Covering Orthogonal Polygons with Star Polygons: The Perfect Graph Approach

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ABSTRACT

We consider the problem of covering simple orthogonal polygons with star polygons. A star polygon contains a point p, such that for every point q in the star polygon, there is an orthogonally convex polygon containing p and q.

In general, orthogonal polygons can have concavities (dents) with four possible orientations. In this case, we show that the polygon covering problem can be reduced to the problem of covering a weakly triangulated graph with a minimum number of cliques. Since weakly triangulated graphs are perfect, we obtain the following duality relationship: the minimum number of star polygons needed to cover an orthogonal polygon P without holes is equal to the maximum number of points of P, no two of which can be contained together in a covering star polygon. Further, the Ellipsoid method gives us a polynomial algorithm for this covering problem.

In the case where the polygon has at most three dent orientations, we show that the polygon covering problem can be reduced to the problem of covering a triangulated (chordal) graph with a minimum number of cliques. This gives us an $O(n^3)$ algorithm.

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1. Introduction

A useful strategy for solving many computational geometry problems on general polygons is to decompose the polygon into simpler polygons, solve the problem on these polygons using a specialized algorithm, and then combine the solutions. The simpler polygons commonly used are convex polygons and star-shaped polygons [12, 17, 21].

A decomposition is called a partitioning, if the polygon is decomposed into non-overlapping pieces. Partitioning problems have received a lot of attention in the literature. See, for instance, [2, 3, 13, 22].

If overlapping pieces are allowed, the decomposition is called a covering. If the polygon has holes, the problems of covering the polygon with a minimum number of convex polygons or a minimum number of star-shaped polygons are NP-hard [22]. Aggarwal [1] later showed that even if the polygon does not contain holes, the problem of finding a minimum covering with star-shaped polygons remains NP-hard. The problem of finding a minimum covering for polygons without holes with convex polygons remains open.

In the light of these NP-hardness results, it becomes important to restrict our attention to orthogonal polygons. Several interesting results have been obtained for covering orthogonal polygons with simpler polygons. Franzblau and Kleitman [6] provide an $O(n^2)$ algorithm for covering a horizontally convex orthogonal polygon with a minimum number of rectangles. Keil [14] has provided an $O(n^2)$ algorithm for covering a horizontally convex orthogonal polygon with a minimum number of orthogonally convex polygons. Reckhow and Culberson [4, 19] extend this, providing an $O(n^2)$ algorithm for covering an orthogonal polygon with concavities (dents) in two directions only with a minimum number of orthogonally convex polygons. Later, Motwani, Raghunathan and Saran [18] and independently, Reckhow [20], obtained polynomial algorithms for

covering an orthogonal polygon with dents in three directions with a minimum number of orthogonally convex polygons. The general case of covering a simple orthogonal polygon with a minimum number of orthogonally convex polygons remains open.

Two kinds of visibility for orthogonal polygons have been studied in the literature [5]. Two points of the polygon are said to be s-visible if there exists an orthogonally convex polygon that contains the two points. Two points of the polygon are said to be r-visible if there exists a rectangle that contains the two points. This gives us two classes of star covers for orthogonal polygons. An s-star polygon contains a point p, such that for every point q in the polygon, there is an orthogonally convex polygon containing p and q. An r-star polygon is similarly defined. Thus, an s-star cover is a cover by s-star polygons and an r-star cover is a cover by r-star polygons.

For r-star covers, Keil [14] has provided an $O(n^2)$ algorithm for covering a horizontally convex orthogonal polygon with a minimum number of r-stars. For s-star covers, Culberson and Reckhow [5] provide an $O(n^2)$ algorithm for covering an orthogonal polygon with only two dent orientations with a minimum number of s-stars. Since this paper does not deal with r-star coverings, the word star will be taken to mean an s-star.

The purpose of this paper is three-fold: we first show that for the case where the orthogonal polygon has all four dent orientations, the covering problem can be formulated and solved as the problem of finding a minimum clique cover for a weakly triangulated graph [11]. Since weakly triangulated graphs are perfect [11], we obtain the following duality relationship: the minimum number of star polygons needed to cover an orthogonal polygon P is equal to the maximum number of points of P, no two of which can be contained together in a covering star polygon. Further, the Ellipsoid Method of Grötschel, Lov´asz and Schrijver [10] gives us a polynomial algorithm for this covering problem. We then show that the problem of covering orthogonal polygons with three dent orientations with a minimum number of star polygons can be formulated and solved by finding a minimum clique cover for triangulated (chordal) graphs [9, 15]. This gives us an $O(n^3)$ algorithm.

Finally, we wish to make the point that perfect graphs play a crucial role in polygon covering problems. In fact, the main emphasis of this paper is to exhibit this relationship rather than provide the most efficient algorithms. For instance, in the case of covering with orthogonally convex polygons, when the orthogonal polygon has only three dent orientations, the underlying graph defined in [18] is perfect. This helps us to solve the problem. However, when the polygon has all four dent orientations, the same graph is no longer perfect, and the problem is still open. We are also convinced that for covering orthogonal polygons with r-stars, the perfect graph approach presented in this paper

would work. This will be reported elsewhere.

This paper is organized as follows:
In Section 2, we discuss the theoretical framework for this problem, as discussed in [5, 18, 19]. In Section 3, we formulate a graph called the star graph of the polygon, and prove that the minimum clique cover of the star graph corresponds exactly to a minimum star cover of the polygon. Section 4 defines a constriction, and proves several properties related to constrictions that help in the proofs presented in later sections. We state our main results concerning general orthogonal polygons in Section 5, and prove our results in Sections 6 and 7. Section 8 deals with the case where the polygon has dents in only three directions. In this case, we obtain a much more efficient algorithm than in the general case. We present our conclusions in Section 9.

2. Preliminaries

An *orthogonal polygon (OP)*, P, is a polygon with all its sides parallel to one of the two co-ordinate axes. In this paper, we are concerned with covering simple, connected OP's. An OP is said to be an orthogonally convex polygon (OCP) if its intersection with every line parallel to one of the co-ordinate axes is either empty or a single line segment.

2.1. Dents

Consider the traversal of the boundary of P in the clockwise direction, ensuring that the interior is always to the right [19]. At each corner (vertex) of P, we either turn 90° right (outside corner) or 90° left (inside corner). A *dent* is an edge of the boundary of P, both of whose endpoints are inside corners. The direction of traversing a dent gives its *orientation*: for instance, a dent traversed from west to east has a **N** orientation. Figure 1 illustrates the **N**, **S**, **E** and **W** dents. A polygon is said to have only three dent orientations if all its dents take on only one of three orientations (see Figure 2).

2.2. Staircase Paths and Visibility

A *staircase path* $p = x_0, x_1, \cdots, x_r = q$ in P is a path joining p and q and completely contained in P such that x_{i-1} and x_i are the endpoints of a segment parallel to one of the co-ordinate axes and with no two consecutive turns to the same side (left or right). We say $p \equiv q$ (read as p *sees* q, or, p is *visible* from q) if there exists a staircase path joining p and q. The following observation from [19] will be useful.

Observation 1. $p \equiv q$ if and only if some OCP includes both p and q.

We say that a staircase path from p to q goes southwest if, in traversing it from p to q, we go west on all the horizontal segments and south on all vertical segments. Thus, staircase paths between p and q can be of four possible orientations: northeast, northwest, southeast, southwest.

We now define a star polygon.

Definition 1. A *star polygon (SP)* P' is an OP such that $\exists\ p \in P'$ with the property that $\forall\ q \in P', p \equiv q$.

A *maximal SP* in P is an SP contained in P, but not contained in any other SP contained in P. The *visibility polygon*, $\nu(p)$, of $p \in P$ is the set of all points $q \in P$, such that $p \equiv q$.

Lemma 1. The boundary between $\nu(p)$ and any connected component of $P - \nu(p)$ is a single line segment (horizontal or vertical).

Proof: The proof is in two parts. We first show that the boundary between $\nu(p)$ and any connected component Q of $P - \nu(p)$ is connected. Assume to the contrary that the boundary is disconnected. Then, there is a region S that is bounded by $[\nu(p) \cup Q]$, and disjoint from $[\nu(p) \cup Q]$. Since P is simple, S is contained in P. For the same reason, the boundary of S with $\nu(p)$ or Q cannot use a polygon edge. By definition no point of S is in $\nu(p)$. Then, $S \cup Q$ is connected and $[S \cup Q] \in P - \nu(p)$, implying that Q is not a connected component of $P - \nu(p)$, a contradiction.

We now show that the boundary between $\nu(p)$ and Q is a single line segment. Assume to the contrary that the boundary contains adjacent line segments ab, that is horizontal, and bc, that is vertical. Assume without loss of generality, that $\nu(p)$ lies to the south of ab. If p lies to the south of ab, then the staircase paths from p to any point on ab goes northeast or northwest. Such a staircase path can be extended vertically upwards to see points in Q, implying that ab is not part of the boundary. Thus, p lies to the north of ab.

