The Dimension of Matrices (Matrix Pencils) with Given Jordan (Kronecker) Canonical Forms

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Abstract

The set of n by n matrices with a given Jordan canonical form defines a subset of matrices in complex n^2 dimensional space. We analyze one classical approach and one new approach to count the dimension of this set. The new approach is based upon and meant to give insight into the staircase algorithm for the computation of the Jordan Canonical Form as well as the occasional failures of this algorithm. We extend both techniques to count the dimension of the more complicated set defined by the Kronecker canonical form of an arbitrary rectangular matrix pencil $A - \lambda B$.

1 Introduction

Given any square matrix A, the set of matrices similar to A forms a manifold in complex n^2 dimensional space. This manifold is, of course, the orbit of A under the action of conjugation:

$$\mathrm{orbit}(A) = \{PAP^{-1} : \det(P) \neq 0\},$$

The matrix pencil analog is to consider any pair of m by n matrices A and B, and define the orbit of the matrix pencil $A - \lambda B$ by the action of multiplication on the left and right by square nonsingular matrices of the appropriate size:

$$\mathrm{orbit}(A - \lambda B) = \{P(A - \lambda B)Q^{-1} : \det(P)\det(Q) \neq 0\},$$

This orbit defines a manifold of pencils in $2mn$ dimensional space. All pencils on this manifold are said to be *equivalent* to $A - \lambda B$. (In matrix theory, a "pencil" refers to a linear matrix polynomial, often in the indeterminate λ. See [4].)

Our concern in this work is to count the (co)dimension of these manifolds as objects in complex Euclidean space. For simplicity of exposition, we sometimes refer to these two problems as counting

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the (co)dimension of a (single) matrix or of a matrix pencil, when more properly, we would refer to counting the (co)dimension of the orbits. We take two approaches, one based on classical techniques that identify the tangent spaces of these manifolds and the other based upon existing numerical algorithms for computing the Jordan and Kronecker forms [7, 8, 9, 10, 12, 13, 16, 17].

The classical approach to solving this problem requires the computation of the tangent space to the orbits. In the single matrix case, the tangent vectors have the form

$$XA - AX, \tag{1}$$

while in the matrix pencil case, the tangents have the form

$$X(A - \lambda B) - (A - \lambda B)Y. \tag{2}$$

Thus the codimension of the single matrix orbit is the number of linearly independent matrices X for which (1) vanishes, while the codimension of the matrix pencil orbit is related to the number of linearly independent matrix pairs X, Y for which (2) vanishes.

Arnold [1] has rederived the formula for the Jordan case for the purpose of defining a particular normal form for deformations of a matrix with a given Jordan form. This form is convenient because of its minimum number of parameters [3]. We are unaware of any general dimension count for matrix pencils in the literature. One partial result of Waterhouse [15] counts the codimension of a singular pair of n by n matrices (i.e. the square case) to be $n + 1$.

Our new approach is based on the so called **staircase algorithms** for the Jordan and Kronecker canonical forms. The staircase algorithm for the Jordan canonical form proceeds by computing the Weyr characteristics of the matrix, while the staircase canonical form proceeds by computing a more complicated set of structural indices.

In this paper we lay the groundwork for a theory that we hope might explain the occasional failures of existing staircase algorithms to find the "right" Jordan or Kronecker form. These algorithms are used in systems and control theory to find the input matrix (or pencil) of highest codimension within a user-supplied distance η of the input data. The structures of these matrices or pencils reflect important physical properties of the systems they model, such as controllability [2, 14]. The user chooses η to measure the uncertainty in the data. The existence of a matrix or pencil with a different structure within distance η of the input means that the actual system may have a different structure than the approximation supplied as input. So the goal of these algorithms is to perturb the input by at most η so as to find the matrix or pencil of as high a codimension as possible. The algorithm is said to **fail** if there is another perturbation of size at most η which would raise the codimension even further. Therefore, we need to understand how the algorithm produces outputs of each codimension, which is explained in this paper, although this is just a first step to explaining the failures. In particular, this is why we need to prove a known result (Theorem 2.1) using a new technique: staircase form. We believe the dimension count for the matrix pencil case (Theorem 2.2) is new.

2 Main Results

Theorem 2.1 *The codimension of the orbit of a given matrix A is*

$$c_{\mathrm{Jor}} = \sum_{\lambda} (q_1(\lambda) + 3q_2(\lambda) + 5q_3(\lambda) + \ldots),$$

where $q_1(\lambda) \geq q_2(\lambda) \geq q_3(\lambda) \geq \ldots$, denotes the sizes of the Jordan blocks of A corresponding to λ.

Theorem 2.2 *The codimension of the orbit of $A - \lambda B$ depends only on its Kronecker structure. This codimension can be computed as the sum of separate codimensions as given in the table below.*

This equation is expressed more compactly in Equation 6 in the next section. Section 7 provides examples of how to use these formulas. Readers already familiar with the Kronecker form may wish to proceed directly to Section 7 before reading the proofs.

BREAKDOWN OF THE CODIMENSION COUNT:

The codimension of the orbit of $A - \lambda B$ depends only on its Kronecker structure. It can be computed as the sum $c_{\text{Total}} = c_{\text{Jor}} + c_{\text{Right}} + c_{\text{Left}} + c_{\text{Jor,Sing}} + c_{\text{Sing}}$, whose components are defined as:

1. The codimension of the Jordan structure:

$$c_{\text{Jor}} = \sum_{\lambda} (q_1(\lambda) + 3q_2(\lambda) + 5q_3(\lambda) + \ldots),$$

   where the sum is over all eigenvalues as in Theorem 2.1, including the infinite eigenvalue if it is present.

2. The codimension of the L singular blocks:

$$c_{\text{Right}} = \sum_{j>k} (j - k - 1),$$

   where the sum is taken over all pairs of blocks L_j and L_k for which $j > k$.

3. The codimension of the L^T singular blocks:

$$c_{\text{Left}} = \sum_{j>k} (j - k - 1),$$

   where the sum is taken over all pairs of blocks L_j^T and L_k^T for which $j > k$.

4. The codimensions due to interactions of the Jordan structure with the singular blocks:

$$c_{\text{Jor,Sing}} = (\text{size of Jordan structure})(\text{number of singular blocks}).$$

   Here the number of singular blocks counts both the left and the right blocks.

