DYNAMICAL SYSTEMS REVISITED: HYBRID SYSTEMS WITH ZENO EXECUTIONS

by

Jun Zhang

Memorandum No. UCB/ERL M00/50

15 December 1999

DYNAMICAL SYSTEMS REVISITED:
HYBRID SYSTEMS WITH ZENO EXECUTIONS

by

Jun Zhang

Memorandum No. UCB/ERL M00/50

15 December 1999

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Dynamical Systems Revisited: Hybrid Systems with Zeno Executions

by Jun Zhang

Research Project

Submitted to the Department of Electrical Engineering and Computer Sciences, University of California at Berkeley, in partial satisfaction of the requirements for the degree of **Master of Science, Plan II**.

Approval for the Report and Comprehensive Examination:

Committee:

Professor Shankar S. Sastry
Research Advisor

Date

* * * * * *

Professor Thomas Henzinger
Second Reader

Date

Abstract

In this thesis, results from classical dynamical systems are generalized to hybrid dynamical systems. Continuous dependence on initial conditions in hybrid systems is established. The concept of ω limit set is introduced for hybrid systems and some important properties are derived, where Zeno and non-Zeno hybrid systems can be treated within the same framework. As an example, LaSalle's Invariance Principle is extended to hybrid systems. The idea of equilibrium set is developed as a generalization of equilibrium point in conventional dynamical systems, and Lyapunov stability of equilibrium set is discussed. Zeno hybrid systems are investigated in detail. The ω limit set of Zeno executions is characterized for a few quite general classes of hybrid systems. Examples of Zeno hybrid automata are studied to illustrate the idea.

Contents

Acknowledgment

First of all, I would like to express my sincere gratitude and appreciation to my research advisor, Professor Shankar Sastry, for his guidance and support. I am thankful for the opportunities and direction he provided. I wish to thank Professor Thomas Henzinger for serving as the second reader for my M.S. research report.

I am also grateful to Dr. Karl Henrik Johansson for providing me invaluable suggestions and helpful discussions. It is his presentation which initialized my interest of this research field. His solid background and knowledge in control systems are always a source of advice and guidance.

I have greatly benefited from interactions with Dr. John Lygeros and Dr. Datta Godbole. They gave me the fundamental idea of the hybrid systems and brought me into this field. I would also like to thank Dr. Slobodan Simić and Magnus Egerstedt for helpful discussions and comments.

This work was supported by Army Research Office (ARO) under the MURI grant DAAH04-96-1-0341, Caltrans Master Award Agreement with PATH 1997-1999, award 65A0013 under MOU 310.

1. Introduction

Hybrid systems, or systems that interacts continuous-time and discrete-time dynamics, have attracted much research attention in these years. They have been used as models in a large variety of applications. The rich structure of such hybrid systems allow them to accurately predict the behavior of quite complex systems. However, the continuous–discrete nature of the system calls for new theoretical tools for modeling, analysis, and design. Intensive recent activity has provided a few such tools, for instance, Lyapunov stability results. However, as will be shown in this thesis, in many cases the results come with assumptions that are not only hard to check but also unnecessary. There are several fundamental properties of hybrid systems that have not been sufficiently studied in the literature. These include questions on existence and uniqueness of executions, which have only recently been addressed in [18, 24]. Another question is when a hybrid system exhibits an infinite number of discrete transitions during a finite time interval, which is referred to as the Zeno phenomenon. The significance of these questions has been pointed out by many researchers, e.g., He and Lemmon [11] wrote "An important issue which is not addressed in this paper concerns necessary and sufficient conditions for a switched system to be live, deadlock free, or nonZeno."

Zeno is an interesting mathematical property of some hybrid systems, which does not occur in conventional dynamical systems. Real physical systems are not Zeno. Models of physical systems may, however, be Zeno due to a too high level of abstraction. It makes hybrid system simulation imprecise and time-consuming. Several hybrid systems simulation packages, such as Dymola [10], Omola [20], and SHIFT [9] get stuck when a large number of discrete transitions appear in a short time interval. Therefore, it is nice if simulation softwares would detect Zeno and either resolve the Zeno phenomenon automatically or with support from the user. Even if Zenoness is an important property of hybrid systems, they have only been studied to some extent [1, 2, 4, 5, 12]. Some researchers took no Zeno execution as a standing assumption in the discussion.

The main contribution of the thesis is to carefully generalize the concepts from classical dynamical systems like ω limit set and invariant set, in a way that Zeno executions are treated within the same framework as regular non-Zeno executions. It is then straightforward to extend existing results, for instance, Lyapunov stability theorems for hybrid systems [6, 26]. We illustrate this by proving LaSalle's Invariance Principle to hybrid systems. In the latter part of the thesis, we characterize Zeno executions and their Zeno states, where the Zeno states are defined as the ω limit points of a Zeno execution. We are able to completely characterize the set of Zeno states for a few classes of hybrid systems. It is shown that the features of the reset maps are important. For example, if the reset relations are identity maps or contracting, the continuous part of the Zeno state is a singleton.

The outline of the thesis is as follows. In Section 2 the formal notation and definition for hybrid automata is presented. Continuous dependence on initial conditions in hybrid systems is introduced in Section 3. Section 4 presents some results in the invariant set and stability of hybrid systems. We propose the idea of equilibrium set of hybrid systems in Section 5 and discuss its Lyapunov stability. Zeno hybrid systems are investigated in detail in Section 6, and two examples are studied in Section 7 to illustrate the idea. Finally, some conclusions and future work are given in Section 8.

Part of this work has been submitted as [27].

2 Hybrid Automata and Executions

For a finite collection V of variables, let $\mathbf{V}$ denote the set of valuations of these variables. We use lower case letters to denote both a variable and its valuation. We refer to variables whose set of valuations is finite or countable as *discrete* and to variables whose set of valuations is a subset of a Euclidean space as *continuous*. For a set of continuous variables X with $\mathbf{X} = \mathbb{R}^n$ for $n \geq 0$, we assume that $\mathbf{X}$ is given the Euclidean metric topology, and use $\|\cdot\|$ to denote the Euclidean norm. For a set of discrete variables Q, we assume that $\mathbf{Q}$ is given the discrete topology (every subset is an open set), generated by the metric $d_D(q, q') = 0$ if $q = q'$ and $d_D(q, q') = 1$ if $q \neq q'$. We denote the valuations of the union $Q \times X$ by $\mathbf{Q} \times \mathbf{X}$, which is given the product topology, generated by the metric $d((q, x), (q', x')) = d_D(q, q') + \|x - x'\|$. Using the metric d, we define the distance between two sets $U_1, U_2 \subseteq \mathbf{Q} \times \mathbf{X}$ by $d(U_1, U_2) = \inf_{(q_i, x_i) \in U_i} d((q_1, x_1), (q_2, x_2))$. We assume that a subset U of a topological space is given the induced topology, and we use $\overline{U}$ to denote its closure, U^o its interior, ∂U its boundary, U^c its complement, $|U|$ its cardinality, and $P(U)$ the set of all subsets of U.

The following definitions are based on [19, 13, 18].

Definition 1 (Hybrid Automaton) *A hybrid automaton H is a collection $H = (Q, X, \mathrm{Init}, f, \mathrm{Inv}, \mathrm{Reset})$, where*

- *Q is a finite collection of discrete variables;*
- *X is a finite collection of continuous variables with $\mathbf{X} = \mathbb{R}^n$;*
- *$\mathrm{Init} \subseteq \mathbf{Q} \times \mathbf{X}$ is a set of initial states;*
- *$f : \mathbf{Q} \times \mathbf{X} \to T\mathbf{X}$ is a vector field;*
- *$\mathrm{Inv} \subseteq \mathbf{Q} \times \mathbf{X}$ is the domain of H;*[1] *and,*
- *$\mathrm{Reset} : \mathbf{Q} \times \mathbf{X} \to P(\mathbf{Q} \times \mathbf{X})$ is a reset relation.*

We refer to $(q, x) \in \mathbf{Q} \times \mathbf{X}$ as the *state* of H. Unless otherwise stated, we introduce the following assumption, to prevent some obvious pathological cases.

Assumption 1 $|\mathbf{Q}| < \infty$ *and f is Lipschitz continuous in its second argument.*

Note that, under the discrete topology on $\mathbf{Q}$, f is trivially continuous in its first argument. Under Assumption 1, a hybrid automaton can be represented by a directed graph $(\mathbf{Q}, E)$, with vertices $\mathbf{Q}$ and edges $E = \{(q, q') \in \mathbf{Q} \times \mathbf{Q} \ : \ \exists x, x' \in \mathbf{X}, (q', x') \in \mathrm{Reset}(q, x)\}$. With each vertex $q \in \mathbf{Q}$, we associate a set of continuous initial states $\mathrm{Init}(q) = \{x \in \mathbf{X} \ : \ (q, x) \in \mathrm{Init}\}$, a vector field $f(q, \cdot)$, and a set $I(q) = \{x \in \mathbf{X} \ : \ (q, x) \in \mathrm{Inv}\}$. With each edge $e = (q, q') \in E$, we associate a guard $G(e) = \{x \in \mathbf{X} \ : \ \exists x' \in \mathbf{X}, (q', x') \in \mathrm{Reset}(q, x)\}$, and a reset relation $R(e, x) = \{x' \in \mathbf{X} \mid (q', x') \in \mathrm{Reset}(q, x)\}$. Since there is a unique graphical

[1] The set Inv is called the invariant set in the hybrid system literature in computer science. Note that Inv is not invariant in the usual dynamical systems sense.

representation for each hybrid automaton, we will use the corresponding graphs as formal definitions for hybrid automata in most examples.

Definition 2 (Hybrid Time Trajectory). *A hybrid time trajectory* $\tau = \{I_i\}_{i=0}^{N}$ *is a finite or infinite sequence of intervals, such that*

- $I_i = [\tau_i, \tau_i']$ *for* $i < N$, *and, if* $N < \infty$, $I_N = [\tau_N, \tau_N']$ *or* $I_N = [\tau_N, \tau_N')$;
- $\tau_i \leq \tau_i' = \tau_{i+1}$ *for* $i \geq 0$.

A hybrid time trajectory is a sequence of intervals of the real line, whose end points overlap. The interpretation is that the end points of the intervals are the times at which discrete transitions take place. Note that $\tau_i = \tau_i'$ is allowed, therefore multiple discrete transitions may take place at the same "time". Since the dynamical systems we will be concerned with will be time invariant we can, without loss of generality, assume $\tau_0 = 0$. Hybrid time trajectories can extend to infinity if τ is an infinite sequence or if it is a finite sequence ending with an interval of the form $[\tau_N, \infty)$. We use $t \in \tau$ as shorthand notation for that there exists i such that $t \in I_i \in \tau$. For a topological space K we use $k : \tau \to K$ as a short hand notation for a map assigning a value from K to each $t \in \tau$; note that k is not a function on the real line, as it assigns multiple values to the same $t \in \mathbb{R}$: $t = \tau_i' = \tau_{i+1}$ for all $i \geq 0$. Each τ is fully ordered by the relation $\prec$ defined by $t_1 \prec t_2$ for $t_1 \in [\tau_i, \tau_i']$ and $t_2 \in [\tau_j, \tau_j']$ if and only if $i < j$, or $i = j$ and $t_1 < t_2$.

Definition 3 (Execution) *An execution* χ *of a hybrid automaton* H *is a collection* $\chi = (\tau, q, x)$ *with* $\tau \in \mathcal{T}$, $q : \tau \to \mathbf{Q}$, *and* $x : \tau \to \mathbf{X}$, *satisfying*

- $\big(q(\tau_0), x(\tau_0)\big) \in \mathrm{Init}$ *(initial condition);*
- *for all* i *with* $\tau_i < \tau_i'$, $q(\cdot)$ *is constant and* $x(\cdot)$ *is a solution*[2] *to the differential equation* $dx/dt = f(q,x)$ *over* $[\tau_i, \tau_i']$, *and for all* $t \in [\tau_i, \tau_i')$, $(q(t), x(t)) \in \mathrm{Inv}$ *(continuous evolution); and*

[2] "Solution" is interpreted in the sense of Caratheodory, i.e., $x(t)$ is C^1 and satisfies $x(t) = x(\tau_i) + \int_{\tau_i}^{t} f(q(s), x(s))\, ds$ for all $t \in [\tau_i, \tau_i']$.