If c lies to the north of b then Q is to the west of bc (see Figure 3). By an argument similar to the one for ab, this implies that p is to the west of bc. Now, the staircase paths from p to a and c, together with ab and bc form an OCP, which must have a non-empty intersection with Q. This implies, by Observation 1, that p sees points in Q, a contradiction.

If c lies to the south of b, then Q is to the east of bc (see Figure 4). As before, p lies to the east of bc. Clearly, the staircase path from p to b must pass through Q, a contradiction.

Q.E.D.

2.3. Dent Lines and Zones

For each dent edge D, we construct a *dent line* $\vec{D}$ by extending D in both directions until it meets the boundary of P. The orientation of $\vec{D}$ is the same as the orientation of D. $\vec{D}$ divides P into three *zones*. Two of these zones (labeled $B_l(\vec{D})$ and $B_r(\vec{D})$ in Figure 5) are said to be *below* the dent, and are the two connected components of P below $\vec{D}$. The third zone, $A(\vec{D})$ is *above* $\vec{D}$, and is the connected component of P above $\vec{D}$. For any $p \in B_l(D), q \in B_r(D)$, $p \not\equiv q$. Thus, if $p \equiv r$ and $q \equiv r$, then $r \in A(D)$. Also, if $p \not\equiv q$, then there is a dent D, such that $p \in B_l(D), q \in B_r(D)$, or vice versa. We say that D *separates* p and q, and D itself is called a separating dent between p and q. Note that when there are several separating dents, we will confine our attention to any one separating dent. The following fact is from [18].

Observation 2. Let L be a southeast staircase path from p to q and M be a northeast staircase path from p to r, where p, q and r are points in P. If $q \not\equiv r$, then an **E** dent D separates q and r. (Similar statements can be made about the other three dent orientations.)

2.4. The Region DAG

The set of all dent lines of P subdivides P into *regions*. Reckhow and Culberson [19] construct a region DAG (directed acyclic graph) as follows: The vertices of the region DAG correspond to the regions, and there is an arc from u to v if u and v share a common border $\vec{D}$ and u is *below* $\vec{D}$. A *source* is a region of zero in-degree in the region DAG. Similarly, a *sink* is a region of zero out-degree in the region DAG. (see Figure 6). Let V denote the set of all sources of P and U denote the set of all sinks of P.

Definition 2. Let u and v be any two regions of the region DAG. Region u is said to *see* region v ($u \equiv v$) if some OCP includes both u and v.

Observation 3. [18] Let u and v be any two regions of the region DAG, and let q_u and q_v be any two points in u and v respectively. Then, $q_u \equiv q_v$ iff $u \equiv v$.

It now follows that for any region u, $v(u)$, the set of all points seen by u, or the visibility polygon of u, is the same as $v(q_u)$, for every $q_u \in u$. Let $N(u)$ denote the set of sources with points in $v(u)$. The following two lemmata from [5] together imply that the minimum set of sinks, $U' \subseteq U$, of P that together see all the sources correspond to a minimum star cover of P. The star polygons that constitute this minimum cover are exactly the visibility polygons of the sinks in U'.

Lemma 2. If β is a set of maximal star polygons that includes every source of P, then β covers every region of P.

Lemma 3. Let $\bigcup_{u \in U'} N(u) = V$. Then, $\{v(u) : u \in U'\}$ is a minimum star cover for P.

Thus, the covering problem has been formulated as the problem of finding the smallest set of sinks that together see every source, which is a set covering problem. Unfortunately, the set covering problem is NP-hard [7], and does not solve the problem for us immediately. However, the geometry of the problem imposes enough structure for it to be solved in polynomial time, as shown in the following sections. Moreover, the advantage of this formulation is that instead of dealing with the visibility of points, we need only consider the visibility of sources and sinks. There are at most $O(n)$ sources and sinks in an OP with three or less dent orientations, and $O(n^2)$ sources and sinks in any simple OP [5].

3. The Star Graph

In order to exploit the geometric structure of the problem, we formulate the set covering problem of the previous section as the problem of finding a minimum clique cover of a graph defined below, called the star graph. Of course, the minimum clique cover problem for general graphs is also NP-hard [7]. However, it turns out that the star graph belongs to a special class of graphs, and this enables us to solve the problem efficiently.

The *star graph* $H = (V, E)$ is defined as follows. The vertices of H are the sources of P. Two vertices, v_1 and v_2 are adjacent in H if there is a sink u that sees both of them, that is, edge $<v_1, v_2> \in E$ if and only if there exists $u \in U$ such that $v_1 \equiv u$ and $v_2 \equiv u$.

For two points p_1, p_2 in P, $p_1 \wedge p_2$ (read as p_1 indirectly sees p_2) if there exists $p \in P$, such that $p_1 \equiv p$ and $p_2 \equiv p$. Let p_1 and p_2 belong to regions r_1 and r_2. We then say that $r_1 \wedge r_2$. We refer to the concatenation of the staircase paths $p_1 p$ and $p\, p_2$ as a *1-bend* path from p_1 to p_2, and p is called the *bend point*. If there does not exist any point p that sees both p_1 and p_2, we say that $p_1 \star p_2$ and $r_1 \star r_2$. As the following lemma shows [5], two vertices of the star graph are adjacent if and only if the corresponding sources indirectly

see each other.

Lemma 4. [5] Two regions indirectly see each other if and only if there exists a sink u in P that sees both.

Clearly, two points of P see each other indirectly if and only if their visibility polygons have at least one point in common. The following two results show that if two points indirectly see each other, then the intersection of their visibility polygons is a simply connected region. Further, if three points indirectly see each other, then there is at least one point that sees all three.

Lemma 5. Let $p, q \in P$ such that $v(p) \cap v(q) \neq \emptyset$. Then, $v(p) \cap v(q)$ is a simply connected polygon.

Proof: Assume to the contrary that $v(p) \cap v(q)$ is not simply connected. Let $W = v(q) - [Ep \cap Eq]$. Since $v(p) \cap v(q)$ is not simply connected, the boundary between $v(q)$ and W is not simply connected. As in the proof of Lemma 1, $v(q) \cup W$, which is $v(p) \cup v(q)$, bounds a region S that is disjoint from $v(q) \cup W$, or $v(p) \cup v(q)$. Every point of S is in P, as P is a simple polygon. Therefore, the boundary of S is composed entirely of the boundary of $v(p) \cup v(q)$, not including edges of P. Let R be the connected component of $P - v(p)$ that contains S. By Lemma 1, the boundary of $v(p)$ with R is a single line segment, implying that the boundary of $v(p)$ with S is a single line segment. Similarly, the boundary of $v(q)$ with S is a single line segment. Since at least four orthogonal line segments are required to enclose a region, we obtain a contradiction.

Q.E.D.

Lemma 6. Let $p, q, r \in P$ such that $v(p) \cap v(q) \neq \emptyset$, $v(q) \cap v(r) \neq \emptyset$ and $v(r) \cap v(p) \neq \emptyset$. Then, $v(p) \cap v(q) \cap v(r) \neq \emptyset$.

Proof: Assume to the contrary that $v(p) \cap v(q) \cap v(r) = \emptyset$. Since $v(p) \cap v(q) \neq \emptyset$, $v(q) \cap v(r) \neq \emptyset$ and $v(r) \cap v(p) \neq \emptyset$, we can show, by a proof similar to that of Lemma 5, that $v(p) \cup v(q) \cup v(r)$ bounds a region, say S, that is disjoint with $v(p) \cup v(q) \cup v(r)$. We can now show, again, by a proof similar to that of Lemma 5, that this is impossible, because three orthogonal lines cannot bound a region. This establishes that $v(p) \cap v(q) \cap v(r) \neq \emptyset$.

Q.E.D.

In what follows, a *cell* is a simply connected compact subset of the plane. The following theorem, due to Moln´ar [16], will be useful.

Theorem 1. (Moln´ar) Let C be a set of cells in the plane. If $C \cap C'$ is a cell for every $C, C' \in$ C and $C \cap C' \cap C'' \neq \emptyset$ for $C, C', C'' \in$ C, then $\cap \{C \in \mathrm{C}\} \neq \emptyset$.

Consider any clique H' in H. The visibility polygon of any source in H' is a cell. The intersection of any two such visibility polygons is non-empty, and a cell by Lemma 5. By Lemma 6, the intersection of any three such visibility polygons is non-empty. Molnar's Theorem now assures us that the intersection of all these visibility polygons is non-empty, implying that there is a point that sees all the sources of H'. By Lemma 4, there is a sink in P that sees all these sources. This results in Theorem 2.

Theorem 2. Let $H' = (V', E')$ be a clique in H. Then, there is a maximal star polygon that covers all the sources in V'.

If v_1 and v_2 are not adjacent in H, they cannot both be covered by the same star polygon. Thus, every maximal star polygon corresponds to a clique in H. We obtain the following corollary.

Corollary 1. A minimum clique cover of H (that is, a minimum cardinality set of cliques of H with every vertex of H belonging to some clique) corresponds exactly to a minimum cover of P by star polygons.

