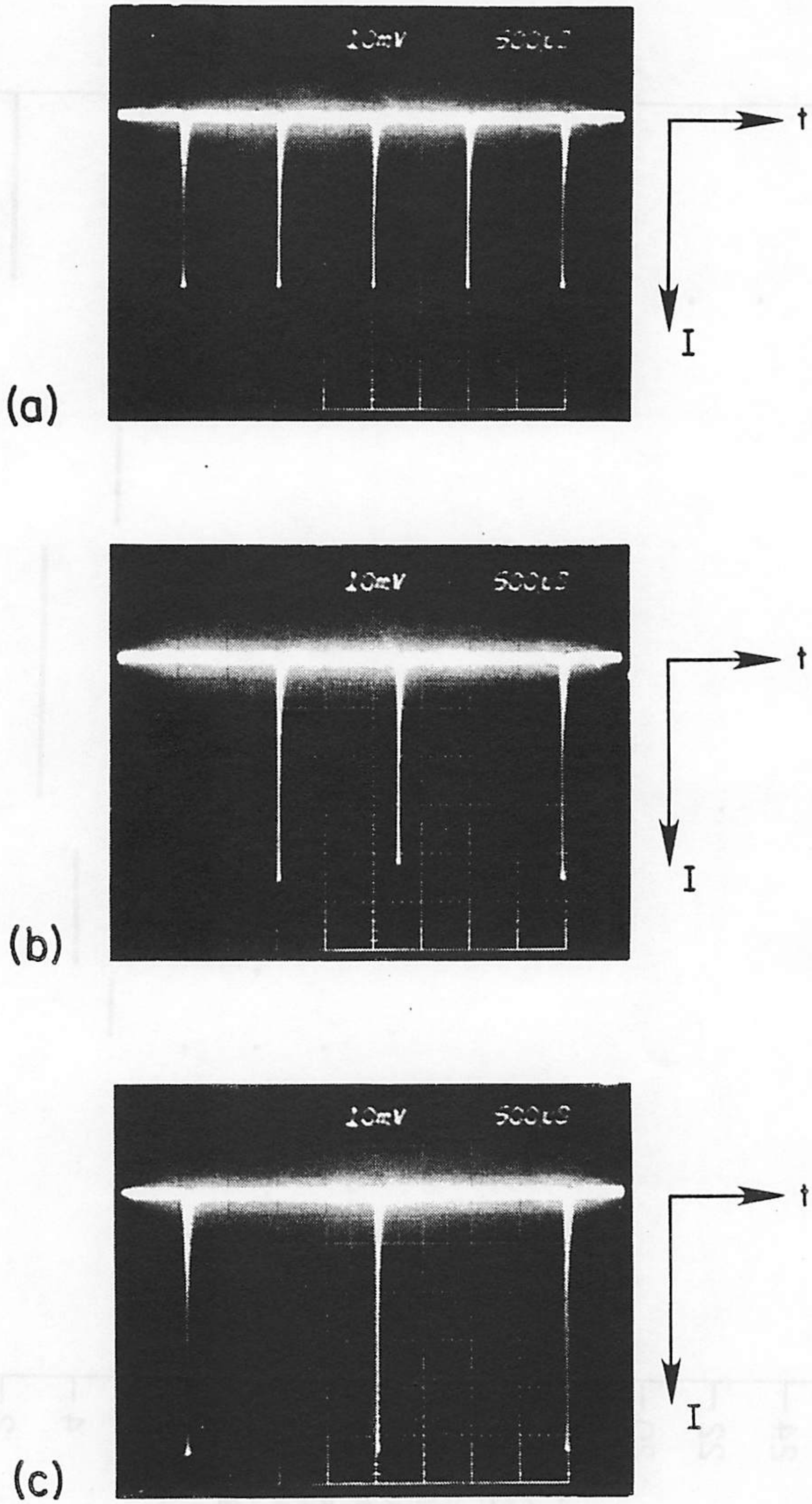

Fig. 4

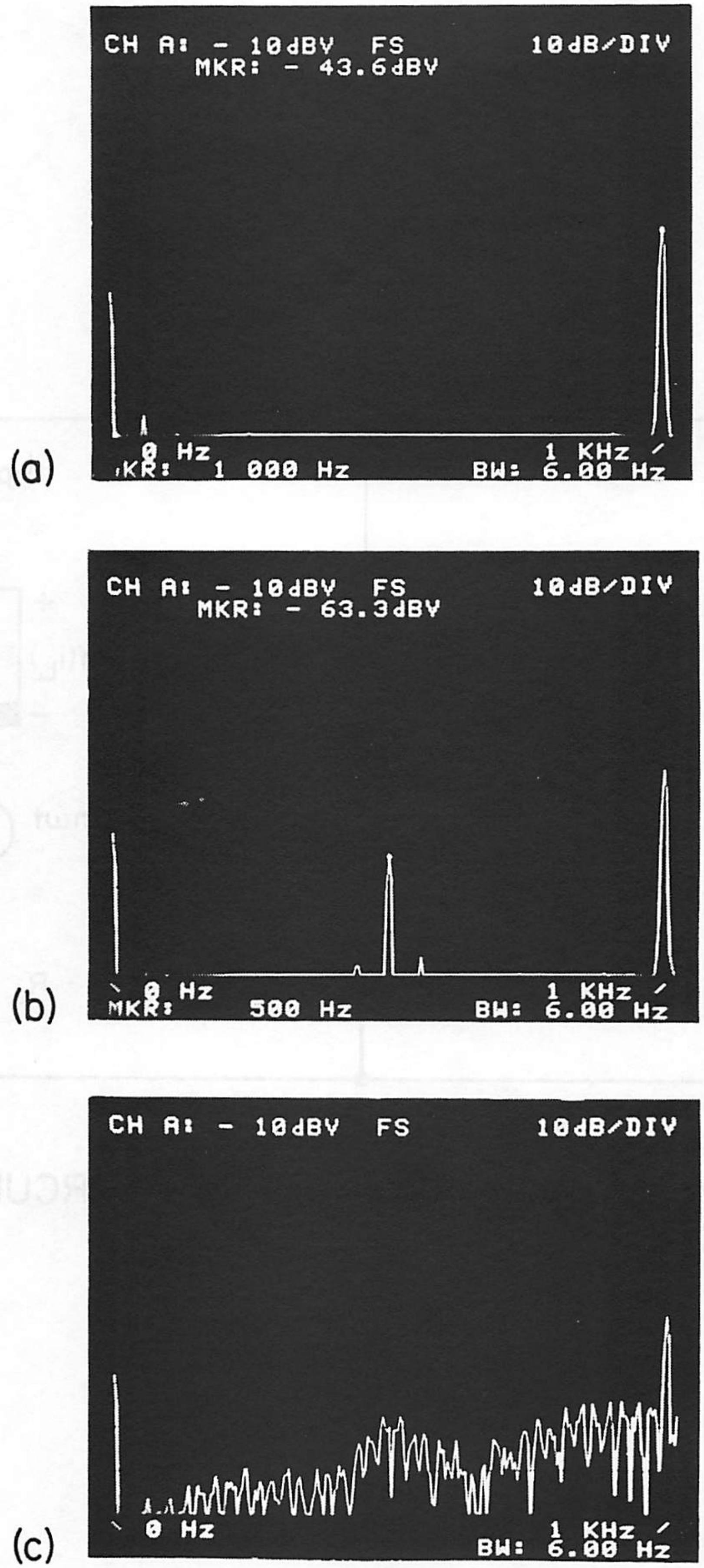
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