Intertwinings of Bessel Processes

By

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1. Introduction.

In this paper, we study particular examples of the *intertwining relation*

(1.a) $$Q_t\Lambda = \Lambda P_t$$

between two Markov semigroups $(P_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ defined respectively on $(E, \mathbf{E})$ and $(F, \mathbf{F})$, via the Markov kernel:

$$\Lambda : (E, \mathbf{E}) \rightarrow (F, \mathbf{F}).$$

A number of examples of (1.a) have already attracted the attention of probabilists for quite some time; see, for instance, Dynkin (1965) and Pitman-Rogers (1981). Some very recent study by Diaconis-Fill (1990) has been carried out in relation with strong uniform times.

In Chapter 2, a general filtering type framework for intertwining is presented which includes a fair proportion of the different examples of intertwinings known up to now.

In Chapter 3, we prove that the relations (1.a) holds when $P_t = P_t^{(d')}$, $Q_t = P_t^{(d)}$, with $0 < d' < d$, $(P_t^{(d)})$, resp: $(P_t^{(d')})$, the semi-group of the square of the Bessel process of dimension d, resp: d′, and $\Lambda \equiv \Lambda_{d',d}$ is defined by:

(1.b) $$\Lambda f(y) = E[f(yZ)] \text{ where } Z \text{ is a beta } \left(\frac{d'}{2}, \frac{d-d'}{2}\right) \text{ random variable}$$

(in the sequel, we shall say that Λ is the multiplication kernel associated with Z).

The intertwining relation:

(1.c) $$P_t^{(d)}\Lambda_{d',d} = \Lambda_{d',d}P_t^{(d')}$$

may then be considered as an extension to the semigroup level of the well-known fact that the product of a beta (a, b) variable by an independent gamma (a + b) variable is a gamma (a) variable

Changing the order in which the product of these two random variables is performed, we show the existence of a semi-group $(\Pi_t^{d',d})$ such that:

(1.d) $$\Pi_t^{d',d}\Lambda_d = \Lambda_d P_t^{(d')} \qquad (0 < d' < d;\ d \geq 2)$$

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where Λ_d is the multiplication kernel associated with a gamma $(\frac{d}{2})$ variable and $\Pi_t^{d',d}$ is the semi-group of a piecewise linear Markov process $Y_{d',d}$ taking values in $\mathbb{R}_+$.

In Chapter 4, it is shown that the $Y_{d',d}$ processes possess a number of properties which are reminiscent of those enjoyed by the squares of Bessel processes $X^{(d')}$.

In Chapter 5, we compare the intertwining relation (1.a) and the notion of duality of two Markov processes with respect to a function h defined on their product space (see Liggett (1985)). The intertwining relationships discussed in Chapter 3 are then translated in terms of this notion of duality. Also, general questions concerning this notion of duality are considered, such as some links with a generalized wave equation.

It would be very interesting to be able, in the examples of intertwinings discussed in this paper (Chapter 3, in particular) to obtain a joint realization of the two Markov processes (X_t) and (Y_t), with respective semi-groups (P_t) and (Q_t) which satisfy (1.a). In many cases (see Siegmund (1976), Diaconis-Fill (1990)), there exists a pathwise construction of Y in terms of X for instance (possibly allowing some extra randomization). So far, we have been able to obtain such a construction of the $Y_{d',d}$ process in terms of $X^{(d')}$ only in the case $d = 2$.

It may well be that, if such a pathwise construction can be obtained for any d, then most of the properties of the $Y_{d',d}$ processes which are being discovered in Chapter 4, mainly by analogy with their Bessel counterparts, will then appear in a more straightforward manner.

A summary, without proofs, of the results contained in this paper has been presented in Yor (1989).

2. A filtering type framework for intertwining.

(2.1) The following set-up provides a fairly general framework for intertwining. $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are two measurable processes, defined on the same probability space $(\Omega, \mathbf{F}, P)$, taking values respectively in E and F, two measurable spaces; furthermore, (X_t) and (Y_t) satisfy the following properties:

1) there exist two filtrations $(\mathbf{G}_t)$ and $(\mathbf{F}_t)$ such that:
   (i) for every t, $\mathbf{G}_t \subset \mathbf{F}_t \subset \mathbf{F}$,
   (ii) (Y_t) is $(\mathbf{G}_t)$ adapted, and (X_t) is $(\mathbf{F}_t)$ adapted;

2) (X_t) is Markovian with respect to $(\mathbf{F}_t)$, with semi-group (P_t),
   (Y_t) is Markovian with respect to $(\mathbf{G}_t)$, with semi-group (Q_t);

3) there exists a Markov kernel Λ: $E \to F$ such that for every $f: E \to \mathbb{R}_+$,

$$E[f(X_t)\,|\,\mathbf{G}_t] = \Lambda f(Y_t) \qquad \text{for every } t \geq 0.$$

We then have

Proposition (2.1): *For every function* $f: E \to \mathbb{R}_+$, *for every* $u, s \geq 0$,

(2.a) $$Q_u \Lambda f(Y_s) = \Lambda P_u f(Y_s).$$

Consequently, under some mild (continuity) assumptions, one obtains the identity:

(2.b) $$Q_u \Lambda = \Lambda P_u \quad (u \geq 0).$$

Proof: The result (2.a) is obtained by computing

$$E[f(X_{u+s}) | G_s]$$

in two different ways.

On one hand, we have:

$$E[f(X_{u+s}) | G_s] = E[E[f(X_{u+s}) | G_{u+s}] | G_s] = E[\Lambda f(Y_{u+s}) | G_s] = Q_u \Lambda f(Y_s).$$

On the other hand,

$$E[f(X_{u+s}) | G_s] = E[E[f(X_{u+s}) | F_s] | G_s] = E[P_u f(X_s) | G_s] = \Lambda P_u f(Y_s).$$ □

(2.2) We now present four classes of examples of intertwining where the hypotheses made in (2.1) are in force.

1) **Dynkin's criterion.**

This is, undoubtedly, one of the best known, and oldest, examples of intertwining between two Markov processes (see Dynkin (1965)). Here, we start with a Markov process $(Y_t, t \geq 0)$ taking its values in a measurable space F; Y is Markovian with respect to (G_t), with semi-group (Q_t). We assume that there exists a measurable application $\phi : F \to E$ such that for every measurable function $f: E \to \mathbb{R}_+$, the quantity:

$$Q_t(f \circ \phi)(y) \text{ only depends, through y, on } \phi(y).$$

Now, if $x = \phi(y)$, we define: $P_t(x,f) = Q_t(f \circ \phi)(y)$. It is now easy to see that the process $(X_t \overset{\text{def}}{=} \phi(Y_t), t \geq 0)$ is Markovian with respect to $(F_t) \equiv (G_t)$, and has semi-group (P_t). Moreover, by definition of (P_t), we have:

$$Q_t \Lambda = \Lambda P_t, \text{ with } \Lambda f(y) = f(\phi(y)),$$

so that the hypotheses in (2.1) are satisfied.

A particularly important example of this situation is obtained by taking Brownian motion in $\mathbb{R}^n$ for (Y_t), and $X_t = |Y_t|$, the radial part of (Y_t), so called: Bessel process with dimension n. Here, $F = \mathbb{R}^n$, $E = \mathbb{R}_+$ and $\phi(y) = |y|$.

2) Filtering theory.

Consider the canonical realization of a nice Markov process (X_t), taking values in E, with semigroup (P_t), and distribution $\mathbb{P}_\mu$ associated with the initial probability measure μ on E. Define

$$\mathbb{P}_\mu = W \otimes \mathbb{P}_\mu$$

where W denotes the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^n)$, which makes (B_t), the process of coordinates on $C(\mathbb{R}_+, \mathbb{R}^n)$, an n-dimensional Brownian motion. Next, define (on the product probability space), the *observation process:*

$$Y_t = B_t + \int_0^t ds\, h(X_s)$$

where $h: E \to \mathbb{R}^n$ is a bounded Borel function.

Define $\mathbf{G}_t = \sigma\{Y_s, s \le t\}$, and the filtering process (Π_t^μ) by:

$$\Pi_t^\mu(f) = \mathbf{E}_\mu[f(X_t) \,|\, \mathbf{G}_t].$$

Then, (Π_t^μ) is a $((\mathbf{G}_t); \mathbb{P}_\mu)$ Markov process, with transition semigroup:

$$Q_t(\nu; \Gamma) = \mathbb{P}_\nu(\Pi_t^\nu \in \Gamma)$$

which satisfies the following intertwining relationship with (P_t):

(2.c) $$Q_t \Lambda = \Lambda P_t, \quad \text{where } \Lambda\phi(\nu) = \langle \nu, \phi \rangle.$$

Proof of (2.c):

$$Q_t \Lambda \phi(\nu) = \mathbf{E}_\nu[\Pi_t^\nu(\phi)] = \mathbf{E}_\nu[\phi(X_t)] = \Lambda P_t \phi(\nu)$$ □

Note: For a more general discussion relating filtering theory and Knight's prediction theory, see Yor (1977).

3) Pitman's representation of BES (3).

Consider $(B_t, t \ge 0)$ a one-dimensional Brownian motion starting from 0. In this example, we take $X_t = |B_t|$, and $Y_t = |B_t| + l_t$, $t \ge 0$, where $(l_t, t \ge 0)$ is the local time at 0 of $(B_t, t \ge 0)$. Then, it follows from Pitman (1975) that $(Y_t, t \ge 0)$ is a 3-dimensional Bessel process starting from 0, and a key to this result is that, if $\mathbf{G}_t = \sigma\{Y_s, s \le t\}$, $t \ge 0$, then, for every Borel function $f: \mathbb{R}_+ \to \mathbb{R}_+$, one has:

$$E[f(X_t) \,|\, \mathbf{G}_t] = \int_0^1 dx\, f(xY_t),$$

so that the hypotheses made in (2.1) are satisfied with: $f(y) = \int_0^1 dx\, f(xy)$. Several variants of this result, in different contexts, have now been obtained, starting with Pitman-Rogers (1981).

4) **Age-processes.**

Let (X_t) be a real-valued diffusion such that 0 is regular for itself, and let $\mathbf{n}$ be the characteristic measure of excursions of X away from 0. Define $g_t = \sup\{s \le t : X_s = 0\}$; $A_t = t - g_t$ $(t \ge 0)$ is called the age-process.
(A_t) is a Markov process in the filtration $\mathbf{G}_t \equiv \mathbf{F}_{g_t}$, and its semigroup $\Pi_t(a, db)$ satisfies

$$\Pi_t \Lambda \;=\; \Lambda P_t, \;\text{ where }\; \Lambda f(a) \;=\; \mathbf{n}(f(X_a) \,|\, V > a).$$

with V, the lifetime of the generic excursion under $\mathbf{n}$. The identity:

$$E[f(X_t) \,|\, \mathbf{F}_{g_t}] \;=\; \Lambda f(A_t)$$

(which corresponds to the third hypothesis in (2.1)) may be proved by excursion theory. In the particular case where (X_t) is a Bessel process with dimension $d < 2$ and index $-\nu$ (the dimension d and the index ν are related by $d = 2(-\nu + 1)$, so that: $0 < \nu < 1$), we shall now identify Λ.
We simply write g for g_1, and define the Bessel meander of index ν, $(m_\nu(u), u \le 1)$, by the formula:

$$m_\nu(u) \;=\; \frac{1}{\sqrt{1-g}} X_{g+u(1-g)} \qquad (u \le 1)$$

(this process is called the Brownian meander in the case $\nu = 1/2$). Then, we have the following

Lemma: *Let* $0 < \nu < 1$.

