# Multi-Vehicle Collision Avoidance via Hamilton-Jacobi Reachability and Integer Linear Programming

*Chia-Yin Shih*

Electrical Engineering and Computer Sciences
University of California, Berkeley

Technical Report No. UCB/EECS-2022-245
http://www2.eecs.berkeley.edu/Pubs/TechRpts/2022/EECS-2022-245.html

December 1, 2022

# Multi-Vehicle Collision Avoidance via Hamilton-Jacobi Reachability and Integer Linear Programming

by

Chia-Yin Shih

A report submitted in partial satisfaction of the

requirements for the degree of

Master of Science

in

Electrical Engineering and Computer Sciences

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Laurent El Ghaoui, Chair
Professor Ruzena Bajscy

Summer 2021

Submitted to the Department of Electrical Engineering and Computer Sciences, University of California Berkeley, in partial satisfaction of the requirements for the degree of **Master of Science, Plan II**. The project report of Chia-Yin Shih, titled Multi-Vehicle Collision Avoidance via Hamilton-Jacobi Reachability and Integer Linear Programming, is approved:

Chair Laurent El Ghaoui ______________________ Date 2/22/2021 __________

Ruzena BAJCSY ______________________ Date 2/24/2021 __________

University of California, Berkeley

Multi-Vehicle Collision Avoidance via Hamilton-Jacobi Reachability and Integer Linear Programming

Abstract

Multi-Vehicle Collision Avoidance via Hamilton-Jacobi Reachability and Integer Linear Programming

by

Chia-Yin Shih

Master of Science in Electrical Engineering and Computer Sciences

University of California, Berkeley

Professor Laurent El Ghaoui, Chair

Multi-agent differential games are important and useful tools for analyzing many practical problems. With the recent surge of interest in using UAVs for civil purposes, the importance and urgency of developing tractable multi-agent analysis techniques that provide safety and performance guarantees is at an all-time high. Hamilton-Jacobi (HJ) reachability has successfully provided safety guarantees to small-scale systems and is flexible in terms of system dynamics. However, the exponential complexity scaling of HJ reachability prevents its direct application to large scale problems when the number of vehicles is greater than two. In this paper, we address the scalability limitations of HJ reachability by using an integer linear program that exploits the properties of HJ solutions to provide higher-level control logic. Our proposed method provides safety guarantee for three-vehicle systems – a previously intractable task for HJ reachability – without incurring significant additional computation cost. Furthermore, our method is scalable beyond three vehicles and performs better than an extension of pairwise collision avoidance to multi-vehicle collision avoidance. We demonstrate our proposed method in simulations.

# Contents

**Contents** **i**

**List of Figures** **ii**

**1 Introduction** **1**

**2 Problem Formulation** **3**

**3 Methodology** **5**

**4 Safety Guarantee For Three Vehicles** **9**

**5 Numerical Simulations** **11**

**6 Conclusions** **17**

**Bibliography** **18**

# List of Figures

5.1 Three vehicles cooperatively resolve conflicts in a cyclic order, $Q_1$ (red) avoids $Q_2$ (green), $Q_2$ (green) avoids $Q_3$ (blue), and $Q_3$ (blue) avoids $Q_1$ (red). . . . . . 14
5.2 Without higher level control logic, the three vehicles are unable to resolve conflicts successfully. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
5.3 Eight vehicles successfully coordinated to resolve conflicts with our algorithm in this challenging scenario. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15
5.4 The lack of coordination using the baseline method results in failure in this challenging eight-vehicle scenario. . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15
5.5 Our scheme outperforms the baseline method substantially in terms of success ratio and aggregate conflict ratio. In particular, we confirmed that for $N = 3$, our method has a success ratio of 1.0 and aggregate conflict ratio of 0.0. . . . . 16

## Preface

The research presented in this report was originally published in the papers Multi-Vehicle Collision Avoidance via Hamilton-Jacobi Reachability and Mixed Integer Programming in the 55th IEEE Conference on Decision and Control (CDC) and Reachability-based Safe Planning for Multi-Vehicle Systems with Multiple Targets in the American Control Conference (ACC) 2021.

In addition to changing some of the wordings in the original CDC paper, I came up with a general guideline for selecting the objective function of the proposed integer program and a proof that demonstrates its validity of guaranteeing safety for three-vehicle systems in the Summer of 2016. The original theorem in the CDC paper assumes a specific choice of the objective function that happens to work through a proof by exhaustive enumeration and brute-force. The new theorem proves a more general result with a more intuitive and general proof. Thus the method and the main result sections in the CDC paper were modified to reflect this. The new general guideline and theorem appeared as part of the paper Reachability-based Safe Planning for Multi-Vehicle Systems with Multiple Targets in ACC 2021.

# Chapter 1

# Introduction

From projects such as Amazon Prime Air and Google Project Wing to other recent uses of unmanned aerial vehicles (UAVs), there is without a doubt an immense interest in using UAVs for civil purposes [11, 2, 1, 4]. Potential uses of UAVs include package delivery, aerial surveillance, and disaster response [28]; future applications of UAVs are only limited by imagination. As a result, government agencies such as the Federal Aviation Administration (FAA) and the National Aeronautics and Space Administration (NASA) are urgently working on UAV-related regulations [17, 24, 19].

Much research has gone into the area of multi-agent systems, which involve aspects of cooperation and asymmetric goals among the agents. In [13, 5], the authors assume that the vehicles will employ certain simple control strategies which induce velocity obstacles that must be avoided in order to maintain safety. Other approaches involved using potential functions to ensure collision avoidance while multiple agents maintain formation to travel along pre-specified trajectories [25, 10]. Although approaches like these provide valuable insight to multi-agent systems, they do not flexibly offer the safety guarantees that are desirable in safety-critical systems.

Multi-agent systems have also been studied in the context of differential games, which are ideal for addressing safety-critical problems such as the ones involving UAVs we now urgently face, because of the safety and performance guarantees that differential game approaches can provide. The HJ formulation of differential games has been studied extensively and successfully applied to small-scale problems involving one or two vehicles [29, 23, 15, 12]. Besides providing safety guarantees, perhaps the most appealing feature of HJ-based methods is its flexibility in terms of the system dynamics. Unfortunately, the computation complexity of HJ-based methods scales exponentially with the number of vehicles in the system, making their direct application to multi-vehicle problems intractable.