4. Constrictions

The main purpose of this section is to provide a theoretical handle on the covering problem. Consider two points, $p, q \in P$, such that they do not have any 1-bend staircase path between them, i.e., $p \not\sim q$. Given such a pair of points, we will identify a connected subset of P, $Cons(p, q)$, with the following properties: (a) there is no staircase path from either p or q to any point in $Cons(p, q)$, and (b) any path in P from p to q must pass through this region. We call this region the constriction between p and q. In a sense, the existence of the constriction is the reason why there is no 1-bend path from p to q. We first give a formal definition of a constriction and then obtain certain useful properties.

Let p and q be points of P, and let $p \not\sim q$. Clearly $\mathrm{v}(p) \cap \mathrm{v}(q) = \emptyset$. Let $Q_1, \cdots, Q_k$ denote the connected components of $Q = P - [\mathrm{v}(p) \cup \mathrm{v}(q)]$. We first claim that there is a unique connected component of Q which shares a boundary with both $\mathrm{v}(p)$ and $\mathrm{v}(q)$.

Lemma 7. There is a unique Q_i for which $v(p), v(q)$ and Q_i form a connected polygon.

Proof: Clearly, $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Since P is connected, this implies that there exists at least one i such that $P_i = Q_i \cup [v(p) \cup v(q)]$ is connected. Now, let there be $i, j, i \neq j$ such that P_i and P_j are connected. Then, $[v(p) \cup v(q)] \cup [Q_i \cup Q_j]$ bounds a region S that is disjoint from it. Since P is simple, every point in S is in P. This implies that Q_i and Q_j were not connected components of $P - [v(p) \cup v(q)]$, a contradiction.

Q.E.D.

Definition 3. Q_i is called the *constriction* between p and q, and is denoted by $Cons(p, q)$. (see Figure 7)

We now state four important properties concerning $Cons(p, q)$.

Property 1. The boundary between $Cons(p, q)$ and $v(p)$ (resp., $v(q)$) is a single line segment (horizontal or vertical), called the p dent line (resp., the q dent line).

Proof: Consider the orthogonal polygon $P - v(p)$. By Definition 3, $Cons(p, q)$ and $v(q)$ lie in the same connected component S of $P - v(p)$. The boundary of S with $v(p)$ is the same as the boundary of $Cons(p, q)$ and $v(p)$, which, by Lemma 1, is a single line segment (horizontal or vertical). Similarly, the boundary of $v(q)$ and $Cons(p, q)$ is a single line segment (horizontal or vertical).

Q.E.D.

Property 2. Let the p dent line of $Cons(p, q)$ be vertical, and let $Cons(p, q)$ lie to its east (west). Let r' be a point on the p dent line, and let r be any point to the east (resp., west) of r', such that $r \equiv r'$. Then, an **E** dent (resp., **W** dent) separates p from r. (Similar statements can be made about the other two dent orientations).

Proof: Since p sees r', and does not see r at an infinitismal distance to the east of r, no staircase from p to r' can travel to the northeast or southeast (see Figure). Thus, r' sees p to its northeast or southeast, and sees r to its east, and $p \not\equiv r$. By Observation 2, there is an **E** dent separating p and r. The proof for the other part of Property 2 is similar.

Q.E.D.

Property 3. Let R be a path in P connecting p and q. Further, let $p \not\sim q$. Then, there is a connected subpath of R that joins a point on the p dent line and a point on the q dent line of $Cons(p, q)$ and is completely contained in $Cons(p, q)$.

Proof: This follows trivially from the definition of $Cons(p, q)$.

Q.E.D.

Property 4. Let $p \star q$, and let $r \wedge p$ and $r \wedge q$. Then, (1) If $r \in Cons(p, q)$, there is a 1-bend path from a point on the p dent line to a point on the q dent line with r as bend point, (2) If $r \notin Cons(p, q)$, r has a single staircase path that meets the p and q dent lines.

Proof: If $r \in Cons(p, q)$, we have that $r \not\equiv p$ and $r \not\equiv q$, implying that there are staircase paths from r to points on the p and q dent lines. If $r \notin Cons(p, q)$, assume, without loss of generality, that r is on the same side of $Cons(p, q)$ as p. Now, clearly, $r \not\equiv q$, implying that there is a staircase path L from r to the q dent line. L has to intersect the p dent line, thus establishing a staircase to both the p and q dent lines.

Q.E.D.

We now define three types of constrictions. The other types of constrictions do not play a part in this paper.

Type I. In a type I constriction, the p and q dent lines are parallel and there exist points r_1 and r_2 on the p and q dent lines respectively such that $r_1 \equiv r_2$.

Type II. In a type II constriction, the p and q dent lines are orthogonal and there exist points r_1 and r_2 on the p and q dent lines respectively such that $r_1 \equiv r_2$.

Type III. In a type III constriction, $Cons(p, q)$ is not of type I or II, and there exist points r_1 and r_2 on the p and q dent lines respectively such that $r_1 \wedge r_2$. Thus, if $r \wedge p$ and $r \wedge q$, then $r \in Cons(p, q)$.

The next four results will be used repeatedly in the following three sections. In what follows, $p, q \in P$, such that $p \star q$.

Lemma 8. If $\exists\ r \in P$, such that $r \wedge p$ and $r \wedge q$, then $Cons(p, q)$ is of type I, II or III.

Proof: Trivial.

Q.E.D.

Lemma 9. Let $r \in Cons(p, q)$, such that $r \wedge p$ and $r \wedge q$. Further, let $s \wedge p$ and $s \wedge q$. Then $r \wedge s$.

Proof: By Property 4, there is a 1-bend path L from r' on the p dent line to r'' on the q dent line with r as bend point. Without loss of generality, let the p dent line be vertical, and let the staircase path from r to r' be southwest (the other cases are handled similarly). Thus, r sees every point below r' on the p dent line (see Figure 8).

Assume to the contrary that $r \star s$. This implies that s does not see r' or any point below r' on the p dent line. Therefore, let s see a point s' on the p dent line that is above r'. Clearly, $r \not\equiv s'$. Since r' sees s' to its north, and r' sees r to its northeast, Observation 2 implies that there is a N dent D separating r and s'. We have $s' \in B_l(D)$ and $r \in B_r(D)$ (see Figure 8). $r' \equiv s'$ and $r' \equiv r$, implying that $r' \in A(D)$. Thus, L crosses

$\vec{D}$ at some point, say l'. By Property 4, there is a path M from s' to s'' on the q dent line that is either a staircase or a 1-bend path with s as the bend point. Since $s'' \notin B_l(D)$, M crosses $\vec{D}$ at m', such that m' is to the west of l'. Hence, $r \equiv m'$, implying that $r \wedge s$.

Q.E.D.

Lemma 10. Let $r \in Cons(p, q)$, such that $r \wedge p$ and $r \wedge q$. Let $R = (p\, r_1, r_1 r_2, \cdots, r_{k-1} r_k, r_k\, q)$ be a sequence of 1-bend paths. Then, $\exists\ i \in \{1, \cdots, k\}$ such that $r \wedge r_i$.

Proof: By a proof very similar to that of Lemma 9, we can establish that r sees some point, say m', of R such that $m' \in Cons(p, q)$. Since R is a sequence of 1-bend paths, and $m' \in Cons(p, q)$, there exists $r_i \in \{r_1, \cdots, r_k\}$, such that $r_i \equiv m'$, implying that $r \wedge r_i$.

Q.E.D.

Lemma 11. Let $Cons(p, q)$ be of type II. Let $r \notin Cons(p, q)$, such that $r \wedge p$ and $r \wedge q$. Let $R = (p\, r_1, r_1 r_2, \cdots, r_{k-1} r_k, r_k\, q)$ be a sequence of 1-bend paths. Then, $\exists\ i \in \{1, \cdots, k\}$ such that $r \wedge r_i$.

Proof: Let r be on the same side of $Cons(p, q)$ as p (the other case is symmetrical). Without loss of generality, assume that the p dent line is vertical and the q dent line is horizontal. By Property 4, there is a staircase path L (southeast, say) from r to r'' on the q dent line that meets the p dent line at r'. Clearly, r sees every point of the p dent line below r' and every point of the q dent line to the right of r'' (see Figure 9). For r not to see some point of R inside $Cons(p, q)$, R has to meet the p dent line above r', and the q dent line to the left of r''. Thus, R meets L inside $Cons(p, q)$, implying that $r \wedge r_i$, for some $r_i \in \{r_1, \cdots, r_k\}$.

Q.E.D.

5. Weakly Triangulated Graphs and the Star Graph

In this section, we assert that the star graph introduced in the previous section belongs to a special class of graphs called perfect graphs [9, 15]. In a perfect graph G, the size of a minimum clique cover of every induced subgraph G' is equal to the size of a maximum independent set of G'. A minimum clique cover of a perfect graph can be found in polynomial time [10]. We first need the following definition.

Definition 4. A graph G is *weakly triangulated* [11] if neither G nor G^c, the complement of G contain induced cycles of length greater than four.

Theorem 3. (Hayward) Weakly triangulated graphs are perfect.

We now state our main result, the proof of which is contained in the next two sections.

Theorem 4. The star graph of an orthogonal polygon P is weakly triangulated.

