Path Transformations Connecting Brownian Bridge, Excursion and Meander

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PATH TRANSFORMATIONS CONNECTING BROWNIAN BRIDGE, EXCURSION AND MEANDER

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ABSTRACT. We present a unified approach to numerous path transformations connecting the Brownian bridge, excursion and meander. Simple proofs of known results are given and new results in the same vein are proposed.

1. INTRODUCTION

Let $B = (B_t : t \geq 0)$ be a standard Brownian motion started at $B_0 = 0$, $B^{br} = (B_t^{br} : 0 \leq t \leq 1)$ a Brownian bridge, $B^{ex} = (B_t^{ex} : 0 \leq t \leq 1)$ a (normalized) Brownian excursion, and $B^{me} = (B_t^{me} : 0 \leq t \leq 1)$ a Brownian meander. That is

$$
\begin{aligned}
B^{br} &\stackrel{d}{=} (B_t : 0 \leq t \leq 1 |\, B_1 = 0) ,\\
B^{ex} &\stackrel{d}{=} (B_t : 0 \leq t \leq 1 |\, B_t > 0 \text{ for } 0 < t < 1 \text{ and } B_1 = 0) ,\\
B^{me} &\stackrel{d}{=} (B_t : 0 \leq t \leq 1 |\, B_t > 0 \text{ for } 0 < t \leq 1) .
\end{aligned}
$$

The symbol $\stackrel{d}{=}$ denotes equality in distribution, referring here to distribution on the space $C[0,1]$. It is well known that the above formal conditioning on events of probability zero can be justified by natural limit schemes, leading to well defined processes with continuous paths. See Durrett et al. [D-I], [D-I-M], Iglehart [Ig] and the references therein, where these processes also appear as weak limits of correspondingly conditioned simple random walks. The scaling property of Brownian motion yields the following elementary construction, see e.g. Biane and Yor [B-Y.1] or Revuz and Yor [R-Y]. Introduce $g = \sup\{t < 1 : B_t = 0\}$ and $d = \inf\{t > 1 : B_t = 0\}$, respectively the last zero of B before time 1, and the first zero of B after time 1. Then

(1-br) $$\left(\frac{1}{\sqrt{g}} B_{gt} : 0 \leq t \leq 1\right) \text{ is a bridge independent of } g,$$

(1-ex) $$\left(\frac{1}{\sqrt{d-g}} |B_{g+(d-g)t}| : 0 \leq t \leq 1\right) \text{ is an excursion independent of } g \text{ and } d,$$

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(1-me) $$\left(\frac{1}{\sqrt{1-g}}|B_{g+(1-g)t}| : 0 \leq t \leq 1\right) \text{ is a meander independent of } g.$$

A recurring feature in the study of these processes is that some functional f of one of them, say B', has the same law as some other functional h of one of the others, say B'':

$$f(B') \stackrel{d}{=} h(B''). \tag{2}$$

Probabilists like to find a 'pathwise explanation' of such identity, meaning a transformation $T : C[0,1] \to C[0,1]$ such that

$$T(B') \stackrel{d}{=} B'' \text{ , and } f = h \circ T. \tag{3}$$

Most often, the discovery of some identity of the form (2) precedes that of the transformation T satisfying (3). But once T is found, (2) is suddenly extended to hold jointly for the infinite collection of all f and h such that $f = h \circ T$.

The purpose of this paper is to present a unified approach to such path transformations connecting the bridge, the excursion and the meander. Known results are reviewed and several new transformations are proposed. Composition of the various mappings described here gives a bewildering variety of transformations which it would be vain to try to exhaust. We have chosen to present only the most significant, usually mapping the bridge into another process. All these transformations can be inverted, though we do not always make the inverse explicit. The main mappings are depicted graphically in figures which should help the reader both in statements and proofs.

We describe essentially four sets of transformations. The first relies on the decomposition of the bridge at its minimum on $[0,1]$ (section 2). The associated mapping from the bridge to the excursion was discovered by Vervaat [Ve]. The mapping from the bridge to the meander was found independently by Bertoin [Be] and Pitman (unpublished). These two results form the starting point of this work and are not re-proved. They will be applied to deduce the other mappings. The second set of transformations is based on the absolute value of the bridge and its local time at 0 (section 3), the third on various types of reflections for the bridge (section 4), and the ultimate on the signed excursions of the bridge away from 0 (section 5).

2. Splitting the bridge at its minimum

Chung [Ch] and Kennedy [Ke] noted that the maximum of the excursion, $\max_{0\leq t\leq 1} B_t^{ex}$, has the same distribution as the amplitude of the bridge, $\max_{0\leq t\leq 1} B_t^{br} - \min_{0\leq t\leq 1} B_t^{br}$. This identity is explained by the path transformation found by Vervaat [Ve]. Take a bridge, split the path at the (a.s. unique) instant when it attains its minimum on $[0,1]$, and paste the pre-minimum part to the end of the post-minimum part (see figure 1). The resulting path is an excursion. This transformation is not one-to-one, and the inverse result, attributed to Vervaat by Imhof [Im.2], and discovered also by Biane [Bi], involves additional randomization.

Theorem 2.1. Bridge $\leftrightarrow$ Excursion. (Vervaat)

(i) Let U be the instant when B^{br} attains its minimum value on $[0,1]$. Then U has a uniform $[0,1]$ distribution, and the process

$$\left(B^{br}_{U+t\ (mod\ 1)} - B^{br}_{U} : 0 \le t \le 1\right)$$

is an excursion independent of U.

(ii) Conversely, if $\bar{U}$ is a uniform $[0,1]$ variable independent of B^{ex}, then

$$\left(B^{ex}_{\bar{U}+t\ (mod\ 1)} - B^{ex}_{\bar{U}} : 0 \le t \le 1\right)$$

is a bridge which attains its minimum at time $U = 1 - \bar{U}$.