Many attempts have also been made to use differential games to analyze larger-scale problems. For example, in works such as [27, 26, 14], the authors discuss various classes of three-player differential game with different assumptions on the role of each agent in non-cooperative settings. For even larger systems, [20, 7, 9, 8] provide promising results when varying degrees of structural assumptions can be made. However, none of these attempts at providing guarantees address the problem of unstructured flight, which may be important in some situations. In addition, having stronger safety guarantees in unstructured environments has the potential to make structured flight of UAVs more resilient to unforeseen circumstances.

In this paper, we build on the HJ-based method for guaranteeing safety when no more than two vehicles are present. We augment the HJ-based method with a higher-level joint cooperative control strategy using an integer linear program (ILP) inspired by the properties of the pairwise safety guarantee. Our proposed ILP scales well with the number of vehicles, provides safety guarantees for three vehicles, and results in much better safety performance for multi-vehicle systems in general compared to when not using the higher-level control logic. We provide a proof for the safety guarantee in a three-vehicle system, and illustrate the safety guarantee and performance benefits through simulations of multi-vehicle systems in various configurations.

# Chapter 2

# Problem Formulation

Consider $N$ vehicles, denoted $Q_i, i = 1, 2, \dots, N$, described by the following ordinary differential equation (ODE)

$$\dot{x}_i = f_i(x_i, u_i), \quad u_i \in \mathcal{U}_i, \quad i = 1, \dots, N \tag{2.1}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state of the $i$th vehicle $Q_i$, and $u_i$ is the control of $Q_i$. Each of the $N$ vehicles may have some objective, such as getting to a set of goal states. Whatever the objective may be, each vehicle $Q_i$ must at all times avoid the *danger zone* $\mathcal{Z}_{ij}$ with respect to each of the other vehicles $Q_j, j = 1, \dots, N, j \neq i$. In general, the danger zones $\mathcal{Z}_{ij}$ may represent any relative configuration between $Q_i$ and $Q_j$ that are considered undesirable, such as collision. In this paper, we make the assumption that $x_{ij} \in \mathcal{Z}_{ij} \Leftrightarrow x_{ji} \in \mathcal{Z}_{ji}$, the interpretation of which is that between a pair of vehicles, a vehicle is in the other vehicle's danger zone if and only if the other vehicle is in its danger zone.

If possible and desired, each vehicle would use a "liveness controller" that helps complete its objective. However, sometimes a "safety controller" must be used in order to prevent the vehicle from entering any danger zones with respect to any other vehicles. Since the danger zones $\mathcal{Z}_{ij}$ are sets of joint configurations, it is convenient to derive the set of relative dynamics between every vehicle pair from the dynamics of each vehicle specified in (2.1). Let the relative dynamics between $Q_i$ and $Q_j$ be specified by the ODE

$$\begin{aligned} &\dot{x}_{ij} = g_{ij}(x_{ij}, u_i, u_j) \\ &u_i \in \mathcal{U}_i, u_j \in \mathcal{U}_j \quad i, j = 1, \dots, N, i \neq j \end{aligned} \tag{2.2}$$

We assume the functions $f_i$ and $g_{ij}$ are uniformly continuous, bounded, and Lipschitz continuous in arguments $x_i$ and $x_{ij}$ respectively for fixed $u_i$ and $(u_i, u_j)$ respectively. In addition, the control functions $u_i(\cdot) \in \mathbb{U}_i$ are drawn from the set of measurable functions[1].

Given the vehicle dynamics in (2.1), some joint objective, the derived relative dynamics in (2.2), and the danger zones $\mathcal{Z}_{ij}, i, j = 1, \dots, N, i \neq j$, we propose a cooperative safety control strategy that performs the following:

[1]A function $f : X \to Y$ between two measurable spaces $(X, \Sigma_X)$ and $(Y, \Sigma_Y)$ is said to be measurable if the preimage of a measurable set in $Y$ is a measurable set in $X$, that is: $\forall V \in \Sigma_Y, f^{-1}(V) \in \Sigma_X$, with $\Sigma_X, \Sigma_Y$ $\sigma$-algebras on $X$,$Y$.

1. detect potential conflict based on the joint configuration of all $N$ vehicles;
2. allow vehicles that are not in potential conflict to complete their objectives using a liveness controller;
3. among the vehicles in potential conflict, attempt to minimize the number of instances in which a vehicle gets into another vehicle's danger zone.

For the case of $N = 3$, we prove that our proposed control strategy guarantees that all vehicles will be able to stay out of all the danger zones with respect to the other vehicles, and thus guaranteeing safety. For all initial configurations in our simulations, all vehicles also complete their objectives.

# Chapter 3

# Methodology

Our proposed method builds on HJ reachability theory, which in the case of $N = 2$ guarantees no vehicle will enter another vehicle's danger zone and that the vehicles will eventually complete their joint objective [23]. HJ reachability becomes computationally intractable for $N > 2$. To provide the same guarantees for $N = 3$, we propose an integer linear program(ILP) motivated by the properties of the HJ pairwise solution to specify a higher level control logic. While unable to provide hard guarantees for $N > 3$, our proposed method is computationally tractable for much larger $N$, and performs much better than applying an extension of the pairwise HJ reachability solution when $N > 3$.

## Hamilton-Jacobi Reachability

HJ reachability has been studied extensively [23, 15, 3, 6, 21] and found many successful applications [23, 12, 7, 22]. Here, we give a brief overview of how to apply HJ reachability to solve a pairwise collision avoidance problem such as the one in [23]. Given the relative dynamics (2.2), we define the target set to be the danger zone $\mathcal{Z}_{ij}$, and compute following the backward reachable set

$$\begin{aligned}\mathcal{V}_{ij}(t) = \{&x_{ij} : \forall u_i \in \mathbb{U}_i, \exists u_j \in \mathbb{U}_j, \\ &x_{ij}(\cdot) \text{ satisfies } (2.2), \exists s \in [0, t], x_{ij}(s) \in \mathcal{Z}_{ij}\}\end{aligned} \tag{3.1}$$