1) m_ν *is independent of* $\mathbf{F}_g$*:*

2) M_ν*, the distribution of* m_ν *on* $C([0,1]; \mathbb{R}_+)$*, and* $P_0^{(\nu)}$*, the distribution of* BES(d) *on* $C([0,1]; \mathbb{R}_+)$*, satisfy the absolute continuity relationship:*

$$\text{(2.d)} \qquad M_\nu \;=\; \frac{c_\nu}{X_1^{2\nu}} \cdot P_0^{(\nu)}, \;\text{ with }\; c_\nu \;=\; \frac{\Gamma(1+\nu)}{2^{(1+\nu)}}.$$

As a consequence of (2.d), it is easily seen that the distributions M_ν are all distinct as ν varies in (0,1), but that, nonetheless, the one-dimensional marginal $X_1(M_\nu)$ does not depend on ν; we have:

$$X_1(M_\nu)(d\rho) \;\equiv\; \mathbb{P}(m_\nu(1) \in d\rho) \;=\; \rho e^{-\rho^2/2} d\rho,$$

so that:

$$\Lambda f(t-g_t) = E[f(\sqrt{t-g_t}\, m_v(1))] = \int_0^\infty d\rho\, \rho e^{-\rho^2/2} f(\sqrt{t-g_t}\,\rho)$$

which yields: $\Lambda f(a) = \int_0^\infty d\rho\, \rho e^{-\rho^2/2} f(\sqrt{a}\,\rho)$.

(2.3) After the presentation of these four classes of examples, the following instructive remark may be made: in the set-up of (2.1), it is wrong to think of (Y_t) as a (Markov) process which would carry less information than the process (X_t), so that one would have:

(2.e) $$\sigma(Y_s, s \le t) \subseteq \sigma(X_s, s \le t).$$

Indeed, in Example 1, it is X which, generally, carries less information than Y; in Example 2, the natural filtrations of X and Y cannot, in general, be compared; in Examples 3 and 4, Y carries less information than X. Instead of (2.e), the important assumption in (2.1) is that X is Markovian with respect to $(\mathbf{F}_t)$, and Y is Markovian with respect to $(\mathbf{G}_t)$, with $(\mathbf{G}_t) \subseteq (\mathbf{F}_t)$; this is quite different from asserting (2.e).

3. The algebra of beta-gamma variables and its relationship with intertwining.

(3.1) The $\beta - \gamma$ algebra

In order to facilitate the reading of the main part (3.2) of this chapter, we need to recall a few well-known facts about beta and gamma distributed random variables.

Let a and b be two strictly positive real numbers. We shall consider three families of random variables, which we denote respectively by Z_a, $Z_{a,b}$, $Z_{a,b}^{(2)}$, and which are distributed as follows:

$$P(Z_a \in dx) = \gamma_a(dx) = x^{a-1} e^{-x} \frac{dx}{\Gamma(a)} \qquad (x > 0)$$

$$P(Z_{a,b} \in dx) = \beta_{a,b}(dx) = x^{a-1}(1-x)^{b-1} \frac{dx}{B(a,b)} \qquad (0 < x < 1)$$

$$P(Z_{a,b}^{(2)} \in dx) = \beta_{a,b}^{(2)}(dx) = \frac{x^{a-1}\,dx}{(1+x)^{a+b} B(a,b)} \qquad (x > 0)$$

(recall that: $B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$)

There exist important algebraic relations between the laws of these different variables. We first remark that:

(3.a) $$Z_{a,b}^{(2)} \overset{(d)}{=} \frac{Z_{a,b}}{1-Z_{a,b}}.$$

The main relation is the following:

$$(3.b) \qquad (Z_{a,b}\,;\, Z_{a+b}) \overset{(d)}{=} \left(\frac{Z_a}{Z_a+Z_b}\,;\, Z_a + Z_b\right)$$

where, on the left-hand side, the two variables are assumed to be independent, while on the right-hand side, Z_a and Z_b are assumed to be independent (and, as a consequence of (3.b), $\frac{Z_a}{Z_a+Z_b}$ and $Z_a + Z_b$ are independent).

Here is an interesting consequence of (3.b): if $Z_{a,b}$ and $Z_{a+b,c}$ are independent, then:

$$(3.c) \qquad Z_{a,b}\, Z_{a+b,c} \overset{(d)}{=} Z_{a,b+c}$$

Proof of (3.c): From (3.b), the pair of variables $(Z_{a,b}\,;\, Z_{a+b,c})$ may be realized as the pair:

$$\left(\frac{Z_a}{Z_a+Z_b}, \frac{Z_a + Z_b}{Z_a + Z_b + Z_c}\right)$$

with Z_a, Z_b, Z_c independent; then:

$$Z_{a,b}\, Z_{a+b,c} \overset{(d)}{=} \frac{Z_a}{Z_a + Z_b + Z_c} \overset{(d)}{=} \frac{Z_a}{Z_a + Z_{b+c}} \overset{(d)}{=} Z_{a,b+c}$$

□

We now remark that, as a consequence of (3.a) and (3.b), we obtain:

$$(3.d) \qquad Z^{(2)}_{a,b} \overset{(d)}{=} \frac{Z_a}{Z_b}$$

where Z_a and Z_b are assumed to be independent.

Finally, we remark that: if $Z_{a,b}$ and $Z^{(2)}_{a+b,c}$ are independent, then:

$$(3.e) \qquad Z_{a,b}\, Z^{(2)}_{a+b,c} \overset{(d)}{=} Z^{(2)}_{a,c}$$

Proof of (3.e): From (3.b) and (3.d), the pair of variables $(Z_{a,b}, Z^{(2)}_{a+b,c})$ may be realized as the pair:

$$\left(\frac{Z_a}{Z_a+Z_b}, \frac{Z_a + Z_b}{Z_c}\right)$$

with Z_a, Z_b, Z_c independent. We then obtain:

$$Z_{a,b}\, Z^{(2)}_{a+b,c} \overset{(d)}{=} \left(\frac{Z_a}{Z_a + Z_b}\right)\left(\frac{Z_a + Z_b}{Z_c}\right) \overset{(d)}{=} \frac{Z_a}{Z_c}$$

□

(3.2) **Notation.**

All the intertwining kernels Λ which will be featured in this Chapter 3 act from $\mathbb{R}_+$ to $\mathbb{R}_+$, and are of the form:

$$\Lambda f(x) = E[f(xZ)]$$

for some random variable Z; it will then be convenient to say that Λ is the kernel of multiplication by Z.

More precisely, we shall encounter the kernels of multiplication listed in the following table:

Z	$2Z_{d/2}$	$Z_{\frac{d'}{2},\frac{d-d'}{2}}$	$1/2Z_{\frac{d-d'}{2}}$	$Z^{(2)}_{\frac{d'}{2},\frac{d-d'}{2}}$
Λ	Λ_d	$\Lambda_{d',d}$	$\tilde{\Lambda}_{d-d'}$	$\Lambda^{(2)}_{d,d-d'}$

(3.3) **Markovian extensions of the $\beta - \gamma$ algebra.**

In this section, $P_t^{(d)}$ denotes the semigroup of the square of the Bessel process of dimension d. Then, we have the following.

Theorem A: *For every* $0 < d' < d$, *and every* t,

(3.f) $$P_t^{(d)}\Lambda_{d',d} = \Lambda_{d',d}P_t^{(d')}.$$

Remarks: 1) The identity (3.f) may be understood as a Markovian extension of the relation (3.b), since we deduce, in particular, from (3.f), that,

$$\Lambda_{d',d}P_t^{(d')}f(0) = P_t^{(d)}\Lambda_{d',d}f(0)$$

which is equivalent to:

(3.g) $$E[f(2tZ_{d'/2})] = E[f(2tZ_{d/2}Z_{\frac{d'}{2},\frac{d-d'}{2}})]$$

where, on the right-hand side, $Z_{d/2}$ and $Z_{\frac{d'}{2},\frac{d-d'}{2}}$ are assumed to be independent.

The relation (3.g) is another way to write the following part of (3.b):

$$Z_a \overset{(law)}{=} Z_{a,b}Z_{a+b}, \text{ for } a = \frac{d'}{2}, \quad \text{and} \quad b = \frac{d-d'}{2}.$$

2) We have already encountered the relation (3.f) in the particular case: $d' = 1$, $d = 3$, in Example 3 of Chapter 2. □

Proof of Theorem A: The identity (3.f) may be obtained as a consequence of Proposition (2.1); indeed, if $(X_t^{(d')})$ and $(X_t^{(d-d')})$ are two independent squares of Bessel processes, with respective dimensions d' and $(d-d')$, starting at 0, then: $X_t^{(d)} = X_t^{(d')} + X_t^{(d-d')}$ is the square of a Bessel process of dimension d, and the hypotheses which are in force in Proposition (2.1) are satisfied, with:

$$F_t = \sigma\{X_s^{(d')}, X_s^{(d-d')};\ s \le t\}, \quad G_t = \sigma\{X_s^{(d)};\ s \le t\}$$

$$X_t = X_t^{(d')}, \quad Y_t = X_t^{(d)}$$

□

We consider again the relation (3.g) which we write in a more concise form as:

$$\Lambda_{d'} = \Lambda_{d',d}\Lambda_d.$$

Since kernels of multiplication commute, we also have:

$$\text{(3.g}')\qquad \Lambda_{d'} = \Lambda_d \Lambda_{d',d}$$

and this identity admits the following Markovian extension:

Theorem B: *Let* $d \ge 2$, $0 < d' < d$. *Define* $k = \frac{d}{2}$, $k' = \frac{d'}{2}$. *Then:*

1) There exists a semi-group on $\mathbb{R}_+$, *which we denote by* $(\Pi_t^{d',d})$ *such that:*

$$\text{(3.h)}\qquad \Pi_t^{d',d}\Lambda_d = \Lambda_d P_t^{(d')}$$

2) This semi-group is characterized by:

$$\text{(3.i)}\qquad \int \Pi_t^{d',d}(y, dz)(1+\lambda z)^{-k} = \frac{(1+\lambda t)^{k-k'}}{(1+\lambda(t+y))^k}$$

3) Let $k > 1$. *Then, every* C^1*-function* $\phi: \mathbb{R}_+ \to \mathbb{R}_+$, *with compact support, belongs to the domain of the infinitesimal generator* $L_{k',k}$ *of* $(\Pi_t^{k',k})$, *and:*

$$L_{k',k}\phi(y) = \phi'(y) + \frac{k-k'}{y}\int_0^1 dz\,(k-1)z^{k-2}(f(zy) - f(y))$$

Comments:

1) The particular case $d = 2$ of the relation (3.h) was already encountered in example 4) in Chapter 2 (up to some elementary modification, since in that example we considered the Bessel process of dimension d', instead of its square). On the

contrary, in the case: $d > 2$, we do not know whether the relation (3.h) may be obtained as a consequence of Proposition (2.1) and our proof of (3.h) consists in showing the existence of $\Pi_t^{d',d}$ via (3.i). The relation (3.i) is deduced from (3.h) by applying both sides to the function $(e^{-\frac{\lambda}{2}y}, y \geq 0)$ and using the relations:

$$\Lambda_d(e^{-\frac{\lambda y}{2}})(z) = c_d(1+\lambda z)^{-k};\ P_t^{(d')}(e^{-\frac{\lambda y}{2}})(z) = (1+\lambda t)^{-k'} \exp - \frac{\lambda z}{2(1+\lambda t)}.$$

2) The third part of the theorem follows from the second when one considers the functions

$$\phi_\lambda(z) = (1+\lambda z)^{-k}.$$

□

3) In the case $d > 2$, the following pathwise description of a Markov process $Y_{d',d}$ with semi-group $\Pi_t^{d',d}$ is easily deduced from part 3) of the theorem.

We now discuss duality properties for the semi-groups $(P_t^{(d')})$ and $(\Pi_t^{d',d})$, which will be important in the sequel, both in order to discover some new intertwining relations (see theorems C and D below) and also to express some results of time reversal for $Y_{d',d}$ (see section (4.5) below). We begin by recalling the

Definition (3.1): *Two Markov semigroups* (P_t) *and* $(\hat{P}_t)$ *on* E *are said to be in duality with respect to a* σ*-finite positive measure* μ *(in short: they are in* μ*-duality) if: for any pair of measurable functions* $f,g: E \to \mathbb{R}_+$,

$$< P_t f, g >_\mu = < f, \hat{P}_t g >_\mu.$$

We now have the following

Theorem (3.2): *Let* $d' > 0$, $\nu' = \frac{d'}{2} - 1$, *and* $\mu(dx) = x^{\nu'} dx$. *Then:*

a) $P_t^{(d')}$ *is self-dual with respect to* μ*;*

2) Let $d > 2$, $0 < d' < d$. *There is a Markovian semi-group* $(\hat{\Pi}_t^{d',d})$ *on* $\mathbb{R}_+$ *which is in* μ*-duality with* $(\Pi_t^{d',d})$*;*

3) Every C^1*-function* $\psi: \mathbb{R}_+ \to \mathbb{R}$ *with compact support belongs to the domain of the infinitesimal generator* $\hat{L}_{k',k}$ *of* $(\hat{\Pi}_t^{d',d})$*, and we have:*

$$\hat{L}_{k',k}\psi(y) = -\psi'(y) + \frac{k-1}{y}\int_0^1 dz\,(k-k')z^{k-k'-1}\left(\psi\left(\frac{y}{z}\right) - \psi(y)\right).$$

From Theorem (3.2), we easily deduce two other intertwining relations, namely (3.j) and (3.k) below.