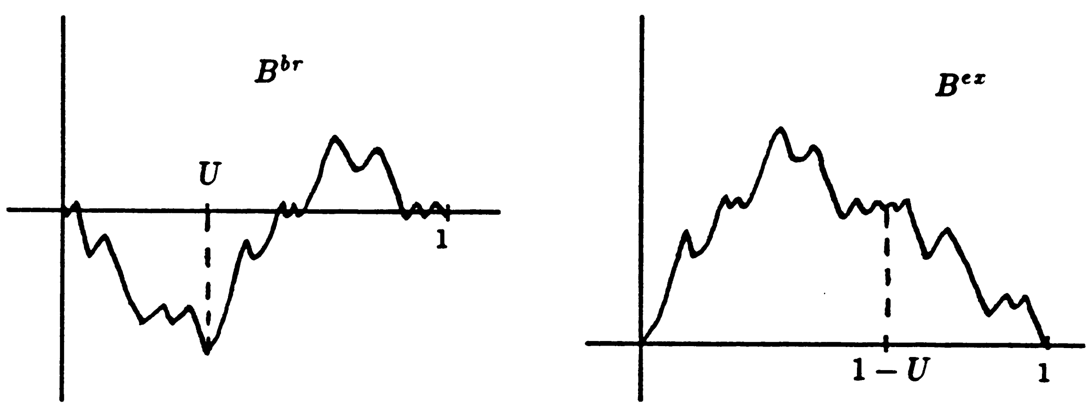

Figure 1: Bridge $\leftrightarrow$ Excursion in Theorem 2.1

A transformation in the same vein, from the bridge to the meander, is described in [Be], Corollary 6: split the bridge at its minimum, time-reverse the pre-minimum part, and then tack on the post-minimum part (see figure 2). This transformation is one-to-one. Here is the formal statement:

Theorem 2.2. Bridge $\leftrightarrow$ Meander. *Notations are as in Theorem 2.1. Put*

$$X_t = \begin{cases} B^{br}_{U-t} - B^{br}_U & \text{for } 0 \le t \le U, \\ B^{br}_t - 2B^{br}_U & \text{for } U \le t \le 1. \end{cases}$$

Then $B^{me} := X$ is a meander. Moreover $U = \sup\{t \le 1 : B^{me}_t = \frac{1}{2}B^{me}_1\}$. In particular, B^{br} can be recovered from B^{me}.

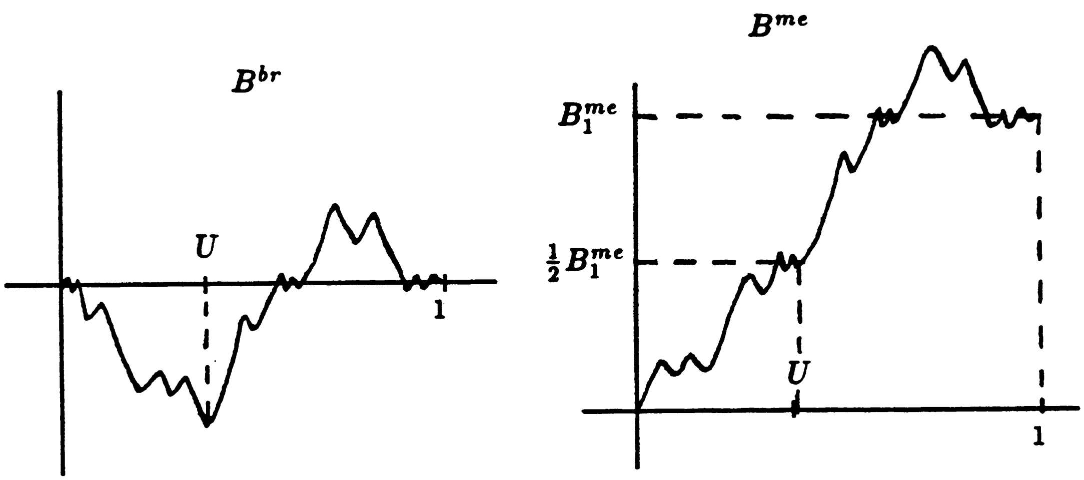

Figure 2: Bridge $\leftrightarrow$ Meander in Theorem 2.2

An immediate combination of Theorems 2.1 and 2.2 yields

Theorem 2.3. Excursion $\leftrightarrow$ Meander. *Let U be a uniform $[0,1]$ variable independent of B^{ex}. Put*

$$X_t = \begin{cases} B^{ex}_t & \text{for } 0 \le t \le U, \\ B^{ex}_U + B^{ex}_{1-(t-U)} & \text{for } U \le t \le 1. \end{cases}$$

Then $B^{me} := X$ is a meander and $U = \sup\{t \le 1 : B^{me}_t = \frac{1}{2}B^{me}_1\}$. In particular, B^{ex} and U can be recovered from B^{me}.

Just as in Vervaat [Ve], Theorem 2.3 also follows by a weak convergence argument from its random walk analog, a simple transformation underlying the classical fluctuation theory of Feller [Fe], vol.1. The details are even easier because there is no difficulty involving ties in the discrete set up.

