General Properties of Landscapes: Vacuum Structure, Dynamics and Statistics

by

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Abstract

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Even the simplest extra-dimensional theory, when compactified, can lead to a vast and complex landscape. To make progress, it is useful to focus on generic features of landscapes and compactifications. In this work we will explore universal features and consequences of (i) vacuum structure, (ii) dynamics resulting from symmetry breaking, and (iii) statistical predictions for low-energy parameters and observations. First, we focus on deriving general properties of the vacuum structure of a theory independent of the details of the geometry. We refine the procedure for performing compactifications by proposing a general gauge-invariant method to obtain the full set of Kaluza-Klein towers of fields for any internal geometry. Next, we study dynamics in a toy model for flux compactifications. We show that the model exhibits symmetry-breaking instabilities for the geometry to develop lumps, and suggest that similar dynamical effects may occur generically in other landscapes. The questions of the observed arrow of time as well as the observed value of the neutrino mass lead us to consider statistics within a landscape, and we verify that our observations are in fact typical given the correct vacuum structure and (in the case of the arrow of time) initial conditions. Finally, we address the question of subregion duality in AdS/CFT, arguing for a criterion for a bulk region to be reconstructable from a given boundary subregion by local operators. While of less direct relevance to cosmological space-times, this work provides an improved understanding of the UV/IR correspondence, a principle that underlies the construction of many holographically-inspired measures used to make statistical predictions in landscapes.
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Chapter 1

Introduction

Our own universe may be part of a larger landscape consisting of many different vacua, each of which realizes different possibilities for low-energy physics. Indeed such a scenario follows naturally from any compactified theory of extra dimensions such as string theory. Different low-energy solutions correspond to different compactified geometries, and from the four-dimensional viewpoint these solutions are connected dynamically through the deformations of moduli governing the size and shape of the extra dimensions. In the case of string theory, where the internal space is a complicated Calabi-Yau manifold with typically hundreds of cycles for flux to wrap in different ways, the resulting low-energy landscape is famously believed to be extremely large, on the order of $10^{500}$ vacua \cite{1,2}. To make progress within such an incredibly complex framework, it is useful to ask questions that rely only on generic properties of a theory rather than intricate details of its solutions. Consequently, this thesis will aim to explore generic properties of landscapes, especially but not necessarily restricted to landscapes resulting from compactification.

One important (and in the case of large landscapes, generic) property of landscapes such as our own that contain de Sitter vacua is eternal inflation \cite{3,4}. We have evidence via the observation of the cosmic microwave background (CMB) and the distribution of galaxies in the sky that our own universe underwent ordinary inflation, a period of accelerated expansion driven by the vacuum energy of a scalar field \cite{5}, and we also know that we live close to the time of matter-$\Lambda$ equality, after which it will again become vacuum energy dominated \cite{6,7}. Within any inflating vacuum there will be pocket regions that exit inflation, for instance by tunneling to a different vacuum in the landscape, and other regions that continue inflating. If the Hubble expansion rate of the false vacuum is large enough there will forever be regions that continue inflating, in other words inflation will be eternal. In a large landscape such as the string landscape, it is reasonable to consider that at least one vacuum will have a Hubble expansion rate large enough compared to its rate to decay out, thus allowing eternal inflation to occur; it should be emphasized that it is only necessary for this to be satisfied by a single vacuum.

Once an eternally inflating phase is entered, infinite volume is produced and not only do all possibilities for different low-energy physics occur, but each occurs infinitely many times.
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One can ask about the relative probability of different low-energy observations, but naively the answer is nonsensical: $\infty$ divided by $\infty$. To make progress one must first regulate these infinities using a measure. In modern cosmology much dissatisfaction has resulted from this realization, since it is true that the choice of measure is highly non-unique; this is the so-called “measure problem.” One may hope that a correct and complete understanding of quantum gravity will come with a prescription for regulating these infinities, but at a point in time where a non-perturbative formulation of string theory remains elusive, this may seem like a far-off optimism.

However, all is not lost because in the meantime it is possible to make progress phenomenologically. Many naive choices of measure can be immediately ruled out for obvious reasons, because they exhibit a youngness paradox [8, 9] or predict that typical observers are Boltzmann brains, freak observers that fluctuate locally from equilibrium and, with overwhelming probability, see only an empty universe [10, 11]. The set of measures is further reduced by dualities that have been discovered between measures that are defined locally, for instance by restricting to the neighborhood of a geodesic, and ones that are defined via some global time cutoff [12, 13]. The remaining set of viable measures tend to be motivated by “physical” principles in the sense that they invoke general facts we have learned thus far from studies of black holes and holography. For example, the black hole no-cloning theorem suggests that quantum gravity should be defined only over the region locally accessible to a single observer [14]. This principle, known as complementarity, has in turn led to the definition of the thus-far phenomenologically successful causal patch measure, which regulates by counting only events that are in the causal past of a geodesic, averaging over all such geodesics [15]. In another successful measure known as the light-cone time cutoff, a global time cutoff is defined holographically via an equivalent cutoff on an infinite future-time surface; this offers a regulation much in the spirit of the UV/IR correspondence of AdS/CFT [16].

Given a choice of measure, it is possible to compare the relative probabilities of different low-energy parameters statistically. A canonical example where this approach has met success is in the explanation of the cosmological constant fine-tuning problem: the question of why the observed value of the accelerated expansion of the universe differs by many orders of magnitude from the theoretical prediction for the size of its individual contributions—for example, shifts in the vacuum energy from cosmological phase transitions in the early universe as well as loops of virtual particles in quantum field theory that contribute to the background vacuum energy. While to-date no other models (for example, from modifications of gravity or extra dimensions) have been able to fully explain this fact, it has been demonstrated that the fine-tuning can be resolved in the context of a large landscape, conditioning on observers, where fundamental parameters such as the value of the cosmological constant vary randomly [17]. Other fundamental parameters beyond the cosmological constant can be varied as additional tests, for example a similar calculation is performed in Chapter 5 of this thesis for the case of the neutrino mass. Successive calculations of this variety provide successive tests of a given measure, and ultimately one hopes that through this “bottom-up” approach it will be possible to hone in on an approximation of the correct measure.
Summary: In Chapter 2 we focus on vacuum structure, specifically the procedure for obtaining the vacuum structure of a theory via dimensional reduction. We refine the toolkit for performing dimensional reduction, both by explicitly invoking a Hodge decomposition theorem for tensors and vectors that organizes the various types of low-energy fields as coefficients of orthogonal components, and by working for the first time completely gauge-invariantly. Our method is generic in the sense that it does not rely on a particular geometry for the internal space. Instead, given a set of fields and interactions in the full space-time, our method provides a straightforward algorithm to obtain the full towers of Kaluza-Klein fields (as well as their interactions, order by order) resulting from compactification. The geometry plays a role only in determining the exact mass spectrum—which depends on the eigenvalues of various Laplacians on the internal space—as well as the number of fields in different sectors. We explicitly work through the procedure in the case of scalars, vectors, $p$-form fields, gravity, and gravity with a flux and cosmological constant on an arbitrary direct-product space-time. Finally, in all of these cases we do an analysis of the perturbative stability of the solutions for these theories. This chapter draws from a paper joint with Kurt Hinterbichler and Janna Levin [18].

In Chapter 3 we focus on dynamics, specifically within a landscape resulting from the compactification of a toy theory known as Einstein-Maxwell theory. The theory includes only gravity, a cosmological constant and a flux form-field, and is defined on a space-time that is the direct product of 4-dimensional Minkowski, de Sitter or Anti-de Sitter space with an extremely simple and symmetric internal space: a sphere of arbitrary dimension. As we will see from the stability analysis in Chapter 2, this theory is sometimes perturbatively unstable to deformations that destroy the perfect symmetry of the sphere, for example by deforming it to be shaped like a “football” or “M&M” or in general to develop a lumpy geometry with reduced symmetry. One might imagine that within the parameter space of perturbatively stable solutions, the geometrical symmetry of the sphere is preserved, but in fact we show that this is not the case. We numerically construct an additional set of “lumpy” warped product solutions and show that they always have lower energy than the “symmetric” ones. Finally, we present several arguments suggesting a particular form for an effective potential in the direction of this shape deformation. From the potential, we see that most of the perturbatively stable symmetric solutions do not remain stable—rather, they exhibit a non-perturbative instability towards lumpiness, either settling in an “M&M”-shaped minimum or exhibiting an infinite runaway towards the football direction. Finally, we suggest that the tendency to dynamically evolve to less symmetrical configurations is not restricted to this model but is in fact generic: a mechanism for (geometrical) symmetry breaking within a landscape. This chapter draws from a paper joint with Alex Dahlen [19].

In Chapter 4 move on to combine statistics in a large landscape with vacuum structure, to address the observed arrow of time: the fact that we observe entropy to increase to our future. This observation has been viewed as a “problem” since it amounts to saying we do not live in an equilibrium state while naively, one might imagine that equilibrium states dominate the phase space. In fact, as we will see, whether or not this is a problem depends intricately on the vacuum structure of a theory. In a landscape consisting of a single de Sitter
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vacuum, the observed arrow of time would indeed be unlikely and thus a “problem,” since most observers would be thermal fluctuations at late times called Boltzmann Brains who see no arrow of time [20]. In a theory with multiple vacua, however, it is not necessarily the case that Boltzmann Brains outnumber “ordinary observers,” observers who do see an arrow of time. The question of which type of observer is more typical is one which must be carefully calculated using a measure. We address this question using the causal patch measure, and we see that having ordinary observers win depends intricately on the vacuum structure as well as the set of initial conditions. Using some assumptions about the spacing of vacua within the string landscape, we find that the arrow of time holds given the “dominant-eigenvector” initial conditions that appear in the definition of global-local measure duality, while the often-studied “Hartle-Hawking” initial conditions violate the arrow of time. This chapter draws from a paper joint with Raphael Bousso [21].

In Chapter 5 we continue with statistics in a large landscape, to compute the probability distribution over the mass of the neutrino. Too large a neutrino mass suppresses structure, since free-streaming neutrinos cause overdense regions to grow more slowly during matter domination. Combined with a prior probability distribution that slightly favors larger neutrino mass, the full resulting probability distribution has a peak at some nonzero value. This value can then be compared to the observed range of neutrino mass, which is currently constrained on the lower end by atmospheric neutrino experiments, and on the upper end by cosmological observations of structure. Using the causal patch measure, we compute this probability distribution. We find that while a slightly larger neutrino mass suppresses structure, past a critical value there is a regime change that actually predicts more structure. However, this transition coincides with the boundary between a cosmology with a hierarchical structure formation scenario, and one where structures form at all scales simultaneously. We argue that due to this dramatic change in structure formation, virialized halos will not be able to cool efficiently to form stars, leading this region of parameter space to be excluded for anthropic reasons. After applying this cooling boundary, the resulting probability distribution agrees well with the observed range within $2\sigma$. This chapter draws from a paper joint with Raphael Bousso and Dan Mainemer Katz [22].

Finally, in Chapter 6 we move on to study subregion duality in AdS/CFT, which is in fact intricately related to the physical motivation of measures used to compute statistical properties in a landscape. As we have seen, certain measures such as light-cone time define bulk cutoffs holographically through short-distance cutoffs at the future boundary surface [16]. To make this precise, it is useful to develop a refined understanding of the UV/IR correspondence which maps a long-distance cutoff of a theory in AdS to a short-distance cutoff of its dual theory on the boundary of AdS. To do this, it is useful to first understand which bulk regions are dual to subregions of the boundary (such as a small region excised by a UV cutoff). We explain how this question relates to known pathologies of smearing functions in Rindler-AdS. Finally, we propose a general criterion for any bulk region to be reconstructable via local operators from a given region on the boundary: all null geodesics that cross through this region must make contact with the boundary. This chapter draws from a paper joint with Raphael Bousso, Ben Freivogel, Stefan Leichenauer and Vladimir Rosenhaus [23].
Chapter 2
Kaluza-Klein Towers on General Manifolds

2.1 Introduction

The universe may conceal curled-up extra dimensions, a topic of fascination since the early work of Kaluza and Klein [24, 25]. Beyond the abstract curiosity of a higher-dimensional universe, there are real implications for our four-dimensional experience if the hidden dimensions are correspondingly real. Extra dimensions, if they exist, might be too small to explore directly, but their existence may be inferred from patterns imprinted on four-dimensional physics. Integration of the fields’ actions over the internal manifold reduces the higher-dimensional laws of physics to effective four-dimensional laws of physics. In addition to the four-dimensional counterparts of the fields, infinite towers of particles of increasing mass appear. The masses are given by the eigenvalues of appropriate Laplacians acting on the internal space.

There have been many studies of the Kaluza-Klein spectrum associated with various compactifications, see for instance some of the classic papers [26], and some of the famous early studies of supergravity compactifications [27, 28]. Here we revisit the derivation of the Kaluza-Klein tower in terms of the eigenvalues of appropriate internal Laplacians. We are motivated to do so by the desire for a complete reference that can later be applied to many possible physical applications, and because we know of no prior treatment that a) applies for any internal manifold (not just spheres or tori or symmetric spaces) of any dimension, b) proceeds entirely at the level of the action, without resort to component-by-component analyses of the equations of motion or propagators, c) is completely gauge invariant and does not require imposition of gauge conditions for infinite towers of gauge fields and gravity, and d) takes account of all subtleties, such as those associated with zero modes, Killing vectors and conformal Killing vectors of the internal space.

The goal of this paper is to describe such a treatment. We cover all the bosonic cases of most interest: scalars, $p$-forms, gravity and flux-compactifications. We include even the
simplest cases, in an attempt to provide a complete, cohesive, self-contained reference. Most important, some physical details are more transparent in the general methodology advocated here. In particular, maintenance of the gauge symmetry is enlightening. With gauge symmetry intact, the higher harmonics of the various Laplacians naturally form St¨ uckelberg fields [29] for the towers of gauge symmetries.\footnote{See [30, 31] for earlier studies of the original 4 + 1 dimensional Kaluza-Klein theory along these lines, and e.g. [32] for more complicated theories.}

Crucial is the combined Hodge and eigenspace decomposition of fields on the internal manifold, which we discuss in detail in the Appendices. The decompositions provide the natural basis to cleave physical fields from St¨ uckelberg fields. Derivation of the four-dimensional Kaluza-Klein action is then a straightforward integration of the decomposed fields over the extra dimensions. The integration is instant since the decomposition is naturally orthonormal. The resulting action is straightforwardly written as a tower of gauge invariant combinations of fields and their St¨ uckelbergs. The mass spectrum is readily read from the action.

We treat here the case of free fields: the scalar in Section 2.2, the Maxwell field in Section 2.3, the abelian $p$-form in Section 2.4, the free graviton in Section 2.5, and in Section 2.6 the case of flux compactifications, where there is both a $p$-form and a graviton with non-trivial mixing. We do not put any constraints on the product manifolds, other than those needed for consistent propagation of the graviton.

In each case, once the spectrum of lower-dimensional fields is attained we address the important question of stability. For a compactification to be stable, the lower-dimensional spectrum must not contain ghosts (particles with a wrong-sign kinetic term), or tachyons (particles with a negative mass squared). In many cases, stability is argued with purely geometrical theorems, such as the Lichnerowicz bound, which apply to the spectrum of the Laplacian or related operators. In other cases there may exist certain eigenvalues that result in instabilities, narrowing down the parameter space of stable compactifications. In particular, we address the question of whether the Kaluza-Klein graviton masses on compactifications to de Sitter space can ever saturate or violate the Higuchi bound, and we find that they cannot.

Conventions: We use mostly plus signature. We are considering fields on a direct product of smooth manifolds $\mathcal{M} \times \mathcal{N}$, where $\mathcal{M}$ is a $d$-dimensional spacetime and $\mathcal{N}$ is a compact $N$-dimensional internal Riemannian manifold. The total spacetime has dimension $D = d + N$. The coordinates on the full product space are $X^A$, with $A, B, \cdots$ running over $D$ values, and the metric is $G_{AB}(X)$. The coordinates on $\mathcal{M}$ are $x^\mu$, with $\mu, \nu, \cdots$ running over $d$ values, and the metric is $g_{\mu\nu}(x)$. The coordinates on $\mathcal{N}$ are $y^m$ with $m, n, \cdots$ running over $N$ values, and the metric is $\gamma_{mn}(y)$. $\mathcal{V}_N = \int d^N y \sqrt{\gamma}$ is the volume of $\mathcal{N}$. The Riemann curvature is defined so that for a vector $V^\mu$, we have $[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma \mu \nu} V^\sigma$. The Ricci curvature is $R_{\mu \nu} = R^\rho_{\mu \rho \nu}$ and the Ricci scalar is $R = R^\mu_{\mu}$. By an Einstein space, we mean any space with a metric satisfying $R_{\mu \nu} = kg_{\mu \nu}$ with $k$ constant. Symmetrization, $(\cdots)$, and anti-symmetrization, $[\cdots]$, of $p$ indices are defined as the sum over (signed) permutations.
with a pre-factor of $1/p!$.

## 2.2 Scalar

As a warm-up, and for completeness, we start with the simplest example of a higher dimensional theory: the free scalar. The higher dimensional action for a free massless scalar $\phi(X)$ on $\mathcal{M} \times \mathcal{N}$ is

$$ S = -\frac{1}{2} \int d^{D}X \sqrt{-G} \partial_{A} \phi \partial^{A} \phi. \quad (2.1) $$

We must expand the scalar in appropriate eigenfunctions of the internal manifold $\mathcal{N}$. The appropriate expansion is the Hodge eigenvalue decomposition, reviewed in Appendix A. In the case of scalars, it reads

$$ \phi(x, y) = \sum_{a} \phi^{a}(x) \psi_{a}(y) + \frac{1}{\sqrt{V_{N}}} \phi^{0}(x), \quad (2.2) $$

where $\psi_{a}(y)$ are a basis orthonormal eigenvectors of the scalar Laplacian on $\mathcal{N}$, $\int d^{N}y \sqrt{\gamma} \psi^{a} \psi_{b} = \delta_{b}^{a}$, and $\Box(y) \psi^{a} = -\lambda_{a} \psi^{a}$, with positive eigenvalues $\lambda_{a} > 0$, labeled by $a$ including multiplicities. The constant piece (where $V_{N} = \int d^{N}y \sqrt{\gamma}$ is the volume of $\mathcal{N}$, ensuring proper normalization) takes account of the zero eigenvalue of the Laplacian.

Although the basis $\psi^{a}$ can always be chosen to be real, we allow it to be complex, since this is often more convenient, e.g. the spherical harmonics on the sphere. The original field $\phi$ is real, so we have the restriction $\phi^{*a} = \bar{\phi}^{a}$ on the lower dimensional fields arising as coefficients in the expansion Eq. (2.2). The bar on the index indicates some involution on the set of indices, e.g. $(\bar{l}, \bar{m}) = (l, -m)$ for the standard spherical harmonics on the two-sphere, see Appendix A for more explanation.

Plugging Eq. (2.2) into Eq. (2.1) and integrating over $\mathcal{N}$, using the orthonormality of the positive eigenfunctions, and the fact that the constant (harmonic) mode is orthogonal to the $\psi^{a}$, we arrive at the $d$-dimensional action

$$ S = \int d^{d}x \sqrt{-g} \left[ -\frac{1}{2} \left( \partial \phi^{0} \right)^{2} - \sum_{a} \frac{1}{2} \left( |\partial \phi^{a}|^{2} + \lambda_{a} |\phi^{a}|^{2} \right) \right]. \quad (2.3) $$

The spectrum is

- One massless scalar,
- One massive scalar for each eigenvector $\psi_{a}$ of the scalar Laplacian with positive eigenvalue $\lambda_{a}$, with mass $m^{2} = \lambda_{a}$.

The spectrum is stable. There are never tachyonic scalars with $m^{2} < 0$ or ghosts with wrong-sign kinetic terms.
The simplest case is \( N = 1 \), for which the internal manifold \( \mathcal{N}_1 \) is the circle. Here the eigenfunctions can be chosen to be \( \psi_a = \frac{1}{\sqrt{2\pi R}} e^{iay/R} \), where \( R \) is the radius of the circle and \( a \) ranges over all the integers, with \( a = 0 \) the zero mode, and \( \bar{a} = -a \). This reproduces the Fourier decomposition of the field over the circle. The eigenvalues in this case are \( \lambda_a = a^2/R^2 \), and the spectrum consists of the massless scalar zero mode and a tower of massive doublets with \( m^2 = a^2/R^2 \).

2.3 Vector

We now proceed to the simplest example of a theory with higher-dimensional gauge invariance: the abelian vector. The higher-dimensional action for an abelian vector field \( A_A(X) \) on \( \mathcal{M} \times \mathcal{N} \) is the Maxwell action,

\[
S = -\frac{1}{4} \int d^D X \sqrt{-G} F_{AB} F^{AB},
\]

(2.4)

where \( F_{AB} \equiv \nabla_A A_B - \nabla_B A_A \) is the usual field strength. The theory is invariant under the gauge transformations

\[
\delta A_A = \partial_A \Lambda ,
\]

(2.5)

where \( \Lambda(X) \) is an arbitrary spacetime function. This higher-dimensional gauge symmetry will appear from the \( d \)-dimensional point of view as an infinite tower of gauge symmetries acting on the Kaluza-Klein tower.

We will use the Hodge decomposition theorem (see Appendix A) to decompose the field into parts which will have a clear physical interpretation from the lower dimensional point of view. We write the vector field as follows,

\[
A_A(x, y) = \left\{ \begin{array}{l}
\sum_a a^a_{\mu}(x) \psi_a(y) + \frac{1}{\sqrt{V_N}} c_{\mu,0}(x) , \\
\sum a^a(x) \partial_n \psi_a(y) + \sum_i b^i(x) Y_{i,n}(y) + \sum \alpha c^\alpha(x) Y_{\alpha,n}(y) .
\end{array} \right.
\]

(2.6)

Here \( \psi_a \) are the eigenmodes of the scalar Laplacian on \( \mathcal{N} \) with positive eigenvalues \( \lambda_a \), the \( Y_{i,n} \) are co-exact (satisfying \( \nabla^n Y_{i,n} = 0 \)) eigenvectors of the vector Laplacian on \( \mathcal{N} \) with positive eigenvalues \( \lambda_i \), and the \( Y_{\alpha,n} \) are harmonic eigenvectors of the vector Laplacian on \( \mathcal{N} \), i.e. those with eigenvalue zero under the vector Laplacian. These all obey the conditions and orthogonality properties described in Appendix A. The field \( A_A \) is real, so we have the restrictions \( a^a_{\mu} = a^a_{\mu}^* \), \( a^a = a^\alpha \), \( b^i = b^i \), \( c^\alpha = c^\alpha \).

Plugging into Eq. (2.4) and integrating over \( y \) using orthogonality of the various eigenspaces, we find:
\[ S = \int d^d x \sqrt{-g} \left[ -\frac{1}{4} f_{\mu \nu,0}^2 + \sum_a \left( -\frac{1}{4} |f_{\mu \nu}^a|^2 - \frac{\lambda_a}{2} |a_{\mu}^a - \partial_{\mu} a_{\nu}^a|^2 \right) \right. \\
- \frac{1}{2} \sum_a |\partial_{\mu} c_{\alpha}^a|^2 - \frac{1}{2} \sum_i \left( |\partial_{\mu} b_i|^2 + \lambda_i |b_i|^2 \right) \right], \tag{2.7} \]

where \( f_{\mu \nu,0} \equiv \nabla_{\mu} c_{\nu,0} - \nabla_{\nu} c_{\mu,0} \) and \( f_{\mu \nu}^a \equiv \nabla_{\mu} a_{\nu}^a - \nabla_{\nu} a_{\mu}^a \).

The gauge parameter can also be expanded over the scalar eigenfunctions,

\[ \Lambda(x, y) = \sum_a \Lambda^a(x) \psi_a(y) + \Lambda_0(x). \tag{2.8} \]

Decomposing Eq. (2.5) and equating coefficients, the original gauge symmetry reduces to an infinite tower of gauge symmetries,

\[ \begin{align*} 
\delta a_{\mu}^a &= \partial_{\mu} \Lambda^a, \\
\delta c_{\mu,0} &= \partial_{\mu} \Lambda_0, \\
\delta a_{\nu}^a &= \Lambda^a. \tag{2.9} 
\end{align*} \]

The action Eq. (2.7) is gauge invariant, as it must be because it is a rewriting of the original gauge invariant higher dimensional action and no gauge has been fixed. We can express it in terms of the gauge invariant combination

\[ \tilde{a}_{\mu}^a = a_{\mu}^a - \partial_{\mu} a_{\nu}^a \tag{2.10} \]

as follows:

\[ S = \int d^d x \sqrt{-g} \left[ -\frac{1}{4} f_{\mu \nu,0}^2 + \sum_a \left( -\frac{1}{4} |f_{\mu \nu}^a|^2 - \frac{\lambda_a}{2} |\tilde{a}_{\mu}^a|^2 \right) \right. \\
- \frac{1}{2} \sum_a |\partial_{\mu} c_{\alpha}^a|^2 - \frac{1}{2} \sum_i \left( |\partial_{\mu} b_i|^2 + \lambda_i |b_i|^2 \right) \right]. \]

The gauge symmetry \( \Lambda^a \) is a Stückelberg symmetry\(^2\) \cite{33}. We can fix it by setting the unitary gauge \( a_{\mu}^a = 0 \), which amounts to setting \( \tilde{a}_{\mu}^a = a_{\mu}^a \), from which we recover the standard action for a massive vectors. The scalars corresponding to the tower of exact one-forms on the internal manifold \( \mathcal{N} \) have become the Stückelberg fields carrying the longitudinal component of the tower of massive vectors associated with the scalar harmonics on \( \mathcal{N} \), and the harmonics of the higher dimensional gauge symmetry have become the Stückelberg symmetries.

