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by

E. Polak* and D. M. Stimler**

Abstract

A methodology is presented for transcribing worst case control system design specifications into tractable semi-infinite inequalities. These inequalities can either be solved directly or they can be incorporated into an optimal design semi-infinite optimization problem.

1. INTRODUCTION

There is a growing realization that many control system design problems can be expressed as optimization problems of some kind, see e.g. [Bec.1, Dav.1, Des.1, Kar.1, Kar.2, Pol.1, Pol.3, Tai.1, Zak.1, Zak.2]. Most often the resulting optimization problems are *semi-infinite*† and therefore require special algorithms for their solution, such as those in [Gon.1, Pol.4].

The most general semi-infinite optimization problems that are solvable by existing algorithms are of the form

$$\min\{f(x) \mid g^j(x) \leq 0,\ j \in J;\ \max_{\nu_k \in N_k} \phi^k(x, \nu_k) \leq 0,\ k \in K\} \tag{1.1}$$

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† Semi-infinite optimization problems are a type of nondifferentiable optimization problem such as (1.1), that are characterized by inequality constraints involving max functions.

where $f : \mathbf{R}^n \to \mathbf{R}$ and the $g^j : \mathbf{R}^n \to \mathbf{R}$ are locally Lipschitz continuous [Cla.1], the $\phi^k : \mathbf{R}^n \times \mathbf{R}^{q_k} \to \mathbf{R}$ are upper semi-continuous in (x, ν_k) and locally Lipschitz continuous in x, while the sets $\mathbf{N}_k \subset \mathbf{R}^{q_k}$ are compact. Furthermore, the use of semi-infinite optimization algorithms is predicated upon the possibility of computing elements in the generalized gradients of $f(\cdot)$, $g^j(\cdot)$ and $\phi^k(\cdot, \nu_k)$.

The amount of computation involved in solving a problem of the form (1.1) depends considerably on the form of the functions $\phi_k(\cdot,\cdot)$ and on the dimensionality of the sets $\mathbf{N}_k$, since global optimization, in the evaluation of the $\max_{\nu_k \in \mathbf{N}_k} \phi^n(x, \nu_k)$, becomes progressively more expensive as the dimension of $\mathbf{N}_k$ increases. Because of this, worst case control system design problems with unstructured plant uncertainty are more difficult to solve than nominal design problems, while worst case design problems with both structured and unstructured plant uncertainty are still more difficult. We shall demonstrate this fact by means of a simple example.

Consider the design of a proportional plus integral compensator $C(x, s) \triangleq x^1 + x^2/s$ for the feedback system in Fig. 1, with $x = (x^1, x^2)$ to be determined by optimization. For the sake of simplicity we assume that $F \equiv 1$. Suppose that one of the design goals is output disturbance suppression over the frequency range $[\omega', \omega'']$. First consider the problem of *nominal* design and suppose that the nominal plant transfer is $P_0(j\omega)$, defined by

$$P_0(s) = \frac{20(s+3)}{(s+8)(s+1-j2)(s+1+j2)} . \tag{1.2}$$

Then the disturbance suppresion requirement can be stated as an inequality of the form

$$\max_{\omega \in [\omega', \omega'']} \{ |H_{yd}(x, j\omega)|^2 - b(\omega) \} \leqslant 0 , \tag{1.3}$$

where $H_{yd}(x, j\omega) = [1 + P_0(j\omega)C(x, j\omega)]^{-1}$ and $b(\omega)$ is an upper semi-continuous function. Clearly, (1.2) is of the form

$$\max_{\nu \in N} \phi(x, \nu) \leqslant 0 , \tag{1.4}$$

with $\nu \triangleq \omega, N \triangleq [\omega', \omega'']$ and $\phi(x, \nu) \triangleq |H_{yd}(x, j\nu)|^2 - b(\nu)$. In this case, $\phi(x, \nu)$ is inexpensive to evaluate for each $\nu \in N$, and the evaluation of $\max_{\nu \in N} \phi(x, \nu)$ can be carried out easily by scanning over a grid of points in the interval $N = [\omega', \omega'']$.

Next, suppose that one must take unstructured plant uncertainty into account, and that the plant transfer function is given by $P(s) = P_0(s)l(s)$, with $P_0(s)$ as in (1.2) and $l(s)$ assumed to be known only to the extent that it is a stable, proper rational function (c.f. [Che.1, Des.1, Doy.1]) satisfying the bounds

$$|l(j\omega)| \in I_m(\omega) \triangleq \left[\frac{20}{|j\omega + 20|}, \frac{1}{20}|j\omega + 20|\right], \ \forall \ \omega \geqslant 0 , \tag{1.5a}$$

$$\arg l(j\omega) \in I_\theta(\omega) \triangleq -\arg(j\omega + 20), \ \arg(j\omega + 20)], \ \forall \ \omega \geqslant 0 . \tag{1.5b}$$

Shifting to polar coordinates, we let $P_0(j\omega) = m_0(\omega)l^{j\theta_c(\omega)}$, $C(x, j\omega) = m_c(x, \omega)l^{j\theta_c(x, \omega)}$. Hence, $l(j\omega) = m_l(\omega)l^{j\theta_l(\omega)}$, and introducing the dependence of the disturbance-to-output transfer function on plant uncertainty, we obtain

$$|H_{yd}(x, j\omega, l(j\omega))|^2 = \{[1 + m_0(\omega)m_l(\omega)m_C(x, \omega)\cos(\theta_0(\omega) + \theta_l(\omega) + \theta_C(x, \omega))]^2$$
$$+ [m_0(\omega)m_l(\omega)m_c(x, \omega)\sin(\theta_0(\omega) + \theta_l(\omega) + \theta_c(x, \omega))]^2\}^{-1} \tag{1.6}$$

If disturbance rejection is to be ensured for all possible plants within the given uncertainty set, we must satisfy (1.4) with $\nu \triangleq \omega, N \triangleq [\omega', \omega'']$, as before, but now

$$\phi(x, \nu) \triangleq \max_{\substack{m \in I_m(\nu) \\ \theta \in I_\theta(\nu)}} \{\{[1 + m_0(\nu)m \ m_c(x, nu)\cos(\theta_0(\nu) + \theta + \theta_c(x, \nu)]^2$$
$$+ m_0(\nu)m \ m_c(x, \nu)\sin(\theta_0(\nu) + \theta + \theta_c(x, \nu)]^2\}^{-1} - b(\nu)\} . \tag{1.7}$$

The evaluation of $\phi(x, \nu)$ is fairly easy if one makes full use of the fact that it is defined as a max over a rectangle in R^2 (see [Pol.2, Sti.1]).

Finally, suppose that there is not only unstructured, but also structured uncertainty in the plant model.[†] In this case, (1.2) has to be replaced by an expression of the form

$$P_0 = \frac{\alpha^0(s + \alpha^1)}{(s + \alpha^2)(s + \alpha^3 - j\alpha^4)(s + \alpha^3 + j\alpha^4)} \tag{1.8}$$

with, say $\alpha^0 \in [12.5, 25]$, $\alpha^1 \in [2, 4]$, $\alpha^2 \in [5, 10]$, $\alpha^3 \in [-1, 0]$, $\alpha^4 \in [1, 2]$. Fig.2 shows the uncertainty rectangles in the complex plane. The vector $\alpha \triangleq (\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4)$. In this case, the magnitude m_0 and the phase θ_0 of the structured part of the plant become functions not only of ω, but also of α and hence, for worst case design purposes (1.7) must be replaced by

$$\phi(x, \nu) \triangleq \max_{\substack{m \in I_m \\ \theta \in I_\theta \\ \alpha \in A}} \{ \{ [1 + m_0(\nu, \alpha) m \; m_c(x, \nu) \cos(\theta_0(\nu, \alpha) + \theta + \theta_c(x, \nu)]^2$$

$$+ m_0(\nu, \alpha) m \; m_c(x, \nu) \sin(\theta_0(\nu, \alpha) + \theta + \theta_c(x, \nu)]^2 \}^{-1} - b(\nu) \} \tag{1.9}$$

where $A \triangleq [12.5, 25] \times [2, 4] \times [5, 10] \times [-1, 0] \times [1, 2] \times I_m \times I_\theta$. We see that even for our low order plant, $\phi(x, \nu)$ is defined as a max over a seven dimensional rectangle and its evaluation is extremely difficult indeed.

The remainder of this paper is devoted to the development of techniques for the reformulation of worst case SISO feedback system design specifications in a computationally efficient form. In particular, it will be shown that, proceeding from a *literal* performance specification function $\phi(x, \nu)$, such as the one in (1.9), it is possible to construct a *majorizing* performance specification function $\tilde{\phi}(x, \nu)$ with the following properties: (i) $\tilde{\phi}(x, \nu)$ can be evaluated through a fairly small number of simple function evaluations; (ii) $\tilde{\phi}(x, \nu) \geqslant \phi(x, \nu)$ for all (x, ν), ensuring that the satisfaction of the more easily verified inequality $\max_{\nu \in N} \tilde{\phi}(x, \nu) \leqslant 0$, implies that $\max_{\nu \in N} \phi(x, \nu) \leqslant 0$ is satisfied; and (iii) $\tilde{\phi}(\cdot, \cdot)$ satisfies the hypotheses imposed by semi-infinite optimization algorithms. Thus we shall see that computational costs of optimal worst case, linear feedback system design.

[†]We refer the reader to [Doy.1, Hor.1, Hor.2, Hor.3, Orl.1] for a discussion of the manner in which the two types of uncertainty arise in models.

can be drastically reduced at the expense of a small amount of conservatism in system performance specification.

Finally, we must point out that we are not the first to propose techniques for the reduction of the computational complexity of semi-infinite inequalites in engineering design. see, e.g., [Hal.1, Heu.1, Sch.1]. Unfortunately, these earlier techniques appear to be unsuitable for optimal worst case control system design. For example, since control system performance specifying functions, such as in (1.7), are not convex in the uncertainty parameters, we are unable to use the results in [Hal.1]. Similarly, we are unable to use the statistical methods proposed in [Heu.1], because they produce a "simple-to-evaluate" approximation $\phi'(x,\nu)$ to $\phi(x,\nu)$, which fails to satisfy the crucial property $\phi'(x,\nu) \geqslant \phi(x,\nu)$ for all (x,ν).

2. A FIRST APPROACH TO DESIGN SPECIFICATION VIA SEMI-INFINITE INEQUALITIES

The transcription of worst case control system design into a semi-infinite optimization problem was discussed at some length in [Pol.2], where we find that many design specifications, such as output disturbance suppression, plant saturation avoidance, input following, etc. result in very similar looking semi-infinite inequalities. Consequently, it is possible to demonstrate fully our computational complexity reduction technique, by applying it to the simple worst case design of the SISO, two degrees of freedom feedback system, shown in Fig. 1, for which the compensators must ensure exponential stability and plant saturation avoidance, in the presence of both parametric and unstructured plant model uncertainty.