- *for all i, $(q(\tau_{i+1}), x(\tau_{i+1})) \in \text{Reset}\big(q(\tau_i'), x(\tau_i')\big)$ (discrete evolution).*

We say a hybrid automaton *accepts* an execution χ or not. For an execution $\chi = (\tau, q, x)$, we use $(q_0, x_0) = \big(q(\tau_0), x(\tau_0)\big)$ to denote the initial state of χ. The *execution time* $\mathcal{T}(\chi)$ is defined as $\mathcal{T}(\chi) = \sum_{i=0}^{N}(\tau_i' - \tau_i)$, where $N+1$ is the number of intervals in the hybrid time trajectory. An execution is *finite* if τ is a finite sequence ending with a compact interval, it is called *infinite* if τ is either an infinite sequence or if $\mathcal{T}(\chi) = \infty$, and it is called *Zeno* if it is infinite but $\mathcal{T}(\chi) < \infty$. We use $\mathcal{E}_H(q_0, x_0)$ to denote the set of all executions of H with initial condition $(q_0, x_0) \in \text{Init}$, $\mathcal{E}_H^\infty(q_0, x_0)$ to denote the set of all infinite executions of H with initial condition $(q_0, x_0) \in \text{Init}$. We define $\mathcal{E}_H = \bigcup_{(q_0,x_0)\in\text{Init}} \mathcal{E}_H(q_0, x_0)$ and $\mathcal{E}_H^\infty = \bigcup_{(q_0,x_0)\in\text{Init}} \mathcal{E}_H^\infty(q_0, x_0)$.

Definition 4 (Non-Blocking and Deterministic Automaton) *A hybrid automaton H is non-blocking if $\mathcal{E}_H^\infty(q_0, x_0)$ is non-empty for all $(q_0, x_0) \in \text{Init}$. It is deterministic if $\mathcal{E}_H^\infty(q_0, x_0)$ contains at most one element for all $(q_0, x_0) \in \text{Init}$.*

Note that if a hybrid automaton is both non-blocking and deterministic, then it accepts a unique infinite execution for each initial condition. In [18] conditions were established that determine whether an automaton is non-blocking and deterministic. The conditions require one to argue about the set of states reachable by a hybrid automaton, and the set of states from which continuous evolution is impossible. A state $(q, x) \in \mathbf{Q} \times \mathbf{X}$ is called *reachable* by H, if there exists a finite execution $\chi = (\tau, q, x)$ with $\tau = \{I_i\}_{i=0}^N$ and $(q(\tau_N'), x(\tau_N')) = (q, x)$. We use Reach_H to denote the set of states reachable by a hybrid automaton, and $\text{Reach}_H(q)$ the projection of Reach_H to discrete state q. We will drop the subscript H whenever the automaton is clear from the context. The set Reach is in general difficult to compute. Fortunately, the conditions of subsequent theorems will not require us to do so: any over-approximation of the reachable set will be sufficient. In [7, 18] methods for computing such over-approximations using simple induction arguments are outlined.

The set of states from which continuous evolution is impossible is given by

$$\text{Out}_H = \{(q^0, x^0) \in \mathbf{Q} \times \mathbf{X} \mid \forall \epsilon > 0,\ \exists t \in [0, \epsilon),\ (q^0, x(t)) \notin \text{Inv}\},$$

where $x(\cdot)$ is the solution to $dx/dt = f(q^0, x)$ with $x(0) = x^0$. Note that if Inv is an open set, then Out is simply Inv^c. If Inv is closed, then Out may also contain parts of the boundary of Inv. In [18] methods for computing Out were proposed, under appropriate smoothness assumptions on f and the boundary of Inv. As before, we will use $\text{Out}_H(q)$ to denote the projection of Out to discrete state q, and drop the subscript H whenever the automaton is clear from the context. With these two pieces of notation one can show the following results [18].

Lemma 1 (Non-blocking) *A (deterministic) hybrid automaton is non-blocking if (and only if) for all* $(q, x) \in \text{Out} \cap \text{Reach}$, $\text{Reset}(q, x) \neq \emptyset$.

Lemma 2 (Deterministic) *A hybrid automaton is deterministic if and only if for all* $(q, x) \in \text{Reach}$, $|\text{Reset}(q, x)| \leq 1$ *and, if* $\text{Reset}(q, x) \neq \emptyset$, *then* $(q, x) \in \text{Out}$.

A hybrid automaton is invariant preserving if the state remains in the closure of the invariant along all executions.

Definition 5 (Invariant Preserving) *A hybrid automaton is invariant preserving if* $\text{Reach} \subseteq \overline{\text{Inv}}$.

Lemma 3 *A hybrid automaton is invariant preserving if and only if* $\text{Init} \subseteq \text{Inv}$ *and for all* $(q, x) \in \text{Inv} \cap \text{Reach}$, $\text{Reset}(q, x) \subseteq \text{Inv}$.

Note that the conditions of the lemma do not depend on the vector field f.

Definition 6 (Transverse Invariants) *A hybrid automaton is said to have transverse invariants if there exists a function* $\sigma : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$ *continuously differentiable in its second argument, such that* $\text{Inv} = \{(q, x) \in \mathbf{Q} \times \mathbf{X} \; : \; \sigma(q, x) \geq 0\}$ *and for all* (q, x) *with* $\sigma(q, x) = 0$, $L_f\sigma(q, x) \neq 0$.

Here $L_f\sigma : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$ denotes the Lie derivative of σ along f defined as

$$L_f\sigma(q, x) = \frac{\partial \sigma}{\partial x}(q, x) \cdot f(q, x)$$

In other words, an automaton has transverse invariants if the set Inv is closed, its boundary is differentiable, and the vector field f is pointing either inside or outside of Inv along the boundary.[3] If H has transverse invariants the set Out_H admits a fairly simple characterization[18]:

$$\mathrm{Out}_H = \{(q,x) \in \mathbf{Q} \times \mathbf{X} \; : \; \sigma(q,x) < 0\} \cup \{(q,x) \in \mathbf{Q} \times \mathbf{X} \; : \; \sigma(q,x) = 0 \text{ and } L_f\sigma(q,x) < 0\}.$$

3 Continuous Dependence on Initial Conditions

Continuity of solutions with respect to initial conditions is a desirable property of many dynamical systems. This is for instance the case in simulation studies. For conventional smooth systems, a Lipschitz condition on the vector field guarantees continuous dependence. For hybrid systems, however, it is not sufficient to require that the vector field in each discrete state is Lipschitz continuous. Some early work on this topic can be found in [8, 23]. The following result lists assumptions that guarantee continuous dependence on initial conditions for a class of hybrid systems.

Theorem 1 *Consider a deterministic hybrid automaton H with transverse invariants. Assume it is invariant preserving and that $f(q,\cdot)$ is C^1 for all $q \in \mathbf{Q}$. Furthermore, assume that for all $e = (q,q') \in E$, $R(e,\cdot)$ is continuous, and $G(e) \cap I(q)$ is an open subset of $\partial I(q)$. Consider a finite execution $\chi = (\tau, q, x) \in \mathcal{E}_H(q_0, x_0)$ with $\tau = \{I_i\}_{i=0}^N$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $(\tilde{q}_0, \tilde{x}_0) \in \mathrm{Init}$ with $d((\tilde{q}_0, \tilde{x}_0), (q_0, x_0)) < \delta$, there exists $T(\tilde{x}_0) > 0$ such that the execution $\tilde{\chi} = (\tilde{\tau}, \tilde{q}, \tilde{x}) \in \mathcal{E}_H(\tilde{q}_0, \tilde{x}_0)$ with $\tilde{\tau} = \{\tilde{I}_i\}_{i=0}^N$ and $\tilde{\tau}'_N = T(\tilde{x}_0)$ satisfies*

- $|\mathcal{T}(\tilde{\chi}) - \mathcal{T}(\chi)| < \epsilon$, *and*
- $d((\tilde{q}(\tilde{\tau}'_N), \tilde{x}(\tilde{\tau}'_N)), (q(\tau'_N), x(\tau'_N))) < \epsilon$.

Remark 1 The result says that for a given execution χ, any execution $\tilde{\chi}$ starting close enough to χ with some appropriate execution time will stay close at the end point. Furthermore, the proof indicates that they have the same sequence of discrete transitions. Note

[3] Under appropriate smoothness assumptions on σ and f the definition of transverse invariants can be relaxed somewhat by allowing $L_f\sigma(q,x) = 0$ on the boundary of Inv and taking higher-order Lie derivatives, until one that is non-zero is found. Even though many of the results presented here extend to this relaxed definition, the proofs are slightly more technical. We will therefore limit ourselves to the notion of transverse invariants given in Definition 6.

that for a given initial state and execution time, the execution $\tilde{\chi}$ is unique by assumption. Also note that it is in general not possible to guarantee the same execution time for $\tilde{\chi}$ and χ.

Remark 2 The assumption on the invariant preserving property can be replaced by the one that $G(q, q') \cap I(q)^c$ is open in $\mathbf{X}$.

Remark 3 If there is only one discrete state and no reset relations, the hybrid automaton, of course, defines a smooth dynamical system. It is easy to see that all assumptions are satisfied for this case. By setting $N = 0$ and $\mathcal{T}(\tilde{\chi}) = \mathcal{T}(\chi)$, we obtain the traditional result of continuous dependence on initial conditions.

The following lemma is used in the proof of the theorem.

Lemma 4 *Consider a hybrid automaton H and an execution χ that satisfy the assumptions of Theorem 1. Let $\phi^i(t, x_0)$ denote the phase flow of the differential equation $dx/dt = f(q(\tau_i), x)$ with initial condition $x(0) = x_0$[4], and $\Delta_i = \tau_i' - \tau_i$. Then, for every $i < N$ with $\Delta_i \neq 0$, there exists a neighborhood[5] $W_0^i \subset I(q(\tau_i))$ of $x(\tau_i)$ and a C^1 function $T^i : W_0^i \to \mathbb{R}^+$, such that for all $y \in W_0^i$,*

- $\phi^i(T^i(y), y) \in \partial I(q(\tau_i))$,
- $\phi^i(t, y) \in I(q(\tau_i))^o$ *for all* $t \in (0, T^i(y))$, *and*
- $\Phi^i : W_0^i \to \partial I(q(\tau_i))$, *defined as* $\Phi^i(y) = \phi^i(T^i(y), y)$, *is continuous.*

Proof: The first part of the lemma is a straightforward application of the Implicit Function Theorem to the function σ, which defines the transverse invariants in Definition 6. Since $\Delta_i \neq 0$, the state $(q(\tau_i'), x(\tau_i'))$ is reached from $(q(\tau_i), x(\tau_i))$ along continuous evolution. By the definition of an execution, it holds that $x(t) \in I(q(\tau_i))$ for all $t \in [\tau_i, \tau_i')$. Therefore, since $I(q(\tau_i))$ is closed by Definition 6, it follows that $x(\tau_i') \in I(q(\tau_i))$. By the definition of an

[4]Note that the phase flow is defined for all $x \in \mathbb{R}^n$ and is not restricted to the domain $I(q(\tau_i))$.

[5]A neighborhood of a continuous state is defined in the subspace topology on the corresponding domain. Here, W_0^i is thus the set of points $z \in I(q(\tau_i))$ such that $\|z - x(\tau_i)\| < r$ for some $r > 0$.

execution, we have $x(\tau_i') \in G(e_i)$, where $e_i = (q(\tau_i'), q(\tau_{i+1}))$. So $\sigma(q(\tau_i), x(\tau_i')) = 0$, because $G(e_i) \cap I(q(\tau_i)) \subset \partial I(q(\tau_i))$ from the assumptions. The composed function $\sigma(q(\tau_i), \phi^i(\cdot,\cdot))$ is C^1 in a neighborhood of $(\Delta_i, x(\tau_i))$ in $\mathbb{R}^+ \times \mathbb{R}^n$, because $\sigma(q(\tau_i), \cdot)$ is C^1 (Definition 6) and $\phi^i(\cdot,\cdot)$ is C^1 in both arguments ($f(q(\tau_i), \cdot)$ is C^1 so the solution is also C^1 [25]). Moreover,

$$\frac{\partial}{\partial t}\sigma(q(\tau_i), \phi^i(t, x_0))\bigg|_{(t,x_0)=(\Delta_i, x(\tau_i))} = \mathcal{L}_f\sigma(q(\tau_i), x(\tau_i)) \neq 0,$$

where the inequality follows from Definition 6. The Implicit Function Theorem is hence applicable. It gives that there exists a neighborhood $\Omega^i \subset \mathbb{R}^+$ of Δ_i and a neighborhood $W_0^i \subset \mathbb{R}^n$ of $x(\tau_i)$, such that for each $y \in W_0^i$ the equation $\sigma(q(\tau_i), \phi^i(t, y)) = 0$ has a unique solution $t \in \Omega^i$. Furthermore, this solution can be given as $t = T^i(y)$, where T^i is a unique C^1 mapping from W_0^i to Ω^i and $\phi^i(T^i(y), y) \in \partial I(q(\tau_i))$.

