Computing the Lines Piercing Four Lines

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Abstract

Given four distinct lines in R^3 there exist zero, one, two, or various infinities of lines incident on the given lines. We wish to characterize and compute the set of incident lines in a numerically stable way. We use the Plücker coordinatization of lines to cast this problem as a null-space computation in R^5, and show how the singular value decomposition (SVD) yields a simple, stable characterization of the incident lines. Finally, we enumerate the types of input degeneracies that may arise, and describe the solution set of lines in each case.

CR Categories and Subject Descriptors: [Computer Graphics]: I.3.5 Computational Geometry and Object Modeling – *geometric algorithms, languages, and systems.*

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1 Introduction

The line is an important primitive element in the geometry of three-space. Lines, however, do not behave as simply as, say, points and planes. For example, three generic points specify a plane by incidence, and three generic planes specify a point; determining either incidence is a linear computation in the input coordinates. In contrast, *four* generic lines are required to specify *two* further lines by incidence (Figure 1). Moreover, computing line incidence is an inherently quadratic problem.

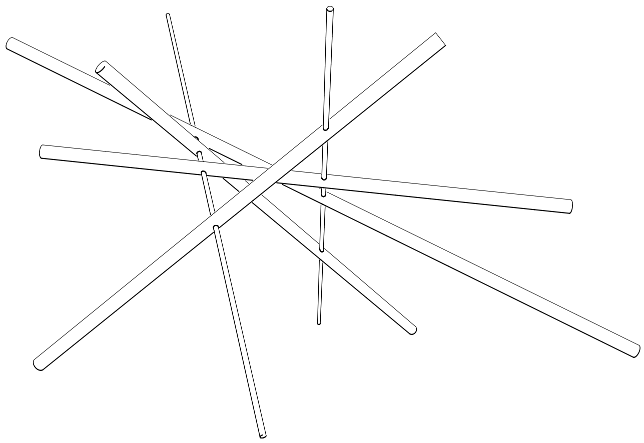

Figure 1: Four (generic) lines determine two further lines by incidence.

We present here an implemented algorithm that, given four arbitrary lines, computes the line or family of lines incident on the input lines in a numerically stable fashion. Generically, four lines induce exactly two incident lines. In practice, however, various input degeneracies may result in zero, one, two, or various infinities of incident lines. We characterize the degeneracies in each case, and describe the computation of the resulting set of lines.

2 Plücker Coordinates

We review the Plücker coordinatization of directed lines in three space [4]. Any ordered pair of distinct points $p = (p_x, p_y, p_z)$ and $q = (q_x, q_y, q_z)$ defines a directed line ℓ in R^3. This line corresponds to a projective six-tuple $\Pi_\ell = (\pi_{\ell 0}, \pi_{\ell 1}, \pi_{\ell 2}, \pi_{\ell 3}, \pi_{\ell 4}, \pi_{\ell 5})$, each component of which is the determinant of a 2×2 minor of the matrix

$$\begin{pmatrix} p_x & p_y & p_z & 1 \\ q_x & q_y & q_z & 1 \end{pmatrix}. \tag{1}$$

There are several conventions dictating the correspondence between the minors of

(1) and the π_i. We define the $\pi_{\ell i}$ as:

$$\begin{aligned}
\pi_{\ell 0} &= p_x q_y - q_x p_y \\
\pi_{\ell 1} &= p_x q_z - q_x p_z \\
\pi_{\ell 2} &= p_x \quad - q_x \\
\pi_{\ell 3} &= p_y q_z - q_y p_z \\
\pi_{\ell 4} &= p_z \quad - q_z \\
\pi_{\ell 5} &= q_y \quad - p_y
\end{aligned}$$

(this somewhat asymmetric order was adopted in [3] to produce positive signs in some identities about Plücker coordinates).