Proof of the discrete analog of Theorem 2.3. Let $S_k = \xi_1 + \cdots + \xi_k$, $k \ge 1$, and $S_0 = 0$, where the ξ_i's are independent with $P(\xi_i = \pm 1) = \frac{1}{2}$. Fix a positive integer n, and let

$\Lambda^+ = \{S_k > 0 \text{ for all } 1 \le k \le 2n\}$, and $\Lambda^{+0} = \{S_k > 0 \text{ for all } 1 \le k < 2n \text{ and } S_{2n} = 0\}$. So, the law of $(S_k : 0 \le k \le 2n)$ conditionally on Λ^+ is the the law of the discrete meander with $2n$-steps, and the law of $(S_k : 0 \le k \le 2n)$ conditionally on Λ^{+0} is the the law of the discrete excursion with $2n$-steps. On the event Λ^+, define $U = \max\{k : 1 \le k < 2n, S_k = S_{2n}/2\}$, and set $X_k = S_k$ for $0 \le k \le U$, $X_k = S_U + S_{2n-(k-U)} - S_{2n}$ for $U < k \le 2n$. Identify the events Λ^+ and Λ^{+0} in the usual way with sets of paths of length $2n$. It is easily verified that

$$(S_k : 0 \le k \le 2n) \to (X_k : 0 \le k \le 2n)$$

induces a mapping from Λ^+ to Λ^{+0} which is $2n-1$ to one: each path in Λ^{+0} comes from exactly $2n-1$ paths in Λ^+, one for each possible value of the cut point U. It follows immediately that, conditionally on Λ^+, the process $(X_k : 0 \le k \le 2n)$ is a discrete excursion independent of U, and that U is uniformly distributed on $\{1, 2, \cdots, 2n-1\}$. □

Remark. The transformation in the discrete analog of Theorem 2.3 is a close relative of the one which Feller [Fe.1], ex III.10.7, attributes to E. Nelson. Let $T = \min\{k > 0 : S_k = 0\}$. Since obviously $P(T = 2n) = 2P(\Lambda^{+0})$ and $P(T > 2n) = 2P(\Lambda^+)$, the transformation implies $P(T = 2n) = (2n-1)P(T > 2n)$. This yields the distribution of T and hence the fundamental formulas of discrete fluctuation theory, see [Fe.1] III.(3-7) and Lemma II.3.2.

As an application of the three preceding theorems, we notice the identity

$$(-B^{br}_{U^{br}}, U^{br}) \stackrel{d}{=} (B^{ex}_{U^{ex}}, U^{ex}) \stackrel{d}{=} (B^{me}_1/2, U^{me}), \tag{4}$$

where U^{br} is the instant when B^{br} attains its minimum on [0,1], U^{ex} is a uniform [0,1] variable independent of B^{ex}, and $U^{me} = \sup\{t \le 1 : B^{me}_t = \frac{1}{2}B^{me}_1\}$. The law of the first component in (4) is the same as $R/2$, where R has the Rayleigh distribution

$$P(R \in dr)/dr = r\exp\{-\frac{1}{2}r^2\}, \qquad r > 0.$$

We refer to [K-S] for an explicit description of the joint law in (4). Futher pairs of random variables associated with the Brownian bridge that have the same distribution as in (4) appear in subsequent identities (9) and (11).

3. Absolute bridge and its local time

Recall Lévy's [Lé] identity

$$(M, M - B) \stackrel{d}{=} (L, |B|), \tag{5}$$

where $M_t = \max_{0 \le s \le t} B_s$ is the maximum process of B, and L the local time process of B at 0. According to Pitman [Pi]:

$$(M, 2M - B) \stackrel{d}{=} (J, BES^3), \tag{6}$$

where BES^3 is the 3-dimensional Bessel process, and $J_t = \min_{t \le s} BES^3_s$ its future minimum process. One deduces from (5) and (6) that

$$(L, |B| + L) \stackrel{d}{=} (J, BES^3). \tag{7}$$

Informally, the meander can be viewed as the Bessel(3) process on [0,1] conditioned by $BES^3_1 = J_1$. More precisely, Imhof [Im.1] showed that the law of the meander is absolutely continuous with respect to the law of the Bessel(3) process on $[0,1]$, with density $\sqrt{\frac{\pi}{2}}/BES^3_1$. Biane and Yor [B-Y.2] used this relation to obtain a conditional form of (7), which provides a transformation from the absolute bridge to the meander. See Theorem 3.1 below and figure 3. The local time process at 0 of the bridge B^{br}, denoted L^{br}, is defined by

$$L^{br}_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|B^{br}_s| < \epsilon\}} ds,$$

where the limit exists $a.s.$ for all $t \in [0,1]$. We denote by $B^{|br|}$ the absolute bridge, that is $B^{|br|} \stackrel{d}{=} |B^{br}|$. Its local time process at 0 is

$$L^{|br|}_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{B^{|br|}_s < \epsilon\}} ds.$$

In particular, if $B^{|br|} = |B^{br}|$, then $L^{|br|} = L^{br}$. Warning: this definition makes $L^{|br|}$ equal *half* the occupation density of $B^{|br|}$ at 0.

Theorem 3.1. |Bridge| $\leftrightarrow$ Meander. (Biane and Yor) *The process*

$$B^{me} := B^{|br|} + L^{|br|}$$

is a meander and

$$L_t^{|br|} = \min_{t \le s \le 1} B_s^{me}.$$

In particular, $B^{|br|}$ can be recovered from B^{me}.