For the second part of the lemma, assume there exists $\hat{t} \in (0, T^i(y))$ such that $\phi^i(\hat{t}, y) \in \partial I(q(\tau_i))$ for some $y \in W_0^i$. Then, $\sigma(q(\tau_i), \phi^i(\hat{t}, y)) = 0$, which contradicts that $T^i(y)$ is the unique solution to the equation $\sigma(q(\tau_i), \phi^i(t, y)) = 0$.

For the third part, note that since $\phi^i(\cdot,\cdot)$ is C^1 in both arguments, it follows that for all $\epsilon > 0$ there exists $\delta_1 > 0$, such that for all t with $||t - T^i(x(\tau_i))|| < \delta_1$ and all $y \in W_0^i$ with $||y - x(\tau_i)|| < \delta_1$,

$$||\phi^i(t, x(\tau_i)) - \phi^i(T^i(x(\tau_i)), x(\tau_i))|| < \epsilon$$
$$||\phi^i(T^i(y), y) - \phi^i(T^i(y), x(\tau_i))|| < \epsilon.$$

By the continuity of T^i, for this particular δ_1, there exists some $\delta_2 > 0$ such that for all $y \in W_0^i$ with $||y - x(\tau_i)|| < \delta_2$, we have $||T^i(y) - T^i(x(\tau_i))|| < \delta_1$. By setting $\delta = \min(\delta_1, \delta_2)$, it follows that for all $y \in W_0^i$ with $||y - x(\tau_i)|| < \delta$,

$$\begin{aligned} ||\Phi^i(y) - \Phi^i(x(\tau_i))|| &= ||\phi^i(T^i(y), y) - \phi^i(T^i(x(\tau_i)), x(\tau_i))|| \\ &\leq ||\phi^i(T^i(y), y) - \phi^i(T^i(y), x(\tau_i))|| + ||\phi^i(T^i(y), x(\tau_i)) - \phi^i(T^i(x(\tau_i)), x(\tau_i))|| \\ &< 2\epsilon, \end{aligned}$$

which proves the continuity of Φ^i. ■

Remark 4 In the case $i = N$, $\Delta_N \neq 0$, and $x(\tau_N') \in \partial I(q(\tau_N))$, by the same argument as in the proof of Lemma 4, the conclusions in the lemma still hold.

Lemma 5 *Consider a hybrid automaton with transverse invariants. For all $q \in \mathbf{Q}$, the domain $I(q) \neq \emptyset$ contains no isolated point.*

Proof: Assume $I(q)$ contains at least one isolated point, say, $\bar{x}$. Then there is an open set $O \subset \mathbb{R}^n$ such that $O \cap I(q) = \{\bar{x}\}$. By Definition 6, $\sigma(q, \bar{x}) = 0$ and $\sigma(q, x) < 0$ for all $x \in O$ and $x \neq \bar{x}$. Since $\sigma(q, \cdot)$ is C^1, $\sigma(q, \cdot)$ attains a local maximum at $\bar{x}$ and thus $\partial\sigma(q, \bar{x})/\partial x = 0$. This, however, implies that $L_f\sigma(q, \bar{x}) = 0$, which contradicts the assumption of transverse invariants. ∎

Now we are ready to prove the theorem.

Proof: We will show that there exists a sequence of sets $\{W^0, V^0, \ldots, W^N, V^N\}$, where $W^i \subset I(q(\tau_i))$ is a neighborhood of $x(\tau_i)$ and $V^i \subset I(q(\tau_i))$ is a neighborhood of $x(\tau_i')$, such that a continuous map Φ^i (which is given by the continuous evolution in $q(\tau_i)$) describes the mapping from W^i into V^i and the continuous reset $R(e_i, \cdot)$ describes the mapping from V^i to W^{i+1}. The composition of the maps is also continuous, which then will be shown to give the result.

Let us define Φ^i properly and simultaneously construct W^i and V^i. We do it recursively and start with $i = N$. Define $V^N = \{x \in I(q(\tau_N)) : \|x - x(\tau_N')\| < \epsilon\}$, which contains no isolated point from Lemma 5. Because H is invariant preserving and $I(q(\tau_N))$ closed, we distinguish the following three cases for the definition of Φ^N: (1) $\Delta_N \neq 0$ and $x(\tau_N') \in \partial I(q(\tau_N))$, (2) $\Delta_N \neq 0$ and $x(\tau_N') \in I(q(\tau_N))^o$, and (3) $\Delta_N = 0$ and $x(\tau_N') \in I(q(\tau_N))$.

(1) From Remark 4, there exists a neighborhood $W_0^N \subset I(q(\tau_N))$ of $x(\tau_N)$ and a C^1 function T^N, such that for all $y \in W_0^N$, $\phi^N(T^N(y), y) \in \partial I(q(\tau_N))$ and $\phi^N(t, y) \in I(q(\tau_N))^o$ for all $t \in (0, T^i(y))$. Define $\Phi^N : W_0^N \to \partial I(q(\tau_N))$ as $\Phi^N(y) = \phi^N(T^N(y), y)$. By continuity of Φ^N, there exists a neighborhood $W^N \subset W_0^N$ of $x(\tau_N)$ such that $\Phi^N(W^N) \subset V^N$. Furthermore, all executions $\tilde{\chi}$ with $\tilde{x}(\tilde{\tau}_N) \in W^N$ fulfills $\tilde{x}(\tilde{\tau}_N') \in V^N$, where $\tilde{\tau}_N' = \tilde{\tau}_N + T^N(\tilde{x}(\tilde{\tau}_N))$.

(2) Define T^N as $T^N(y) \equiv \Delta_N$. Let $W_0^N \subset I(q(\tau_N))$ be a neighborhood of $x(\tau_N)$ such that for all $y \in W_0^N$ and $t \in (0, \Delta_N)$, $\phi^N(t, y) \in I(q(\tau_N))^o$. Such a neighborhood W_0^N

exists, because for all $t \in (0, \Delta_N)$, $\phi^N(t, x(\tau_N))$ belongs to the interior of $I(q(\tau_N))$. Define $\Phi^N : W_0^N \to I(q(\tau_N))$ as $\Phi^N(y) = \phi^N(T^N(y), y)$. By continuous dependence on initial conditions for differential equations, there exists a neighborhood $W^N \subset W_0^N$ of $x(\tau_N)$ such that both $\Phi^N(W^N) \subset V^N$ and all executions with $\tilde{x}(\tilde{\tau}_N) \in W^N$ fulfills $\tilde{x}(\tilde{\tau}'_N) \in V^N$, where $\tilde{\tau}'_N = \tilde{\tau}_N + T^N(\tilde{x}(\tilde{\tau}_N))$.

(3) Define T^N as $T^N(y) \equiv 0$, $W^N = V^N$ and Φ^N to be the identity map. Then, $\Phi^N(W^N) = V^N$.

Next let us define V^{N-1} and let $e_i = (q(\tau'_i), q(\tau_{i+1}))$. Note that the domain of $R(e_{N-1}, \cdot)$ is $G(e_{N-1}) \cap I(q(\tau_{N-1}))$, which follows from the definitions of the reset and the guard and that H is invariant preserving. Since $R(e_{N-1}, \cdot)$ is continuous, there exists a neighborhood $V_0^{N-1} \subset I(q(\tau_{N-1}))$ of $x(\tau'_{N-1})$ such that $R(e_{N-1}, V_0^{N-1} \cap G(e_{N-1})) \subset W^N$, where W^N is given by any of the three cases above. By assumption, $G(e_{N-1}) \cap I(q(\tau_{N-1}))$ is an open subset of $\partial I(q(\tau_{N-1}))$, so there exists a neighborhood $V^{N-1} \subset V_0^{N-1}$ of $x(\tau_{N-1})$ such that $V^{N-1} \cap \partial I(q(\tau_{N-1})) \subset G(e_{N-1}) \cap I(q(\tau_{N-1}))$. Since H is deterministic, it then follows that all executions with $\tilde{x}(\tilde{\tau}'_{N-1}) \in V^{N-1} \cap \partial I(q(\tau_{N-1}))$ fulfills $\tilde{x}(\tilde{\tau}_N) \in W^N$.

To define Φ^{N-1}, we distinguish two cases: (1) $\Delta_{N-1} \neq 0$ and (2) $\Delta_{N-1} = 0$.

(1) Define T^{N-1} and Φ^{N-1} as in (1) above. Again there exists a neighborhood $W^{N-1} \subset I(q(\tau_{N-1}))$ of $x(\tau_{N-1})$ such that $\Phi^{N-1}(W^{N-1}) \subset V^{N-1} \cap \partial I(q(\tau_{N-1}))$. Moreover, all executions $\tilde{\chi}$ with $\tilde{x}(\tilde{\tau}_{N-1}) \in W^{N-1}$ fulfills $\tilde{x}(\tilde{\tau}'_{N-1}) \in V^{N-1} \cap \partial I(q(\tau_{N-1}))$, where $\tilde{\tau}'_{N-1} = \tilde{\tau}_{N-1} + T^{N-1}(\tilde{x}(\tilde{\tau}_{N-1}))$.

(2) As in (3) above, let $T^{N-1}(y) \equiv 0$, $W^{N-1} = V^{N-1}$ and Φ^{N-1} be the identity map. Hence, $\Phi^N(W^{N-1}) = V^{N-1}$.

Proceeding further, we get a sequence of sets $\{W^0, V^0, \ldots, W^N, V^N\}$[6] as well as continuous functions $T^i : W^i \to \mathbb{R}^+$ and $\Phi^i : W^i \to V^i$ for $i = 0, \ldots, N$. Now, for $k = 1, \ldots, N$, define

[6] It is assumed that W^0 is a subset of $\mathrm{Init}(q(\tau_0))$. If this is not the case, one should replace W^0 by $W^0 \cap \mathrm{Init}(q(\tau_0))$ here and in the following.

the function $\Psi^k : W^0 \to W^k$ recursively as

$$\Psi^0(\tilde{x}_0) = \tilde{x}_0$$
$$\Psi^k(\tilde{x}_0) = R(e_{k-1}, \Phi^{k-1}(\Psi^{k-1}(\tilde{x}_0)))$$

and for $k = 1, \ldots, N+1$, define the function $\gamma^k : W^0 \to \mathbb{R}^+$ as

$$\gamma^k(\tilde{x}_0) = \sum_{\ell=1}^{k} T^{\ell-1}(\Psi^{\ell-1}(\tilde{x}_0)).$$

Then, $\Psi^k(\tilde{x}_0) = \tilde{x}(\tilde{\tau}'_k)$ and $\gamma^k(\tilde{x}_0) = \tilde{\tau}'_k - \tilde{\tau}_0$ for execution $\tilde{\chi} = (\tilde{\tau}, \tilde{q}, \tilde{x})$ with $(\tilde{q}_0, \tilde{x}_0) \in q_0 \times W^0$. The functions Ψ^k and γ^k are continuous by construction. Particularly, by the continuity of γ^{N+1}, there exists $\delta_1 > 0$ such that for all $\tilde{x}_0$ with $\|\tilde{x}_0 - x_0\| < \delta_1$, we have $|\gamma^{N+1}(\tilde{x}_0) - \gamma^{N+1}(x_0)| < \epsilon$. The latter inequality is equivalent to $|\sum_{i=0}^{N} \tilde{\Delta}_i - \sum_{i=0}^{N} \Delta_i| < \epsilon$. Finally, the continuity of Φ^N implies that there exists $\delta_2 > 0$ such that for all $y \in W^N$ with $\|y - x(\tau_N)\| < \delta_2$, $\|\Phi^N(y) - x(\tau'_N)\| < \epsilon$. Hence, by the continuity of Ψ^N, there exists $\delta_3 > 0$ such that for all $\tilde{x}_0 \in W^0$ with $\|\tilde{x}_0 - x_0\| < \delta_3$, it holds that $\|\Psi^N(\tilde{x}_0) - x(\tau_N)\| < \delta_2$. Since $\Phi^N(\Psi^N(\tilde{x}_0)) = \tilde{x}(\tilde{\tau}'_N)$, we have $\|\tilde{x}(\tilde{\tau}'_N) - x(\tau'_N)\| < \epsilon$. The proof is completed by setting $\delta = \min(\delta_1, \delta_3)$. ∎

4 LaSalle's Invariance Principle

We first recall some standard concepts from dynamical system theory, and discuss how they are generalized to hybrid automata.