5. The codimensions due to interactions between L and L^T singular blocks:

$$c_{\text{Sing}} = \sum_{j,k} (j + k + 2),$$

   where the sum is taken over all pairs of blocks L_j and L_k^T.

These are complex codimensions, but the answers are correct for real codimensions when the matrices or matrix pencils have real Jordan or Kronecker forms. For the rest of this paper all dimensions will be complex dimensions (half the number of real dimensions).

3 Mathematical Preliminaries and Notation

3.1 Matrix Canonical Forms

The basic notation in this area has been reinvented by many authors. So as to make this work self-contained and also to fix notation, we review the basic definitions. Further information may be found in standard matrix theory texts such as [4] or [11].

Given a matrix A that has only one eigenvalue λ it is always possible to find a similarity that transforms A into the form

$$J^{\lambda}(A) = \mathrm{diag}(J^{\lambda}_{q_1}, J^{\lambda}_{q_2}, \ldots) \tag{3}$$

where J^{λ}_{q} is a q by q matrix with λ on the diagonal and 1 on the superdiagonal known as a **Jordan block**.

For an arbitrary matrix, it is always possible to find a similarity that transforms A into a union of blocks of the form (3):

$$J(A) = \mathrm{diag}(J^{\lambda_1}(A), J^{\lambda_2}(A), \ldots), \tag{4}$$

where λ_1, λ_2 denotes the distinct eigenvalues of A.

To fix the order of the Jordan blocks within (3), we assume

$$q_1(\lambda) \geq q_2(\lambda) \geq \ldots,$$

but we do not fix the order of the eigenvalues:

Definition 3.1 *The matrix $J(A)$ defined up to eigenvalue orderings is known as the* **Jordan Canonical Form** *of A.*

Definition 3.2 *The sequence of numbers $(q_i(\lambda))$ defined above gives the sizes of the Jordan blocks for the eigenvalue λ. They are known as the* **Segre characteristics** *of A relative to λ.*

It is sometimes convenient to think of this as an infinite sequence with $q_j(\lambda) = 0$ for $j >$(the number of Jordan blocks corresponding to λ).

Definition 3.3 *The* **elementary divisors** *of the matrix $A - xI$ are the polynomials $(\lambda - x)^{q_i(\lambda)}$ in the indeterminate x, where λ is an eigenvalue of A and $q_i(\lambda)$ is a Segre characteristic corresponding to λ.*

Definition 3.4 *The* **invariant factors** *of the matrix $A - xI$ are the polynomials $P_i(x) = \prod_{\lambda}(\lambda - x)^{q_i(\lambda)}$. It follows that if we let p_i denote the degree of the ith invariant factor then*

$$p_i = \sum_{\lambda} q_i(\lambda).$$

Of course $n = \sum p_i$ because this counts the sizes of all the Jordan blocks of all the eigenvalues of A.

Some authors (see [11] pages 43 and 93) consider the quantity m_i defined as the degree of the greatest common divisor of all the i by i minors of the linear matrix polynomial $A - \lambda I$. It can be shown that $m_i = p_{n+1-i} + \ldots + p_n$.

Definition 3.5 *The* **nullity** *of an n by n matrix A is $n - \mathrm{rank}(A)$. For m by n matrices the row nullity and the column nullity are $m - \mathrm{rank}(A)$ and $n - \mathrm{rank}(A)$ respectively.*

Definition 3.6 *Let $w_j(\lambda)$ denote the difference*

$$\text{nullity}(A-\lambda I)^j - \text{nullity}(A-\lambda I)^{j-1} \equiv \text{rank}(A-\lambda I)^{j-1} - \text{rank}(A-\lambda I)^j$$

The numbers $w_j(\lambda)$ are the **Weyr characteristics** *of A relative to λ. The number of blocks $J_q(\lambda)$ with $q \geq j$ is exactly $w_j(\lambda)$. The dimension of the nullspace of $(A-\lambda I)$ is $w_1(\lambda)$.*

The following lemma is critical for the construction of the staircase algorithm.

Lemma 3.1 *Let Q be any unitary matrix whose first w_1 columns span the nullspace of $A-\lambda I$. Then*

$$Q^T A Q = \begin{array}{c} \begin{array}{cc} w_1 & n-w_1 \end{array} \\ \begin{pmatrix} \lambda I & S \\ 0 & \hat{A} \end{pmatrix} \end{array}$$

where $\hat{A}$ is an $n-w_1$ by $n-w_1$ matrix. With the deletion of $w_1(\lambda)$, the Weyr characteristics of $\hat{A}$ are the same as that of A. In particular, the Weyr characteristics of the other eigenvalues are unchanged.

Proof The Jordan structure of $\hat{A}$ is the same as the Jordan structure of A except that every Jordan block of $\hat{A}$ corresponding to the eigenvalue λ is exactly one dimension smaller. $\square$

Let $A-\lambda B$ be an m by n matrix pencil. (When discussing the Kronecker case, λ is always an indeterminate.) It is possible to find an equivalent pencil $\text{Kron}(A-\lambda B)$ in the Kronecker Form:

$$\text{Kron}(A-\lambda B) = \text{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_g}, L_{\eta_1}^T, \ldots, L_{\eta_h}^T, J, J^\infty). \tag{5}$$

The L_ϵ blocks are ϵ by $\epsilon+1$ rectangular blocks with λ on the diagonal and 1 on the superdiagonal. The L_η^T blocks are $\eta+1$ by η, with λ on the diagonal, and 1 on the subdiagonal. The ϵ and η can be 0, leading to 0 columns and rows respectively. The J block is of the form (4) with the addition of λI. This constitutes the Jordan structure of the finite eigenvalues. The J^∞ block is the union of blocks of size $q_i(\infty)$ each of which has 1 on the main diagonal and λ on the superdiagonal. This constitutes the Jordan structure corresponding to the infinite eigenvalue. Frequently there will be no need to distinguish between the finite and infinite eigenvalues. Indeed, with an appropriate Möbius transformation sending $A-\lambda B$ to $(\alpha A+\beta B)-\lambda(\gamma A+\delta B)$, all eigenvalues may be assumed finite.

The L and L^T blocks constitute the **singular** part of the pencil. The Jordan structure for finite and infinite eigenvalues constitutes the **regular** part of the pencil. The Segre characteristics remain well defined for a matrix pencil, but we must include the characteristics for the infinite eigenvalue as well.