The spectrum is now manifest, we have

\(^2\)See Section 4 of \cite{33} for a review of the Stückelberg formalism applied to massive vectors.
• One massless vector,
• One massive vector for each eigenvector of the scalar Laplacian with positive eigenvalue, with mass \( m^2 = \lambda_a \),
• One massless scalar for each harmonic one-form,
• One massive scalar for each co-exact eigenvector of the vector Laplacian, with mass \( m^2 = \lambda_i \).

The spectrum is stable. There are never tachyonic particles with \( m^2 < 0 \) or ghosts with wrong-sign kinetic terms.

In the simplest case \( N = 1 \), where the internal manifold \( \mathcal{N}_1 \) is the circle, there are no co-exact one-forms so the \( i \) index is empty and there are no massive scalars. There is only one massless scalar, corresponding to the single harmonic one-form which is a constant vector over the circle, a massless vector corresponding to the harmonic scalar, and a tower of massive vector doublets, with masses \( m^2 = a^2 / R^2 \), \( a = 1, 2, 3, \cdots \), where \( R \) is the radius of the circle.

### 2.4 \( p \)-form

The vector field generalizes to a \( p \)-form. The higher dimensional action for a \( p \)-form gauge field \( A_{A_1 \cdots A_p}(X) \) on \( \mathcal{M} \times \mathcal{N} \) is

\[
S = - \frac{1}{2(p+1)!} \int d^D X \sqrt{-G} F_{A_1 \cdots A_{p+1}} F^{A_1 \cdots A_{p+1}},
\]

(2.11)

where \( F_{A_1 \cdots A_{p+1}} = (p+1) \nabla_{[A_1} A_{A_2 \cdots A_{p+1}]} \) is the field strength. The theory is invariant under the gauge transformations

\[
\delta A_{A_1 \cdots A_p} = p \nabla_{[A_1} \Lambda_{A_2 \cdots A_p]},
\]

(2.12)

where \( \Lambda_{A_1 \cdots A_{p-1}}(X) \) is an arbitrary \((p-1)\)-form.

We use the Hodge decomposition theorem reviewed in Appendix A to write the components of the form field in terms of exact, co-exact and harmonic forms,
A_{A_1 \cdots A_p}(x, y) = \begin{cases} 
\sum_{i_0} a_{\mu_1 \cdots \mu_p}^{i_0} Y_{i_0} + c_{\mu_1 \cdots \mu_p}, \\
\sum_{i_1} a_{\mu_1 \cdots \mu_{p-1}}^{i_1} Y_{i_1} + \sum_{i_0} b_{\mu_1 \cdots \mu_{p-1}}^{i_0} (dY_{i_0}) + \sum_{\alpha_1} c_{\mu_1 \cdots \mu_{p-1}}^{\alpha_1} Y_{\alpha_1}, \\
\sum_{i_2} a_{\mu_1 \cdots \mu_{p-2}}^{i_2} Y_{i_2, n_1 n_2} + \sum_{i_1} b_{\mu_1 \cdots \mu_{p-2}}^{i_1} (dY_{i_1}) + \sum_{\alpha_2} c_{\mu_1 \cdots \mu_{p-2}}^{\alpha_2} Y_{\alpha_2, n_1 n_2}, \\
\vdots \\
\sum_{i_q} a_{\mu_1 \cdots \mu_{p-q}}^{i_q} Y_{i_q, n_1 \cdots n_q} + \sum_{i_{q-1}} b_{\mu_1 \cdots \mu_{p-q}}^{i_{q-1}} (dY_{i_{q-1}}) + \sum_{\alpha_q} c_{\mu_1 \cdots \mu_{p-q}}^{\alpha_q} Y_{\alpha_q, n_1 \cdots n_q}, \\
\vdots \\
\sum_{i_p} a_{\mu_1 \cdots \mu_{p-1}}^{i_p} Y_{i_p, n_1 \cdots n_p} + \sum_{i_{p-1}} b_{\mu_1 \cdots \mu_{p-1}}^{i_{p-1}} (dY_{i_{p-1}}) + \sum_{\alpha_p} c_{\mu_1 \cdots \mu_{p-1}}^{\alpha_p} Y_{\alpha_p, n_1 \cdots n_p}.
\end{cases}

(2.13)

Here, $i_q$ indexes a basis of co-exact (transverse) $q$-forms $Y_{i_q, n_1 \cdots n_q}$ which are eigenvalues of the Hodge Laplacian Eq. (A.5) with eigenvalue $\lambda_{i_q}$. The $\alpha_q$ index a basis of harmonic $q$-forms $Y_{\alpha_q, n_1 \cdots n_q}$. (Note that $i_0$ corresponds to the index $a$, and $\alpha_0$ corresponds to the single value 0. The $Y_{i_0} = \psi_a$ are just the positive scalar eigenvalues of the Laplacian.) These satisfy the conditions and orthogonality properties described in Appendix A. The field $A_{A_1 \cdots A_p}$ is real, so we have the reality conditions $a_{\mu}^{i_q} = a_{\mu}^{i_q}$, $b_{\mu}^{i_q} = b_{\mu}^{i_q}$, $c_{\mu}^{\alpha_q} = c_{\mu}^{\alpha_q}$.

Plugging Eq. (2.13) into Eq. (2.11) and integrating over $y$ using orthogonality of the various eigenspaces, we find the Lagrangian\footnote{It is easiest to start by writing}

$$F_{A_1 \cdots A_{p+1}}^2 = F_{\mu_1 \cdots \mu_{p+1}}^2 + \cdots + \left( \frac{p+1}{q} \right) F_{\mu_1 \cdots \mu_{p+1-q} n_1 \cdots n_q}^2 + \cdots + F_{n_1 \cdots n_{p+1}}^2. \quad (2.14)$$

Each term of Eq. (2.14) then becomes a line in (2.15).
\[ \mathcal{L} = - \frac{1}{2(p+1)!} \left( \sum_{i_0} |f^{i_0}_{\mu_1 \cdots \mu_{p+1}}|^2 + f^{\alpha_0}_{\mu_1 \cdots \mu_{p+1}} \right) \]

\[ - \frac{1}{2p!} \left( \sum_{i_0} \lambda_{i_0} \left| a^{i_0}_{\mu_1 \cdots \mu_{p+1}} + (-1)^p (db^{i_0})_{\mu_1 \cdots \mu_{p+1}} \right|^2 + \sum_{i_1} |f^{i_1}_{\mu_1 \cdots \mu_{p}}|^2 + \sum_{\alpha_1} |f^{\alpha_1}_{\mu_1 \cdots \mu_{p}}|^2 \right) \]

\[ - \frac{1}{2(p+1-q)!} \left( \sum_{i_{q-1}} \lambda_{i_{q-1}} \left| a^{i_{q-1}}_{\mu_1 \cdots \mu_{p+1-q}} + (-1)^{p+1-q} (db^{i_{q-1}})_{\mu_1 \cdots \mu_{p+1-q}} \right|^2 + \sum_{i_q} |f^{i_q}_{\mu_1 \cdots \mu_{p+1-q}}|^2 + \sum_{\alpha_q} |f^{\alpha_q}_{\mu_1 \cdots \mu_{p+1-q}}|^2 \right) \]

\[ \vdots \]

\[ - \frac{1}{2} \left( \sum_{i_{p-1}} \lambda_{i_{p-1}} \left| a^{i_{p-1}}_{\mu_1} \right|^2 + \sum_{i_p} |f^{i_p}_{\mu_1}|^2 + \sum_{\alpha_p} |f^{\alpha_p}_{\mu_1}|^2 \right) \]

\[ - \frac{1}{2} \sum_{i_p} \lambda_{i_p} \left| a^{i_p} \right|^2 , \]

where \( f \equiv da \) is the field strength of the form with the corresponding index.

The gauge parameter can also be expanded over the eigenforms,

\[ \Lambda_{A_1 \cdots A_{p-1}}(x, y) = \begin{cases} 
\sum_{i_0} \Lambda^{i_0}_{\mu_1 \cdots \mu_{p-1}} Y_{i_0} + \zeta^{\alpha_0}_{\mu_1 \cdots \mu_{p-1}} , \\
\sum_{i_1} \Lambda^{i_1}_{\mu_1 \cdots \mu_{p-2}} Y_{i_1,n} + \sum_{i_0} \omega^{i_0}_{\mu_1 \cdots \mu_{p-2}} (dY_{i_0})_n + \sum_{\alpha_1} \zeta^{\alpha_1}_{\mu_1 \cdots \mu_{p-2}} Y_{\alpha_1,n} , \\
\vdots \\
\sum_{i_q} \Lambda^{i_q}_{\mu_1 \cdots \mu_{p-1-q}} Y_{i_q,n_1 \cdots n_q} + \sum_{i_{q-1}} \omega^{i_{q-1}}_{\mu_1 \cdots \mu_{p-1-q}} (dY_{i_{q-1}})_{n_1 \cdots n_q} + \sum_{\alpha_q} \zeta^{\alpha_q}_{\mu_1 \cdots \mu_{p-1-q}} Y_{\alpha_q,n_1 \cdots n_q} , \\
\vdots \\
\sum_{i_{p-1}} \Lambda^{i_{p-1}}_{n_1 \cdots n_{p-1}} Y_{i_{p-1},n_1 \cdots n_{p-1}} + \sum_{i_{p-2}} \omega^{i_{p-2}}_{n_1 \cdots n_{p-1}} (dY_{i_{p-2}})_{n_1 \cdots n_{p-1}} + \sum_{\alpha_{p-1}} \zeta^{\alpha_{p-1}}_{n_1 \cdots n_{p-1}} Y_{\alpha_{p-1},n_1 \cdots n_{p-1}} . 
\end{cases} \]

Expanding the gauge transformation law Eq. (2.12) and equating coefficients, the component fields get the transformation laws:
CHAPTER 2. KALUZA-KLEIN TOWERS ON GENERAL MANIFOLDS

\begin{align*}
\delta a_{\mu_1 \cdots \mu_p}^{i_0} &= (d\Lambda)_{\mu_1 \cdots \mu_p}^{i_0}, \\
\delta c_{\mu_1 \cdots \mu_p}^{\alpha_0} &= (d\zeta)_{\mu_1 \cdots \mu_p}^{\alpha_0}, \\
\delta c_{\mu_1 \cdots \mu_{p-1}}^{\alpha} &= (d\zeta)_{\mu_1 \cdots \mu_{p-1}}^{\alpha},
\end{align*}

\begin{align*}
\delta a_{\mu_1 \cdots \mu_{p-1}}^{i_1} &= (d\Lambda)_{\mu_1 \cdots \mu_{p-1}}^{i_1}, \\
\delta b_{\mu_1 \cdots \mu_{p-1}}^{i_0} &= (d\omega)_{\mu_1 \cdots \mu_{p-1}}^{i_0} + (-1)^{p+1} \Lambda_{\mu_1 \cdots \mu_{p-1}}^{i_0}, \\
\delta c_{\mu_1 \cdots \mu_{p-1}}^{\alpha_0} &= (d\zeta)_{\mu_1 \cdots \mu_{p-1}}^{\alpha_0},
\end{align*}

\begin{align*}
\delta a_{\mu_1 \cdots \mu_{p-1}}^{i_1} &= (d\Lambda)_{\mu_1 \cdots \mu_{p-1}}^{i_1}, \\
\delta b_{\mu_1 \cdots \mu_{p-1}}^{i_0} &= (d\omega)_{\mu_1 \cdots \mu_{p-1}}^{i_0} + (-1)^{p+1} \Lambda_{\mu_1 \cdots \mu_{p-1}}^{i_0}, \\
\delta c_{\mu_1 \cdots \mu_{p-1}}^{\alpha_0} &= (d\zeta)_{\mu_1 \cdots \mu_{p-1}}^{\alpha_0}.
\end{align*}

We form the gauge invariant combinations

\begin{align*}
\tilde{a}_{\mu_1 \cdots \mu_p}^{i_0} &= a_{\mu_1 \cdots \mu_p}^{i_0} + (-1)^p (db)_{\mu_1 \cdots \mu_p}^{i_0}, \\
\vdots \\
\tilde{a}_{\mu_1 \cdots \mu_{p-1}}^{i_0} &= a_{\mu_1 \cdots \mu_{p-1}}^{i_0} + (-1)^p (db)_{\mu_1 \cdots \mu_{p-1}}^{i_0}, \\
\vdots \\
\tilde{a}_{\mu_1}^{i_0} &= a_{\mu_1}^{i_0} - (db)_{\mu_1}^{i_0},
\end{align*}

in terms of which the Lagrangian is

\begin{align*}
\frac{\mathcal{L}}{\sqrt{-g}} &= -\frac{1}{2(p+1)!} \left( \sum_{i_0} |\tilde{f}_{\mu_1 \cdots \mu_{p+1}}^{i_0}|^2 + (p+1) \lambda_{i_0} |\tilde{a}_{\mu_1 \cdots \mu_p}^{i_0}|^2 + f_{\mu_1 \cdots \mu_{p+1}}^{\alpha_0} \right) \\
&-\frac{1}{2p!} \left( \sum_{i_q} |\tilde{f}_{\mu_1 \cdots \mu_p}^{i_q}|^2 + p \lambda_{i_q} |\tilde{a}_{\mu_1 \cdots \mu_p}^{i_q}|^2 + \sum_{\alpha_1} |f_{\mu_1 \cdots \mu_p}^{\alpha_1}|^2 \right) \\
&-\frac{1}{2(p+1-q)!} \left( \sum_{i_q} |\tilde{f}_{\mu_1 \cdots \mu_{p+1-q}}^{i_q}|^2 + (p+1-q) \lambda_{i_q} |\tilde{a}_{\mu_1 \cdots \mu_{p+1-q}}^{i_q}|^2 + \sum_{\alpha_q} |f_{\mu_1 \cdots \mu_{p+1-q}}^{\alpha_q}|^2 \right) \\
&-\frac{1}{2} \left( \sum_{\alpha_p} |f_{\mu_1}^{\alpha_p}|^2 + \lambda_{i_p} |a^{i_p}|^2 + \sum_{\alpha_p} |f_{\mu_1}^{\alpha_p}|^2 \right).
\end{align*}

The gauge symmetries $\omega^{i_q}$ are higher order gauge symmetries, and can be absorbed by redefining $\Lambda^{i_q} = \Lambda^{i_q} + (-1)^{p+q+1}d\omega^{i_q}$. Then the gauge symmetries $\tilde{\Lambda}^{i_q}$ are St"uckelberg. We can fix them by setting $\tilde{a}^{i_q} = 0$, which amounts to setting $\tilde{a}^{i_q} = a^{i_q}$ ($q = 0, \cdots, p-1$).
The spectrum is now manifest:

- Massless $q$-forms for each harmonic $(p - q)$-form, indexed by $\alpha_{p-q}$, $q = 0, \ldots, p$.

- Massive $q$-forms for each co-exact form $(p - q)$-form, indexed by $i_{p-q}$, $q = 0, \ldots, p$ (including the positive eigenvalues of the scalar Laplacian at $q = p$), with mass $m^2 = \lambda_{i_{p-q}}$.

The spectrum is stable. There are never tachyonic particles with $m^2 < 0$ or ghosts with wrong-sign kinetic terms.

Recall that in $d$ dimensions a massless $p$-form can be dualized into a $(d - p - 2)$-form, and a massive $p$-form can be dualized into a $(d - p - 1)$-form, so in any given example these dualities can be used to reformulate the $d$-dimensional action. Note that many of the ingredients may be non-dynamical for low dimensions. A massless $p$-form field in $d$ dimensions, and a $(d - p - 2)$-form, which can be dualized to a $p$-form, are non-dynamical for $p \geq d - 1$. A massive $p$-form field in $d$ dimensions, and a $(d - p - 1)$-form, which can be dualized to a $p$-form, are non-dynamical for $p \geq d$.

Hodge duality (reviewed in Appendix A) tells us that the spectrum of harmonic $p$-forms on $\mathcal{N}$ is identical to the spectrum of harmonic $(N - p)$-forms. Thus the number of massless $q$-forms is the same the number of massless $(2p - N - q)$-forms.

In the simplest case $N = 1$, where the internal manifold $\mathcal{N}_1$ is the circle, there are no co-exact one-forms so the $i$ indices are empty and there are no massive forms of degree $< p$. There is only one massless $(p - 1)$-form, corresponding to the single harmonic one-form which is a constant vector over the circle, a massless $p$-form corresponding to the harmonic scalar, and a tower of massive $p$-form doublets, with masses $m^2 = a^2/R^2$, $a = 1, 2, 3, \ldots$, where $R$ is the radius of the circle.

## 2.5 Graviton

In the cases of the scalars, vectors, and $p$-forms, we were free to choose the background Kaluza-Klein manifold as we pleased. This is no longer the case for a graviton. For a graviton to consistently propagate on a background spacetime, that background must be a solution to Einstein’s equations \cite{34, 35}. Thus we must first find backgrounds in the form of a $D = (d + N)$-dimensional product space $\mathcal{M} \times \mathcal{N}$ which satisfy Einstein’s equations.

### Background

The action for gravity in $D$ dimensions with a $D$-dimensional cosmological constant $\Lambda_{(D)}$ and $D$-dimensional Planck mass $M_P$ is

$$S = \frac{M_P^{D-2}}{2} \int d^D X \sqrt{-G} \left( R_{(D)} - 2\Lambda_{(D)} \right). \quad (2.20)$$
The Einstein equations for the metric are
\[ R_{AB} - \frac{1}{2} R_{(D)} G_{AB} + \Lambda_{(D)} G_{AB} = 0 . \] (2.21)

Taking the trace and solving for the Ricci curvature, these can be equivalently written as
\[ R_{AB} = \frac{R_{(D)}}{D} G_{AB} \,, \quad R_{(D)} = \frac{2D}{D-2} \Lambda_{(D)} . \] (2.22)

Breaking Eq. (2.22) into its $\mathcal{M}$ components and $\mathcal{N}$ components, we find that both factors must be Einstein spaces,
\[ R_{\mu\nu} = \frac{R_{(d)}}{d} g_{\mu\nu} \,, \quad R_{(d)} \text{ constant} \,, \] (2.23)
\[ R_{mn} = \frac{R_{(N)}}{N} \gamma_{mn} \,, \quad R_{(N)} \text{ constant} \,, \] (2.24)
where the curvatures on $\mathcal{M}$ and $\mathcal{N}$ are given by,
\[ R_{(d)} = \frac{2d}{d + N - 2} \Lambda_{(D)} \,, \quad R_{(N)} = \frac{2N}{d + N - 2} \Lambda_{(D)} . \] (2.25)

We have the useful relations
\[ \frac{R_{(d)}}{d} = \frac{R_{(N)}}{N} \,, \quad \Lambda_{(D)} = \frac{1}{2} \left( 1 - \frac{1}{d} \right) R_{(d)} + \frac{1}{2} \left( 1 - \frac{1}{N} \right) R_{(N)} . \] (2.26)

(In the case $D = 2$ (i.e. $d = N = 1$), we must have $\Lambda_{(D)} = 0$, but in this case gravity is topological and the action for fluctuations is a total derivative, so there is nothing to Kaluza-Klein reduce.)

**Linear Action**

We now write the full metric as
\[ G_{AB} + \frac{2}{M_{P}^{2} - 1} H_{AB} , \] (2.27)

where $G_{AB}$ satisfies the background equations of Section 2.5, and $H_{AB}$ is the fluctuation. We expand the action Eq. (2.20) to second order in $H_{AB}$. The result is the standard action for linearized gravity on a curved background,
\[ S = \int d^{D} X \sqrt{-G} \left[ -\frac{1}{2} \nabla_{C} H_{AB} \nabla^{C} H^{AB} + \nabla_{C} H_{AB} \nabla^{B} H^{AC} - \nabla_{A} H_{(D)} \nabla_{B} H^{AB} + \frac{1}{2} \nabla_{A} H_{(D)} \nabla^{A} H_{(D)} + \frac{R_{(D)}}{D} \left( H^{AB} H_{AB} - \frac{1}{2} H^{2}_{(D)} \right) \right] . \] (2.28)
Here the metric, covariant derivatives and curvature $R_{(D)}$ are those of the background, and satisfy the background equations of motion. Indices are always moved with the background metric.

The linear action Eq. (2.28) is invariant under gauge transformations which are the linearized diffeomorphisms of GR,

$$\delta H_{AB} = \nabla_A \Xi_B + \nabla_B \Xi_A ,$$

with $\Xi^A(X)$ the vector gauge parameter.

**Reduction of Fluctuations**

We proceed to split the components of the metric fluctuation $H_{AB}$ into their lower dimensional pieces. The components $H_{\mu\nu}$ are scalars in the internal dimensions, and should, like the scalar field in Section 2.2, be split into eigenmodes of the scalar Laplacian. The components $H_{\mu n}$ are vectors in the internal dimensions, and should be split according to the vector Hodge decomposition, analogously to the internal components of the vector field in Section 2.3. The new ingredient that does not appear in the case of the $p$-forms is the components $H_{mn}$, which are symmetric tensors in the extra dimensions. The best way to split these is to use the symmetric tensor version of the Hodge decomposition, reviewed in Appendix D.

This leads to the following ansatz

$$H_{\mu\nu} = \sum_a h^a_{\mu\nu} \psi_a + \frac{1}{\sqrt{V}} h^0_{\mu\nu}$$

$$H_{\mu n} = \sum_i A^i_\mu Y_{n,i} + \sum_a A^a_\mu \nabla_n \psi_a$$

$$H_{mn} = \sum_I \phi^T h_{mn,I} + \sum_{i \neq \text{Killing}}^i \phi^i (\nabla_m Y_{n,i} + \nabla_n Y_{m,i})$$

$$+ \sum_{a \neq \text{conformal}} \phi^a \left( \nabla_m \nabla_n \psi_a - \frac{1}{N} \nabla^2 \psi_a \gamma_{mn} \right) + \sum_a \frac{1}{N} \phi^a \psi_a \gamma_{mn} + \frac{1}{N} \sqrt{V} \phi^0 \gamma_{mn} .$$

(2.30)

Several comments are in order, starting at the top: the $H_{\mu\nu}$ components are split just as in Section 2.2, the $\psi^a$ are positive orthonormal eigenmodes of the scalar Laplacian, and there is a zero mode $h^0_{\mu\nu}$. In the split of the $H_{\mu n}$ components, the $Y_{n,i}$ are orthonormal co-closed (i.e. transverse) eigenvectors of the vector Laplacian. In other words, in contrast to Eq. (2.6) in the case of the vector field, here we have combined the co-exact and harmonic forms together in the single index $i$. This is because the harmonic forms will play no special role in the case of pure gravity (as they did for the vector, where they correspond to massless scalars in $d$ dimensions), so they need not be indexed and carried around separately. Instead, it is the
Killing vectors which will be important, and will correspond to massless vector modes in \( d \) dimensions.

Moving to the split of the \( H_{mn} \) components, the \( h^T_{mn} \) are symmetric transverse traceless orthonormal eigenfunctions of the Lichnerowicz operator Eq. (D.2), which is the natural Laplacian on the space of symmetric tensors on an Einstein space. Special care has been extended to those co-exact vectors that are Killing vectors (which we denote \( i = \text{Killing} \)), and those scalars that are conformal scalars (which we denote \( a = \text{conformal} \)). The Killing vectors are precisely the co-exact one-forms that have eigenvalue \( \lambda_i = \frac{2R(N)}{N-1} \), which is the lowest possible eigenvalue for such forms (see Appendix B, which collects facts about Killing vectors on closed Einstein manifolds). Conformal scalars are those scalars whose gradients are conformal Killing vectors which are not Killing. Conformal scalars exist only on the sphere, and are the scalar eigenfunctions with the lowest non-zero eigenvalue, \( \lambda_a = \frac{R(N)}{N-1} \) (see Appendix C, which collects facts about conformal scalars on closed Einstein manifolds). The \( \phi^i \) scalars are not present in the decomposition when the index \( i \) takes a value corresponding to an eigenmode which is a Killing vector. Thus we explicitly exclude the Killing vectors from the sums in the expression for \( H_{mn} \). We will find that there is \( d \)-dimensional vector field for each co-exact one-form labeled by \( i \), with mass \( m_i^2 = \lambda_i - \frac{2R(d)}{d} \). The vector is massless only for those \( i \) which are Killing vectors, for which \( \lambda_i = \frac{2R(N)}{N-1} = \frac{2R(d)}{d} \). The \( \phi^i \) will be the longitudinal mode of the massive vectors, and the longitudinal mode does not exist for the case when \( i \) is a Killing vector, corresponding to massless vector in \( d \) dimensions. Similarly, the \( \phi^a \) scalars are not present when the index \( a \) takes a value corresponding to an eigenmode that is a conformal scalar. Thus we explicitly exclude the conformal scalars from the sums in the expression for \( H_{mn} \). We will see that there is a tower of massive scalars corresponding to the positive eigenmodes of scalar Laplacian, with masses \( m_a^2 = \lambda_a - \frac{2R(d)}{d} \). There will be no such scalar, however, for the case where \( a \) corresponds to a conformal scalar.