Definition 7 (Invariant Set) *A set $M \subset \mathbf{Q} \times \mathbf{X}$ is called invariant if for all $(q_0, x_0) \in M$, for all $(\tau, q, x) \in \mathcal{E}_H(q_0, x_0)$ and all $t \in \tau$, it holds that $(q(t), x(t)) \in M$.*

Invariant set is such that all executions starting in the set remain in the set for ever.[7] The class of invariant set is closed under arbitrary unions and intersections. We are interested in studying the stability of invariant set, i.e., determine whether all trajectories that start close to an invariant set remain close to it. More formally:

[7] Strictly speaking, we need to assume that $M \subset \mathrm{Init}$.

Definition 8 (Stable Invariant Set) *An invariant set $M \subset \mathbf{Q} \times \mathbf{X}$ is called stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $(q_0, x_0) \in \mathbf{Q} \times \mathbf{X}$ with $d((q_0, x_0), M) < \delta$, for all $\chi = (\tau, q, x) \in \mathcal{E}_H(q_0, x_0)$, and for all $t \in \tau$, $d((q(t), x(t)), M) < \epsilon$. An invariant set is called (locally) asymptotically stable if it is stable and in addition there exists $\Delta > 0$ such that for all (q_0, x_0) with $d((q_0, x_0), M) < \Delta$ and all $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$, $\lim_{t \to \mathcal{T}(\chi)} d((q(t), x(t)), M) = 0$.*

Note that since τ is fully ordered the above limit is well defined.

The asymptotic behavior of an infinite execution is captured in terms of its ω limit set.

Definition 9 (ω Limit Set) *The ω limit point $(\hat{q}, \hat{x}) \in \mathbf{Q} \times \mathbf{X}$ of an infinite execution $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty$ is a point for which there exists a sequence $\{\theta_n\}_{n=0}^\infty$ with $\theta_n \in \tau$ such that as $n \to \infty$, $\theta_n \to \mathcal{T}(\chi)$ and $(q(\theta_n), x(\theta_n)) \to (\hat{q}, \hat{x})$. The ω limit set $S_\chi \subset \mathbf{Q} \times \mathbf{X}$ is the set of all ω limit points of an execution χ.*

The following proposition establishes a relation between ω limit set and invariant set.

Lemma 6 *Consider a hybrid automaton H that satisfies the conditions of Theorem 1. Then for any execution $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty$, if $x(\cdot)$ is bounded, the ω limit set S_χ is (i) nonempty, (ii) compact, and (iii) invariant. Furthermore, (iv) for all $\epsilon > 0$ there exists $T \in \tau$ such that $d((q(t), x(t)), S_\chi) < \epsilon$ for all $t \geq T$.*

Proof: The proofs of (i) and (ii) are similar to the corresponding proofs for continuous dynamical systems [25]. We include them here for completeness.

(i) Recall that $\mathbf{Q} \times \mathbf{X}$ is a metric space. If x is bounded, χ is contained in a compact subset of that space. Therefore, it has a limit point, by the Bolzano–Weierstrass property [21]. Hence, $S_\chi \neq \emptyset$.

(ii) To show that S_χ is compact, it suffices to show that it is closed, since x is assumed to be bounded. Consider an arbitrary $(\hat{q}, \hat{x}) \in S_\chi^c$. Then there exists an open neighborhood, U, of $(\hat{q}, \hat{x})$ and a $T \in \tau$, such that $(q(t), x(t)) \notin U$ for all $t > T$. Therefore, $U \cap S_\chi = \emptyset$ and, since

$(\hat{q}, \hat{x})$ is arbitrary, S_χ^c is open.

(iii) To show that S_χ is invariant, take an arbitrary $(\hat{q}, \hat{x}) \in S_\chi$, it suffices to show that for all $\bar{\chi} = (\bar{\tau}, \bar{q}, \bar{x}) \in \mathcal{E}_H(\hat{q}, \hat{x})$ with $\bar{\tau} = \{\bar{I}_i\}_{i=0}^N$ and $\bar{\Delta}_i = \bar{\tau}_i' - \bar{\tau}_i$, we have $(\bar{q}(\bar{\tau}_N'), \bar{x}(\bar{\tau}_N')) \in S_\chi$. If the hybrid automaton is blocking at $(\bar{q}(\bar{\tau}_N'), \bar{x}(\bar{\tau}_N'))$, or it will jump between discrete states with no continuous evolution, the property is trivially satisfied. Then we only need to consider the case $\mathcal{T}(\bar{\chi}) > 0$. Note that since $(\hat{q}, \hat{x}) \in S_\chi$, there exists a sequence $\{\theta_n\}_{n=0}^\infty$ with $\theta_n \in \tau$ such that as $n \to \infty$, $\theta_n \to \mathcal{T}(\chi)$ and $(q(\theta_n), x(\theta_n)) \to (\hat{q}, \hat{x})$. The last condition implies that there exists $N_1 > 0$ such that $q(\theta_n) = \hat{q}$ for all $n > N_1$. From the continuous dependence on initial conditions in Section 3, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $(\tilde{q}_0, \tilde{x}_0)$ with $d((\tilde{q}_0, \tilde{x}_0), (\hat{q}, \hat{x})) < \delta$, there exists $T(\tilde{x}_0) > 0$, such that the execution $\tilde{\chi} = (\tilde{\tau}, \tilde{q}, \tilde{x}) \in \mathcal{E}_H(\tilde{q}_0, \tilde{x}_0)$ with $\tilde{\tau} = \{\tilde{I}_i\}_{i=0}^N$, $\tilde{\tau}_N' = T(\tilde{x}_0)$ and $\tilde{\Delta}_i = \tilde{\tau}_i' - \tilde{\tau}_i$ satisfies

- $|\mathcal{T}(\tilde{\chi}) - \mathcal{T}(\bar{\chi})| < \epsilon$, and
- $d((\tilde{q}(\tilde{\tau}_N'), \tilde{x}(\tilde{\tau}_N')), (\bar{q}(\bar{\tau}_N'), \bar{x}(\bar{\tau}_N'))) < \epsilon$.

For this particular δ, there exists $N_2 > N_1$ such that for all $n > N_2$, $d((q(\theta_n), x(\theta_n)), (\hat{q}, \hat{x})) < \delta$. Therefore, the execution $\chi^n = (\tau^n, q^n, x^n) \in \mathcal{E}_H(q(\theta_n), x(\theta_n))$ with $\tau^n = \{I_i^n\}_{i=0}^N$, $\tau_N'^n = T(x(\theta_n))$, and $\Delta_i^n = \tau_i'^n - \tau_i^n$ satisfies

- $|\mathcal{T}(\chi^n) - \mathcal{T}(\bar{\chi})| < \epsilon$, and
- $d((q^n(\tau_N'^n), x^n(\tau_N'^n)), (\bar{q}(\bar{\tau}_N'), \bar{x}(\bar{\tau}_N'))) < \epsilon$.

By determinism and time invariant properties,

$$(q^n(\tau_N'^n), x^n(\tau_N'^n)) = (q(\theta_n + \tau_N'^n - \tau_0^n), x(\theta_n + \tau_N'^n - \tau_0^n)).$$

Summarizing, there exists a sequence $\{\theta_n'\} = \{\theta_n + \tau_N'^n - \tau_0^n\}_{n=0}^\infty$ such that $\theta_n' \to \mathcal{T}(\chi) + \mathcal{T}(\bar{\chi})$, and $(q(\theta_n'), x(\theta_n')) \to (\bar{q}(\bar{\tau}_N'), \bar{x}(\bar{\tau}_N'))$. Therefore, $(\bar{q}(\bar{\tau}_N'), \bar{x}(\bar{\tau}_N')) \in S_\chi$.

(iv) The proof is similar to the continuous case [22, 16]. Assume, for the sake of contradiction, that there exists $\epsilon > 0$ such that for all $T \in \tau$, $d((q(t), x(t)), S_\chi) \geq \epsilon$ for some $t \geq T$.

Then there exists a sequence $\{\theta_n\}_{n=0}^{\infty}$ with $\theta_n \in \tau$ such that as $n \to \infty$, $\theta_n \to \tau_\infty$ and $d(q(\theta_n), x(\theta_n)), S_\chi) \geq \epsilon$. This sequence is bounded, therefore, by the Bolzano–Weierstrass property it has a limit point $(\hat{q}, \hat{x})$. Moreover, $(\hat{q}, \hat{x}) \in S_\chi$, by the definition of S_χ. But, by construction of the sequence, $d((\hat{q}, \hat{x}), S_\chi) \geq \epsilon$, which is a contradiction. ∎

LaSalle's principle is a useful tool when studying the stability of conventional, continuous dynamical systems. Lemma 6 allows us to extend this tool to hybrid systems.

Theorem 2 (LaSalle's Invariance Principle) *Consider a hybrid automaton H that satisfies the conditions of Theorem 1. Assume there exists a compact invariant set $\Omega \subset \mathbf{Q} \times \mathbf{X}$, and define $\Omega_1 = \Omega \cap \mathrm{Out}^c$ and $\Omega_2 = \Omega \cap \mathrm{Out}$. Furthermore, assume there exists a continuous function $V : \Omega \to \mathbb{R}$, such that*

- *for all $(q, x) \in \Omega_1$, V is continuously differentiable with respect to x and $L_f V(q, x) \leq 0$; and*
- *for all $(q, x) \in \Omega_2$, $V(\mathrm{Reset}(q, x)) \leq V(q, x)$.*

Define $S_1 = \{(q, x) \in \Omega_1 : \; L_f V(q, x) = 0\}$ and $S_2 = \{(q, x) \in \Omega_2 : \; V(\mathrm{Reset}(q, x)) = V(q, x)\}$ and let M be the largest invariant subset of $S_1 \cup S_2$. Then, for all $(q_0, x_0) \in \Omega$ the execution $(\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$ approaches M as $t \to \tau_\infty$.

Proof: Consider an arbitrary state $(q_0, x_0) \in \Omega$ and let $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$. Since Ω is invariant, $(q(t), x(t)) \in \Omega$ for all $t \in \tau$. Since Ω is compact and V is continuous, $V(q(t), x(t))$ is bounded from below. Moreover, $V(q(t), x(t))$ is a non-increasing function of $t \in \tau$ (recall that τ is fully ordered), so therefore the limit $c = \lim_{t \to \tau_\infty(\chi)} V(q(t), x(t))$ exists.

Since Ω is bounded, x is bounded, and therefore the ω limit set S_χ is nonempty. Moreover, since Ω is closed, $S_\chi \subset \Omega$. By definition, for any $(\hat{q}, \hat{x}) \in S_\chi$, there exists a sequence $\{\theta_n\}_{n=0}^{\infty}$ with $\theta_n \in \tau$ such that as $n \to \infty$, $\theta_n \to \tau_\infty$ and $(q(\theta_n), x(\theta_n)) \to (\hat{q}, \hat{x})$. Moreover, $V(\hat{q}, \hat{x}) = V(\lim_{n\to\infty}(q(\theta_n), x(\theta_n)) = \lim_{n\to\infty} V(q(\theta_n), x(\theta_n)) = c$, by continuity of V. Since S_χ is invariant (Lemma 6), it follows that $L_f V(\hat{q}, \hat{x}) = 0$ if $(\hat{q}, \hat{x}) \notin \mathrm{Out}$, and $V(\mathrm{Reset}(\hat{q}, \hat{x})) = V(\hat{q}, \hat{x})$ if $(\hat{q}, \hat{x}) \in \mathrm{Out}$. Therefore, $S_\chi \subset S_1 \cup S_2$, which implies that $S_\chi \subset M$ since S_χ is invariant. Moreover, by (iv) in Lemma 6, the execution χ approaches S_χ, and hence M, as $t \to \tau_\infty$. ■

Note that since the class of invariant set is closed under arbitrary unions a unique largest invariant set M exists.

5 Equilibrium Set and Lyapunov Stability

This section is concerned with the stability of hybrid automata. In conventional dynamical systems, equilibrium point is an important concept in the Lyapunov stability theory. All the solutions starting nearby some equilibrium points stay nearby forever, and at the equilibrium point there is no more motion. Some researchers have studied the extension of Lyapunov theory to hybrid systems [6, 26, 15]. However, the idea of equilibrium points cannot be generalized to hybrid systems immediately. In hybrid systems, when the continuous trajectory converges to some points in the state space, those points are not necessarily the equilibrium points of the corresponding vector field, as illustrated by Example 1. Here we introduce equilibrium set as a counterpart of equilibrium point for hybrid systems. Equilibrium set will be shown to play a similar role as equilibrium points in Lyapunov stability analysis of hybrid systems.