Definition 3.7 *Let*

$$0 \leq \epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_g$$

denote the sizes of the g L blocks of a pencil, and let

$$0 \leq \eta_1 \leq \eta_2 \leq \ldots \leq \eta_h$$

denote the sizes of the h L^T blocks. Then the numbers ϵ_i are known as the **column minimal indices***, while the η_i are the* **row minimal indices***.*

We can now recast Theorem 2.2 using the notation from the previous definitions. The codimension of the orbit of $A - \lambda B$ can be written compactly as

$$\boxed{\begin{aligned} \mathrm{cod}(\mathrm{orbit}(A-\lambda B)) &= (p_1 + 3p_2 + 5p_3 + \ldots) + (g+h)\sum p_i \\ &+ \sum_{i>j}(\epsilon_i - \epsilon_j - 1) + \sum_{i>j}(\eta_i - \eta_j - 1) + \sum_{i,j}(\epsilon_i + \eta_j + 2), \end{aligned} \qquad (6)}$$

where the p_i include any infinite eigenvalue blocks.

3.2 Conjugate Partitions

The Weyr characterists and the Segre characterists of a matrix for a given eigenvalue are closely related.

Definition 3.8 *Let $k_1 \geq k_2 \geq k_3 \geq \ldots \geq 0$ be a partition of the positive integer k (i.e. $k = k_1 + k_2 + \ldots$). Let l_j denote the number of k_i that are greater than or equal to j. Then the l_j form a partition of k known as the* **conjugate partition** *of the k_i.*

It is easy to verify that the property of being a conjugate partition is symmetric. For example, 17=6+6+3+1+1=5+3+3+2+2+2 are conjugate partitions of 17. This is easy to verify by reading the diagram below (known as a Ferrers diagram) vertically and horizontally:

$$\begin{array}{c|c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{1} \\ \cline{2-6}
5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cline{2-6}
3 & \cdot & \cdot & \cdot & \multicolumn{2}{c}{} \\ \cline{2-4}
3 & \cdot & \cdot & \cdot & \multicolumn{2}{c}{} \\ \cline{2-4}
2 & \cdot & \cdot & \multicolumn{3}{c}{} \\ \cline{2-3}
2 & \cdot & \cdot & \multicolumn{3}{c}{} \\ \cline{2-3}
2 & \cdot & \cdot & \multicolumn{3}{c}{} \\ \cline{2-3}
\end{array}$$

The idea of the conjugate partition is very simple, yet very powerful. It allows the interchange of summations:

$$\sum_i \sum_j^{k_i} f(i,j) = \sum_j \sum_i^{l_j} f(i,j),$$

where $f(i,j)$ is any function of i and j, and the k_i and l_j are conjugate partitions.

Lemma 3.2 *The Weyr characteristics and the Segre characteristics of a matrix corresponding to a particular eigenvalue are conjugate partitions.*

The proof of this lemma is evident from the Jordan form of the matrix.

3.3 A Fundamental Codimension Count

Our codimension counts for the Jordan and Kronecker form are built up from the fundamental Lemma 3.3. To state it, we need to introduce a little notation from manifold theory.

Definition 3.9 *The set of k dimensional subspaces of n dimensional space along with its natural manifold structure forms the* **Grassmann manifold** *denoted $G_k(n)$.*

The Grassman manifold and its dual $G_{n-k}(n)$ are isomorphic of dimension $k(n-k)$. In Lemma 3.3 we will need a **full-rank parameterization** for $G_{n-k}(n)$, which we construct as follows. (Recall that a **chart** for a complex d-dimensional manifold M is an open neighborhood U in C^d plus a homeomorphism from M into U. A full rank parameterization is the inverse of this homeomorphism.) Because of the action of the unitary group, it suffices to specify a local full rank parameterization near any one element, say E_k, the one generated by the first k coordinate vectors. We create a parameterization from unitary matrices of the form

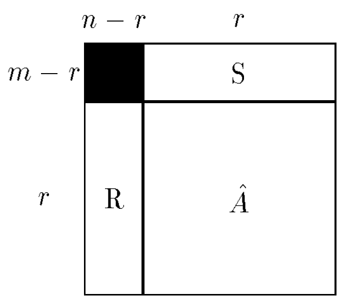
$$Q_0 = \begin{pmatrix} I & -R^* \\ R & I \end{pmatrix} \begin{pmatrix} I + R^*R & 0 \\ 0 & I + RR^* \end{pmatrix}^{-1/2}, \tag{7}$$

where R is $n-k$ by k. The homeomorphism maps complex $n-k$ by k matrices R to the span of the first k columns of Q_0. If Q is any fixed unitary matrix, the homeomorphism from $R \in C^{n-k \times k}$ to the space spanned by the first k columns of QQ_0 provides the parameterization mapping from a neighborhood of the origin in $C^{n-k\times k}$ to a neighborhood in $G_k(n)$ of the space spanned by the first k columns of Q.

Lemma 3.3 *The set of m by n matrices with rank r is a manifold with codimension $(m-r)(n-r)$.*

Proof We construct a parameterization whose image is a neighborhood of a particular m by n rank r matrix A as follows. A neighborhood of the origin in the product space $C^{r\times n-r} \times C^{m\times r}$ will serve as a domain for the parameterization. Let Q be any unitary matrix whose first $n-r$ columns span the nullspace of A, so that $AQ = [0M]$ is zero in its first $n-r$ columns and its last r columns M have full rank. Let Q_0 be as in (7), with $k = n-r$. Then the map from $(R,T) \in C^{r\times n-r} \times C^{m\times r}$ to $[0, M+T]Q_0^*Q^*$ is the desired homeomorphism. If $m = n$, then we may equally well use the homeomorphism mapping (R,T) to $QQ_0[0, M+T]Q_0^*Q^*$. Thus the dimension is $r(n-r) + mr$, and the codimension is $mn - r(n-r) - mr = (m-r)(n-r)$. □

We graphically depict the independent parameters as follows:



(8)

Here R refers to the coordinates that define the null space, while $T = [S^T, \hat{A}^T]^T$ is the matrix in $C^{m\times r}$. The black square in the upper left clearly indicates the codimension of $(m-r)(n-r)$.