If a and b are two directed lines, and Π_a, Π_b their corresponding Plücker mappings, a relation $side(a, b)$ can be defined as the permuted inner product

$$\Pi_a \odot \Pi_b = (\pi_{a0}\pi_{b4} + \pi_{a1}\pi_{b5} + \pi_{a2}\pi_{b3} + \pi_{a4}\pi_{b0} + \pi_{a5}\pi_{b1} + \pi_{a3}\pi_{b2}). \tag{2}$$

This sidedness relation has a geometric interpretation analogous to the well-known "right-hand rule" (Figure 2): if the thumb of one's right hand is directed along a, then $side(a, b)$ is positive (negative) if b goes by a with (against) one's fingers. If a and b are coplanar (i.e., intersect or are parallel), $side(a, b)$ is zero.

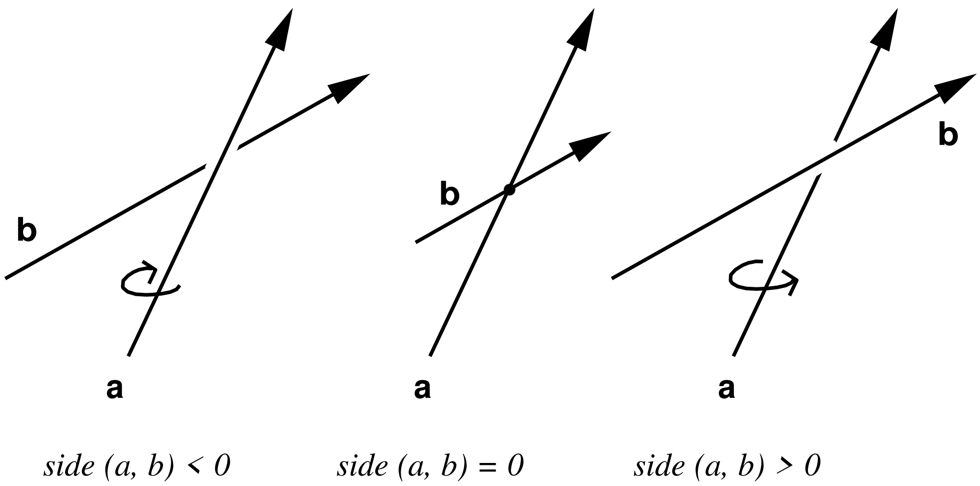

Figure 2: The right-hand rule in the context of the relation $side(a, b)$.

Thus, the six-tuple Π_l can be treated either as a (homogeneous) point $\Pi_\ell = (\Pi_{\ell 0} \dots \Pi_{\ell 5})$ in P^5 or, after permutation, as the coefficients of a 5-dimensional hyperplane $(\pi_4, \pi_5, \pi_3, \pi_0, \pi_1, \pi_2)$. The advantage of transforming lines to Plücker coordinates is that detecting incidence of lines in R^3 is equivalent to computing the inner product of a homogeneous point (the mapping of one line) with a hyperplane (the mapping of the other).

Plücker coordinates simplify computations on lines by mapping them to points and hyperplanes, which are familiar objects. However, although every directed line in

R^3 maps to a point in Plücker coordinates, not every six-tuple of Plücker coordinates corresponds to a *real line*. Only those points Π satisfying the quadratic relation

$$\Pi \odot \Pi = 0 \tag{3}$$

correspond to real lines in R^3. The remainder of the points correspond to *imaginary lines*.

The Plücker coordinates of a real line are not independent. First, since they describe a projective space, they are distinct only to within a scale factor. Second, they must satisfy Equation 3. Thus, the six Plücker coordinates describe a four-parameter space. This confirms basic intuition: one could parametrize all lines in R^3 in terms of, for example, their intercepts on two standard planes.

The set of points in P^5 satisfying Equation 3 is called the *Plücker quadric* [4]. One might visualize this set as a four-dimensional ruled surface embedded in P^5 that is analogous to a quadric hyperboloid of one sheet in R^3 (Figure 3).