Definition 10 (Equilibrium Set) *A finite collection of states $S = \{(\hat{q}_j, \hat{x}_j)\}_{j=1}^N$ is an equilibrium set of a hybrid automaton if every $\hat{q}_j$ belongs to a cycle $\{(q_1, q_2), (q_2, q_3), \cdots, (q_m, q_1)\}$, where $q_i \in \{\hat{q}_j\}_{j=1}^N$, and for every q_i there exists some x_i such that $(q_i, x_i) \in S$, and*

- $(q_{i+1}, x_{i+1}) \in \mathrm{Reset}(q_i, x_i)$, *for all* $i = 1, 2, \ldots, m-1$,
- $(q_1, x_1) \in \mathrm{Reset}(q_m, x_m)$.

For the ease of notation, we let $(q_0, x_0) = (q_m, x_m)$ and $(q_{m+1}, x_{m+1}) = (q_1, x_1)$. We know that every hybrid automaton can be associated with a directed graph $(\mathbf{Q}, E)$. If this graph contains no cycle, obviously the hybrid automaton has no equilibrium set.

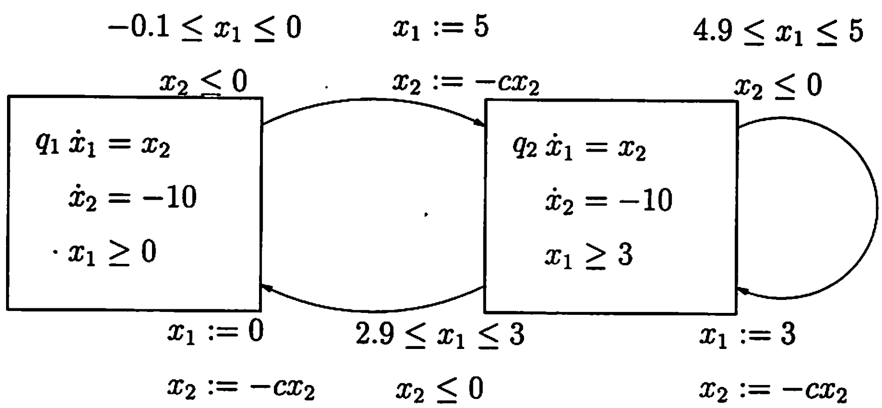

Figure 1: Example 1

Remark 5 An equilibrium set is an invariant set. If some states of an equilibrium set is reachable, then the hybrid automaton accepts a Zeno execution.

Example 1

Consider the hybrid automaton in Figure 1, we have

$$R(q_1, q_2, x) = Ax + b_1, \quad R(q_2, q_2, x) = Ax + b_2, \quad R(q_2, q_1, x) = Ax + b_3,$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}, \quad b_1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned} R_1(x_1) &= A^3 x_1 + A^2 b_1 + A b_2 + b_3 = x_1 \\ R_2(x_2) &= A^3 x_2 + A^2 b_2 + A b_3 + b_1 = x_2 \\ R_3(x_3) &= A^3 x_3 + A^2 b_3 + A b_1 + b_2 = x_3, \end{aligned}$$

which yields the unique solution as

$$x_1 = (0,0)^T, x_2 = (5,0)^T, x_3 = (3,0)^T.$$

And it is easy to check that

$$x_1 \in G(q_1, q_2), \quad x_2 \in G(q_2, q_2), \quad x_3 \in G(q_2, q_1).$$

Therefore, $\{(q_1, x_1), (q_2, x_2), (q_2, x_3)\}$ is an equilibrium set.

Definition 11 (Stable Equilibrium Set) *An equilibrium set $S = \{(\hat{q}_i, \hat{x}_i)\}_{i=1}^N$ of a hybrid automaton H is stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all (q_0, x_0) with $d((q_0, x_0), S) < \delta$, for all $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$ and for all $t \in \tau$, $d((q(t), x(t)), S) < \epsilon$.*

In [6] multiple Lyapunov functions was introduced as a tool for analyzing Lyapunov stability of hybrid systems. Here we borrow this idea and apply it to equilibrium set to get the following theorem. In addition, we improve the theorem in [6] by getting rid of the restriction on reset map and the assumption on non-Zenoness.

Theorem 3 *Consider a hybrid automaton H with equilibrium set $S = \{(\hat{q}_j, \hat{x}_j)\}_{j=1}^N$, a collection of open sets $\{D_j\}_{j=1}^N$ with $\hat{x}_j \in D_j \subseteq I(\hat{q}_j)$, and functions $V(\hat{q}_j, \cdot) : D_j \to \mathbb{R}$ continuously differentiable in x such that for all j:*

1. *$V(\hat{q}_j, \hat{x}_j) = 0$,*
2. *$V(\hat{q}_j, x) > 0$ for all $x \in D_j \setminus \{\hat{x}_j\}$, and*
3. *$\frac{\partial}{\partial x} V(\hat{q}_j, x) f(\hat{q}_j, x) \leq 0$ for all $x \in D_j$.*

If all the reset relations are continuous functions and for all the execution $(\tau, q, x) \in \mathcal{E}_H^\infty$ and for all j, the sequence $\{V(q(\tau_k), x(\tau_k)) : q(\tau_k) = \hat{q}_j\}$ is non-increasing, then S is a stable equilibrium set of H.

Proof: The proof is similar to that in [6]. For simplicity, we do the proof only for the case $N = 2$. The proof is similar if $N > 2$. Given $\epsilon > 0$, choose $R \in (0, \epsilon)$ such that $B(\hat{x}_j, R) \subseteq D_j$ for all j, where $B(\hat{x}_j, R) = \{x \in \mathbb{R}^n : \|x - \hat{x}_j\| \leq R\}$. Set $c_j(R) = \min_{x \in \partial B(\hat{x}_j, R)} V(q_j, x)$, $\Omega_r^j = \{x \in D_j : V(q_j, x) \leq r\}$. Pick $\alpha_j \in (0, c_j(R))$, for q_1, choose $\beta_1 \in (0, \alpha_1)$ such that $R(q_1, q_2, \Omega_{\beta_1}^1) \subseteq \Omega_{\alpha_2}^2$; for q_2, choose $\beta_2 \in (0, \alpha_2)$ such that $R(q_2, q_1, \Omega_{\beta_2}^2) \subseteq \Omega_{\alpha_1}^1$. Pick $\delta > 0$ such that $B(\hat{x}_1, \delta) \subseteq \Omega_{\beta_1}^1$ and $B(\hat{x}_2, \delta) \subseteq \Omega_{\beta_2}^2$. Assume without loss of generality that $q(\tau_0) = q_1$. Take any execution (τ, q, x) with $\|x(\tau_0) - \hat{x}_1\| < \delta$. By a continuous Lyapunov argument and the sequence $\{V(q(\tau_k), x(\tau_k)) : q(\tau_k) = \hat{q}_1\}$ is non-increasing, $x(t) \subseteq \Omega_{\beta_1}^1$ for all t until some discrete transition (q_1, q_2) takes place at $t = \tau_{i-1}'$. And then $x(\tau_i) = R(q_1, q_2, x(\tau_{i-1}')) \in \Omega_{\alpha_2}^2$. By same argument again, $x(t) \subseteq \Omega_{\alpha_2}^2$ for all t until discrete transition (q_2, q_1) takes place at $t = \tau_{j-1}'$. By assumption, $x(\tau_j) = R(q_2, q_1, x(\tau_{j-1}'))$, and $V(q_1, x(\tau_j)) \leq V(q_1, x(\tau_0))$. Therefore $x(\tau_j) \in \Omega_{\beta_1}^1$, and $x(t) \subseteq \Omega_{\beta_1}^1$ for all its stay in q_1. The claim follows by induction.

■

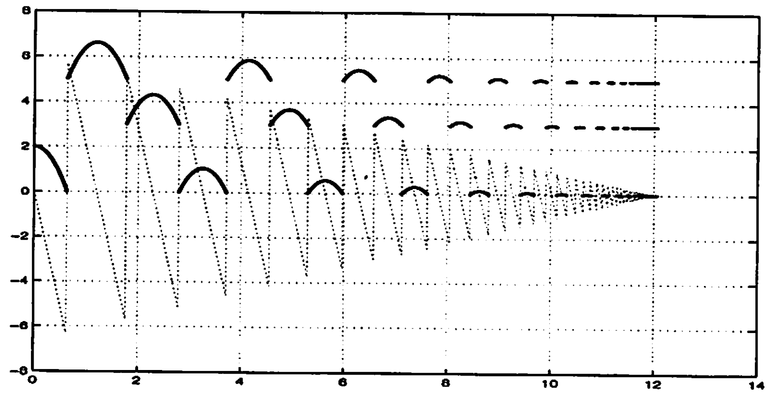

Figure 2: Simulation for Example 1: x_1 solid, x_2 dotted

6 Zeno Hybrid Automaton

Zeno hybrid automata accept executions with infinitely many discrete transitions within a finite time interval. Such systems are hard to analyze and simulate in a way that gives constructive information about the behavior of the real system. It is therefore important to be able to determine if a model is Zeno and in applicable cases remove Zenoness. These problems have been discussed in [13, 14]. In this section, some further characterization of Zeno executions are made. Recall that an infinite execution χ is Zeno if $\mathcal{T}(\chi) = \sum_{i=0}^{\infty}(\tau_i' - \tau_i)$ is bounded, and we also use τ_∞ to denote the Zeno time of execution χ whenever the context is clear.

Definition 12 (Zeno Hybrid Automaton) *A hybrid automaton H is Zeno if there exists* $(q_0, x_0) \in \text{Init}$ *such that all executions in* $\mathcal{E}_H^\infty(q_0, x_0) \neq \emptyset$ *are Zeno.*

Example 2

The hybrid automaton in Figure 1 is Zeno. This is easily checked by explicitly deriving the time intervals $\tau_i' - \tau_i$, which in this case gives a converging geometric series. Figure 2 shows an execution accepted by the hybrid automaton.

Our interest is to study the properties of Zeno execution. It is clear that Zeno execution is determined by the vector fields, reset relations as well as the guards. In Example 1, should the reset map of x_2 be replaced by $x_2 := \dfrac{x_2}{dx_2 - 1}$, where $d = 1/\sqrt{20x_1(\tau_0)}$, it's easy to verify that $\{\Delta_i\}_{i=1}^{\infty}$ has the same diverging rate as $\{\frac{1}{i}\}_{i=1}^{\infty}$, hence the hybrid automaton will not have any Zeno execution.

Now we will introduce Zeno state set to investigate the properties of Zeno hybrid automaton.

Definition 13 (Zeno State Set) *The ω limit point of a Zeno execution is called its Zeno state.*

We use $Z_\infty \subset \mathbf{Q} \times \mathbf{X}$ to denote the set of Zeno states. In other words, Z_∞ consists of all cluster points of sequence $\{(q(\theta_n), x(\theta_n))\}_{n=0}^{\infty}$ with $\theta_n \in \tau$ such that $\theta_n \to \tau_\infty$ as $n \to \infty$. In Example 1, the Zeno state set is $\{(q_1, (0,0)^T), (q_2, (3,0)^T), (q_2, (5,0)^T)\}$. One straightforward necessary condition for the existence of Zeno executions is that $(\mathbf{Q}, E)$ contains a cycle. We write $\mathbf{Q}_\infty$ for the discrete part of Zeno states set:

$$\mathbf{Q}_\infty = \{q \in \mathbf{Q} : \exists x \in \mathbf{X}, (q, x) \in Z_\infty\},$$

and E_∞ for the edges in $\mathbf{Q}_\infty$.

We will give some examples to illustrate the notion of Zeno state set. In Example 1, we have already observed that one discrete Zeno state may correspond to multiple continuous Zeno states. Besides, the Zeno state set could be a finite, a countable, or even an uncountable set.