Fig. 5

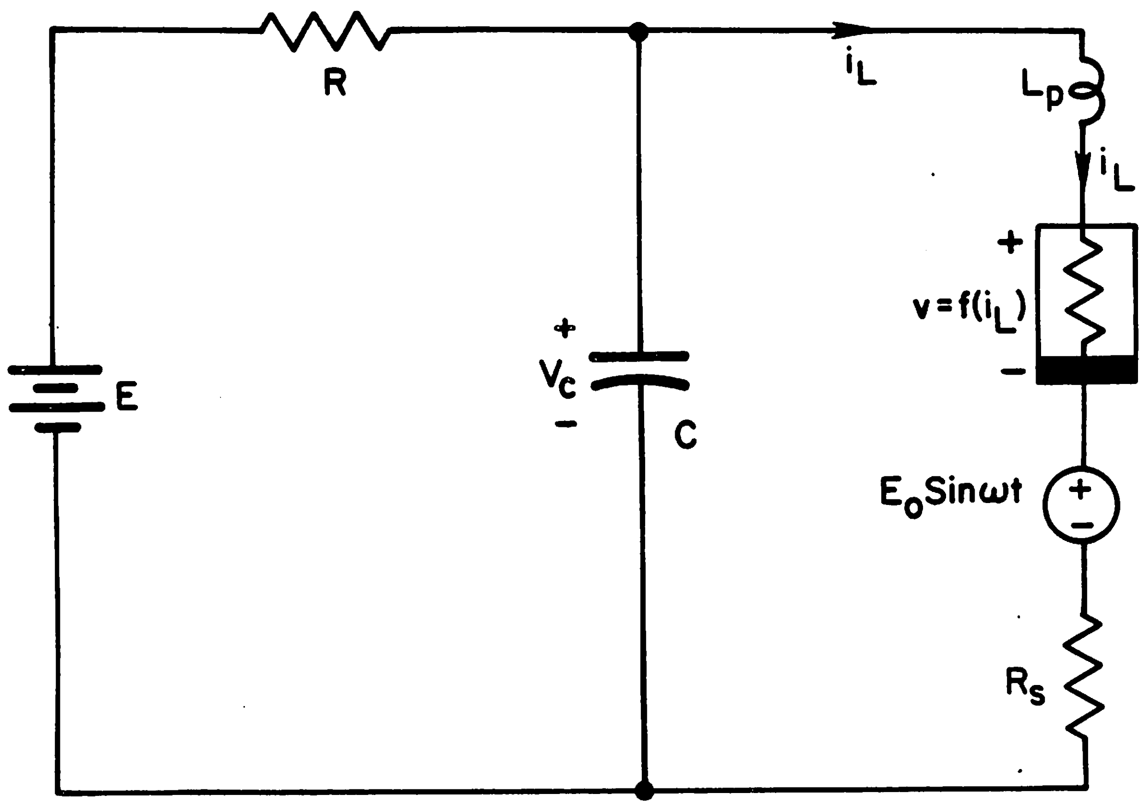
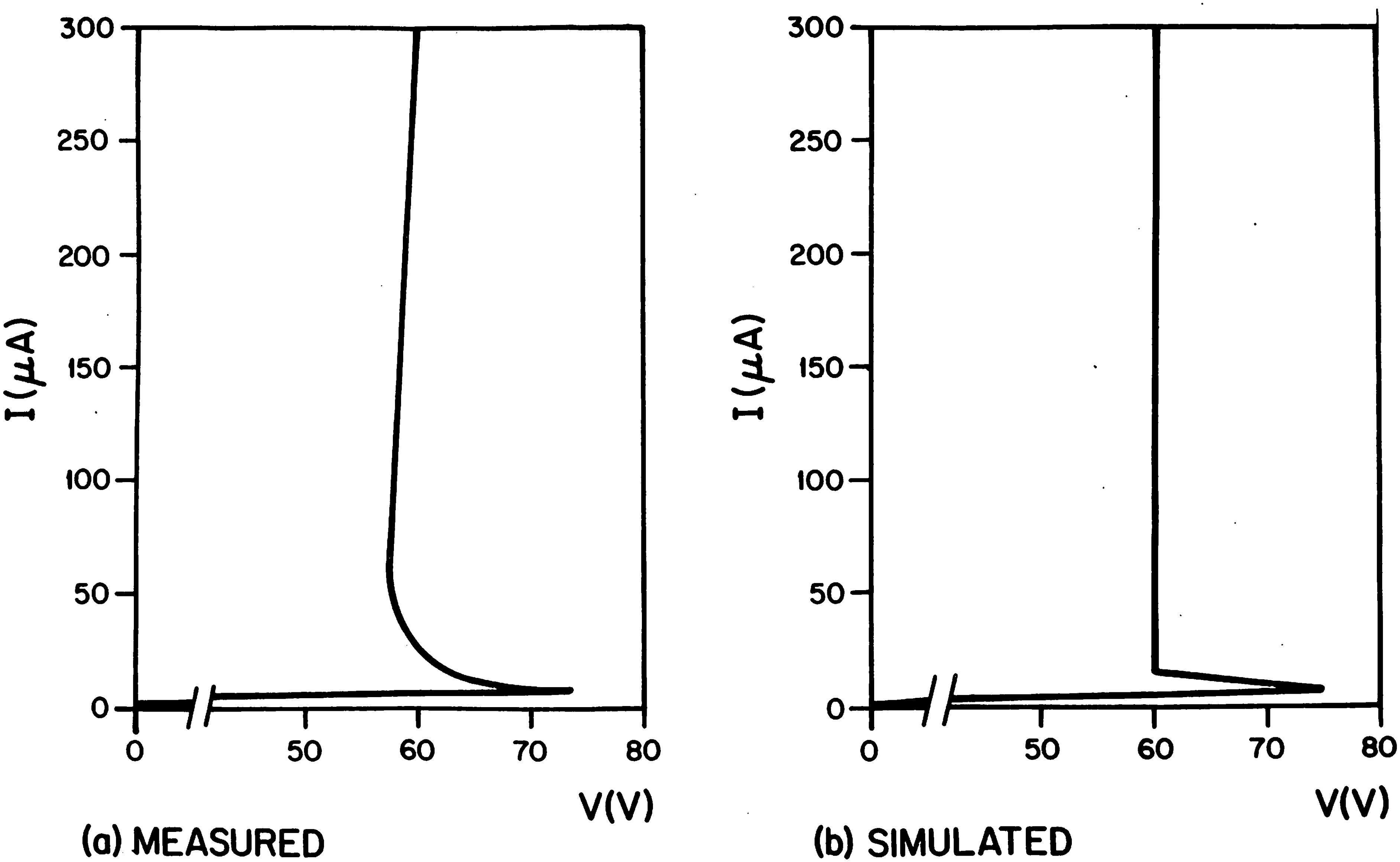


Fig. 6 VAN DER POL'S