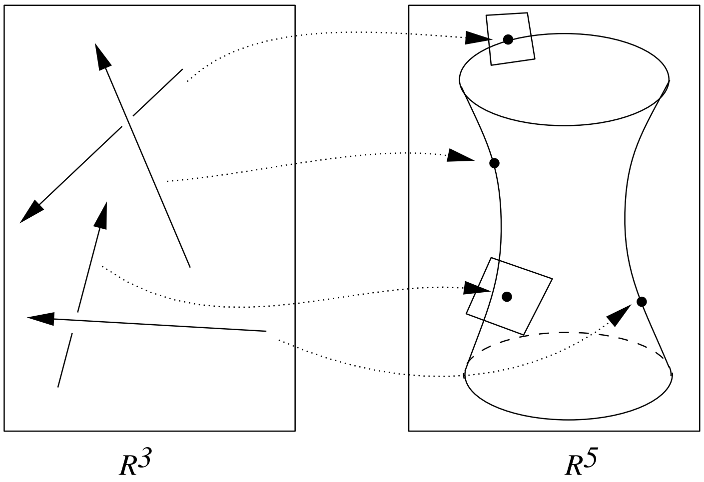

Figure 3: Real lines map to points on, or hyperplanes tangent to, the Plücker quadric.

Henceforth, we use the notation $\mathbf{\Pi} : l \rightarrow \Pi_l$ to denote the map $\mathbf{\Pi}$ that takes a directed line l to the Plücker point (hyperplane) Π_l, and the notation $\mathcal{L} : \Pi \rightarrow l_\Pi$ to denote the map that takes any point Π on the Plücker quadric and constructs the corresponding real directed line l_Π in R^3.

3 Computing the Incident Lines

Suppose we are given four lines $l_k, 1 \leq k \leq 4$ in R^3, and wish to compute all further lines that are *incident* on, or intersect, the l_k. By the sidedness relation above, we wish to find all lines s such that $\text{side}(s, l_k) = 0$ for all k. Each line l_k, under the Plücker mapping, is mapped to a hyperplane Π_k in P^5. Four such hyperplanes intersect in a *line* $\mathbf{L}$ in P^5. In Plücker coordinates, $\mathbf{L}$ contains the images under $\mathbf{\Pi}$ of all lines, real or imaginary, incident on the four l_k. To find the *real* incident lines in R^3, we must intersect $\mathbf{L}$ with the Plücker quadric (Figure 4). As in three space, a line-quadric intersection may contain 0, 1, 2, or (since the quadric is ruled) an infinite number of points.

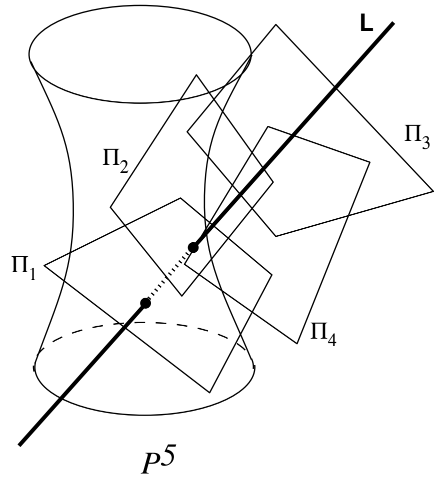

Figure 4: The four Π_k determine a line to be intersected with the Plücker quadric.

Thus the incidence computation has two parts. The first is an intersection of four hyperplanes Π_k, to form a line (the *null space*) of the Π_k. The second is an intersection of this line with a quadric surface to produce a discrete result. We have implemented this computation in the C language using a FORTRAN singular value decomposition package from Netlib [1]. Figure 1 depicts the algorithm applied to four generic lines. Note that the input lines (thick) are mutually skew, and that each of the two solution lines (thin) pierces the input lines in a distinct order.