Later, we will take advantage of this construction to recursively construct further submanifolds by placing analogous rank constraints on $\hat{A}$, so that $\hat{A}$ still lies in a small neighborhood of the origin. Therefore, it will be easy to see that we need merely add the codimensions of our constraints at each level in order to compute the overall codimension of the final submanifold.

4 Proofs of Theorem 2.1 (Codimension Count for Jordan Form)

4.1 Classical Proof

Consider conjugating the matrix A by $I + \delta X$, where δ is a small scalar. This yields

$$A + \delta(XA - AX) + O(\delta^2),$$

from which it is evident that the tangent space to orbit(A) at A consists of the matrices of the form $XA - AX$. The dimension of the orbit is equal to the dimension of the tangent space so that the codimension of the orbit is equal to the dimension of the nullspace of the mapping that sends X to $XA - AX$. The codimension of the orbit is then the number of linearly independent solutions to $AX = XA$. This number of solutions is well known to be

$$p_1 + 3p_2 + 5p_3 + \dots.$$

(See page 222 of volume 1 of [4].) $\square$

An alternative expression for the number of solutions to $AX = XA$ is

$$n + 2(m_1 + \ldots + m_{n-1})$$

as given in [11]. According to the remark following Definition 3.4, these expressions are identical.

4.2 Outline of the Staircase Algorithm

The staircase algorithm for the computation of the Jordan Canonical Form appears in [5, 6, 9, 10, 12]. It is built recursively upon the idea in the proof of Lemma 3.1:

Staircase algorithm for computing the Jordan form for eigenvalue λ

$i = 0$
$A_{tmp} = A - \lambda I$
while A_{tmp} not full rank
    $i = i + 1$
    Let $n' = \sum_{j=1}^{i-1} w_j$ and $n_{tmp} = n - n' = dim(A_{tmp})$
    Compute an n_{tmp} by n_{tmp} unitary matrix Q whose leading w_i columns span the null space of A_{tmp}
    $A = \mathrm{diag}(I_{n'}, Q^*) \cdot A \cdot \mathrm{diag}(I_{n'}, Q)$
    Let A_{tmp} be the lower right $n_{tmp} - w_i$ by $n_{tmp} - w_i$ corner of A
    $A_{tmp} = A_{tmp} - \lambda I$
endwhile

The final A is easily seen to be unitarily similar to the initial A. The final A is in staircase form, as illustrated with the following example:

$$\begin{array}{c|c|c|c|c|c|}
 & w_1 & w_2 & w_3 & w_4 & n' \\
\hline
w_1 & \lambda I & A_{12} & * & * & * \\
\cline{2-6}
w_2 & & \lambda I & A_{23} & * & * \\
\cline{3-6}
w_3 & & & \lambda I & A_{34} & * \\
\cline{4-6}
w_4 & & & & \lambda I & * \\
\cline{5-6}
n' & & & & & A' \\
\hline
\end{array}$$

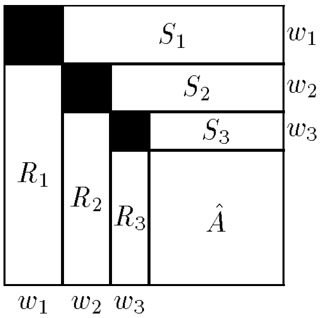
Here, the superdiagonal blocks $A_{i,i+1}$ (the "stairs") and also $A' - \lambda I$ are of full column rank, while the staircase region in the lower triangle is entirely 0. If A has only one eigenvalue λ then n' is 0 and the last block row and block column do not appear. If A has other eigenvalues λ', then the staircase form corresponding to the remaining eigenvalues may be extracted by applying the same algorithm to A'.

An easy observation is that

Lemma 4.1 *The w_i computed by the staircase algorithm for the eigenvalue λ are the Weyr characteristics corresponding to the eigenvalue λ.*

4.3 Second Proof of Theorem 2.1

Let A be any matrix. We will show that the staircase algorithm, in effect, creates a parameterization for an open neighborhood $N(A)$ of A on the manifold orbit(A). Let λ be an eigenvalue of A. Then $A - \lambda I$ has rank $n - w_1$. The independent parameters portrayed in (8) may be used as a parameterization for a neighborhood of $A - \lambda I$ on the manifold of rank $n - w_1$ matrices. Lemma 3.1 tells us that we have a parameterization for orbit(A) if we make further assumptions on the Jordan structure of $\hat{A}$. Notice that in a small enough neighborhood of A, the last $n - r$ columns of the staircase form are full rank. It is important to observe the independence of the $w_1(n - w_1)$ parameters in R_1 from the $w_1(n - w_1)$ parameters of S_1 and the as of yet uncounted parameters in $\hat{A}$. The first eigenvalue λ is "fully parameterized" when $\hat{A} - \lambda I$ has full rank. The parameters are pictorially depicted below in an example that recurs two more times before $\hat{A} - \lambda I$ has full rank.



This parameterization process is repeated on $\hat{A}$ with a new eigenvalue shift in an identical manner. This repetition continues until $\hat{A}$ does not exist. The areas of the black squares in the figure above indicate the codimension that we might attribute to the eigenvalue λ. This codimension is then

$$\begin{aligned} \sum_i w_i^2 &= \sum_i \sum_{k=1}^{w_i} (2k-1) \\ &= \sum_k \sum_{i=1}^{q_k} (2k-1) \\ &= \sum_k (2k-1) q_k, \end{aligned}$$

using the fact that the Weyr and Segre characteristics are conjugate partitions.

The total codimension for the entire Jordan structure of A is obtained by summing over all the eigenvalues because of the independence of the parameters. □

5 Tangent Space Proof of Theorem 2.2

We include two proofs both of which we believe to be new. The first proof requires counting the number of independent solutions to two simultaneous matrix equations derived by analyzing the tangent space, while the second proof (in Section 6) requires an analysis of the staircase algorithms for the Kronecker canonical form.

Consider an orbit preserving transformation of the m by n pencil $A-\lambda B$ obtained by multiplying on the left by $I+\delta X$ and the right by $I-\delta Y$, where δ is a small scalar. This yields $A-\lambda B+\delta(X(A-\lambda B)-(A-\lambda B)Y)+O(\delta^2)$, from which it is evident that the tangent space to the orbit of the pencil consists of the pencils that can be represented in the form

$$f(X,Y)=X(A-\lambda B)-(A-\lambda B)Y, \tag{9}$$

where X is an m by m matrix and Y is an n by n matrix.