NEON BULB CIRCUIT - COMPUTER MODEL



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V(V)
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Fig. 7

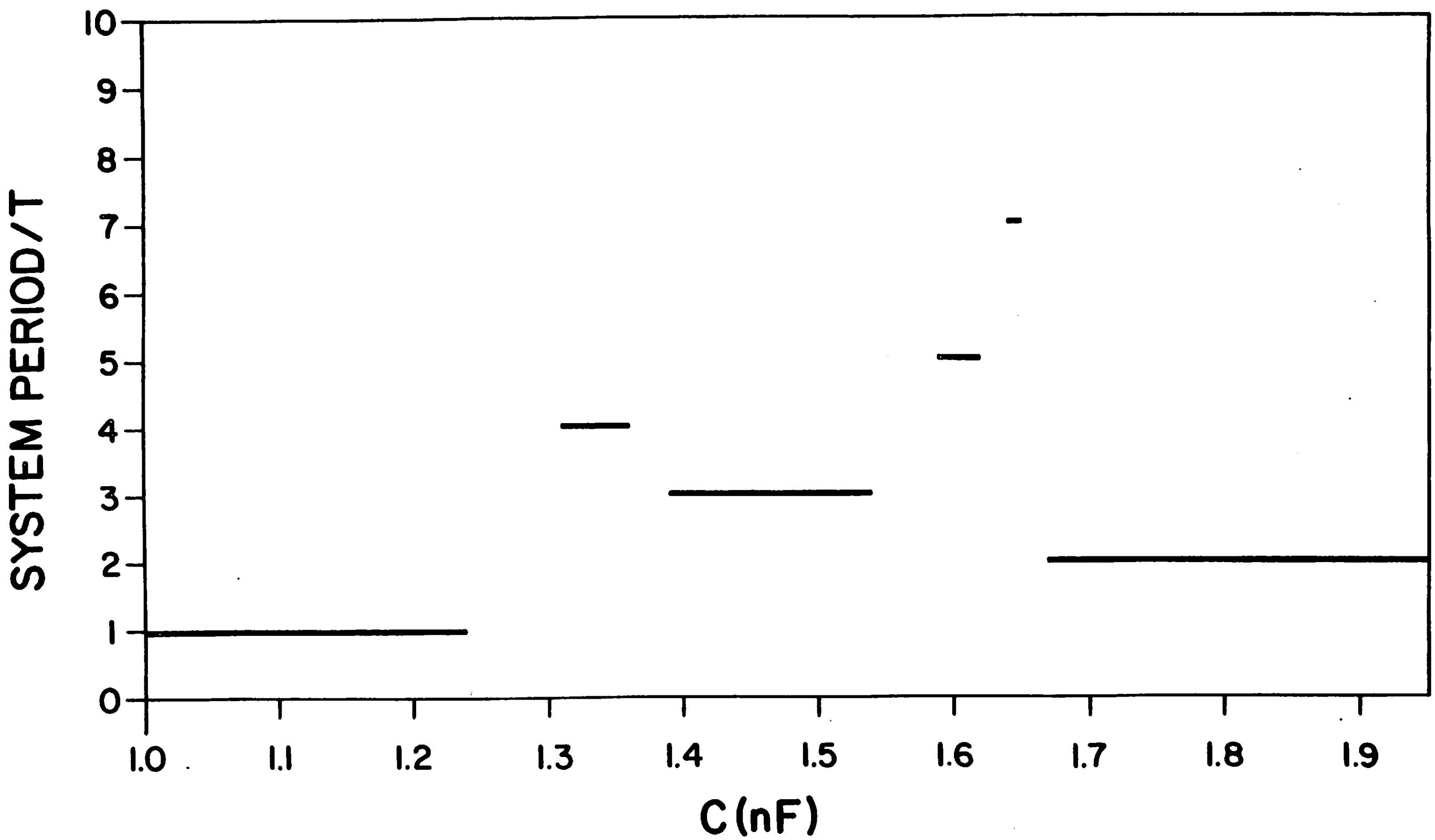
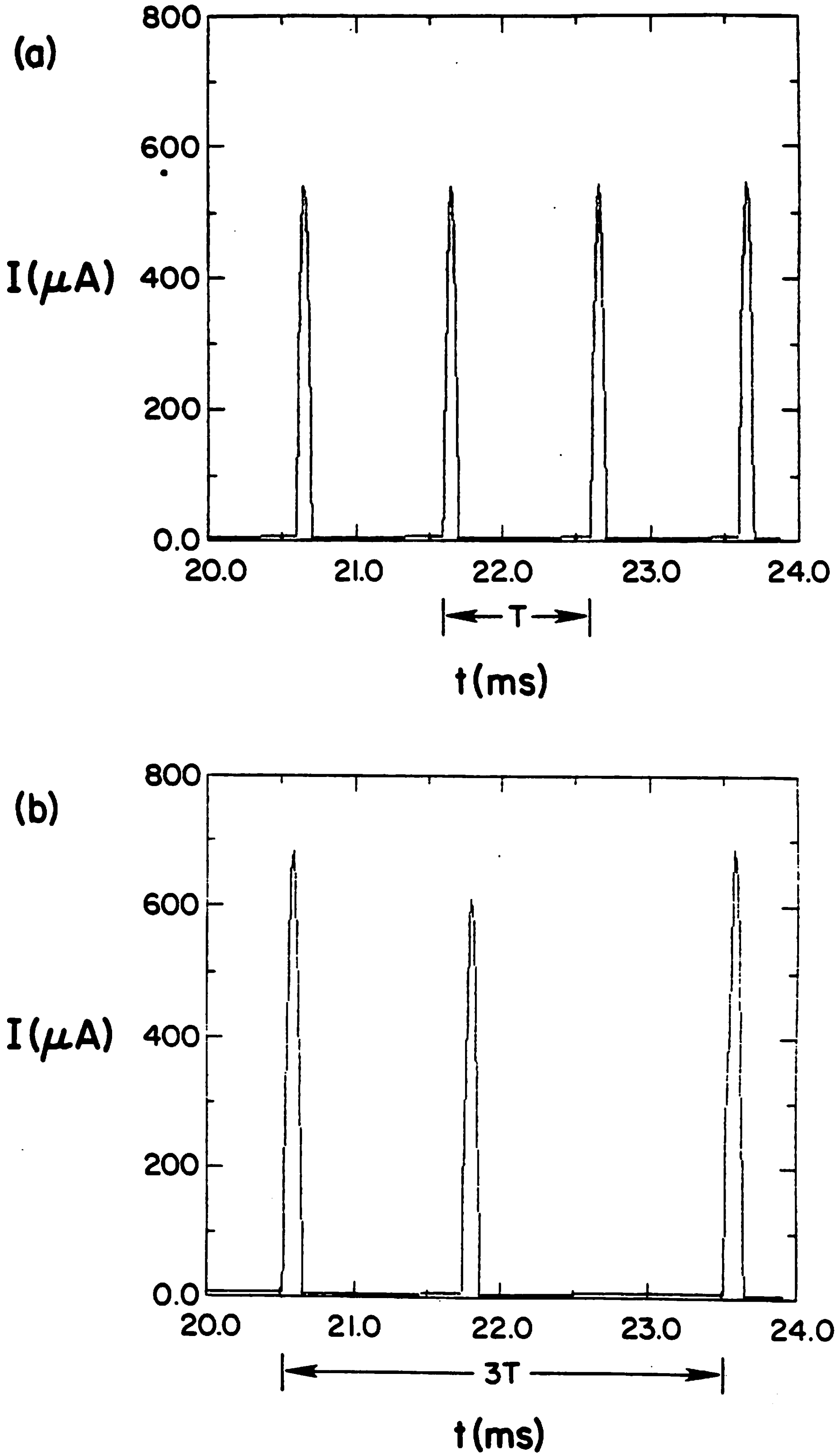


Fig. 8