We formulate the first part of the problem as a singular value decomposition. Each of the lines l_k corresponds to a six-coefficient hyperplane Π_k under the Plücker mapping. Thus, we must find the null-space of the matrix

$$\mathbf{M} = \begin{pmatrix} \Pi_{04} & \Pi_{05} & \Pi_{03} & \Pi_{00} & \Pi_{01} & \Pi_{02} \\ \Pi_{14} & \Pi_{15} & \Pi_{13} & \Pi_{10} & \Pi_{11} & \Pi_{12} \\ \Pi_{24} & \Pi_{25} & \Pi_{23} & \Pi_{20} & \Pi_{21} & \Pi_{22} \\ \Pi_{34} & \Pi_{35} & \Pi_{33} & \Pi_{30} & \Pi_{31} & \Pi_{32} \end{pmatrix}.$$

By the singular value decomposition theorem [2], this 4×6 real matrix can be written as the product of three matrices, $\mathbf{U} \in \mathbf{R}^{4\times4}$, $\mathbf{\Sigma} \in \mathbf{R}^{4\times6}$, and $\mathbf{V} \in \mathbf{R}^{6\times6}$, with

$\mathbf{U}$ and $\mathbf{V}$ orthogonal, and $\boldsymbol{\Sigma}$ zero except along its diagonal:

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \begin{pmatrix} u_{00} & \cdots & u_{03} \\ \vdots & & \vdots \\ u_{30} & \cdots & u_{33} \end{pmatrix} \begin{pmatrix} \sigma_0 & & & & 0 & 0 \\ & \sigma_1 & 0 & & 0 & 0 \\ & & \sigma_2 & & 0 & 0 \\ 0 & & & \sigma_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{00} & \cdots & \cdots & v_{05} \\ \vdots & & & \vdots \\ & & & \\ \vdots & & & \vdots \\ v_{50} & \cdots & \cdots & v_{55} \end{pmatrix}.$$

The σ_i can be ordered by decreasing magnitude, and comprise the *singular values* of $\mathbf{M}$; the number of non-zero σ_i equals the rank of $\mathbf{M}$. Each zero or elided σ_i corresponds to a row of $\mathbf{V}$; collectively, these rows form the *null space* of $\mathbf{M}$.

If $\sigma_3 \neq 0$ then the null space of $\mathbf{M}$ is spanned by the vectors comprising the last two rows of $\mathbf{V}$. Call these rows $\mathbf{F}$ and $\mathbf{G}$.

Consider the map $\boldsymbol{\Lambda} : \mathbf{P} \rightarrow \mathbf{P}^5$ defined by

$$\boldsymbol{\Lambda}(t) \equiv \mathbf{F} + t\mathbf{G}.$$

The null space property implies that

$$\boldsymbol{\Lambda}(t) \odot \Pi_k = 0, \qquad 0 \leq k \leq 3, \forall t.$$

Since $\boldsymbol{\Lambda}$ is injective, it is an isomorphism between $\mathbf{P}$ and the set of all lines (real and imaginary) tight on the l_k. Thus there is a one-to-one correspondence between the real lines incident to the l_k and the roots of

$$\boldsymbol{\Lambda}(t) \odot \boldsymbol{\Lambda}(t) = 0.$$

This is a quadratic equation in t:

$$\mathbf{F} \odot \mathbf{F}t^2 + 2\mathbf{F} \odot \mathbf{G}t + \mathbf{G} \odot \mathbf{G} = 0,$$

or

$$at^2 + 2bt + c = 0,$$

where $a = \mathbf{F} \odot \mathbf{F}$, $b = \mathbf{F} \odot \mathbf{G}$, and $c = \mathbf{G} \odot \mathbf{G}$. This is an "even" quadratic with the discriminant $b^2 - ac$, rather than the more familiar $b^2 - 4ac$ [5].

If $a^2 + b^2 + c^2 = 0$, all t are solutions, and the null-space line in P^5 lies in the (ruled) Plücker quadric. In this case any linear combination of $\mathbf{F}$ and $\mathbf{G}$ corresponds to a real line incident on the l_k.