If $x_{ij}$, the relative state of $Q_i$ and $Q_j$, is outside of $\mathcal{V}_{ij}$ for all $j$, then $Q_i$ is free to use a liveness controller to make progress towards its objective. If $x_{ij}$ is on the boundary of $\mathcal{V}_{ij}$ for a single $j$, then danger can be averted, regardless of the action of $Q_j$, by using the optimal control denoted $u^*_{ij}$, which can be obtained from the gradient of the value function $V_{ij}(t, x_{ij})$ representing $\mathcal{V}_{ij}(t)$. For details on obtaining $V_{ij}$, see [23]; for this paper, it is sufficient to note that $\mathcal{V}_{ij} = \{x_{ij} : V_{ij}(x_{ij}) \leq 0\}$ where we assume $t \to \infty$ and write $V_{ij}(x_{ij}) = \lim_{t\to\infty} V_{ij}(t, x_{ij})$. The interpretation is that $Q_i$ is guaranteed to be able to avoid collision with $Q_j$ over an infinite time horizon as long as the optimal control $u^*_{ij}$ is applied as soon as the potential conflict occurs.

If $x_{ij}$ is in $\mathcal{V}_{ij}$ for more than one $j$, then the pairwise optimal controls $u^*_{ij}$ cannot guarantee safety. However, in this case, our proposed cooperative control strategy, which uses an ILP to provide a higher level control logic, can provide safety guarantees when $N = 3$.

## The Integer Linear Program

For the $N > 2$ case, we use an ILP to provide higher level control logic to synthesize a cooperative safety controller. We first note two properties of the pairwise solution:

1. If every vehicle pair stays out of each other's danger zones, then the entire set of $N$ vehicles would be out of each other's danger zones.
2. Since the solution is pairwise, the safety controller derived from HJ reachability can only guarantee that some vehicle $i$ can avoid the danger zone with respect to a single other vehicle $j$.

Intuitively, a higher level control logic is needed to provide a far-sighted avoidance maneuver; without this higher level logic, pairwise avoidance maneuvers between two vehicles $Q_i$ and $Q_j$ may lead to unavoidable dangerous configurations with respect to a third vehicle $Q_k$.

**Definition 1.** ***Control logic matrix****: Let $\hat{U} \in \{0,1\}^{N\times N}$ be the control logic matrix specifying the joint cooperative control of the $N$ vehicles. Denote the element of $\hat{U}$ in position $(i,j)$ to be $\hat{u}_{ij}$. If $\hat{u}_{ij} = 1$, then the control logic stipulates that vehicle $Q_i$ must execute the pairwise optimal control $u^*_{ij}$ to avoid vehicle $Q_j$.*

**Definition 2.** ***Reward coefficient matrix****: Let $C \in \mathbb{R}^{N\times N}$ be the reward coefficient matrix with elements $c_{ij}$. Each $c_{ij}$ specifies the "reward" for choosing to have vehicle $i$ avoiding vehicle $j$, or in other words, choosing $\hat{u}_{ij} = 1$.*

Motivated by the above two properties, and using the above definitions, we arrive at the following ILP:

$$
\begin{aligned}
\max_{\hat{u}_{ij}} & \sum_{i,j} c_{ij}\hat{u}_{ij} & & \\
\text{subject to } & \hat{u}_{ij} + \hat{u}_{ji} \le 1 & \forall i,j, i \ne j \quad & (3.2a) \\
& \sum_{j} \hat{u}_{ij} \le 1 & \forall i \quad & (3.2b) \\
& \hat{u}_{ij} \in \{0,1\} & \forall i,j \quad & (3.2c)
\end{aligned}
\tag{3.2}
$$

At a given time, the vehicles' joint state determines $C$, which forms the objective of (3.2). Thus, the interpretation of the objective of (3.2) depends on the choice of the reward coefficient matrix $C$. A large $c_{ij}$ encourages $\hat{u}_{ij}$ to be 1, causing vehicle $Q_i$ to avoid $Q_j$. The decision variables consist of the elements of $\hat{U}$, which provides the high level control logic. This is captured by constraint (3.2c).

The pairwise HJ optimal control guarantees that a vehicle $Q_i$ can remain safe with respect to another vehicle $Q_j$ regardless of the action of $Q_j$. Therefore, in every pair $(Q_i, Q_j)$, if either $Q_i$ or $Q_j$ is avoiding the other, there is no need for the other vehicle to also be avoiding the first. The constraint (3.2a) states that out of every vehicle pair, at most one vehicle should avoid the other so that no control authority is wasted by having both vehicles avoid each other. The other vehicle then could use its control authority to avoid a third vehicle with whom it may come into conflict.

Finally, since the control logic ultimately results in vehicles performing pairwise optimal controls, each vehicle is only guaranteed to be able to avoid at most one other vehicle. The constraint (3.2b) encodes this limitation.

Note that the proposed integer linear program (3.2) can be solved through standard off-the-shelf optimization problem solvers such as Gurobi [16].

## Design of the Objective Function

The objective function in (3.2) can be designed by choosing the reward coefficient matrix $C$. In general, $C$ should depend on the vehicles' safety levels and *avoidance priority*, discussed below. In this paper, we propose a general guideline for choosing $C$ in order to guarantee safety for three vehicles.

Given the form of the objective function, the first obvious choice for some of the elements of $C$ would be $c_{ii} = -1, \forall i$. This forces $\hat{u}_{ii} = 0 \ \forall i$, which states that a vehicle $Q_i$ does not need to avoid itself. Before designing the rest of $C$, we need to define the notion of a safety level.