We begin with a description of a convenient compensator and plant transfer function parametrization and the "standard" method of transcribing performance specifications into semi-infinite inequalities.

We parametrize the compensators in factored form:

$$C(x,s) \triangleq \frac{n_C(x,s)}{d_C(x,s)} = \frac{K_C(s + d_C^1)\prod_{i=1}^{N_C'}(s_2 + 2a_C^i\, s + b_C^i)}{(s + d_C^2)\prod_{i=N_C'+1}^{N_C}(s^2 + 2a_C^i\, s + b_C^i)} \tag{2.1a}$$

$$F(x,s) \triangleq \frac{n_F(x,s)}{d_F(x,s)} = \frac{K_F(s + d_F^1)\prod_{i=1}^{N_F'}(s_2 + 2a_F^i\, s + b_F^i)}{(s + d_F^2)\prod_{i=N_F'+1}^{N_F}(s^2 + 2a_F^i\, s + b_F^i)} \tag{2.1b}$$

where $N_C \geqslant 2N_C'$, $N_F \geqslant 2N_F'$ to ensure the properness of the compensators.† The vector x is made up of all the compensation coefficients, viz., $x = (K_C, K_F, a_C, b_C, d_C, a_F, b_F, d_F)$, where $a_c = (a_c^1, \ldots, a_c^{N_c})$, etc.

Next we we assume that plant transfer function is of the form‡

$$P(s,\alpha,l) = P_0(s,\alpha)\, l(s) \tag{2.2a}$$

where $\alpha \in \mathbf{R}^{n_\alpha}$ is the (structured) *parametric uncertainty* vector and $l : \mathbf{C} \to \mathbf{C}$ is an *unstructured uncertainty* rational function. The structured part, P_0, has the form

$$P_0(s,\alpha) = \frac{n_0(s,\alpha)}{d_0(s,\alpha)} \triangleq \frac{K_p\prod_{i=1}^{M}(s + z^i)\prod_{i=M+1}^{M_p}(s + z^i)(z + z^{i*})}{\prod_{i=1}^{N}(s + p^i)\prod_{i=N+1}^{N_p}(s + p^i)(s + p^{i*})} \tag{2.2b}$$

with $2M_p - M \leqslant 2N_p - N$, to ensure that P_0 is proper. The uncertainty vector $\alpha \in \mathbf{R}^{n_\alpha}$ consists of the gain and all the poles and zeros in (2.26) which are known only to the extent that they are contained in intervals in $\mathbf{R}$ or $\mathbf{C}$. Hence, if all the parameters in P_0 are uncertain, $\alpha = (K_p, z^1, z^2, \ldots, z^{N_p}, p^1, p^2, \ldots, p^{N_p})$ and

†As stated, the compensator numerators and denominators are of odd degree. When this is not desired, the appropriate terms in (2.1a, b) may be deleted, provided properness is not violated.

‡Although unstructured multiplicative uncertainty is commonly expressed as $P_0(s)[1 + \tilde{l}(s)]$, see e.g. [Doy.1], for our purposes the equivalent form in (2.2a) is more convenient.

$$K_p \in [\underline{K}_p, \overline{K}_p], \tag{2.2c}$$

$$z^i \in [\underline{z}^i, \overline{z}^i], i = 1, 2, \ldots, N_p, \tag{2.2d}$$

$$p^i \in [\underline{p}^i, \underline{p}^i], \ i = 1, 2, \ldots, M_p, \tag{2.2e}$$

where a complex "interval" $[\underline{z}^i, \overline{z}^i]$ in $\mathbb{C}$ is defined by

$$[\underline{z}^i, \overline{z}^i] \triangleq \{z \in \mathbb{C} \mid \mathrm{Re}(\underline{z}^i) \leqslant \mathrm{Re}(z^i) \leqslant \mathrm{Re}(\overline{z}^i), \mathrm{Im}(\underline{z}^i) \leqslant \mathrm{Im}(z^i) \leqslant \mathrm{Im}(\overline{z}^i)\}. \tag{2.3}$$

A typical pole zero uncertainty diagram for the transfer function for the example considered in (1.8), which corresponds to $z^1 \in [2, 4]$, $p^1 \in [5, 10]$ and $p^2 \in [-1 + j1, 0 + j2]$, was shown in Fig. 2.

The uncertainty intervals (2.2c)-(2.2d) define a *parametric uncertainty set* which we shall denote by A. Clearly, A is given by

$$\mathrm{A} \triangleq [\underline{K}_p, \overline{K}_p] \times \prod_{z=1}^{N_p} [\underline{z}^i, \overline{z}^i] \times \prod_{z=1}^{N_p} [\underline{p}_i, \overline{p}_i] \tag{2.4}$$

The unstructured part of the plant model, $l(\cdot)$, will be assumed to be known only to the extent that it is a member of the family of functions L, defined as follows.

Definition 2.1. We shall denote by L the family of *unstructured uncertainty* rational functions $l : \mathbb{C} \to \mathbb{C}$ which have equal numerator and denominator degrees and satisfy the two inequalities (see Fig. 3)

$$\underline{l}_m(\omega) \leqslant |l(j\omega)| \leqslant \overline{l}_m(\omega), \ \forall \omega \geqslant 0, \tag{2.5a}$$

$$\underline{l}_A(\omega) \leqslant \arg l(j\omega) \leqslant \overline{l}_A(\omega), \ \forall \omega \geqslant 0, \tag{2.5b}$$

where the bound functions $\underline{l}_M, \overline{l}_M, \underline{l}_A, \overline{l}_A : \mathrm{R}_+ \to \mathrm{R}$ are continuously differentiable and satisfy

$$0 < \underline{l}_M(\omega) \leqslant 1 \leqslant \overline{l}_M(\omega), \ \forall \omega \geqslant 0, \tag{2.6a}$$

$$\underline{l}_A(\omega) \leqslant 0 \leqslant \overline{l}_A(\omega), \forall \ \omega \geqslant 0. \tag{2.6b}$$

■

We now introduce an assumption which is designed to eliminated the ill-posed problem of stabilizing a plant with uncertainty about the number of unstable poles (see [Zam.1, Zam.2]) and to ensure that there are no $j\omega$-axis pole-zero cancellations.

Assumption 2.1. The plant transfer function $P(\cdot, \alpha, l)$ has the same number of $\mathbb{C}_+$ poles for all $\alpha \in \mathbf{A}$ and $l \in \mathbf{L}$.

■

Note: Since $l(s) \equiv 1$ is a function in $\mathbf{L}$, it follows that the structured part of P, $P_0(\cdot, \alpha)$, has the same number of $\mathbb{C}_+$ poles for all $\alpha \in \mathbf{A}$.

Next we turn to the transcription of our plant saturation avoidance and bibo stability requirements into semi-infinite inequality form. We begin with saturation avoidance, which is the simpler of the two. The most direct formulation of this requirement is†

$$\max_{\substack{\omega \in [\omega', \omega''] \\ \alpha \in \mathbf{A} \\ l \in \mathbf{L}}} \{ |H_{vr}(x, j\omega, \alpha, l(j\omega))|^2 - b_{vr}(\omega)\} \leq 0, \tag{2.7}$$

where $[\omega', \omega'']$ is the expected bandwidth of the system, $b_{vr} : \mathbf{R}_+ \to \mathbf{R}$ is a continuously differentiable bound function, and the reference input to plant input transfer function, H_{vr}, is given by

$$H_{vr}(x, j\omega, \alpha, l(j\omega)) = F(x, j\omega)\, C(x, j\omega)[1 + P(j\omega, \alpha, l)C(x, j\omega)]^{-1}. \tag{2.8}$$

We find it convenient to rewrite (2.7) with all transfer functions in polar coordinates: $F(x, j\omega) = m_F(x, \omega)e^{j\theta_F(x,\omega)}$, $C(x, j\omega) = m_C(x, \omega)e^{j\theta_C(x,\omega)}$, $P_0(j\omega, \alpha) = m_0(\omega, \alpha)e^{j\theta_0(\omega,\alpha)}$ and $l(j\omega) = m_l(\omega)l^{j\theta_l(\omega)}$. For $\omega \in \mathbf{R}_+$, let $\mathbf{S}(\omega) \subset \mathbf{R}^2$ be defined by

$$\begin{aligned} \mathbf{S}(\omega) \triangleq \{(m, \theta) \in \mathbf{R}^2 | m &= m_0(\omega, \alpha)m_l(\omega), \\ \theta &= \theta_0(\omega, \alpha) + \theta_l(\omega), \alpha \in \mathbf{A}, l \in \mathbf{L}\}. \end{aligned} \tag{2.9}$$

Then (2.7) becomes

$$\begin{aligned} \max_{\omega \in [\omega', \omega'']} \max_{(m, \theta) \in \mathbf{S}(\omega)} \{ & m_F(x, \omega)^2\, m_C(x, \omega)^2 \times \\ & [m^2\, m_C(x, \omega)^2 + 2m\, m_C(x, \omega) \cos(\theta + \theta_C(x, \omega)) + 1]^{-1} - b_{vr}(\omega)\} \leq 0. \end{aligned} \tag{2.10}$$

We see that (2.10) is of the form

†We use $|H_{vr}|^2$ rather than $|H_{vr}|$ in (2.7) so as to remove the "corner" or the magnitude function at the origin.

$$\max_{\omega \in [\omega', \omega'']} \phi_{vr}(x, \omega) \leqslant 0, \tag{2.11a}$$

where ϕ_{vr} is of the form

$$\phi_{vr}(x, \omega) = \max_{\eta \in S(\omega)} \xi_{vr}(x, \omega, \eta) \tag{2.11b}$$

with $\eta \triangleq (m, \theta)$ and

$$\xi_{vr}(x, \omega, \eta) \triangleq m_F(x, \omega)^2 m_c(x, \omega)^2 [m^2 m_c(x, \omega)^2 + 2m\, m_c(x, \omega) \cos(\theta + \theta_c(x, \omega)) + 1]^{-1} - b_{vr}(\omega). \tag{2.11c}$$

The computational difficulty of checking inequality (2.7) is the same as that of (2.10). The form (2.10) gives an *illusion* of computational complexity simplification because S(ω) is only a subset of $\mathbb{R}^2$. The problem is that the difficulty has been merely shifted to the very complex description of S(ω). Nevertheless, as we shall see in the next section, (2.10) eventually leads to a computationally more tractable form of saturation avoidance specification.