Example 3

Consider a Zeno execution with $Z_\infty = \{(\hat{q}, \hat{x})\}$, modify this hybrid automaton by extending three extra continuous states (x_e, x_f, x_g) with

$$\dot{x}_e = 0, \quad \dot{x}_f = 0, \quad \dot{x}_g = 0,$$

and reset maps

$$\begin{pmatrix} x_e \\ x_f \end{pmatrix} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_e \\ x_f \end{pmatrix},$$
$$x_g := \tan(\frac{\pi}{2} x_f),$$

where $\theta/2\pi$ is irrational, and the initial condition $x_e(\tau_0) = 1$, $x_f(\tau_0) = 0$, $x_g(\tau_0) = 0$. Then the Zeno state set is

$$Z_\infty = \{\hat{q}\} \times \{(\hat{x}, x_e, x_f, x_g) : (x_e, x_f)^T \in S^1, x_g \in \mathbb{R}\}.$$

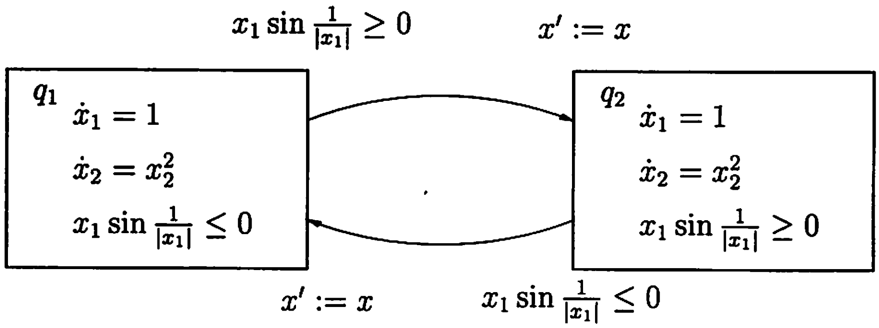

Figure 3: Example 4

A Zeno hybrid automaton may have Zeno executions with no Zeno state. Note that the vector field in the following example is not Lipschitz continuous.

Example 4

Consider the hybrid automaton H in Figure 3. The execution of H with initial state $(q_1, (-1, 1)^T)$ exhibits an infinite number of discrete transitions by $\tau_\infty = 1$, and $x_2(t) = 1/(1-t)$, for all $t \in [0, \tau_\infty)$. However, for all $\{\theta_i\}_{i=0}^\infty$, $\theta_i \to 1$, the sequence $\{x_2(\theta_i)\}_{i=0}^\infty$ is strictly monotonic increasing and unbounded. Therefore, H has no Zeno state.

Example 5

Similar to Example 3, if we have a Zeno execution with $Z_\infty = \{(\hat{q}, \hat{x})\}$, augment an extra continuous state x_e with trivial continuous dynamics $\dot{x}_e = 0$, reset map $x_e(\tau_{i+1}) = 2x_e(\tau_i')$ and the initial condition $x_e(0) = 1$. For all $\{\theta_i\}_{i=0}^\infty$, $\theta_i \to 1$, $\{x_e(\theta_i)\}_{i=0}^\infty$ has no cluster point. Evidently, the modified hybrid automaton has no Zeno state.

In Example 4, the Zeno execution has no Zeno state because f is not Lipschitz continuous, which yields the finite escape time. And in Example 5, it is due to the exploding reset map. It turns out that both continuous dynamics and reset map are crucial to the Zeno state.

By definition the discrete part of the Zeno state will be visited infinitely often, but the discrete state being visited infinitely often is not necessarily in $\mathbf{Q}_\infty$. This can be observed by modifying the vector field and reset map in Example 4 as:

- $f(q, x) = (1, 0)^T$, for all $(q, x) \in \mathbf{Q} \times \mathbf{X}$;
- $R(q_1, q_2, x) = (x_1, \exp\frac{1}{|x_1|\pi})^T$, $R(q_2, q_1, x) = (x_1, 1)$.

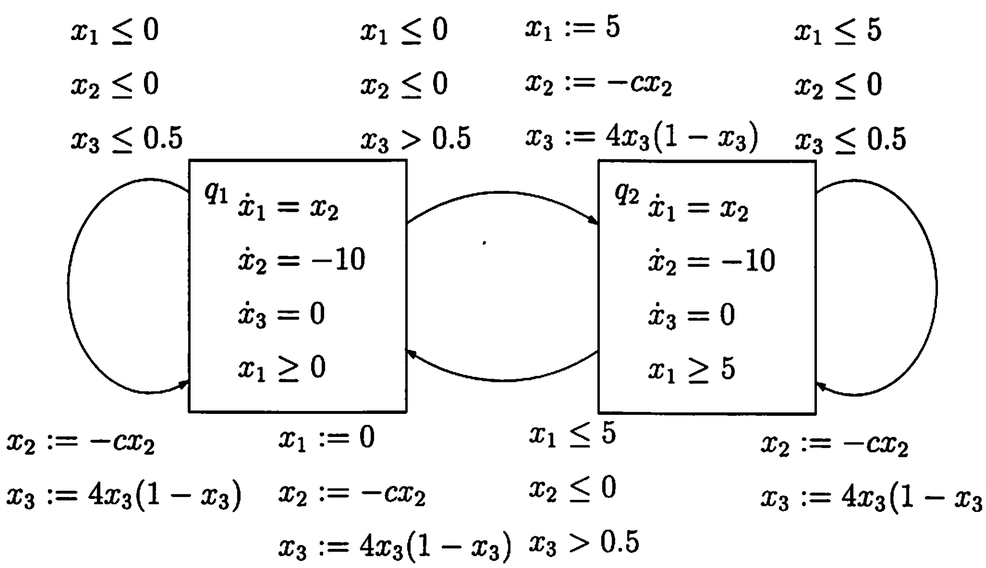

Figure 4: Example 6

The execution of H with initial state $(q_1, (-1,1)^T)$ is a Zeno execution with $\tau_\infty = 1$, and $(q_1, (0,1)^T)$ is the only Zeno state. The discrete state q_2 is visited infinitely often, but it is not in $\mathbf{Q}_\infty$ since x_2 blows up in state q_2, so here we actually have $E_\infty = \emptyset$. In Lemma 8, we will give conditions under which the discrete state being visited infinitively often is in $\mathbf{Q}_\infty$.

As pointed out before, in most cases discrete execution sequence q has at least one loop which corresponds to a cycle in directed graph $(\mathbf{Q}, E)$. Let us consider a hybrid automaton with only two discrete states q_1 and q_2. If denoting q_1 as 0 and q_2 as 1, the discrete execution sequence can be described as a binary number $0.0110010\ldots$ in $[0,1]$. One interesting question is whether this number is rational or irrational. Most of the hybrid systems behave like a rational number, which means the discrete execution sequence will repeat itself periodically. However, there exist hybrid systems that do not periodically jump between the two discrete states.

Example 6

Consider the hybrid automaton in Figure 4 with Init $= \{q_1\} \times \{x \in \mathbb{R}^3 : 0 \leq x_1 < 5, x_2 = 0, x_3 = 0.9\}$. A simulation is presented in Figure 5, where x_1 and x_2 are shown. The reset map of x_3 is the logistic map with initial condition $x_3(\tau_0) = 0.9$, and the iteration of this map will take any value in $(0,1)$ [22].

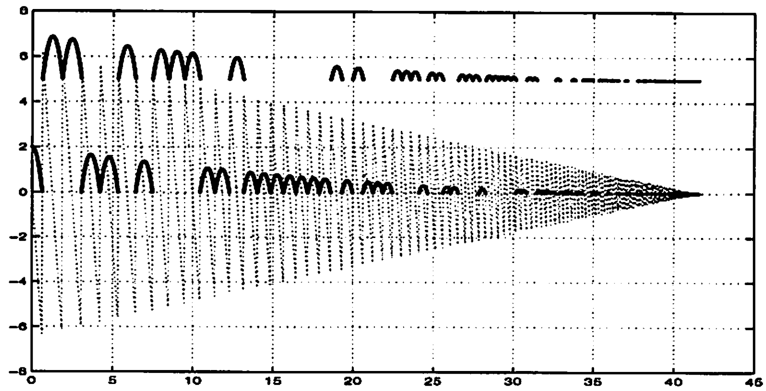

Figure 5: Simulation for Example 6: x_1 solid, x_2 dotted

Our objective is to study the Zeno hybrid automaton via Zeno state. The hybrid trajectory will approach the Zeno state set when time tends to Zeno time. We will show that under certain conditions the Zeno state set is the equilibrium set of the hybrid automaton. Before doing that, let us first introduce some notions about reset relations.

Reset relation $R(q, q', \cdot)$ is called *non-expanding* if there exists some $\delta \in [0, 1]$ such that for all $x \in G(q, q')$, all $x' \in R(q, q', x)$,

$$\|x'\| \leq \delta \|x\|;$$

non-contracting if there exists some $\delta > 1$ such that for all $x \in G(q, q')$, all $x' \in R(q, q', x)$,

$$\|x'\| \geq \delta \|x\|;$$

and *contracting* if there exists some $\delta \in [0, 1)$ such that for all $x, y \in G(q, q')$, $x' \in R(q, q', x)$, $y' \in R(q, q', y)$,

$$\|x' - y'\| \leq \delta \|x - y\|.$$

Note that contracting reset relation is actually a function: if we pick up $x = y$, then $R(q, q', x) = R(q, q', y)$.

Lemma 7 (Rate of Growth/Decay) *Consider a hybrid automaton H, there exists some $c > 0$ such that for all execution $\chi = (\tau, q, x) \in \mathcal{E}_H$, all $t \in \tau$,*

i) if for all $(q, q') \in E$, $R(q, q', \cdot)$ is non-expanding, then

$$\|x(t)\| \leq (\|x(\tau_0)\| + 1)e^{c(t-\tau_0)} - 1;$$

ii) if for all $(q,q') \in E$, $R(q,q',\cdot)$ is non-contracting, then

$$\|x(t)\| \geq \left(\|x(\tau_0)\| + 1\right)e^{-c(t-\tau_0)} - 1.$$

Proof: There exists some $c > 0$ such that for all i, all $t \in [\tau_i, \tau_i']$,

$$\|f(q(\tau_i), x(t))\| \leq c\left(\|x(t)\| + 1\right).$$

Since $\|x\|^2 = x^T x$ it follows that

$$\left|\frac{d\|x\|^2}{dt}\right| = 2\|x\|\left|\frac{d\|x\|}{dt}\right| = 2|x^T\dot{x}| \leq 2\|x\|\|\dot{x}\|$$

so that

$$\left|\frac{d\|x\|}{dt}\right| \leq \|\dot{x}\| = \|f(q,x)\| \leq c(\|x\| + 1),$$

$$-c\left(\|x\| + 1\right) \leq \frac{d\|x\|}{dt} \leq c\left(\|x\| + 1\right).$$

Applying Bellman-Gronwall lemma [22] twice, we have

$$\left(\|x(\tau_i)\| + 1\right)e^{-c(t-\tau_i)} \leq \|x(t)\| + 1 \leq \left(\|x(\tau_i)\| + 1\right)e^{c(t-\tau_i)}, \quad t \in [\tau_i, \tau_i'].$$

i) Since by non-expanding assumption, $\|x(\tau_i)\| \leq \|x(\tau_{i-1}')\|$, which yields

$$\begin{aligned}\|x(t)\| + 1 &\leq \left(\|x(\tau_{i-1}')\| + 1\right)e^{c(t-\tau_i)} \\ &\leq \left(\|x(\tau_{i-1})\| + 1\right)e^{c(\tau_{i-1}'-\tau_{i-1})}e^{c(t-\tau_i)}.\end{aligned}$$

Proceeding further,

$$\|x(t)\| + 1 \leq \left(\|x(\tau_0)\| + 1\right)e^{c(t-\tau_0)},$$

that is,

$$\|x(t)\| \leq \left(\|x(\tau_0)\| + 1\right)e^{c(t-\tau_0)} - 1.$$

The proof of ii) is similar to i). ■

In the conventional systems dynamics analysis, Lipschitz continuity assumption on the vector field excludes the possibility for finite escape time. For the hybrid system, when the reset relation is non-expanding, we can conclude that hybrid execution has no finite escape time. An important implication of this lemma is, for Zeno hybrid automaton, since $t \in [\tau_0, \tau_\infty]$, then $x(\cdot)$ is bounded on $[\tau_0, \tau_\infty]$. Hence for all $q_i \in \mathbf{Q}$, all $t \in [\tau_0, \tau_\infty]$, there exists some $K > 0$ such that $\|f(q_i, x(t))\| \leq K_i$. That is, the vector field of the Zeno hybrid automaton is bounded along the execution, provided the reset relation is non-expanding.

Now we are ready to give the relationship between Zeno state set and equilibrium set of the hybrid system.