Theorem C: *Let* $0 < d' < d$, *and* $d > 2$. *Then, we have:*

(3.j) $$P_t^{(d')}\tilde{\Lambda}_{d-d'} = \tilde{\Lambda}_{d-d'}\hat{\Pi}_t^{d',d}$$

where $\tilde{\Lambda}_\delta g(y) = E[g(y/2Z_{\delta/2})]$.

Proof: We start from (3.h): $\Pi_t^{d',d}\Lambda_d = \Lambda_d P_t^{(d')}$, and consider the adjoint operators in $L^2(\mu)$, where $\mu(dx) = x^{v'}dx$, as in Theorem (3.2). We obtain:

$$\hat{\Lambda}_d\hat{\Pi}_t^{d',d} = P_t^{(d')}\hat{\Lambda}_d$$

since $P_t^{(d')}$ is self-adjoint with respect to μ. It remains to compute explicitly $\hat{\Lambda}_d$; one finds:

$$\hat{\Lambda}_d g(y) = \frac{\Gamma(k-k')}{2^{k'}\Gamma(k)} E[g(y/2Z_{\frac{d-d'}{2}})] \equiv c_{k,k'}\tilde{\Lambda}_{d-d'}g(y)$$ □

Theorem D: *Let* $0 < d' < d$, *and* $d > 2$. *Then, we have:*

(3.k) $$\Pi_t^{d',d}\Lambda^{(2)}_{d,d-d'} = \Lambda^{(2)}_{d,d-d'}\hat{\Pi}_t^{d',d}$$

where $\Lambda^{(2)}_{d,d-d'}f(x) = E[f(xZ^{(2)}_{\frac{d}{2},\frac{d-d'}{2}})]$

Proof: Remark that, from (3.d): $\Lambda^{(2)}_{d,d-d'} = \Lambda_d\tilde{\Lambda}_{d-d'}$. The result (3.k) now follows immediately from the intertwining relations (3.h) and (3.j). □

As was already pointed out, Theorems A and B may be understood as Markovian extensions of the relation (3.b). Likewise, the next Theorem E is a Markovian extension of the relation

(3.c) $$Z_{a,b}Z_{a+b,c} \overset{(d)}{=} Z_{a,b+c}$$

with the notation of section (3.1).

Theorem E: *Let* $0 < d_1 < d_2 < d_3$, *and* $2 \le d_2 < d_3$. *Then:*

(3.l) $$\Pi_t^{d_1,d_2}\Lambda_{d_2,d_3} = \Lambda_{d_2,d_3}\Pi_t^{d_1,d_3}$$

and

(3.m) $$\hat{\Pi}_t^{d_1,d_3}\tilde{\Lambda}_{d_2-d_1,d_3-d_1} = \tilde{\Lambda}_{d_2-d_1,d_3-d_1}\hat{\Pi}_t^{d_1,d_2}$$

where $\tilde{\Lambda}_{\delta',\delta}\, g(y) \;=\; E[\, g(y/Z_{\frac{\delta'}{2},\frac{\delta-\delta'}{2}})]\qquad (0<\delta'<\delta)$

Proof: 1) Since the kernel Λ_{d_3} is determining, it suffices, in order to prove (3.1), to show the relation:

$$(3.1)' \qquad \Pi_t^{d_1,d_2}\Lambda_{d_2,d_3}\Lambda_{d_3} \;=\; \Lambda_{d_2,d_3}\Pi_t^{d_1,d_3}\Lambda_{d_3}.$$

Now, the left-hand side of (3.1)′ is equal to: $\Pi_t^{d_1,d_2}\Lambda_{d_2}$, with the help of (3.g) (or (3.g)′).

The right-hand side of (3.1)′ is equal to:

$$\Lambda_{d_2,d_3}\Lambda_{d_3}P_t^{(d_1)} \;=\; \Lambda_{d_2}P_t^{(d_1)} \;=\; \Pi_t^{d_1,d_2}\Lambda_{d_2},$$

using first Theorem B, then (3.g), and again Theorem B.

2) To prove (3.m), we consider the adjoint operators in $L^2(\mu)$, where $\mu(dx) = x^{\nu_1}dx$, of the kernels featured in (3.1).

By Theorem (3.2), the adjoint of $\Pi_t^{d_1,d_i}$ $(i=2,3)$ is $\hat{\Pi}_t^{d_1,d_i}$, and it is easily shown that the adjoint of Λ_{d_2,d_3} is a multiple of $\tilde{\Lambda}_{d_2-d_1,\,d_3-d_1}$. The relation (3.m) is now proved. □

Remarks: 1) Assuming that the different intertwining relations obtained in this chapter may be realized in such a way that they fit into the filtering framework discussed in section (2.1), Theorem E suggests that, for d′ fixed, and as d increases, the process $(Y_{d',d}(t), t\ge 0)$ is Markovian with respect to a filtration $(F_t^{(d)}, t\ge 0)$ which increases with d; roughly speaking, more information seems to be required as d increases in order to construct $Y_{d',d}$, and the case $d=\infty$ corresponds to BESQ (d′); see Theorem (4.6) for a more precise result formulated as a limit in law.

2) Transforming the relation (3.k) in Theorem D by duality with respect to the measure $\mu(dx) = x^{\nu'}dx$ does not yield any new relation since $\Lambda^{(2)}_{d,d-d'}$ is its own adjoint (up to a multiplicative constant) □

(3.4) Explicit computation of the semigroup $\Pi_t^{d',d}$.

We first reduce the problem to the inversion of a certain Laplace transform. Let t, y be given, and define $\alpha = \dfrac{t}{t+y}$. Then, from formula (3.i), there exists a measure $\mu^\alpha(du)$ on $\mathbb{R}_+$ which depends only on α (and k, k′) such that:

$$\int \Pi_t^{d',d}(y;dz)\, f(z) \;=\; \int \mu^\alpha(du)\, f((t+y)u)$$

and, from formula (3.i) again, μ^α is the only probability measure on $\mathbb{R}_+$ such that, for every $\lambda \geq 0$:

(3.n) $$\int_0^\infty \mu^\alpha(du)\,(1+\lambda u)^{-k} = \frac{(1+\lambda\alpha)^{k-k'}}{(1+\lambda)^k}.$$

In fact, from the comments following Theorem B, we see that μ^α must be carried by [0,1].

We shall then deduce from formula (3.n) the following Laplace transform identity:

(3.o) $$\int_{[0,1]} \mu^\alpha(du)\,\frac{1}{u^k}\,e^{-s(\frac{1}{u}-1)} = \frac{\Gamma(k)}{\Gamma(k')}\left[\frac{\alpha}{s}\right]^{k-k'}\Phi\left(k'-k,k';-\frac{\bar{\alpha}}{\alpha}s\right)$$

where $\bar{\alpha} = 1-\alpha = \frac{y}{t+y}$, and $\Phi(a,b;z)$ is the confluent hypergeometric function defined by:

$$\Phi(a,b;z) = \sum_k \frac{(a)_k}{(b)_k}\frac{z^k}{k!}$$

where: $(a)_k = a(a-1)\cdots(a-k+1)$. The hypergeometric function

$$F(a,b,c;z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

shall also play a prominent role in the sequel. Now, the key to the explicit computation of $\Pi_t^{d',d}$ is the

Proposition (3.3): *Let* $0 < k' < k$ *and* $k > 1$. *Then:*

1) there exists a unique function $g_{k',k}: \mathbb{R}_+^* \to \mathbb{R}_+$ *such that for all* $s \geq 0$*:*

$$1 + \int_0^\infty du\, g_{k',k}(u)\,e^{-su} = \frac{\Gamma(k)}{\Gamma(k')}\,s^{k'-k}\,\Phi(k'-k,k';-s)$$

2) the function $g_{k',k}$ *may be expressed as follows in terms of* F*:*

$$g_{k',k}(u) = \begin{cases} c_+\,u^{k-k-1}\,F(k'-k,1+k'-k,k';\frac{1}{u}) & (u>1)\\ c_-(F(2-k,k'-k+1,2;u) & (u<1)\end{cases}$$

where

$$c_+ = \frac{1}{B(k',k-k')} \quad \textit{and} \quad c_- = (k-1)(k-k').$$

It is now easy to express μ^α and $\Pi_t^{d',d}(y;dz)$ in terms of $g_{k',k}$. We obtain the:

Theorem (3.4): *Let* $0 < k' < k$ *and* $k > 1$. *Then:*

$$\mu^{\alpha}(du) = \bar{\alpha}^{k-k'}\varepsilon_1(du) + \left[\frac{\bar{\alpha}}{\alpha}\right]\bar{\alpha}^{k-k'} g_{k',k}\left[\frac{\alpha\bar{u}}{\bar{\alpha}u}\right]\frac{du}{u^{2-k}}1_{(0<u<1)}$$

and the semigroup $\Pi_t^{d',d}(y\,;dz)$ *is given by the formula:*

$$\int \Pi_t^{d',d}(y\,;dz)\,f(z)$$

$$= \left[\frac{y}{t+y}\right]^{k-k'} f(t+y) + \int_0^1 \frac{du}{u^{2-k}}\left[\frac{y}{t+y}\right]^{k-k'} g_{k',k}\left(\frac{t}{y}\left(\frac{1}{u}-1\right)f((t+y)u).\right.$$

For the sake of clarity, we have postponed the proofs of formula (3.o) and Proposition (3.3) until now.

Proof of formula (3.o): If we apply the formula:

$$\frac{1}{a^k} = \frac{1}{\Gamma(k)}\int_0^\infty dx\, x^{k-1}e^{-ax}$$

to $a = 1 + \lambda u$, the left-hand side of (3.n) becomes:

$$\frac{1}{\Gamma(k)}\int_{[0,1]} d\mu^{\alpha}(u)\int_0^\infty dx\, x^{k-1}e^{-x-\lambda ux}$$

$$= \frac{1}{\Gamma(k)}\int_0^\infty d\xi\, e^{-\lambda\xi}\xi^{k-1}\int_{[0,1]} d\mu^{\alpha}(u)\frac{1}{u^k}e^{-\xi/u}.$$

We shall now identify the right-hand side of (3.n) as a Laplace transform in λ. Since formula (3.i) follows from (3.h), we know that:

(3.p)
$$\frac{(1+\lambda t)^{k-k'}}{(1+\lambda(t+y))^k} = E\left[P_t^{(d')}((2y)Z_k\,;e^{-\frac{\lambda}{2}\cdot})\right]$$

where, keeping with our notation, Z_k is a gamma variable with parameter k. We introduce the density $p_t^{(d')}(a,b)$ of $P_t^{(d')}(a\,;db)$ which is known to be (see Molchanov (1967)):

(3.q)
$$p_t^{(d')}(a,b) = \frac{1}{2t}\left(\frac{b}{a}\right)^{\frac{k'-1}{2}}\exp-\left(\frac{a+b}{2t}\right)I_{k'-1}\left(\frac{\sqrt{ab}}{t}\right)$$

Making an elementary change of variables in (3.p), we obtain the identity:

$$\frac{(1+\lambda\alpha)^{k-k'}}{(1+\lambda)^k} = 2E\left[\int_0^\infty dz\, p_t^{(d')}(2yZ_k\,;2z)\,e^{-\frac{\lambda z}{t+y}}\right] \qquad (\text{recall}: \alpha = \frac{t}{t+y})$$

$$= 2(t+y)\int_0^\infty d\xi\, e^{-\lambda\xi}E\left[p_t^{(d')}(2y\,Z_k\,;2(t+y)\xi)\right]$$

Comparing the new forms we have just obtained for the two sides of (3.n), we get the identity:

(3.r) $$\frac{\xi^{k-1}}{\Gamma(k)} \int_{[0,1]} d\mu^{\alpha}(u) \frac{1}{u^k} e^{-\xi/u} = 2(t+y) E[p_t^{(d')}(2yZ_k; 2(t+y)\xi]$$

Using formula (3.q), we obtain:

$$2(t+y)E[p_t^{(d')}(2y Z_k; 2(t+y)\xi]$$

$$= \frac{1}{\alpha(\bar{\alpha})^{\frac{k'-1}{2}}} E[(\frac{\xi}{Z_k})^{\frac{k'-1}{2}} \exp -(\frac{\bar{\alpha} Z_k + \xi}{\alpha}) I_{k'-1}(\frac{2}{\alpha}\sqrt{\bar{\alpha} Z_k \xi})]$$

and, developing this expectation, we find that formula (3.r) may be written as:

(3.r)' $$\frac{1}{\Gamma(k)} \xi^{k-1} \int_{[0,1]} d\mu^{\alpha}(u) \frac{1}{u^k} e^{-\xi/u}$$

$$= \frac{\xi^{\frac{k'-1}{2}}}{\alpha(\bar{\alpha})^{\frac{k'-1}{2}}} e^{-\xi/\alpha} \int_0^{\infty} d\eta\, \eta^{k-\frac{k'-1}{2}} e^{-\eta/\alpha} I_{k'-1}(\frac{2}{\alpha}\sqrt{\bar{\alpha}\xi\eta})$$

Now, with the help of the integral representation:

$$\Phi(a,b;z) = \frac{\Gamma(b)}{\Gamma(b-a)} e^z z^{(1-b)/2} \int_0^{\infty} dt\, e^{-t} t^{\frac{1}{2}(b-1)-a} J_{b-1}(2\sqrt{zt})$$

which is valid for: $\mathrm{Re}(b-a) > 0$, $|\arg z| < \pi$, $b \neq 0,1,2,\ldots$ (see Lebedev (1972), p. 278) together with the relation: $I_{\nu}(\xi) = e^{-\frac{i\pi\nu}{2}} J_{\nu}(\xi e^{i\frac{\pi}{2}})$, we find that (3.r)' may be written as:

$$\int_{[0,1]} \mu^{\alpha}(du) e^{-\xi/u} \frac{1}{u^k} = \frac{\Gamma(k)}{\Gamma(k')} (\frac{\alpha}{\xi})^{k-k'} e^{-\xi} \Phi(k'-k, k'; -\frac{\bar{\alpha}}{\alpha}\xi)$$

which is obviously equivalent to (3.o).

Proof of Proposition (3.3):

i) The case when $k - k'$ is an integer n is easy, since then $\Phi(-n, k'; -s)$ is a polynomial of degree n in s and the inversion of the Laplace transform:

$$s^{-n} \Phi(-n, k'; -s)$$

is elementary;

ii) It then remains to prove the Proposition when $k > k' > k-1$, and then, when: $k-1 > k' > k-2$, etc...

In fact, from the definition of $g_{k',k}$ as presented in part 1) of Proposition (3.3), we shall deduce the recurrence relation:

$$\text{(3.s)} \qquad g_{k',k}(x) = \frac{x^{k-k'-1}}{B(k',k-k')} + (k-k')\int_1^\infty \frac{dt}{t^{k-k'}}\, g_{k'+1,k}(tx)$$

(more precisely, assuming that $g_{k'+1,k}$ exists, then if we define $g_{k',k}$ by (3.s), it satisfies part 1) of the proposition).

On the other hand, we also show that the expression of $g_{k',k}$, as presented in part 2) of Proposition (3.3) satisfies the same recurrence relation; consequently, using a recurrence argument, it will be sufficient to prove the proposition in the case $k > k' > k-1$.

iii) We start with the proof of the recurrence relation (3.s). We denote by $g^*_{k',k}(x)$ the right-hand side of (3.s). We easily obtain the formula:

$$1 + \int_0^\infty du\, g^*_{k,k'}(u)\, e^{-su} = \frac{\Gamma(k)}{\Gamma(k')}\frac{1}{s^{k-k'}} + \frac{(k-k')\Gamma(k)}{s^{k-k'}\Gamma(k'+1)}\int_0^s dv\, \Phi(k'+1-k,k'+1;-v)$$

and, in order to prove (3.s), it suffices to show that the right-hand side in the last equality is, in fact:

$$\frac{\Gamma(k)}{\Gamma(k)}\frac{1}{s^{k-k'}}\Phi(k'-k,k';-s)$$

or, equivalently:

$$\Phi(k'-k,k';-s) = 1 + \frac{k-k'}{k'}\int_0^s dv\, \Phi(k'+1-k,k'+1;-v).$$

But, this follows from the identity:

$$\frac{d}{dx}\Phi(k'-k,k';x) = \frac{k'-k}{k'}\Phi(k'+1-k,k'+1;x)$$

(see Lebedev (1972), formula (9.9.4), p. 261).

iv) We now prove the same recurrence relation (3.s) between $\tilde{g}_{k',k}$ (:the function defined in part 2) of the Proposition in terms of F) and $\tilde{g}_{k'+1,k}$. It is elementary to transform the desired relation (3.s) between $\tilde{g}_{k',k}$ and $\tilde{g}_{k'+1,k}$ into the following equivalent relation:

$$\text{(3.s)}' \quad \tilde{g}_{k',k}\left(\frac{1}{y}\right) = \frac{1}{y^{k-k'-1}}\left(\frac{1}{B(k',k-k')} + (k-k')\int_0^y d\eta\, \eta^{k-k'-2}\tilde{g}_{k'+1,k}\left(\frac{1}{\eta}\right)\right).$$

Consequently, in order to prove (3.s)' for $y < 1$, we need to verify the identity:

$$F(k'-k,1+k'-k,k';y) = 1 + \frac{B(k',k-k')(k-k')}{B(k'+1,k-(k'+1))}\int_0^y d\eta\, F(1+k'-k,2+k'-k,1+k';\eta)$$

which follows from the classical identity:

$$\frac{d}{dz} F(\alpha, \beta, \gamma; z) = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; z)$$

(see Lebedev (1972), formula (9.2.2), p. 241) considered for: $\alpha = k' - k$, $\beta = 1 + k' - k$, $\gamma = k'$.

At this point, it remains to verify the relation (3.s) between $\tilde{g}_{k',k}$ and $\tilde{g}_{k'+1,k}$ only for $x < 1$. We write (3.s) in the equivalent form:

$$\tilde{g}_{k',k}(x) = x^{k-k'-1}\left(\frac{1}{B(k', k-k')} + (k-k')\int_x^{\infty} \frac{d\xi}{\xi^{k-k'}} \tilde{g}_{k'+1,k}(\xi)\right)$$

which implies:

$$\tilde{g}'_{k',k}(x) = \frac{k-k'-1}{x} \tilde{g}_{k',k}(x) - \frac{k-k'}{x} \tilde{g}_{k'+1,k}(x).$$

Since the value of $\tilde{g}_{k',k}(1)$ is known, the above differential equation determines $\tilde{g}_{k',k}$ uniquely. Hence, all we have to verify is the following relationship:

$$c_- \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; x)$$

$$= c_- \frac{(k-k'-1)}{x} F(\alpha, \beta, \gamma; x) - \frac{(k-k')}{x}(k-1)(k-k'-1) F(\alpha, \beta + 1, \gamma; x),$$

where: $c_- = (k-1)(k-k')$; $\alpha = 2 - k$; $\beta = k' - k + 1$; $\gamma = 2$. This relationship is equivalent to:

$$\frac{\alpha x}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; x) = -F(\alpha, \beta; x) + F(\alpha, \beta + 1, \gamma; x)$$

which is precisely formula (9.2.13), p. 243, in Lebedev (1972).

v) We finally prove the Proposition when $k > k' > k - 1$. Define $a = k' - k$; it satisfies: $a + 1 > 0$ and $1 - a > 0$. The first part of the proposition will now follow from the relationship:

$$\frac{d}{ds}(s^a \Phi(a, k'; -s)) = (-a) s^{a-1} \Phi(a + 1, k'; -s)$$

and the integral representations:

$$\Gamma(1-a) y^{a-1} = \int_0^{\infty} dt\, e^{-yt} t^{-a} \text{ and } \Phi(a + 1, k'; -y) = \frac{1}{B(a + 1, k - 1)} \int_0^1 dt\, e^{-yt} t^a (1-t)^{k-2}.$$

We now obtain that part 1) of the proposition is satisfied with the function $g = g_{k',k}$ defined by:

$$u g(u) = \frac{c(-a)}{B(a + 1, k - 1)\Gamma(1-a)} h(u),$$

where: $h(u) = \int_0^{u\wedge 1} dt\, t^a (1-t)^{k-2} (u-t)^{-a}$ and $c = \frac{\Gamma(k)}{\Gamma(k')}$. The expression of h, hence of g, in terms of F, is then deduced from the integral representation:

$$F(\alpha,\beta,\gamma;\, u) \;=\; \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 dt\, t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-ut)^{-\alpha}$$

which is valid for: $\mathrm{Re}\gamma > \mathrm{Re}\beta > 0$ and $u < 1$ (see Lebedev (1972), formula (9.1.4), p. 239).

4. Some properties of the $Y_{d',d}$ processes.

The family of the $Y_{d',d}$ processes enjoys a number of properties which are the counterparts of properties of the squares of Bessel processes. In the eight following sections, we shall compare such properties for both classes of processes.

(4.1) Absolute continuity relations.

Fix $x > 0$. As d varies in $[2,\infty[$, the laws $P_x^{(d)}$ of $BESQ_x(d)$ are locally mutually equivalent. The following explicit formula holds:

$$\text{(4.a)} \qquad P_x^{(d)}|_{F_t} \;=\; \left(\frac{X_t}{x}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\right) \cdot P_x^{(2)}|_{F_t} \qquad \left(\nu \equiv \frac{d}{2} - 1\right).$$

From this relation, one deduces the important formula:

$$\left(\frac{y}{x}\right)^{\nu/2} E_x^{(2)}\left[\exp -\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s} \,|\, X_t = y\right] \;=\; \frac{p_t^{(d)}(x,y)}{p_t^{(2)}(x,y)}$$

which implies:

$$E_x^{(2)}\left[\exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\right) | X_t = y\right] \;=\; \frac{I_{|\nu|}}{I_0}\left(\frac{\sqrt{xy}}{t}\right).$$

This formula plays a key role in the study of the winding number of complex Brownian motion around 0 (see Spitzer (1958), Yor (1980), for instance). The counterpart of (4.a) for the laws $\Pi_y^{k',k}$ of the $Y_{d',d}$ processes starting from y is the following

Theorem 4.1: *Let* $\lambda \geq 0$, $k_\lambda = k + \lambda$, $k_\lambda'(k_\lambda - 1) = \lambda k_\lambda + (k-1)(k'+\lambda)$. *Then, one has:*

$$\Pi_y^{k_\lambda',k_\lambda}|_{G_t} \;=\; \left(\frac{Y_t}{y}\right)^{\lambda} \exp\left(-\mu\int_0^t \frac{ds}{Y_s}\right) \cdot \Pi_y^{k',k}|_{G_t}, \text{ where } \mu = \lambda \cdot \frac{k'-1+\lambda}{k-1+\lambda}.$$

(4.2) **Time-changing**

a) Here are two transformations of Bessel processes which are most useful in some computations:

i) if $(R_t)_{t\geq 0}$ is BES(d), with $d \geq 2$, starting at $r_0 > 0$, there exists a real-valued Brownian motion $(\beta_t)_{t\geq 0}$ such that:

$$\log R_t = (\beta_u + \nu u)\Big|_{u=\int_0^t \frac{ds}{R_s^2}} \quad , \text{ where } \nu = \frac{d}{2} - 1 .$$

In the literature, this relation is also found in the form of a representation of the geometric Brownian motion with drift ν, i.e.: $(\exp(\beta_u + \nu u), u \geq 0)$, in terms of a Bessel process with dimension $d = 2(1 + \nu)$, as follows:

$$\exp(\beta_u + \nu u) = R\Big(\int_0^u ds \exp 2(\beta_s + \nu_s)\Big), \quad u \geq 0 .$$

(see, for example, D. Williams (1974(a)), and for some applications, Yor [(1992a) and (1992b)]).

ii) for convenience, $(R_\mu(t), t \geq 0)$ now denotes the Bessel process with index μ. Let p and q such that: $\frac{1}{p} + \frac{1}{q} = 1$.