Since \( H_{AB} \) is real, we have the reality conditions \( h_{\mu\nu}^a = h_{\nu\mu}^a, A^a_{\mu} = A^a_{\mu}, \phi^a = \phi^a, \bar{\phi}^a = \bar{\phi}^a, A^{i*}_\mu = A^{i}_{\mu}, \phi^{i*} = \phi^i, \bar{\phi}^{i*} = \bar{\phi}^i \).

For the gauge parameters, we expand as

\[
\Xi_\mu = \sum_a \xi_\mu^a \psi_a + \xi_\mu^0, \quad (2.31)
\]

\[
\Xi_n = \sum_i \xi_i Y_{n,i} + \sum_a \xi_a \nabla_n \psi_a. \quad (2.32)
\]

Expanding the gauge transformation Eq. (2.84) and equating components, we find the following gauge transformations for the lower-dimensional fields,
\[\delta h^a_{\mu\nu} = \nabla_\mu \xi^a_\nu + \nabla_\nu \xi^a_\mu, \quad \delta \phi^T = 0,\]
\[\delta h^0_{\mu\nu} = \nabla_\mu \xi^0_\nu + \nabla_\nu \xi^0_\mu, \quad \delta \phi^i = \xi^i, \quad i \neq \text{Killing}\]
\[\delta A^i_\mu = \nabla_\mu \xi^i, \quad \delta \phi^a = 2 \xi^a, \quad a \neq \text{conformal}\]
\[\delta A^a_\mu = \xi^a_\mu + \nabla_\mu \xi^a, \quad \delta \tilde{\phi}^a = -2 \lambda_a \xi^a, \]
\[\delta \phi^0 = 0.\]  

(2.33)

First, we see that the field \( \phi^i \) (which only exists when \( i \) is not a Killing vector) is pure Stückelberg, and we can define the following gauge invariant combination

\[\tilde{A}^i_\mu = A^i_\mu - \partial_\mu \phi^i, \quad i \neq \text{Killing}.\]  

(2.34)

Next, we see that the fields \( A^a_\mu \), and one combination of \( \phi^a, \tilde{\phi}^a \) are pure Stückelberg, and it will be possible to gauge them away. However, when the index \( a \) is that of a conformal scalar, the field \( \phi^a \) does not exist. We must treat this case separately. First consider the case where \( a \) is not conformal. We can define the following convenient gauge invariant combinations,

\[F^a = \lambda_a \phi^a + \tilde{\phi}^a, \quad a \neq \text{conformal},\]  

(2.35)
\[\tilde{h}^a_{\mu\nu} = h^a_{\mu\nu} - (\nabla_\mu A^a_\nu + \nabla_\nu A^a_\mu) + \nabla_\mu \nabla_\nu \phi^a, \quad a \neq \text{conformal}.\]  

(2.36)

In the case where \( a \) is conformal, \( \phi^a \) does not exist, so there is no gauge invariant scalar combination. Thus we have only

\[\tilde{h}^a_{\mu\nu} = h^a_{\mu\nu} - (\nabla_\mu A^a_\nu + \nabla_\nu A^a_\mu) - \frac{1}{\lambda_a} \nabla_\mu \nabla_\nu \tilde{\phi}^a, \quad a = \text{conformal}.\]  

(2.37)

The gauge invariant scalar \( F^a \) exists only for the non-conformal modes.

We are now ready to perform the decomposition. We express our result in terms of \( \epsilon(h) \), which denotes the \( d \)-dimensional massless graviton action,

\[\epsilon(h) = -\frac{1}{2} \nabla_\lambda h^*_{\mu\nu} \nabla^\lambda h^{\mu\nu} + \nabla_\lambda h^*_{\mu\nu} \nabla^\nu h^{\mu\lambda} - \nabla_\mu h^* h^{\nu} - \nabla_\nu h^* h^{\mu} + \frac{1}{2} \nabla_\mu h^* \nabla^\mu h + \frac{R(d)}{d} \left( h^{\mu\nu} h_{\mu\nu} - \frac{1}{2} |h|^2 \right) + c.c.\]  

(2.38)

This kinetic term is gauge invariant, so that \( \epsilon(h^a) = \epsilon(\tilde{h}^a) \).

Inserting the decomposition Eq. (2.30) into Eq. (2.28), and doing the integrals over the extra dimensions \( \mathcal{N} \) using the orthogonality of the various parts of the decomposition Eq. (2.30), we find the action

\[S = \int d^d x \left( \mathcal{L}_0 + \mathcal{L}_a + \mathcal{L}_i + \mathcal{L}_I \right).\]  

(2.39)
where

\[
\frac{\mathcal{L}_0}{\sqrt{-g}} = \varepsilon(h_0^0) + h^{\mu\nu,0}(\nabla_\mu \nabla_\nu \phi^0 - \Box \phi^0 g_{\mu\nu}) - \frac{R_{(d)}}{d} h^0 \phi^0 \\
+ \frac{1}{2} \left(1 - \frac{1}{N}\right)(\partial \phi^0)^2 - \frac{1}{2} \left(1 - \frac{2}{N}\right) \frac{R_{(d)}}{d} (\phi^0)^2 ,
\]

(2.40)

\[
\frac{\mathcal{L}_a}{\sqrt{-g}} = \sum_{a \neq \text{conformal}} \varepsilon(\tilde{h}^a) - \frac{1}{2} \lambda_a \left(\tilde{h}_a^a|^2 - |\tilde{h}^a|^2\right) \\
+ \frac{1}{2} \left\{h^{\mu\nu,a^*}(\nabla_\mu \nabla_\nu F^a - \Box F^a g_{\mu\nu}) + \left[\left(1 - \frac{1}{N}\right) \lambda_a - \frac{R_{(d)}}{d}\right] \tilde{F}^{a*} F^a + \text{c.c.}\right\} \\
+ \frac{1}{2} \left(1 - \frac{1}{N}\right) |\partial F^a|^2 + \frac{1}{2} \left(1 - \frac{2}{N}\right) \left[\left(1 - \frac{1}{N}\right) \lambda_a - \frac{R_{(d)}}{d}\right] |F^a|^2 \\
+ \sum_{a = \text{conformal}} \varepsilon(\tilde{h}^a) - \frac{1}{2} \lambda_a |\tilde{h}_a^a|^2 + \frac{1}{2} \lambda_a |\tilde{h}^a|^2 ,
\]

(2.41)

\[
\frac{\mathcal{L}_i}{\sqrt{-g}} = \sum_{i \neq \text{Killing}} -\frac{1}{2} |\tilde{F}_{\mu\nu}^i|^2 - \left(\lambda_i - \frac{2R_{(d)}}{d}\right) |\tilde{A}_i^\mu|^2 + \sum_{i = \text{Killing}} -\frac{1}{2} |F_{\mu\nu}^i|^2 ,
\]

(2.42)

\[
\frac{\mathcal{L}_T}{\sqrt{-g}} = \sum_{T} -\frac{1}{2} |\partial \phi^T|^2 - \frac{1}{2} \left(\lambda_T - \frac{2R_{(d)}}{d}\right) |\phi^T|^2 ,
\]

(2.43)

where \(\tilde{F}_{\mu\nu}^i, F_{\mu\nu}^i\) are the standard Maxwell field strengths of the corresponding vectors.

This \(d\)-dimensional action is manifestly gauge invariant, since the massive modes are written completely in terms of the gauge invariant variables. The \(A_i^a\) and a gauge non-invariant combination of \(\phi^a, \tilde{\phi}^a\) become the longitudinal St"uckelberg fields [29] for the tower of massive gravitons, and the \(\phi^0\) become St"uckelberg fields for a tower of massive vectors. The higher harmonics of the higher-dimensional diffeomorphism symmetry becomes the St"uckelberg symmetry. We can now, if we like, impose a unitary gauge, by setting \(\phi^0 = A_i^0 = \phi^i = 0\), after which the action reads the same as above with \(\tilde{h}_{\mu\nu} \rightarrow h_{\mu\nu}^a, \tilde{A}_i^\mu \rightarrow A_i^\mu\) and \(F_{\mu\nu} \rightarrow \tilde{\phi}^a\).

Part of the spectrum can now be read off. From \(\mathcal{L}_T\) we find a tower of scalars, one for each transverse traceless tensor mode of the Lichnerowicz operator, with mass \(m_T^2 = \lambda_T - \frac{2R_{(d)}}{d}\).

These are massless precisely when \(\lambda_T = \frac{2R_{(d)}}{d}\). As reviewed in Appendix D, this is precisely the case where \(h_{\mu\nu}^{TT}\) corresponds to a deformation which preserves the condition that the internal metric be Einstein, and which cannot be undone by a diffeomorphism or change in volume, i.e. it corresponds to a direction in the moduli space of Einstein structures. From \(\mathcal{L}_i\) we find a tower of massive vectors, one for each non-Killing co-closed one-form labeled by \(i\), with mass \(m_i^2 = \lambda_i - \frac{2R_{(d)}}{d}\), and we find precisely one massless vector for each Killing vector. The remaining parts \(\mathcal{L}_0\) and \(\mathcal{L}_a\) are mixed and must be diagonalized.

\[\text{4See [36] and Section 4 of [33] for reviews of the St"uckelberg formalism applied to massive gravitons.}\]
CHAPTER 2. KALUZA-KLEIN TOWERS ON GENERAL MANIFOLDS

Diagonalization

We now diagonalize the $L_0$ and $L_a$ parts of the action. We first assume $d \geq 3$. There are subtleties for the lower dimensional cases where gravity is non-dynamical, so we will treat them separately.

We start with $L_0$. The zero mode graviton-scalar terms are diagonalized by making the linearized conformal transformation

$$h_0^{\mu \nu} = h_0^{\mu \nu} - \frac{1}{d-2} \phi^0 g_{\mu \nu}.$$  \hfill (2.45)

This gives

$$\frac{L_0}{\sqrt{-g}} = \varepsilon(h_0) + \frac{d + N - 2}{2N(d-2)} \left( -(\partial \phi^0)^2 + \frac{2R(d)}{d}(\phi^0)^2 \right).$$

This is the action for a massless graviton and a scalar of mass $m_0^2 = -\frac{2R(d)}{d}$. This scalar is the volume modulus of the internal manifold. In the positively curved case, the mass term is tachyonic and there is an instability.

Now turn to $L_a$. We can diagonalize the graviton-scalar cross-terms by the following transformation

$$h_a^{\mu \nu} = h'_a^{\mu \nu} - \frac{d}{d(d-1)} \left( 1 - \frac{1}{N} \right) R(d) F^a g_{\mu \nu} + \frac{d}{d(d-1)} \left( 1 - \frac{d-2}{N} \right) \partial_\mu \partial_\nu F^a, \quad a \neq \text{conformal}.$$  \hfill (2.46)

Note that by the Lichnerowicz bound $\lambda_a \geq \frac{\lambda}{d-1}$ (see Appendix C) combined with the curvature relation Eq. (2.26), the denominator is never zero so this is always a valid transformation.

The result in terms of $h_a^{\mu \nu}$ is

$$\frac{L_a}{\sqrt{-g}} = \varepsilon(h_0) - \frac{1}{2} \lambda_a \left( \frac{1}{2} |h_a^{\mu \nu}|^2 - |h_a^{\mu \nu}|^2 \right) + \frac{\left( 1 + \frac{d-2}{N} \right)}{2 \left( d(d-1) \lambda_a + (d-2)R(d) \right)} \left( -|F^a|^2 - \left( \lambda_a - \frac{2R(d)}{d} \right) |F^a|^2 \right), \quad a \neq \text{conformal}.$$  \hfill (2.47)

There is a Fierz-Pauli massive graviton $\tilde{h}_a^{\mu \nu}$ [37] for each positive eigenvector with $m^2 = \lambda_a$, and a massive scalar $F^a$ for each non-conformal positive eigenvector with $m^2 = \lambda_a - \frac{2R(d)}{d}$.

5 Under a conformal transformation

$$h_{\mu \nu} = h'_{\mu \nu} + \pi g_{\mu \nu},$$  \hfill (2.44)

for any scalar $\pi$, the graviton action $\varepsilon(h)$ in (2.38), including the curvature terms, transforms as

$$2\varepsilon(h) = 2\varepsilon(h') + \left[ (d-2)h'^{\mu \nu} (\nabla_\mu \nabla_\nu \pi - \Box \pi g_{\mu \nu}) + \frac{(d-1)(d-2)}{2} |\partial \pi|^2 - \frac{d-2}{d} R(d) \left( h'^{\nu \pi} + \frac{d}{2} |\pi|^2 \right) + \text{c.c.} \right].$$

6 See Section 2 of [33] for a review of the Fierz-Pauli action.
Using the Lichnerowicz bound Eq. (C.14), it is possible to see that the kinetic terms for the scalars are always positive, so the scalars are never ghostly.

\(d = 2\) case:
In two dimensions, \(d = 2\), the kinetic part of the graviton action, \(\sqrt{-g}\epsilon(h)\) with \(\epsilon(h)\) as in Eq. (2.38), is a total derivative so we may drop it. The remaining gravitons have no derivatives and are therefore auxiliary fields and we should solve for them through their equations of motion.

Starting with \(L_0\), the zero mode sector after reduction now has the Lagrangian

\[
L_0 \sqrt{-g} = h^{\mu\nu,0}(\nabla_\mu \nabla_\nu \phi^0 - \Box \phi^0 g_{\mu\nu}) - \frac{R(2)}{2} h^0 \phi^0 + \frac{1}{2} \left(1 - \frac{1}{N}\right) (\partial \phi^0)^2 - \left[\frac{1}{2} \left(1 - \frac{2}{N}\right) \frac{R(2)}{2}\right] (\phi^0)^2 .
\]

(2.48)

This is the linearization of a version of dilaton gravity in two dimensions [38].

Turning to \(L_a\), the part of the reduced action corresponding to scalar eigenfunctions of the Laplacian is now

\[
\frac{L_a}{\sqrt{-g}} = \sum_{a \neq \text{conformal}} -\frac{1}{2} \lambda_a \left( |\tilde{h}_a^{\mu\nu}|^2 - |h^a|^2 \right) + \frac{1}{2} \left(1 - \frac{1}{N}\right) |\partial F^a|^2 + \frac{1}{2} \left(1 - \frac{2}{N}\right) \left[\left(1 - \frac{1}{N}\right) \lambda_a - \frac{R(d)}{d}\right] |F^a|^2 + \sum_{a = \text{conformal}} -\frac{1}{2} \lambda_a \left( |\tilde{h}_a^{\mu\nu}|^2 - |h^a|^2 \right) .
\]

(2.49)

For the non-conformal part, varying with respect to \(h^{\mu\nu,a}\) gives an expression for the non-dynamical graviton,

\[
h^{a}_{\mu\nu} = \frac{1}{\lambda_a} \nabla_\mu \nabla_\nu F^a - \frac{1}{\lambda_a} \left[\left(1 - \frac{1}{N}\right) \lambda_a - \frac{R(2)}{2}\right] F^a g_{\mu\nu} , \ a \neq \text{conformal} .
\]

(2.50)

Substituting this back into the action, we find

\[
\frac{\mathcal{L}}{\sqrt{-g}} = \sum_{a \neq \text{conformal}} -\frac{1}{2} \left[\left(1 - \frac{1}{N}\right) - \frac{R(2)}{2\lambda_a}\right] (|\partial F^a|^2 + (\lambda_a - R(2)) |F^a|^2) .
\]

(2.51)

There is a tower of scalars with masses \(m_a^2 = \lambda_a - \frac{2R(d)}{d}\), the same expression as in the \(d \geq 3\) case. Again the Lichnerowicz bound Eq. (C.14) guarantees that the kinetic term never has the wrong sign, i.e. there are no ghosts.
For the conformal scalars, $a = \text{conformal}$, the equations of motion give $h^a_{\mu\nu} = 0$, and the conformal part of the Lagrangian reduces to zero.

In summary, for $d = 2$ the spectrum contains the same ingredients as the $d \geq 3$ case, with the exception of the zero mode gravitons and the scalar corresponding to the volume modulus, which take the form of dilaton gravity when $d = 2$.

$d = 1$ case:
The $d = 1$ case is somewhat trivial: here $R(d) = 0$, and all the graviton kinetic terms, Fierz-Pauli mass terms, and vector kinetic terms vanish identically. All that remains of $L_a$ is a kinetic term for the volume modulus $\phi^0$. All that remains of the graviton in $L_a$ is a cross term $\sim hF$, so $h^a$ is a multiplier which sets $F^a = 0$. Nothing dynamical remains of $L_i$, since the vector kinetic terms vanish. Thus the spectrum for $d = 1$ is just the zero mode and the scalars of $L_T$.

Spectrum
Collecting all the ingredients, we now summarize the Kaluza-Klein spectrum of pure gravity:

- One massless graviton (linearized dilaton gravity for $d = 2$),
- A tower of Fierz-Pauli massive gravitons, one for each eigenvector of the scalar Laplacian with eigenvalue $\lambda_a > 0$, with masses $m^2_a = \lambda_a$,
- A massless vector for each Killing vector,
- A tower of massive vectors, one for each non-Killing co-exact one-form labeled by $i$, with mass $m^2_i = \lambda_i - \frac{2R(d)}{d}$, $i \neq \text{Killing},$
- One scalar for the volume modulus, with a curvature-dependent mass $m^2_0 = -\frac{2R(d)}{d}$ (part of dilaton gravity for $d = 2$),
- A tower of massive scalars, one for each non-conformal eigenvalue of the scalar Laplacian, with masses $m^2_a = \lambda_a - \frac{2R(d)}{d}$, $a \neq \text{conformal}$ (not dynamical for $d = 1$),
- A tower of scalars for the transverse traceless tensor modes of the Lichnerowicz operator, with mass $m^2_T = \lambda_T - \frac{2R(d)}{d}$. They are massless when $\lambda_T = \frac{2R(d)}{d}$, i.e. for each modulus of the Einstein structure.

Many of the ingredients are in fact non-dynamical for low dimensions. For example, massless gravitons are not dynamical for $d \leq 3$, the massive gravitons and massless vectors are non-dynamical in $d \leq 2$, and the vector fields are non-dynamical for $d = 1$.

Finally, let us comment on the case $N = 1$, the original case of Kaluza and Klein, where $N$ is the circle. The extra dimensional metric has only one component $\gamma_{yy} = 1$. There are no co-exact vectors other than the single Killing vector which is constant around the circle. There are no transverse traceless tensors, and no non-conformal scalars (every vector is a
conformal Killing vector for \( N = 1 \). The scalar eigenfunctions are simply the Fourier modes,
\[
\psi_a = \frac{1}{\sqrt{2\pi R}} e^{iay/R},
\]
where \( R \) is the radius of the circle and \( a \) ranges over all the integers, with \( a = 0 \) the zero mode, and the eigenvalues are \( \lambda_a = a^2/R^2 \). In this case, the ansatz Eq. (2.30) simplifies to a simple Fourier expansion,
\[
\begin{align*}
H_{\mu\nu} &= \frac{1}{\sqrt{2\pi R}} \left[ h^0_{\mu\nu} + \sum_{a=1}^{\infty} h^a_{\mu\nu} e^{iay/R} + \text{c.c.} \right], \\
H_{\mu y} &= \frac{1}{\sqrt{2\pi R}} \left[ A^0_\mu + \sum_{a=1}^{\infty} \left( \frac{ia}{R} A^a_\mu e^{iay/R} + \text{c.c.} \right) \right], \\
H_{yy} &= \frac{1}{\sqrt{2\pi R}} \left[ \phi^0 + \sum_{a=1}^{\infty} (\bar{\phi}^a e^{iay/R} + \text{c.c.}) \right].
\end{align*}
\]
(2.52)

In this case, we recover a massless graviton \( h^0_{\mu\nu} \), a massless vector \( A^0_\mu \), the massless dilaton \( \phi^0 \), and a tower of massive graviton doublets \( h^a_{\mu\nu} \) with longitudinal modes \( A^a_\mu, \bar{\phi}^a \) and masses \( m^2 = a^2/R^2 \).

**Stability**

The Kaluza-Klein spectrum of fluctuations contains the information required to determine stability of the compactification. Since none of the fields are ever ghost-like, the threat to stability comes from fields that may have masses which become tachyonic.

**Gravitons:** For gravitons on a flat or negatively curved Einstein space, \( R_{(N)} \leq 0 \), instability occurs when the Fierz-Pauli mass squared is negative, which never happens, because the mass squared is given by the non-zero scalar eigenvalues \( \lambda_a \) which are always positive.

For positively curved Einstein spaces, \( R_{(N)} > 0 \), gravitons are stable as long as their masses are above the Higuchi bound \([39]\),
\[
m^2 \geq \frac{d-2}{d(d-1)} R_{(d)}. \tag{2.53}
\]

The bound of Lichnerowicz (see Appendix C) tells us that on the internal manifold, the smallest non-zero eigenvalue \( \lambda_a \) of the scalar Laplacian is bounded from below,
\[
\lambda_a \geq \frac{R_{(N)}}{N-1}. \tag{2.54}
\]

Since the mass squared of the graviton is just the eigenvalue, \( m^2 = \lambda_a \), to violate or saturate the Higuchi bound we would need
\[
\lambda_a \leq \frac{d-2}{d(d-1)} R_{(d)} = \frac{d-2}{N(d-1)} R_{(N)}, \tag{2.55}
\]
where we have used Eq. (2.26). This is consistent with Eq. (2.54) only if \( \frac{N}{N-1} \leq \frac{d-2}{d-1} \), which is impossible since \( \frac{N}{N-1} > 1 \) and \( \frac{d-2}{d-1} < 1 \). Thus the gravitons are always stable.

A massive graviton that saturates the Higuchi bound develops a scalar gauge symmetry and propagates one less degree of freedom than a massive graviton above the Higuchi bound. Such a graviton is known as a partially massless graviton [40, 41, 42, 43]. As of this writing, there is no known consistent self-interacting theory that contains a stable background propagating a partially massless graviton, and attempts to find one have run into obstructions [44, 45, 46, 47]. Here, we see that Kaluza-Klein reductions are no exception: a partially massless graviton can never arise in a pure Kaluza-Klein expansion. The removal of the scalars \( \phi^a \) in the decomposition Eq. (2.30) when there are conformal Killing vectors does not result in the removal of the longitudinal modes of the corresponding massive gravitons, it instead results in the removal of the corresponding scalars \( F^a \).

**Vectors:** For vectors, instability occurs when the mass squared is negative. The mass squared is given by \( \lambda_i - \frac{2R(d)}{d} \), which can never be negative by the arguments of Appendix B. Thus the vectors are never unstable.

**Scalars:** The scalars are the only potential sources of instabilities. A scalar is stable for flat and dS space if its mass squared is non-negative. For AdS spaces, the mass squared may be negative as long as it satisfies the Breitenlohner-Friedman bound [48, 49],

\[
m^2 \geq \frac{d-1}{4d} R(d).
\]

(2.56)

Our only condition on the \( d \)-dimensional space was that it be Einstein, which is more general than the maximally symmetric Minkowski, dS and AdS spaces for which these bounds strictly apply. However, since we know of no more general bounds, we will take these as our criteria for stability.

- First consider the zero mode scalar \( \phi^0 \) corresponding to the volume modulus, which has a mass squared \( m^2 = \frac{-2R(d)}{d} \). This is tachyonic for positively curved manifolds, so compactifications of pure gravity to positively curved spaces, e.g. de Sitter, are always unstable. For flat manifolds, it is massless. For negatively curved spaces, \( \phi^0 \) is safely massive. In the case \( d = 2 \) we have dilaton gravity, which in fact propagates no local degrees of freedom [50], so we may call this stable in all cases if we are only concerned with local degrees of freedom.

- Next consider the tower of non-conformal scalars \( F^a \), present when \( N \geq 2 \), with masses \( m^2 = \lambda_a - \frac{2R(d)}{d} \). Since \( \lambda_a > 0 \), for negative or flat curvature these fields all have \( m^2 > 0 \). For positive curvature, there may in principle be tachyonic instabilities coming from these scalars if the internal manifold has low lying eigenvalues that are too small.

- Finally, consider the scalars \( \phi^I \) coming from the eigenstates of the Lichnerowicz operator, which have masses \( m^2 = \lambda_I - \frac{2R(d)}{d} \), which may potentially become tachyonic. We know of no universal bounds on the spectrum of the Lichnerowicz operator, so we cannot say in general when a Kaluza-Klein compactification is stable.
There are known cases where some of the \( \phi^I \) are tachyonic. For example, compactifying on a space that is a product of two spheres, there can be a Lichnerowicz zero mode which corresponds to blowing up one of the spheres while shrinking the other in such a way that the total volume is fixed [51, 52, 53]. This corresponds in the large \( d \) dimensions to a tachyon with mass \( m^2 = -\frac{2R_{(d)}}{d} \), so the instability scale is of order the \( d \)-dimensional Hubble scale.