Theorem 4, together with Hayward's theorem, provides us with the following duality relationship:

Corollary 2. (The Duality Relationship) The minimum number of star polygons needed to cover an orthogonal polygon P is equal to the maximum number of points of P, no two of which see each other by 1-bend paths.

We can now use any algorithm that would cover the vertices of a weakly triangulated graph with a minimum number of cliques to cover an orthogonal polygon with a minimum number of star polygons. The best known algorithm to find minimum clique cover of a perfect graph, which is due to Grötschel, Lov´asz and Schrijver, is based on the Ellipsoid method, and it runs in polynomial time [10]. Reckhow and Culberson [5] provide a polynomial algorithm to compute the instance of the set covering problem in Section 2. The star graph can easily be obtained from this. Thus, there is a polynomial algorithm for finding a minimum star cover for P.

6. Induced Cycles of the Star Graph

In this section, we establish one part of the proof of Theorem 4, namely that the star graph contains no induced cycles of length greater than four.

Lemma 12. The star graph H does not contain an induced cycle of length greater than four.

Proof: Assume to the contrary that $C = (V', E')$ is such an induced cycle, $|V'| > 4$. For convenience, let $V' = \{1, 2, \cdots, n\}, n > 4$, and let $< i, i + 1 \bmod n > \in E'$. By assumption, edge $< 1, 3 > \notin E$, implying that $1 \not\wedge 3$. Hence, we have the constriction $Cons\,(1, 3)$.

We now assert that 2 cannot be in $Cons\,(1, 3)$. Assume to the contrary that $2 \in Cons\,(1, 3)$. $R = C - \{2\}$ is a sequence of 1-bend paths connecting 1 and 3, and hence by Lemma 10, $2 \wedge i$, for some $i \in \{4, \cdots, n\}$, thus establishing a chord.

This also implies that $Cons\,(1, 3)$ is not of type III. Further, by Lemma 11, if $Cons\,(1, 3)$ is of type II, $2 \wedge i$, for some $i \in \{4, \cdots, n\}$, establishing a contradiction.

Now, let *Cons* (1, 3) be of type I, and let the two dent lines be vertical (the other case is similar). Further, let 2 be on the same side of *Cons* (1, 3) as 1. By the proof of Property 4, there is a staircase L (southeast, say) to the 3 dent line, meeting the 1 and 3 dent lines at $2'$ and $2''$, respectively. Therefore, 2 sees every point below $2'$ and $2''$ on the 1 and 3 dent lines, respectively (see Figure 10). To prevent chords, R needs to meet the 1 and 3 dent lines (at z and y) above $2'$ and $2''$.

Case 1: $2' \not\equiv y$ (see Figure 11).
$2''$ sees y to its north, and sees $2'$ to its northwest. By Observation 2, there is a N dent D separating $2'$ and y. We have $2' \in B_l(D)$ and $y \in B_r(D)$. $2'' \equiv 2'$ and $2'' \equiv y$, implying that $2'' \in A(D)$. Thus, L crosses $\vec{D}$ at some point, say l'. Since z is to the north of $2'$, we have that $z \in B_l(D)$. Thus, R is a sequence of 1-bend paths from a point in $B_l(D)$ to a point in $B_r(D)$. This implies that R cross $\vec{D}$ at x to the east of l'. Hence, $2 \equiv x$, implying that $2 \wedge i$, for some $i \in \{4, \cdots, n\}$, a contradiction.

Case 2: $2' \equiv y$ (see Figure 12).
If the staircase from $2'$ to y is southeast, then the concatenation of this staircase with the southeast staircase from 2 to $2'$ gives us a southeast staircase from 2 to y, implying that $2 \wedge i$, for some $i \in \{4, \cdots, n\}$, a contradiction. Therefore, the staircase from $2'$ to y is northeast. Let R' be such a staircase from $2'$ to y such that no other such staircase lies entirely to its north. Thus, some horizontal edge of R' touches a polygon edge. R', L and the 3 dent line between y and $2''$ together form an OCP, S. We now assert that R has a point in S. Assume to the contrary that R does not have a point in S. Then $z \notin S$. This means that z is above R' on the 1 dent line. By our choice of R', y and z are now in different connected components of the portion of P that is bounded by R' and the two dent lines (see Figure 11). Since this portion of P borders S at R', and R is a connected path, R must have a point in S.

Since R is a sequence of 1-bend paths, and S is in *Cons* (1, 3), one of the points of R in S is either $i \in \{4, \cdots, n\}$ or a bend point x such that x sees some $j \in \{5, \cdots, n\}$ (every bend point of R that is in *Cons* (1, 3) sees two points in $\{4, \cdots, n\}$. If $i \in S$, then the vertical line from i to L exists in S (S is an OCP), thus establishing a chord between 2 and i. If there is no $i \in \{4, \cdots, n\}$ in S, then we have a bend point, $x \in S$, such that $x \equiv j$, for some $j \in \{5, \cdots, n\}$. By our choice of S, j is above R' in *Cons* (1, 3). Since R' is a northeast staircase from $2'$ to y, the staircase from j to x can only travel southwest, southeast or northeast. If it is southwest or southeast, then $2 \wedge j$ by using a vertical line from x to L. If it is northeast, then $3 \wedge j$ by using a horizontal line from x to the 3 dent line, implying that $3 \wedge j$, a contradiction.

Q.E.D.

7. Induced Cycles in the Complement of the Star Graph

We establish in this section the other part of the proof of Theorem 4, namely that the complement $H^c = (V, F)$ of the star graph does not have induced cycles of length greater than 4.

Lemma 13. H^c does not contain an induced cycle of length greater than 4.

Proof: Since H does not contain an induced 5 cycle, and the complement of a 5 cycle is a 5 cycle, H^c cannot contain an induced 5 cycle. Assume to the contrary that there exists an induced cycle $C = (V', F')$ in H^c, with $|V'| > 5$. For convenience, let $V' = \{1, 2, \cdots, n\}, n > 5$, and let $<i, i+1 \bmod n> \in H'$. Since $1 \not\wedge 2$, we have the constriction $Cons(1,2)$. Since edge $<i, j> \notin F$, for $j \neq i+1 \bmod n$, we have that $i \wedge j$.

We now assert that $4, 5, \cdots, n-1$ cannot be in $Cons(1,2)$. Suppose 4 were in $Cons(1,2)$. $4 \wedge 1$ and $4 \wedge 2$, and $5 \wedge 1$ and $5 \wedge 2$. From Lemma 9, $4 \wedge 5$, a contradiction. Similar arguments establish that $5, \cdots, n-1$ cannot be in $Cons(1,2)$. It is further clear that $Cons(1,2)$ is not of type III, else $4, 5, \cdots, n-1$ have to be inside $Cons(1,2)$ in order to see 1 and 2. Suppose $Cons(1,2)$ is of type II. Then, by Lemma 11, $4 \wedge 5$, a contradiction.

We now have that all of $Cons(i, i+1 \bmod n)$ must be of type I. Let us now further classify type I constrictions as type IA (where the two dent lines are vertical) and type IB (where the two dent lines are horizontal). We now assert that $Cons(1,2)$ and $Cons(i, i+1)$ cannot both be of type IA or IB (IA, say), for some $i \in \{4, \cdots, n-2\}$. To see this, we reason as follows: Neither the i dent line nor the $i+1$ dent line of $Cons(i, i+1)$ can be inside $Cons(1,2)$, since i and $i+1$ see both 1 and 2. Now, let the i dent line be outside $Cons(1,2)$, implying that $Cons(i, i+1)$ and $Cons(1,2)$ are on different sides of the i dent line. This would imply that that $i+1$ does not indirectly see 1 and 2, a contradiction. A simple combinatorial argument now shows that it is impossible to obtain an induced 7 cycle using only type IA and IB constrictions for cycle edges. Thus, let $n = 6$.

It is clear that at least two of $Cons(1,2)$, $Cons(3,4)$ and $Cons(5,6)$ must be of the same type (IA or IB). Assume, without loss of generality, that $Cons(1,2)$ and $Cons(3,4)$ are both of type IA. We now assert that this would imply that $Cons(2,3)$ must be of type IA and that $Cons(1,2)$ is contained in $Cons(3,4)$ (see Figure 13 (a)). Thus, the only way that one could possibly obtain an induced 6 cycle is by using type IA constrictions for three consecutive edges and type IB consttrictions for the other three edges. Let edges $<1,2>$, $<2,3>$, and $<3,4>$ use type IA constrictions and let $<4,5>$, $<5,6>$ and $<6,1>$ use type IB constrictions. The arrangement is shown in Figures 13 (a) and 13 (b).

Since 6 is not in *Cons* (2,3), and $6 \wedge 2$ and $6 \wedge 3$, Property 4 implies that there is a single staircase path K from 6 to the 2 and 3 dent lines of *Cons* (2,3), and passing through *Cons* (2,3). Since the 1 dent line of *Cons* (1,2) is in *Cons* (2,3), (see Figure 13 (a)), K intersects the 1 dent line of *Cons* (1,2), implying that $6 \wedge 1$. But <6,1> is a cycle edge of H^c, a contradiction.

Q.E.D.