Fig. 9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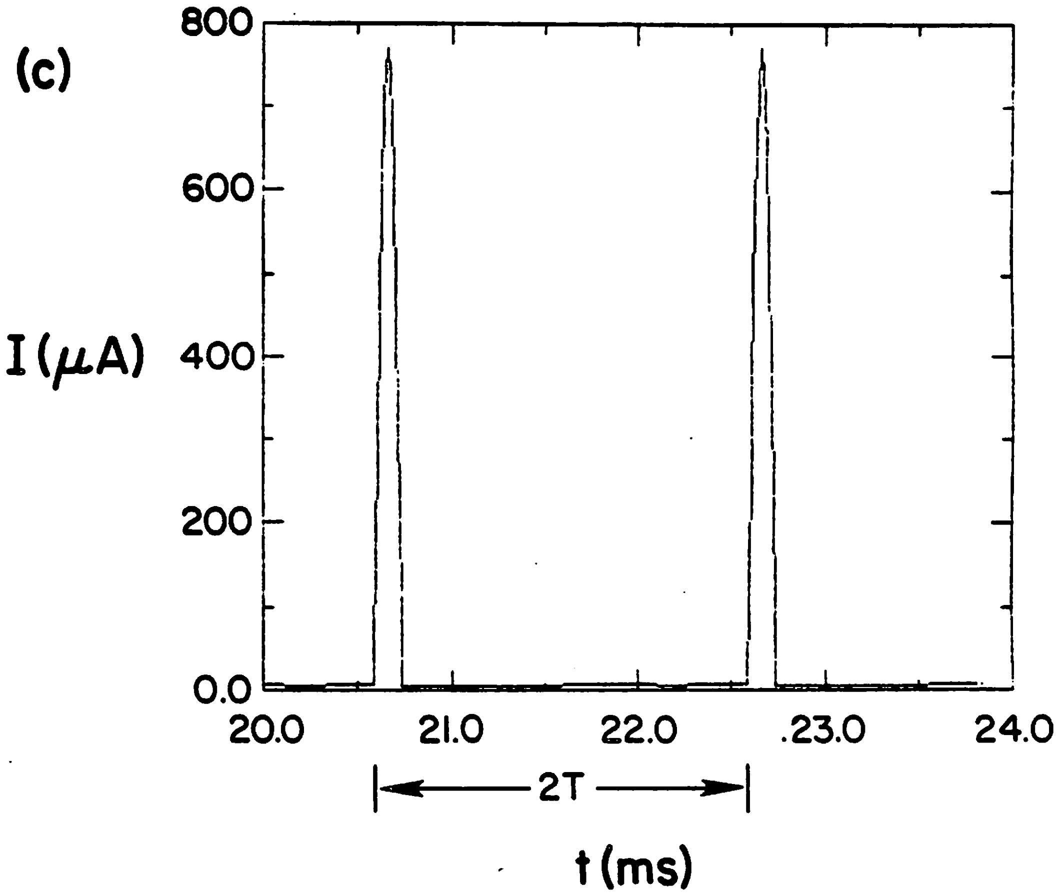
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