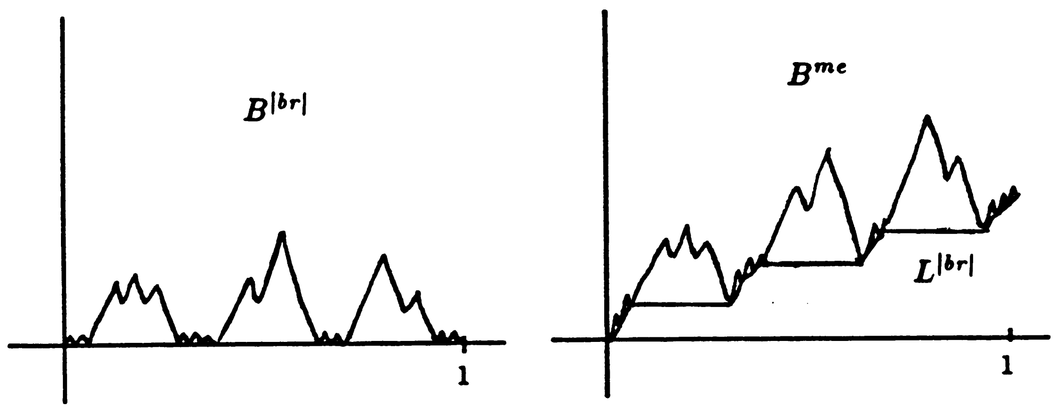

Figure 3: |Bridge| $\leftrightarrow$ Meander in Theorem 3.1

Theorem 3.1 can also be deduced from elementary time-reversal arguments as follows.

Proof of Theorem 3.1. It follows from Lévy's identity (5) and (1-me) that

$$B^{me} \stackrel{d}{=} \left(\frac{1}{\sqrt{1-\rho}} (M_\rho - B_{\rho+(1-\rho)t}) : 0 \le t \le 1 \right),$$

where ρ is the instant when B attains its maximum on [0,1]. Since the reversed Brownian motion $(B_1 - B_{1-t} : 0 \le t \le 1)$ is again a Brownian motion, we deduce that

$$(J^{me}, B^{me} - J^{me}) \stackrel{d}{=} \left(\frac{1}{\sqrt{\rho}} (M_\rho - M_{\rho(1-t)}, M_{\rho(1-t)} - B_{\rho(1-t)}) : 0 \le t \le 1 \right),$$

where $J_t^{me} = \min_{t \le s \le 1} B_s^{me}$. By Lévy's identity (5), the right-hand side has the same law as

$$\left(\frac{1}{\sqrt{g}} (L_g - L_{g(1-t)}, |B_{g(1-t)}|) : 0 \le t \le 1 \right),$$

where g is the last zero of B before 1. According to (1-br), and to the invariance in law under time-reversal for the bridge, the above pair has the same distribution as $(L^{br}, |B^{br}|)$. This establishes the Theorem. □

The next result transforms an absolute bridge into an excursion (see figure 4).

Theorem 3.2. |Bridge| ↔ Excursion. *Notations are as in Theorem 3.1. Let* $U = \sup\{t \le 1 : L_t^{|br|} = \frac{1}{2} L_1^{|br|}\}$. *Then U is uniformly distributed on* $[0,1]$. *Put*

$$K_t = \begin{cases} L_t^{|br|} & \text{for } 0 \le t \le U, \\ L_1^{|br|} - L_t^{|br|} & \text{for } U \le t \le 1. \end{cases}$$

Then

$$B^{ex} := K + B^{|br|}$$

is an excursion independent of U. Moreover,

$$K_t = \begin{cases} \min_{t \le s \le U} B_s^{ex} & \text{for } 0 \le t \le U, \\ \min_{U \le s \le t} B_s^{ex} & \text{for } U \le t \le 1. \end{cases}$$

In particular, $B^{|br|}$ can be recovered from B^{ex} and U.

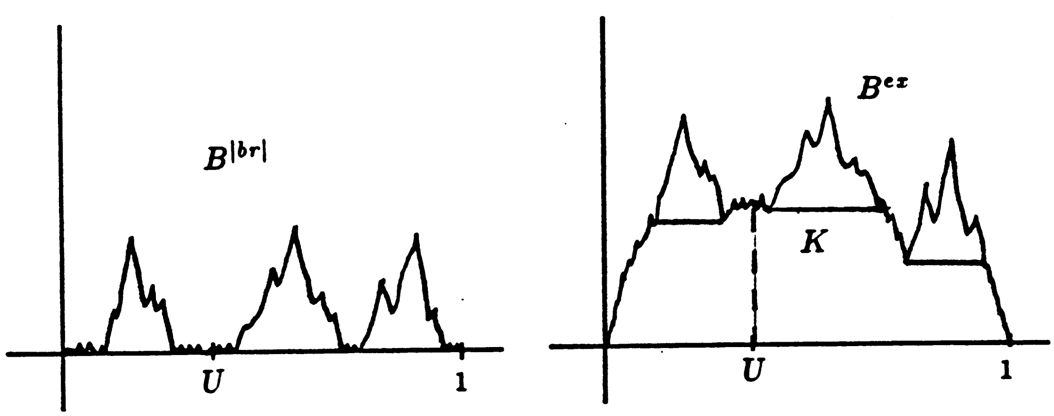

Figure 4: |Bridge| ↔ Excursion in Theorem 3.2

This result comes from the combination of Lemma 3.3 below and Theorems 3.1 and 2.3.

Lemma 3.3. **|Bridge| $\leftrightarrow$ |Bridge|.** *Notations are as in Theorem 3.2. Put*

$$X_t = \begin{cases} B_t^{|br|} & \text{for } 0 \le t \le U, \\ B_{1-(t-U)}^{|br|} & \text{for } U \le t \le 1. \end{cases}$$

Then X is an absolute bridge. Moreover, if L^X stands for its local time process at 0, then $U = \sup\{t \le 1 : L_t^X = \frac{1}{2}L_1^X\}$, and

$$L_t^X = \begin{cases} L_t^{|br|} & \text{for } 0 \le t \le U, \\ L_U^{|br|} + L_1^{|br|} - L_{1-(t-U)}^{|br|} & \text{for } U \le t \le 1. \end{cases}$$

Proof. The lemma holds in general for any diffusion bridge, and is intuitively obvious. We just sketch the proof and leave details to the reader. First, one observes (by excursion theory) that

(8-a) $\qquad (B_t^{|br|} : 0 \le t \le U)$ and $(B_{U+t}^{|br|} : 0 \le t \le 1-U)$ have the same law

(where U is as in Theorem 3.2), and that

(8-b) $\qquad$ the processes in (8-a) are independent conditionally on $(U, L_1^{|br|})$.