8. An $O(n^3)$ Algorithm for Polygons with Three Dent Orientations

In this section, we show that if P has only three dent orientations, then the star graph H is triangulated [9, 15]. This would give us an $O(n^3)$ algorithm for finding the star cover for P.

Definition 5. [9, 15] A graph is said to be triangulated (or chordal) if it contains no induced cycles of length greater than three.

In general, the star graph H is not triangulated: there exist induced 4 cycles in H (see Figure 14). For the case where P has only three dent orientations, H is triangulated, as the following theorem shows.

Theorem 5. The star graph H of an orthogonal polygon with at most three dent orientations is triangulated.

Proof: By Lemma 12, we have that H does not contain induced cycles of length greater than four. It now suffices to show that H cannot have an induced 4 cycle.

Assume to the contrary that <1,2>, <2,3>, <3,4> and <4,1> is an induced 4 cycle in H. Since <1,3> is not present, $1 \not\wedge 3$. Consider *Cons* (1,3). Neither 2 nor 4 is in *Cons* (1,3), else by Lemma 9, $2 \wedge 4$. This also implies that *Cons* (1,3) is not of type III.

If *Cons* (1,3) is of type II, then by Lemma 11, $2 \wedge 4$, a contradiction.

Thus, *Cons* (1,3) is a type I constriction. Without loss of generality, let the two dent lines be vertical and let the 1 dent line be to the west of the 3 dent line (see Figure 15 (a)). By Property 2, we have established the presence of an E dent and a W dent.

Without loss of generality, let 2 be on the same side of *Cons* (1,3) as 1. By Property 4, there is a staircase L (southeast, say) from 2 to 2″ on the 3 dent line, intersecting the 1 dent line at 2′. 2 sees every point below 2′ on the 1 dent line and every point below 2″ on the 3 dent line.

Case 1: Let 4 be on the same side of *Cons* (1,3) as 1.
Then, every staircase from 4 to 4″ on the 3 dent line, and hence to 4′ on the 1 dent line,

has to be northeast (else, $4 \wedge 2$) (see Figure 15 (a)). Since 2 does not see 4′, 2′ sees 4′ to its north, and 2′ sees 2 to its northwest, Observation 2 implies that a N dent separates 2 and 4′. By a similar argument, a S dent separates 4 from 2′. This establishes that there exist 4 different dent orientations, a contradiction.

Case 2: Let 4 be on the same side of *Cons* (1, 3) as 3.
Then the staircase from 4 to 4′ on the 1 dent line, and hence to 4″ on the 3 dent line, has to be northwest (else, $4 \wedge 2$) (see Figure 15 (b)) Since 2 does not see 4′, 2′ sees 4′ to its north, and 2′ sees 2 to its northwest, Observation 2 implies that a N dent separates 2 and 4′. By a similar argument, a S dent separates 4 from 2″. This establishes the existence of four different dent orientations, a contradiction.

Q.E.D..

Reckhow and Culberson [19] show that if P has only three dent orientations, the number of sources and the number of sinks are both $O(n)$. For a source v and sink u, it is easy to figure out in $O(n)$ time if $u \equiv v$ [5]. Thus, in $O(n^3)$ time, we can list the sources seen by each sink. Now, for each pair of sources, we can figure out in $O(n)$ time if they see each other indirectly, thus constructing the sink graph H in $O(n^3)$ time (H has $O(n^2)$ edges). Gavril's algorithm [8] now gives, in $O(n^3)$ time, the minimum clique cover of H, implying that the the star cover of P can be obtained in $O(n^3)$ time.

9. Conclusions

We have shown that the minimum clique cover of the star graph of a simple orthogonal polygon corresponds exactly to a minimum star cover of the polygon. We have further shown that this graph is weakly triangulated. By Hayward's Theorem, weakly triangulated graphs are perfect [11]. This implies the following duality relationship: the minimum number of star polygons needed to cover an orthogonal polygon is equal to the maximum number of points in the polygon, no two of which can be contained together in a single star polygon. Further, The Ellipsoid method of Grötschel, Lov´asz and Schrijver provides a polynomial algorithm to find a minimum clique cover of perfect graphs, and hence to cover such polygons with a minimum number star polygons.

In the case where the polygon has only three dent orientations, we have shown that the star graph is triangulated. By Gavril's algorithm, we can find a minimum star cover for such polygons in $O(n^3)$ time.

Acknowledgements

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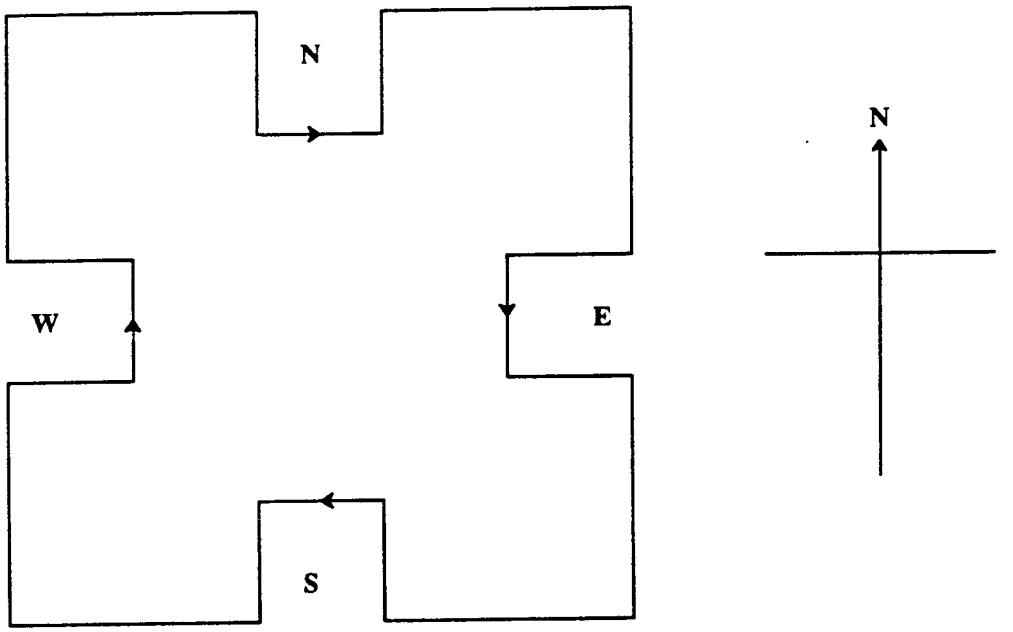