**Definition 3.** ***Safety level***: *Given $x_{ij}$, the state of vehicle $Q_i$ with respect to vehicle $Q_j$, define the safety level to be $V_{ij}(x_{ij})$. For convenience, let $s_{ij} = V_{ij}(x_{ij})$.*

**Proposition 1.** *Suppose $s_{ij} > 0$ at some time $t = t_0$. If $Q_i$ chooses the control $u^*_{ij}$, then $s_{ij} > 0 \ \forall t > t_0$.*

*Proof.* Based on the definitions of $s_{ij}$, $\mathcal{V}_{ij}$, and $V_{ij}(x_{ij})$, we have that if $s_{ij} > 0$ at $t = t_0$, then the control $u^*_{ij}$ guarantees collision avoidance for an infinite time horizon. This implies $s_{ij} > 0$ for all time. For a more involved discussion, see reference [23]. □

**Corollary 1.** *Between the pair $(Q_i, Q_j)$, if $s_{ij} > 0$ or $s_{ji} > 0$, then there exists a joint control strategy $(u_i, u_j)$ to ensure neither vehicle enters the danger zone of the other.*

*Proof.* If $s_{ij} > 0$, then safety is guaranteed if $Q_i$ chooses $u_i = u^*_{ij}$ to avoid $Q_j$. If instead $s_{ij} \leq 0$ and $s_{ji} > 0$, simply swap the indices $i$ and $j$. □

Let $K$ be a safety level threshold where $K > 0$. We say $Q_i$ is in *potential conflict* with $Q_j$ if $s_{ij} \leq K$. Based on this safety level threshold, we set $c_{ij} = -1$ whenever $s_{ij} > K$. A large value of $c_{ij}$ indicates that $Q_i$ should avoid $Q_j$ with a high priority. In order to impose such a priority, we propose a general way of setting $c_{ij}$ whenever $s_{ij} \leq K$. We will show that as long as we choose $c_{ij}$ such that the following inequalities hold, safety can be guaranteed for $N = 3$. We will use the notation $c_{ij}^+ = max(c_{ij}, 0)$.

$$\forall 1 \leq i < N,\ s_{i,i+1} \leq K \Rightarrow c_{i,i+1} > c^+_{N-1,1} + c^+_{N,2} + \sum_{j=1}^{N-2} c^+_{j,j+2}, \tag{3.3a}$$

$$s_{N1} \leq K \Rightarrow c_{N1} > c^+_{N-1,1} + c^+_{N,2} + \sum_{j=1}^{N-2} c^+_{j,j+2}. \tag{3.3b}$$

Specifically, for the case $N = 3$, the above conditions translate to

$$s_{12} \leq K \Rightarrow c_{12} > c^+_{13} + c^+_{21} + c^+_{32}, \tag{3.4a}$$

$$s_{23} \leq K \Rightarrow c_{23} > c^+_{13} + c^+_{21} + c^+_{32}, \tag{3.4b}$$

$$s_{31} \leq K \Rightarrow c_{31} > c^+_{13} + c^+_{21} + c^+_{32}. \tag{3.4c}$$

**Remark 1.** *Avoidance priority is important for guaranteeing safety even when $N = 2$. Consider the scenario where vehicle $Q_1$ applies the control $u^*_{12}$ to avoid $Q_2$, but $Q_2$ does not try to avoid $Q_1$. As long as $Q_1$ continues to avoid $Q_2$, the two vehicles can avoid each other's danger zones.*

*While $Q_1$ is avoiding $Q_2$, $s_{12}$ is guaranteed to remain positive; however, since $Q_2$ is not avoiding $Q_1$, $s_{21}$ could become negative. When $s_{21} < 0$, safety cannot be guaranteed if $Q_1$ does not continue to avoid $Q_2$ to keep $s_{12} > 0$. An appropriate avoidance priority encourages that some $Q_j$ does not try to avoid $Q_i$ when $s_{ji} < 0$. Instead, the responsibility of avoidance would remain with $Q_i$, which continues to avoid $Q_j$ so that $s_{ij} > 0$.*

# Chapter 4

# Safety Guarantee For Three Vehicles

The method for constructing a joint safety controller via an ILP described in Chapter 3 guarantees safety when $N = 3$. We now formally state this guarantee and prove the result.

First, we prove an intermediate result:

**Lemma 1.** *If $c_{mn} < 0$, then the optimal solution $\hat{U}^*$ of the integer linear program* (3.2) *satisfies $\hat{u}^*_{mn} = 0$.*

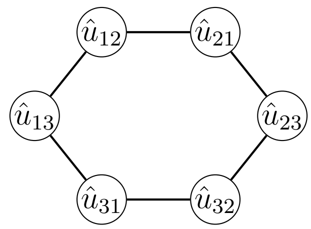
*Proof:* We will prove this by contradiction. Suppose $\hat{u}^*_{mn} = 1$, and let $\hat{U}'$ be a different feasible point to ILP (3.2) such that $\hat{u}'_{ij} = 0$ when $i = m, j = n$ and $\hat{u}'_{ij} = \hat{u}^*_{ij}$ otherwise. Then the values of the objective function based the points $\hat{U}'$ and $\hat{U}^*$ have the following relationship:

$$\begin{aligned} c_{mn}\hat{u}^*_{mn} + \sum_{i,j:(i,j)\neq(m,n)} c_{ij}\hat{u}^*_{ij} &= c_{mn} + \sum_{i,j:(i,j)\neq(m,n)} c_{ij}\hat{u}^*_{ij} \\ &< 0 + \sum_{i,j:(i,j)\neq(m,n)} c_{ij}\hat{u}^*_{ij} \quad (\text{since } c_{mn} < 0) \\ &= c_{mn}\hat{u}'_{mn} + \sum_{i,j:(i,j)\neq(m,n)} c_{ij}\hat{u}'_{ij} \quad (\text{since } \hat{u}'_{mn} = 0) \end{aligned}$$

which contradicts the assumption that $\hat{U}^*$ is a maximizer. Thus, it must be the case that $\hat{u}^*_{mn} = 0$. ■

**Theorem 1.** *Suppose $N = 3$. If $s_{12}, s_{23}, s_{31} > 0$ at some time $t = t_0$, then the joint control strategy from the ILP* (3.2) *with the reward coefficient matrix elements $c_{ij}$ satisfying the relationship described in Chapter 3 guarantees that $s_{12}, s_{23}, s_{31} > 0$ for all $t > t_0$.*

*Proof:* Observe that we can use a graph to represent the constraints in the ILP (3.2).




Each vertex represents a variable $\hat{u}_{ij}, i \neq j$ in the optimization problem. An edge between node $\hat{u}_{ij}$ and $\hat{u}_{kl}$ exists if and only if the constraint $\hat{u}_{ij} + \hat{u}_{kl} \leq 1$ is in the linear constraints in ILP (3.2) when we eliminate considering $\hat{u}_{ii}$ as its optimal value is 0 for all $i$ trivially.