Since the time-reversed bridge is again a bridge, we deduce from (8-a) that

$$(B_{U+t}^{|br|} : 0 \le t \le 1-U) \stackrel{d}{=} (B_{1-t}^{|br|} : 0 \le t \le 1-U).$$

Observe that the two processe above have the same lifetime, $1-U$, and the same local time at 0, $\frac{1}{2}L_1^{|br|}$. Therefore, the preceding identity in law also holds conditionally on $(U, L_1^{|br|})$. Going back to (8-a,b), this establishes the first part of the Lemma. The second follows from the additive property of the local time. □

We conclude this section with the observation that the pair

(9) $$\left(\frac{1}{2}L_1^{|br|}, U^{|br|}\right), \text{ where } U^{|br|} = \inf\{t : L_t^{|br|} = \frac{1}{2}L_1^{|br|}\},$$

can be added to the list of identically distributed pairs in (4).

4. Reflecting the bridge

In this section, we present three transformations of the bridge by reflection. The first can be viewed as a bridge analogue of Lévy's identity (5) (see figure 5).

Theorem 4.1. Bridge $\leftrightarrow$ |Bridge|. *Let σ^{br} be the (a.s. unique) instant when B^{br} attains its maximum on $[0,1]$, and*

$$N_t^{br} = \begin{cases} \max_{0\le s\le t} B_s^{br} & \text{for } 0 \le t \le \sigma^{br}, \\ \max_{t\le s\le 1} B_s^{br} & \text{for } \sigma^{br} \le t \le 1. \end{cases}$$

Then the process

$$B^{|br|} := N^{br} - B^{br}$$

is an absolute bridge, and its local time process at 0, $L^{|br|}$, is specified by the relations

$$N_t^{br} = \begin{cases} L_t^{|br|} & \text{for } 0 \le t \le \sigma^{br}, \\ L_1^{|br|} - L_t^{|br|} & \text{for } \sigma^{br} \le t \le 1. \end{cases}$$

In particular, $\sigma^{br} = \inf\{t \le 1 : L_t^{|br|} = \frac{1}{2}L_1^{|br|}\}$, and B^{br} can be recovered from $B^{|br|}$.

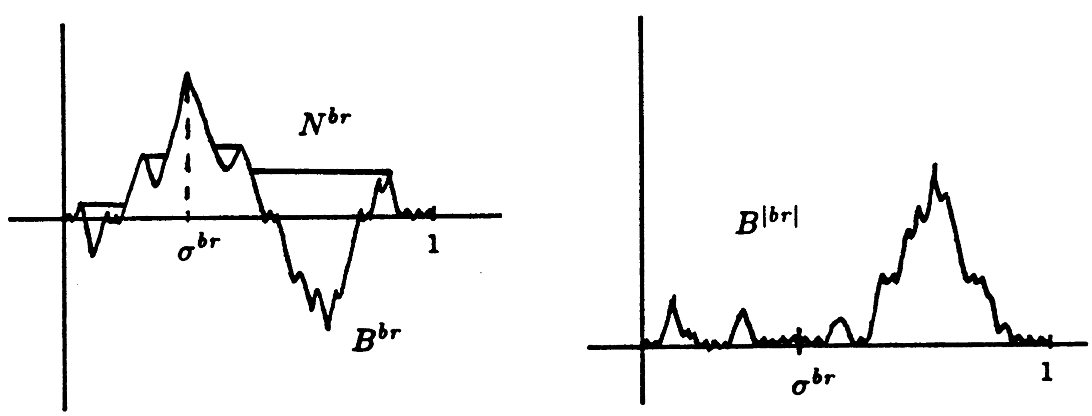

Figure 5: Bridge $\leftrightarrow$ | Bridge | in Theorem 4.1

Proof. First, we observe an identity for the absolute bridge, similar to Lemma 3.3. Put $U := U^{|br|} = \inf\{t \le 1 : L_t^{|br|} = \frac{1}{2}L_1^{|br|}\}$. Then

(10)
$$(B^{|br|}_{U+t\ (mod\ 1)} : 0 \le t \le 1) \text{ is an absolute bridge, and its local time at 0 equals}$$
$$L^{|br|}_{U+t} - L^{|br|}_U \text{ for } 0 \le t \le 1-U \text{ and } L^{|br|}_{t-1+U} + L^{|br|}_U \text{ for } 1-U \le t \le 1.$$

The first assertion comes from (8), and the second from the additive property of the local time.

We now deduce the Theorem by composition of the successive transformations

$$Bridge \leftrightarrow Bridge \leftrightarrow Excursion \leftrightarrow |Bridge| \leftrightarrow |Bridge|,$$

where the first consists of taking the opposite, the second is given in Theorem 2.1.i, the third is the inverse transformation described in Theorem 3.2, and the last is given by (10). □

Combining Theorems 4.1 and 3.2 (respectively 4.1 and 3.1), we deduce the following bridge analogs of Pitman's identity (6). The first transformation is depicted in figure 6.

Theorem 4.2. Bridge ↔ Excursion. *Notations are as in Theorem 4.1. The process*

$$B^{ex} := 2N^{br} - B^{br}$$

is an excursion independent of σ^{br}. Moreover,

$$N_t^{br} = \begin{cases} \min\limits_{t \le s \le \sigma^{br}} B_s^{ex} & \text{for } 0 \le t \le \sigma^{br}, \\ \min\limits_{\sigma^{br} \le s \le t} B_s^{ex} & \text{for } \sigma^{br} \le t \le 1. \end{cases}$$

Therefore, B^{br} can be recovered from B^{ex} and σ^{br}.