Figure 1: Orientation of dents

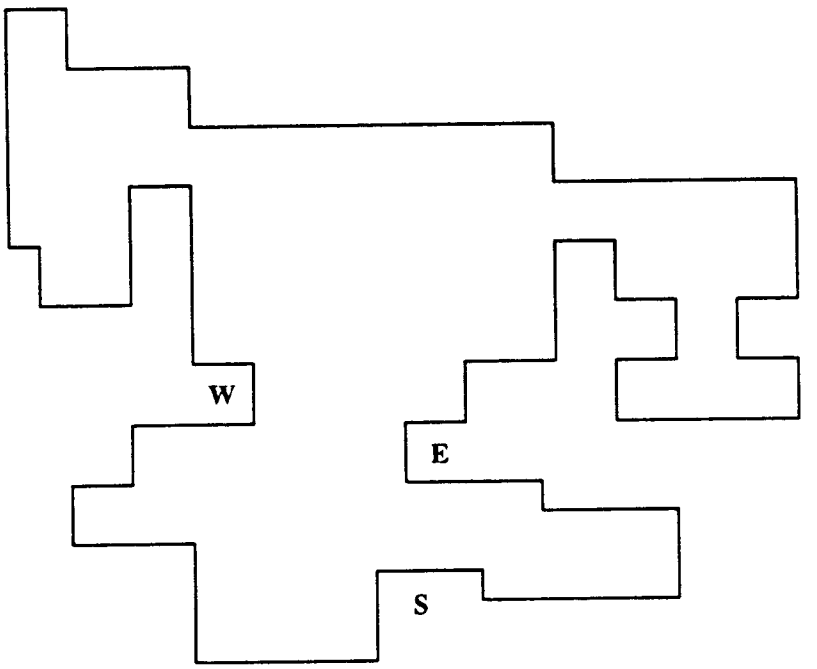

Figure 2: A Polygon with only 3 dent orientations

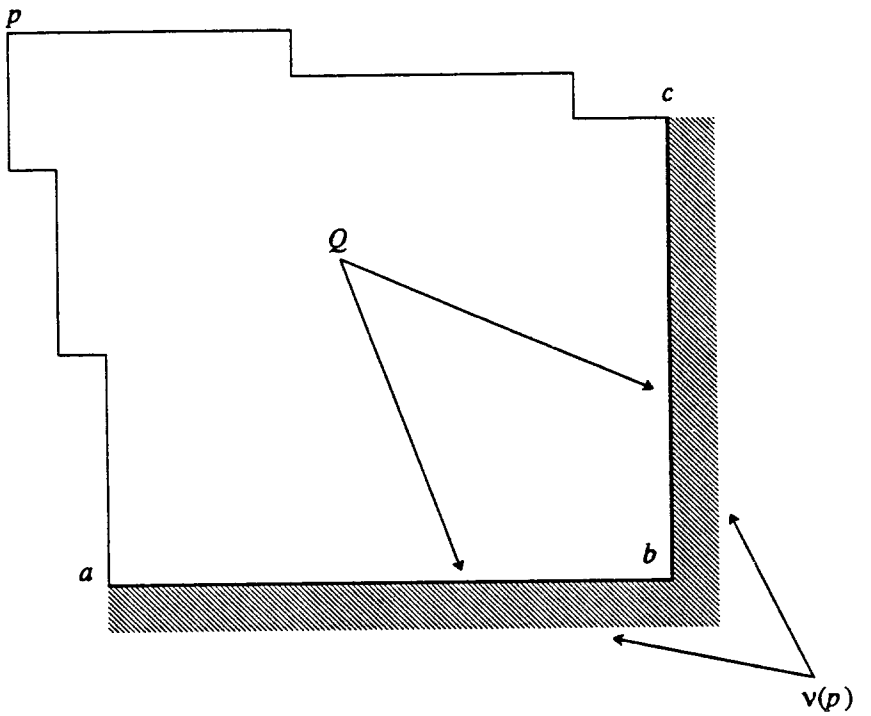

Figure 3: Proof of Lemma 1

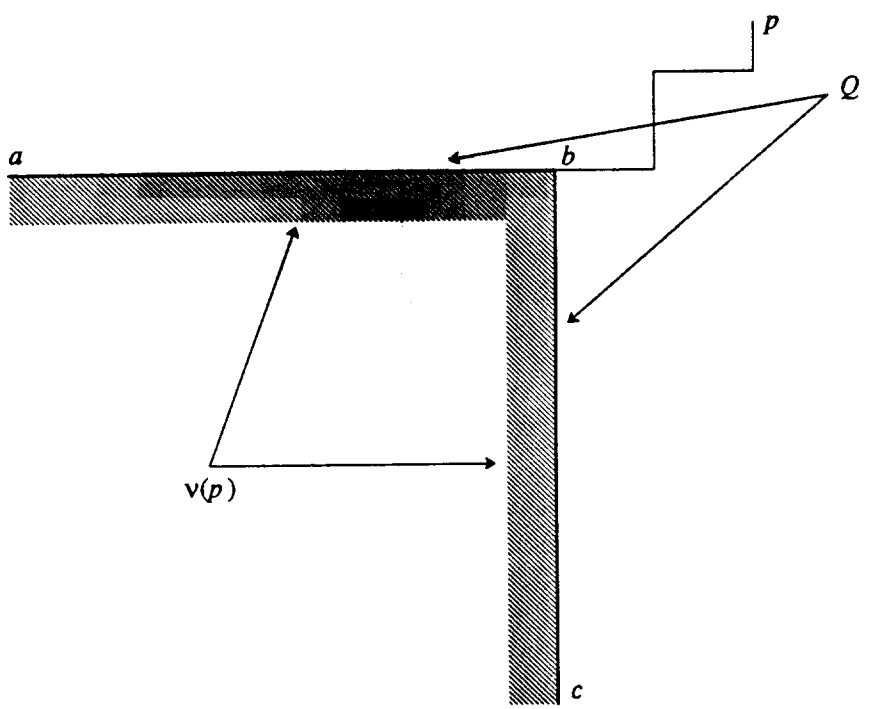

Figure 4: Proof of Lemma 1

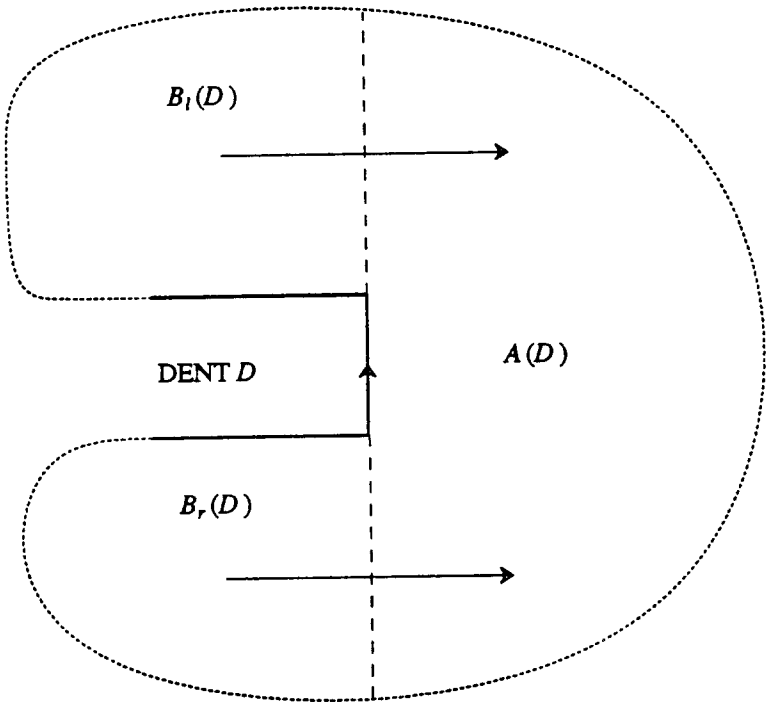

Figure 5: Dent lines and Zones

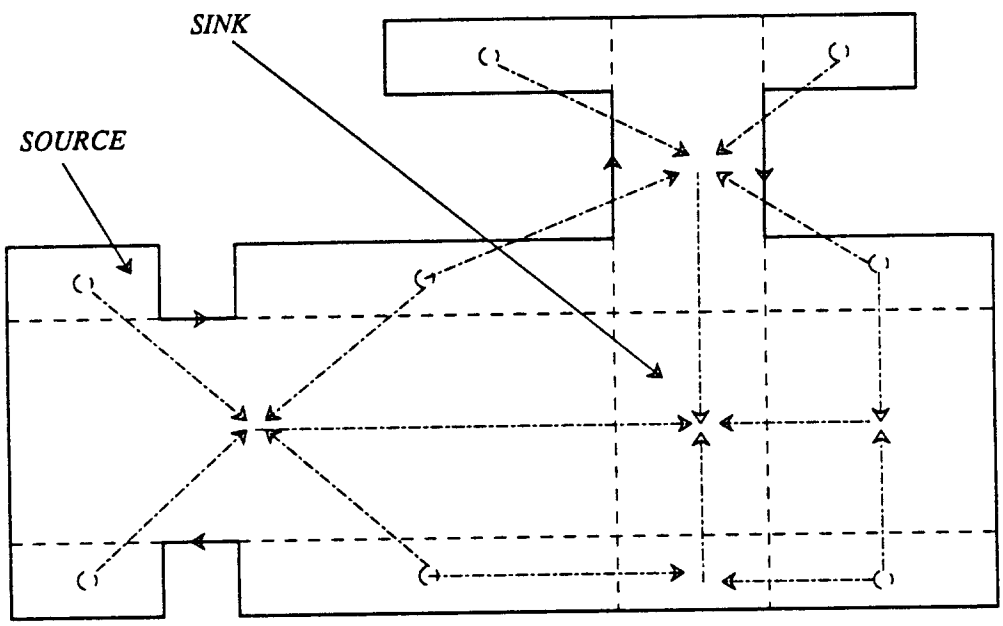

Figure 6: Identifying Sources and Sinks