Maximally symmetric spaces: We can say more in the case where the internal manifold is a quotient of a maximally symmetric space. In this case, \( R_{mnpq} = \frac{R_{(N)}}{N(N-1)} (\gamma_{mp}\gamma_{nq} - \gamma_{mq}\gamma_{np}) \), and the Lichnerowicz operator on transverse traceless tensors becomes \( \Delta_L h_{mn} = -\nabla^2 h_{mn} + \frac{2R_{(N)}}{N-1} h_{mn} \). For \( R_{(N)} = 0 \), the spectrum is non-negative, and the zero modes are the massless scalar moduli. For \( R_{(N)} < 0 \), i.e. quotients of hyperbolic space, the spectrum is positive; there are no massless modes for which \( \lambda_\alpha = \frac{2R_{(N)}}{N} \) (there are no moduli, a consequence of Mostow-Prasad rigidity [54, 55]). For \( R_{(N)} > 0 \), i.e. spheres, the spectrum of the “rough” Laplacian, \( -\nabla^2 \), on symmetric rank \( r \) tensors on the unit \( N \)-sphere is \( l(l+N-1)-r \), where \( l = 0, 1, 2, \cdots \) is an integer [56]. Using this for \( r = 2 \), restoring the radius of the sphere and adding the curvature parts, we find the Lichnerowicz spectrum \( \lambda_\alpha = [l(l + N - 1) - 2] \frac{R_{(N)}}{N(N-1)} + \frac{2R_{(N)}}{N-1} \). The masses of the scalars are then \( m^2 = \lambda_\alpha - \frac{2R_{(N)}}{N} = l(l + N - 1) \frac{R_{(N)}}{N(N-1)} > 0 \), so we have no tachyons or moduli from the \( \phi^I \) sector in reductions on spheres.

Moving to the \( F^a \) scalars, the only dangerous case was \( R_{(N)} > 0 \). In the case the maximally symmetric internal manifold is a sphere and the scalar eigenvalues are \( \lambda_\alpha = \frac{l(l+N-1)}{N(N-1)} \frac{R_{(N)}}{N} \). The \( l = 1 \) modes are the conformal modes, and for \( l \geq 2 \) the eigenvalues are \( > \frac{2R_{(N)}}{N} = \frac{2R_{(d)}}{d} \), so none of the \( F^a \) states are tachyonic.

Summary of stability results:

- Compactifications with positive curvature are all unstable in the volume modulus. They are stable in the non-conformal scalars \( F^a \) if and only if the scalar Laplacian has no eigenvalues below \( \frac{2R_{(N)}}{N} \). They are stable in the Lichnerowicz scalars if and only if the spectrum of the Lichnerowicz operator has no eigenvalues below \( \frac{2R_{(N)}}{N} \).

- Compactifications with zero or negative curvature are always stable in the volume modulus and in the scalars \( F^a \). They are stable in the Lichnerowicz scalars (and hence completely stable) if and only if the spectrum of the Lichnerowicz operator has no eigenvalues below \( \frac{2R_{(N)}}{N} \).

- Compactifications on quotients of maximally symmetric spaces are always stable in the non-conformal scalars and in the Lichnerowicz scalars.

Recall that for pure gravity, the \( d \)-dimensional curvature \( R_{(d)} \), the \( N \)-dimensional curvature \( R_{(N)} \), and the higher-dimensional cosmological constant \( \Lambda \) all have the same sign or, in the flat case, all vanish, so the above results can be read in terms of the \( d \)-dimensional cosmological constant or in terms of the \( D \)-dimensional cosmological constant.
2.6 Flux Compactifications

We now proceed to consider the case of Freund-Rubin-type flux compactifications [57]. These arise from gravity plus a possible cosmological constant in a product space $\mathcal{M} \times \mathcal{N}$, together with an $N$-form flux wrapping the internal space.\footnote{For the case of an internal manifold that is a product manifold, with a collection of flux fields that individually wrap each component, see [58].}

Background

The action is Einstein gravity in $D$ dimensions with a $D$-dimensional cosmological constant $\Lambda_{(D)}$ and $D$-dimensional Planck mass $M_P$, minimally coupled to an $(N-1)$-form potential $A_{A_1\ldots A_{N-1}}(X)$,

$$S = \frac{M_{P}^{D-2}}{2} \int d^Dx \sqrt{-G} \left( R_{(D)} - 2\Lambda_{(D)} - \frac{1}{N!} F_{A_1\ldots A_N}^2 \right), \quad (2.57)$$

where the $N$-form field strength is $F_{A_1\ldots A_N} = (dA)_{A_1\ldots A_N} = N \nabla_{[A_1} A_{A_2\ldots A_N]}$.

The Einstein equations of motion for the metric are

$$R_{AB} - \frac{1}{2} R_{(D)} G_{AB} + \Lambda_{(D)} G_{AB} = \frac{1}{(N-1)!} F_{AA_2\ldots A_N} F_B^{A_2\ldots A_N} - \frac{1}{2N!} G_{AB} F_{A_1\ldots A_N}^2, \quad (2.58)$$

and the equations of motion for the $(N-1)$-form are

$$\nabla^{A_1} F_{A_1 A_2\ldots A_N} = 0. \quad (2.59)$$

We allow the $N$-form flux to take values only in the internal space, $F_{\mu A_1\ldots A_{N-1}} = 0$, and its value on the internal space is proportional to the volume form on $\mathcal{N}$,

$$F_{n_1\ldots n_N} = Q \epsilon_{n_1\ldots n_N}. \quad (2.60)$$

This ansatz automatically solves the $(N-1)$-form equation of motion Eq. (2.59). The constant of proportionality $Q$ has mass dimension one.

We now plug the ansatz Eq. (2.60) into Eq. (2.58), take the trace and use it to solve for the Ricci tensor:

$$R_{AB} = \frac{R_{(D)}}{D} G_{AB}, \quad R_{(D)} = \frac{2D}{D-2} \Lambda_{(D)} + \frac{d-N}{D-2} Q^2. \quad (2.61)$$

Breaking Eq. (2.61) into its $d$- and $N$-dimensional components, with metrics $g_{\mu\nu}$ and $\gamma_{mn}$ respectively, we find that both factors of the product space must be Einstein,

$$R_{\mu\nu} = \frac{R_{(d)}}{d} g_{\mu\nu}, \quad (2.62)$$

$$R_{mn} = \frac{R_{(N)}}{N} \gamma_{mn}, \quad (2.63)$$
with the following relations for the scalar curvatures on $\mathcal{M}$ and $\mathcal{N}$,

\[
R_{(d)} = \frac{2d}{d + N - 2} \Lambda_{(D)} - \frac{d(N - 1)}{d + N - 2} Q^2, \quad R_{(N)} = \frac{2N}{d + N - 2} \Lambda_{(D)} + \frac{N(d - 1)}{d + N - 2} Q^2.
\] (2.64)

These can be solved to obtain the higher dimensional cosmological constant in terms of the lower dimensional curvatures,

\[
\Lambda_{(D)} = \frac{1}{2} \left( 1 - \frac{1}{d} \right) R_{(d)} + \frac{1}{2} \left( 1 - \frac{1}{N} \right) R_{(N)},
\] (2.65)

and the flux in terms of the lower dimensional curvatures,

\[
Q^2 = \frac{R_{(N)}}{N} - \frac{R_{(d)}}{d}.
\] (2.66)

Note that when the flux is non-vanishing, the total background product space with metric $G_{AB}$ is not required to be Einstein. The relation Eq. (2.66) will be the most useful during the Kaluza-Klein reduction process.

Quantum mechanically, there is often a Dirac quantization condition imposed on the flux strength $Q^2$. A $p$-form couples to $(p - 1)$-branes through the worldvolume action $S = M_{P}^{p-1} e \int A$, where $e$ is a coupling constant of dimension $p - (D - 2)/2$. Demanding that the path integral integrand $e^{iS}$ be independent of which choice of gauge potential is used to represent the background flux can lead to quantization conditions. For example, when the internal manifold $\mathcal{N}$ has the topology of a sphere, the condition is [59],

\[
Q = \frac{2\pi n}{M_{P}^{p-1} e V_{N}}, \quad n \in \text{integers}.
\] (2.67)

The quantity $Q$ is really a flux density. We may define the flux as $q \equiv Q V_{N}$, which has dimension $1 - N$ and is the volume-independent quantity that is quantized in units of the fundamental constants. Since we are only working classically, we will not impose these quantization conditions, though they should be kept in mind because it is through these conditions that quantum mechanics can restrict the class of allowed solutions.

**Fluctuations**

Now we write the full metric and form field respectively as

\[
G_{AB} + \frac{2}{M_{P}^{p-1}} H_{AB},
\] (2.68)

\[
A_{A_1...A_{N-1}} + \frac{1}{M_{P}^{p-1}} \tilde{A}_{A_1...A_{N-1}},
\] (2.69)
where \( G_{AB} \) and \( A_{A_1...A_{N-1}} \) solve the background equations of motion from Section 2.6, and \( H_{AB}, \tilde{A}_{A_1...A_{N-1}} \) are the small fluctuations of the metric and form respectively. The field strength is then \( F_{A_1...A_N} + \frac{1}{M_p^2} \tilde{F}_{A_1...A_N} \), where \( F_{A_1...A_N} \) is the background field strength and the fluctuation is \( \tilde{F}_{A_1...A_N} = (d\tilde{A})_{A_1...A_N} = N\nabla_{[A_1} \tilde{A}_{A_2...A_N]} \).

Expanding up to quadratic order in the fluctuations, and dropping linear terms which vanish identically by the equations of motion, the linearized action is

\[
\frac{\mathcal{L}}{\sqrt{-G}} = -\frac{1}{2} \nabla_C H_{AB} \nabla^C H^{AB} + \nabla_C H_{AB} \nabla^B H^{AC} - \nabla_A H_{(D)} \nabla_B H^{AB} + \frac{1}{2} \nabla_A H_{(D)} \nabla^A H_{(D)}
\]

\[\quad - \left( \frac{1}{2} R_{(D)} - \Lambda_{(D)} \right) \left( H^{AB} H_{AB} - \frac{1}{2} H_{(D)}^2 \right) + 2 R^{AB} \left( H_A^C H_{BC} - \frac{1}{2} H_{AB} H_{(D)} \right)
\]

\[\quad + \frac{1}{2N!} \left( H^{AB} - \frac{1}{2} H_{(D)}^2 \right) F_{A_1...A_N}^2 - \frac{1}{(N-2)!} H_{A_1B_1} H_{A_2B_2} F_{A_1...A_N}^{A_3...A_N}
\]

\[\quad + \frac{2}{(N-1)!} H_{A_1B_1} \tilde{F}_{A_1A_2...A_N} F_{B_1}^{A_2...A_N} - \frac{1}{N!} H_{(D)} \tilde{F}_{A_1...A_N} F_{A_1...A_N}^2 . \tag{2.70}
\]

The first two lines reduce to Eq. (2.28), the linearized action for the graviton with no flux, if it is additionally imposed that the full space be Einstein.

\[\text{d} + \text{N Split and Ansätze}\]

There are four different parts to the expression Eq. (2.70): the kinetic terms for the graviton (the first line), the non-derivative quadratic terms for the graviton (the second, third and fourth lines), the kinetic terms for the form fluctuations (the fifth line), and the mixing terms between graviton and flux (the sixth line).

Using the value for the background flux and the definition of the Hodge star (reviewed in Appendix A), the mixing terms (the sixth line of Eq. (2.70)) simplify to

\[
(-1)^{N-1} Q \left[ 2 H^{nm} \nabla_m (\ast \tilde{A})_n + (-1)^N 2 H^{nm} \nabla^m (\ast \tilde{A})_n \right] + \left( H_{(N)} - H_{(d)} \right) \nabla^n (\ast \tilde{A})_n , \tag{2.71}
\]

where \( H_{(N)} \equiv \gamma^{mn} H_{mn} \) and \( H_{(d)} \equiv g^{\mu\nu} H_{\mu\nu} \), and the non-kinetic terms for the graviton (the second, third and fourth lines of Eq. (2.70)) reduce to

\[
\frac{R_{(d)}}{d} \left( H_{\mu\nu}^{\mu\nu} - \frac{1}{2} H_{(d)}^2 \right) + \frac{2 R_{(d)}}{d} \left( H_{\mu\nu} - \frac{1}{2} H_{(d)} H_{(N)} \right) + \left( \frac{R_{(d)}}{d} + Q^2 \right) \left( H_{mn}^2 - \frac{1}{2} \left( \frac{R_{(d)}}{d} + 2 Q^2 \right) H_{(N)}^2 \right) . \tag{2.72}
\]
The ansatz for the graviton we will use is

\[ H_{\mu\nu} = \sum_a h^a_{\mu\nu} \psi_a + \frac{1}{\sqrt{V_N}} h^0_{\mu\nu}, \]

\[ H_{\mu\nu} = \sum_i A^i_\mu Y_{n,i} + \sum_a A^a_\mu \nabla_n \psi_a + \sum_\alpha A^\alpha_\mu Y_{n,\alpha}, \]

\[ H_{mn} = \sum_I \phi_I^T h_{mn,I} + \sum_{i \neq \text{Killing}} \phi^i (\nabla_m Y_{n,i} + \nabla_n Y_{m,i}) + \sum_{\alpha \neq \text{Killing}} \phi^\alpha (\nabla_m Y_{n,\alpha} + \nabla_n Y_{m,\alpha}) \]

\[ + \sum_{a \neq \text{conf.}} \phi^a \left( \nabla_m \nabla_n \psi_a - \frac{1}{N} \nabla^2 \psi_a \gamma_{mn} \right) + \sum_a \frac{1}{N} \phi^a \psi_a \gamma_{mn} + \frac{1}{N} \frac{1}{\sqrt{V_N}} \phi^0 \gamma_{mn}. \]

(2.73)

Here there is a difference compared to the pure graviton case Eq. (2.30). We have separated the harmonic one-forms from the co-closed one-forms, so that \( i \) now ranges only over the co-exact forms and \( \alpha \) ranges over the harmonic forms (i.e. we are using the original Hodge decomposition for one-forms). We do this because the fields corresponding to the harmonic forms will mix with those coming from the \( p \)-forms, and we will have to treat them separately.

There are three distinct cases we are attempting to treat simultaneously: a positively curved internal manifold, \( R_{(N)} > 0 \), a negatively curved internal manifold \( R_{(N)} < 0 \), and a flat internal manifold \( R_{(N)} = 0 \). In the positively curved case, there are no harmonic one-forms (see the arguments in Appendix B), so the set of \( \alpha \) is empty. In the negatively curved case, there are no Killing vectors (see the arguments in Appendix B), so the sets of \( i \neq \text{Killing} \), \( \alpha \neq \text{Killing} \) are the same as the sets of \( i \), \( \alpha \). In the flat case, the Killing vectors and harmonic forms are one and the same, so the set of \( \alpha \neq \text{Killing} \) is empty.

We expand the \((N-1)\)-form fluctuation \( \tilde{A}_{A_1 \ldots A_{N-1}} \) as in Eq. (2.13). For reducing Eq. (2.71) we need to take the Hodge star of the relevant parts of the flux decomposition Eq. (2.13),

\[ (*\tilde{A})_n = \sum_{i_{N-1}} a^{i_{N-1}}_n (*Y_{i_{N-1}})_n + \sum_{i_{N-2}} b^{i_{N-2}}_n (*dY_{i_{N-2}})_n + \sum_{\alpha_{N-1}} c^{\alpha_{N-1}}_n (*Y_{\alpha_{N-1}})_n, \]

(2.74)

\[ (*\tilde{A})_{mn} = \sum_{i_{N-2}} a^{i_{N-2}}_{\mu} (*Y_{i_{N-2}})_{mn} + \sum_{i_{N-3}} b^{i_{N-3}}_{\mu} (*dY_{i_{N-3}})_{mn} + \sum_{\alpha_{N-2}} c^{\alpha_{N-2}}_{\mu} (*Y_{\alpha_{N-2}})_{mn}. \]

(2.75)

We will use some facts, reviewed in Appendix A, to map these basis vectors onto ones that have simple overlap with the eigenfunctions in the graviton decomposition. The Hodge star provides an isomorphism between the space of harmonic \( p \)-forms and the space of harmonic \((N-p)\)-forms. Thus the indices \( \alpha \) (which is \( \alpha_1 \)) and \( \alpha_{N-1} \) take values over the same set. Similarly, the Hodge star provides an isomorphism between the space of co-exact \( p \)-forms and the space of exact \((N-p)\)-forms, so the indices \( a \) (which is \( i_0 \), the case \( p = 0 \)) and \( i_{N-2} \) take values over the same set and the indices \( i \) (which is \( i_1 \), the case \( p = 1 \)) and \( i_{N-1} \) take values over the same set. For \( N > 2 \), we may define our bases as in Eqs. (A.38) and (A.40).
in Appendix A:

\[ Y_{i_{N-1}} = \frac{1}{\sqrt{\lambda_i}} \ast dY_i , \quad (2.76) \]

\[ Y_{i_{N-2}} = \frac{1}{\sqrt{\lambda_i}} \ast dY_i , \quad (2.77) \]

\[ Y_{\alpha_{N-1}} = \ast Y_{\alpha} . \quad (2.78) \]

For \( N = 2 \), the Hodge star maps the space of harmonic one-forms labeled by \( \alpha \) into itself. In this case, we cannot choose all the basis vectors in accord with Eq. (2.78) (although Eqs. (2.76) and (2.76) need not be modified). Instead, the index \( \alpha \) of the \( Y_{\alpha} \) is chosen so that it can be divided into imaginary self dual and imaginary anti-self dual sets, \( \alpha_+ \) and \( \alpha_- \). The sets \( Y_{\alpha \pm} \) then transform under the Hodge star as (see Appendix A),

\[ \ast Y_{\alpha \pm} = \pm i Y_{\alpha \pm} . \quad (2.79) \]

The extreme case \( N = 1 \), the case where the internal manifold is a circle and the \( p \)-form is just a scalar, goes through as well. In this case, Eq. (2.77) is empty, and Eq. (2.78) is a simple identity between the single constant function and the single constant vector. Eq. (2.76) relates the space of scalar eigenfunctions to itself and so may have to be supplemented by an additional unitary transformation depending on the choice of basis, which may then be reabsorbed by a field redefinition of \( a^a \). We say more about the \( N = 1 \) case at the end of Section 2.6, and point out as we go along how the lower dimensional cases differ.

Using the inverse of Eqs. (2.76)-(2.78) obtained by taking the Hodge star of both sides, we can write Eq. (2.74) and Eq. (2.75) as

\[
(\ast \tilde{A})_n = (-1)^{N-1} \left[ \sum_a \frac{1}{\sqrt{\lambda_a}} a^a (d\psi_a)_n + \sum_i \sqrt{\lambda_i} b^i Y_{i,n} + \sum_{\alpha} c^\alpha Y_{\alpha,n} \right],
\]

\[
(\ast \tilde{A}_\mu)_{mn} = \sum_i \frac{1}{\sqrt{\lambda_i}} a^i_\mu (dY_i)_{mn} + \ldots,
\]

where the ellipsis denotes contributions that are orthogonal to all terms in the graviton decomposition, and hence do not contribute in Eq. (2.71).

\section*{Gauge Symmetries and Gauge Invariant Combinations}

As in Section 2.4, the theory is invariant under the \((N - 2)\)-form gauge transformations acting on the form field fluctuations,

\[
\delta \tilde{A}_{A_1\ldots A_{N-1}} = (N - 1) \nabla_{[A_1} A_{A_2\ldots A_{N-1}]} ,
\]

\section*{Renaming the labels on the coefficients in Eqs. (2.74) and (2.75),}

\[
a^{i_{N-1}} \leftrightarrow a^a , \quad b^{i_{N-2}} \leftrightarrow b^i , \quad c^{\alpha_{N-1}} \leftrightarrow c^\alpha , \quad a^{i_{N-2}} \leftrightarrow a^i , \quad \ast Y_{\alpha_\pm} = \pm i Y_{\alpha_\pm} .
\]

and using the inverse of Eqs. (2.76)-(2.78) obtained by taking the Hodge star of both sides, we can write Eq. (2.74) and Eq. (2.75) as
where $\Lambda_{A_1\ldots A_{N-2}}(X)$ is an arbitrary $(N - 2)$-form. As in Section 2.5, the diffeomorphism symmetries acting on the graviton fluctuations are

$$\delta H_{AB} = \nabla_A \Xi_B + \nabla_B \Xi_A .$$

(2.84)

The diffeomorphism symmetries act on the form field fluctuations as well. Acting on the field strength (which is gauge invariant under Eq. (2.83) so that the transformation is unambiguous), we have

$$\delta \tilde{F}_{A_1\ldots A_N} = \mathcal{L}_\Xi F_{A_1\ldots A_N} \equiv \Xi^A \nabla_A F_{A_1\ldots A_N} + (\nabla_A \Xi^A) F_{A_2\ldots A_N} + \ldots + (\nabla_A \Xi^A) F_{A_1\ldots A_{N-1}A} .$$

(2.85)

For the diffeomorphism symmetries, we expand the gauge parameter over the eigenforms,

$$\Xi_\mu = \sum_a \xi^a_\mu \psi_a + \xi^0_\mu ,$$

(2.86)

$$\Xi_n = \sum_i \xi^i Y_{n,i} + \sum_a \xi^a \nabla_n \psi_a + \sum_\alpha \xi^\alpha Y_{n,\alpha} .$$

(2.87)

By expanding out Eq. (2.84) and equating coefficients, we find how the diffeomorphisms act on the lower dimensional fields in the graviton decomposition,

$$\delta h^a_{\mu\nu} = \nabla_\mu \xi^a_{\nu} + \nabla_\nu \xi^a_{\mu} ,$$

$$\delta \phi^T = 0 ,$$

$$\delta \phi^i = \xi^i , \quad i \neq \text{Killing} ,$$

$$\delta \phi^a = 2\xi^a , \quad a \neq \text{conformal} ,$$

$$\delta \phi^0 = 0 ,$$

$$\delta \phi^\alpha = \xi^\alpha , \quad \alpha \neq \text{Killing} .$$

(2.88)

We also have the $(N - 1)$-form gauge transformations descending from Eq. (2.83) with the expansion Eq. (2.16),

$$\delta a^a = 0 ,$$

$$\delta a^i_\mu = \partial_\mu \Lambda^i ,$$

$$\delta b^i = \Lambda^i ,$$

$$\delta c^\alpha = 0 .$$

(2.89)

Descending from Eq. (2.85), we find the following diffeomorphism transformations for the form fields$^8$,

$$\delta a^a = 2Q \sqrt{\lambda^a_\mu} \xi^a \, ,$$

$$\delta (a^i_\mu - \partial_\mu b^i) = -\frac{2Q}{\sqrt{\lambda^i_\mu}} \partial_\mu \xi^i \, ,$$

$$\delta c^\alpha = 2Q \xi^\alpha .$$

(2.90)

$^8$The factors of 2 are from the factor of 2 in the canonical normalization of $H_{AB}$ in Eq. (2.68).
We define the quantities,

\[
\tilde{h}_{\mu\nu}^a = \begin{cases} 
  h_{\mu\nu}^a - (\nabla_\mu A_\nu^a + \nabla_\nu A_\mu^a) + \nabla_\mu \nabla_\nu \phi^a, & a \neq \text{conformal} , \\
  h_{\mu\nu}^a - (\nabla_\mu A_\nu^a + \nabla_\nu A_\mu^a) - \frac{1}{\sqrt{\lambda^a}} \nabla_\mu \nabla_\nu \phi^a, & a = \text{conformal} , 
\end{cases}
\]  

(2.91)

\[
F^a = \lambda_a \phi^a + \bar{\phi}^a, \quad a \neq \text{conformal} ,
\]  

(2.92)

\[
K^a = \begin{cases} 
  -Q \phi^a + \frac{1}{\sqrt{\lambda^a}} a^a, & a \neq \text{conformal} , \\
  a^a + \frac{Q}{\sqrt{\lambda^a}} \bar{\phi}^a, & a = \text{conformal} , 
\end{cases}
\]  

(2.93)

\[
\tilde{A}_\mu^i = \begin{cases} 
  A_\mu^i - \partial_\mu \phi^i, & i \neq \text{Killing} , \\
  A_\mu^i, & i = \text{Killing} , 
\end{cases}
\]  

(2.94)

\[
\tilde{a}_\mu^i = \begin{cases} 
  a_\mu^i - \partial_\mu b^i + \frac{2Q}{\sqrt{\lambda^i}} \partial_\mu \phi^i, & i \neq \text{Killing} , \\
  a_\mu^i - \partial_\mu b^i, & i = \text{Killing} , 
\end{cases}
\]  

(2.95)

\[
\tilde{A}_\mu^\alpha = \begin{cases} 
  A_\mu^\alpha - \partial_\mu \phi^\alpha, & \alpha \neq \text{Killing} , \\
  A_\mu^\alpha - \frac{1}{2Q} \partial_\mu c^\alpha, & \alpha = \text{Killing} , 
\end{cases}
\]  

(2.96)

\[
\tilde{c}^\alpha = c^\alpha - 2Q \phi^\alpha, \quad \alpha \neq \text{Killing} .
\]  

(2.97)

These are all gauge invariant, with the exception of \(\tilde{A}_\mu^i\) and \(\tilde{a}_\mu^i\) for \(i = \text{Killing}\), which transform as \(\tilde{A}_\mu^i = \partial_\mu \xi^i\), \(\delta \tilde{a}_\mu^i = -\frac{2Q}{\sqrt{\lambda^i}} \partial_\mu \xi^i\). One linear combination of these is gauge invariant and will yield a massive vector, the other transforms as a Maxwell field and will give a massless vector with a gauge invariant Maxwell action.