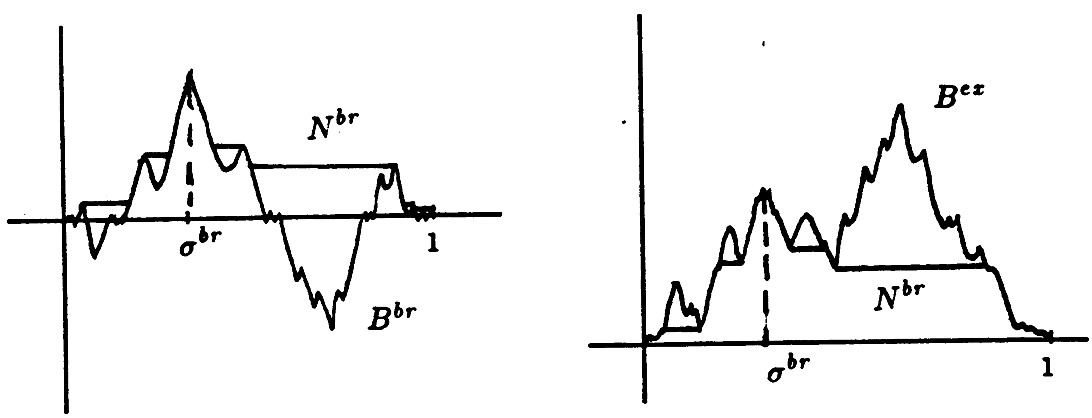

Figure 6: Bridge ↔ Excursion in Theorem 4.2

Theorem 4.3. Bridge $\leftrightarrow$ Meander. *Let $M_t^{br} = \max_{0\le s\le t} B_s^{br}$. Then the process*

$$B^{me} := 2M^{br} - B^{br}$$

is a meander. Moreover, the instant when B^{br} attains its maximum on $[0,1]$ is

$$\sigma^{br} = \sup\{t \le 1 : B_t^{me} = \frac{1}{2} B_1^{me}\},$$

and

$$M_t^{br} = \min_{t\le s\le 1} B_s^{me} \qquad t \le \sigma^{br}.$$

Therefore, B^{br} can be recovered from B^{me}.

Just as Theorem 3.1, Theorem 4.3 can be viewed as a conditional version of Pitman's identity (6). More precisely, recall that B is a Brownian motion with maximum process M, and put $BES^3 := 2M - B$ and $J := M$. Then $U := J_1/BES_1^3 = M_1/(2M_1 - B_1)$ is a uniform $[0,1]$ variable independent of the process $(BES_t^3 : 0 \le t \le 1)$. Note that for every $\epsilon > 0$,

$$\{B_1 \in [-\epsilon, \epsilon]\} \quad = \quad \{2U - 1 \in [-\epsilon/BES_1^3, \epsilon/BES_1^3]\},$$

and that $2U-1$ has a uniform $[-1,1]$ distribution. Conditioning by the above event and then letting ϵ go to 0, we deduce that the law of $(2M_t - B_t : 0 \le t \le 1)$ conditionally on $B_1 = 0$, that is the law of $(2M_t^{br} - B_t^{br} : 0 \le t \le 1)$, is absolutely continuous with respect to the law of $(BES_t^3 : 0 \le t \le 1)$, with density $\sqrt{\pi/2}/BES_1^3$. According to Imhof [Im.1], $2M^{br} - B^{br}$ is a meander.

Remark. The above argument also shows that if the bridge B^{br} is replaced by a Brownian bridge ending at $a \ne 0$, that is $(B_t : 0 \le t \le 1 | B_1 = a)$, then the path transformation of Theorem 4.3 yields a meander conditioned on $B_1^{me} > |a|$.

Here is an example of particular interest, due to Aldous [Al], equation (21), of a transformation by reflection for the excursion.

Theorem 4.4. Excursion $\leftrightarrow$ Excursion. (Aldous) *Let U be a uniform $[0,1]$ variable, independent of B^{ex}, and*

$$I_t^{ex} \quad = \quad \begin{cases} \min_{t\le s\le U} B_s^{ex} & \text{for } 0 \le t \le U, \\ \min_{U\le s\le t} B_s^{ex} & \text{for } U \le t \le 1. \end{cases}$$

Then the process

$$X = (B_U^{ex} + B_{U+t \ (mod\ 1)}^{ex} - 2I_{U+t \ (mod\ 1)}^{ex} : 0 \le t \le 1),$$

is an excursion independent of U. Moreover, B^{ex} can be recovered from X and U.

Aldous discovered this result as a projection of very natural symmetry of his compact continuum random tree. In the present setting, this transformation is identified as follows

$$Excursion \leftrightarrow Bridge \leftrightarrow Bridge \leftrightarrow Excursion,$$

where the first transformation is described in Theorem 2.1.ii, the second consists of taking the opposite, and the third is given in Theorem 4.2.

To conclude this section, we mention that Biane and Yor [B-Y.1], Theorem 7.1, describe a transformation from the bridge to the meander by an infinite sequence of reflections. This mapping explains the identity due to Kennedy [Ke], that the maximum of the meander, $\max_{0\le t\le 1} B_t^{me}$, has the same distribution as twice the maximum of the absolute bridge, $2\max_{0\le t\le 1} B_t^{|br|}$. But this transformation does not seem to be closely related to those of the present paper.