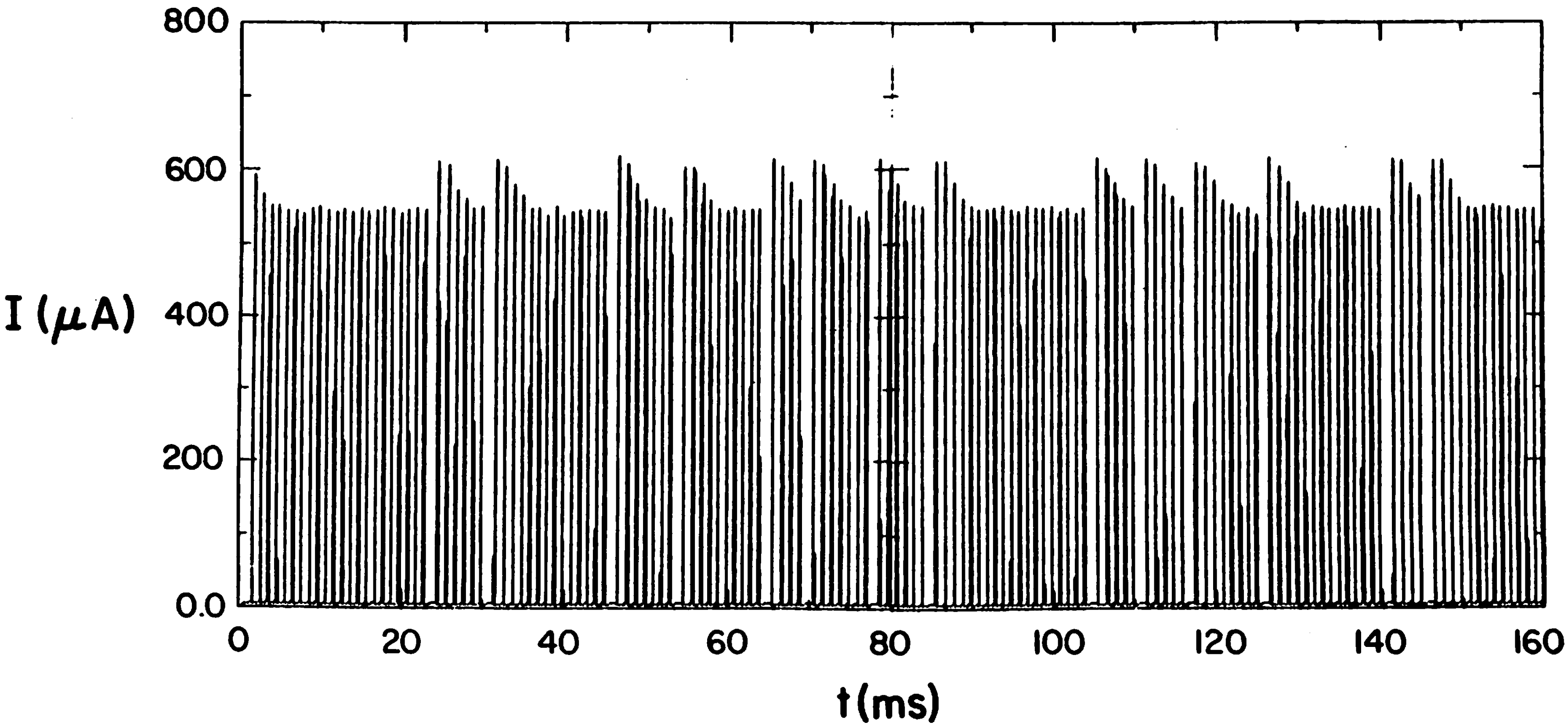

Fig. 10

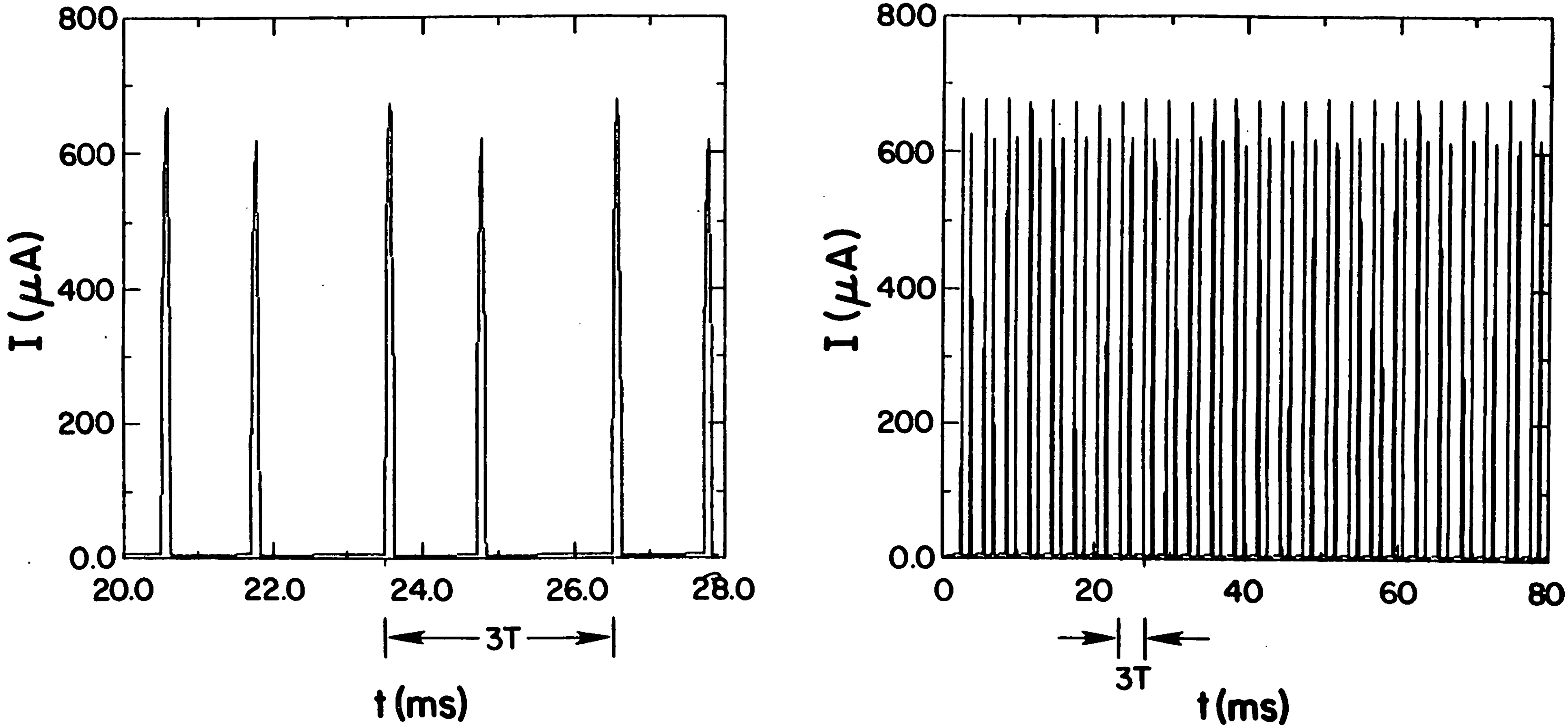