Since (9) maps a space of dimension m^2+n^2 linearly into a space of dimension $2mn$, the dimension of the image space is m^2+n^2-d, where d is the dimension of the kernel of $f(X,Y)$, and so the codimension is

$$2mn-(m^2+n^2-d)=d-(m-n)^2. \tag{10}$$

The term $(m-n)^2$ represents extra baggage due to our consideration of rectangular pencils. As in the Jordan case, we need to calculate d, the number of linearly independent solutions to $f(X,Y)=0$. This can be written as the two simultaneous equations

$$XA=AY \text{ and } XB=BY. \tag{11}$$

Unfortunately, we can not simply quote a classical count of the number of independent solutions to (11) as we were able to do in Section 4.1. However since

$$Pf(X,Y)Q^{-1}=(PXP^{-1})P(A-\lambda B)Q^{-1}-P(A-\lambda B)Q^{-1}(QYQ^{-1}),$$

it follows that the number of linearly independent solutions to $f(X,Y)=0$ depends only on the Kronecker structure of $A-\lambda B$. Thus, we assume that $A-\lambda B$ is already in Kronecker canonical form $M=\mathrm{diag}(M_1,M_2,\ldots)$. The Kronecker case is more complicated than the Jordan case due to the greater number of possibilities for the Kronecker structure M.

We partition the equation $XM=MY$ conformally with $M=\mathrm{diag}(M_1,M_2,\ldots)$ so that $X_{ij}M_j=M_iY_{ij}$, where M_k is m_k by n_k, X_{ij} is m_i by m_j, and Y_{ij} is n_i by n_j:

$$\begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} m_1 & m_2 \end{array} \\ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \end{array} \begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix} \end{array} = \begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix} \end{array} \begin{array}{c} \\ n_1 \\ n_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \end{array}$$

The next lemma allows us to compute the quantity d mentioned before Equation (11) as the sum of the number d_{ij} of independent solutions of $X_{ij}M_j=M_iY_{ij}$ in the variables X_{ij} and Y_{ij}.

Lemma 5.1 *In terms of the above notation*

$$d=\sum_{i,j} d_{ij}.$$

Proof As is evident from the example

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix} = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

the equations $X_{ij}M_j = M_iY_{ij}$, $i = 1, 2, \ldots$, $j = 1, 2, \ldots$ are all mutually independent.

$\square$

Given any two blocks, M_i and M_j (we allow $i = j$ here) we define their **interaction** and the **cointeraction**:

Definition 5.1 *Let M_i be $m_i \times n_i$ and let M_j be $m_j \times n_j$. Let X be an arbitrary $m_j \times m_i$ matrix and Y be an arbitrary $n_j \times n_i$ matrix. We define the* **interaction** d_{ij} *of M_i with M_j as the dimension of the linear space $\{X, Y\}$ such that $XM_j = M_iY$. We define the* **cointeraction** *of M_i with M_j as $c_{ij} = d_{ij} - (m_i - n_i)(m_j - n_j)$. We also consider the* **combined cointeraction** *which we define as $c_{ij} + c_{ji}$ when $i \neq j$, and simply c_{ii} when $i = j$.*

Notice that the combined cointeraction has a different definition depending on whether M_i and M_j are distinct blocks (even if they happen to be equal) on one hand, or if $i = j$ on the other hand. Strictly speaking the combined cointeraction is a function of M_i, M_j, and the Kronecker delta δ_{ij}.

Lemma 5.2 *The codimension of a matrix pencil M with Kronecker structure* $\mathrm{diag}(M_1, M_2, \ldots)$ *is the sum of cointeractions of M_i with M_j for all combinations of i and j.*

Proof The sum of the cointeractions is

$$\sum_{i,j} \{d_{ij} - (m_i - n_i)(m_j - n_j)\} = d - (m - n)^2$$

as in Equation (10). $\square$

We must now count the number of linearly independent solutions (and the associated combined cointeractions) to the following equations:

- $XL_j = L_kY$ and $XL_j^T = L_k^TY$
- $XL_j = L_k^TY$ and $XL_j^T = L_kY$
- $XJ = L_jY$ and $XL_j = JY$ and related structures
- $XJ = JY$

where J denotes the non-singular structure of the pencil.

5.1 $XL_j = L_kY$ and $XL_j^T = L_k^TY$

Consider the equation $XL_j = L_kY$, where X is an unknown k by j matrix and Y is an unknown $k+1$ by $j+1$ matrix. This equation is equivalent to the two equations

$$X[0\ I_j] = [0\ I_k]Y$$
$$X[I_j\ 0] = [I_k\ 0]Y,$$

where 0 denotes a column of zeros. These two equations are in turn equivalent to the conditions

$$X_{\alpha,\beta} = Y_{\alpha,\beta},\ \alpha = 1,\ldots,k,\ \beta = 1,\ldots,j$$
$$Y_{\alpha,\beta} = Y_{\alpha+1,\beta+1},\ \alpha = 1,\ldots,k,\ \beta = 1,\ldots,j$$
$$Y_{\alpha+1,1} = Y_{\alpha,j+1} = 0,\ \alpha = 1,\ldots,k$$

If $j < k$ there is only the trivial solution $X = 0$ and $Y = 0$. The interaction is 0, so that the cointeraction is $0 - (j - (j+1))(k - (k+1)) = -1$.

If $j \geq k$ then there are non-trivial solutions: Y can be any upper triangular Toeplitz matrix with $1 + j - k$ diagonals starting from the main diagonal. X is then obtained from Y by omitting the first row and column. The interaction of L_j with L_k is $1 + j - k$ so that the cointeraction is $(1 + j - k) - 1 = j - k$.

We conclude that the combined cointeraction of L_j and L_j is 0, while if $j > k$ then the combined cointeraction of L_j with L_k is $j - k - 1$.

Taking the transpose and interchanging the roles of j and k, we see that the same result holds for blocks of the form L_j^T. We also remark that the analysis is correct even if j or k is 0.