Next we turn to the stability requirement. For the two degrees of freedom system in Fig. 1 to be exponentially stable, both the precompensator F and the unity feedback closed loop system around PC, must be table. Because of the parametrization in (2.1b), the precompensator F will be exponentially stable if for a chosen $\epsilon > 0$, the following inequalities are satisfied:

$$\epsilon - d_F^2 \leqslant 0 \tag{2.12a}$$

$$\epsilon - a_F^i \leqslant 0,\ i = N_F', N_F'+1, \ldots, N_F, \tag{2.12b}$$

$$\epsilon - b_F^i \leqslant 0,\ i = N_F', N_F'+1, \ldots, N_F. \tag{2.12c}$$

To ensure worst case stability for the closed loop system, we propose to further extend the extended Nyquist criterion described in [Pol.2]. This criterion requires a normalizing polynomial $d(s)$, of degree $2N_p + 2N_c - 2N_c^1 - N + 1$, such that all of its zero are in $\overset{\circ}{\mathbb{C}}$. The degree of $d(s)$ is be equal to the degree of the characteristic polynomial of the closed loop subsystem in Fig. 1, for $l(x) \equiv 1$. It can be deduced from [Pol.2] that worst case exponential stability of the closed loop subsystem in Fig. 1 is ensured if and only if the locus of

$$T(x, j\omega, \alpha, l) \triangleq \frac{n_c(x, j\omega) n_0(j\omega, \alpha) l(j\omega) + d_c(x, j\omega) d_0(j\omega, \alpha)}{d(j\omega)}, \tag{2.13}$$

traced out for $-\infty \leqslant \omega \leqslant \infty$, does not encircle the origin for all $\alpha \in \mathrm{A}$ and for all $l \in \mathrm{L}$. A sufficient condition for this to hold is that the locus of $T(x, j\omega, \alpha, l)$ stays out of a parabolic region enclosing the origin, as shown in Fig. 4, i.e. that

$$\max_{\substack{\omega \in [0, \omega_s] \\ \alpha \in \mathrm{A} \\ l \in \mathrm{L}}} \{\mathrm{Im}[T(x, j\omega, \alpha, l)] - k_1 \{\mathrm{Re}[T(x, j\omega, \alpha, l)]\}^2 + k_2\} \leqslant 0, \tag{2.14}$$

where $k_1, k_2 > 0$ determine the parabola $v = k_1 v^2 - k_2$, in Fig. 4, and ω_s is sufficiently large to cover the frequency range where encroachment into the parabolic region might take place.

Next, let $n_c(x, j\omega)/ d(j\omega) = r^1(x, \omega) e^{j\beta^1(x, \omega)}$ and $d_c(x, j\omega)/ d(j\omega) = r^2(x, \omega) e^{j\beta^2(x, \omega)}$. We define the set $S_s(\omega) \subset \mathrm{R}^4$ by

$$\begin{aligned} S_s(\omega) \triangleq \{ (m^1, m^2, \theta^1, \theta^2) \in \mathrm{R}^4 \mid m^1 = |n(j\omega, \alpha) l(\omega)|,\ m^2 = |d(j\omega, \alpha)|, \\ \theta^1 = \arg n_0(j\omega, \alpha) + \arg l(j\omega),\ \theta^2 = \arg d_0(j\omega, \alpha),\ \alpha \in \mathrm{A},\ l \in \mathrm{L} \}. \end{aligned} \tag{2.15}$$

Then (2.14) can be rewritten in the equivalent form

$$\max_{\omega \in [0, \omega_s]} \phi_s(x, \omega) \leqslant 0 \tag{2.16a}$$

with

$$\phi_s(x, \omega) \triangleq \max_{\eta \in S_s(\omega)} \xi_s(x, \omega, \eta), \tag{2.16b}$$

where $\eta \triangleq (m^1, m^2, \theta^1, \theta^2)$ and

$$\xi_s(x, \omega, \eta) \triangleq \sum_{i=1}^{2} m^i r^i \sin(\theta^i + \beta^i) - k_1 \left[\sum_{i=1}^{2} m^i r^i \sin(\theta^i + \beta^i) \right]^2 + k_2 . \tag{2.16c}$$

We are now ready to proceed with the development of computationally more tractable design specification inequalities.

3. COMPLEXITY REDUCTION VIA MAJORIZATION

Our technique for the development of computationally efficient replacements for performance specification inequalities, such as (2.11a) and (2.16a) is based on two simple observations. The first one is obvious:

Proposition 3.1. Let Ω be a compact subset of $\mathbb{R}$ and let $\phi, \phi^{\sim} : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}$ be piecewise continuous functions such that for every $x \in \mathbb{R}^{n_x}$

$$\phi(x, \omega) \leqslant \phi^{\sim}(x, \omega), \ \forall \ \omega \in \Omega. \tag{3.1}$$

If

$$F \triangleq \{x \in \mathbb{R}^{n_x} \mid \sup_{\omega \in \Omega} \phi(x, \omega) \leqslant 0\}, \quad F^{\sim} \triangleq \{x \in \mathbb{R}^{n_x} \mid \sup_{\omega \in \Omega} \phi^{\sim}(x, \omega) \leqslant 0\}. \tag{3.2}$$

then $F^{\sim} \subset F$.

■

Definiton 3.1. Whenever (3.1) holds, we shall say that the (function $\phi^{\sim}$) inequality $\sup_{\omega \in \Omega} \phi^{\sim}(x, \omega) \leqslant 0$, *majorizes* the (function ϕ) inequality $\sup_{\omega \in \Omega} \phi(x, \omega) \leqslant 0$.

■

Now suppose that we wish to solve a problem P, of the form $\min\{f(x) \mid x \in F\}$, with F as in (3.2), and that $\phi(x, \omega)$ is very difficult to evaluate. Then we may elect to solve the more conservative problem $P^{\sim}$, $\min\{f(x) \mid x \in F^{\sim}\}$, with $F^{\sim}$ as in (3.2), provided $\phi^{\sim}, \phi$ satisfy (3.1a) and $\phi^{\sim}(x, \omega)$ in much easier to evaluate then $\phi(x, \omega)$. Clearly, any solution $x^{\sim *}$ of $P^{\sim}$ is feasible for P and, if $F^{\sim}$ is not much smaller than F, then $f(x^{\sim *})$ may be close to the optimal value of P.

We shall now show in two steps that we can construct simple *majorizing* functions $\phi^{\sim}_{vr}$ and $\phi^{\sim}_{s}$, to be used as replacements for ϕ_{vr} and ϕ_s, defined in (2.11b), (2.16b) respectively. We begin with ϕ_{vr}. Referring to (2.9), for any $\omega \geqslant 0$, let

$$\underline{m}(\omega) \triangleq \min\{m \mid (m, \theta) \in S(\omega)\}, \quad \overline{m}(\omega) \triangleq \sup\{m \mid (m, \theta) \in S(\omega)\}, \tag{3.3a}$$

$$\underline{\theta}(\omega) \triangleq \min\{\theta \mid (m, \theta) \in S(\omega)\}, \overline{\theta}(\omega) \triangleq \max\{\theta \mid (m, \theta) \in S(\omega)\}. \tag{3.3b}$$

Next, for any $\omega \geqslant 0$, let

$$\mathbf{R}_0(\omega) \triangleq \{(m,\theta) \in \mathbf{R}^2 \,|\, \underline{m}(\theta) \leqslant m \leqslant \overline{m}(\omega), \underline{\theta}(\omega) \leqslant \theta \leqslant \overline{\theta}(\omega)\} . \tag{3.4}$$

Then $S(\omega) \subset \mathbf{R}_0(\omega)$ for all $\omega \geqslant 0$, see Fig. 5, and hence, with $\eta = (m,\theta)$, the function $\overline{\phi} : \mathbf{R}^{n_x} \times \mathbf{R}_+ \to \mathbf{R}$, defined by

$$\overline{\phi}_{vr}(x,\omega) \triangleq \max_{\eta \in \mathbf{R}_0(\omega)} \xi_{vr}(x,\omega,\eta), \tag{3.5}$$

satisfies $\overline{\phi}_{vr}(x,\omega) \geqslant \phi_{vr}(x,\omega)$ for all $x \in \mathbf{R}^{n_x}$ and $\omega \geqslant 0$. Next we note that the max in (3.5) is easy to evaluate because $\mathbf{R}_0(\omega)$ is a rectangle. As a result, the evaluation of $\overline{\phi}_{vr}(x,\omega)$ involves only the evaluation of $\xi_{vr}(x,\omega,\eta)$ at the four vertices η_i of $\mathbf{R}_0(\omega)$ and of the finitely many zeros of its reduced gradient in $\mathbf{R}_0(\omega)$. Closed form expressions for the zeros of the reduced gradient can be found in [Pol.2, Sti.1]; hence their evaluation is simple. Furthermore, it seems that the difference between $\overline{\phi}_{vr}(x,\omega)$ and $\phi_{vr}(x,\omega)$ need not be very large (see Fig. 5). Hence $\overline{\phi}_{vr}$ is a very attractive candidate majorizing function for ϕ_{vr}, provided the construction of the sect $\mathbf{R}_0(\omega)$ is not expensive. Fortunately, the product form (2.2b) makes the computation of the bounds $\underline{m}(\omega)$, $\overline{m}(\omega)$, $\underline{\theta}(\omega)$ and $\overline{\theta}(\omega)$ quite easy because mathematical programming decomposition theory (see e.g. [Las.1]) can be used to obtain the following nice result.

Theorem 3.1. For all $\omega \geqslant 0$ such that $|P(j\omega,\alpha,l)|$ is finite for all $\alpha \in \mathbf{A}, l \in \mathbf{L}$,

$$\underline{m}(\omega) \triangleq \min_{\substack{\alpha \in \mathbf{A} \\ l \in \mathbf{L}}} |P(j\omega,\alpha,l)|$$

$$= \frac{\underline{l}_M(\omega)\underline{K}_p \prod_{i=1}^{M} \min_{z^i \in [\underline{z}^i, \overline{z}^i]} |j\omega + z^i| \prod_{i=M+1}^{M_p} \min_{z^i \in [\underline{z}^i, \overline{z}^i]} |j\omega + z^i|^2}{\prod_{i=1}^{N} \max_{p^i \in [\underline{p}^i, \overline{p}^i]} |j\omega + p^i| \prod_{i=N+1}^{N_p} \max_{p^i \in [\underline{p}^i, \overline{p}^i]} |j\omega + p^i|^2} . \tag{3.6a}$$

$$\overline{m}(\omega) \triangleq \max_{\substack{\alpha \in \mathbf{A} \\ l \in \mathbf{L}}} |P(j\omega,\alpha,l)|$$

$$= \frac{\bar{l}_M(\omega)\bar{K}_p \prod_{i=1}^{M} \max_{z^i \in [\underline{z}^i, \bar{z}^i]} |j\omega + z^i| \prod_{i=M+1}^{M_p} \max_{z^i \in [\underline{z}^i, \bar{z}^i]} |j\omega + z^i|^2}{\prod_{i=1}^{N} \min_{p^i \in [\underline{p}^i, \bar{p}^i]} |j\omega + p^i| \prod_{i=N+1}^{N_p} \min_{p^i \in [\underline{p}^i, \bar{p}^i]} |j\omega + p^i|^2}, \qquad (3.6b)$$

$$\underline{\theta}(\omega) \triangleq \min_{\substack{\alpha \in A \\ l \in L}} \arg P(j\omega, \alpha, l)$$