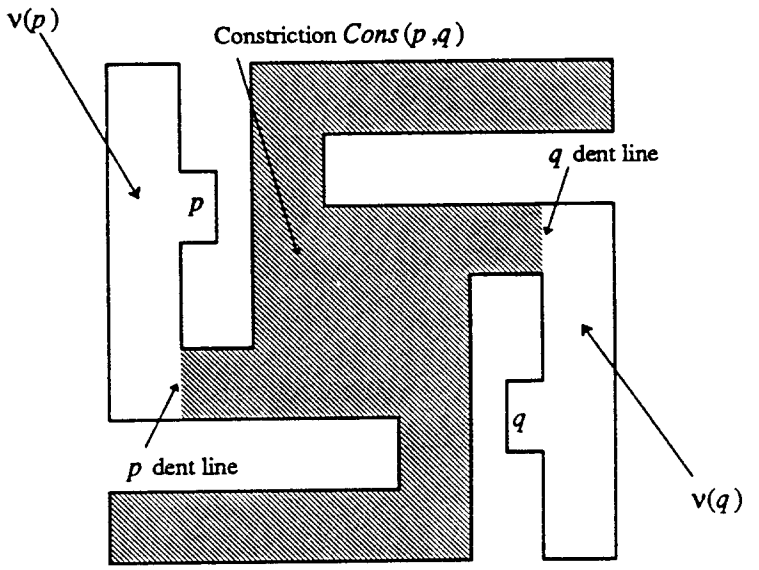

Figure 7: A constriction between points p and q

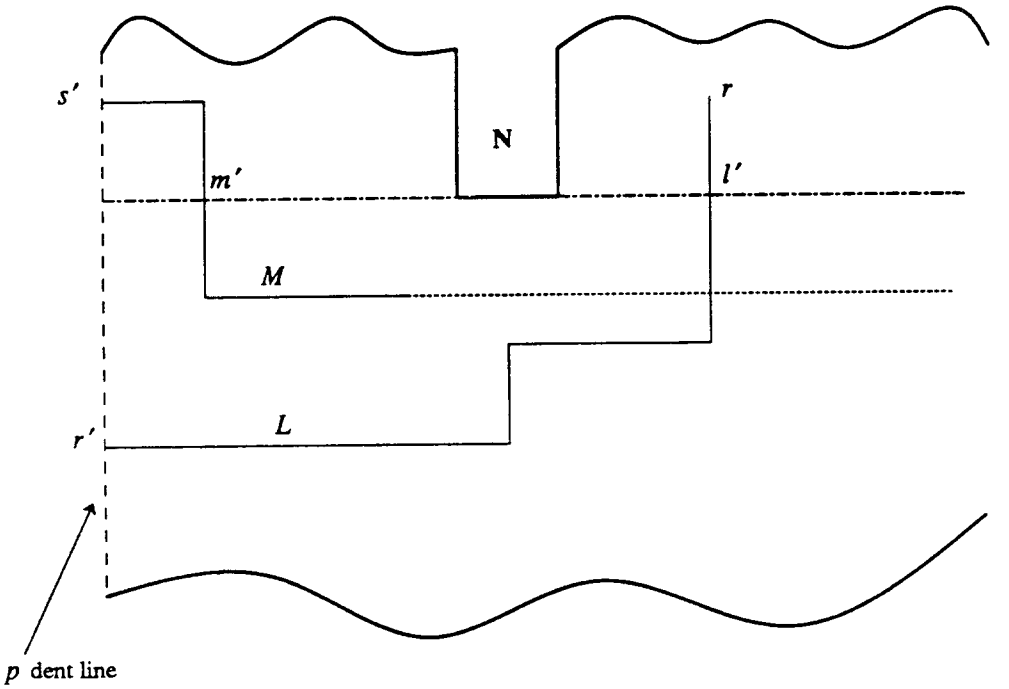

Figure 8: Proof of Lemma 9

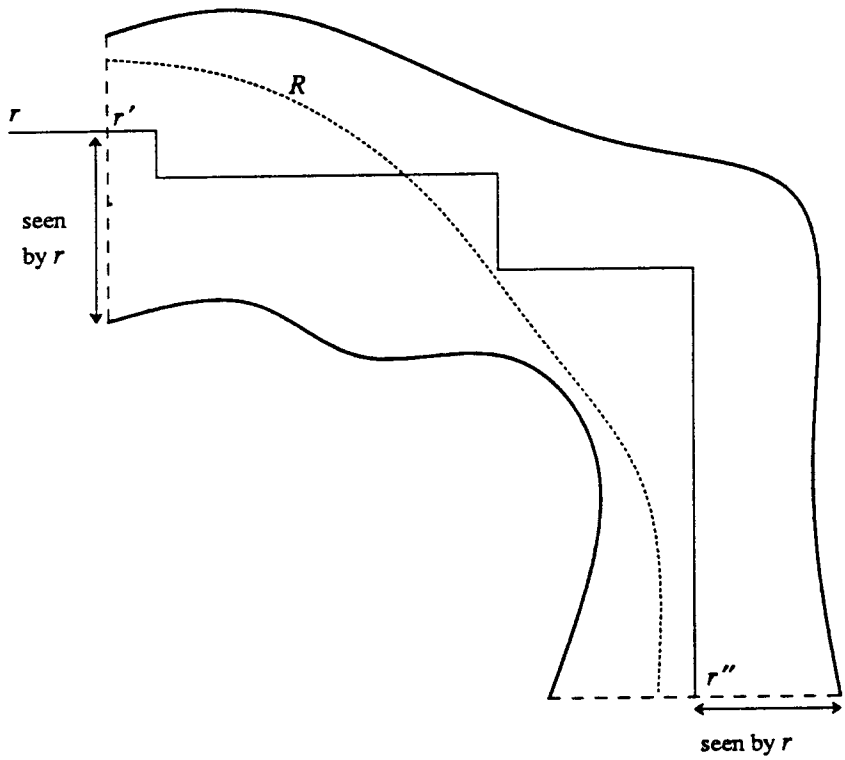

Figure 9: Proof of Lemma 11

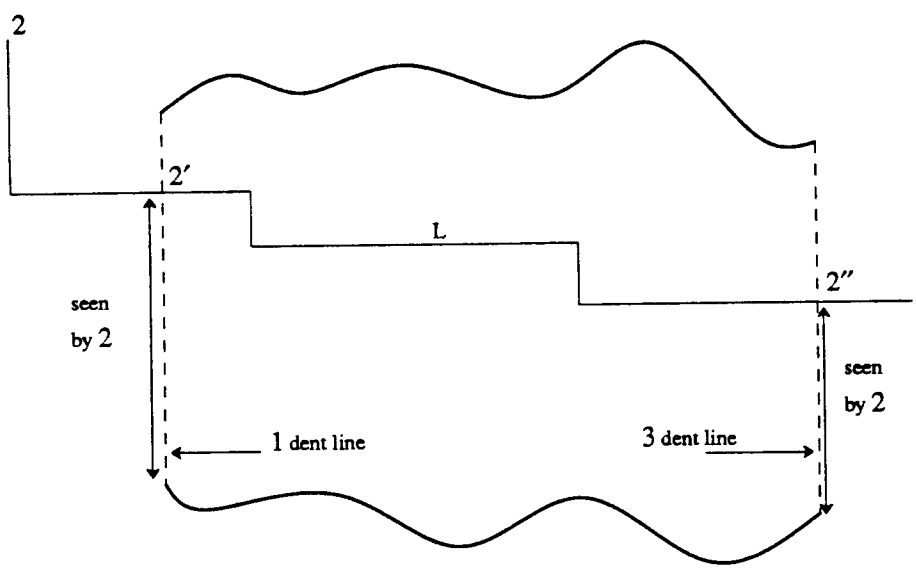

Figure 10: Proof of Lemma 12

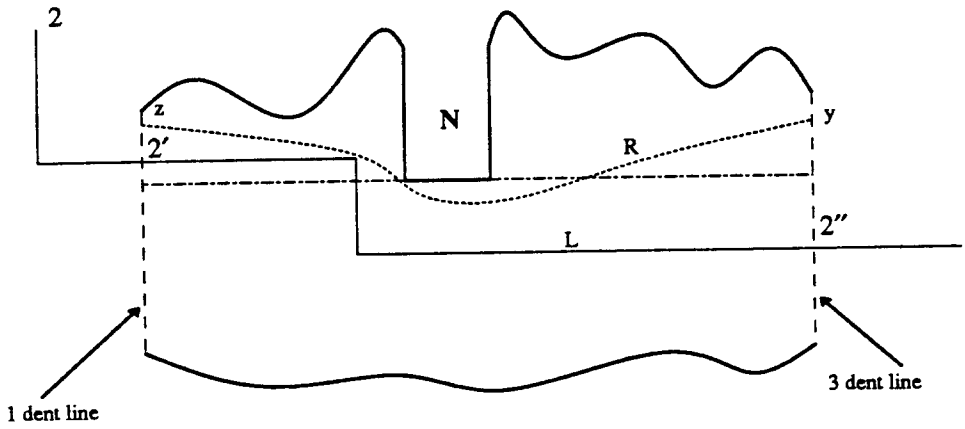

Figure 11: Proof of Lemma 12, Case 1

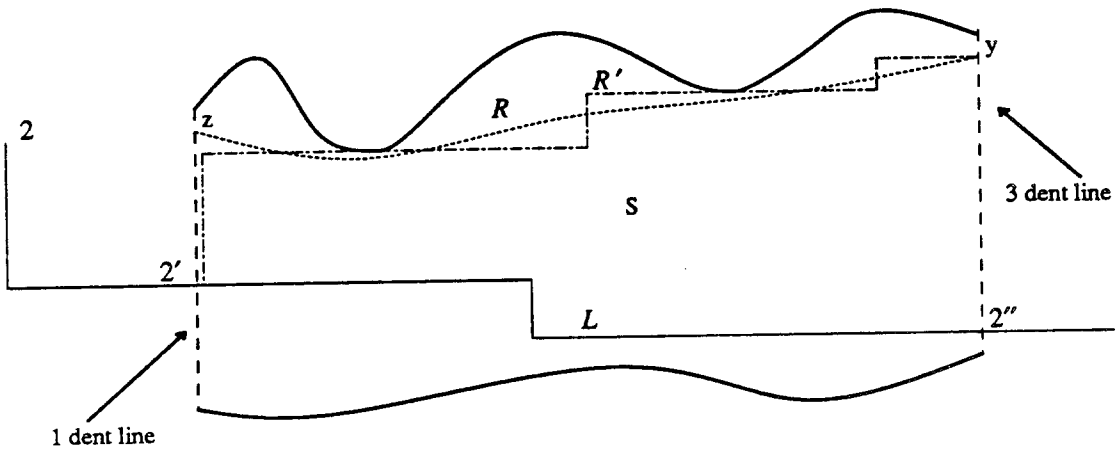