There is a curiosity that takes place when the internal manifold is flat, \(R_{(N)} = 0\). As reviewed in Appendix B, this is the only case for which there can be harmonic forms which are also Killing, so that \(\alpha = \text{Killing}\) is non-empty, and in this case the harmonic forms are precisely the Killing vectors. As indicated in Eq. (2.96), the gauge invariant combination is \(A_\mu^\alpha - \frac{1}{2Q} \partial_\mu c^\alpha\), which combines the zero-modes of the flux field with the zero modes of the Kaluza-Klein photon coming from the graviton (the gravi-photon). This combination will appear in \(d\) dimensions as a massive vector, whose mass can be interpreted as arising from a kind of Anderson-Higgs mechanism; the gravi-photon eats the zero-mode of the form field to become massive. This only happens for non-zero flux \(Q^2 > 0\), so that by Eq. (2.66) the \(d\)-dimensional space is negatively curved, \(R_{(d)} < 0\). When \(Q^2 = 0\), we see from Eq. (2.90) that the zero mode becomes gauge invariant and is no longer combined with the gravi-photon. The scalar and the gravi-photon then become a separate massless vector and scalar.

### Reduced Action

Using orthogonality of the various eigenfunctions to integrate out the internal dimensions, combining all the different terms and regrouping them into the combinations Eqs. (2.91)-
(2.97), we find,
\[
\frac{\mathcal{L}_0}{\sqrt{-g}} = \varepsilon(h^0) + h^{\mu\nu}\delta(\nabla_\mu \nabla_\nu \phi^0 - \Box \phi^0 g_{\mu\nu}) - \frac{R(d)}{d} h^0 \phi^0 \\
+ \frac{1}{2} \left(1 - \frac{1}{N}\right) (\partial \phi^0)^2 - \left[\frac{1}{2} \left(1 - \frac{2}{N}\right) \frac{R(d)}{d} + \left(1 - \frac{1}{N}\right) Q^2\right] (\phi^0)^2,
\]
\[
\frac{\mathcal{L}_a}{\sqrt{-g}} = \sum_{\alpha \neq \text{conformal}} \varepsilon(\tilde{h}^a) - \frac{1}{2} \lambda_\alpha \left(\tilde{h}^a_{\mu\nu} - |\tilde{h}^a|^2\right) \\
+ \frac{1}{2} \left\{\tilde{h}^{\mu\nu,a} (\nabla_\mu \nabla_\nu F^a - \Box F^a g_{\mu\nu}) + \left[\left(1 - \frac{1}{N}\right) \lambda_\alpha - \frac{R(d)}{d}\right] \tilde{h}^{a*} F^a + Q \lambda_\alpha \tilde{h}^{a*} K^a + \text{c.c.}\right\} \\
+ \frac{1}{2} \left(1 - \frac{1}{N}\right) |\partial F^a|^2 + \frac{1}{2} \left(1 - \frac{2}{N}\right) \left\{\left(1 - \frac{1}{N}\right) \lambda_\alpha - \frac{R(d)}{d}\right\} |F^a|^2 - \left(1 - \frac{1}{N}\right) Q^2 |F^a|^2 \\
- \frac{1}{2} \lambda_\alpha |\partial K^a|^2 - \frac{1}{2} \lambda^2_\alpha |K^a|^2 - \frac{1}{2} \lambda_\alpha \{K^{a*} F^a + \text{c.c.}\} \\
+ \sum_{\alpha = \text{conformal}} \varepsilon(\tilde{h}^a) - \frac{1}{2} \lambda_\alpha \left(\tilde{h}^a_{\mu\nu} - |\tilde{h}^a|^2\right) + \frac{1}{2} \left\{Q \sqrt{\lambda_\alpha} \tilde{h}^{a*} K^a + \text{c.c.}\right\} - \frac{1}{2} |\partial K^a|^2 - \frac{1}{2} \lambda_\alpha |K^a|^2,
\]
\[
\frac{\mathcal{L}_i}{\sqrt{-g}} = \sum_i -\frac{1}{4} |\tilde{F}^i_{\mu\nu}|^2 - \frac{1}{4} |\tilde{F}^i_{\mu\nu}|^2 - \left(\lambda_i - \frac{2R(d)}{d} - 2Q^2\right) |\tilde{A}^i_\mu|^2 - 2 |Q \tilde{A}^i_\mu + \frac{1}{2} \sqrt{\lambda_i} \tilde{a}^i_\mu|^2,
\]
\[
\frac{\mathcal{L}_a}{\sqrt{-g}} = \sum_{\alpha \neq \text{Killing}} -\frac{1}{2} |\tilde{F}^a_{\mu\nu}|^2 + \frac{2}{2} \left\{\frac{R(d)}{d} + Q^2\right\} |\tilde{A}^a_\mu|^2 - 2 |Q \tilde{A}^a_\mu - \frac{1}{2} \partial_\mu c^a|^2 \\
+ \sum_{\alpha = \text{Killing}} -\frac{1}{2} |\tilde{F}^a_{\mu\nu}|^2 + \frac{2R(d)}{d} |\tilde{A}^a_\mu|^2,
\]
\[
\frac{\mathcal{L}_I}{\sqrt{-g}} = \sum_I -\frac{1}{2} |\partial \phi^I|^2 - \frac{1}{2} \left(\lambda_I - \frac{2R(d)}{d} - 2Q^2\right) |\phi^I|^2,
\]
where \(\tilde{F}^i_{\mu\nu}, \tilde{F}^a_{\mu\nu}\) are the standard Maxwell field strengths of the corresponding vectors.

In addition, we have terms that we have not displayed coming from the kinetic terms for the form fluctuations (the third line of Eq. (2.70)). These are unaffected by the mixing with gravity, and look identical to Eq. (2.19) for \(p = N - 1\). The only exceptions, which are affected and which we have included here, are the parts in Eqs. (2.74) and (2.75); in particular, the kinetic term for these parts coming from the third line of Eq. (2.70) give the terms in Eqs. (2.99), (2.100) and (2.101) that are squares of the \(p\)-form components.

For the case \(N = 2\), there was the subtlety that the one-forms are mapped to themselves under the Hodge star, so that we had to split the index \(\alpha\) into imaginary self-dual and imaginary anti-self dual parts, \(\alpha_\pm\), as in Eq. (2.79). From the first term of Eq. (2.71), we get the cross term \(\sum_\alpha Q_i \left(A^\alpha_{\mu} \partial^\mu c^\alpha - A^{\alpha*}_{\mu} \partial^\mu c^{\alpha*}\right) + \text{c.c.}\) After a field re-definition \(c^\alpha \rightarrow -ic^{\alpha*}\).
$e^\alpha \to ie^{\alpha}$, the $\alpha^+$ and $\alpha^-$ indices can be recombined into $\alpha$, giving the cross terms of Eq. (2.101) which are of the same form as $N > 2$. The part of the spectrum that can be immediately read off is $L_T$. This gives a tower of massive scalars with mass $m_I^2 = \lambda_I - \frac{2R(d)}{d} - 2Q^2 = \lambda_I - \frac{2R(N)}{N}$. In terms of $R(N)$ this is the same mass spectrum as for pure gravity. In particular, the scalars are massless precisely when $\lambda_l = \frac{2R(N)}{N}$, i.e. for each modulus of the Einstein structure. For the rest, we must diagonalize the action.

**Diagonalization**

We now diagonalize the various parts of the action that are mixed. The case $d = 2$ is special due to the fact that gravitons have a vanishing kinetic term in this dimension, so we first restrict to the case $d \geq 3$. (As with the pure graviton, the case $d = 1$ is trivial: $R(d) = 0$, every scalar indexed by $a$ is conformal, and everything that might need diagonalization vanishes or is set to zero by equations of motion.)

**Zero modes:** We start with the $L_0$ sector in Eq. (2.98). We can diagonalize this graviton-scalar sector by the conformal transformation

$$h_0^{\mu
u} = h_0^{\mu
u} - \frac{1}{d-2}\phi^0 g_{\mu
u}, \quad (2.103)$$

giving

$$\frac{L_0}{\sqrt{-g}} = \varepsilon(h^0) + \frac{d + N - 2}{2N(d-2)}\left( - (\partial \phi^0)^2 + \frac{2R(d)}{d}(\phi^0)^2 \right) - \left( 1 - \frac{1}{N} \right)Q^2(\phi^0)^2. \quad (2.104)$$

This describes a massless graviton, and a non-ghost scalar with mass,

$$m_0^2 = -\frac{2R(d)}{d} + \frac{2(N-1)(d-2)}{d + N - 2}Q^2, \quad (2.105)$$

Using Eq. (2.66) we may write this as $m_0^2 = -\frac{2R(N)}{N} + \frac{2N(d-1)}{d+N-2}Q^2$, and we see that the volume modulus for positively curved $N$ may be stabilized if the flux is sufficiently large.

**a modes:** We next turn to the $L_a$ sector in Eq. (2.99).

First consider the non-conformal sector. We can diagonalize by the transformation

$$h_a^{\mu
u} \to h_a^{\mu
u} - \frac{d(1 - \frac{1}{N})}{d(d-1)\lambda_a - (d-2)R(d)} F_a g_{\mu\nu} + \frac{d(1 + \frac{d-2}{N})}{d(d-1)\lambda_a - (d-2)R(d)} \partial_{\mu} \partial_{\nu} F_a^a - \frac{dQ\lambda_a}{d(d-1)\lambda_a - (d-2)R(d)} K^a g_{\mu\nu} - \frac{d(d-2)Q}{d(d-1)\lambda_a - (d-2)R(d)} \partial_{\mu} \partial_{\nu} K^a, \quad a \neq \text{conformal}. \quad (2.106)$$

The Lichnerowicz bound Eq. (C.14) combined with the positivity of $Q^2$ in Eq. (2.66) implies that the denominator is strictly positive, so the transformation is always valid (except in the case $d = 1$ where $R(d) = 0$, which we will treat separately).
This decouples the graviton, leaving a Fierz-Pauli massive graviton of mass squared $\lambda_a$. The remaining action for $F^a$ and $K^a$ is still kinetically mixed, and the mixing between the kinetic terms of the $F^a$ and $K^a$ scalars can be removed by the transformation,

$$F^a \rightarrow F'^a - \frac{d \lambda_a Q}{d \left(1 - \frac{1}{N}\right) \lambda_a - R(d)} K^a , \ a \neq \text{conformal} . \quad (2.107)$$

Again the denominator is never zero. The resulting mass matrix mixing the $F^a$ and $K^a$ scalars has eigenvalues

$$m^2_{a\pm} = \lambda_a + A \pm \sqrt{A^2 + 4\lambda_a B} , \ a \neq \text{conformal} , \text{ where } \begin{cases} A \equiv \frac{(d-2)(N-1)Q^2}{d+N-2} - \frac{R(d)}{d} , \\ B \equiv \frac{(d-1)(N-1)Q^2}{d+N-2} . \end{cases} \quad (2.108)$$

Now consider the conformal sector. We can diagonalize by the transformation

$$h^a_{\mu\nu} \rightarrow h'^a_{\mu\nu} - \frac{d \sqrt{\lambda_a} Q}{d(d-1)\lambda_a - (d-2)R(d)} K^a g_{\mu\nu} - \frac{d(d-2)Q}{d \sqrt{\lambda_a}} \partial_{\mu} \partial_{\nu} K^a , \ a = \text{conformal} ,$$

which is valid since the denominator is always strictly positive by $\lambda_a = \frac{R(d)}{N-1}$ for conformal scalars, combined with the positivity of $Q^2$ in Eq. (2.66). This gives the final Lagrangian for the conformal part,

$$\frac{\mathcal{L}_a}{\sqrt{-g}} \supset \sum_{a=\text{conformal}} \varepsilon(h^a) - \frac{1}{2} \lambda_a (|h^a_{\mu\nu}|^2 - |h^a|^2)$$

$$+ \frac{d(d + N - 2) \left(Q^2 + \frac{R(d)}{d}\right)}{(d + N - 2)R(d) + Nd(d - 1)Q^2} \left[ - \frac{1}{2} |\partial K^a|^2 - \frac{1}{2} \left( \lambda_a + \frac{2N(d - 1)}{d + N - 2} Q^2 \right) |K^a|^2 \right] .$$

The denominator of the kinetic term for the scalars is always strictly positive. Also, recall that $Q^2 + \frac{R(d)}{d} = \frac{R(N)}{N}$ by Eq. (2.66). Thus, there are no ghosts for a positively curved internal space, $R(N) > 0$. For a flat or negatively curved internal space, $R(N) = 0$ or $R(N) < 0$, no conformal scalars exist (see Appendix C).

The masses of these scalars are

$$m^2_a = \lambda_a + 2N\frac{(d - 1)}{d + N - 2} Q^2 , \ a = \text{conformal} . \quad (2.111)$$

The mass is always strictly positive, which is seen by using $\lambda_a = \frac{R(N)}{N-1}$ combined with the positivity of $Q^2$ in Eq. (2.66).

\textbf{i modes:} The Lagrangian $\mathcal{L}_i$, (2.100), describes two vectors for each $i$. The mass matrix has eigenvalues,

$$m^2_{i\pm} = \lambda_i - \frac{R(d)}{d} \pm \sqrt{\left( \frac{R(d)}{d} \right)^2 + 2Q^2 \lambda_i} . \quad (2.112)$$
The positive root is always greater than zero. The negative is always greater than or equal to zero, and is equal to zero precisely when $i$ corresponds to a Killing vector. Recall from Appendix B that $\lambda_i \geq \frac{2R(N)}{N}$, with equality if and only if $i$ corresponds to a Killing vector. In the case $R(N) > 0$, we may have such Killing vectors, and the negative root of Eq. (2.112) will give one massless vector for each of the Killing vectors. For the Killing vectors, the positive root remains massive, corresponding to the gauge invariant combination of the two vectors $\tilde{A}_i^\mu$ and $\tilde{a}_i^\mu$, and the negative root corresponds to a gauge non-invariant combination which transforms as a massless vector should.

In the case $R(N) \leq 0$, we do not have Killing vectors among the $i$’s, because they don’t exist when $R(N) < 0$, and because they are harmonic and hence included among the $\alpha$’s when $R(N) = 0$. The bound $\lambda_i \geq 0$ on the vector Laplacian ensures that the squared masses Eq. (2.112) are not less than 0.

Finally, let us analyze the spectrum of the harmonic sector $L_\alpha$ in Eq. (2.101). For $\alpha \neq $ Killing, we diagonalize by the transformation
\[
\tilde{A}_\mu^\alpha \rightarrow \tilde{A}_\mu^\alpha - \frac{dQ}{2R(d)} \partial_\mu \tilde{c}^\alpha, \quad \alpha \neq \text{Killing}.
\]

Note that we do not have to worry about $R(d) = 0$ in the denominator of the transformation Eq. (2.113), because in this case Eq. (2.66) tells us that $R(N) > 0$, which by the arguments of Appendix B implies that there are no harmonic vectors. We have,
\[
\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2} |\tilde{E}_\mu^\alpha|^2 + \frac{2R(N)}{d} |A_\mu^\alpha|^2 - \frac{1}{2} \left(1 + \frac{d}{R(d)} Q^2\right) |\partial_\mu \tilde{c}^\alpha|^2, \quad \alpha \neq \text{Killing}.
\]

We find massless scalars when $R(N) > 0$, which are never ghostly since $1 + \frac{d}{R(d)} Q^2 > 0$ when $R(N) > 0$, and vectors of mass squared $m^2 = -\frac{2R(N)}{d} = 2Q^2 - \frac{2R(N)}{N}$. This mass squared is always greater than zero, since harmonic vectors do not exist on positively curved manifolds.

The case $R(N) = 0$ is the case when any harmonic vectors are also Killing, so we do not need to diagonalize in this case, and we have only massive vectors of mass $m^2 = -\frac{2R(N)}{d}$, which is always positive because by Eq. (2.66) $R(d) < 0$ when $R(N) = 0$ and $Q^2 > 0$.

$d = 2$ case:

In $d = 2$, gravitons are non-dynamical, so we must solve for the gravitons as auxiliary fields through their equations of motion. The kinetic part of the graviton, $\sqrt{-g} \epsilon(h)$ with $\epsilon(h)$ as in Eq. (2.38), is a total derivative. Dropping this total derivative, the zero mode sector after reduction has the Lagrangian
\[
\frac{\mathcal{L}_0}{\sqrt{-g}} = h^{\mu\nu,0} (\nabla_\mu \nabla_\nu \phi^0 - \Box \phi^0 g_{\mu\nu}) - \frac{R(2)}{2} h^0 \phi^0
\]
\[
+ \frac{1}{2} \left(1 - \frac{1}{N}\right) (\partial \phi^0)^2 - \frac{1}{2} \left(1 - \frac{2}{N}\right) \frac{R(2)}{2} + \left(1 - \frac{1}{N}\right) Q^2 \right)(\phi^0)^2.
\]

This is just a linearization of a version of dilaton gravity in two dimensions [38].
The part of the reduced action corresponding to scalar eigenfunctions of the Laplacian becomes

\[
\frac{\mathcal{L}_a}{\sqrt{-g}} = \sum_{\alpha \neq \text{conformal}} -\frac{1}{2} \lambda_a |\tilde{h}^{a}_{\mu
u}|^2 + \frac{1}{2} \lambda_a |\tilde{h}^{a}|^2 \\
+ \frac{1}{2} \left\{ \tilde{h}^{\mu\nu, a*} (\nabla_{\mu} \nabla_{\nu} F^a - \Box F^a g_{\mu\nu}) + \left[ \left( 1 - \frac{1}{N} \right) \lambda_a - \frac{R(2)}{2} \right] \tilde{h}^{a*} F^a + Q \lambda_a \tilde{h}^{a*} K^a + \text{c.c.} \right\} \\
+ \frac{1}{2} \left( 1 - \frac{1}{N} \right) |\partial F^a|^2 + \frac{1}{2} \left( 1 - \frac{2}{N} \right) \left[ \left( 1 - \frac{1}{N} \right) \lambda_a - \frac{R(2)}{2} \right] |F^a|^2 - \left( 1 - \frac{1}{N} \right) Q^2 |F^a|^2 \\
- \frac{1}{2} \lambda_a |\partial K^a|^2 - \frac{1}{2} \lambda_a^2 |K^a|^2 - \frac{1}{2} \lambda_a Q \{ K^{a*} F^a + \text{c.c.} \} \\
+ \sum_{\alpha = \text{conformal}} -\frac{1}{2} \lambda_a |\tilde{h}^{a}_{\mu
u}|^2 + \frac{1}{2} \lambda_a |\tilde{h}^{a}|^2 + \frac{1}{2} \left\{ Q \sqrt{\lambda_a} \tilde{h}^{a*} K^a + \text{c.c.} \right\} - \frac{1}{2} |\partial K^a|^2 - \frac{1}{2} \lambda_a |K^a|^2 .
\]

(2.116)

For the non-conformal part, varying with respect to \( h^{\mu\nu, a} \) gives us an expression for the non-dynamical graviton,

\[
h^{a}_{\mu\nu} = \frac{1}{\lambda_a} \nabla_{\mu} \nabla_{\nu} F^a - \frac{1}{\lambda_a} \left[ \left( 1 - \frac{1}{N} \right) \lambda_a - \frac{R(2)}{2} \right] F^a g_{\mu\nu} + Q K^a g_{\mu\nu} , \quad a \neq \text{conformal} .
\]

(2.117)

Substituting this back into the action, we find

\[
\frac{\mathcal{L}_a}{\sqrt{-g}} \supset \sum_{\alpha \neq \text{conformal}} -\frac{1}{2} \left[ \left( 1 - \frac{1}{N} \right) - \frac{R(2)}{2\lambda_a} \right] |\partial F^a|^2 - \frac{1}{2} \left( 1 - \frac{R(2)}{\lambda_a} \right) \left[ \left( 1 - \frac{1}{N} \right) \lambda_a - \frac{R(2)}{2} \right] |F^a|^2 \\
- \left( 1 - \frac{1}{N} \right) Q^2 |F^a|^2 - \frac{1}{2} \lambda_a (\partial K^a)^2 - \frac{1}{2} \left( \lambda_a + 2Q^2 \right) \lambda_a |K^a|^2 \\
+ \frac{1}{2} \left[ \left( -3 + \frac{2}{N} \right) \lambda_a + R(2) \right] Q K^{a*} F^a - Q \partial_{\mu} F^{a*} \partial^{\mu} K^a + \text{c.c.} \right\} .
\]

(2.118)

The kinetic cross-terms can be eliminated by the transformation,

\[
F^a \rightarrow F^{a'} - \frac{Q \lambda_a}{\left( 1 - \frac{1}{N} \right) \lambda_a - \frac{R(2)}{2}} K^a , \quad a \neq \text{conformal} .
\]

(2.119)

As with the previous cases, the denominator is never zero. After canonically normalizing the kinetic terms, the resulting mass matrix has eigenvalues

\[
m^2_{\pm} = \lambda_a - \frac{R(2)}{2} \pm \sqrt{\frac{R^2(2)}{4} + 4 \left( 1 - \frac{1}{N} \right) Q^2 \lambda_a} , \quad a \neq \text{conformal} .
\]

(2.120)

This is exactly the same as the earlier formula Eq. (2.108) restricted to \( d = 2 \).
For the conformal sector, the equations of motion give
\[ h^a_{\mu\nu} = \frac{Q}{\sqrt{\lambda_a}} K^a g_{\mu\nu} \], \quad a = \text{conformal} \quad (2.121)\]

Substituting back into the action gives
\[ \mathcal{L}_a \supset \sum_{a=\text{conformal}} \left( -\frac{1}{2} |\partial K^a|^2 - \frac{1}{2} (\lambda_a + 2Q^2) |K^a|^2 \right), \quad a = \text{conformal} \quad (2.122) \]

\[ \text{d = 1 case:} \]
In \( d = 1 \) we have \( R_{(d)} = 0 \). All the graviton kinetic terms, Fierz-Pauli mass terms, and vector kinetic terms vanish identically. All that remains of \( \mathcal{L}_0 \) is the volume modulus \( \phi^0 \) with mass \( m^2 = 2 \left( 1 - 1/N \right) Q^2 \). All that remains of the graviton in \( \mathcal{L}_a \) for the non-conformal modes are cross terms ~ \( hF, hK \), so \( h^a \) is a multiplier which sets \( K^a = -N^{-1} F^a \). Plugging back in, we find \( F^a \) remains as a free scalar with mass \( m^2 = \lambda_a - 2Q^2 \). In the case of the conformal modes, all that remains of the graviton is a cross term ~ \( hK \), so \( h^a \) is a multiplier which sets \( K^a = 0 \). Nothing dynamical remains of the vectors in \( \mathcal{L}_i \), since the vector kinetic terms vanish. \( \mathcal{L}_\alpha \) is empty because \( R_{(d)} = 0 \) means \( R_{(N)} > 0 \) so there are no harmonic vectors. Thus the spectrum for \( d = 1 \) is just the zero mode, the conformal scalars of \( \mathcal{L}_a \), and the scalars of \( \mathcal{L}_T \).

**Spectrum**

In summary, the spectrum is

- One massless graviton (dilaton gravity in \( d = 2 \)),
- One zero mode scalar for the volume modulus with a curvature-dependent mass \( m_0^2 = -\frac{2R_{(N)}}{N} + \frac{2N(d-1)}{d+N-2} Q^2 \) (part of dilaton gravity in \( d = 2 \)),
- A tower of Fierz-Pauli massive gravitons, one for each eigenvector of the scalar Laplacian with eigenvalue \( \lambda_a > 0 \), with mass \( m_a^2 = \lambda_a \),
- A tower of massive scalars, two for each non-conformal eigenvalue of the scalar Laplacian labeled by \( a \), with masses given by Eq. (2.108) (one of the pair is not dynamical for \( d = 1 \)),
- A tower of massive scalars for each conformal scalar when \( N \geq 2 \), with masses given by \( m_a^2 = \lambda_a + \frac{2N(d-1)}{d+N-2} Q^2 \) where \( \lambda_a = \frac{R_{(N)}}{N-1} \) (not dynamical for \( d = 1 \)),
- A tower of vectors for each co-exact form labeled by \( i \), with masses given by Eq. (2.112). One is massless for each Killing vector,
- Vectors with mass \( m_{\alpha}^2 = -\frac{2R_{(d)}}{d} \) for each harmonic form,
• A massless scalar for each non-Killing harmonic form labeled by $\alpha$,
• A tower of massive scalars for the transverse traceless tensor modes of the Lichnerowicz operator, in general massive with mass $m^2_I = \lambda_I - \frac{2R(N)}{N}$. They are massless when $\lambda_I = \frac{2R(N)}{N}$, i.e. for each modulus of the Einstein structure.

In addition, we have the remaining parts of the $(N-1)$-form field decomposition which do mix with the graviton.

Many of the ingredients are non-dynamical for low dimensions. For instance, massless gravitons are non-dynamical for $d \leq 3$, the massive gravitons and massless vectors are non-dynamical in $d \leq 2$, and the vector fields are non-dynamical for $d = 1$.