Proposition 1 *Consider a hybrid automaton with Zeno execution $\chi = (\tau, q, x)$. If it has finite Zeno state set $Z_\infty = \{(q_k, x_k)\}_{k=1}^m$, and $(\mathbf{Q}_\infty, E_\infty)$ is a cycle graph, and for all $e \in E$, $G(e)$ is closed and $R(e, \cdot)$ is continuous, then Z_∞ is equilibrium set of the hybrid automaton.*

Proof: Take (q_1, x_1), there exists a sequence $\{\theta_i\}_{i=0}^\infty$, $\theta_i \to \tau_\infty$ such that $q(\theta_i) \to q_1$ and $x(\theta_i) \to x_1$. Since the discrete states are finite, when i large enough, $q(\theta_i) = q_1$. Suppose $\theta_i \in [\tau_{ni}, \tau'_{ni}]$, by Lipschitz continuity vector field assumption, when $i \to \infty$,

$$\|x(\tau_{ni})' - x_1\| \leq \|x(\tau_{ni})' - x(\theta_i)\| + \|x(\theta_i) - x_1\| \to 0,$$

and $x(\tau'_{ni}) \in G(q_1, q_{ni})$ for some $q_{ni} \in \mathbf{Q}_\infty$. Let q_2 be the accumulation point of $\{q_{ni}\}_{i=0}^\infty$, then $\{x(\tau'_{ni})\}_{i=0}^\infty$ has a subsequence in the guard $G(q_1, q_2)$. For the ease of notation, assume it is just $x(\tau'_{ni})$ itself. Since all the guards are closed and $x(\tau'_{ni}) \to x_1$ as $i \to \infty$, we have $x_1 \in G(q_1, q_2)$. And also,

$$x(\tau_{ni+1}) = R(q_1, q_2, x(\tau'_{ni})),$$

$$x(\tau_{ni+1}) \to x_2,$$

hence $x_2 = R(q_1, q_2, x_1)$. Applying the same argument again and again and since there are only finite many discrete states, we can conclude that Z_∞ is equilibrium of the hybrid automaton. ∎

When $x(\cdot)$ is bounded, Weierstrass theorem says that $\{q(\theta_i), x(\theta_i)\}_{i=0}^\infty$, $\theta_i \in [\tau_i, \tau'_i]$, has at least one cluster point. Therefore when the Zeno hybrid automaton has the non-expanding reset map, it has at least one Zeno state. Specifically, we are interested in the case that Zeno state set is a finite set in the hybrid state space.

Proposition 2 *Consider a hybrid automaton with $R(q, q', x) = \{x\}$ for all $(q, q') \in E$. For* *every Zeno execution* $\chi = (\tau, q, x)$, *it holds that* $Z_\infty = \mathbf{Q}_\infty \times \{\hat{x}\}$ *for some* $\mathbf{Q}_\infty \subseteq \mathbf{Q}$ *and* $\hat{x} \in \mathbf{X}$.

Proof: For all $\{\theta_i\}_{i=0}^\infty$, $\theta_i \to \tau_\infty$, suppose $\theta_i \in [\tau_{ni}, \tau'_{ni}]$, $ni \to \infty$ as $i \to \infty$, we have

$$\begin{aligned} x(\theta_i) &= x(\tau_{ni}) + \int_{\tau_{ni}}^{\theta_i} f\big(q(\tau_{ni}), x(\tau)\big)\, d\tau \\ &= x(\tau_{ni}) + (\theta_i - \tau_{ni})\left(f_1(q(\tau_{ni}), x(\xi_{ni}^1)), \ldots, f_n(q(\tau_{ni}), x(\xi_{ni}^n))\right)^T \end{aligned}$$

for some $\xi_{ni}^1, \ldots, \xi_{ni}^n \in [\tau_{ni}, \tau'_{ni}]$. Hence for all $k > l \geq 0$,

$$\begin{aligned} x(\theta_k) = x(\theta_l) &+ (\tau'_{nl} - \theta_l)\left(f_1(q(\tau_{nl}), x(\xi_{nl}^1)), \ldots, f_n(q(\tau_{nl}), x(\xi_{nl}^n))\right)^T \\ &+ \sum_{i=nl+1}^{nk-1} (\tau'_i - \tau_i)\left(f_1(q(\tau_i), x(\xi_i^1)), \ldots, f_n(q(\tau_i), x(\xi_i^n))\right)^T \\ &+ (\theta_k - \tau_{nk})\left(f_1(q(\tau_{nk}), x(\xi_{nk}^1)), \ldots, f_n(q(\tau_{nk}), x(\xi_{nk}^n))\right)^T \end{aligned}$$

which gives that

$$\|x(\theta_k) - x(\theta_l)\| \leq K \sum_{i=nl}^{nk} (\tau'_i - \tau_i).$$

By the fact that $\sum_{i=0}^\infty (\tau'_i - \tau_i) < \infty$, we know that $\{x(\theta_i)\}_{i=0}^\infty$ is a Cauchy sequence. The space $\mathbf{X} = \mathbb{R}^n$ is complete, so the sequence has a limit $\hat{x} = \lim_{i\to\infty} x(\theta_i)$. For any other sequence $\{\alpha_i\}_{i=0}^\infty$ with $\alpha_i \in \tau$, $\alpha_i \to \tau_\infty$, from the same argument as above, it is clear that $\|x(\theta_i) - x(\alpha_i)\| \to 0$ as $i \to \infty$, thus $\hat{x}$ is the unique limit point. Therefore, $\hat{x}$ is the unique limit point. ■

The merit of the above proposition is that as far as R is an identity map, the continuous part of Zeno states set is a singleton. Further, this singleton will be just the origin if all the reset relation is contracting and has 0 as fixed point.

Proposition 3 *Consider a hybrid automaton with* $R(q, q', \cdot)$ *contracting and* $R(q, q', 0) = 0$ *for all* $(q, q') \in E$, *then for every Zeno execution* $\chi = (\tau, q, x)$, *it holds that* $Z_\infty = \mathbf{Q}_\infty \times \{0\}$ *for some* $\mathbf{Q}_\infty \subseteq \mathbf{Q}$.

Proof: For all $\{\theta_i\}_{i=0}^\infty$, $\theta_i \to \tau_\infty$, suppose $\theta_i \in [\tau_{ni}, \tau'_{ni}]$, $ni \to \infty$ as $i \to \infty$, we have

$$\begin{aligned} \|x(\theta_i)\| &\leq \|x(\tau_{ni})\| + \left\|\int_{\tau_{ni}}^{\theta_i} f\big(q(\tau_{ni}), x(\tau)\big)\, d\tau\right\| \\ &\leq \|x(\tau_{ni})\| + K(\tau'_{ni} - \tau_{ni}). \end{aligned}$$

Using the fact that $\|x(\tau_{ni})\| \leq \delta \|x(\tau'_{ni-1})\|$, it follows that

$$\begin{aligned}\|x(\theta_i)\| &\leq \delta \|x(\tau'_{ni-1})\| + K(\tau'_{ni} - \tau_{ni}) \\ &= \delta \|x(\tau_{ni-1}) + \int_{\tau_{ni-1}}^{\tau'_{ni-1}} f(q(\tau_{ni-1}), x(\tau))\, d\tau\| + K(\tau'_{ni} - \tau_{ni}) \\ &\leq \delta \|x(\tau_{ni-1})\| + K\delta(\tau'_{ni-1} - \tau_{ni-1}) + K(\tau'_{ni} - \tau_{ni}).\end{aligned}$$

By induction,

$$\|x(\theta_i)\| \leq \delta^{ni} \|x(\tau_0)\| + K \sum_{m=0}^{ni} \delta^{ni-m}(\tau'_m - \tau_m),$$

and

$$\sum_{ni=0}^{\infty} K \sum_{m=0}^{ni} \delta^{ni-m}(\tau'_m - \tau_m) = K \sum_{m=0}^{\infty}(\tau'_m - \tau_m) \sum_{ni=0}^{\infty} \delta^{ni} = \frac{K\tau_\infty}{1-\delta} < \infty.$$

Therefore, $K \sum_{m=0}^{ni} \delta^{ni-m}(\tau'_m - \tau_m) \to 0$ as $ni \to \infty$, which yields that $\|x(\theta_i)\| \to 0$ as $i \to \infty$, hence $\hat{x} = 0$. ∎

In the above proposition, 0 is the common fixed point of all the reset function, and it is also the continuous part of the set of Zeno states. This is not a coincidence, actually we have the following general result which guarantees that the set of Zeno states will be just $Z_\infty = \mathbf{Q}_\infty \times \{x^*\}$ provided all the contracting reset functions share the same fixed point x^*.

Proposition 4 *Consider a hybrid automaton with $R(q, q', \cdot)$ contracting and there exists $x^* \in \mathbf{X}$ such that $R(q, q', x^*) = x^*$ for all $(q, q') \in E$. For every Zeno execution $\chi = (\tau, q, x)$, it holds that $Z_\infty = \mathbf{Q}_\infty \times \{x^*\}$ for some $\mathbf{Q}_\infty \subseteq \mathbf{Q}$.*

The proof is very similar to the last proposition. The only thing worth mentioning is that we can't say that all the vector field along the hybrid trajectory are bounded directly from Lemma 7, however, the same proof technique still works.

Lemma 8 *For a Zeno hybrid automaton with non-empty Zeno states set, if all the reset relations are non-expanding, then there exists some $N \geq 0$ such that for all $i \geq N$, $q(\tau_i) \in \mathbf{Q}_\infty$.*

Proof: Suppose for all $n \geq 0$ there exist some $i_n \geq n$ such that $q(\tau_{i_n}) \notin \mathbf{Q}_\infty$. By assumption $\mathbf{Q}$ is finite, hence sequence $\{(q(\tau_{i_n}), q(\tau_{i_n+1}))\}_{n=0}^{\infty}$ has a subsequence $\{(q(\tau_{i_{n_m}}), q(\tau_{i_{n_m}+1}))\}_{m=0}^{\infty}$

where $q(\tau_{i_{nm}}) = \hat{q} \notin \mathbf{Q}_\infty$ and $q(\tau_{i_{nm}+1}) = \hat{q}'$. By Lemma 7, for all $q_i \in \mathbf{Q}$, $\|f(q_i, x(t))\|$ is bounded by some $K > 0$. Hence,

$$\|x(\tau'_{i_{nm}}) - x(\tau_{i_{nm}})\| \leq K\|\tau'_{i_{nm}} - \tau_{i_{nm}}\| \to 0,$$

there exists some $\hat{x} \in \mathbf{X}$ such that $\lim_{m\to\infty} x(\tau_{i_{nm}}) = \hat{x}$. Then by the definition of Zeno state, $(\hat{q}, \hat{x}) \in Z_\infty$, which gives the contradiction. ∎

The next result is about the location of Zeno state. For an invariant preserving hybrid automaton, the continuous trajectory is always wondering inside some $I(q)$. Under certain conditions, it can be shown that continuous part of the Zeno state lies in the boundary of $I(q)$, for some $q \in \mathbf{Q}_\infty$.

Proposition 5 *Consider an invariant preserving Zeno hybrid automaton with all reset relation non-expanding and* $Z_\infty = \{(q_i, x_i)\}_{i=1}^N$. *If* $G(q, q') \cap I(q)^o = \emptyset$ *for all* $q, q' \in \mathbf{Q}_\infty$ *and* $(q, q') \in E$, *then* $x_i \in \partial I(q_i)$ *for all* i. *Furthermore, if* $x_i = x$ *for all* i, *it holds that* $x \in \bigcap_{i=1}^N \partial I(q_i)$.

Proof: Note that for all (q_i, x_i), there exists a sequence $\{\theta_{i_n}\}_{n=0}^\infty$, $\theta_{i_n} \to \tau_\infty$ such that $q(\theta_{i_n}) \to q_i$ and $x(\theta_{i_n}) \to x_i$. Suppose $\theta_{i_n} \in [\tau_{k(i_n)}, \tau'_{k(i_n)}]$, since

$$\|x(\theta_{i_n}) - x(\tau_{k(i_n)})\| \leq K\|\theta_{i_n} - \tau_{k(i_n)}\| \to 0,$$

we have $\|x(\tau_{k(i_n)}) - x_i\| \to 0$. By assumption the automaton is invariant preserving, $x(\tau_{k(i_n)}) \in \overline{I(q_i)}$. And from Lemma 8, when n is large enough, all the discrete transitions will take place in $\mathbf{Q}_\infty$. So there exists some $q_i' \in \mathbf{Q}_\infty$ such that $x(\tau_{k(i_n)}) \in G(q_i, q_i')$, hence $x(\tau_{k(i_n)}) \in G(q_i, q_i') \cap \overline{I(q_i)}$. Since $G(q_i, q_i') \cap I(q_i)^o = \emptyset$, we have $x(\tau_{k(i_n)}) \in \partial I(q_i)$. Notice that $\partial I(q_i)$ is a closed set, so that $x = \lim_{n\to\infty} x(\tau_{k(i_n)}) \in \partial I(q_i)$. The second part is obvious. ∎

From the above result, the following propositions can be obtained.