5.2 $XL_j = L_k^T Y$ and $XL_j^T = L_k Y$

We proceed in a manner similar to the previous case. Consider the equation $XL_j = L_k^T Y$, where X is an unknown $k+1$ by j matrix and Y is an unknown k by $j+1$ matrix. The equations are equivalent to

$$X_{\alpha,\beta} = Y_{\alpha,\beta},\ \alpha = 1,\ldots,k,\ \beta = 1,\ldots,j$$
$$Y_{\alpha+1,\beta} = Y_{\alpha,\beta+1},\ \alpha = 1,\ldots,k,\ \beta = 1,\ldots,j$$
$$Y_{\alpha,1} = Y_{\alpha,j+1} = 0,\ \alpha = 1,\ldots,k$$
$$X_{k+1,\beta} = 0, \beta = 1,\ldots,j$$

This has only the trivial solution $X = Y = 0$ so that the interaction of L_j with L_k^T is 0 and the cointeraction is $0 - (-1)(1) = 1$.

A similar examination of the equation $XL_j^T = L_k Y$ shows that the interaction of L_j^T with L_k is $j+k$ so the cointeraction is $j+k-(1)(-1) = j+k+1$. We conclude that the combined cointeraction is $j + k + 2$.

5.3 Jordan Blocks and Singular Blocks

In one way, the computation involving Jordan blocks is easier since the interaction is equal to the cointeraction. (This is true simply because the Jordan block is square.) However, we must now allow for arbitrary eigenvalues.

Assume that J_k is a single Jordan block of size k corresponding to the finite eigenvalue e. (We use e here so that there is no confusion with the indeterminate λ.) We consider solutions to $XJ_k = L_j Y$. The reader can verify that the dimension of the space of solutions is k. Indeed the first row of the $j+1$ by k matrix Y can be chosen arbitrarily and this determines the remaining elements as follows: $Y_{\alpha 1} = Y_{11}e^{\alpha-1}$, X is obtained from Y by deleting the last row, and $eY_{\alpha,\beta} + Y_{\alpha,\beta-1} = Y_{\alpha+1,\beta}$. An analogous, though simpler argument shows that the case of infinite eigenvalues gives the same answer. (We can also resort to a Möbius transformation as well.) We conclude that the interaction of J_k with L_j is k.

The interaction of L_j with J_k is readily shown to be 0. From the equation $XL_j = J_kY$, we can conclude that X is obtained from Y by deleting the last column, that the last column of Y is zero, and if the mth column of Y is 0, then so is the $m-1$st column of X and hence so is the $m-1$st column of Y.

The cases $XL_j^T = JY$ and $XJ = L_j^TY$ can be reduced to the previous cases by remembering that if J is a Jordan block, $J^T = PJP$ where P is the permutation that renumbers indices in backwards order. For example, the number of independent solutions to $XL_j^T = JY$ is the same as the number of solutions to $(Y^TP)(PJ^TP) = (L_jX^TP)$.

5.4 Jordan Blocks with other Jordan Blocks

Let J be the entire non-singular portion of the Kronecker structure. If we assume that there are no infinite eigenvalues, then the equation $XJ = JY$ implies $X = Y$ and we are reduced to the case in Theorem 2.1. We remark that Theorem 2.1 tells us that there is no interaction among Jordan blocks with different eigenvalues.

We omit the tedious algebra, but it is possible to show that an infinite eigenvalue behaves exactly as if it were finite. (A simpler argument would point out that we can rotate the Riemann sphere to insure that all the eigenvalues are finite, without changing the codimension count.) We conclude that the combined cointeractions of the non-singular portion of the pencil is exactly as in Theorem 2.1.

5.5 Proof of Theorem 2.2

The proof follows from the analysis of the cases presented in Sections 5.1.1 through 5.1.4. $\square$

6 Proof of Theorem 2.2 Based on the Staircase Algorithm

We begin by reviewing the staircase algorithm. The version we use has three passes. Let $A - \lambda B$ be an m by n matrix pencil. The first pass produces two sequences of numbers s_i and r_i and returns a pencil $A' - \lambda B'$ with no L_j blocks and no zero eigenvalues. The sequence satisfies

$$s_0 \geq r_0 \geq s_1 \geq r_1 \geq s_2 \geq \ldots,$$

where

- $s_i - r_i =$ the number of L_i blocks and
- $r_i - s_{i+1} =$ the number of J_{i+1}^0 blocks.

The algorithm is as follows.

Staircase algorithm for computing the Kronecker form for the 0 eigenvalue and L_j blocks

> $i = -1$
> $A_{tmp} = A$
> while A_{tmp} not full rank
> $i = i + 1$
> Let $n' = \sum_{j=0}^{i-1} s_j$ and $n_i = n - n' = \#cols(A_{tmp})$
> Let $m' = \sum_{j=0}^{i-1} r_j$ and $m_i = m - m' = \#rows(A_{tmp})$
> Compute an n_i by n_i unitary matrix Q whose leading $s_i = \text{nullity}(A_{tmp})$ columns span
>   the right null space of A_{tmp}
> Let $A = A \cdot \text{diag}(I_{n'}, Q)$ and $B = B \cdot \text{diag}(I_{n'}, Q)$
> $B_{tmp} = B(m'+1 \ : \ m \ , \ n'+1 \ : \ n'+s_i)$
> Compute an m_i by m_i unitary matrix P whose first $r_i = \text{rank}(B_{tmp})$ rows span
>   the column space of B_{tmp}
> Let $A = \text{diag}(I_{m'}, P) \cdot A$ and $B = \text{diag}(I_{m'}, P) \cdot B$
> Let A_{tmp} be the last $m_i - r_i$ rows and $n_i - s_i$ columns of A
> endwhile

It is easy to see the final $A - \lambda B$ is unitarily equivalent to the initial $A - \lambda B$. We illustrate the final form of $A - \lambda B$ with the following small example:

	s_0	s_1	s_2	s_3	n'
r_0	$0 - \lambda B_{00}$	$A_{01} - \lambda B_{01}$	$*$	$*$	$*$
r_1		$0 - \lambda B_{11}$	$A_{12} - \lambda B_{12}$	$*$	$*$
r_2			$0 - \lambda B_{22}$	$A_{23} - \lambda B_{23}$	$*$
m'					$A' - \lambda B'$

On completion, the B_{ii} blocks have full row rank, and the $A_{i,i+1}$ blocks have full column rank.