Then, under suitable conditions on μ and p, we have:

$$q R_\mu^{1/q}(t) = R_{\mu q}\Big(\int_0^t ds\, R_\mu^{-2/p}(s)\Big)$$

(see Biane-Yor (1987), lemma (3.1) and Revuz-Yor (1991), Chapter XI).

b) Here are some similar results for the $Y_{d',d}$ processes.

Theorem (4.2): *i) If* $Y \equiv Y_{d',d}$ *starts from* $y > 0$, *and* $d' \geq 2$, *there exists a process with stationary independent increments* $(\xi(u), u \geq 0)$ *such that:*

$$\log Y_t = \xi\Big(\int_0^t \frac{ds}{Y_s}\Big) \qquad (t \geq 0).$$

The generator of $\xi(u) \equiv \xi_{d',d}(u)$ *is given by:*

$$L_{k',k}\phi(z) = \phi'(x) + (k - k')\int_0^\infty dy(k-1)e^{-y(k-1)}(\phi(x-y) - \phi(x))$$

ii) Let $\alpha > 0$. *Then:*

$$Y_t \equiv Y_{k',k}(t) = Y_{k_{(\alpha)}', k_{(\alpha)}}\Big(\int_0^t du\, \alpha Y_u^{\alpha-1}\Big)$$

where: $k'_{(\alpha)} = \dfrac{k'-1}{\alpha} + 1$, *and* $k_{(\alpha)} = \dfrac{k-1}{\alpha} + 1$

Remarks: 1) *Beware:* the notation $(k'_{(\alpha)}, k_{(\alpha)})$ has nothing to do with the notation $(k'_\lambda k_\lambda)$ introduced in Theorem (4.1);

2) There are some similar results for the $\hat{Y}$ processes as introduced via Theorem (3.2), the discussion of which is postponed until subsection (4.4).

c) In fact, both Bessel processes and the processes Y are examples of a particular class of $\mathbb{R}_+$-valued Markov processes X which enjoy the following scaling property: there exists $\alpha > 0$ such that, for $a \geq 0$, the law of $(X_{\lambda t}, t \geq 0)$ under P_a is that of

$$(\lambda^\alpha X_t, t \geq 0) \text{ under } P_{(a/\lambda^\alpha)}.$$

Lamperti (1972) has studied these processes, which he calls semi-stable Markov processes and has shown that, if P_a a.s., $(X_t, t \geq 0)$ does not visit 0, then one has:

$$\log X_t = \xi\left(\int_0^t \frac{du}{X_u}\right), \quad t \geq 0$$

(here, we have assumed, for simplicity, $\alpha = 1$) for some process ξ with stationary independent increments. Several studies of such processes have been made in recent years.

(4.3) **First passage times.**

a) *First passage times for* BESQ(d).

If (X_t) denotes BESQ(d), we recall (Kent (1978), Getoor-Sharpe (1979), Pitman-Yor (1981)) that:

$$\phi(\lambda X_t)e^{-\lambda t} \text{ is a local martingale, for } \phi = \phi_+ \text{ or } \phi_-,$$

with

$$\phi_+(x) = x^{-\nu/2} I_\nu(\sqrt{2x}) \text{ and } \phi_-(x) = x^{-\nu/2} K_\nu(\sqrt{2x}).$$

This implies:

$$E_a[e^{-\lambda T_b}] = \frac{\phi(\lambda a)}{\phi(\lambda b)}, \text{ with } \phi = \begin{cases} \phi_+, \text{ if } a < b \\ \phi_-, \text{ if } a > b. \end{cases}$$

b) *Intertwining and martingales.*

The following lemma will be useful in the sequel:

Lemma (4.3): *Assume that* $Q_t \Lambda = \Lambda P_t$. *Then:*

1) if $\phi(X_t)e^{-\lambda t}$ *is a* (P_x) *martingale, for every* x, *then:*

$$\Lambda\phi(Y_t)e^{-\lambda t} \text{ is a } (Q_y) \text{ martingale, for every } y,$$

2) More generally, if L, *resp:* $\tilde{L}$ *denotes the infinitesimal generator of* X, *resp:* Y, *then:* $\tilde{L}\Lambda = \Lambda L$, *and: if* $f \in \mathbf{D}(L)$, *then:* $\Lambda f \in \mathbf{D}(\tilde{L})$ *and* $f(X_t) - \int_0^t (Lf)(X_s)\,ds$ *is a* P_x-*martingale, while:*

$$\Lambda f(Y_t) - \int_0^t ds\,\Lambda(Lf)(Y_s) \text{ is a } Q_y\text{–martingale.}$$

Remark: The first result may be understood as a particular case of the second one, since the function ϕ satisfies: $L\phi = \lambda\phi$, and hence: $\tilde{L}(\Lambda\phi) = \lambda(\Lambda\phi)$.

c) *First passage times for* $Y_{d',d}$.

From the above paragraphs a) and b), we deduce that:

$$\Lambda_{d',d}\,\phi_{\pm}(\lambda Y_t)\,e^{-\lambda t} \text{ is a } \Pi_y^{d',d}\text{–martingale,}$$

which yields:

$$\Phi(k,k';\lambda Y_t)e^{-\lambda t} \text{ and } \Psi(k,k';\lambda Y_t)e^{-\lambda t} \text{ are } \Pi_y^{d',d}\text{–martingales.}$$

Hence:

$$\text{(4.b)} \qquad \Pi_a^{d',d}(e^{-\lambda T_a}) = \frac{H(k,k';\lambda_a)}{H(k,k';\lambda b)}, \text{ where } H = \begin{cases} \Phi, \text{ if } a < b \\ \Psi, \text{ if } a > b. \end{cases}$$

In the particular case a = 0, b = 1, we obtain:

$$\Pi_0^{d',d}(e^{-\lambda T_1}) = 1/\Phi(k,k';\lambda).$$

Hence, the function: $\log\Phi(k,k';\lambda)$ admits the Lévy-Khintchine representation:

$$\log\Phi(k,k';\lambda) = c\lambda + \int_0^\infty d\nu(x)(1-e^{-\lambda x}),$$

for some measure ν to be determined. Taking derivatives with respect to λ, and using the relations:

$$\frac{d}{d\lambda}\Phi(k,k';\lambda) = \frac{k}{k'}\Phi(k+1,k'+1;\lambda) = \Phi(k,k';\lambda) + \frac{k-k'}{k}\Phi(k,k'+1;\lambda)$$

(see Lebedev (1972), (9.9.13), p. 262), we obtain:

$$1 + \left(\frac{k-k'}{k}\right)\frac{\Phi(k,k'+1;\lambda)}{\Phi(k,k';\lambda)} = c + \int_0^\infty d\nu(x)\,xe^{-\lambda x}.$$

From the asymptotic result (see Lebedev (1972), (9.12.8), p. 271):

$$\Phi(k,k';\lambda) \sim C_{k,k'} e^{\lambda} \lambda^{-(k'-k)} \qquad (\lambda \to \infty),$$

we deduce that c = 1 and there exists a probability $\mu(dx)$ on $\mathbb{R}_+$ such that

$$\text{(4.c)} \qquad \frac{\Phi(k,k'+1;\lambda)}{\Phi(k,k';\lambda)} = \int_0^\infty \mu(dx)\, e^{-\lambda x}, \text{ and } : \mu(dx) = \left(\frac{k}{k-k'}\right) x\nu(dx).$$

Another interpretation of the probability μ shall be given in section (4.7).

d) *First passage times for* $\xi_{d',d}$.

The results in this paragraph follow essentially from the absolute continuity relation obtained in Theorem (4.2) for the ξ processes.

First, we have:

$$E_0[e^{\lambda \xi_t}] = e^{t\psi(\lambda)}, \text{ where } \psi(\lambda) \doteq \lambda \frac{k'-1+\lambda}{k-1+\lambda} = \lambda\left(\frac{\alpha-\beta+\lambda}{\alpha+\lambda}\right)$$

and we have defined $\alpha = k-1$ and $\beta = k-k'$.

We then deduce from this (or we could appeal again to Theorem (4.2)) that, with the notation $\sigma_a = \inf\{u : \xi_u = a\}$,

$$E_0[e^{-\mu\sigma_a}] = e^{-a\psi^{-1}(\mu)}$$

where $\psi^{-1}(\mu) = \frac{1}{2}\{\mu - (\alpha-\beta) + ((\mu-(\alpha-\beta))^2 + 4\alpha\mu)^{1/2}\}$.

It is interesting to study the Lévy-Khintchine representation of ψ^{-1}: we find

$$\psi^{-1}(\mu) = \mu + \int_0^\infty \nu(du)(1-e^{-\mu u}), \quad \text{where:}$$

$$\text{(4.d)} \qquad \nu(du) = \frac{\sqrt{\alpha\beta}}{u} I_1(2\sqrt{\alpha\beta}u)\, e^{-(\alpha+\beta)u}.$$

Proof of formula (4.d): We first remark that:

$$(\mu-(\alpha-\beta))^2 + 4\alpha\mu = (\mu+\alpha+\beta)^2 - 4\alpha\beta.$$

We now seek a constant c and a positive measure ν on $\mathbb{R}_+$ such that:

$$\mu - \alpha + \beta + ((\mu+\alpha+\beta)^2 - 4\alpha\beta)^{1/2} = 2\left(c\mu + \int_0^\infty \nu(du)(1-e^{-\mu u})\right).$$

Taking derivatives of both sides with respect to μ, we obtain:

$$1 + \frac{\mu+\alpha+\beta}{((\mu+\alpha+\beta)^2-4\alpha\beta)^{1/2}} = 2\left(c + \int_0^\infty \nu(du)\, u e^{-\mu u}\right)$$

from which we deduce, by letting $\mu \to \infty$, that: $c = 1$. It remains to find the measure ν which is specified by the equality:

$$\frac{\mu+\alpha+\beta}{((\mu+\alpha+\beta)^2-4\alpha\beta)^{1/2}} - 1 = 2\int_0^\infty \nu(du)\, u e^{-\mu u}.$$

Making the change of variables: $\mu+\alpha+\beta = 2\sqrt{\alpha\beta}\,\eta$, and using the following relation, valid for $\eta \ge 1$ (see Feller (1966), p.414):

$$\frac{\eta}{\sqrt{\eta^2-1}} - 1 = \int_0^\infty dx\, I_1(x)\, e^{-\eta x}$$

we obtain:

$$\begin{aligned}\frac{\mu+\alpha+\beta}{((\mu+\alpha+\beta)^2-4\alpha\beta)^{1/2}} - 1 &= \frac{\eta}{(\eta^2-1)^{1/2}} - 1 = \int_0^\infty dx\, I_1(x)\, e^{-\eta x} \\ &= \int_0^\infty dx\, I_1(x)\, e^{-\frac{\mu+\alpha+\beta}{2\sqrt{\alpha\beta}}x} \\ &= 2\sqrt{\alpha\beta}\int_0^\infty dy\, I_1(2\sqrt{\alpha\beta}y)\, e^{-\mu y}\, e^{-(\alpha+\beta)y}\end{aligned}$$

and formula (4.d) follows.

Note: These computations appear to be closely related to recent work by J. Pellaumail (1991) in Queuing Theory.

e) *Laguerre polynomials and hypergeometric polynomials.*

e.i) Let (X_t) denote the square of BES(d'), with $d' = 2(\nu'+1)$. (X_t) may be characterized (in law) as the unique solution of the martingale problem:

(4.e) for every $\lambda > 0$, $\phi(\lambda X_t)\, e^{-\lambda t}$ is a martingale, where $\phi(x) = x^{-\nu'/2} I_{\nu'}(\sqrt{2x})$.