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Fig. 11 (a)

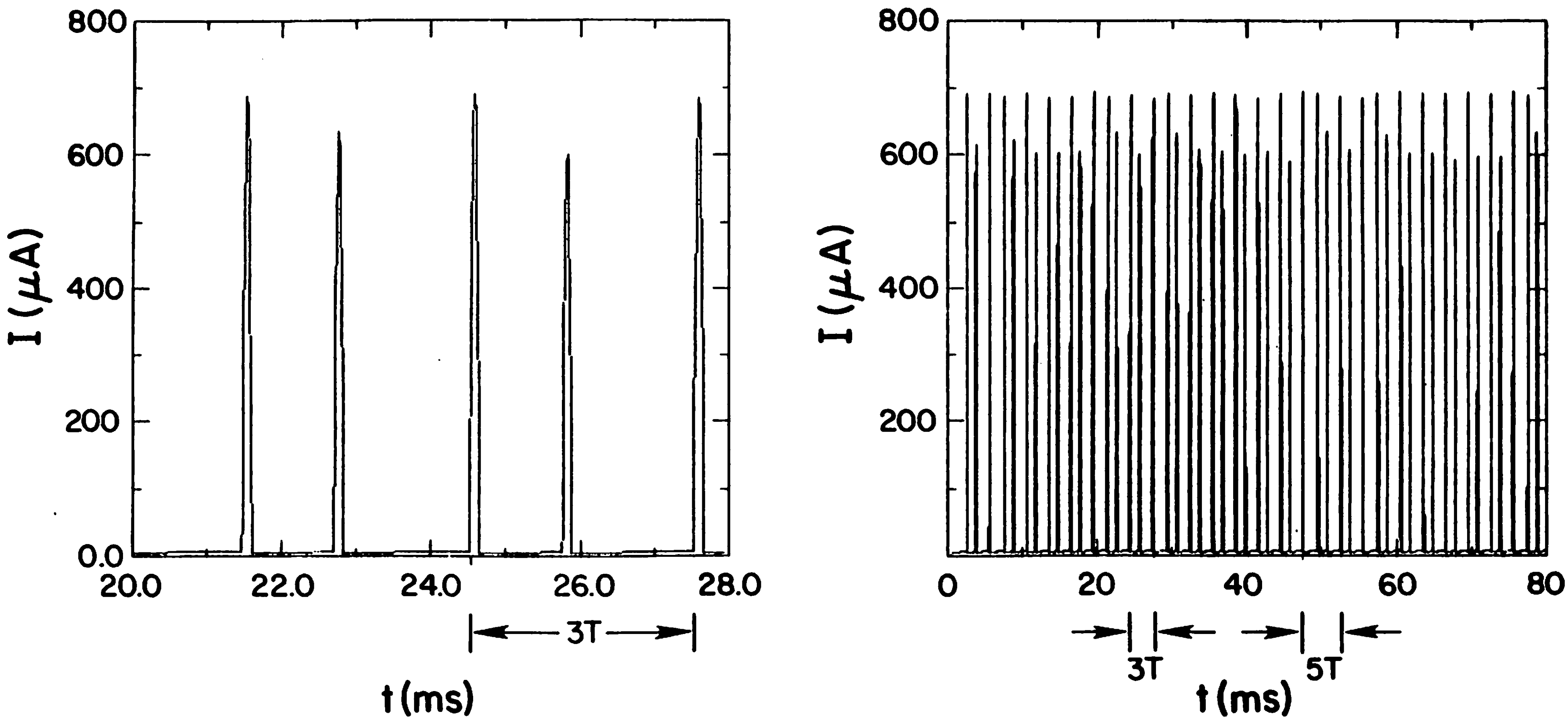
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