It suffices to show that $0 < s_{12}, s_{23}, s_{31} \leq K$ at $t = t_0$ implies $s_{12}, s_{23}, s_{31} > 0\ \forall t > t_0$. Let $\hat{U}^*$ denote the optimal solution to ILP (3.2), and assume $s_{12}, s_{23}, s_{31} > 0$ at time $t = t_0$. Based on Proposition 1, our goal is to have

$$\begin{aligned} s_{12} \leq K &\Rightarrow \hat{u}^*_{12} = 1, \\ s_{23} \leq K &\Rightarrow \hat{u}^*_{23} = 1, \\ s_{31} \leq K &\Rightarrow \hat{u}^*_{31} = 1. \end{aligned}$$

Without loss of generality (WLOG), we prove that $\hat{u}^*_{12} = 1$ whenever $s_{12} \leq K$. Consider the following three cases when $s_{12} \leq K$:

- $s_{23}, s_{31} \leq K$: By (3.4), we have $c_{12}, c_{23}, c_{31} > c^+_{13} + c^+_{21} + c^+_{32}$. From the constraint graph, one can see that the maximum number of non-neighboring variables that can take on values of 1 is three. Thus it's clear that $\hat{u}^*_{12} = \hat{u}^*_{23} = \hat{u}^*_{31} = 1, \hat{u}^*_{13} = \hat{u}^*_{21} = \hat{u}^*_{32} = 0$ yields the largest possible objective while being feasible.

- Exactly one of the inequalities $s_{23} \leq K$, $s_{31} \leq K$ is true: Assume WLOG that $s_{23} \leq K$ and $s_{31} > K$. Applying Lemma 1, we have $\hat{u}^*_{31} = 0$. With $\hat{u}^*_{31} = 0$, regardless of the values of $c_{13}, c_{21}, c_{32}$, we always have $\hat{u}^*_{12} = \hat{u}^*_{23} = 1$. This is because first, $c_{12} + c_{23}$ is always greater than $c_{12}$ or $c_{23}$ alone as they are both positive. Second, $c_{12} + c_{23}$ is also always greater than the sum of any feasible combination of $c_{13}, c_{21}, c_{32}$ and the sum of exactly one of $c_{12}, c_{23}$ plus any feasible combination of $c_{13}, c_{21}, c_{32}$ due to the condition that $c_{12}, c_{23} > c^+_{13} + c^+_{21} + c^+_{32}$.

- $s_{23}, s_{31} > K$: Applying Lemma 1, we have $\hat{u}^*_{23} = \hat{u}^*_{31} = 0$. Based on (3.4), it's clear that the optimizer always has $\hat{u}^*_{12} = 1$ because regardless of the values of $c_{13}, c_{21}, c_{32}$, $c^+_{13} + c^+_{21} + c^+_{32}$ is always less than $c_{12}$.

In summary, when $s_{12} \leq K$, we always have $\hat{u}^*_{12} = 1$. By a similar argument, $s_{23} \leq K \Rightarrow \hat{u}^*_{23} = 1$ and $s_{31} \leq K \Rightarrow \hat{u}^*_{31} = 1$ hold.

**Corollary 2.** *By Theorem 1 and Corollary 1, if $N = 3$ and each vehicle $Q_i$ uses the pairwise safety controller $u^\star_{ij}$ with respect to $Q_j$ whenever $\hat{u}^*_{ij} = 1$, then no vehicle will enter any other vehicle's danger zone.*

# Chapter 5

# Numerical Simulations

In this chapter, we illustrate our proposed method through simulations and compare our method with a baseline pairwise method that uses solely the HJ pairwise optimal control solution in which each agent $Q_i$ avoids the agent $Q_j$ in the potential conflict set $\mathcal{J}_i$ with the smallest pairwise safety value $s_{ij}$. Compared with our ILP formulation (3.2), the baseline can be thought of as a different ILP that

- omits constraint (3.2a), making the vehicles unable to coordinate among each other, and
- assumes $\forall i, c_{ij} = 1$ if $Q_i$ has the lowest safety value with respect to $Q_j$ out of all $j$ for $j \in \mathcal{J}_i$, and $c_{ij} = -1$ otherwise, making the vehicles lack a notion of global avoidance priority.

Such a baseline is chosen to illustrate the benefits of our design considerations, which are important features of our proposed method. For illustration purposes, we assumed that the dynamics of each vehicle is given by

$$\begin{aligned} \dot{p}_{x,i} &= v \cos \theta_i \\ \dot{p}_{y,i} &= v \sin \theta_i \\ \dot{\theta}_i &= \omega_i, \quad |\omega_i| \leq \bar{\omega}. \end{aligned} \tag{5.1}$$

where the state variables $p_{x,i}, p_{y,i}, \theta_i$ represent the $x$ position, $y$ position, and heading of vehicle $Q_i$. Each vehicle travels at a constant speed of $v = 5$, and chooses its turn rate $\omega_i$, constrained by some maximum $\bar{\omega} = 1$. The danger zone for HJ computation between $Q_i$ and $Q_j$ is defined as

$$\mathcal{Z}_{ij} = \{[p_{x,ij}, p_{y,ij}, \theta_{ij}] : (p_{x,j} - p_{x,i})^2 + (p_{y,j} - p_{y,i})^2 \leq R_c^2\}, \tag{5.2}$$

whose interpretation is that $Q_i$ and $Q_j$ are considered to be in each other's danger zone if their positions are within distance $R_c$ of each other. In our examples, we chose $R_c = 5$. Here, we define $p_{x,ij} \equiv p_{x,j} - p_{x,i}$, $p_{y,ij} \equiv p_{y,j} - p_{y,i}$, and $\theta_{ij} \equiv \theta_j - \theta_i$.

To obtain safety levels and the optimal pairwise safety controller, we compute the BRS (3.1) with the relative dynamics

$$\begin{aligned} \dot{q}_{x,ij} &= -v + v\cos\theta_{ij} + \omega_i q_{y,ij} \\ \dot{q}_{y,ij} &= v\sin\theta_{ij} - \omega_i q_{x,ij} \\ \dot{\theta}_{ij} &= \omega_j - \omega_i, \quad |\omega_i|, |\omega_j| \leq \bar{\omega} \end{aligned} \tag{5.3}$$

where $[q_{x,ij},\ q_{y,ij}]$ is $[p_{x,ij},\ p_{y,ij}]$ rotated clockwise by $\theta_i$ around the origin on the 2D plane. By defining $x_{ij} \equiv [q_{x,ij}, q_{y,ij}, \theta_{ij}]$, we can similarly define the danger zone as $\{x_{ij} : (q_{x,j} - q_{x,i})^2 + (q_{y,j} - q_{y,i})^2 \leq R_c^2\}$ because the norm of a vector is invariant under rotations around the origin.