Let us comment on the case $N = 1$, where $N$ is the circle and the extra dimensional metric has only one component $\gamma_{yy} = 1$. There are no co-exact vectors other than the single Killing vector which is constant around the circle. This is also the only harmonic one-form. There are no transverse traceless tensors, and no non-conformal scalars (every vector is a conformal Killing vector for $N = 1$). The scalar eigenfunctions can be chosen to be simply the Fourier modes, $\psi_a = \frac{1}{\sqrt{2\pi R}} e^{iay/R}$, where $R$ is the radius of the circle and $a$ ranges over all the integers, with $a = 0$ the zero mode. The eigenvalues are $\lambda_a = a^2/R^2$. In this case, the graviton ansatz Eq. (2.73) simplifies to the simple Fourier expansion Eq. (2.52). The $p$-form field is simply a scalar, $A$, with a background value $\partial_y A = Q$. The $d$-dimensional space is negatively curved, with curvature $R(d) = -Q^2d$. The expansion Eq. (2.13) of the fluctuation of $A$ becomes a simple Fourier transform,

$$\tilde{A} = \frac{1}{\sqrt{2\pi R}} \left[ e^0 + \sum_{a=1}^{\infty} (a^a e^{iay/R} + c.c.) \right].$$

(2.123)

As for the duality relations, Eq. (2.77) is empty, and Eq. (2.76) relates the space of scalar eigenfunctions to itself and for our choice of basis must be supplemented by an additional factor of $i$ to remain true, $\psi_a = \frac{i}{\sqrt{\lambda_a}} * d\psi_a$. Then Eq. (2.81) remains true and reads just like Eq. (2.123) if we redefine $a^a \rightarrow -ia^a$.

In this case, we recover a massless graviton $h^0_{\mu\nu}$, a massive vector $A^0_{\mu}$ with mass squared $-2R(d)/d$, the massive dilaton $\phi^0$ with mass squared $-2R(d)/d$, a tower of massive graviton doublets $h^a_{\mu\nu}$ with longitudinal modes $A^a_{\mu}$, $\phi^a$, and a tower of scalars $K^a$. As mentioned at the end of Section 2.6, the presence of flux can be interpreted as an Anderson-Higgs mechanism: the zero mode of the zero-form field is eaten by the Kaluza-Klein vector, causing it to become massive.

**Stability**

**Gravitons:** The graviton spectrum is identical to the case of pure gravity with no flux. For gravitons on a flat or negatively curved space, stability occurs only when the Fierz-Pauli
mass term is negative, which never happens. For positively curved spaces, gravitons are stable as long as their masses are above the Higuchi bound [39],

\[ m^2 \geq \frac{d - 2}{d(d - 1)} R_{(d)}. \]  

(2.124)

Since the mass squared of the graviton is just the eigenvalue, \( m^2 = \lambda_a \), to violate or saturate the Higuchi bound we would need to have

\[ \lambda_a \leq \frac{d - 2}{d - 1} \left( \frac{R_{(N)}}{N} - Q^2 \right) \leq \frac{d - 2}{d - 1} \frac{R_{(N)}}{N}. \]  

(2.125)

This is inconsistent with the Lichnerowicz bound Eq. (C.14), \( \lambda_a \geq \frac{R_{(N)}}{N - 1} \). Thus the gravitons are always stable, the Higuchi bound is never saturated, and there is never a partially massless graviton in the spectrum.

**Vectors:** Vectors are unstable only when their mass squared is negative, which never occurs, so no instability arises from the vectors.

**Scalars:** Any instabilities always arise from the scalar modes, which may potentially become tachyonic.

- The volume modulus \( \phi^0 \) is always stable for \( R_{(d)} \leq 0 \). For a positively curved internal manifold, \( R_{(d)} > 0 \), there is a window of stability,

\[ \frac{2N(d - 1)}{d + N - 2} Q^2 \geq \frac{2R_{(N)}}{N}. \]  

(2.126)

Recall that this sector is always unstable for positive curvature in the pure graviton case, \( Q^2 = 0 \), so the flux has a stabilizing effect. In the case \( d = 2 \), where we have dilaton gravity, there are no local degrees of freedom [50], so we may call this stable in all cases if we are only concerned with local degrees of freedom.

- There may be instabilities in the \( \phi^I \), the sector corresponding to transverse traceless tensor modes of the Lichnerowicz operator. In terms of the curvature of the internal manifold, the condition for stability is identical to that of the pure gravity case: the mass squared is given by \( m^2_\lambda = \lambda_\lambda - \frac{2R_{(N)}}{N} \). These are never unstable in the case of compactifications on spheres or other quotients of maximally symmetric spaces.

- Additional instabilities can occur in the scalar sector that corresponds to scalar eigen-modes of the Laplacian. The regime of stability is governed by Eq. (2.108). In the following subsection, we will look more closely at this spectrum in a specific and familiar setup, namely for compactifications on spheres.
CHAPTER 2. KALUZA-KLEIN TOWERS ON GENERAL MANIFOLDS

Spheres

While the spectrum of scalars in the scalar eigenmode sector is complicated in general, it is simple to make contact with known stability regimes of standard Freund-Rubin compactifications when the case where the internal manifold is a sphere (see e.g. [60]). Here we summarize some results from these formulae in the case of spheres for \( d, N \geq 2 \). The scalar eigenmodes are \( \lambda_a = \frac{l(l+N-1)}{R^2} \), where \( l = 0, 1, 2, \cdots \) and \( R \) is the radius of the sphere and is related to the curvature by \( R(N) = \frac{N(N-1)}{R^2} \). The \( l = 0 \) mode is the zero modes, and its stability is governed by Eq. (2.105). The \( l = 1 \) modes are the conformal scalars, with mass squared given by Eq. (2.111), and they are always stable. The \( l \geq 2 \) modes are the non-conformal scalars, and their stability is governed by the lesser of Eq. (2.108). The \( l = 0 \) and \( l \geq 2 \) modes are the potential sources of instability. We have:

\( d > 2 \):

- \( R(d) < 0 \): The \( l = 0 \) modes are always stable. For the \( l \geq 2 \) modes, the negative solution must be compared to the Breitenlohner-Friedmann bound Eq. (2.56) to assess stability. We have,
  - \( N = 2, 3 \): Stable,
  - \( N \) even, \( N \geq 4 \): Unstable to one or more \( l \leq \frac{N}{2} \) modes when \( \Lambda \) is above some critical value which is \( > 0 \). Stable when \( \Lambda \) is below this critical value, in particular, stable when \( \Lambda \leq 0 \).
  - \( N \) odd, \( N \geq 5 \): Entire \( \Lambda > 0 \) regime is unstable to one or more \( l \leq \frac{N-1}{2} \) modes. \( \Lambda = 0 \) is stable, with \( l = \frac{N-1}{2} \) exactly saturating the Breitenlohner-Friedmann bound. \( \Lambda < 0 \) is stable.

- \( R(d) > 0 \): These solutions always have \( \Lambda > 0 \). The \( l = 0 \) modes are now stable in the range \( \frac{2(d-2)}{d-1} \Lambda \geq \frac{d+N-2}{d}R(d) \), and so are unstable below some critical value for \( \Lambda \). For the \( l \geq 2 \) modes, we have
  - \( N = 2, 3 \): Stable.
  - \( N = 4 \): Unstable to the \( l = 2 \) mode above a critical value of \( \Lambda \). There is a window of stability between this critical value and the critical value for the zero mode.
  - \( N \geq 5 \): Unstable to one or more \( l \geq 2 \) modes above a critical value of \( \Lambda \), with no window of stability between this critical value and the critical value for the zero mode.

- \( R(d) = 0 \): These solution satisfy \( \Lambda = \frac{1}{2} \left( 1 - \frac{1}{N} \right) R(N) > 0 \). The \( l = 0 \) modes are always stable. For the higher modes we have,
  - \( N = 2, 3 \): Stable,
\(- \quad N \geq 4: \) Unstable to one or more \( l \geq 2 \) modes.

In \( d = 2 \), the decoupled zero mode with mass Eq. (2.105) is replaced by a dilation gravity theory, which carries no local degrees of freedom, so there is no local instability here \([50]\). Thus, any instabilities come from only for the \( l \geq 2 \) modes. Apart from this, the main differences occur in the case \( N = 4 \). We have,

\( d = 2 \):
- \( R(d) < 0 \): An additional window of stability opens for \( N = 4 \) above a new critical \( \Lambda > 0 \). Otherwise the same as \( d > 2 \).
- \( R(d) > 0 \): We have,
  - \( N = 2, 3, 4 \): Stable,
  - \( N \geq 5 \): Unstable to one or more \( l \geq 2 \) modes when \( \Lambda \) is larger than some critical value.
- \( R(d) = 0 \): We have,
  - \( N = 2, 3, 4 \): Stable,
  - \( N \geq 5 \): Unstable to one or more \( l \geq 2 \) modes.

2.7 Discussion

We have derived the full Kaluza-Klein tower of fluctuations for scalars, vectors, \( p \)-forms, linearized gravity and flux compactifications, on a general smooth manifold of arbitrary dimension, using Hodge decomposition theorems. We have performed these calculations in a fully gauge invariant way, at the level of the action.

The dimensional reduction of linear theories has been studied in the past for large classes of compactifications, but we feel the calculations presented here, based on the action and performed in a gauge invariant manner, are more streamlined than those given previously based on the equations of motion and/or restricted to specific gauges.

For instance, we can see at the level of the action the instabilities occurring in \([60, 61, 62, 63]\). They are a general feature of Kaluza-Klein reductions; for example, any reduction of gravity on a positively curved spacetime will feature instabilities coming from the volume modulus.

We have worked with only linearized theories, so our calculations ignore any effects due to non-linear interactions. However, another virtue of our method is that it is straightforward, if tedious, to extend to interactions. One simply plugs the ansätze we have given for the fields into the full non-linear action. The interaction terms mixing the modes in \( d \) dimensions
will then have coefficients containing triple and higher overlap integrals involving the various pieces of the Hodge decompositions.
Chapter 3

Flux Compactifications Grow Lumps

3.1 Introduction

We study the \((D = p + q)\)-dimensional action

\[
S = \int d^p x \, d^q y \sqrt{-g} \left[ \frac{1}{2} M_D^{D-2} \mathcal{R} - \Lambda_D - \frac{1}{2} \frac{1}{q!} F_q^2 \right], \tag{3.1}
\]

where \(M_D\) is the \(D\)-dimensional Planck mass, \(F_q\) is a \(q\)-form flux, and \(\Lambda_D\) is a higher-dimensional cosmological constant. This action admits a simple product compactification on a \(q\)-sphere that is uniformly wrapped by the \(q\)-form flux; the remaining extended \(p\) dimensions form a maximally symmetric spacetime, either AdS, Minkowski, or de Sitter. These Freund-Rubin solutions, as they are called, are considered the simplest models of stabilized extra dimensions and have a long pedigree: they were first discussed in 1980 [57], and were soon generalized to non-zero \(\Lambda_D\) [64]. There has been a resurgence of interest lately because they provide a simple model for string compactifications [2, 65, 66, 67].

It is known that Freund-Rubin compactifications are sometimes perturbatively unstable, and also that there exist other static extrema of Eq. (3.1): [53, 60, 18, 58] showed that symmetry-breaking perturbations to the internal shape can sometimes have a negative mass squared; and [68, 69, 70] found warped static solutions in which the internal manifold is lumpy. Our goal in this paper is to give a more complete story that links these observations, and to present a phase diagram of compactified solutions and their instabilities. In particular, we argue that:

- When \(q \geq 3\), each Freund-Rubin solution is accompanied by a large number of warped solutions where the internal manifold is lumpy;
- Often, at least one of these lumpy solutions has lower vacuum energy, larger entropy, and is more stable than the symmetric solution;
- Perturbatively stable Freund-Rubin vacua have a previously undiscovered non-perturbative instability to quantum mechanically sprout lumps. We will argue that this new
Figure 3.1: A cartoon of the phase diagram of ellipsoidal solutions for the case $\Lambda_D > 0$ and $q \geq 4$. For each value of the conserved flux number $n$, there are two solutions: a symmetric solution, where the internal manifold is a perfect sphere, and a lumpy one, where the internal manifold is either a prolate or an oblate ellipsoid. There are two critical values of $n$: first, $n = n_c$, at which the warped solution crosses through the symmetric one; second, $n = n_{\text{max}}$, at which both solutions disappear spontaneously, indicated by the star. Left: a plot of ellipticity against $n$. Arrows indicate directions of decreasing free energy; they point towards the solution with smaller $\Lambda_{\text{eff}}$ and away from the solution with larger $\Lambda_{\text{eff}}$. Right: A plot of the effective cosmological constant $\Lambda_{\text{eff}}$ against $n$. The dominant solution, the one with smaller $\Lambda_{\text{eff}}$, is shown with a solid line and the subdominant solution is shown with a dashed line; the two solutions switch at $n = n_c$.

decay is often the fastest decay, proceeding exponentially faster than the two previously studied instabilities of Freund-Rubin vacua, which are flux tunneling [71, 72, 73, 74] and decompactification [75, 76, 77].

For example, Fig. 1 shows a sample partial phase diagram of solutions where the internal manifold is ellipsoidal. For each value of the conserved flux number $n$, there are two solutions: the symmetric Freund-Rubin solution, and a lumpy solution where the internal manifold is deformed away from spherical. In this case, there is a critical value $n = n_c$ at which the lumpy solution and the symmetric solution cross: for $n < n_c$, the lumpy solution is oblate (M&M-shaped); for $n > n_c$, when the lumpy solution exists it is prolate (football-shaped). The arrows in Fig. 1 show the directions along which the free energy is decreasing; arrows point towards the dominant vacuum and away from the subdominant vacuum.

The structure of this phase diagram can be captured by the cartoon effective potential in Fig. 3.2. In the effective potential picture, we imagine treating the ellipticity of the internal manifold as a $p$-dimensional field living in an effective potential; static solutions correspond to extrema and the value of the potential at an extremum is the effective cosmological constant $\Lambda_{\text{eff}}$. The $n < n_c$ behavior is shown in the left panel: the symmetric solution
symmetric solution unstable
\((n < n_c)\)

\[ \frac{V_{\text{eff}}}{M_p^p} \]

ellipticity

\[ M&M \]

spherical

football

symmetric solution stable
\((n_c < n < n_{\text{max}})\)

\[ \frac{V_{\text{eff}}}{M_p^p} \]

ellipticity

\[ M&M \]

spherical

football

Figure 3.2: A cartoon of the effective potential in the ellipticity direction. We will argue that the effective potential tends to \(+\infty\) as the internal manifold becomes increasingly M&M-shaped, and to \(-\infty\) as the internal manifold becomes increasingly football-shaped. The symmetric solution is always an extremum of this effective potential. Whether the warped solution is football- or M&M-shaped, and whether it is dominant or subdominant is determined solely by the classical stability of the symmetric solution. When the symmetric solution has negative mass squared (left panel), the warped solution has lower free energy and is M&M-shaped; when the symmetric solution has positive mass squared (right panel), the warped solution has higher free energy and is football-shaped. All these solutions have an instability (either perturbative or non-perturbative) to becoming increasingly football-shaped.

has a negative mass squared; the lumpy solution is M&M-shaped and has lower \(\Lambda_{\text{eff}}\) than the symmetric solution. The \(n > n_c\) behavior is shown in the right panel: the symmetric solution has positive mass squared; the lumpy solution is football-shaped and has higher \(\Lambda_{\text{eff}}\) than the symmetric solution. Three quantities are linked—the sign of the mass squared of the Freund-Rubin solution, whether the lumpy solution is prolate or oblate, and whether the lumpy solution is dominant or subdominant. If you know one of these quantities, you know the other two. In this paper, we will study different values of \(\Lambda_D\), different numbers of dimensions \(p\) and \(q\), and even higher-\(\ell\) deformations, and we will consistently find this same connection. We argue that the essential physics of all of the lumpy compactifications is captured by the two effective potentials of Fig. 3.2. (The effective potential is drawn for \(\ell = 2\) deformations, which lead to ellipsoidal solutions; we will also study higher-\(\ell\) deformations, which lead to even lumpier solutions.)

The effective potential of Fig. 3.2 also implies that these solutions have a non-perturbative instability to tunnel in the football-shaped direction. We will provide an estimate of the rate of such a decay, and argue that it is generically the fastest non-perturbative decay of the
perturbatively stable Freund-Rubin vacua.

In Sec. 2, we review the symmetric Freund-Rubin solutions and their perturbative stability. In Secs. 3-5, we numerically construct the lumpy solutions and give the full phase diagram of solutions, first focusing on the case of the $\ell = 2$ instability and the corresponding ellipsoidal solutions (Sec. 4) and then broadening our study to include higher-$\ell$ instabilities (Sec. 5). In Sec. 6, we argue that the effective potential is that given in Fig. 3.2; we also discuss shape-mode tunneling, estimate its rate, and show that it is often the fastest known decay. What happens to the compactification solution as it rolls down the effective potential becoming increasingly football-shaped is unclear—we speculate in the final section.

Preliminaries

The Einstein equations that follow from the action Eq. (3.1) are

$$M_D^{D-2} G_{MN} = T_{MN} = \frac{1}{(q-1)!} F_{M P_2 \cdots P_q} F_N^{P_2 \cdots P_q} - \frac{1}{2} F_q^2 g_{MN} - \Lambda g_{MN}, \quad (3.2)$$

where $G_{MN} \equiv \mathcal{R}_{MN} - \frac{1}{2} \mathcal{R} g_{MN}$ is the Einstein tensor, and capital Roman indices run over the full $(D = p + q)$-dimensional solution. The Maxwell equations are

$$(d F_q)_{P_2 \cdots P_q} = \nabla^M F_{M P_2 \cdots P_q} = 0. \quad (3.3)$$

The $q$-form flux $F_q$ is the exterior derivative of a flux potential $A_{q-1}$, so $F_q = d A_{q-1}$ and $d F_q = 0$.

We are interested in compactified solutions where the internal manifold is topologically a $q$-sphere and the full $D$-dimensional theory is reduced down to $p$ dimensions, with $p \geq 3$ and $q \geq 2$. We will use indices $\mu, \nu, \ldots$ that run over the $p$ dimensions and $\alpha, \beta, \ldots$ that run over the $q$ dimensions.

3.2 The Symmetric Solution

We begin by reviewing the symmetric Freund-Rubin solutions, where the $p$ extended dimensions are maximally symmetric and the $q$ internal dimensions form a $q$-sphere uniformly wrapped by flux. The defining features of this solution follow from the fact that it is a direct product compactification. For product compactifications, Maxwell’s equations force the flux to be uniform in the extra dimensions:

$$F_q = \rho \, \text{vol}_{S^q}, \quad (3.4)$$

where $\rho$ is the flux density and $\text{vol}_{S^q}$ is the volume form on the internal $q$-sphere, which is proportional to the Levi-Civita tensor. The direct product condition also guarantees that
the $p$ extended dimensions form an Einstein space. Restricting to the case of a maximally symmetric extended space-time, the metric takes the form:

$$ds^2 = L^2 \left(-dt^2 + \cosh^2 t d\Omega_{p-1}^2\right) + R^2 d\Omega_q^2,$$

where $L$ is the curvature length of the extended dimensions and $R$ is the radius of the internal sphere. The Ricci tensor is:

$$\mathcal{R}_{\mu\nu} = \frac{p-1}{L^2} g_{\mu\nu}, \quad \mathcal{R}_{\alpha\beta} = \frac{q-1}{R^2} g_{\alpha\beta}.$$

(3.6)

When $L^{-2} > 0$, the extended dimensions form a dS$_p$ with Hubble scale $H^2 = L^{-2}$; when $L^{-2} < 0$, analytically continuing one of the angular coordinates in the $d\Omega_{p-1}^2$ reveals that the extended dimensions form an AdS$_p$ with curvature length $\ell^2_{\text{AdS}} = -L^2$. When $L^{-2} = 0$, the extended dimensions are Minkowski.

Einstein’s equations, Eq. (3.2), enforce a relation between the two curvature lengths, $L$ and $R$, and the flux density, $\rho$:

$$\frac{\Lambda_D}{M_D^p} = \frac{(p-1)^2}{2} (M_D L)^{-2} + \frac{(q-1)^2}{2} (M_D R)^{-2},$$

(3.7)

$$\frac{\rho^2}{M_D^p} = -(p-1) (M_D L)^{-2} + (q-1) (M_D R)^{-2}.$$  

(3.8)

The effective $p$-dimensional cosmological constant (measured in units of the $p$-dimensional Planck mass $M_p$) is

$$\frac{\Lambda_{\text{eff}}}{M_p^p} \equiv \frac{(p-1)(p-2)}{2} (M_p L)^{-2} = \frac{(p-1)(p-2)}{2} (M_D L)^{-2} \left( \frac{1}{M_D^q \text{Vol}_S^q} \right)^{2/(p-2)},$$

(3.9)

where in the last equality, we have used the definition $M_p^{p-2} \equiv M_D^{D-2} \times \text{Vol}_S^q$, and $\text{Vol}_S^q \sim R^q$ is the total internal volume.

The flux density $\rho$ is not a conserved quantity, but the total number of flux units $n$ is; $n$ is defined by integrating over the internal $q$-cycle:

$$n \equiv M_D^{(q-p)/2} \int_{S_q} \mathbf{F}_q = \left( \frac{\rho}{M_D^{D/2}} \right) (M_D^q \text{Vol}_S^q).$$

(3.10)

The first equality is the definition of $n$; the second is specific to the Freund-Rubin vacua. The factor of $M_D^{(p-q)/2}$ is inserted to make $n$ dimensionless.

For the special case $\Lambda_D = 0$, these relations imply the following scalings with $n$:

$$M_D R \sim n^{\frac{1}{q-1}}, \quad \text{and} \quad \frac{\Lambda_{\text{eff}}}{M_p^p} \sim -n^{-\frac{2(D-2)}{(p-2)(q-1)}}.$$  

(3.11)
Figure 3.3: The Freund-Rubin solutions. Left: When $\Lambda_D < 0$, there is a single solution for each value of $n$, and it is always AdS$_p$ ($\Lambda_{\text{eff}} < 0$). Increasing $n$ increases $\Lambda_{\text{eff}}$ towards zero from below and causes the flux density $\rho$ to fall towards an asymptote. Right: When $\Lambda_D > 0$, there are either two solutions or no solutions, depending on $n$. The small-volume branch is drawn in green; it always has smaller $\Lambda_{\text{eff}}$, and is stable against total-volume perturbations. It is AdS$_p$ for $n < n_{\text{Mink}}$ and dS$_p$ for $n_{\text{Mink}} < n < n_{\text{max}}$. The large-volume branch is drawn in orange; it is unstable to total-volume perturbations and is always dS$_p$. As $n$ is increased past $n_{\text{max}}$, the two branches merge, annihilate, and disappear. As $n \to 0$, the green lines tend to $\Lambda_{\text{eff}} \to -\infty$. The behavior of the solution in this limit is independent of $\Lambda_D$; we refer to this limit as the ‘nothing state’.

For all $n$, the $p$-dimensional spacetime is AdS ($\Lambda_{\text{eff}} < 0$). As $n \to \infty$, $R \to \infty$ and $\Lambda_{\text{eff}}$ approaches 0 from below. As $n \to 0$, both the internal volume and the effective $p$-dimensional curvature length go to zero ($R \to 0$ and $\Lambda_{\text{eff}} \to -\infty$). We will refer to the $n \to 0$ limit as the ‘nothing state’, following the terminology of [78]. The flux density $\rho \sim nR^{-q} \sim n^{-1/(q-1)}$ is inversely proportional to the number of flux units $n$. Adding flux causes the internal radius to swell so much that the density of flux decreases.

Figure 3.3 shows the behavior of $\Lambda_{\text{eff}}$ and $\rho$ as a function of $n$ when $\Lambda_D \neq 0$. The behavior is qualitatively different depending on the sign of $\Lambda_D$. When $\Lambda_D < 0$, the $p$-dimensional
spacetime is always AdS; increasing $n$ causes the internal volume to grow without bound and $\Lambda_{\text{eff}}$ to approach 0 from below. As with the $\Lambda_D = 0$ case, the flux density $\rho$ is a falling function of $n$, except that instead of asymptoting to $\rho = 0$ as $n \to 0$, $\rho$ approaches the nonzero value $\rho_{\text{asymptote}} = \sqrt{-2\Lambda_D/(p-1)}$. There are no solutions for smaller $\rho$. When $\Lambda_D > 0$, there is no longer only a single solution for each value of $n$. Instead, there is a critical value $n = n_{\text{max}}$ at which the number of solutions changes discontinuously. Below $n_{\text{max}}$ there are two solutions: a small-volume solution and a large-volume solution. The small-volume solution always has the lower value of $\Lambda_{\text{eff}}$ and (as we will see in the next subsection) is always stable against total-volume fluctuations. The large-volume solution is unstable to total-volume fluctuations: it can decrease its effective potential either by shrinking towards the small-volume branch or by expanding towards decompactification. As $n$ is raised through $n_{\text{max}}$, the small-volume solution and the large-volume solution merge, annihilate, and disappear. There are no solutions for larger $n$.