We recall the hypergeometric functions notation (see Lebedev (1972), p. 275):

$${}_0F_1(-, 1+\nu'; z) = \Gamma(\nu') z^{-\nu'/2} I_{\nu'}(2\sqrt{z})$$

which implies:

$$\text{(4.f)} \qquad {}_0F_1\left(-, 1+\nu'; \frac{z}{2}\right) = c_{\nu'}\, \phi(z), \quad \text{where } c_{\nu'} = \Gamma(\nu') 2^{\nu'/2}.$$

The Laguerre polynomials with parameter ν': $L_n^{(\nu')}(x)$ may be defined as the coefficients of the generating function:

$${}_0F_1(-, 1+\nu'; -xy)\, e^y = \sum_{n=0}^\infty \frac{L_n^{(\nu')}(x)\, y^n}{(1+\nu')_n}$$

(see McBride (1971), p. 39).

It then follows from formula (4.f) that:

(4.g) $$c_{\nu'}\,\phi(\lambda x)\,e^{-\lambda t} = \sum_{n=0}^{\infty} \frac{1}{(1+\nu')_n} L_n^{(\nu')}\left(\frac{x}{2t}\right)(-\lambda t)^n = \sum_{n=0}^{\infty} \lambda^n P_n(x,t)$$

where we have defined:

$$P_n(x,t) = \frac{(-t)^n}{(1+\nu')_n} L_n^{(\nu')}\left(\frac{x}{2t}\right) = \frac{(-t)^n}{n!}\Phi\left(-n, \nu'+1; \frac{x}{2t}\right),$$

since the expression of $L_n^{(\nu')}$ in terms of the confluent hypergeometric function Φ is:

$$L_n^{(\nu')}(z) = \frac{(\nu'+1)_n}{n!}\Phi(-n, \nu'+1; z) \qquad \text{(Lebedev (1972), p.273).}$$

(we recall that, with our notation, $k' = \nu' + 1$). We deduce from (4.e) and (4.g) that:

(4.h) for every $n \in \mathbb{N}$, $\left(t^n L_n^{(\nu')}\left(\frac{X_t}{2t}\right),\ t \geq 0\right)$ is a martingale.

e.ii) We shall now discuss similar results for the process $Y = Y_{d',d}$. This process may be characterized as the unique solution of the martingale problem:

(4.i) for every $\lambda > 0$, $\Lambda_d[\phi(\lambda\,\cdot)](Y_t)\,e^{-\lambda t}$ is a martingale.

Define $\psi(y) = \Lambda_d\phi(y) \equiv \frac{1}{\Gamma(k)} \int_0^{\infty} da\, a^{k-1} e^{-a} \phi(2ya) \qquad (k = \frac{d}{2})$

and $Q_n(y,t) = \frac{1}{c_\nu} \Lambda_d(P_n(\cdot, t))(y)$.

It follows from (4.g) that:

(4.j) $$c_{\nu'}\,\psi(\lambda y) e^{-\lambda t} = \sum_{n=0}^{\infty} \lambda^n Q_n(y, t).$$

We now identify ψ and Q_n.

We remark that, in general, if $F(z) = \sum_{p=0}^{\infty} f_p z^p$ (with $f_p \geq 0$, for every p), then:

$$F^{(d)}(z) \overset{\text{def}}{=} \Lambda_d(F)(z) = \sum_{p=0}^{\infty} (k)_p f_p z^p.$$

In particular, the application: $F \rightarrow F^{(d)}$ transforms ${}_pF_q(\alpha_r, \gamma_s; z)$ into:

$${}_{p+1}F_q(k, \alpha_r, \gamma_s; z).$$

Consequently, we obtain:

$$\psi(y) = \Lambda_d\phi(y) = \frac{1}{c_{\nu'}}\, {}_0F_1(-, k'; \cdot)^{(d)}(z) \qquad \text{(from (4.g))}$$

$$= \frac{1}{c_{v'}} \Phi(k, k'; z).$$

Likewise,

$$Q_n(y,t) = \frac{1}{c_{v'}} \Lambda_d(P_n(\cdot, t))(y) = \frac{(-t)^n}{c_{v'} n!} \Phi(-n, k'; \frac{\cdot}{t})^{(d)}(y)$$

$$= \frac{(-t)^n}{c_{v'} n!} F(-n, k, k'; \frac{y}{t}).$$

Hence, the series (4.j) may be written in the form:

(4.k) $$\Phi(k, k'; \lambda y) e^{-\lambda t} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (-t)^n F(-n, k, k'; \frac{y}{t});$$

the polynomials $F(-n, k, k'; y)$ are the so-called hypergeometric polynomials.

The assertions similar to (4.e) and (4.h) are:

(4.l) for every $\lambda > 0$, $\Phi(k, k', \lambda Y_t) e^{-\lambda t}$ is a martingale;

(4.m) for every $n \in \mathbb{N}$, $t^n F(-n, k, k'; \frac{Y_t}{t})$ is a martingale.

e.iii) We have just seen that, in analytic terms, the intertwining of the processes $X_{d'}$ and $Y_{d',d}$ with respect to the kernel Λ_d translates as the transformation of Laguerre polynomials $\Phi(-n, k'; \cdot)$ into hypergeometric polynomials $F(-n, k, k'; \cdot)$ via the formula:

$$F(-n, k, k'; y) = \frac{1}{\Gamma(k)} \int_0^\infty da\, a^{k-1} e^{-a} \Phi(-n, k'; ay).$$

Likewise, the intertwining of the processes $X_{d'}$ and X_d $(d' < d)$ with respect to the kernel $\Lambda_{d',d}$ translates, in analytic terms, as the transformation of Laguerre polynomials with parameter $\nu' = k' - 1$: $L_n^{(\nu')}(x)$ into Laguerre polynomials with parameter $\nu = k - 1$: $L_n^{(\nu)}(x)$ via *Koshlyakov's formula* (see Lebedev (1972), p. 94):

$$L_n^{(\nu)}(x) = \frac{\Gamma(n+k)}{\Gamma(k-k')\Gamma(n+k')} \int_0^1 dt\, t^{k-1} (1-t)^{k-k'-1} L_n^{(\nu')}(xt).$$

In the same spirit, the integral relation (see Lebedev (1972), p.277)

$$F(\alpha, \beta, \gamma; z) = \frac{1}{B(c, \gamma - c)} \int_0^1 dt\, t^{c-1} (1-t)^{\gamma-c-1} F(\alpha, \beta, c; zt)$$

may be considered as a translation, in analytic terms, of the intertwining relations which hold between the different processes $Y_{d',d}$ (Theorem E above).

e.iv) We now consider two other fundamental generating functions for $(L_n^{(\nu')}(x), n \geq 0)$ and $(F(-n,k,k';z), n \geq 0)$ respectively, which have a clear meaning in terms of martingale properties of $(X_{d'}(t))$ and $(Y_{d',d}(t))$. These generating functions are:

$$(4.n)\qquad \begin{aligned} (1-t)^{-(\nu'+1)} e^{-xt/1-t} &= \sum_{n=0}^{\infty} L_n^{(\nu')}(x) t^n \\ (1-t)^{k-k'}(1-t+xt)^{-k} &= \sum_{n=0}^{\infty} \frac{(k')_n}{n!} F(-n,k,k';x) t^n \end{aligned}$$

(see Lebedev (1972), p.77 and 277 respectively).

Let $t = \frac{\lambda s}{1+\lambda}$, with $s < 1$, $x = \frac{z}{2s}$, and $u(\lambda) = (1+\lambda)^{-k'}$. The two left hand sides of (4.n) become:

$$u(\lambda)(1+\lambda-\lambda s)^{-k'} e^{-\frac{\lambda z}{2(1+\lambda(1-s))}}$$

and

$$u(\lambda)(1+\lambda-\lambda s)^{k-k'}(1+\lambda(1-s+z))^{-k};$$

both expressions played a key role in the explicit computation of $\Pi_t^{d',d}$ (see formula (3.i)). Indeed, these expressions are in fact respectively equal to:

$$u(\lambda) P_{1-s}^{(d')}(e^{-\frac{\lambda y}{2}})(z) \equiv u(\lambda) \sum_{n=0}^{\infty} L_n^{(\nu')}(\frac{z}{2s}) s^n (\frac{\lambda}{1+\lambda})^n$$

and

$$u(\lambda) \Pi_{1-s}^{d',d}((1+\lambda\xi)^{-k})(z) \equiv u(\lambda) \sum_{n=0}^{\infty} \frac{(k')_n}{n!} F(-n,k,k';\frac{z}{2s}) s^n (\frac{\lambda}{1+\lambda})^n$$

Now, replacing z respectively by $X_{d'}(s)$ and $Y_{d',d}(s)$ $(s < 1)$, we obtain two martingales which are in correspondence via the intertwining kernel Λ_d, since:

$$\Lambda_d(e^{-\frac{\lambda y}{2}})(\xi) = c_d(1+\lambda\xi)^{-k}$$

(4.4) **Time reversal.**

In this section, we shall apply the following general result on time-reversal successively to $X^{(d')}$, a BESQ(d') process, and $Y_{d',d}$, at their last exit time from $b > 0$, when $d' > 2$.

Theorem (4.4): (Nagasawa (1964); see also Sharpe (1980) for another proof) *Let* X *and* $\hat{X}$ *be standard Markov processes in* E, *which are in duality with respect to* μ *(cf: Definition (3.1)).*

Let u(x,y) *denote the potential kernel density of* X *relative to* μ, *so that:*

$$E_x[\int_0^\infty f(X_t)\,dt] = \int u(x,y)\,f(y)\,\mu(dy).$$

Let L *be a cooptional time for* X, *that is a positive random variable satisfying:* $L \le \zeta$ *and* $L \circ \theta_t = (L-t)^+$. *Define* $\tilde{X}_t$ *by:*

$$\tilde{X}_t = \begin{cases} X_{(L-t)-}, \text{ on } 0 < L < \infty, \text{ for } 0 < t < L \\ \Delta, \quad \text{otherwise.} \end{cases}$$

Then, for any initial law λ, *the process* $(\tilde{X}_t)_{t>0}$, *under* P_λ, *is homogeneous Markov, with transition semi-group* $(\tilde{P}_t)$ *given by:*

$$\tilde{P}_t f(y) = \begin{cases} \hat{P}_t(fv)(y)/v(y), & \text{if } 0 < v(y) < \infty \\ 0 & , \text{ if } v(y) = 0 \text{ or } \infty. \end{cases}$$

In case $\lambda = \varepsilon_x$, $v(y) = u(x,y)$.

For our application, we take: $x = 0$, $d' > 2$, L the last exit time from $b > 0$, for either $X^{(d')}$ or $Y_{d',d}$, and $\mu(dx) = x^{\nu'}dx$. Then, according to Theorem (3.2), $P_t^{(d')}$ is self-dual with respect to μ, and the semi-group $(\Pi_t^{d',d})$ is in μ-duality with $(\hat{\Pi}_t^{d',d})$. Furthermore, we remark that, in both cases, thanks to the scaling properties enjoyed by the processes, we have: $v(y) = c/y^{\nu'}$, for some constant c.

Indeed, dealing with $Y(t) \equiv Y_{d',d}(t)$, for instance, we remark that:

$$E_0[\int_0^\infty dt\,f(Y_t)] = E_0[\int_0^\infty dt\,f(tY_1)] = \int_0^\infty du\,f(u)\,E_0[\frac{1}{Y_1}] = \int_0^\infty dy\,y^{\nu}(\frac{c}{y^{\nu'}})\,f(y)$$

which yields the desired result.

We now have the following

Theorem (4.5): *Let* $2 < d' < d$, *and* $(X_t^{(d')})$ *and* $(Y_{d',d}(t))$ *start at 0; then, for* $b > 0$*:*

a) $(X_t^{(d')};\, t \le L_b) \overset{(d)}{=} (X_t^{(4-d')};\, t \le T_0)$

b) $(Y_{d',d}(t);\, t \le L_b) \overset{(d)}{=} (\hat{Y}_{4-d',d+2-d'}(t);\, t \le T_0)$

where, on both right-hand sides, it is assumed that the processes start at b.

Remark: The result b) is probably better understood by looking at the infinitesimal generators L and $\tilde{L}$ of, respectively, the left-and right-hand sides; one has:

$$L\phi(y) = \phi'(y) + \frac{k-k'}{y}\int_0^1 dz\,(k-1)\,z^{k-2}\,(f(zy) - f(y))$$

and

$$\tilde{L}\phi(y) = -\phi'(y) + \frac{k-k'}{y}\int_0^1 dz\,(k-1)\,z^{k-2}\,(f(\frac{y}{z}) - f(y))$$

and it is clear that this time reversal result could have been proved simply by considering the pathwise description given in the Comments after Theorem B.