5. Signed excursions of the bridge

Sparre-Andersen [S-A] discovered the following identity for finite chains with exchangeable increments. The index of the maximum of the chain has the same distribution as the number of steps in the positive half-line. Feller illuminated Sparre-Andersen's identity with a simple chain transformation, see [Fe.2], Lemma 3 in Section XII-8. A continuous time analogue of Feller's transformation for the Brownian bridge was obtained by Karatzas and Shreve [K-S] (see figure 7). To describe their result, let $I_+ = (0,\infty)$, $I_- = (-\infty, 0)$, and for $\pm \in \{+,-\}$, let

$$A_t^{\pm} = \int_0^t \mathbf{1}_{\{B_s^{br}\in I_\pm\}} ds \qquad \text{for } 0\le t\le 1,$$

the time spent by B^{br} in $I_\pm$ before the instant t, and

$$\alpha_s^{\pm} = \inf\{t\le 1 : A_t^{\pm} = s\} \qquad \text{for } 0\le s\le A_1^{\pm},$$

the inverse of $A^{\pm}$. The time-substitution by α^+ consists of erasing the negative excursions of B^{br} and then closing up the gaps. Similarly, α^- erases the positive excursions of B^{br} and closes up the gaps.

Theorem 5.1. Bridge $\leftrightarrow$ Bridge. (Karatzas and Shreve) *Let*

$$X_t = \frac{1}{2} L^{br}_{\alpha_t^+} - B^{br}_{\alpha_t^+} \qquad \text{for } 0 \le t \le A_1^+,$$
$$X_{1-t} = \frac{1}{2} L^{br}_{\alpha_t^-} + B^{br}_{\alpha_t^-}, \qquad \text{for } 0 \le t \le A_1^-.$$

Then $\tilde{B}^{br} := X$ is a bridge that attains its maximum at time A_1^+. Moreover, for $\tilde{N}^{br}$ derived from $\tilde{B}^{br}$ as in Theorem 4.1,

$$\tilde{N}^{br}_t = \frac{1}{2} L^{br}_{\alpha_t^+} \qquad \text{for } 0 \le t \le A_t^+,$$
$$\tilde{N}^{br}_{1-t} = \frac{1}{2} L^{br}_{\alpha_t^-} \qquad \text{for } 0 \le t \le A_t^-.$$

Finally, B^{br} can be recovered from $\tilde{B}^{br}$.

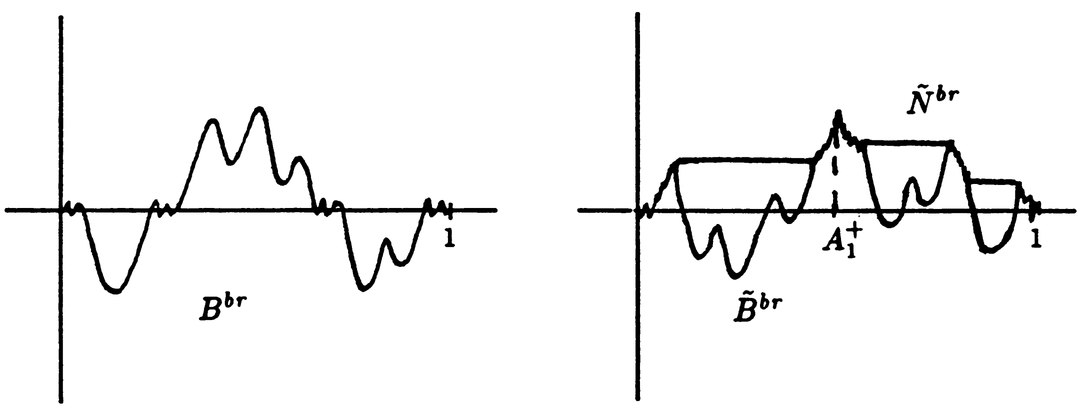

Figure 7: Bridge $\leftrightarrow$ Bridge in Theorem 5.1

In connection with (4), one deduces the identity in distribution

$$(11) \qquad (-B^{br}_{U^{br}}, U^{br}) \stackrel{d}{=} (L_1^{br}/2, A_1^+).$$

Karatzas and Shreve first noticed the identity (11), and then explained it through Theorem 5.1. In our setting, Theorem 5.1 comes from Lemma 5.2 below, Lemma 3.3 and Theorem 4.1.

Lemma 5.2. Bridge $\leftrightarrow$ |Bridge|. *Let*

$$Y_t = B^{br}_{\alpha_t^+} \qquad \text{for } 0 \le t \le A_1^+,$$
$$Y_{t+A_1^+} = -B^{br}_{\alpha_t^-} \qquad \text{for } 0 \le t \le A_1^-.$$

Then $B^{|br|} := Y$ is an absolute bridge, and its local time at 0 , $L^{|br|}$, is given by

$$L_t^{|br|} = \frac{1}{2} L^{br}_{\alpha_t^+} \qquad \text{for } 0 \le t \le A_1^+,$$
$$L^{|br|}_{t+A_1^+} = \frac{1}{2} L_1^{br} + \frac{1}{2} L^{br}_{\alpha_t^-} \qquad \text{for } 0 \le t \le A_1^-.$$

In particular, $A_1^+ = \inf\{t \le 1 : L_t^{|br|} = \frac{1}{2} L_1^{|br|}\}$. Finally, B^{br} can be recovered from $B^{|br|}$.

Proof. The Lemma holds in general for any diffusion bridge which has the same law as its opposite. Here is an elementary proof in the Brownian case that uses the scaling property. Let $\mathfrak{e}$ be an exponential variable, independent of the Brownian motion B, and $g(\mathfrak{e}) = \inf\{t < \mathfrak{e} : B_t = 0\}$ be the last zero of B on $[0, \mathfrak{e}]$. The excursion process of $(B_t : t \le g(\mathfrak{e}))$ (in the sense of Itô [It]) is a Poisson point process killed at the independent time $L_\mathfrak{e}$. Its characteristic measure is clearly invariant under the mapping $\omega \rightarrowtail -\omega$.