The Effective Potential

Another way to understand the Freund-Rubin solutions is in terms of a $p$-dimensional effective theory. The radius $R$ is treated as a $p$-dimensional radion field, living in an effective potential given schematically by

$$\frac{V_{\text{eff}}(R)}{M_p^p} \sim \left(\frac{1}{M_D R}\right)^{2q/(p-2)} \left[\frac{n^2}{(M_D R)^{2q}} - \frac{1}{(M_D R)^2} + \frac{\Lambda_D}{M_D^2}\right]. \quad (3.12)$$

The three terms in square brackets represent the energy density in flux, curvature, and higher-dimensional vacuum, respectively. The multiplicative factor outside the square brackets is related to the unit conversion from $D$-dimensional Planck units $M_D$ to $p$-dimensional Planck units $M_p$—the same factor that appeared in Eq. (3.9). The flux term dominates at small $R$; flux lines repel and push the sphere out to larger radius. The curvature is an attractive term and the two terms can interact to form a minimum of the potential. The Freund-Rubin solutions are solutions in which the scalar field remains static at an extremum of this effective potential, and the value of the potential at that extremum is $\Lambda_{\text{eff}}$.

This effective potential is plotted in Fig. 3.4 for various values of $n$. The qualitative behavior depends on the sign of $\Lambda_D$. When $\Lambda_D \leq 0$, there is only ever a single extremum, which is always an AdS minimum; increasing $n$ shifts the minimum to larger values of $V_{\text{eff}}$ and to larger values of $R$, in agreement with the results of Fig. 3. When $\Lambda_D > 0$, there are two extrema—a minimum and a maximum which come together, merge, and annihilate as $n$ is increased through $n_{\text{max}}$.

At small $n$ (specifically $n \ll \left(\Lambda_D/M_D^D\right)^{(q-1)/2}$), the behavior of the minimum becomes independent of $\Lambda_D$. This is because, at such small $n$, the minimum sits at a value of $R$ that is hierarchically smaller than the higher-dimensional Hubble length $H_D^{-1} \sim M_D(\sqrt{\Lambda_D/M_D^D})^{-1}$. This hierarchy means that higher-dimensional curvature cannot affect the compactification.

To arrive at this effective potential, we treated the shape of the internal sphere as fixed, and the radius of the sphere as a dynamic field. This assumption, however, proves too
restrictive: minima of the effective potential in Fig. 4 can be unstable saddle points in additional directions in field-space that correspond to shape-mode fluctuations.

**Stability**

The full perturbative spectrum of these Freund-Rubin solutions was computed in [53, 60, 18, 58], and some diagonalized fluctuations were shown to have a negative mass squared. In this subsection, we will review the relevant parts of the calculation (deferring difficulties to those papers) and extract some information that will be relevant for us. In particular, we will leave all fluctuations turned off except for two scalar perturbations with angular momentum $\ell \geq 2$: a shape mode in which the internal manifold deforms away from sphericality and a flux mode in which the flux distribution deforms away from uniformity; that these two modes decouple from all other fluctuations is proven in [53, 60, 18, 58].

The shape mode is a fluctuation of the metric of the form:

$$
\delta g_{\mu\nu} = -\frac{1}{p-2}g_{\mu\nu}h(x)Y_{\ell}(\theta) , \quad \delta g_{\alpha\beta} = \frac{1}{q}g_{\alpha\beta}h(x)Y_{\ell}(\theta) .
$$

(3.13)

The second half of this equation says the internal sphere is deformed along the $Y_{\ell}$ spherical harmonic by an amount proportional to the $p$-dimensional field $h(x)$. The first half says that the $p$-dimensional metric $g_{\mu\nu}$ adjusts by $-hY_{\ell}/(p-2)$, introducing warping into the compactification; this adjustment is required to decouple the fluctuation from the $p$-dimensional
ized to form two linearly independent fluctuations $\psi^{\pm}$ is necessary to enforce Bianchi’s identity. Both $h$ and $a$ are dimensionless as defined.

These two fluctuations decouple from all other fluctuation modes; they can be diagonalized to form two linearly independent fluctuations $\psi^{\pm}$ which satisfy

$$\Box_x \psi^\pm = m_{\pm}^2 \psi^\pm ,$$

with

$$\psi^\pm = \left( A \pm \sqrt{A^2 + 4\lambda_q B} \right) h + 2 \frac{q - 1}{q} \rho^2 a , \quad m_{\pm}^2 = A + \lambda_\ell \pm \sqrt{A^2 + 4\lambda_q B} ,$$

and

$$A = \frac{q(p - 1)}{D - 2} \frac{\rho^2}{M_D^2} - \frac{q - 1}{M_D^2 R^2} , \quad B = \frac{(p - 1)(q - 1)}{D - 2} \frac{\rho^2}{M_D^2} .$$

Equations (3.16)-(3.18) are only valid for angular momenta $\ell \geq 2$. This is because the $\ell = 0$ flux mode is gauge (adding a constant to $A_{\mu-1}$ does not change $F_q$) and the $\ell = 1$ shape mode is gauge (perturbing a sphere by its $\ell = 1$ harmonic shifts it but does not change the induced metric on it). For $\ell \geq 2$, however, neither mode is gauge, so both $\psi^+$ and $\psi^-$ are physical fluctuations. The mode $\psi^+$ always has a positive mass squared; $\psi^-$ on the other hand is the danger mode, as it can sometimes have a negative mass squared. For this danger mode, $h$ and $a$ shift in opposite directions (sign $h = -\text{sign} a$); this means that wherever the radius gets larger, the flux density also gets larger, and vice versa (sign $\delta F_{a_1...a_q} = \text{sign} \delta g_{a\beta}$).

For instance, in the unstable $\ell = 2$ direction, when the internal manifold becomes football-shaped, the flux concentrates at the poles, and when the internal manifold becomes M&M-shaped, the flux concentrates at the equator. The $\psi^+$ mode, the safe mode, has the opposite behavior.

The $\ell$th danger mode has a negative mass squared ($m_{\pm}^2 < 0$) if and only if

$$\frac{\rho^2 R^2}{M_D^{D-2}} = \frac{1}{p - 1} \left[ (q - 1)(D - 2) - 2 \frac{\Lambda_D R^2}{M_D^{D-2}} \right] > \frac{D - 2}{2(p - 1)(q - 2)} \left[ \ell(\ell + q - 1) - 2(q - 1) \right] .$$

The implications of Eq. (3.19) are plotted in Fig. 5, and can be summarized as follows:
• When $\Lambda_D = 0$, the $\ell = (q - 1)$ mode is exactly massless; all modes between $\ell = 2$ and $\ell = (q - 2)$ have negative mass squared and all higher-$\ell$ modes have positive mass squared.

• When $\Lambda_D \neq 0$, the $n \to 0$ limit has the same stability properties as the $\Lambda_D = 0$ case. In this limit (which we label as ‘nothing’ in Fig. 5) $R \to 0$, $\rho \to \infty$, and the value of $\rho^2 R^2$ in Eq. (3.19) becomes independent of $\Lambda_D$.

• When $\Lambda_D > 0$, there are fewer $\ell \geq 2$ modes that have a negative mass squared, and increasing $R$ tends to make more modes stable. The ‘Nariai’ solution, with $\rho = n = 0$, is unstable to the total-volume mode (which has $\ell = 0$) but stable to all modes with $\ell \geq 2$.

• When $\Lambda_D < 0$, there are more $\ell \geq 2$ modes that have a negative mass squared, and increasing $R$ tends to make more modes unstable. As $n$ and $R$ go to infinity, eventually all danger modes will develop a negative mass squared.

For example, let us look in detail at the case $p = q = 4$. When $\Lambda_D = 0$, the $\ell = 2$ mode has a negative mass squared, the $\ell = 3$ mode is perfectly massless, and all higher modes are massive. When $\Lambda_D < 0$, increasing $n$ makes higher and higher $\ell$ modes develop

Figure 3.5: Equation (3.19) gives the critical value of $\rho^2 R^2$ at which the $\ell$th spherical harmonic develops a negative mass squared. The larger $\ell$ is, the larger the value of $\rho^2 R^2$ at which the mode becomes negative. When $\Lambda_D = 0$, there is a fluctuation with $\ell = q - 1$ that is exactly massless, and there are fluctuations with $\ell = 2$ through $\ell = q - 2$ that have negative mass squareds. When $\Lambda_D > 0$, increasing $R$ tends to push more modes to positive mass squareds. When $\Lambda_D < 0$, increasing $R$ tends to push more modes to negative mass squareds; in fact, as $R \to \infty$, there are negative mass squared modes for all values of $\ell$. 
a negative mass squared. When $\Lambda_D > 0$, there are two critical values of $n$. First, there is the value $n = n_{\text{max}} \equiv 81 \pi^2 / [\sqrt{2} (\Lambda_8 / M_8^8)^{3/2}]$; at this value the small-volume and large-volume branches of the Freund-Rubin solutions merge and annihilate. Second, there is the value $n = n_c \equiv 32 \sqrt{3} \pi^2 / (\Lambda_8 / M_8^8)^{3/2} \sim 0.97 n_{\text{max}}$; at this value, the $\ell = 2$ fluctuation about the small-volume branch is perfectly massless. For all $n < n_{\text{max}}$, the large-volume branch is unstable to the total-volume mode, but stable to all higher-$\ell$ modes. The small-volume branch is always stable to the total-volume mode and to all modes with $\ell \geq 3$, but the mass squared of the $\ell = 2$ danger mode changes at $n = n_c$. When $n < n_c$, the danger mode has a negative mass squared, and when $n_c < n < n_{\text{max}}$, the danger mode has a positive mass squared. Only the small-volume solutions with $n_c < n < n_{\text{max}}$ are completely perturbatively stable.

(AdS compactifications can tolerate modes with small negative mass squareds and still remain stable, as long as the mass squared is above the BF bound [48]. The results summarized above concern where the modes develop a negative mass squared, not where they go unstable. The critical value where $m^2 = 0$ is the more important one for this paper; for results on stability, see [53, 60, 18, 58].)

### 3.3 The Lumpy Solutions

In the previous section, we investigated the Freund-Rubin solutions, where the internal manifold is a $q$-sphere uniformly wrapped by $q$-form flux; we saw that symmetry-breaking perturbations can sometimes have a negative mass squared. The fact that there is a critical value of $n$ at which a perturbation develops a negative mass squared implies that there must be other lumpy solutions to the same equations of motion. (Under smooth deformations that preserve the asymptotics, a local minimum of a one-dimensional function cannot become a local maximum without ejecting other extrema.) The goal of this paper is to construct these warped, lumpy solutions and to understand their properties.

These additional solutions do not have perfect symmetry and uniformity, and therefore they necessarily include warping. For simplicity, we investigate only solutions that are warped along a single internal direction—in other words, solutions that break the $\text{SO}(q + 1)$ symmetry of the Freund-Rubin compactifications down to an $\text{SO}(q)$ symmetry. We additionally assume that the internal manifold has a symmetry under exchange of the north and south poles, so the full internal symmetry will be $\text{SO}(q) \times \mathbb{Z}_2$. Our symmetry ansatz means that we study spherical harmonics $Y_{\ell \ell}$ with $\ell$ even and zero azimuthal part; a more complete study would further break this internal symmetry, but even with this restrictive ansatz, we still find a large number of lumpy solutions.

Our metric ansatz is:

$$ds^2 = \Phi(\theta)^2 \left[ -dt^2 + \cosh^2 t d\Omega_p^{2} - 1 \right] + R(\theta)^2 d\Omega_q^{2},$$

(3.20)

where $\theta$ is the angular direction singled out for warping, and $d\Omega_q^{2} = d\theta^2 + \sin^2 \theta d\Omega_{q-1}^{2}$. The
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flux is also taken to be non-uniformly distributed in the \( \theta \)-direction:

\[
F_{\alpha_1 \ldots \alpha_q} = M_D^{(q-p)/2} Q \Phi^{-p} (\theta) \epsilon_{\alpha_1 \ldots \alpha_q},
\]

where \( Q \) is a constant, and the factor of \( M_D^{(q-p)/2} \) is included to make \( Q \) dimensionless. This flux ansatz automatically satisfies both Maxwell’s equations Eq. (3.3) and Bianchi’s identity \( \text{d}F = 0 \).

Plugging Eqs. (3.20) and (3.21) into Eq. (3.2) gives one constraint equation,

\[
\frac{2 \Lambda_D R^2}{M_D^{p-2}} = \left( \frac{Q R}{M_D^{p-1} \Phi^p} \right)^2 + p(p-1) \frac{R^2 - \Phi'^2}{\Phi^2} - 2p(q-1) \frac{R \cot \theta + R' \Phi'}{\Phi} + (q-1)(q-2) \frac{R^2 - 2RR' \cot \theta - R'^2}{R^2},
\]

and two dynamic equations of motion,

\[
(D-2) \frac{R'}{R} = -p \left( \frac{Q R}{M_D^{p-1} \Phi^p} \right)^2 - p(p-1) \frac{R^2 - \Phi'^2}{\Phi^2} + p(q-p) \frac{R \cot \theta + R' \Phi'}{\Phi} + (D-2) \frac{R^2 - 2RR' \cot \theta + R'^2}{R^2},
\]

\[
(D-2) \frac{\Phi'}{\Phi} = (q-2) \left( \frac{Q R}{M_D^{p-1} \Phi^p} \right)^2 + (p-1)(q-2) \frac{R^2 - \Phi'^2}{\Phi^2} + (q-1)(p-q+2) \frac{R \cot \theta + R' \Phi'}{\Phi} + (D-2) \frac{R \Phi'}{R\Phi} - (q-1)(q-2) \frac{R^2 - 2RR' \cot \theta - R'^2}{R^2}.
\]

These equations admit a constant solution where \( R = R_0 \) and \( \Phi = \Phi_0 \); this is the Freund-Rubin solution of Sec. 2 with \( L = \Phi_0 \) and \( \rho = M_D^{(q-p)/2} Q L^{-p} \). But constants are not the only solutions.

We find non-constant solutions numerically; such lumpy solutions were first found in [68], and sample solutions that we found are shown in Fig. 3.6. The number of extrema that a solution has between \( \theta = 0 \) and \( \theta = \pi \) will turn out to be an important classifier. In Sec. 3.4, we will study solutions with one extremum, such as the middle two lines of Fig. 3.6; these solutions are ellipsoidal in shape and will turn out to be related to \( \ell = 2 \) deformations of the Freund-Rubin solutions. In Sec. 3.5, we will study solutions with even more extrema, such as the bottom line of Fig. 3.6; these solutions are even lumpier than ellipsoids and will turn out to be related to higher-\( \ell \) harmonics. The remainder of this section is devoted to a discussion of a few properties of the equations of motion, Eqs. (E.10)-(3.24).

The Case \( q = 2 \): When the internal manifold is topologically a 2-sphere, the constraint equation enforces that the Freund-Rubin solution is the only solution to the equations of motion. There are no lumpy solutions that fit our metric ansatz unless \( q \geq 3 \).

Scaling Properties: Equations (E.10)-(3.24) have a scaling symmetry that relates solutions at different values of \( \Lambda_D \). The equations of motion are invariant under the transformation

\[
\Lambda_D \rightarrow \alpha^{-2} \Lambda_D, \quad R(\theta) \rightarrow \alpha R(\theta), \quad \Phi(\theta) \rightarrow \alpha \Phi(\theta), \quad Q \rightarrow \alpha^{(p-1)} Q.
\]
This means that there is no need to study different values of $\Lambda_D$, because the physics is simply related by scaling. The one restriction on this argument is that $\alpha^2$ is necessarily positive, so this scaling transformation does not map positive $\Lambda_D$ to negative $\Lambda_D$. There are thus three cases to consider: $\Lambda_D < 0$, $\Lambda_D = 0$, and $\Lambda_D > 0$.

When $\Lambda_D = 0$, the equations have an additional scaling symmetry that relates solutions at different values of $Q$. The $\Lambda_D = 0$ version of the equations of motion are invariant under the transformation

$$Q \rightarrow \beta^{p-1} Q , \quad \Phi(\theta) \rightarrow \beta \Phi(\theta) , \quad R(\theta) \rightarrow \beta R(\theta) \quad \text{(when } \Lambda_D = 0).$$

(3.26)

In this case, we need only consider a single value of $Q$, because solutions with different amounts of flux are related by rescaling the field profiles $R(\theta)$ and $\Phi(\theta)$. In other words, when $\Lambda_D = 0$, adding more flux increases the total internal volume, but does not affect the shape or flux distribution of the solution.

**Boundary Conditions:** Regularity at the north and south poles demands that $R'(0) = \Phi'(0) = R'(\pi) = \Phi'(\pi) = 0$; these conditions are also implied by the equations of motion at the poles. As discussed, we also assume the internal manifold has a $\mathbb{Z}_2$ symmetry that relates the north and south poles; this extra symmetry implies that both $R$ and $\Phi$ are even functions about the equator $\theta = \pi/2$.

To find solutions, we use a shooting technique. We set initial conditions for $R$ and $\Phi$ at the equator, and numerically evolve to one of the poles. For most initial conditions, regularity is not satisfied at the poles, but there is a measure zero set of initial conditions at $\theta = \pi/2$ that gives the appropriate behavior at $\theta = 0$ and $\pi$: these are our solutions. Initial conditions at the equator are $R(\pi/2)$ and $\Phi(\pi/2)$; the constraint equation solves for $R(\pi/2)$, so for each value of $Q$ and $\Lambda_D$, we scan over values of $\Phi(\pi/2)$ to find solutions—there can be several for each set of $Q$ and $\Lambda_D$.

**Computing $\Lambda_{\text{eff}}$:** To compute $\Lambda_{\text{eff}}/M_p$ for a numeric solution, we use the action density. Numeric solutions are plugged into the action Eq. (3.1), which is then integrated over the internal dimensions. The result is compared against the formula for a $p$-dimensional maximally symmetric spacetime with cosmological constant $\Lambda_{\text{eff}}$ and Planck mass $M_p$:

$$S_E = \int d^p x M_p \sqrt{g_p} \frac{2}{p - 2} \left( \frac{\Lambda_{\text{eff}}}{M_p^p} \right).$$

(3.27)

For the symmetric solutions, this definition agrees with the formula Eq. (3.9). The smaller $\Lambda_{\text{eff}}/M_p$, the smaller the free energy; the solution with the lowest value of $\Lambda_{\text{eff}}$ at a given value of the conserved flux number $n$ is the dominant vacuum.

**Embedding:** As a visualization tool, we will sometimes plot the internal metric as an embedding in $q+1$ Euclidean dimensions; the right column of Fig. 3.6 gives such an embedding. The internal metric $d s^2 = R(\theta)^2 d \Omega_q^2$ can be realized as the induced metric on the surface...
Figure 3.6: Four sample solutions to Eqs. (E.10)-(3.24), all with \( p = q = 4 \). The top solution, labeled ‘symmetric’ \([\Lambda_D/M_D^D = 1, Q = 446 \text{ and } M_D\Phi(\pi/2) = 5.10]\) is of the Freund-Rubin type, where \( R \) and \( \Phi \) are constant. The next two solutions, labeled ‘football’ \([\Lambda_D/M_D^D = 1, Q = 34.4 \text{ and } M_D\Phi(\pi/2) = 3.36]\) and ‘M&M’ \([\Lambda_D/M_D^D = 1, Q = 58.6 \text{ and } M_D\Phi(\pi/2) = 2.55]\), both have a single extremum between \( \theta = 0 \) and \( \theta = \pi \). These solutions are ellipsoidal in shape and related to the \( \ell = 2 \) instability. The last solution, labeled ‘Higher \( \ell \)’ \([\Lambda_D/M_D^D = -1, Q = 5.83 \text{ and } M_D\Phi(\pi/2) = 1.37]\) has three extrema between \( \theta = 0 \) and \( \theta = \pi \). There can be many solutions with the same value of \( n \)—the top three solutions all come from the same theory, and have \( n = 0.91n_{\text{max}} \). The symmetric solution lies on the small-volume Freund-Rubin branch of Fig. 3.8, the M&M lies on the small-volume lumpy branch, and the football lies on the large-volume lumpy branch. The third column gives the internal geometry as an embedding space: when the plotted curve is rotated around the \( X_1 \)-axis, the induced metric on the resulting surface is equal to the metric of the lumpy internal manifold. Football solutions are prolate and \( R \) reaches a minimum at the equator (\( \theta = \pi/2 \)); M&M solutions are oblate and \( R \) reaches a maximum at the equator. The field \( \Phi \) always has the opposite behavior, so that the flux density always gets larger where \( R \) gets smaller, and vice versa.
\[ X_1 = \int_{\pi/2}^{\theta} d\theta' \sqrt{R(\theta')^2 - \left[ \partial_{\theta'} (R(\theta') \sin \theta') \right]^2} \]  
\[ X_2 = R(\theta) \sin \theta \cos \theta_2 \]  
\[ X_3 = R(\theta) \sin \theta \sin \theta_2 \cos \theta_3 \]  
\[ \vdots \]  
\[ X_q = R(\theta) \sin \theta \sin \theta_2 \cdots \sin \theta_{q-1} \cos \theta_q \]  
\[ X_{q+1} = R(\theta) \sin \theta \sin \theta_2 \cdots \sin \theta_{q-1} \sin \theta_q \]  

in \((q+1)\)-dimensional flat space with coordinates \((X_1, \ldots, X_{q+1})\). Where available, we will give embedding surfaces because they provide an intuitive way to picture the internal geometry. When \(R(\theta)^2 < \left[ \partial_{\theta} (R(\theta) \sin \theta) \right]^2\), i.e. when \(R(\theta)\) is far from constant, the embedding surface is no longer real; instead it pivots into the complex \(X_1\)-plane. In this case, analytically continuing \(X_1 \rightarrow iX_1\) realizes the metric as an embedding in \((q+1)\)-dimensional Minkowski space, though perhaps this final step belies any gains in intuition.

### 3.4 Ellipsoidal Solutions and the \(\ell = 2\) Instability

We first examine the ellipsoidal solutions, which have a single extremum between \(\theta = 0\) and \(\theta = \pi\). The middle two rows in Fig. 3.6 are examples of ellipsoids, and we will argue that these solutions are related to the \(\ell = 2\) perturbations of the Freund-Rubin solution.

The first clue that they are related to the \(\ell = 2\) instability comes from the shape of these single-extremum solutions. Not only is the field profile \(R(\theta)\) roughly ellipsoidal, but the flux is distributed on the ellipsoid in the appropriate way. We saw in Sec. 2.2 that the unstable danger mode had shape mode \(h\) and flux mode \(a\) inversely correlated; likewise, all of the solutions we find have \(R\) and \(\Phi\) inversely correlated, as can be seen in Fig. 3.6. The flux density gets larger wherever the radius of the sphere gets larger, so that the flux is concentrating at the tips of the footballs and around the equator of the M&M's. This makes intuitive sense because, for these solutions, the regions with larger radius have higher curvature and it takes a larger flux density to support a region of higher curvature against collapse.

For each of these lumpy solutions, we define an order parameter that quantifies the ellipticity:

\[ \varepsilon \equiv \frac{2 \times \text{distance from north to south pole}}{\text{distance around equator}} = \frac{2 \int_0^{\pi} R(\theta) d\theta}{2\pi \int_0^{\pi/2} R(\theta = \pi/2)}. \]  

Figure 3.7 shows the definition of \(\varepsilon\) pictorially. Football-shaped solutions have \(\varepsilon > 1\) and M&M-shaped solutions have \(\varepsilon < 1\). (We opted for this definition of ellipticity because it is intrinsic to the internal manifold, rather than the standard definition of ellipticity which appeals to embedding space.)
Figure 3.7: The order parameter $\varepsilon$, which measures the lumpiness of the solution, is defined by the length of the blue curve that runs along a line of longitude divided by the length of the red curve that runs around the equator. M&M-shaped solutions have $\varepsilon < 1$; football-shaped solutions have $\varepsilon > 1$; Freund-Rubin solutions have $\varepsilon = 1$ by definition.

In the remainder of this section, we will present phase diagrams of ellipsoidal solutions. We will see that there are three simple principles that determine the shape of the phase diagrams:

- Every Freund-Rubin solution is accompanied by a single ellipsoidal solution with the same value of $n$.
- Whenever the $\ell = 2$ ‘danger mode’ has a negative mass squared, the ellipsoidal solution is M&M-shaped and energetically favored [$\varepsilon < 1$ and $(\Lambda_{\text{eff}}/M_p^p)^{\text{lumpy}} < (\Lambda_{\text{eff}}/M_p^p)^{\text{symmetric}}$]. The opposite is also true: Whenever the $\ell = 2$ ‘danger mode’ has a positive mass squared, the ellipsoidal solution is football-shaped and energetically disfavored [$\varepsilon > 1$ and $(\Lambda_{\text{eff}}/M_p^p)^{\text{symmetric}} < (\Lambda_{\text{eff}}/M_p^p)^{\text{lumpy}}$].
- The $n \rightarrow 0$ behavior is independent of the value of $\Lambda_D$.

**Ellipsoidal Solutions: the Case $p = q = 4$ and $\Lambda_D > 0$**

Let us first look in detail at the special case of compactifications of 8-dimensional de Sitter space down to 4 dimensions, where the internal manifold is topologically a 4-sphere. This is an interesting case study because, as we saw in the Sec. 2.2, it features a phase transition in the Freund-Rubin solutions from stability to instability.