(4.6) **Some limit theorems.**

In this section, we obtain several limit theorems concerning the processes $Y_{d',d}$ and $\xi_{d',d}$, some of which are then applied to the study of the asymptotics of the functional $\int_0^t \frac{ds}{Y_{d',d}(s)}$, as $t \to \infty$, when $Y_{d',d}(0) \neq 0$.

In the sequel, we use the notation (fd) to denote the convergence in law of finite-dimensional distributions of processes indexed by $\mathbb{R}_+$. Moreover, in this section, we shall use the notation δ, instead of d, for the second "dimension" parameter of the process Y, since, in integrals such as the one we have just written, d stands for the differential in ds, as well as for the second "dimension" parameter in $Y_{d',d}(s)$, which might create some confusion.

a) The main result in this section is the following

Theorem (4.6): *Let* $0 < d' < \delta$, *and* $\delta > 2$. *Define* $\nu' = \frac{d'}{2} - 1$, $\nu = \frac{\delta}{2} - 1$, *and let* $(X_t^{(d')}, t \geq 0)$ *denote a* BESQ(d'), *and* $(\beta_t, t \geq 0)$ *a 1-dimensional* BM. *Then:*

i) *for fixed* d', $(Y_{d',\delta}(\delta t), t \geq 0) \underset{\delta\to\infty}{\overset{(fd)}{\to}} (X_t^{(d')}, t \geq 0)$

ii) *for fixed* d', $(\xi_{d',\delta}(\delta t), t \geq 0) \underset{d\to\infty}{\overset{(fd)}{\to}} (2(\beta_t + \nu' t)\,;\, t \geq 0)$

iii) *for fixed* d' *and* δ *with* $2 < d' < \delta$, $\frac{1}{\lambda}\xi_{d',\delta}(\lambda t) \underset{\lambda\to\infty}{\overset{(P)}{\to}} \frac{\nu'}{\nu} t$

iv) *for fixed* $\delta > 2$, $(\frac{1}{\sqrt{\lambda}}\xi_{2,\delta}(\lambda t), t \geq 0) \underset{\lambda\to\infty}{\overset{(fd)}{\to}} ((\frac{2}{\nu})^{1/2}\beta_t, t \geq 0)$

Remarks: 1) The result ii) is in agreement with i) and the time-change formula (see Theorem (4.2) in section (4.2)):

$$\log Y_{d',\delta}(t) = \xi_{d',\delta}\left(\int_0^t \frac{ds}{Y_{d',\delta}(s)}\right);$$

hence, we have: $\log Y_{d',\delta}(\delta t) = \xi_{d',\delta}\left(\delta \cdot \int_0^t \frac{du}{Y_{d',\delta}(\delta u)}\right)$ and we remark that the result i) fits in well with the time-change representation of $(\log X_t^{(d')}, t \geq 0)$ as:

$$\log X_t^{(d')} = 2(\beta_u + \nu' u)\Big|_{u=\int_0^t \frac{ds}{X_s^{(d')}}}.$$

2) In the case where $Y_{d',\delta}(0) = 0$, the following scaling property holds:

$$(4.o) \qquad (Y_{d',\delta}(\lambda t), t \geq 0) \overset{(d)}{=} (\lambda Y_{d',\delta}(t), t \geq 0)$$

and we may write i) in the equivalent form:

$$(\delta Y_{d',\delta}(t), t \geq 0) \underset{\delta\to\infty}{\overset{(fd)}{\to}} (X_t^{(d')}, t \geq 0).$$

The result for one-dimensional marginals is easily understood; indeed, for $t = 1$, we know that the law of $Y_{d',\delta}(1)$ is $\beta(\frac{d'}{2}, \frac{\delta - d'}{2})$, hence:

$$Y_{d',\delta}(1) \overset{(d)}{=} \frac{X_1^{(d')}}{X_1^{(d')} + X_1^{(\delta-d')}}$$

where $X^{(d')}$ and $X^{(\delta-d')}$ are independent BESQ processes with respective dimensions d' and $\delta - d'$. We then deduce from the law of large numbers that $\frac{\delta}{X_1^{(d')} + X_1^{(\delta-d')}}$ converges in probability to 1, as $\delta \to \infty$, which implies the desired result.

3) iv) is obviously a refinement of iii) in the case $d' = 2$ (which implies $\nu' = 0$).

4) Inspection of infinitesimal generators easily yields the following identity in law:

$$(4.p) \qquad (\frac{1}{\lambda}\xi_{k',k}(\lambda t), t \geq 0) \overset{(d)}{=} (\xi_{k'_\lambda, k_\lambda}(t), t \geq 0)$$

where the couple (k_λ', k_λ) is defined by:

$$k_\lambda - k_\lambda' = k - k'; \quad k_\lambda - 1 = \lambda(k-1)$$

or, in terms of indices instead of dimensions:

$$\nu_\lambda = \lambda\nu \quad \text{and} \quad \nu_\lambda' = \nu' + (\lambda - 1)\nu.$$

The identity in law (4.p) allows to recast the limit results ii), iii), iv) in terms of ξ-processes, both indices of which increase to $+\infty$, as $\lambda \to \infty$, in the manner we have just indicated.

Proof of Theorem (4.6): 1) The infinitesimal generator of $(Y_{d',\delta}(\delta t), t \geq 0)$, applied to $\phi \in C^1(\mathbb{R}_+)$, is, in terms of k' and k:

$$2k\{\phi'(y) + \frac{k-k'}{y}\int_0^1 dz\,(k-1)\,z^{k-2}(\phi(zy) - \phi(y))\}$$

$$= 2k\{\phi'(y) + \frac{k-k'}{y}\int_0^\infty dv\, e^{-v}(\phi(ye^{-\frac{v}{k-1}}) - \phi(y))\},$$

after an elementary change of variables.

It is now easy to justify that, as k′ is fixed, and $k \to \infty$, we may replace:

$$\phi(ye^{-\frac{v}{k-1}}) - \phi(y)$$

by: $$y\phi'(y)(e^{-\frac{v}{k-1}} - 1) + \frac{y^2}{2}\phi''(y)(e^{-\frac{v}{k-1}} - 1)^2.$$

Then, the coefficients of $\phi'(y)$, resp: $\phi''(y)$ converge, as $k \to \infty$, towards: d′, resp: 2y, which implies i).

2) The same sort of arguments may be applied to prove the results ii), iii) and iv). We only give the details for ii):

The infinitesimal generator of $(\xi_{d',\delta}(\delta t), t \geq 0)$, applied to $\phi \in C^1(\mathbb{R})$, is, in terms of k′ and k:

$$2k\{\phi'(y) + (k-k')\int_0^\infty du\,(k-1)e^{-u(k-1)}(\phi(y-u) - \phi(y))\}.$$

We then replace: $\phi(y-u) - \phi(y)$ by: $-u\phi'(y) + \frac{u^2}{2}\phi''(y)$; then, the coefficients of $\phi'(y)$, resp: $\phi''(y)$, are:

$$\frac{2k}{k-1}(k'-1), \quad \text{resp:} \quad \frac{2k(k-k')}{(k-1)^2}$$

and they converge, as $k \to \infty$, to: $2\nu'$, resp: 2, which implies ii).

b) We begin by recalling the following asymptotic results for the BESQ(d′) process $X^{(d')}$, when $X_0^{(d')} \neq 0$:

(4.q) $$\frac{4}{(\log t)^2}\int_0^t \frac{ds}{X_s^{(2)}} \underset{t\to\infty}{\overset{(d)}{\to}} \sigma$$

where $\sigma = \inf\{t : \beta_t = 1\}$, and β is a 1-dimensional BM starting from 0, and, when $d' > 2$:

(4.q)′ $$\frac{2}{\log t}\int_0^t \frac{ds}{X_s^{(d')}} \underset{t\to\infty}{\overset{a.s.}{\to}} \frac{1}{\nu'}$$

We shall now prove similar results for the $Y_{d',\delta}$ processes:

Theorem (4.7): *We consider the process* $Y_{d',\delta}$ *with* $d' \geq 2$ *and* $Y_{d',\delta}(0) \neq 0$. *Then:*

i) if $d' = 2$, $\frac{1}{(\log t)^2} \int_0^t \frac{du}{Y_{2,\delta}(u)} \underset{t\to\infty}{\overset{(d)}{\rightarrow}} \frac{\nu}{2}\sigma$

where $\nu = \frac{\delta}{2} - 1$, *and* $\sigma = \inf\{u : \beta_u = 1\}$, *with the same notation as in (4.q) above;*

ii) if $d' > 2$, $\frac{1}{\log t} \int_0^t \frac{ds}{Y_{d',\delta}(s)} \underset{t\to\infty}{\overset{a.s.}{\rightarrow}} \frac{\nu}{\nu'}$, *where:* $\nu = \frac{\delta}{2} - 1$ *and* $\nu' = \frac{d'}{2} - 1$.

At least, 3 different proofs of (4.q) are known; they hinge respectively on:

a) Laplace's asymptotic method (Durrett (1982), Yor (1985), Le Gall-Yor (1986)),

b) a pinching argument (D. Williams (1974), Messulam-Yor (1982)), and, finally:

c) the explicit computation of the law of $\int_0^t \frac{ds}{X_s^{(2)}}$ (Spitzer (1958), Itô-McKean (1965), Yor (1980)).

We shall now see that the 3 methods admit variants from which part i) of Theorem (4.7) follows.

a) *Laplace's method.*

For simplicity, we write Y_δ instead of $Y_{2,\delta}$, resp: ξ_δ instead of $\xi_{2,\delta}$. From the formula:

$$\log Y_\delta(t) = \xi_\delta\left(\int_0^t \frac{du}{Y_\delta(u)}\right),$$

we deduce:

$$\int_0^t \frac{du}{Y_\delta(u)} = \inf\{v : \int_0^v ds \exp(\xi_\delta(s)) > t\}.$$

Let $\lambda = \log t$. We have after some elementary transformations:

$$\frac{1}{\lambda^2}\int_0^t \frac{du}{Y_\delta(u)} = \inf\{u : \frac{1}{\lambda} \log \int_0^{\lambda^2 u} ds \exp(\xi_\delta(s)) > 1\}$$

$$\text{(4.n)} \qquad = \inf\{u : \frac{1}{\lambda} \log(\lambda^2 \int_0^k ds \exp(\lambda \frac{1}{\lambda} \xi_\delta(\lambda^2 s)) > 1\}.$$

Using part iv) of Theorem (4.6), we now deduce from (4.n) that:

$$\frac{1}{\lambda^2}\int_0^t \frac{du}{Y_\delta(u)} \underset{t\to\infty}{\overset{(d)}{\rightarrow}} \inf\{u : (\frac{2}{\nu})^{1/2} \beta_u > 1\} \overset{(d)}{=} (\frac{\nu}{2})\sigma,$$

which proves part i) of Theorem (4.7).

b) *Pinching method* (D. Williams (1974)).

Let $T_a = \inf\{t: Y_\delta(t) = a\}$ and $\tau_b = \inf\{t: \xi_\delta(t) = b\}$. The main ingredients of the proof are:

$$\text{(4.s)} \qquad \frac{1}{(\log t)^2} \int_t^{T_t} \frac{du}{Y_\delta(u)} \underset{t\to\infty}{\overset{(P)}{\to}} 0,$$

and

$$\int_0^{T_t} \frac{du}{Y_\delta(u)} = \tau_{(\log t)}.$$

The latter equality is immediate from the time change formula:

$$\log(Y_\delta(t)) = \xi_\delta\left(\int_0^t \frac{du}{Y_\delta(u)}\right).$$

Moreover, from part iv) in Theorem (4.6), we obtain:

$$\frac{1}{(\log t)^2} \tau_{(\log t)} \underset{t\to\infty}{\overset{(d)}{\to}} \frac{\nu}{2}\sigma$$

(this could also be deduced from the explicit formula:

$$E[\exp(-\mu\tau_b)] = \exp -\frac{b}{2}\{\mu + (\mu^2 + 2\nu\mu)^{1/2}\};$$

see subsection (4.3), d).)