$$= \underline{l}_A(\omega) + \sum_{i=1}^{M} \min_{z^i \in [\underline{z}^i, \bar{z}^i]} \arg(j\omega + z^i) + \sum_{i=M+1}^{M_p} \min_{z^i \in [\underline{z}^i, \bar{z}^i]} \{\arg(j\omega + z^i)$$

$$+ \arg(j\omega + z^{i*})\} - \sum_{i=1}^{N} \max_{p^i \in [\underline{p}^i, \bar{p}^i]} \arg(j\omega + p^i) - \sum_{i=N+1}^{N_p} \max_{p^i \in [\underline{p}^i, \bar{p}^i]} \{\arg(j\omega + p^i)$$

$$+ \arg(j\omega + p^{i*})\} \qquad (3.6c)$$

$$\bar{\theta}(\omega) \triangleq \max_{\substack{\alpha \in A \\ l \in L}} \arg P(j\omega, \alpha, l)$$

$$= \bar{l}_A(\omega) + \sum_{i=1}^{M} \max_{z^i \in [\underline{z}^i, \bar{z}^i]} \arg(j\omega + z^i) + \sum_{i=M+1}^{M_p} \max_{z^i \in (\underline{z}^i, \bar{z}^i]} \{\arg(j\omega + z^i)$$

$$+ \arg(j\omega + z^{i*})\} - \sum_{i=1}^{N} \min_{p^i \in [\underline{p}^i, \bar{p}^i]} \arg(j\omega + p^i) - \sum_{i=N+1}^{N_p} \min_{p^i \in [\underline{p}^i, \bar{p}^i]} \{\arg(j\omega + p^i)$$

$$+ \arg(j\omega + p^{i*})\}. \qquad (3.6d)$$

The one term extremizers in (3.6a - 3.6d) can be easily computed using closed form expressions given in [Pol.2]. For example, for the structured part P_0 in (1.8), with the bounds given for it, we get

$$\bar{m}(\omega) = \begin{cases} 25\bar{l}_M(\omega)(\omega^2 + 16)^{1/2}(l - \omega^2)^{-1}(\omega^2 + 25)^{-1/2}, & \forall\, \omega \in [0,1] \\ \infty, & \forall\, \omega \in (1,2], \\ 25\bar{l}_M(\omega)(\omega^2 + 16)^{1/2}(\omega^2 - 4)^{-1}(\omega^2 + 25)^{-1/2}, & \forall\, \omega \in (2,\infty) \end{cases}, \qquad (3.7a)$$

$$\underline{m}(\omega) = \begin{cases} 12.5\underline{l}_M(\omega)(\omega^2 + 4)^{1/2}[(5 - \omega^2)^2 + 4\omega^2]^{-1/2}(\omega^2 + 100)^{-1/2}, & \forall\, \omega \in [0\ \sqrt{3.5}] \\ 12.5\underline{l}_M(\omega)(\omega^2 + 4)^{1/2}[(2 - \omega^2)^2 + 4\omega^2]^{-1/2}(\omega^2 + 100)^{-1/2}, & \forall\, \omega \in (\sqrt{3.5}, \infty) \end{cases}$$

(3.7b)

$$\overline{\theta}(\omega) = \begin{cases} \overline{l}_A(\omega) + \tan^{-1}\omega/2 + \tan^{-1}\sqrt{(1+\omega)/(1-\omega)} - \\ \qquad \tan^{-1}\sqrt{(1-\omega)/(1+\omega)} - \tan^{-1}\omega/10, \ \omega \in [0, 1], \\ \overline{l}_A(\omega) + \tan^{-1}\omega/2 + \pi - \tan^{-1}\omega/10, \ \omega \in (1, \infty) \end{cases} \tag{3.7c}$$

$$\underline{\theta}(\omega) = \begin{cases} \underline{l}_A(\omega) + \tan^{-1}\omega/4 - \tan^{-1}\omega/5, \ \forall\, \omega \in [0, 2] \\ \underline{l}(\omega) + \tan^{-1}\omega/4 + \tan^{-1}(\omega-2) - \tan^{-1}\omega/5, \ \forall\, \omega \in (2, \infty) \end{cases} \tag{3.7d}$$

We see from this example that the functions $\overline{m}(\cdot)$, $\underline{m}(\cdot)$, $\overline{\theta}(\cdot)$, $\underline{\theta}(\cdot)$ need not turn out to be continuous. In fact, referring to (3.7a - d) we see that for the plant discussed in Example 2.1, $\overline{m}(\cdot)$ has discontinuities at $\omega = 1,2$ and it is infinite for $\omega \in [1,2]$, while $\overline{\theta}(\omega)$ has a jump increase of $\pi/2$ at $\omega = 1$. It is therefore far from obvious whether, for the general case, the function $\overline{\phi}(\cdot,\cdot)$, defined by (3.5), is upper semi-continuous in (x, ω), locally Lipschitz continuous in x, nor what is the formula for its partial generalized gradient, w.r.t.x, if it exists, all of which are required by semi-infinite optimization algorithms. In the remainder of this section we shall show that simple transformations lead from the functions $\overline{\phi}_{vr}$ and $\overline{\phi}_s$ to functions $\tilde{\phi}_{vr}$ and $\tilde{\phi}_{vs}$ which have all of the required properties. First we introduce a change of variable. Thus, for all $m \geqslant 0$ let

$$\mu \triangleq \frac{m}{1+m}. \tag{3.8}$$

Next, with $\nu \triangleq (\mu, \theta)$, let $\tilde{\xi}_{vr} : \mathbf{R}^{n_x} \times \mathbf{R}_+ \times [0,1] \to \mathbf{R}$ be defined by

$$\tilde{\xi}_{vr}(x, \omega, \nu) \triangleq \xi_{vr}(x, \omega, (\mu/(1-\mu), \theta), \tag{3.9}$$

and let

$$\underline{\mu}(\omega) = \frac{\underline{m}(\omega)}{1+\underline{m}(\omega)}, \quad \overline{\mu}(\omega) = \frac{\overline{m}(\omega)}{1+\overline{m}(\omega)}. \tag{3.10}$$

Theorem 3.2. (a) Let (x, ω, ν) be such that $x \in \mathbf{R}^{n_x}$, $\omega \geqslant 0$, $\nu \in [0, 1] \times \mathbf{R}$ and $\tilde{\xi}_{vr}(x, \omega, \nu) < \infty$ Then $\tilde{\xi}_{vr}(\cdot,\cdot,\cdot)$ is continuous at (x, ω, ν) (with respect to $\mathbf{R}^{n_x} \times \mathbf{R}_+ \times [0, 1] \times \mathbf{R}$). (b) The bound functions $\underline{\mu}$, $\overline{\mu} : \mathbf{R}_+ \to [0, 1]$ are continuous. (c) The bound functions $\underline{\theta}$, $\overline{\theta} : \mathbf{R}_+ \to \mathbf{R}$ (defined in (3.3b)) are piecewise continuous, with bounded discontinuities occuring in the set Ω_A defined by

$$\Omega_4 \triangleq \{\omega \in \mathbf{R}_+ | \omega = \underline{z}^i \text{ and } \mathrm{Re}\, \underline{z}^i = 0, \text{ or } \omega = \bar{z}^i$$
$$\text{and } \mathrm{Re}\, \bar{z}^i = 0,\ i = 1, \ldots, M_p, \text{ or } \omega = \underline{p}^i \text{ and } \mathrm{Re}\, \underline{p}^i = 0, \text{ or } \omega = \bar{p}^i$$
$$\text{and } \mathrm{Re}\, \bar{p}^i = 0,\ i = 1, \ldots, N_p\}. \tag{3.11}$$

Proof: a) From (2.11c),

$$\tilde{\xi}_{vr}(x, \omega, \nu) = m_F(x, \omega)_2 m_c(x, \omega)^2 \left[(\mu/(1-\mu))^2 m_c(x, \omega)^2\right.$$
$$\left. + ((2\mu/(1-\mu)) m_C(x, \omega) \cos(\theta + \theta_C(x, \omega)) + 1\right]^{-1} - b_{vr}(\omega)$$
$$= m_F(x, \omega)^2 m_C(x, \omega)^2 (1-\mu)^2 [\mu^2 m_C(x, \omega)^2$$
$$+ 2\mu(1-\mu) m_C(x, \omega) \cos(\theta + \theta_C(x, \omega)) + (1-\mu)^2]^{-1} - b_{vr}(\omega). \tag{3.12}$$

Hence the continuity of $\tilde{\xi}_{vr}(\cdot, \cdot, \cdot)$ follows by inspection.

b) Referring to (2.2b), for $\omega \geqslant 0$, let

$$\bar{n}(\omega) \triangleq \max_{\alpha \in A} |n_0(j\omega, \alpha)| \bar{l}_M(\omega), \quad \underline{n}(\omega) \triangleq \min_{\alpha \in A} |n_0(j\omega, \alpha)| \underline{l}_M(\omega). \tag{3.13a}$$

$$\bar{d}(\omega) \triangleq \max_{\alpha \in A} |d_0(j\omega, \alpha)|, \quad \underline{d}(\omega) \triangleq \min_{\alpha \in A} |d_0(j\omega, \alpha)|. \tag{3.13b}$$

Then $\bar{n}(\cdot), \underline{n}(\cdot), \bar{d}(\cdot), \underline{d}(\cdot)$ are continuous by the maximum theorem in [Ber.1] and $\bar{m} = \bar{n} / \underline{d}$, $\underline{m} = \underline{n} / \bar{d}$. Hence

$$\underline{\mu}(\omega) = \frac{\underline{m}(\omega)}{1 + \underline{m}(\omega)} = \frac{\underline{n}(\omega)}{\underline{n}(\omega) + \bar{d}(\omega)}, \quad \bar{\mu}(\omega) = \frac{\bar{m}(\omega)}{1 + \bar{m}(\omega)} = \frac{\bar{n}(\omega)}{\bar{n}(\omega) + \underline{d}(\omega)}. \tag{3.13c}$$

and the continuity of $\underline{\mu}, \bar{\mu}$ follows from the continuity of $\underline{n}, \bar{n}, \underline{d}, \bar{d}$ and the fact that $\bar{n}(\omega) + \underline{d}(\omega) > 0$ and $\underline{n}(\omega) + \bar{d}(\omega) > 0$ for all $\omega > 0$ because of Assumption 2.1.

c) It follows directly from the Maximum Theorem [Ber.1] and (3.6c), (3.6d) that $\bar{\theta}, \underline{\theta}$ must be piecewise continuous, with bounded discontinuities which can occur only in the set Ω_4.