In our simulation, we choose the reward coefficient matrix in definition 2 by first assigning the non-diagonal elements in descending order according to Sarrus' rule [18] then squaring each of the elements. Then if $s_{ij} > K$, we assign $c_{ij} = -1$. For example, in the case of $N = 3$, if $s_{ij} \leq K\ \forall i, j$, then we would have

$$C = \begin{bmatrix} -1 & 36 & 9 \\ 4 & -1 & 25 \\ 16 & 1 & -1 \end{bmatrix} \tag{5.4}$$

As another example, if $N = 3, s_{ij} \leq K\ \forall i, j$ except $s_{13}, s_{32} > K$, then we would have

$$C = \begin{bmatrix} -1 & 36 & -1 \\ 4 & -1 & 25 \\ 16 & -1 & -1 \end{bmatrix} \tag{5.5}$$

Observe that when $N = 3$, the above reward coefficient matrix assignment method always satisfies the conditions 3.4. Hence safety can be guaranteed for the 3-vehicle simulation.

In our examples, we chose $K = 1.5$. Whenever $\hat{u}^*_{ij} = 0\ \forall j$, $Q_i$ applies the optimal control to reach its destination[1]. Otherwise, $Q_i$ uses the control specified by the joint cooperative safety controller that we propose in this paper.

Simulations for $N = 3$ and $N = 8$ are presented in detail for our method and the baseline method. Each vehicle aims to reach the circular target of matching color while avoiding other vehicles' danger zones. The vehicles keep traveling at constant speed even if they enter the danger zones of other vehicles until they reach their targets. The $s_{ij} = 0, K$ safety level sets are plotted for some pairs of vehicles. When a vehicle is inside the $K$ safety level set (outer boundary), plotted in the same color as the vehicle, it is in potential conflict with the vehicle around which the level set is plotted. However, as long as the vehicle stays outside of the 0 safety level set (inner boundary), the pair of vehicles will be able to avoid entering each other's danger zones.

[1]This optimal control can be computed by solving a reachability problem using the dynamics (5.1), but for brevity we will not go into the details here.

Fig. 5.1 illustrates how our joint collision avoidance method cooperatively resolves conflicts for three vehicles. The vehicles start outside of each others' $K$ safety level sets. Each of them performs optimal control to reach their respective targets. On the way, $Q_2$ (green) and $Q_3$ (blue) come in conflict with each other. Cooperatively, $Q_2$ avoids $Q_3$ while $Q_3$ heads to the target since $Q_2$ is already resolving the pairwise conflict. At time $t = 0.8$, all vehicles come in conflict with each other, and our proposed algorithm advises that $Q_1$ (red) avoids $Q_2$, $Q_2$ avoids $Q_3$, and $Q_3$ avoids $Q_1$, efficiently utilizing their control authorities for avoidance. At time $t = 1.5$, the conflicts are resolved as each vehicle's safety level rises to above $K = 1.5$ with respect to the others. Eventually, all vehicles reach their targets without any entering each other's danger zones.

Fig. 5.2 illustrates the pitfall of using the baseline method. Here, each vehicle avoids the vehicle with the smallest pairwise safety value. At $t = 0.6$, all vehicles come in conflict with each other, and without higher level logic, $Q_1$ (red) avoids $Q_3$ (blue), $Q_2$ (green) avoids $Q_1$, and $Q_3$ avoids $Q_1$. By avoiding each other, $Q_1$ and $Q_3$ waste control authority that can be used to prevent $Q_2$ and $Q_3$ from going closer to each other. When $Q_2$ and $Q_3$ come closer to each other, they begin avoiding each other, leading to $Q_1$ and $Q_3$ coming closer to each other. The lack of coordination causes this behavior to repeat, bringing them closer and closer together ($t = 0.9$), and eventually leading them into each other's danger zones at $t = 1.6$. This alternating avoidance behavior also highlights the importance of imposing avoidance priority.

Fig. 5.3 illustrates a difficult eight-vehicle scenario that our cooperative algorithm successfully resolves. The safety level sets are plotted for each avoidance pair. At $t = 2.7$, multiple vehicles are in conflict with each other. Notice that no redundant control is used (a pair of vehicles avoiding each other). Instead one vehicle in a given conflict pair can free up its control to avoid another agent. Fig. 5.4 shows the result of applying the baseline approach, which is unable to resolve the multiple conflicts. In particular, at $t = 1.7$ (top right), multiple vehicle pairs avoid each other during the conflicts. In addition, at $t = 11.5$ (bottom right), two vehicles end up in a "limbo" state where they alternate between avoiding each other and trying to get closer to their targets, continually going in a direction that is *further* from their targets.

Additionally, we compare our method with the baseline method for $N = 3, 4, 5, 6, 7, 8$ vehicles by performing 200 simulations with randomized initial conditions for each case, and show that our algorithm performs substantially better than the baseline pairwise approach. We initialized each vehicle by placing each of them symmetrically on a circle of radius $10 + 2 \times (N - 3)$ facing the center of the circle, and then adding random perturbations to its initial state. We define the two performance metrics below. The average over the 200 trials for each case are presented in Fig. 5.5.