The first pass through the inner loop of the algorithm postmultiplies A and B by a unitary Q so A's leading $s_0 = \text{nullity}(A)$ columns are 0, and then premultiplies A and B by a unitary P so that B_{00}, the leading r_0 by s_0 submatrix of B, is full rank, and the remaining rows of the first s_0 columns of B are zero. We then repeat the process on the trailing $m - r_0$ by $n - s_0$ submatrix of $A - \lambda B$ to get s_1 and r_1. We continue until the trailing block of A has full rank (or is null).

Just as with the Jordan form, each step of the algorithm incrementally builds a parameterization for the set of matrices of a given Kronecker form. Each step of the algorithm restricts the Kronecker form of the pencil to a set of higher codimension. The restrictions imposed at each step are independent for the same reason they were in the Jordan case, so we can just add codimensions. The increase in codimension at each step is given by Lemma 3.3, as the sum of the products of the row and column nullities of submatrices of A and B. Specifically the m_i by n_i submatrix of A has column nullity s_i, rank $n_i - s_i$, row nullity $m_i + s_i - n_i$, and so by Lemma 3.3 codimension $(m_i + s_i - n_i)s_i$. Similarly the codimension due to B at step i is $(m_i - r_i)(s_i - r_i)$. The first pass through the algorithm determines the L and J^0 blocks so that the codimension due to these blocks is given by

$$\sum_i \{(m_i + s_i - n_i)s_i + (m_i - r_i)(s_i - r_i)\}. \tag{12}$$

We proceed to show that (12) is the formula given in Theorem 2.2.

For convenience we list our notation:

m_i	number of rows in the lower right subpencil at step $i = m - \sum_{k=0}^{i-1} r_k$
n_i	number of columns in the lower right subpencil at step $i = n - \sum_{k=0}^{i-1} s_k$
s_i	column nullity of A_{tmp} at step i
r_i	row rank of B_{tmp} at step i
l_i	number of L_i blocks in the original pencil
l'_i	number of L_i^T blocks in the original pencil
t_i	number of J_i^0 blocks in the original pencil
u	size of the regular structure corresponding to $\lambda \neq 0$.

6.1 Only left singular blocks

We begin by assuming that our pencil only contains left singular blocks. Let l_i denote the number of L_i blocks. It is easy to show by induction that the algorithm computes

$$
\begin{aligned}
m_i &= \sum_{j=i}^{\infty}(j-i)l_j \\
n_i &= \sum_{j=i}^{\infty}(1+j-i)l_j \\
s_i &= \sum_{j=i}^{\infty} l_j \\
r_i &= \sum_{j=i+1}^{\infty} l_j.
\end{aligned}
$$

Thus for this case expression (12) evaluates to

$$\alpha = \sum_{i=0}^{\infty} l_i \sum_{j=i+1}^{\infty} (j-i-1)l_j. \tag{13}$$

This is exactly $\sum_{i>j}(\epsilon_i - \epsilon_j - 1)$ as in (6).

6.2 Left singular blocks and J^0 blocks

We now add the assumption that there are J^0 blocks as well. Let t_i be the number of J_i^0 blocks, i.e., Jordan blocks of size i corresponding to a zero eigenvalue. Again by induction it is possible to show

$$
\begin{aligned}
m_i &= \sum_{j=i}^{\infty}(j-i)(l_j+t_j) \\
n_i &= m_i + \sum_{j=i}^{\infty} l_j \\
s_i &= \sum_{j=i}^{\infty} l_j + \sum_{j=i+1}^{\infty} t_j \\
r_i &= \sum_{j=i+1}^{\infty} (l_j+t_j).
\end{aligned}
$$

Now for this case Expression (12) evaluates to

$$\beta = \sum_{i=0}^{\infty} \left\{ \left(\sum_{j=i+1}^{\infty} t_j \right) \left(\sum_{j=i}^{\infty} l_j + \sum_{j=i+1}^{\infty} t_j \right) + l_i \sum_{j=i+1}^{\infty} (j-i-1)(l_j + t_j) \right\} \tag{14}$$

which can readily be manipulated to be

$$\beta = \alpha + \sum_{i=0}^{\infty} (\sum_{j=i+1}^{\infty} t_j)^2 + \sum_{i=0}^{\infty} \left\{ \sum_{j=i}^{\infty} l_j \sum_{k=i+1}^{\infty} t_k + l_i \sum_{j=i+1}^{\infty} (j-i-1) t_j \right\},$$

where α is the same interaction among the left singular blocks as in Equation (13). We recognize from Definition 3.6 that $(\sum_{j=i+1}^{\infty} t_j)^2$ is w_{i+1}^2, the square of the $i+1$'st Weyr characteristic of the zero eigenvalue. From our new proof of Theorem 2.1 we know that $\sum_{i=0}^{\infty} w_{i+1}^2$ is the codimension due to the zero eigenvalue alone.

Lastly, we must evaluate

$$\begin{aligned}
&\sum_{i=0}^{\infty} \left\{ \sum_{j=i}^{\infty} l_j \sum_{k=i+1}^{\infty} t_k + l_i \sum_{j=i+1}^{\infty} (j-i-1) t_j \right\} \\
&= \sum_{i=0}^{\infty} l_i (\sum_{j=0}^{i} \sum_{k=j+1}^{\infty} t_k + \sum_{k=i+1}^{\infty} (k-i-1) t_k) \\
&= \sum_{i=0}^{\infty} l_i \left\{ \sum_{k=1}^{i} \sum_{j=0}^{k-1} t_k + \sum_{k=i+1}^{\infty} \sum_{j=0}^{i} t_k + \sum_{k=i+1}^{\infty} (k-i-1) t_k \right\} \\
&= (\sum_{i=0}^{\infty} l_i)(\sum_{k=1}^{\infty} k t_k) \\
&= (\text{size of Jordan structure for } \lambda = 0)(\text{number of left singular blocks}).
\end{aligned}$$

Therefore $\beta = \alpha + g \sum_i q_i^0$.

6.3 Arbitrary singular blocks and arbitrary Jordan structure

We complete the first pass through the algorithm by defining l_i' to denote the number of L_i^T blocks, and u to be the size of the regular Jordan structure for $\lambda \neq 0$. Thus, $u = \sum_i (p_i - q_i^0)$ includes the structure for $\lambda = \infty$ which plays no special role during the first pass through the algorithm.