■

Example 3.1. We now illustrate what can happen as a result of discontinuities, such as those in the phase bounds $\underline{\theta}(\cdot), \bar{\theta}(\cdot)$. Let $\xi(x, \omega, \eta) \triangleq e^{-\eta x^2}$, with $x, \eta \in \mathbf{R}$, and let

$$Q(\omega) \triangleq \begin{cases} [1-\omega,\, 1], & \text{for } \omega \in [0,\, 1) \\ [1,\, \omega+1], & \text{for } \omega \geqslant 1 \end{cases} \tag{3.14a}$$

Clearly $\xi(\cdot,\cdot,\cdot)$ is continuous, $Q(\omega)$ is compact for each ω and it is piecewise continuous, with a single discontinuity at $\omega = 1$. However, it is not upper semi-continuous. Let $\phi(x,\omega) \triangleq \max_{\eta \in Q(\omega)} \xi(x,\omega,\eta)$. Then it is easy to see that

$$\phi(x,\omega) = \begin{cases} e^{-(1-\omega)x^2} & \text{for } \omega \in [0,\, 1) \\ e^{-x^2} & \text{for } \omega \geqslant 1 \end{cases} \tag{3.14b}$$

and hence that $\phi(\cdot,\cdot)$ is lower semi-continuous. Hence there is no $\hat{\omega} \in [0,\, 2]$ such that for $x \neq 0$.

$$1 = \psi(x) \triangleq \sup_{\substack{\omega \in [0,2] \\ \eta \in Q(\omega)}} \xi(x,\omega,\eta) = \phi(x,\hat{\omega}) = \sup_{\omega \in [0,2]} \phi(x,\omega). \tag{3.14c}$$

However, if we redefine $Q(\omega)$ at $\omega = 1$ to be the union of its limits as $\omega \to 1$, i.e., if we set $Q(1) = [0,2]$, then $Q(\cdot)$ becomes upper semi-continuous. Consequently $\phi(\cdot,\cdot)$ becomes upper semi-continuous, and the sup in (3.14b) is attained at $\hat{\omega} = 1$, while the value of $\psi(x)$ remains the same for all x.

■

Since in the general case the set value map

$$Q(\omega) \triangleq \{(\mu,\theta) \in \mathbf{R}^2 \mid \underline{\mu}(\omega) \leqslant \mu \leqslant \overline{\mu}(\omega),\ \underline{\theta}(\omega) \leqslant \theta \leqslant \overline{\theta}(\omega)\}\,. \tag{3.15}$$

is piecewise continuous, but not upper semi-continuous, the function $\xi(x,\omega) = \max_{\eta \in Q(\omega)} \xi_{vr}(x,\omega,\eta)$ is not upper semi-continuous. Hence, as was done in Example 3.1, we will redefine $Q(\omega)$ at its discontinuity points so as to generate a new upper semi-continuous set valued map, to be used in place of $Q(\omega)$. We therefore define the functions $\tilde{\theta},\, \underline{\theta} : \mathbf{R}_+ \to \mathbf{R}$ by

$$\tilde{\theta}(\omega) \triangleq \overline{\lim_{\omega' \to \omega}}\, \overline{\theta}(\omega),\quad \underline{\theta} \triangleq \underline{\lim_{\omega' \to \omega}}\, \underline{\theta}(\omega), \tag{3.16}$$

and the set valued map $\mathbf{R} : \mathbf{R}_+ \to 2^{\mathbf{R}^2}$ by

$$\mathrm{R}(\omega) \triangleq \{(\mu,\theta) \in \mathrm{R}^2 \mid \underline{\mu}(\omega) \leqslant \mu \leqslant \overline{\mu}(\omega),\ \underline{\theta}(\omega) \leqslant \theta \leqslant \tilde{\theta}(\omega)\}\,. \tag{3.17a}$$

Finally, with $\eta \triangleq (\mu,\theta)$, we define $\tilde{\phi} : \mathrm{R}^{n_x} \times \mathrm{R} \to \mathrm{R}$ by

$$\tilde{\phi}(x,\omega) \triangleq \max_{\eta \in \mathrm{R}(\omega)} \tilde{\xi}_{vr}(x,\omega,\eta)\,. \tag{3.17b}$$

The following result follows directly from the continuity of the bounds $\underline{\mu}(\cdot)$, $\overline{\mu}(\cdot)$ and the piecewise continuity of the bounds $\underline{\theta}(\cdot)$, $\overline{\theta}(\cdot)$.

Proposition 3.2. (a) The set valued map $\mathrm{R}(\cdot)$ is upper semi-continuous in the sense of Berge [Ber.1]; (b) $\mathrm{R}(\omega) \supset \mathrm{Q}(\omega)$ for all $\omega \geqslant 0$ and $\mathrm{R}(\omega) = \mathrm{Q}(\omega)$ for all $\omega \geqslant 0$, $\omega \notin \Omega_4$.

■

Hence we are lead to the following result.

Corollary 3.1. Let $\mathrm{X} \subset \mathrm{R}^{n_x}$ be defined by

$$\mathrm{X} \triangleq \{x \in \mathrm{R}^{n_x} \mid \tilde{\xi}_{vr}(x,\omega,\eta) < \infty,\ \forall\ \omega \in \mathrm{R}_+,\ \forall\ \eta \in \mathrm{R}(\omega)\}\,. \tag{3.18}$$

Then (a) $\tilde{\phi}(\cdot,\cdot)$ is upper semi-continuous and $\tilde{\phi}(\cdot,\omega)$ is locally Lipschitz continuous on $\mathrm{X} \times \mathrm{R}_+$, and (b) for $x \in \mathrm{X}$

$$\tilde{\psi}_{vr}(x) \triangleq \sup_{\substack{\omega \in [\omega',\omega''] \\ \eta \in \mathrm{R}_0(\omega)}} \xi_{vr}(x,\omega,\eta) = \max_{\omega \in [\omega',\omega'']} \tilde{\phi}_{vr}(x,\omega). \tag{3.19}$$

Proof: 0 (a) Since $\mathrm{R}(\cdot)$ is upper semi-continuous, it follows from the maximum theorem [Ber.1] that $\tilde{\phi}_{vr}(\cdot,\cdot)$ is upper semi-continuous. Since $\tilde{\xi}$ is differentiable in x and since the max in (3.17b) does not involve x, it follows that $\tilde{\phi}_{vr}(\cdot,\omega)$ is locally Lipschitz continuous. (b) Since $\tilde{\phi}_{vr}(\cdot,\cdot)$ is upper semi-continuous, the maximum in the right hand side of (3.19) is achieved. Finally, it follows directly from the definition of the set valued map $\mathrm{R}(\omega)$ that, with $\mathrm{Q}(\cdot)$ as in (3.15),

$$\sup_{\substack{\omega \in [\omega',\omega''] \\ \eta \in \mathrm{R}_0(\omega)}} \xi_{vr}(x,\omega,\eta) = \sup_{\substack{\omega \in [\omega',\omega''] \\ \eta \in \mathrm{Q}(\omega)}} \tilde{\xi}_{vr}(x,\omega,\eta)$$

$$= \max_{\substack{\omega \in [\omega',\omega''] \\ \eta \in \mathrm{R}(\omega)}} \tilde{\xi}_{vr}(x,\omega,\eta) = \max_{\omega \in [\omega',\omega'']} \tilde{\phi}_{vr}(x,\omega)\,. \tag{3.20}$$

■

Thus we have finally succeeded in constructing a majorizing function for $\phi_{vr}(\cdot, \cdot)$, which meets both the requirement of easy evaluation and the requirements imposed by semi-infinite optimization algorithms. We shall derive the required formula for the generalized gradient of $\widetilde{\psi}_{vr}(\cdot)$ in the next section.

Next we turn to the stability constraint (2.16a)-(2.16c). Let

$$\overline{m}^1(\omega) \triangleq \max_{\alpha \in A} |n_0(j\omega, \alpha)| \overline{l}_M(\omega), \quad \underline{m}^1(\omega) \triangleq \min_{\alpha \in A} |n_0(j\omega, \alpha)| \underline{l}_M(\omega), \tag{3.21a}$$

$$\overline{m}^2(\omega) \triangleq \max_{\alpha \in A} |d_0(j\omega, \alpha)|, \quad \underline{m}^2(\omega) \triangleq \min_{\alpha \in A} |d_0(j\omega, \alpha)|, \tag{3.21b}$$

$$\overline{\theta}^1(\omega) \triangleq \max_{\alpha \in A} \arg n_0(j\omega, \alpha) + \overline{l}_A(\omega), \quad \underline{\theta}^1(\omega) \triangleq \min_{\alpha \in A} \arg n_0(j\omega, \alpha) + \underline{l}_A(\omega), \tag{3.21c}$$

$$\overline{\theta}^2(\omega) \triangleq \max_{\alpha \in A} \arg d_0(j\omega, \alpha), \quad \underline{\theta}^2(\omega) \triangleq \min_{\alpha \in A} \arg d_0(j\omega, \alpha). \tag{3.21d}$$

Clearly, making use of a result analogous to Theorem 3.1, one can compute the above quantities quite easily. Next, let

$$\begin{aligned} \mathbf{R}_{s0}(\omega) \triangleq \{m^1, m^2, \theta^1, \theta^2) \in \mathbf{R}^4 | \underline{m}^i(\omega) \leqslant m^i \leqslant \overline{m}^i(\omega), \\ \underline{\theta}^i(\omega) \leqslant \theta^i \leqslant \overline{\theta}^i(\omega), i = 1, 2)\} \end{aligned} \tag{3.22}$$

Then $\mathbf{R}_s(\omega) \supset \mathbf{S}_s(\omega)$, where $\mathbf{S}_s(\omega)$ was defined in (2.15), and hence (see (2.16c)) the function

$$\overline{\phi}_s(x, \omega) \triangleq \max_{\eta \in \mathbf{R}_{s0}(\omega)} \xi_s(x, \omega, \eta). \tag{3.23}$$

majorizes the function $\phi_s(x, \omega)$ defined by (2.16b).

The following result from [Pol.2, Sti.1] shows that the evaluation of $\overline{\phi}_s(x, \omega)$ reduces to the solution of four simple maximization problems in $\mathbf{R}_2$.

Theorem 3.3. Let $\overline{\phi}_s(x, \omega)$ be defined as in (3.23). Then

$$\overline{\phi}_s(x, \omega) = \max_{\substack{\theta^1 \in [\underline{\theta}^1(\omega), \overline{\theta}^1(\omega)] \\ \theta^2 \in [\underline{\theta}^2(\omega), \overline{\theta}^2(\omega)]}} \max\{$$

$$\xi_s(x, \omega, \underline{m}^1(\omega), \underline{m}^2(\omega), \theta^1, \theta^2), \xi_s(x, \omega, \underline{m}^1(\omega), \bar{m}^2(\omega), \theta^1, \theta^2),$$

$$\xi_s(x, \omega, \bar{m}^1(\omega), \underline{m}^2(\omega), \theta^1, \theta^2), \xi_s(x, \omega, \bar{m}^1(\omega), \bar{m}^2(\omega), \theta^1, \theta^2)\} . \tag{3.24}$$

■

We note again that because $[\underline{\theta}^1(\omega), \bar{\theta}^1(\omega)] \times [\underline{\theta}^2(\omega), \bar{\theta}^2(\omega)]$ is a rectangle in $\mathbb{R}^2$, each maximation in (3.24) is easy to perform. In particular, it was shown in [Pol.2, Sti.1] that one only need to consider the vertices of the rectangles $[\underline{\theta}^i(\omega), \bar{\theta}^i(\omega)]$, $i = 1, 2$, and the feasible zeros of $\nabla_{\theta^1, \theta^2} \xi_s(x, \omega, m^1, m^2, \cdot, \cdot)$.