Figure 12: Proof of Lemma 12, Case 2

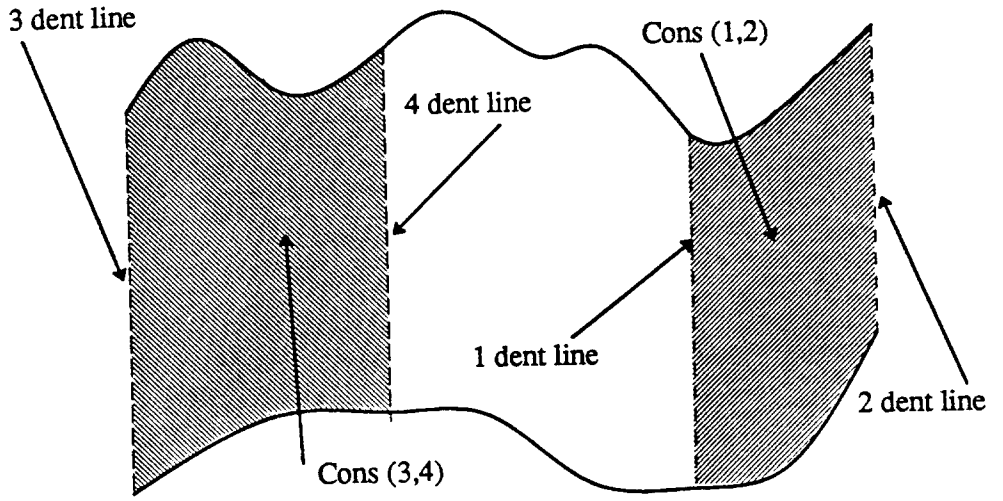

Figure 13 (a): Proof of Lemma 13

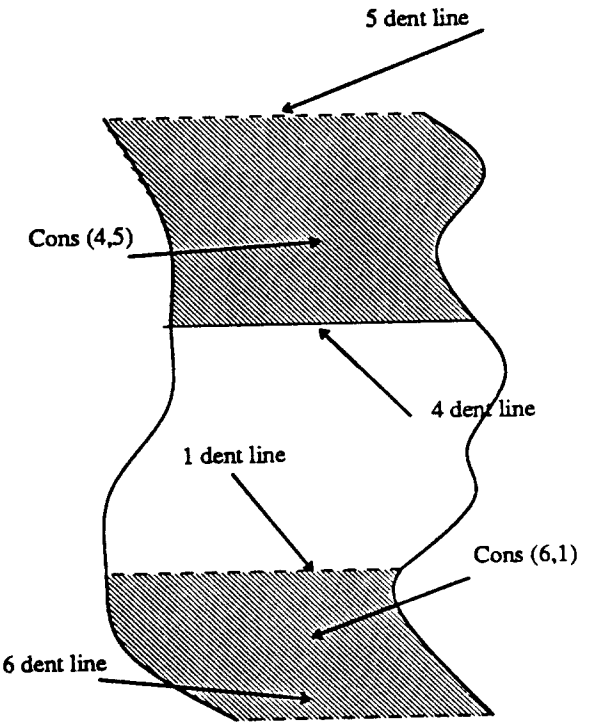

Figure 13 (b): Proof of Lemma 13

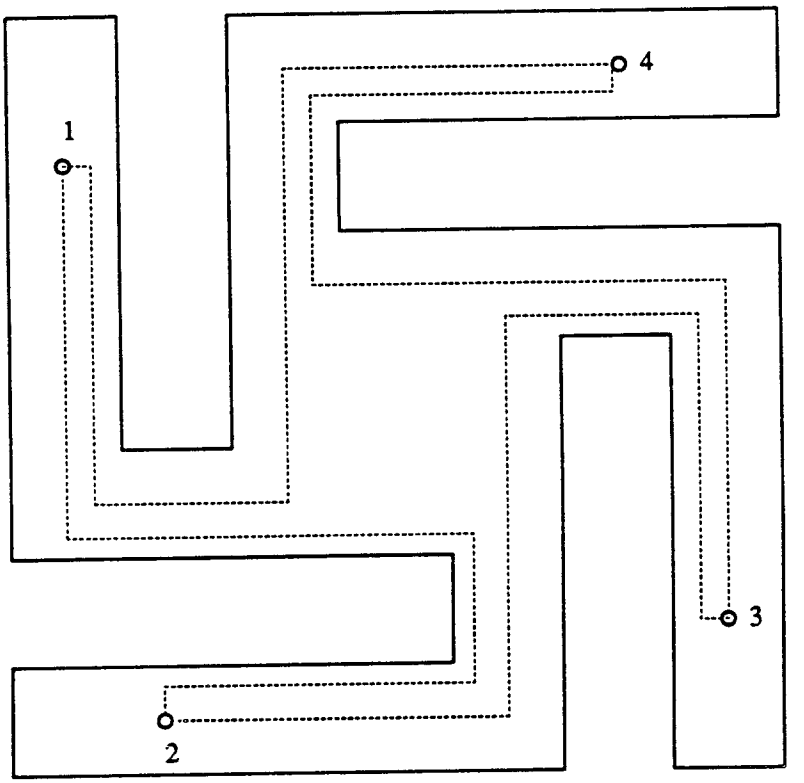

Figure 14: A 4-cycle with no chords

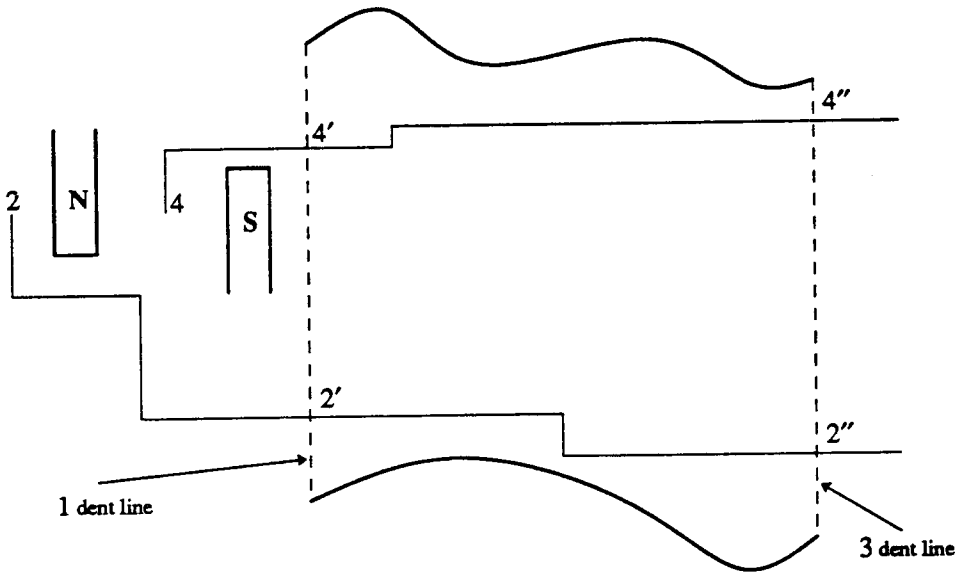

Figure 15 (a): Proof of Theorem 5, Case 1

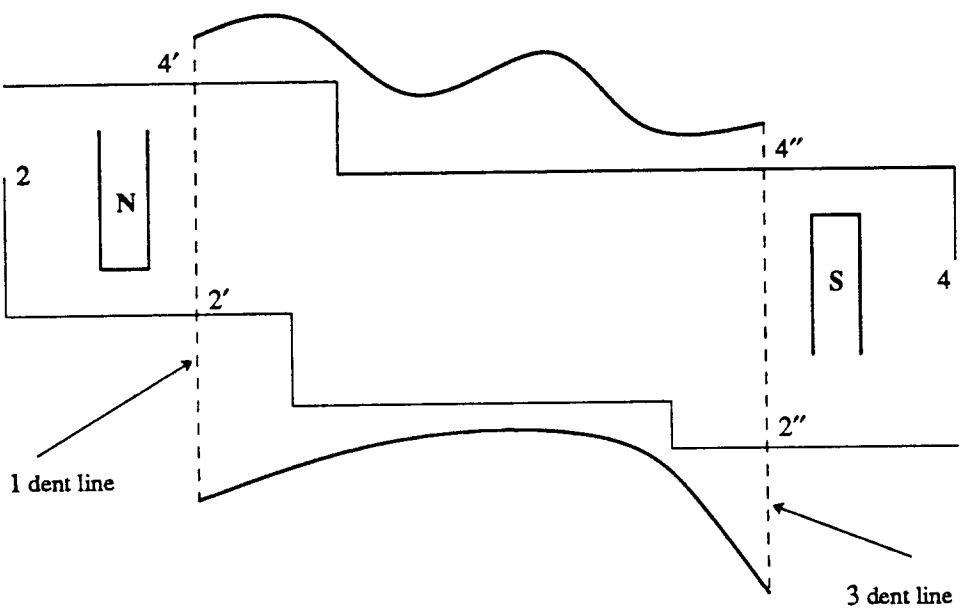

Figure 15 (b): Proof of Theorem 5, Case 2