We once again omit the details, but it is possible to show by induction that the algorithm computes

$$\begin{aligned}
m_i &= m_i^0 + \sum_{j=0}^{\infty} (j+1) l_j' + u \\
n_i &= n_i^0 + \sum_{j=0}^{\infty} j l_j' + u \\
s_i &= s_i^0 \\
r_i &= r_i^0,
\end{aligned}$$

where the superscript 0 indicates no right singular structure and no non-zero regular structure, i.e. as in the notation of Section 6.2.

We now have that the codimension expression in (12) is

$$\gamma = \beta + \sum_{i=0}^{\infty} \left\{ (\sum_{j=0}^{\infty} l'_j)(\sum_{j=i}^{\infty} l_j + \sum_{j=i+1}^{\infty} t_j) + l_i(\sum_{j=0}^{\infty}(j+1)l'_j + u) \right\},$$

where β is as in (14). With some algebraic manipulation, we obtain

$$\gamma = \beta + \sum_{i,j=0}^{\infty} l_i l'_j(i+j+2) + u\sum_{i=0}^{\infty} l_i + (\sum_{i=0}^{\infty} l'_i)(\sum_{k=1}^{\infty} k t_k).$$

The terms here are the terms

$$\gamma = \beta + \sum_{i,j}(\epsilon_i + \eta_j + 2) + g\sum_i(p_i - q_i^0) + h\sum_i q_i^0.$$

6.4 Second and third passes through algorithm

The first pass through the algorithm gives us a pencil $A' - \lambda B'$, which may have only L_j^T blocks and nonzero eigenvalues. We then run the algorithm on $(B' - \lambda A')^T$, so that the indices that gave the right singular blocks before now give the left singular blocks. The indices that described $\lambda = 0$ now describe $\lambda = \infty$. This algorithm returns a pencil with only a regular part that has no zero or infinite eigenvalues.

If we reinvoke the previous results, we see that the second pass through the algorithms nearly completes the entire expression (6). The only gap is

$$\sum_{\lambda \notin \{0,\infty\}} (q_1^\lambda + 3q_2^\lambda + 5q_3^\lambda + \ldots).$$

This is just the Jordan structure of the regular part other than the zero and infinite eigenvalues. This is covered in the third phase of the algorithm, completing the proof. $\square$

7 Examples, Observations About Genericity, and Applications to the Waterhouse Theorems

We illustrate how these theorems may be used with a number of examples:

1. Let A be a matrix all of whose eigenvalues are λ. The most generic such matrix, whose orbit has codimension n, is a single Jordan block. The least generic such matrix, with codimension $1 + 3 + 5 + \ldots = n^2$, i.e. dimension 0, is the single point λI.

2. Let A be a matrix with no multiple eigenvalues. The codimension of its orbit is then $\sum_\lambda 1$ or n. One might intuitively think of this as having specified the n eigenvalues, but no other information about the matrix. Indeed, if you do not wish to specify the value of an eigenvalue, the correct codimension for this unspecified eigenvalue is one less:

   $$-1 + q_1(\lambda) + 3q_2(\lambda) + 5q_3(\lambda) + \ldots.$$

   In the Kronecker algorithm one sometimes specifies that that the eigenvalues are 0, ∞ or "other". It would therefore be correct to subtract one for eigenvalues classified as "other".

3. Let the Kronecker structure of a particular 8 by 12 pencil be $\text{diag}(L_0, L_2, L_3, L_3)$. Since this pencil has only L_j blocks, the entire codimension is to be found in c_{right}. It is $1+2+2=5$. Notice that the interactions of two L_j blocks that are equal or differ by only one, make no contribution to the codimension. If a pencil contains only blocks of the form L_α or $L_{\alpha+1}$, the codimension is 0. We have therefore observed

   Corollary 7.1 *The generic Kronecker structure for a matrix pencil with $d = n - m > 0$ is*

   $$\text{diag}(L_\alpha, \ldots, L_\alpha, L_{\alpha+1}, \ldots, L_{\alpha+1}),$$

   where $\alpha = \lfloor m/d \rfloor$, the total number of blocks is d, while the number of $L_{\alpha+1}$ blocks is given by $m \bmod d$ (which is 0 when d divides m).

   The same statement holds when $d = m - n > 0$ if we replace the L_α and $L_{\alpha+1}$ blocks by their transposes. Corollary 7.1 was obtained by Van Dooren, Wilkinson, and Wonham as discussed on page 3.55 of [13].

4. Let an n by n matrix pencil have the Kronecker structure $\text{diag}(L_j, L^T_{n-j-1})$, where $0 \le j < n$. From the c_{Sing} portion of the codimension, we learn that the orbit has codimension $j + (n - j + 1) + 2 = n + 1$. If a square pencil has any singular part at all, it is fairly easy to check that the smallest possible codimension is $n + 1$ and it must be of this form. We have thus reproduced a result of Waterhouse([15]:

   Corollary 7.2 *The generic singular pencils of size n by n have Kronecker structures*

   $$\text{diag}(L_j, L^T_{n-j-1}),$$

   where $j = 0, \ldots, n-1$.

   Intuitively, we might think of this as the $n+1$ conditions on the coefficients of λ that $\det(A - \lambda B) = 0$.)

   More generally, [15] has shown that if a square matrix has one L_r block and one L^T_s block and otherwise has a generic $n - r - s - 1 \times n - r - s - 1$ block (eigenvalues unspecified), then the codimension is $(r + s + 2) + 2(n - r - s - 1) = 2n - (r + s)$. This too readily follows from our results.

5. If an 11 by 12 pencil has the Kronecker form $\text{diag}(L_1, L^T_1, L_3, J^\infty_5)$, where here J^∞_5 denotes a single 5 by 5 Jordan block with eigenvalue ∞, then $c_{\text{Jor}} = 5$, $c_{\text{Right}} = 1$, $c_{\text{Jor,Sing}} = 5 \times 3 = 15$, and $c_{\text{Sing}} = 4 + 6 = 10$ giving a total codimension of 31.

6. The 0 pencil has a Kronecker structure consisting of m L^T_0 blocks and n L_0 blocks. The codimension from c_{Sing} only is $2mn$, i.e. the dimension is 0.

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