- Success ratio = fraction of runs such that all vehicles reach their targets without ever entering others' danger zones
- Aggregate conflict ratio = $\frac{\text{total \# of danger zone violations}}{\text{\# of time steps} \times C_2^N}$. The denominator is the maximum

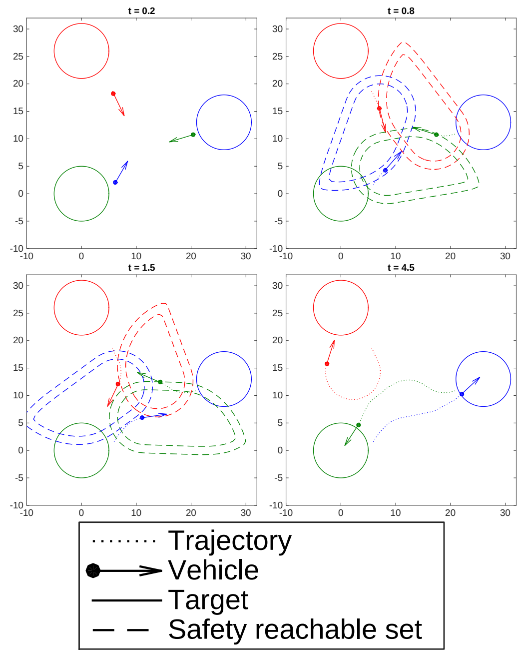



Figure 5.1: Three vehicles cooperatively resolve conflicts in a cyclic order, $Q_1$ (red) avoids $Q_2$ (green), $Q_2$ (green) avoids $Q_3$ (blue), and $Q_3$ (blue) avoids $Q_1$ (red).

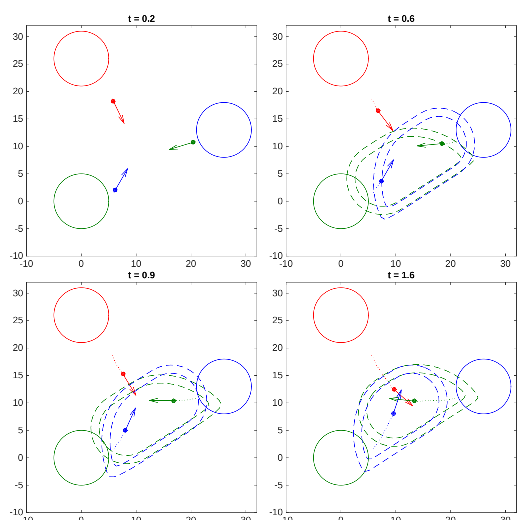



Figure 5.2: Without higher level control logic, the three vehicles are unable to resolve conflicts successfully.

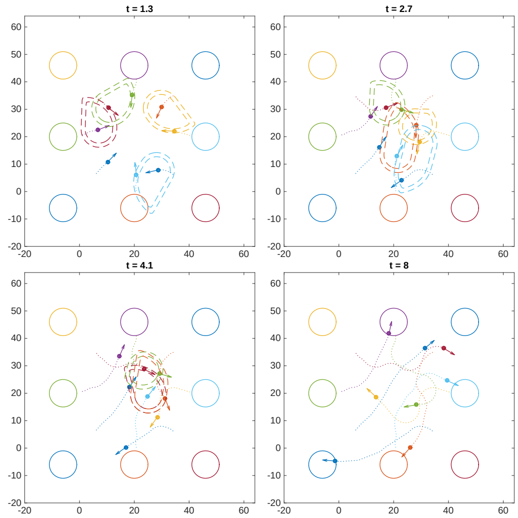



Figure 5.3: Eight vehicles successfully coordinated to resolve conflicts with our algorithm in this challenging scenario.

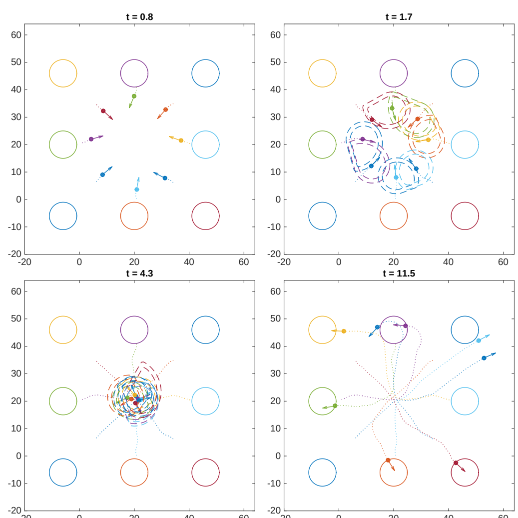



Figure 5.4: The lack of coordination using the baseline method results in failure in this challenging eight-vehicle scenario.

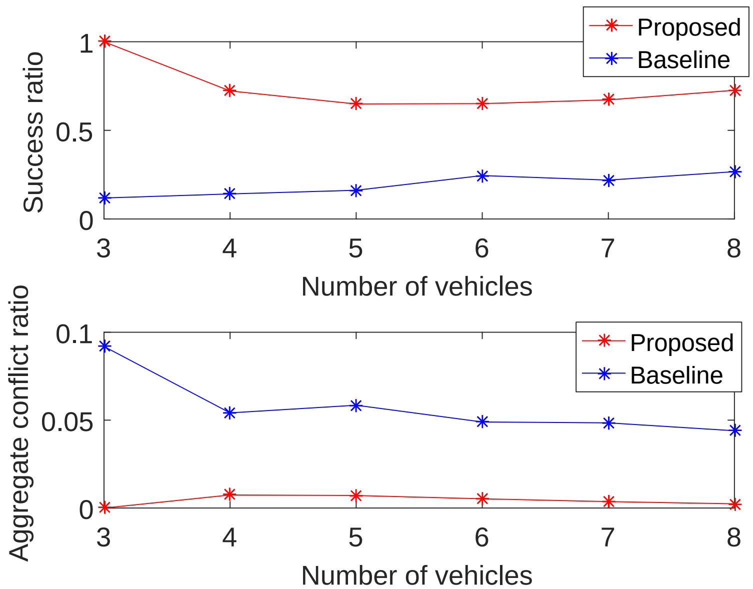



Figure 5.5: Our scheme outperforms the baseline method substantially in terms of success ratio and aggregate conflict ratio. In particular, we confirmed that for $N = 3$, our method has a success ratio of 1.0 and aggregate conflict ratio of 0.0.

possible number of danger zone violations that could occur, which is the number of time steps times $C_2^N$ ($N$ choose 2).

With our proposed method, the average computation time per simulation is 4.1 seconds for $N = 3$ and 25.5 seconds for $N = 8$; this time includes the time needed to solve the ILP (3.2). With the baseline method, the average computation time for the same simulations is 5.9 seconds for $N = 3$ and 22.9 seconds for $N = 8$. Both methods require the same BRS, which takes approximately 1 minute to compute. All computations were done on a MacBookPro 11.2 laptop with an Intel Core i7-4750 processor.