Referring to (3.21a-d), we see that the magnitude bounds $\underline{m}^1, \bar{m}^1, \underline{m}^2, \bar{m}^2$ are continuous by the Maximum Theorem in [Ber.1]. However, the phase bounds may be discontinuous in the set Ω_4 and hence $\bar{\phi}(\cdot, \cdot)$ need not be upper semi-continuous, as required by semi-infinite optimization theory.

We rectify this situation just as we did for the saturation constraint, by augmenting the set $\mathbf{R}_{s0}(\omega)$ at the discontinuity points, as follows. For any $\omega \geqslant 0$, let

$$\underset{\sim}{\theta}^i(\omega) \triangleq \varliminf_{\omega' \to \omega} \underline{\theta}^i(\omega'), i = 1, 2, \quad \tilde{\theta}^i(\omega) \triangleq \varlimsup_{\omega' \to \omega} \bar{\theta}^i(\omega'), i = 1, 2, \tag{3.25}$$

and let

$$\mathbf{R}_s(\omega) \triangleq \{(m^1, m^2, \theta^1, \theta^2) \in \mathbf{R}^4 \mid \underline{m}^i(\omega) \leqslant m^i \leqslant \bar{m}(\omega), \ \underset{\sim}{\theta}^i(\omega) \leqslant \theta^i \leqslant \tilde{\theta}^i(\omega), i = 1,2\} \tag{3.2.6}$$

The the following result is easily established.

Proposition 3.3. (a) For all $\omega \geqslant 0$, $\mathbf{R}_s(\omega) \supset \mathbf{R}_{s0}(\omega)$; (b) $\mathbf{R}_s(\omega) = \mathbf{R}_{s0}(\omega)$ for all $\omega \geqslant 0, \omega \notin \Omega_4$; (c) $\mathbf{R}_s(\cdot)$ is upper semi-continuous in the sense of Berge [Ber.1].

■

Corollary 3.2. 0 Let $\tilde{\phi}_s : \mathbf{R}^{n_x} \times \mathbf{R}_+ \to \mathbf{R}$ be defined by

$$\tilde{\phi}_s(x, \omega) \triangleq \max_{\eta \in \mathbf{R}_s(\omega)} \xi_s(x, \omega, \eta) \tag{3.27}$$

with $\eta \triangleq (m^1, m^2, \theta^1, \theta^2)$. Then, (a) $\tilde{\phi}_s(\cdot, \cdot)$ is upper semi-continuous, $\tilde{\phi}_s(\cdot, \omega)$ is locally Lipschitz continuous, and (b)

$$\widetilde{\psi}_s(x) \triangleq \sup_{\substack{\omega \in [0,\omega_s] \\ \eta \in \mathbf{R}_{s0}(\omega)}} \xi_s(x,\omega,\eta) = \max_{\omega \in [0,\omega_s]} \widetilde{\phi}_s(x,\omega). \tag{3.28}$$

■

Thus $\widetilde{\phi}_s(\cdot,\cdot)$ is a satisfactory majorizing function for the function $\phi_s(\cdot,\cdot)$ defined in (2.16b). We shall derive a formula for the generalized gradient of $\widetilde{\psi}_s(\cdot)$ in the next section.

4. LIPSCHITZ CONTINUITY AND GENERALIZED GRADIENTS

To complete our demonstration that the inequalities $\widetilde{\psi}_{vr}(x) \leqslant 0$ and $\widetilde{\psi}_s(x) \leqslant 0$ are efficient, majorizing substitutes for the inequalities (2.7) and (2.14), respectively, we must show that the functions $\widetilde{\psi}_{vr}(\cdot)$ and $\widetilde{\psi}_s(\cdot)$ are locally Lipschitz continuous and we must obtain simple formulas for their generalized gradients. Both of these required results follow from the general theorem below, which extends a theorem by F. Clarke [Cla.2].

Theorem 4.1. Let $\Omega \subset \mathbf{R}^m$ be compact and suppose that $\xi : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$ and $\mathbf{R} : \mathbf{R}^m \to 2^{\mathbf{R}^p}$ have the following properties:

(a) $\xi(\cdot,\cdot,\cdot)$ is upper semi-continuous;

(b) $\xi(\cdot,\omega,\eta)$ is locally Lipschitz continuous, uniformly for (ω,η) in compact sets;

(c) the directional derivative of $\xi(\cdot,\omega,\eta)$, at x, in the direction h, $d_x\xi(x,\omega,\eta;h)$, satisfies $d_x\xi(x,\omega,\eta;h) = \max\{\langle y,\eta\rangle \mid y \in \partial_x\xi(x,\omega,\eta)\}$, where $\partial_x\xi(x,\omega,\eta)$ is the generalized gradient of $\xi(\cdot,\omega,\eta)$ at x.

(d) The partial generalized gradient $\partial_x\xi(\cdot,\cdot,\cdot)$ is upper semi-continuous;

(e) The set valued map $\mathbf{R}(\cdot)$ is upper semi-continuous.

If we define

$$\psi(x) \triangleq \max_{\substack{\omega \in \Omega \\ \eta \in \mathbf{R}(\omega)}} \xi(x,\omega,\eta). \tag{4.1}$$

then,

(i) $\psi(\cdot)$ is locally Lipschitz continuous;

(ii) The generalized gradient of $\psi(\cdot)$, is given by

$$\partial\psi(x) = \text{co}\{\partial_x\xi(x, \omega, \eta) \mid (\omega, \eta) \in \text{M}(x)\}, \tag{4.2}$$

where co denotes the convex well of the set in braces and

$$\text{M}(x) \triangleq \{(\omega, \eta) \in \Omega\times\text{R}(\omega) \mid \xi(x, \omega, \eta) = \psi(x)\}. \tag{4.3}$$

Proof: First we must show that $\psi(x)$ is well defined. Let $\bar{x} \in \text{R}^{n_x}$ ber arbitrary. Since $\text{R}(\cdot)$ is upper semi-continuous and Ω is compact and $\xi(\cdot,\cdot,\cdot)$ is upper semi-continuous, $\psi(\bar{x}) = \sup_{\substack{\omega\in\Omega \\ \eta\in\text{R}(\omega)}} \xi(\bar{x}, \omega, \eta)$ is well defined and hence there exist sequences $\{\omega_i\} \subset \Omega$, $\{\eta_i\}$, with $\eta_i \in \text{R}(\omega_i)$, such that $\xi(\bar{x}, \omega_i, \eta_i) \geqslant \psi(\bar{x}) - 1/i$, $i = 1, 2, \ldots$ Since Ω is compact and $\text{R}(\cdot)$ is upper semi-continuous, there exist subsequences $\{\omega_i\}_{i\in K}$, $\{\eta_i\}_{i\in K}$, $K \subset \text{N}_+$ such that $\omega_i \to \bar{\omega} \in \Omega$ and $\eta_i \xrightarrow{K} \bar{\eta} \in \text{R}(\bar{\omega})$. Therefore, since $\xi(\bar{x},\cdot,\cdot)$ is continuous,

$$\xi(\bar{x}, \bar{\omega}, \bar{\eta}) \geqslant \overline{\lim_{i\in K}}\, \xi(\bar{x}, \omega_i, \eta_i) \geqslant \psi(\bar{x}), \tag{4.4}$$

which shows that the maximum in (4.1) is achieved, i.e. that $\psi(x)$ is well defined for all $x \in \text{R}^n$.

Next, we show that there exists a compact set $\text{B} \subset \text{R}^p$ such that $\text{R}(\omega) \subset \text{B}$ for all $\omega \in \Omega$ i.e. $\text{R}(\Omega) \subset \text{B}$. For suppose that $\text{R}(\Omega)$ is unbounded. Then there exist sequences $\{\omega_i\} \subset \Omega$ and $\{\eta_i\}$, with $\eta_i \in \text{R}(\omega_i)$ such that $\|\eta_i\| \geqslant i$, $i = 1, 2, \ldots$. However, since Ω is compact, there exists a subsequence $\{\omega_i\}_{i\in K}$, $K \subset \text{N}_+$, such that $\omega_i \to \hat{\omega} \in \Omega$ and since $\text{R}(\cdot)$ is upper semi-continuous, $\text{R}(\hat{\omega})$ is compact and, given any bounded open set $\text{O} \supset \text{R}(\hat{\omega})$, there exists an i_0 such that $\text{R}(\omega_i) \subset \text{O}$ for all $i \geqslant i_0$, $i \in K$. But this contradicts the assumption that $\|\eta_i\| \geqslant i$ for all $i \in \text{N}_+$. We conclude that $\text{R}(\Omega)$ is bounded and hence that there exists a compact set B containing $\text{R}(\Omega)$.

Since $\Omega\times\text{R}(\Omega) \subset \Omega\times\text{B}$ and since local Lipschitz continuity of $\xi(\cdot, \omega, \eta)$ is uniform for $(\omega, \eta) \in \Omega\times\text{B}$, given any bounded set $\text{S} \subset \text{R}^n$ there exists a Lipschitz constant K for $\xi(\cdot, \omega, \eta)$, for $x \in \text{S}$ and $(\omega, \eta) \in \Omega\times\text{B}$. Let $x_1, x_2 \in \text{S}$ and $(\omega_1, \eta_1) \in \text{M}(x_1)$. Then

$$\psi(x_1) = \xi(x_1, \omega_1, \eta_1) \leqslant \xi(x_2, \omega_1, \eta_1) + K\|x_1 - x_2\|$$

$$\psi(x_1) = \xi(x_1, \omega_1, \eta_1) \leqslant \xi(x_2, \omega_1, \eta_1) + K\|x_1 - x_2\|$$
$$\leqslant \psi(x_2) + K\|x_1 - x_2\| \tag{4.5}$$

Since the relation (4.5) is symmetric in x_1, x_2, (i) is established.