If $b^2 - ac < 0$ there are no real lines incident on the l_k. If $b^2 - ac = 0$, there is a single line incident on the l_k given by $\mathbf{L}(t), t = \frac{-b}{a}$. If $b^2 - ac > 0$ there are two real lines corresponding to

$$t = \frac{-b \pm \sqrt{b^2 - ac}}{a}. \qquad (4)$$

It may be the case that some of the σ_i are zero. If n of the σ_i are zero, then the set of real and imaginary lines incident on the l_k are spanned by the vectors comprising the last $n+2$ rows of $\mathbf{V}$. This set of lines can be parametrized by $\mathbf{P}^{n+1}$. The real lines in this set must satisfy the Plücker relationship (Eq. 3), inducing a quadratic constraint on $\mathbf{P}^{n+1}$. This is a quadratic equation on the line for $n=0$; a conic in the plane for $n=1$; and a quadric surface in projective space for $n \geq 2$. The solution to the quadratic equation can be all of $\mathbf{P}^{n+1}$, empty, reducible or, when $n>1$, irreducible. If it is reducible, each component can be parametrized by $\mathbf{P}^n$; otherwise it is irreducible, and the entire set of lines can be parametrized by $\mathbf{P}^n$. In the following we consider the various special cases that arise.

If $\sigma_3 = 0$ and $\sigma_2 \neq 0$ then $\mathbf{M}$ has rank three. That is, only three rows of $\mathbf{M}$ are linearly independent. In this case, the set of lines incident on the l_k are those lines whose Plücker coefficients are orthogonal to the first three rows of $\mathbf{V}$. Thus, by the SVD, the last three rows of $\mathbf{V}$ span the space of lines (real and imaginary) incident on the l_k. Consider the map $\mathbf{\Lambda}(u,v) : \mathbf{P}^2 \rightarrow \mathbf{P}^5$ given by

$$\mathbf{\Lambda}(u,v) = u\mathbf{F} + v\mathbf{G} + \mathbf{H}$$

where $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are the last three rows of $\mathbf{V}$. $\mathbf{\Lambda}(u,v)$ parametrizes the real and imaginary incident lines. The real lines incident on the l_k must satisfy

$$\mathbf{\Lambda} \odot \mathbf{\Lambda} = 0.$$

This is a quadratic equation $q(u,v)=0$ in the variables u and v. If the solution is a pair of lines, the set of lines incident on the l_k comprise two 1-parameter families of lines. Otherwise the conic can be parametrized by a single variable t. Thus if $u(t), v(t)$ satisfy $q(u(t),v(t)) = 0$, the incident lines are given by $\mathcal{L}(u(t)\mathbf{F} + v(t)\mathbf{G} + \mathbf{H})$.

If $\sigma_3 = 0$, $\sigma_2 = 0$, and $\sigma_1 \neq 0$, the set of real and imaginary lines incident on the l_k can be parametrized by

$$\mathbf{\Lambda}(u,v,w) = u\mathbf{F} + v\mathbf{G} + w\mathbf{H} + \mathbf{I},$$

where $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{I}$ are the last four rows of $\mathbf{V}$. Again, the real lines satisfy a quadratic equation $q(u,v,w) = 0$ in $\mathbf{P}^3$. The zero surface of this equation can be parametrized by the projective plane.

4 Implicitizing a One-Parameter Line Family

Four lines are in general position if no two are coplanar, no three are coconical or cocylindrical, and the four are not *cohyperbolic*, i.e., do not lie on the same ruled quadric surface.

A common one-parameter line family, the *regulus* (or hyperboloid of one sheet), arises often in line computations, either because four input lines are found to be cohyperbolic, or because only three line constraints are active, implying a one-parameter family $\mathbf{\Lambda}(t)$ of incident lines as in the previous section. In either case, three lines are the "generators" for a regulus (Figure 5). If line segments $\overline{\mathbf{pq}}$, $\overline{\mathbf{rs}}$, and $\overline{\mathbf{tu}}$ define the