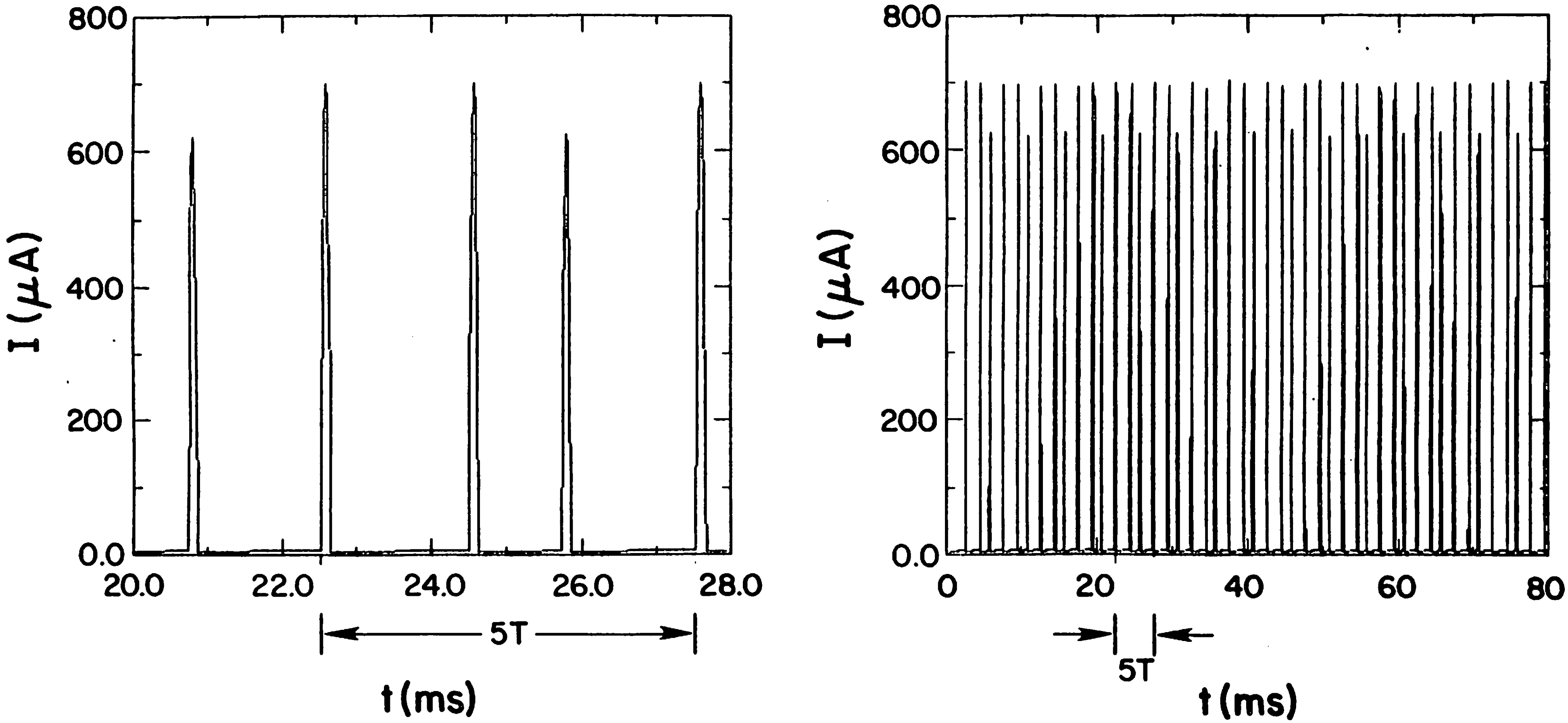
Fig. 11 (b)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t (ms)

Fig. 11 (c)