Next, by (c), for any $(\omega, \eta) \in M(x)$ and any $h \in R^n$,

$$\max\{\langle y, h\rangle \mid y \in \partial_x\xi(x, \omega, \eta)\} =$$
$$d_x\xi(x, \omega, \eta; h) = \lim_{t \downarrow 0} \frac{1}{t}[\xi(x + th, \omega, \eta) - \xi(x, \omega, \eta]$$
$$\leqslant \overline{\lim_{t \downarrow 0}} \frac{1}{t}[\psi(x + th) - \psi(x)]$$
$$\leqslant \max\{\langle y, h\rangle \mid h \in \partial\psi(x)\}, \tag{4.6}$$

by the definition of the generalized directional derivative [Cla.1]. Making use of a well known property of support functions [Roc.1], we conclude that

$$\mathrm{co}\{\partial_x\xi(x, \omega, \eta)\}_{(\omega,\eta)\in M(x)} \subset \partial\psi(x). \tag{4.7}$$

Next we show that $M(\cdot)$ is upper semi-continuous. Since $M(x) \subset \Omega \times B$ for all $x \in R^n$, it is unformly bounded. Now if $x_i \to x$ and $(\omega_i, \eta_i) \in M(x_i)$, with $(\omega_i, \eta_i) \to (\omega, \eta)$, we have, because $\psi(\cdot)$ is continuous and $\xi(\cdot, \cdot, \cdot)$ is upper semi-continuous that

$$\psi(x) = \lim \psi(x_i) = \lim \xi(x_i, \omega_i, \eta_i) \leqslant \xi(x, \omega, \eta). \tag{4.8}$$

Since Ω is compact and $R(\cdot)$ is upper semi-continuous, $\omega \in \Omega$ and $\eta \in R(\omega)$, which leads to the conclusion that $\psi(x) = \xi(x, \omega, \eta)$, i.e., that $(\omega, \eta) \in M(x)$, and hence that $M(\cdot)$ is upper semi-continuous. Now, by [Cla.1],

$$\partial\psi(x) = \mathrm{co}\{\lim_{n_i \to x} \nabla\psi(x_i)\}, \tag{4.9}$$

with $\{x_i\} \subset R^n$ arbitrary, but such that $x_i \to x$, and $\nabla\psi(x_i)$ and $\lim_{n_i \to x} \nabla\psi(x_i)$ exist. By (4.7) for any x_i such that $\nabla\psi(x_i)$ exists $\partial_x\xi(x_i, \omega_i, \eta_i) = \{\nabla\psi(x_i)\}$, for any $(\omega_i, \eta_i) \in M(x_i)$. Hence, since $M(\cdot)$ is upper semi-continuous, from above, and $\partial_x\xi(\cdot, \cdot, \cdot)$ is upper semi-continuous by (d), it follows that (4.2) holds.

■

Referring to (2.16c) is obvious that $\xi_s(\cdot,\cdot,\cdot)$ is continuous and its gradient $\nabla_x \xi_s(\cdot,\cdot,\cdot)$ exists and is continuous, on $\mathbf{R}^{n_x} \times \mathbf{R}_+ \times \mathbf{R}^4$, while from (3.12) it follows that $\tilde{\xi}_{vr}(\cdot,\cdot,\cdot)$ is continuous and its gradient $\nabla_x \tilde{\xi}_{vr}(\,,\,,\,)$ exists and is continuous on $X \times \mathbf{R}_+ \times \mathbf{R}^2$, where X was defined in (3.28). Hence we get the following obvious

Corollary 4.1. Let $\tilde{\psi}_{vr} : \mathbf{R}^{n_x} \rightarrow \mathbf{R}$ and $\tilde{\psi}_s : \mathbf{R}^{n_x} \rightarrow \mathbf{R}$ be defined as in (3.19), (3.28) respectively. Then $\tilde{\psi}_{vr}(\cdot)$ and $\tilde{\psi}_s(\cdot)$ are both locally Lipschitz continuous and their generalized gradients are given by

$$\partial \tilde{\psi}_{vr}(x) = \mathrm{co}\{\nabla_x \tilde{\xi}_{vr}(\omega,\omega,\eta)\}_{(\omega,\eta)\in \mathbf{M}_{vr}(x)}, \tag{4.10a}$$

where

$$\mathbf{M}_{vr}(\omega) \triangleq \{(\omega,\eta) \in \mathbf{R}_+ \times \mathbf{R}^2 | \omega \in [\omega',\omega], \eta \in \mathbf{R}(\omega), \tilde{\xi}_{vr}(x,\omega,\eta) = \tilde{\psi}_{vr}(x)\}\,. \tag{4.10b}$$

and

$$\partial \tilde{\psi}_s(x) = \omega\{\nabla_x \xi_s(x,\omega,\eta)\}_{(\omega,\eta)\in \mathbf{M}_s(x)} \tag{4.11a}$$

where

$$\mathbf{M}_s(x) \triangleq \{(\omega,\eta) \in \mathbf{R}_+ \times \mathbf{R}^4 | \omega \in [0,\omega_s], \eta \in \mathbf{R}_s(\omega), \xi_s(x,\omega,\eta) = \tilde{\psi}_s(x)\} \cdot \tag{4.11b}$$

■

To conclude this section, we shall obtain explicit formula for $\mathbf{R}(\omega)$ and $\mathbf{R}_s(\omega)$. It was shown in [Pol.2, Sti.1] that at the points of discontinuity, $\omega_A \in \Omega_A$, the phase functions, $\underline{\theta}, \overline{\theta}$ defined by (3.3c-d) satisfy

$$\underline{\theta}(\omega_A) = \lim_{\omega \uparrow \omega_A} \underline{\theta}(\omega) \text{ and } \overline{\theta}(\omega_A) = \lim_{\omega \uparrow \omega_A} \overline{\theta}(\omega) \tag{4.12}$$

i.e. the functions are given by their left hand limits at the points of discontinuity. It now follows from Theorem 3.1. (c) that $\underset{\sim}{\theta}, \tilde{\theta}$ defined by (3.16a-b) are given by $\underset{\sim}{\theta}(\omega) = \min\{\lim_{\omega' \downarrow \omega} \underline{\theta}(\omega), \lim_{\omega' \uparrow \omega} \underline{\theta}(\omega)\}, \tilde{\theta}(\omega) = \max\{\lim_{\omega' \downarrow \omega} \overline{\theta}(\omega), \lim_{\omega' \uparrow \omega} \overline{\theta}(\omega)\}$ and hence that

$$\underset{\sim}{\theta}(\omega) = \begin{cases} \underline{\theta}(\omega) & \text{if } \omega \in \{\mathbf{R}_+ - \Omega_A\} \\ \min\{\underline{\theta}(\omega), \lim\limits_{\omega' \downarrow \omega} \underline{\theta}(\omega')\} & \text{if } \omega \in \Omega_A \end{cases} \tag{4.13a}$$

$$\theta^{\sim}(\omega) = \begin{cases} \overline{\theta}(\omega) & \text{if } \omega \in \{\mathbf{R}_+ - \Omega_A\} \\ \max\{\overline{\theta}(\omega), \lim\limits_{\omega' \downarrow \omega} \overline{\theta}(\omega)\} & \text{if } \omega \in \Omega_A . \end{cases} \tag{4.13b}$$

Similar results to the above hold for $\underset{\sim}{\theta}^{i}, \theta^{\sim i}, i = 1, 2$ defined by (3.25a-b). From the above, it is easy to establish the following formula for determining the sets $\mathbf{R}(\omega)$ and $\mathbf{R}_s(\omega)$:

Proposition 4.2. (a) The point to set map $\mathbf{R} : \mathbf{R}_+ \to 2^{[0,1] \times \mathbf{R}}$ defined by (3.17a) is given by

$$\mathbf{R}(\omega) = \begin{cases} \{(\mu, \theta) \in \mathbf{R}^2 \mid \underline{\mu}(\omega) \leqslant \mu \leqslant \overline{\mu}(\omega),\ \underline{\theta}(\omega) \leqslant \theta \leqslant \overline{\theta}(\omega)\}, & \text{if } \omega \in \{\mathbf{R}_+ - \Omega_A\}, \\ \{(\mu, \theta) \in \mathbf{R}^2 \mid \underline{\mu}(\omega) \leqslant \mu \leqslant \overline{\mu}(\omega), \\ \quad \min\{\underline{\theta}(\omega), \lim\limits_{\omega' \downarrow \omega} \underline{\theta}(\omega')\} \leqslant \theta \leqslant \max\{\overline{\theta}(\omega), \lim\limits_{\omega' \downarrow \omega} \overline{\theta}(\omega')\}\}, & \text{if } \omega \in \Omega_A . \end{cases} \tag{4.14a}$$

(b) The point to set map, $\mathbf{R}_s : \mathbf{R}_+ \to 2^{\mathbf{R}^4}$ defined by (3.26) is given by

$$\mathbf{R}_s(\omega) = \begin{cases} \{(m^1, m^2, \theta^1, \theta^2) \in \mathbf{R}^4 \mid \underline{m}^i(\omega) \leqslant m^i \leqslant \overline{m}^i(\omega), \\ \quad \underline{\theta}^i(\omega) \leqslant \theta^i \leqslant \overline{\theta}^i(\omega), i = 1,2\}, \text{ if } \omega \in \{\mathbf{R}_+ - \Omega_A\}, \\ \{(m^1, m^2, \theta^1, \theta^2) \in \mathbf{R}^4 \mid \underline{m}^i(\omega) \leqslant m^i \leqslant \overline{m}^i(\omega), \\ \quad \min\{\underline{\theta}^i(\omega), \lim\limits_{\omega' \downarrow \omega} \underline{\theta}^i(\omega')\} \leqslant \theta^i \leqslant \max\{\overline{\theta}^i(\omega), \lim\limits_{\omega' \downarrow \omega} \overline{\theta}^i(\omega')\}, \\ \quad i = 1,2\}, \text{ if } \omega \in \Omega_A . \end{cases} \tag{4.14b}$$

5. CONCLUSION

Plant models containing a description of plant uncertainty have been used for some time in the design of linear, time invariant feedback systems, see, e.g. [Des.1, Doy.1, Hal.1, Hor.1, Hor.2, Hor.3]. More recently, such models have started to appear in the adaptive control literature, see, e.g., [Orl.1]. In this paper, we have examined the problem of computationally efficient formulation of a class of optimal worst case control system design problems in which the plant model contains a description of the modeling uncertainty. In particular, we have shown that a literal translation of common frequency domain design requirements into inequalities results in inequalities involving max functions that are computationally prohibitively costly because they involve global maximization over polyhedral multidimensional sets, or alternatively, over two dimensional sets of highly complex description. As a way out of this predicament, we have presented a methodology for translating design requirements into majorizing inequalities which are somewhat more stringent than the original design requirements, but which involve max functions that are very easy to evaluate because they involve maximization over 2-D rectangles only. These rectangles contain the above mentioned, 2-D sets of complex description. Should the designer feel that the use of rectangles leads to excessive conservatism in design, he/she has the option of replacing the rectangles with convex 2-D polyhedra which contain the complex sets more "tightly" than the rectangles do. Though the evaluation of the resulting max functions will be only slightly more costly than when rectangles are used, the computation of such polyhedra is a subject for future research.

An important aspect of our work was to demonstrate that the majorizing design inequalities which we propose satisfy a number of hypotheses which ensure that the inequalities can be solved by nondifferentiable optimization algorithms. Since transfer functions can have poles on the $j\omega$ axis, it turns out that the required properties are not satisfied everywhere in the design parameter space, by the inequalities obtained either by literal translation of design requirements or by majorization techniques. Hence it is necessary to modify standard, two phase algorithms, such as those in [Gon.1, Pol.4], to obtain a special, three phase algorithm capable of solving problems involving inequalities on the majorizing functions we have constucted. We shall present such an algorithm in a paper to follow.