It follows now from the independence of the positive and negative excursions and excursion theory that the process Z given by

$$Z_t = B_{\alpha_t^+} \qquad \text{for } 0 \le t \le A^+_{g(\mathfrak{e})},$$
$$Z(t + A^+_{g(\mathfrak{e})}) = -B_{\alpha_t^-} \qquad \text{for } 0 \le t \le A^-_{g(\mathfrak{e})}$$

has the same law as $(|B_t| : 0 \le t \le g(\mathfrak{e}))$. Morover, its local time at 0, L^Z, is given by

$$L_t^Z = \frac{1}{2} L_{\alpha_t^+} \qquad \text{for } 0 \le t \le A^+_{g(\mathfrak{e})},$$
$$L^Z(t + A^+_{g(\mathfrak{e})}) = \frac{1}{2} L_{g(\mathfrak{e})} + \frac{1}{2} L_{\alpha_t^-} \qquad \text{for } 0 \le t \le A^-_{g(\mathfrak{e})}.$$

The first part of the lemma follows now from (1-ex). Finally, B^{br} can be recovered from the excursion process of Y in a similar way as described in Pitman and Yor [P-Y], p. 747. □

We deduce now from Theorems 5.1 and 4.2 the following (see figure 8).

Theorem 5.3. Bridge $\leftrightarrow$ Excursion. *Let*

$$Y_t = \frac{1}{2} L^{br}_{\alpha_t^+} + B^{br}_{\alpha_t^+} \qquad \text{for } 0 \le t \le A_1^+,$$
$$Y_{1-t} = \frac{1}{2} L^{br}_{\alpha_t^-} - B^{br}_{\alpha_t^-} \qquad \text{for } 0 \le t \le A_1^-.$$

Then $B^{ex} := Y$ is an excursion, $U := A_1^+$ is a uniform $[0,1]$ variable, and B^{ex} and U are independent. Moreover (with the same notation as in Theorem 4.4),

$$I^{ex} = \frac{1}{2} L^{br}_{\alpha_t^+} \qquad \text{for } 0 \le t \le A_1^+,$$
$$I^{ex}_{1-t} = \frac{1}{2} L^{br}_{\alpha_t^-} \qquad \text{for } 0 \le t \le A_1^-.$$

Finally, B^{br} can be recovered from U and B^{ex}.

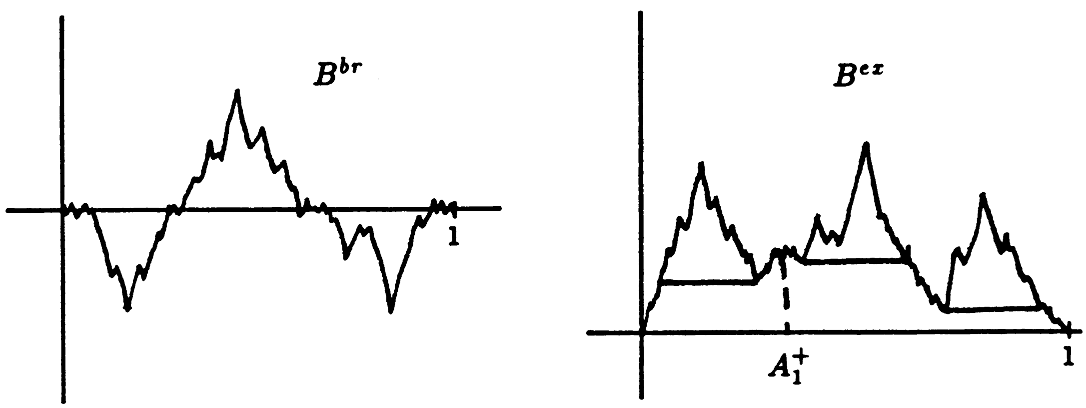

Figure 8: Bridge $\leftrightarrow$ Excursion in Theorem 5.3

Theorems 5.3 and 2.1.i yield a transformation from the bridge to itself which is given in Corollary 5 of [Be]. The formulation of this mapping in the present setting is left to the reader. Finally, here is the analogue of Theorem 5.3 for the meander.

Theorem 5.4. Bridge $\leftrightarrow$ Meander. *Let*

$$Y_t = \frac{1}{2} L^{br}_{\alpha_t^+} + B^{br}_{\alpha_t^+} \qquad \text{for } 0 \le t \le A_1^+,$$
$$Y_{t+A_1^+} = \frac{1}{2} L^{br}_1 + \frac{1}{2} L^{br}_{\alpha_t^-} - B^{br}_{\alpha_t^-} \qquad \text{for } 0 \le t \le A_1^-.$$

Then $B^{me} := Y$ is a meander. Moreover, $A_1^+ = \sup\{t \le 1 : B^{me}_t = \frac{1}{2} B^{me}_1\}$, and

$$\min_{t \le s \le 1} B^{me}_s = \frac{1}{2} L^{br}_{\alpha_t^+} \qquad \text{for } 0 \le t \le A_1^+,$$
$$\min_{t+A_1^+ \le s \le 1} B^{me}_s = \frac{1}{2} L^{br}_1 + \frac{1}{2} L^{br}_{\alpha_t^-} \qquad \text{for } 0 \le t \le A_1^+.$$

Finally, B^{br} *can be recovered from* B^{me}.

Proof. The result follows by inspection of the successive transformations provided by Theorems 5.3 and 3.2 (modulo time-reversal and change of sign). □

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