# Chapter 6

# Conclusions

By exploiting properties of pairwise optimal collision avoidance, our proposed integer linear program method guarantees collision avoidance of three vehicle systems and performs well for larger multi-vehicle systems. Future work includes guaranteeing safety for a larger number of vehicles and improving the safety performance for large numbers of vehicles.

# Bibliography


[1] Amazon.com, Inc. *Amazon Prime Air*. 2016. URL: `http://www.amazon.com/b?node=8037720011` (visited on 02/16/2016).

[2] AUVSI News. *UAS Aid in South Carolina Tornado Investigation*. 2016. URL: `http://www.auvsi.org/blogs/auvsi-news/2016/01/29/tornado` (visited on 02/16/2016).

[3] E. N. Barron. "Differential Games with Maximum Cost". In: *Nonlinear analysis: Theory, methods & applications* (1990), pp. 971–989.

[4] BBC Technology. *Google plans drone delivery service for 2017*. 2016. URL: `http://www.bbc.com/news/technology-34704868` (visited on 02/16/2016).

[5] Jur van den Berg, Ming C. Lin, and Dinesh Manocha. "Reciprocal Velocity Obstacles for real-time multi-agent navigation". In: *IEEE International Conference on Robotics and Automation*. May 2008, pp. 1928–1935. DOI: `10.1109/ROBOT.2008.4543489`.

[6] Olivier Bokanowski, Nicolas Forcadel, and Hasnaa Zidani. "Reachability and minimal times for state constrained nonlinear problems without any controllability assumption". In: *SIAM Journal on Control and Optimization* (2010), pp. 1–24.

[7] Mo Chen, Zhengyuan Zhou, and Claire J. Tomlin. "Multiplayer Reach-Avoid Games via Low Dimensional Solutions and Maximum Matching". In: *Proceedings of the American Control Conference*. 2014.

[8] Mo Chen et al. "Safe Platooning of Unmanned Aerial Vehicles via Reachability". In: *IEEE Conference on Decision and Control*. 2015.

[9] Mo Chen et al. "Safe Sequential Path Planning of Multi-Vehicle Systems via Double-Obstacle Hamilton-Jacobi-Isaacs Variational Inequality". In: *European Control Conference*. 2015.

[10] Yao-Li Chuang et al. "Multi-Vehicle Flocking: Scalability of Cooperative Control Algorithms using Pairwise Potentials". In: *IEEE International Conference onRobotics and Automation*. Apr. 2007, pp. 2292–2299. DOI: `10.1109/ROBOT.2007.363661`.

[11] Wesley M. Debusk. "Unmanned Aerial Vehicle Systems for Disaster Relief: Tornado Alley". In: *Infotech@Aerospace Conferences*. 2010.

[12] Jerry Ding et al. "Reachability Calculations for Automated Aerial Refueling". In: *IEEE Conference on Decision and Control*. Cancun, Mexico, 2008.

[13] Paolo Fiorini and Zvi Shillert. "Motion Planning in Dynamic Environments using Velocity Obstacles". In: *International Journal of Robotics Research* 17 (1998), pp. 760–772.

[14] Jaime F. Fisac and S. Shankar Sastry. "The Pursuit-Evasion-Defense Differential Game in Dynamic Constrained Environments". In: *IEEE Conference on Decision and Control*. 2015.

[15] Jaime F. Fisac et al. "Reach-Avoid Problems with Time-Varying Dynamics, Targets and Constraints". In: *18th International Conference on Hybrid Systems: Computation and Controls*. 2015.

[16] LLC Gurobi Optimization. *Gurobi Optimizer Reference Manual*. 2016. URL: `http://www.gurobi.com`.

[17] Jointed Planning and Development Office (JPDO). *Unmanned Aircraft Systems (UAS) Comprehensive Plan – A Report on the Nation's UAS Path Forward*. Tech. rep. Federal Aviation Administration, 2013.

[18] Dinesh Khattar. *The Pearson Guide to Complete Mathematics for AIEEE*. 3rd ed. Pearson Education India, 2010.

[19] Parimal Kopardekar et al. "UAS Traffic Management (UTM) Concept of Operations to Safely Enable Low Altitude Flight Operations". In: *AIAA Aviation Technology, Integration, and Operations Conference*. 2016.

[20] Wei Lin. "Differential Games for Multi-agent Systems under Distributed Information". PhD thesis. University of Central Florida, 2013.

[21] K. Margellos and J. Lygeros. "Hamilton-Jacobi Formulation for Reach-Avoid Differential Games". In: *IEEE Transactions on Automatic Control* 56.8 (Aug. 2011). ISSN: 0018-9286.

[22] K. Margellos and J. Lygeros. "Toward 4-D Trajectory Management in Air Traffic Control: A Study Based on Monte Carlo Simulation and Reachability Analysis". In: *IEEE Transactions on Control Systems Technology* 21.5 (Sept. 2013). ISSN: 1063-6536. DOI: `10.1109/TCST.2012.2220773`.

[23] I.M. Mitchell, A.M. Bayen, and C.J. Tomlin. "A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games". In: *IEEE Transactions on Automatic Control* 50.7 (2005), pp. 947–957. DOI: `10.1109/TAC.2005.851439`.

[24] National Aeronautics and Space Administration. *Challenge is On to Design Sky for All*. 2016. URL: `http://www.nasa.gov/feature/challenge-is-on-to-design-sky-for-all` (visited on 02/12/2016).

[25] Reza Olfati-Saber and Richard M. Murray. "Distributed Cooperative Control of Multiple Vehicle Formations Using Structural Potential Functions". In: *IFAC World Congress*. 2002.

[26] Mao Su, Yong-ji Wang, and Lei Liu. “Bounded guidance law based on differential game for three-player conflict”. In: *IEEE Conference on Modeling, Identification, and Control*. 2014.

[27] S. Tanimoto. “On a class of three-player differential games”. In: *Journal of Optimization Theory and Applications* 25.3 (1978), 469?473. ISSN: 1573-2878.

[28] Brian P. Tice. “Unmanned Aerial Vehicles – The Force Multiplier of the 1990s”. In: *Airpower Journal* (1991).

[29] Emmanuil Mordukhovich Vaisbord and Vladislav Iosifovich Zhukovskii. *Introduction to Multi-player Differential Games and Their Applications*. Routledge, 1988.