Finally, it should be apparent, that the complexity reduction techniques proposed in this paper have some applications to multivariable design as well, though not with as dramatic simplification as in the SISO case.

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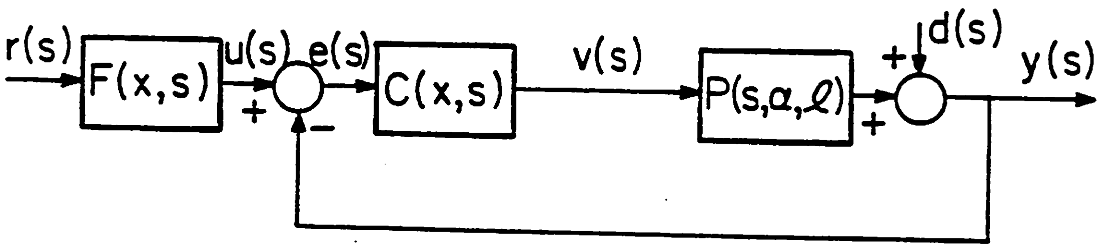

Fig. 1. Two degrees of freedom control structure.

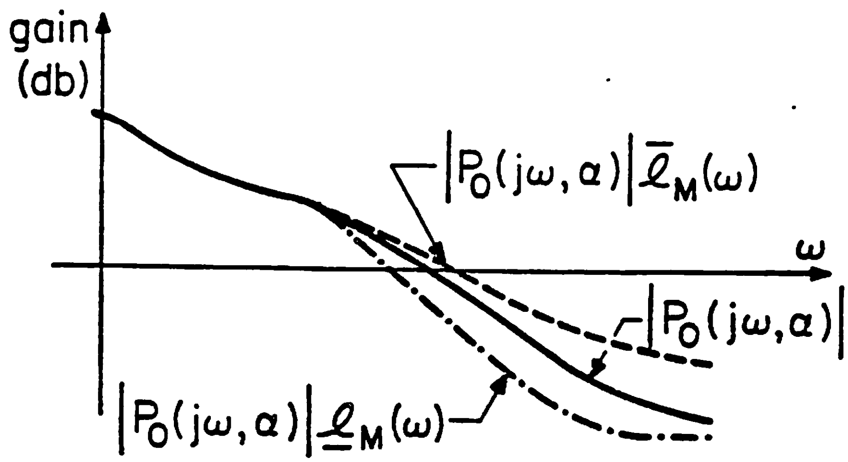

Fig. 2a. Envelope of plant magnitude uncertainty with respect to multiplicative perturbations.

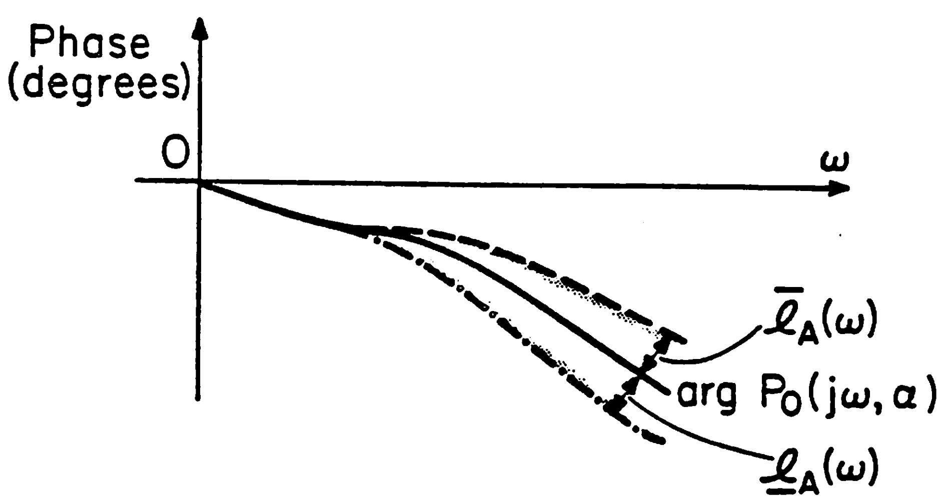

Fig. 2b. Envelope of plant phase uncertainty with respect to multiplicative perturbations.

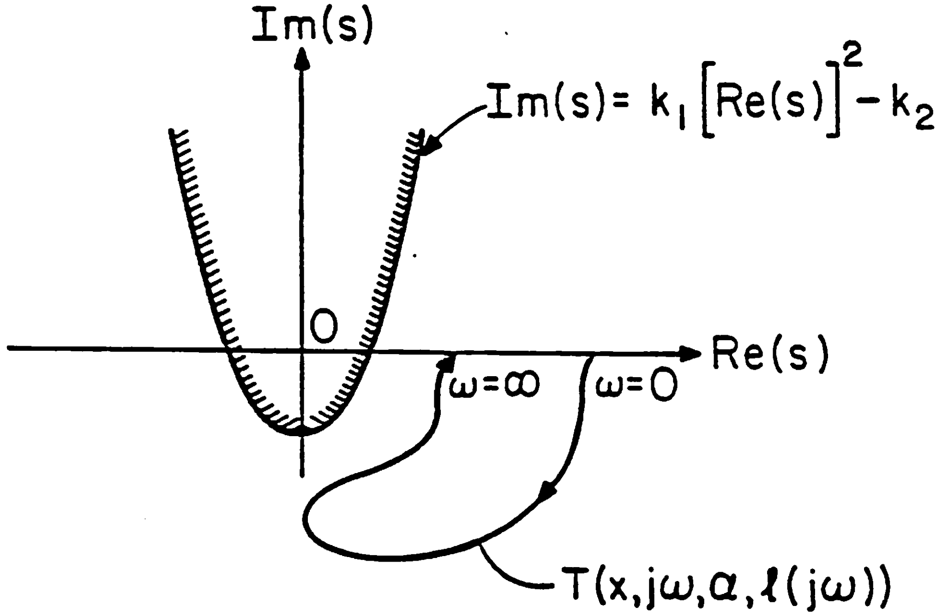

Fig. 3. Exclusion region for stability test.

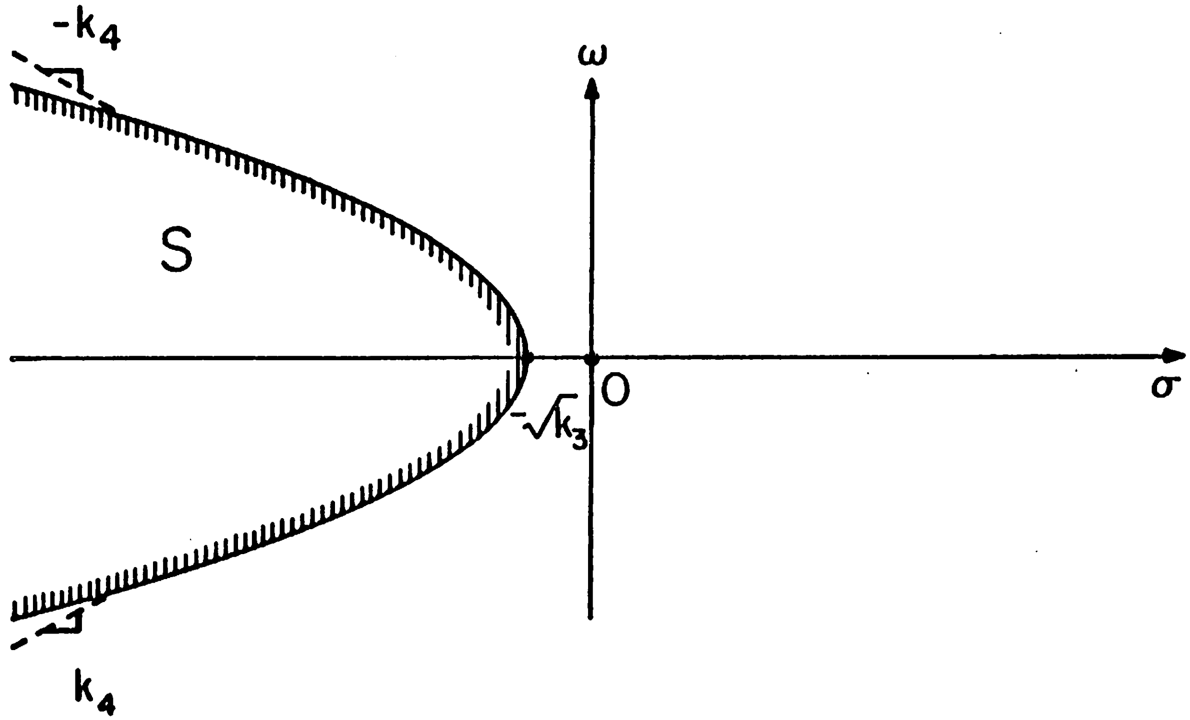

Fig. 4. Parabolic inclusion region for pole placement.

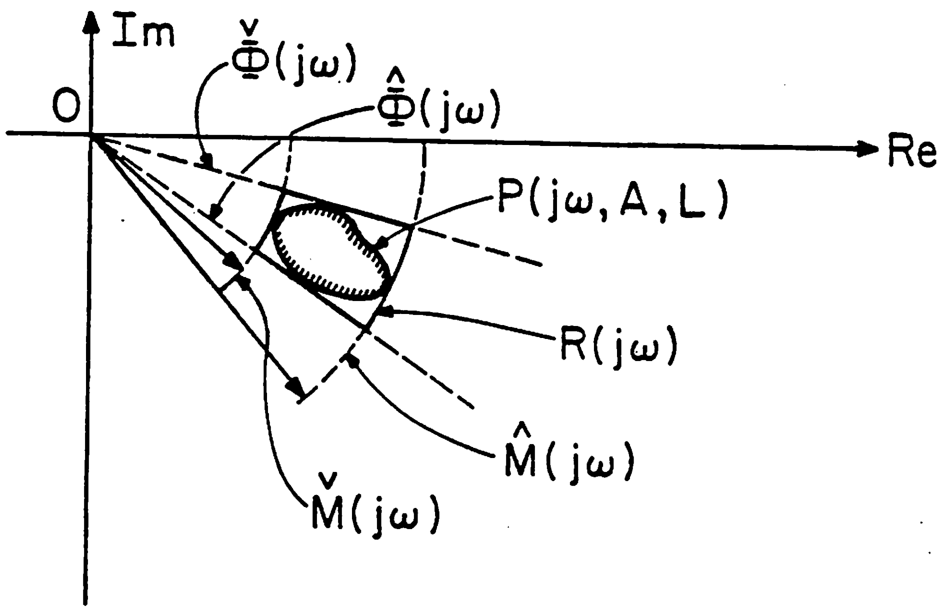

Fig. 5. Construction of rectangular approximation to the set of plant variations.