Proposition 6 *Consider an invariant preserving hybrid automaton with identity reset relations. It has no Zeno execution if*

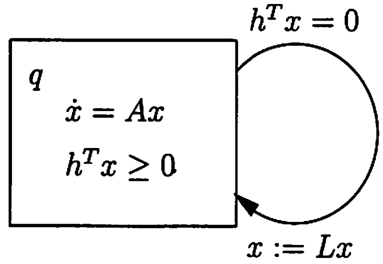

Figure 6: Case study I

- $G(q, q') \cap \overline{I(q)} \subseteq \partial I(q)$ *for all* $(q, q') \in E$,
- $\bigcap_{i=1}^{N} \partial I(q_i) = \emptyset$.

Proof: Assume the hybrid automaton has Zeno execution $\chi = (\tau, q, x)$, from Proposition 2, we know that $Z_\infty = \mathbf{Q}_\infty \times \{\hat{x}\}$ for some $\mathbf{Q}_\infty \subseteq Q$ and $\hat{x} \in \mathbf{X}$. By virtue of Proposition 5, we have $\hat{x} \in \bigcap_{i=1}^{N} \partial I(q_i) = \emptyset$, which gives the contradiction. ∎

7 Examples of Zeno Hybrid Automata

We characterized some theoretical properties of Zeno hybrid automaton in the last section. It would be interesting to investigate the existence of Zeno execution in some special hybrid systems. In this section, we study two hybrid automata both with one discrete state and planar vector field.

Case I

Consider the hybrid automaton in Figure 6 with linear vector field, linear reset map and $I(q) = \{x \in \mathbb{R}^2 : h^T x \geq 0\}$, where $h^T = (1, h_1)$.

If $x(\tau_0) = 0$, the hybrid automaton will take discrete transitions back and forth forever. There is no continuous evolution. To exclude this pathological case, we assume that $x(\tau_0) \neq$

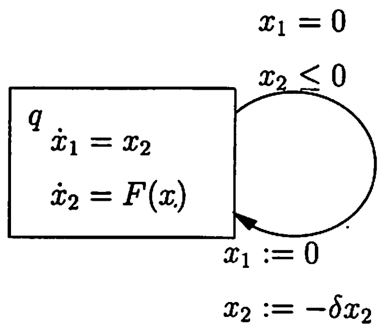

Figure 7: Case study II

0. The solution of the continuous system is

$$x(t) = e^{A(t-\tau_i)}x(\tau_i).$$

At $t = \tau'_{i+1}$, the guard $h^T x(\tau'_{i+1}) = 0$ is satisfied, therefore

$$h^T e^{A(\tau'_{i+1}-\tau_{i+1})}x(\tau_{i+1}) = 0.$$

Let $\Delta_i = \tau'_i - \tau_i$, and recall that $x(\tau_{i+1}) = Lx(\tau'_i)$, we have

$$h^T e^{A\Delta_{i+1}}Lx(\tau'_i) = 0.$$

At $t = \tau_i$, it also holds true that $h^T x(\tau'_i) = 0$, that is, $x_1(\tau'_i) + h_1 x_2(\tau'_i) = 0$. It follows that

$$h^T e^{A\Delta_{i+1}}L \begin{pmatrix} -h_1 \\ 1 \end{pmatrix} = 0,$$

Now it is clear that for all $i \in \mathbb{N}$, Δ_i satisfies the same equation, so this hybrid automaton spends exactly the same time in its stay in every discrete state, so there is no possibility for Zeno execution.

This example can be generalized to hybrid automaton with n discrete state each of them has second order LTI vector field. However, for the general LTI systems in $\mathbb{R}^n$, we don't have any result now.

Case II

In Example 1, we have a ball bouncing in the gravity field. Here we consider hybrid automaton whose vector field is conservative system with one degree of freedom [3]. Intuitively, it describes the movement of a ball in some other fields.

Definition 14 *A system is called a conservative system with one degree of freedom if it is described by the differential equation*

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = F(x),$$

where F is a differentiable function.

We also have the following terminology in literature:

- $E_k = \frac{1}{2}x_2^2$, the kinetic energy
- $E_p = -\int_{x_0}^{x} F(\xi)d\xi$, the potential energy
- $E_n = E_k + E_p$, the total energy.

Note that

$$\frac{d}{dt}E_n(t) = x_2\dot{x}_2 - F(x)\dot{x}_1 = 0,$$

hence total energy E_n keeps constant along the continuous trajectory. For this hybrid automaton, assume

- $I(q) = \{x \in \mathbb{R}^2 : x_1 \geq 0\}$,
- $G(q,q) = \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\}$,
- $R(q,q,x) = (0, -\delta x_2)$, $\delta > 0$.

If the potential energy is in the form

$$E_p = kx_1^\alpha + d, \quad k > 0, \alpha > 0,$$

and $x(\tau_0) = (0,c)^T$, then the level curves of the total energy is

$$E_n = \frac{1}{2}x_2^2 + kx_1^\alpha + d,$$

therefore

$$\frac{1}{2}x_2^2 + E_p(x_1) = E_p(x_1(\frac{\Delta_i}{2})).$$

Now the continuous dynamics reads

$$\dot{x}_1 = \sqrt{2\left(E_p(x_1(\frac{\Delta_i}{2})) - E_p(x_1)\right)},$$

hence

$$\Delta_i = \sqrt{2}\int_0^{x_1(\frac{\Delta_i}{2})} \frac{d\xi}{\sqrt{E_p(x_1(\frac{\Delta_i}{2})) - E_p(\xi)}}.$$

Since

$$\frac{1}{2}x_2^2(\tau_i) + d = kx_1^\alpha(\frac{\Delta_i}{2}) + d,$$

then

$$x_1(\frac{\Delta_i}{2}) = \left(\frac{1}{2k}x_2^2(\tau_i)\right)^{\frac{1}{\alpha}} > 0,$$

From the reset relation we have

$$x_2(\tau_i) = -\delta x_2(\tau'_{i-1}) = \delta x_2(\tau_{i-1}).$$

By induction,

$$x_2(\tau_i) = \delta^i x_2(\tau_0),$$
$$x_1(\frac{\Delta_i}{2}) = \left(\frac{1}{2k}\delta^{2i}x_2^2(\tau_0)\right)^{\frac{1}{\alpha}} = \delta^{\frac{2}{\alpha}}x_1(\frac{\Delta_{i-1}}{2}).$$

Now

$$\Delta_i = \sqrt{2}\int_0^{\delta^{\frac{2}{\alpha}}x_1(\frac{\Delta_{i-1}}{2})} \frac{d\xi}{\sqrt{k\delta^2 x_1^\alpha(\frac{\Delta_{i-1}}{2}) - k\xi^\alpha}},$$
$$=\sqrt{2}\int_0^{x_1(\frac{\Delta_{i-1}}{2})} \frac{\delta^{\frac{2}{\alpha}}d\xi}{\delta\sqrt{kx_1^\alpha(\frac{\Delta_{i-1}}{2}) - k\xi^\alpha}} = \delta^{\frac{2}{\alpha}-1}\Delta_{i-1}.$$

1. If $0 < \alpha \leq 2 \wedge \delta \geq 1$ or $\alpha \geq 2 \wedge 0 \leq \delta < 1$, then $\delta^{\frac{2}{\alpha}-1} \geq 1$, and $\tau_\infty = \sum_{i=0}^{\infty}\Delta_i$ diverges, there is no Zeno execution. Especially, when $\delta = 1$, the hybrid system has a closed orbit.

2. If $0 < \alpha < 2 \wedge 0 < \delta < 1$ or $\alpha > 2 \wedge \delta > 1$, then $\delta^{\frac{2}{\alpha}-1} < 1$, and $\tau_\infty = \sum_{i=0}^{\infty}\Delta_i$ converges, so we have a Zeno execution in this case.

8. Conclusions and Future Work

Motivated by numerous assumptions like "In this paper, we assume that the switched system is live and nonZeno" [11] and suggestions like "Additional work is needed in determining the role that Zeno-type control might play in hybrid system supervision" [17], we have extended some classical results to hybrid systems, using tools that capture also the features of Zeno executions. We have tried to illustrate some of the nature of Zeno by characterizing Zeno executions and Zeno states for a few quite general classes of hybrid systems.

There are several problems that need further investigations in the area of hybrid systems. One important issue is to carefully generalize the results in dynamical systems to hybrid systems. In the thesis, we study some fundamental properties of hybrid systems. The rich content in dynamical systems is always a resource for further research. The other issue is Zeno. Physical systems are not Zeno, but due to modeling simplification, models of real systems can be Zeno. We are interested in developing methods to automatically detect Zeno hybrid automata and to extend the simulation of the automaton beyond the Zeno time.

References

[1] R. Alur and D. L. Dill. Automata for modeling real-time systems. In *ICALP '90*, Lecture Notes in Computer Science 443, pages 322–335. Springer-Verlag, 1990.

[2] R. Alur and T. A. Henzinger. Modularity for timed and hybrid systems. In *CONCUR 97: Concurrency Theory*, Lecture Notes in Computer Science 1243, pages 74–88. Springer-Verlag, 1997.

[3] V. I. Arnold. *Ordinary differential equations.* Cambridge: MIT Press, 1978.

[4] B. Bérard, P. Gastin, and A. Petit. On the power of non observable actions in timed automata. In *Actes du STACS '96*, Lecture Notes in Computer Science 1046, pages 257–268. Springer-Verlag, 1996.

[5] M. Branicky, V. Borkar, and S. Mitter. A unified framework for hybrid control. In *IEEE Conference on Decision and Control*, pages 4228–4234, 1994.

[6] M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, April 1998.

[7] M. S. Branicky, E. Dolginova, and N. Lynch. A toolbox for proving and maintaining hybrid specifications. In A. Nerode P. Antsaklis, W. Kohn and S. Sastry, editors, *Hybrid Systems IV*, number 1273 in LNCS, pages 18–30. Springer Verlag, 1997.

[8] M. Broucke. Regularity of solutions and homotopy equivalence for hybrid systems. In *IEEE Conference on Decision and Control*, Tampa, FL, 1998.

[9] A. Deshpande, A. Gollu, and L. Semenzato. The SHIFT programming language for dynamic networks for hybrid automata. *IEEE Transactions on Automatic Control*, 43(4):584–587, April 1998.

[10] H. Elmqvist. *Dymola—Dynamic Modeling Language, User's Manual.* Dynasim AB, Sweden, 1994.

[11] K. X. He and M. D. Lemmon. Lyapunov stability of continuous-valued systems under the supervision of discrete-event transition systems. In *Hybrid Systems: Computation and Control*, volume 1386 of *Lecture Notes in Computer Science.* Springer-Verlag, Berling, 1998.

[12] T. A. Henzinger. The theory of hybrid automata. In *Annual Symposium on Logic in Computer Science*, pages 278–292. IEEE Computer Society Press, 1996.

[13] K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry. Regularization of Zeno hybrid automata. *Systems & Control Letters*, 1999. To appear.

[14] K. H. Johansson, J. Lygeros, S. Sastry, and M. Egerstedt. Simulation of Zeno hybrid automata. In *IEEE Conference on Decision and Control*, Phoenix, AZ, 1999.

[15] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):555–559, April 1998.

[16] H. K. Khalil. *Nonlinear Systems.* Prentice Hall, 1996.

[17] M. D. Lemmon, K. X. He, and I Markovsky. Supervisory hybrid systems. *IEEE Control Systems Magazine*, 19(4):42–55, 1999.

[18] J. Lygeros, K. H. Johansson, S. Sastry, and M. Egerstedt. On the existence of executions of hybrid automata. In *IEEE Conference on Decision and Control*, Phoenix, AZ, 1999.

[19] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35(3), March 1999.

[20] S. E. Mattsson, M. Andersson, and K. J. Åström. Object-oriented modelling and simulation. In D. A. Linkens, editor, *CAD for Control Systems*, chapter 2, pages 31–69. Marcel Dekker Inc., New York, 1993.

[21] J. R. Munkres. *Topology: A First Course.* Prentice Hall, New Jersey, 1975.

[22] S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control.* Springer-Verlag, New York, 1999.

[23] L. Tavernini. Differential automata and their discrete simulators. *Nonlinear Anal., Theory, Methods, Appl.*, 11(6):665–683, 1987.

[24] A. J. van der Schaft and J. M. Schumacher. Complementarity modeling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):483–490, April 1998.

[25] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer-Verlag, New York, 1990.

[26] H. Ye, A. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Transactions on Automatic Control*, 43(4):461–474, April 1998.

[27] J. Zhang, K. H. Johansson, J. Lygeros, and S. Sastry. Dynamical systems revisited: hybrid systems with Zeno executions. submitted to HSCC'00.