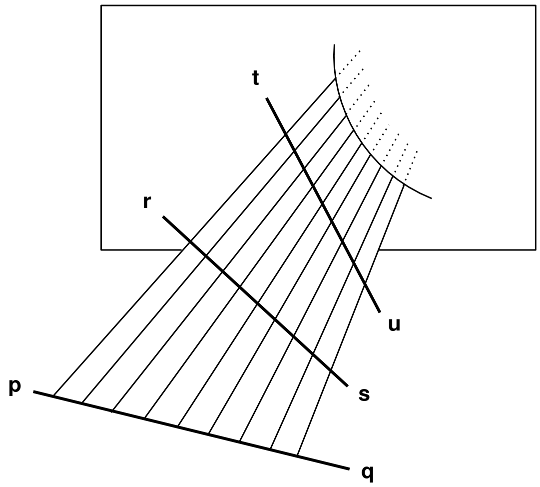

Figure 5: Three generator lines and part of the induced regulus of incident lines.

three lines, the implicit equation of the regulus through the lines is [4]:

$$|\mathbf{pqrx}|\,|\mathbf{stux}| - |\mathbf{pqsx}|\,|\mathbf{rtux}| = 0 \tag{5}$$

where $\mathbf{x} = (X, Y, Z, 1)$ is the unknown point, and the expression $|\mathbf{abcd}|$ denotes the determinant of a 4×4 matrix.

This can be rewritten in a more familiar way as the quadratic form

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix} \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0,$$

or

$$Ax^2 + Ey^2 + Hz^2 + 2Bxy + 2Cxy + 2Fyz + 2Dx + 2Gy + 2Iz + J = 0,$$

where

$$\begin{aligned}
A &= \alpha_X\beta_X - \gamma_X\delta_X \\
E &= \alpha_Y\beta_Y - \gamma_Y\delta_Y \\
H &= \alpha_Z\beta_Z - \gamma_Z\delta_Z \\
B &= (\gamma_X\delta_Y + \gamma_Y\delta_X - \alpha_X\beta_Y - \alpha_Y\beta_X)/2 \\
C &= (\alpha_X\beta_Z + \alpha_Z\beta_X - \gamma_X\delta_Z - \gamma_Z\delta_X)/2 \\
F &= (\gamma_Y\delta_Z + \gamma_Z\delta_Y - \alpha_Y\beta_Z - \alpha_Z\beta_Y)/2 \\
D &= (\gamma_X\delta_1 + \gamma_1\delta_X - \alpha_X\beta_1 - \alpha_1\beta_X)/2 \\
G &= (\alpha_Y\beta_1 + \alpha_1\beta_Y - \gamma_Y\delta_1 - \gamma_1\delta_Y)/2 \\
I &= (\gamma_Z\delta_1 + \gamma_1\delta_Z - \alpha_Z\beta_1 - \alpha_1\beta_Z)/2 \\
J &= \alpha_1\beta_1 - \gamma_1\delta_1,
\end{aligned}$$

and where the matrices $\alpha, \beta, \gamma, \delta$ correspond, respectively, to the terms in Equation 5, and subscription denotes the determinant of the relevant 3×3 minor; e.g.,

$$\alpha_X = \begin{vmatrix} p_y & p_z & 1 \\ q_y & q_z & 1 \\ r_y & r_z & 1 \end{vmatrix}; \ \beta_Y = \begin{vmatrix} s_x & s_z & 1 \\ t_x & t_z & 1 \\ u_x & u_z & 1 \end{vmatrix}; \ \gamma_Z = \begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ s_x & s_y & 1 \end{vmatrix}; \ \delta_1 = \begin{vmatrix} r_x & r_y & r_z \\ t_x & t_y & t_z \\ u_x & u_y & u_z \end{vmatrix}.$$

When the three generator lines are not mutually skew, the coefficients degenerate to those of a cylinder or double plane.

Conclusion

Using a duality relationship connecting directed lines in R^3 and hyperplanes in R^5, we have described an algorithm that computes, in a numerically stable fashion, the set of lines incident on four given lines. The computation amounts to a null-space determination of the line incident on four hyperplanes in R^5, and an intersection of this line with a quadric surface. The null-space determination can be cast as a singular value decomposition. The line-quadric intersection amounts to finding the roots of a quadratic equation. Finally, we describe the types of incident line families that result from degenerate input.

Acknowledgments

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