It now remains to prove the convergence result (4.s). We have:

$$\int_t^{T_t} \frac{du}{Y_\delta(u)} = \int_1^{\tilde{T}_1} \frac{dv}{\tilde{Y}_\delta(v)}, \quad \text{where:} \quad \tilde{Y}_\delta(v) = \frac{1}{t} Y_\delta(tv),$$

which, thanks to the scaling property of Y_δ, converges in law, as $t \to \infty$, towards: $(Y_\delta^\#(v), v \geq 0)$, a Y_δ process starting from 0. Consequently, we have:

$$\int_t^{T_t} \frac{du}{Y_\delta(u)} \underset{t\to\infty}{\overset{(d)}{\to}} \int_1^{T_1^\#} \frac{dv}{Y_\delta^\#(v)}$$

and the result (4.s) follows a fortiori.

c) *Explicit computation.*

In the Bessel case, this computation follows from the conditional expectation formula given in (4.1), as a consequence of the Girsanov relationship (4.a). Likewise, for the $Y_{d',\delta}$ processes, we deduce from Theorem 4.1 the following:

$$\Pi_t^{k_\lambda bakc40',k_\lambda}(y, dz) = \Pi_y^{k',k}[\exp(-\mu\int_0^t \frac{ds}{Y_s}) | Y_t = z] (\frac{z}{y})^\lambda \Pi_t^{k',k}(y, dz)$$

where $\mu = \lambda \cdot \dfrac{k' - 1 + \lambda}{k - 1 + \lambda}$.

Then, using the explicit forms of the semigroups $\Pi_t^{d',d}(y, dz)$ presented in (3.4), we obtain a closed form expression for the above conditional expectation, from which one should be able to deduce the limit results announced in Theorem 4.7.

(4.7) **A Ciesielski-Taylor type theorem.**

a) Let $X^{(d')}$ and $X^{(d'+2)}$ be two squares of Bessel processes with respective dimensions d' and $d'+2$, with $d' > 0$, starting from 0. Let $T_{(d')} = \inf\{u : X_u^{(d')} \geq 1\}$ and $S_{(d'+2)} = \int_0^\infty du\, 1_{(X_u^{(d'+2)} \leq 1)}$. Ciesielski and Taylor (1962) (see also Getoor-Sharpe (1979)) have proved that:

$$T_{(d')} \overset{\text{(law)}}{=} S_{(d'+2)}. \tag{4.t}$$

For an extension of this result to a large class of functionals, see Biane (1985).

b) We shall now prove a result similar to (4.t) when the Bessel processes are replaced by the processes $Y_{d',d}$, with $0 < d' < d$ and $d > 2$.

Theorem (4.8): *We simply note* Y *for* $Y_{d',d}$, *starting from 0, and define* $T_y = \inf\{u : Y_u \geq y\}$. *Then:*

a)
$$E[\exp(-\lambda T_y)] = 1/\Phi(k, k'; \lambda y);$$

b)
$$\text{if } k' > 1,\ E[\exp -\lambda \int_0^\infty ds\, 1_{(Y_s \leq y)}] = 1/\Phi(k, k'-1; \lambda y)$$

Consequently, for every $y \geq 0$, *we have:*

$$T_y^{d',d} \overset{\text{(law)}}{=} \int_0^\infty ds\, 1_{(Y_s^{d'+2,d} \leq y)}. \tag{4.u}$$

Proof: Part a) was already proved in subsection (4.3), c).

To prove part b), we may take $y = 1$, using the scaling property. We now remark that, if there exists a C^1-function $(u(x), x \geq 0)$ such that: $L_{d',d}\, u(x) = \lambda\, 1_{(x \leq 1)} u(x)$, then:

$$E[\exp - \lambda \int_0^{T_a} du\, 1_{(Y_u \leq 1)}] = \frac{u(0)}{u(a)},$$

so that, letting a increase to $+\infty$, we obtain:

$$E[\exp - \lambda \int_0^\infty du\, 1_{(Y_u \leq 1)}] = \frac{u(0)}{u(\infty)}. \tag{4.v}$$

The function: $u(x) = \begin{cases} \Phi(k,k';\lambda x) & (x<1) \\ \alpha+\beta x^{1-k'} & (x>1) \end{cases}$ satisfies:

$$L_{d',d}\, u(x) = \lambda\, 1_{(x\le 1)}\, u(x) \quad \text{on} \quad]0,1[\quad \text{and} \quad]1,\infty[.$$

It remains to find α and β such that u is C^1. This will be so iff:

$$\text{(4.w)} \qquad \begin{cases} \alpha+\beta = \Phi(k,k';\lambda) \\ (1-k')\beta = \lambda \dfrac{k}{k'} \Phi(k+1,k'+1;\lambda) \end{cases}$$

where, in order to find the second equality, we have used:

$$\frac{d}{dx} \Phi(k,k',x) = \frac{k}{k'} \Phi(k+1,k'+1;x)$$

(see Lebedev (1972), (9.9.4), p. 261). The solution of the system (4.w) is:

$$\beta_\lambda = \frac{k\lambda}{k'(1-k')} \Phi(k+1,k'+1;\lambda),\ \alpha_\lambda = \Phi(k,k';\lambda) + \frac{k\lambda}{k'(k'-1)} \Phi(k+1,k'+1;\lambda).$$

Hence, we have: $u(0)=1$, $u(\infty)=\alpha_\lambda$, so that, from (4.v):

$$E\Big[\exp -\lambda \int_0^\infty du\, 1_{(Y_u\le 1)}\Big] = 1/\alpha_\lambda.$$

We shall now show, with the help of the recurrence relations satisfied by Φ, that $\alpha_\lambda = \Phi(k,k'-1;\lambda)$, which implies b). Indeed, we find in Lebedev (1972), (9.9.12), that:

$$\frac{\lambda}{k'} \Phi(k+1,k'+1;\lambda) = \Phi(k+1,k';\lambda) - \Phi(k,k';\lambda)$$

whence:

$$\begin{aligned} \alpha_\lambda &= \Phi(k,k';\lambda) + \frac{k}{k'-1}\{\Phi(k+1,k';\lambda) - \Phi(k,k';\lambda)\} \\ &= \frac{1}{k'-1}\{k\Phi(k+1,k';\lambda) + (k'-k-1)\Phi(k,k';\lambda)\} \\ &= \Phi(k,k'-1;\lambda), \text{ from Lebedev (1972), (9.9.6)} \end{aligned}$$

□

We now remark, using jointly parts a) and b) of Theorem (4.8) that:

$$\Pi_y^{d'+2,d}\Big(\exp -\lambda \int_0^\infty ds\, 1_{(Y_s\le y)}\Big) = \frac{\Phi(k,k'+1;y)}{\Phi(k,k';y)}$$

so that the probability measure μ defined in subsection (4.3), c) now appears as the distribution of $\int_0^\infty ds\, 1_{(Y_s\le y)}$ under $\Pi_y^{d'+2,d}$.

Again, there exist similar results for Bessel processes (see Pitman-Yor (1981), Getoor-Sharpe (1979)) and Bessel functions (see Kent (1978)).

Note: An explanation of the Ciesielski-Taylor identity (4.t) is given in Yor (1991), using jointly Ray-Knight theorems for local times of Bessel processes and a stochastic integration by parts formula.

It would be interesting to attempt such an approach to explain the identity in law (4.u).

(4.8) **Affine boundaries.**

a) Let $d' > 0$, and consider $\tilde{T}_c = \inf\{u : X_u^{(d')} = c\,(1 + u)\}$, where $X^{(d')}$ is a $BESQ_a(d')$, with $a < c$.

Following a method due to Shepp (1967) in the case $d' = 1$, it has been shown in Yor (1984) that:

$$E_a^{(d')}\,[\,(1 + \tilde{T}_c)^{-\alpha}\,] = \frac{\Phi\,(\alpha, k'; \frac{a}{2})}{\Phi\,(\alpha, k'; \frac{c}{2})} \tag{4.x}$$

Remark: It may be interesting to compare this formula with:

$$\Pi_a^{d',d}\,(e^{-\lambda T_c}) = \frac{\Phi\,(k, k'; \lambda a)}{\Phi\,(k, k'; \lambda c)} \tag{4.y}$$

a formula obtained in the above subsection (4.3), c.

b) We shall now obtain a formula similar to (4.x) for $\tilde{T}_c = \inf\{u : Y_{d',d}\,(u) = c(1 + u)\}$, when $Y_{d',d}\,(0) = a$, and $a < c < 1$.

Under these conditions, we prove the formula:

$$\Pi_a^{d',d}\,((1 + \tilde{T}_c)^{-\alpha}) = \frac{F\,(\alpha, k, k'; a)}{F\,(\alpha, k, k'; c)} \tag{4.z}$$

Proof: Following Shepp (1967) again, we use the two next arguments jointly (we drop the superscripts (d', d) since there is no risk of confusion).

i) $\Phi\,(k, k'; \lambda Y_t) e^{-\lambda t}$ is a martingale;

ii) $F\,(\alpha, k, k'; y) = \dfrac{1}{\Gamma\,(\alpha)} \displaystyle\int_0^\infty d\lambda\,\lambda^{\alpha-1}\,e^{-\lambda}\,\Phi\,(k, k'; \lambda y)$

From (i), we deduce:

$$\Pi_a\,(\Phi\,(k, k'; \lambda c\,(1 + \tilde{T}_c))\,e^{-\lambda \tilde{T}_c}) = \Phi\,(k, k'; \lambda a)$$

and then, integrating both sides with respect to $\dfrac{d\lambda}{\Gamma\,(\alpha)}\,\lambda^{\alpha-1}\,e^{-\lambda}$, we obtain:

$$\Pi_a \{ \frac{1}{\Gamma(\alpha)} \int_0^\infty d\lambda \, \lambda^{\alpha-1} e^{-\lambda(1+\tilde{T}_c)} \Phi(k, k'; \lambda c (1 + \tilde{T}_c)) \} = F(\alpha, k, k'; a)$$

Making the change of variables $\xi = \lambda(1 + \tilde{T}_c)$ in the above integral in $(d\lambda)$, we obtain formula (4.z). □

5. Some general remarks about duality and intertwining.

(5.1) μ-duality and h-duality.

There are presently, in the Markovian literature, two notions of duality which have little in common, they are:

a) the notion of duality of two Markov semi-groups (P_t) and $(\hat{P}_t)$ on E, with respect to a σ-finite measure μ on E:

this notion, which has already been presented in Definition (3.1) above plays, as we have seen in section (4.5), a crucial role in time reversal;

b) the notion of duality of two Markov semi-groups (R_t) and (S_t) on E and F respectively, with respect to a function $h: E \times F \to \mathbb{R}_+$; we borrow this notion from Liggett (1985): (R_t) and (S_t) are said to be in h-duality if: for every $(\xi, \eta) \in E \times F$,

$$R_t(h_\eta)(\xi) = S_t(h^\xi)(\eta) \tag{5.a}$$

where: $h_\eta(\xi) = h^\xi(\eta) \equiv h(\xi, \eta)$.

(5.2) Comparison of intertwining and h-duality.

The following proposition shows, under adequate assumptions, the equivalence between a property of intertwining and a property of h-duality.

Proposition (5.1): *Suppose that the semi-groups* (S_t) *and* $(\hat{S}_t)$ *are in* μ*-duality. Then:*

1) if the semi-groups (R_t) *and* (S_t) *are in* h*-duality, then:*

$$R_t H_\mu = H_\mu \hat{S}_t;$$

2) conversely, if $R_t \Lambda = \Lambda \hat{S}_t$, *with* $\Lambda f(\xi) = \int d\mu(\eta) \lambda(\xi, \eta) f(\eta)$, *then* (R_t) *and* (S_t) *are in almost* λ*-duality.*

6. Temporary conclusion (August 1992).

A more complete list of intertwinings of Markov processes has now been established in joint work with Ph. Carmona and F. Petit (1992), making important use of the reflecting Brownian motion $(|B_t|, t \geq 0)$ perturbed by a multiple of its local time $(l_t^0, t \geq 0)$ at 0, i.e: $(|B_t| - \lambda l_t^0, t \geq 0)$, for some $\lambda > 0$.

The new Markov processes are constructed explicitly in terms of this perturbed reflecting Brownian motion, which gives more hope that the intertwining relations described in the present paper and in Carmona-Petit-Yor (1992) may have a pathwise interpretation.

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