As a reminder, there are two critical values of the flux number $n$ that arise in this case: $n_{\text{max}}$ and $n_c \sim 0.97 n_{\text{max}}$. When $n < n_c$, the small-volume branch is unstable to becoming ellipsoidal while the large-volume branch is stable to it; when $n_c < n < n_{\text{max}}$, both branches are stable to it; when $n > n_{\text{max}}$, there are no Freund-Rubin solutions.
Figure 3.8: A phase diagram of the ellipsoidal solutions for the case \( p = q = 4 \) and \( \Lambda_D > 0 \). Symmetric solutions have \( \varepsilon = 1 \); for \( n < n_{\text{max}} \) there are two symmetric solutions—a small-volume solution and a large-volume solution—and for \( n > n_{\text{max}} \) there are no symmetric solutions. We find that each symmetric solution is accompanied by an ellipsoidal solution with \( \varepsilon \neq 1 \), which has roughly the same internal volume. Whenever the ellipsoidal solution has \( \varepsilon > 1 \), it also has \( \Delta(\Lambda_{\text{eff}}/M_p^p) \equiv (\Lambda_{\text{eff}}/M_p^p)_{\text{lumpy}} - (\Lambda_{\text{eff}}/M_p^p)_{\text{symmetric}} > 0 \) and vice versa.

To produce this phase diagram, we numerically found lumpy solutions at a large number of values of \( Q \), and then numerically integrated each solution to find \( n \) and \( \Lambda_{\text{eff}} \). (The region with very small \( Q \) is hard to access numerically; this region corresponds to the large-volume ellipsoidal solutions with small \( n \). We were not able to find solutions in this region, and so we have drawn a straight dashed line that continues the trend.)

The results of our analysis are given in Fig. 3.8. We find that there are four solutions when \( n < n_{\text{max}} \) and zero solutions when \( n > n_{\text{max}} \); both the small-volume and the large-volume Freund-Rubin vacua are accompanied by their own lumpy branch of solutions. The large-volume lumpy branch is always football-shaped (\( \varepsilon > 1 \)) and always has a larger value of \( \Lambda_{\text{eff}}/M_p^p \). The shape of the small-volume lumpy branch depends on \( n \). As anticipated, it crosses through the Freund-Rubin solution at \( n = n_c \). When \( n < n_c \), the small-volume lumpy solution is M&M-shaped and has a smaller value of \( \Lambda_{\text{eff}}/M_p^p \), and when \( n_c < n < n_{\text{max}} \), the
The phase diagram of ellipsoidal solutions. In all cases, solutions have $\varepsilon < 1$ if and only if $\Delta(\Lambda_{\text{eff}}/M_p) < 0$ and they have $\varepsilon > 1$ if and only if $\Delta(\Lambda_{\text{eff}}/M_p) > 0$. When $\Lambda_D < 0$, increasing $n$ makes the solution more and more M&M-shaped; when $\Lambda_D = 0$ the additional scaling symmetry means that $\varepsilon$ stays constant with $n$ and only the total internal volume changes; when $\Lambda_D > 0$, there are two ellipsoidal solutions that come together, merge, and annihilate as $n$ is increased. The case $q = 3$ is special because there is no ellipsoidal solution when $\Lambda_D = 0$; this is related to the fact that the $\ell = 2$ ‘danger mode’ is precisely massless. For all higher $q$, the $\Lambda_D = 0$ ellipsoidal solution is M&M-shaped. As $n \to 0$, all three cases merge to a single solution whose volume and shape is independent of $\Lambda_D$. To make this figure, we numerically found solutions for a large number of values of $Q$ with $q = 3$, 4 and 5, and with $\Lambda_D < 0$, $\Lambda_D = 0$ and $\Lambda_D > 0$. The specific plots shown here are for $p = 4$ and $q = 3$ on the left and $p = 4$ and $q = 4$ on the right. We conjecture that this qualitative behavior continues for $q \geq 4$. (As with Fig. 8, numerics prevented us from finding solutions with very small $Q$ and $\Lambda_D > 0$, and so as before we have drawn a straight dashed line that continues the trend. Also, in the $\Lambda_D > 0$ case, the symmetric branch ends spontaneously at the same $n = n_{\text{max}}$ as the lumpy branch.)

small-volume lumpy solution is football-shaped and has a larger value of $\Lambda_{\text{eff}}/M_p$.

These results are consistent with the three principles listed at the start of this section.

Ellipsoidal Solutions: the General Case

Figure 3.9 represents the phase diagram of ellipsoidal solutions in a wider number of cases, including different numbers of internal dimensions and different signs of $\Lambda_D$. In all these cases, the three principles listed at the start of this section operate: there is a single ellipsoidal solution for each Freund-Rubin solution, and it has $\varepsilon < 1$ if and only if $\Delta(\Lambda_{\text{eff}}/M_p) < 0$, and vice versa.

The three cases $\Lambda_D > 0$, $\Lambda_D = 0$ and $\Lambda_D < 0$ behave qualitatively differently. When
$\Lambda_D > 0$, there are two branches of lumpy solutions—a small-volume branch and large-volume branch—that come together, merge, and annihilate as $n$ is increased; this is the behavior seen in the explicit example of the previous subsection. When $\Lambda_D = 0$, there is a single ellipsoidal solution for each value of $n$, and its ellipticity is constant in $n$. Increasing $n$ increases the internal volume, but does not change the shape. This is related to the extra scaling symmetry of the $\Lambda_D = 0$ version of the equations of motion, as discussed in the previous section. Finally, when $\Lambda_D < 0$, increasing $n$ makes the solution more and more M&M-shaped.

As $n \to 0$, the ellipsoidal solutions converge to a single value $\varepsilon$, independent of $\Lambda_D$. This is consistent with the fact, discussed in Sec. 2, that when $n$ is very small, the solution has a very small internal volume and there is a large separation of scales between the compactification scale and the higher-dimensional Hubble scale.

The case $q = 3$ is qualitatively different from the case $q \geq 4$. When $q = 3$ and $\Lambda_D = 0$, we find no ellipsoidal solution, whereas for all larger values of $q$, there is an ellipsoidal solution and it is M&M-shaped. The reason for this absence can be traced back to the mass squared of the $\ell = 2$ ‘danger mode’: when $q = 3$, this deformation is exactly massless for $\Lambda_D = 0$, and so we find no ellipsoid. However, when $q \geq 4$, this deformation has negative mass squared for $\Lambda_D = 0$, and so we find an M&M-shaped solution.

Once again, these results are all consistent with our three principles.

## 3.5 Higher-$\ell$ Solutions

In the previous section, we numerically studied the lumpy solutions with a single extremum between $\theta = 0$ and $\theta = \pi$. These solutions are ellipsoidal in shape and are associated with the $\ell = 2$ spherical harmonic. In this section, we report on the behavior of lumpier solutions with additional extrema between $\theta = 0$ and $\theta = \pi$. A sample lumpier solution is given in the bottom row of Fig. 3.6. These solutions with extra extrema were first found in [70]; they have much in common with the oscillating bounces of [79, 80, 81, 82]. (Our assumption of $\mathbb{Z}_2$ symmetry relating the north and south hemispheres limits our study to solutions with an odd number of extrema.)

As before, let us begin with a special case. Consider compactifying 8-dimensional AdS ($\Lambda_8 < 0$) down to 4 dimensions. This case features a phase transition where the $\ell = 4$ mode develops a negative mass squared. (In fact, increasing $n$ eventually makes perturbations with arbitrarily high $\ell$ develop negative mass squareds). The critical value of $n$ in this case is $n_{c, \ell=4} = 10 \sqrt{330} \pi^2 / |\Lambda_8/M_8|^3/2$. When $n < n_{c, \ell=4}$, the $\ell = 4$ perturbation has positive mass squared and when $n > n_{c, \ell=4}$, it has negative mass squared. Figure 3.10 shows a lumpy solution associated with the $\ell = 4$ instability for $n < n_{c, \ell=4}$, $n = n_{c, \ell=4}$ and $n > n_{c, \ell=4}$. As with the ellipsoidal solutions, we find a connection between the shape of the solution and whether it is dominant or subdominant to the Freund-Rubin solution. In particular, we find solutions that are diamond-shaped and subdominant for small $n$ and box-shaped and dominant for large $n$. 
Figure 3.10: A branch of lumpy solutions associated with $\ell = 4$ deformations of the Freund-Rubin solutions; the plots of $R(\theta)$ and $\Phi(\theta)$ for these solutions have three extrema. The geometry of the internal manifold is shown in embedding coordinates: rotating the surface around the $X_1$-axis gives a surface with an induced metric that matches the lumpy internal metric. These solutions are for the case $p = q = 4$ and $\Lambda_D < 0$. This case has a critical value of $n = n_c, \ell = 4$ at which the $\ell = 4$ danger mode develops a negative mass squared. When $n < n_c, \ell = 4$, the solutions we find always are energetically subdominant and diamond-shaped; when $n > n_c, \ell = 4$, the solutions are energetically favored and box-shaped. This type of transition is qualitatively similar to that observed for the ellipsoids of the previous section.

Our investigation of these lumpier solutions was less extensive than for the ellipsoidal solutions; we constructed solutions with one, three, and five extrema for a variety of values of $q$ and $\Lambda_D$. The behavior of these solutions always exhibits the same connection, which is the reason we conjecture that:

- Each Freund-Rubin solution is accompanied by a single solution with $k$ extrema for all $k$.
- These lumpy solutions are related to fluctuations with $\ell = k + 1$. When the $\ell$th danger mode has a negative mass squared, the lumpy solution is dominant and perturbed in one direction along the spherical harmonic $Y_{\ell}(\theta)$. When the danger mode has a positive mass squared, the lumpy solution is subdominant and perturbed in the other direction.

A more thorough treatment of these lumpier solutions would be interesting, in order to further test these principles.
3.6 The Effective Potential for Lumpiness

We have just constructed a phase diagram of solutions to Eq. (3.2); we found many branches of lumpy solutions that cross the Freund-Rubin solution. Despite this complexity, we argued that the important features are controlled by simple rules: the classical and thermodynamic stability of the Freund-Rubin solutions determines the shape and behavior of the lumpy solutions.

We will now give an effective potential description that neatly encapsulates these rules. In Sec. 2.1 we reviewed a p-dimensional effective theory that is a dimensional reduction of Eq. (3.1); we treated the shape of the internal manifold as a rigid sphere and its radius as a dynamical radion field. In the effective theory, the radion lives in the effective potential of Eq. (3.12). Extrema correspond to Freund-Rubin vacua, and the value of the potential at an extremum is $\Lambda_{\text{eff}}$. However, we have seen that the shape of the internal manifold does not always remain rigid: we must allow it to vary as well. In addition to the radion, we must treat the ellipticity $\varepsilon$ as a p-dimensional field and extend the effective potential in the $\varepsilon$-direction. (For the $\ell > 2$ modes, we should parametrize lumpiness by other order parameters and extend the effective potential in those new directions as well.)

Because the symmetric Freund-Rubin compactifications are solutions to the equations of motion, the effective potential necessarily has an extremum at $\varepsilon = 1$, so that $\partial_\varepsilon V_{\text{eff}}(\varepsilon = 1) = 0$. Whether the extremum is a local minimum or maximum follows from the calculation of the mass squared in Sec. 2.2. In fact, we saw that varying $n$ can bring about a phase transition—the Freund-Rubin solution can change from a local minimum in the $\varepsilon$-direction to a local maximum in the $\varepsilon$-direction.

In the spirit of Landau, to understand this phase transition, we look to the next higher-order term in the effective potential: the $(\varepsilon - 1)^3$ term. The presence or absence of locally cubic behavior in the potential dictates the structure of the phase transition. The potential we sketched in Fig. 3.2 exhibits such behavior, and we have seen repeatedly that the resulting phase transition fully captures the physics of the phase diagrams we constructed numerically. Had there been no cubic behavior, the phase transition would have proceeded qualitatively differently: two lumpy branches would have merged and annihilated at the symmetric solution when $n = n_c$. The cubic behavior determines the physics.

Because prolate spheroids are different from oblate spheroids, there is no symmetry forcing $V_{\text{eff}}$ to be even about $\varepsilon = 1$ and no reason for a cubic term to be absent. In fact, it can be computed explicitly in the regime near $n \sim n_c$, where the quadratic term is tuned to be small and therefore the coupling between the $\ell = 2$ mode and the higher-$\ell$ modes is also small. This technique is analogous to that used by Gubser [83] to study the lumpy black strings related to the onset of the Gregory-Laflamme instability [84].

So far, we have been considering the effective potential in a Taylor series about $\varepsilon = 1$. We will now speculate about what happens further from sphericality. There are three possibilities: first, the effective potential could continue its trend, asymptoting to $V \to +\infty$ in the infinite M&M-direction and to $V \to -\infty$ in the infinite football-direction; second, it could turn over in one or both directions, introducing more extrema; third, it could asymptote...
to a finite value. However, scanning over \( \Phi(\pi/2) \) did not reveal any such additional solutions, which we consider an argument against option two. Also, option three implies that the potential has a long flat section without an obvious symmetry protecting it, which we find problematic. This leaves option one.

Moreover, we can give an intuitive argument for option one. In the direction of increasing \( \varepsilon \), the \((q - 1)\)-sphere at the equator is shrinking to zero radius while the flux is clearing away and concentrating at the poles. Without flux to buttress it, the \((q - 1)\)-sphere’s tendency is to collapse to zero radius; the sphere’s curvature term drives the effective potential to \(-\infty\) as \( R \to 0 \). (This can be seen from the \( n = 0 \) version of Eq. (3.12).) On the other hand, in the direction of decreasing \( \varepsilon \), the \((q - 1)\)-sphere around the equator is growing to larger size, and the flux is concentrating there. The north and south poles are moving towards each other in embedding space, but this does not contribute a curvature term to the effective potential. This argument relies on the curvature term of the equatorial \((q - 1)\) sphere; when \( q = 2 \), this term is 0, and indeed we found no lumpy solutions in that case.

**Tunneling in the Football-Direction**

Not only does the effective potential explain the details of the phase transition, it also predicts a non-perturbative instability for the perturbatively stable Freund-Rubin vacua with \( n > n_c \). The minimum in the right panel of Fig. 3.2 is unstable to the quantum nucleation of bubbles, where the inside of the bubble has football-shaped internal geometry, and the outside has spherical internal geometry. These Freund-Rubin vacua were already known to have two other non-perturbative instabilities: decompactification, in which the internal manifold swells over the potential barrier in Fig. 4, and flux tunneling, in which flux discharges through a Schwinger process. A rough estimate suggests that shape-mode tunneling proceeds exponentially fastest for all but the highest de Sitter vacua, where decompactification is so fast that it is no longer semiclassical.

The fact that shape-mode tunneling typically proceeds faster than decompactification can be seen from the scale of the potential: the effective potential in the shape-mode direction is far weaker than it is in the total-volume direction. This manifests in two ways. First, characteristic values of \((\Lambda_{\text{eff}}/M_p^p)_{\text{lumpy}} - (\Lambda_{\text{eff}}/M_p^p)_{\text{small-vol FR}}\) are about two orders of magnitude smaller than values of \((\Lambda_{\text{eff}}/M_p^p)_{\text{large-vol FR}} - (\Lambda_{\text{eff}}/M_p^p)_{\text{small-vol FR}}\). Second, typical mass squareds in the shape direction are also about two orders of magnitude smaller than mass squareds in the total-volume direction. The fact that the scale of the potential is smaller suggests that shape-mode tunneling will typically beat volume-mode tunneling. We can make this statement more quantitative by using the Hawking-Moss instanton to estimate the decay rate [85]. In the Hawking-Moss process, the instanton perches uniformly on the saddle point that separates the true and false vacua, and the decay rate is given by

\[
\Gamma \sim e^{-\Delta S_E}, \quad \Delta S_E = S_{E, \text{saddle}} - S_{E, \text{instanton}}, \quad \text{where} \quad S_E \sim -\left(\frac{1}{\Lambda_{\text{eff}}/M_p^p}\right)^{(p-2)/2}.
\]

(3.34)
This gives an estimate of the true decay rate that is increasingly accurate for higher de Sitter vacua, which are also the few that are perturbatively stable to begin with. For the case $p = q = 4$ and $\Lambda_D > 0$, we compared Hawking-Moss decay rates and found that 99% of perturbatively stable vacua prefer to decay by shape-mode tunneling over decompactification. The rate of flux tunneling depends on the mass of the brane that discharges the flux. If this mass is tuned to be very small, the rate of flux tunneling can be made arbitrarily fast. However, for extremal branes whose mass is set by their charge, the no-backreaction estimate of the flux tunneling shows that it proceeds even slower than decompactification. Shape-mode tunneling is typically the fastest decay. (For those keeping track, this means that, for $p = q = 4$ and $\Lambda_D > 0$, 97% of the Freund-Rubin vacua are perturbatively unstable, 2.97% decay to footballs, and .03% decompactify non-semiclassically; the Freund-Rubin branch is decimated.)

### 3.7 Discussion

The lumpiness instability of Freund-Rubin solutions has a lot in common with the Jeans instability of uniform dust: it is a classical, symmetry-breaking instability in which energy density concentrates in regions of stronger gravity. (Other analogous examples include the Gregory-Laflamme instability of [84] and the striped-phase instability of [86, 87].) The Jeans instability is ultimately cut off by non-linear terms—collapsing dust forms stars. Does something similar happen for the classical instability of the Freund-Rubin solution?

We have investigated perturbations that break the $\text{SO}(q + 1)$ symmetry of Freund-Rubin down to $\text{SO}(q) \times \mathbb{Z}_2$. Within the dominion of this symmetry, this instability can lead in one of two directions: the flux can either concentrate at points at the poles of the solution, or it can concentrate in bands along lines of constant latitude. (M&M-shaped solutions, for example, have a band of flux around the equator and football-shaped solutions have points of flux at the poles. Higher-$\ell$ solutions combine these features; for instance the $\ell = 4$ solution in the left-most panel of Fig. 10 has flux concentrated both at the poles and in a band around the equator.) We will discuss in turn both possibilities, as parametrized by whether $\epsilon$ gets bigger or smaller from 1.

When the Freund-Rubin solution is perturbed to smaller $\epsilon$ in the left panel of Fig. 3.2, the flux starts to concentrate in a band. In this case, we found a lumpy endpoint: the M&M-shaped solution. Consistent with the correlated stability conjecture [88], classical instability is accompanied by thermodynamic instability, and the M&M’s, when they exist, always have $\Delta(\Lambda_{\text{eff}}/M_p) < 0$. However, they are not necessarily the final endpoint of the instability; they might be saddle points with further instabilities that lead to the true endpoint. Indeed, taking the analogy with the Jeans instability seriously suggests that the M&M solutions we found, and indeed all the higher-$\ell$ solutions with bands of flux, are likely unstable to perturbations that further break the internal symmetry—if the energy density is trying to clump, it won’t be satisfied by a uniform band, it will want to concentrate to a point. (Specifically, while the mechanism we’ve detailed in this paper explicitly stabilizes the harmonics with zero
azimuthal part, the harmonics with non-zero azimuthal part, which break the remaining symmetry, could still be unstable.

When the Freund-Rubin solution is perturbed to larger $\varepsilon$ in the left panel of Fig. 3.2, the flux starts to concentrate at the tips. In this case, we did not find an endpoint; instead, we argued that the effective potential continues its downward trend forever. What happens to the solution as it rolls down the effective potential, becoming increasingly football-shaped, is unclear; we highlight two possibilities. The first is analogous to the endpoint of the Jeans instability: as the flux concentrates at the poles, it can collapse to form a soliton supported by non-linear terms in the potential—such solutions might resemble the rugby-ball solutions of [89, 90]. The second is more radical: as the flux concentrates at the poles, the equatorial $(q-1)$-sphere is unsupported by flux; when a sphere is unbuttressed, it can shrink to zero size in finite time, pinching off in a process described in [91, 78]. Perhaps becoming football-shaped is the first step towards the sphere ripping itself in two.

The perfect symmetry of the Freund-Rubin solutions is not enough to ensure their stability. Unstable shape-mode perturbations break the internal symmetry spontaneously. An important question is to what extent these Freund-Rubin solutions serve as toy models for the far more complex string compactifications. For vacua not protected by supersymmetry [92, 93, 94, 95, 96, 97], one may worry that the lesson here carries over: symmetries may be broken, lumps may grow.
Chapter 4

Multi-Vacuum Initial Conditions and the Arrow of Time

4.1 Introduction

The entropy in our past light-cone has increased by $\Delta S > 10^{103}$ since the era of big-bang nucleosynthesis. Moreover, the universe is still far from a state of maximum entropy today. The arrow-of-time problem is the challenge of explaining these observations.

Naively, it seems sufficient to posit an initial state of low entropy, at the earliest time when a semi-classical description of the universe is possible. However, it has recently become clear that this assumption is neither necessary nor sufficient for the observed arrow of time. The vacuum structure of the underlying theory plays a crucial part. This is particularly important if the theory contains vacua with positive cosmological constant, as it must [7, 6].

In a stable de Sitter vacuum, no arrow of time is predicted, independently of initial conditions [20]. Conversely, there are vacuum landscapes such as the string landscape, which lead to an arrow of time even if the initial entropy is arbitrarily larger than the entropy at the time of nucleosynthesis [98]. This result builds on earlier analyses of the possible predominance of Boltzmann brains [20, 99, 100]. A Boltzmann brain is an observer produced by a minimal fluctuation from equilibrium. Such observers see only an arrow of time large enough for their own existence. They do not see a highly ordered world around them. The dominance of Boltzmann brains is equivalent to the absence of an arrow of time.

All previous analyses [20, 99, 100, 98] relied on the simplifying assumption that initial conditions have support only in a single vacuum, in identifying conditions for the dominance or absence of Boltzmann brains. In general, however, a theory of initial conditions may assign nonzero probability to a number of different vacua. Initial conditions may even have some support in excited states above the metastable vacua. In this paper, we will explore how generalized initial conditions affect the conditions under which an arrow of time emerges.

There is a reason to suspect that a small correction to single-vacuum initial conditions might destroy the prediction of an arrow of time, even if nearly all of the initial probability
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is concentrated in one vacuum. A necessary condition for an arrow of time is the complete inability of the initial vacuum to produce Boltzmann brains directly. This is not a problem in a realistic landscape, since one expects that the overwhelming majority of vacua, including the one selected by initial conditions, will not give rise to the fine-tuned low-energy physics that allows for complex structures such as observers to exist, whether they form by classical evolution or by fluctuations. However, if all or most vacua have at least a tiny probability to be the initial state, then this will include fine-tuned vacua which can produce Boltzmann brains, such as our own vacuum. A more careful quantitative analysis is needed to understand how much initial probability in Boltzmann-producing vacua can be tolerated such that Boltzmann brains remain nevertheless suppressed.

A realistic vacuum structure must contain our own vacuum. This means it must contain at least one de Sitter vacuum that is stable over cosmological timescales. Moreover, observation tells us that the age of the universe is comparable to the timescale set by the cosmological constant. This implies that the universe is eternally inflating unless the decay rate of our vacuum is fine-tuned [101]. Eternal inflation requires a regulator, or measure. Here, we use the causal patch measure [15, 102]. Closely related local measures, such as the fat geodesic [99, 103] or the Hubbletube [13] are equivalent for the purpose of the (relatively crude) question of whether an arrow of time is predicted. Finally, there also exist global measures, such as light-cone time [104, 12, 16] and scale factor time [105]. They are exactly equivalent to local measures with a particular choice of initial conditions. (Indeed, the initial conditions we study in Sec. 4.6 are those dictated by this equivalence. They are dominated by the longest-lived vacuum, and we determine whether they can be approximated by it for the purpose of the arrow of time.)

We assume that singularities are terminal (see Ref. [106] for an alternative assumption). Worldlines that enter vacua with negative cosmological constant end at the big crunch. The importance of terminal vacua for the existence of some arrow of time was recently emphasized in Refs. [107, 108]. In this paper, our interest is in a stronger condition required for agreement with observation: that the flow towards terminal vacua is strong enough to avoid the dominance of Boltzmann brains over ordinary observers.

Outline and Summary: We establish notation and briefly review important approximation techniques in Sec. 4.2. In Sec. 4.3, we gain some intuition by considering two toy landscapes; for each, we study conditions for the generalization from single- to multiple-vacuum initial conditions to destroy the prediction of an arrow of time.

In Sec. 4.4, we consider a general landscape subject to certain assumptions about its structure; the string landscape is expected to satisfy these assumptions. We briefly review the analysis of the arrow of time in the case of single-vacuum initial conditions [98]. We then consider multiple-vacuum initial conditions. We identify sufficient conditions both for the absence, and for the presence of an arrow of time.

In the remaining sections, we consider two specific proposals for multiple-vacuum initial conditions that may give large weight to vacua with small cosmological constant. (This is the only class of proposals of for which the question is nontrivial. With initial conditions in
vacua with large cosmological constant—e.g., Refs. [109, 110]—an arrow of time is typically predicted.)

In Sec. 4.5, we show that Hartle-Hawking initial conditions lead to Boltzmann brain dominance unless the initial vacuum that dominates paths to ordinary observers is extremely long-lived. This is a significant deviation from the result that would be obtained in the single-initial-vacuum approximation, even though the initial probability distribution is overwhelmingly dominated by a single vacuum.\(^1\)

In Sec. 4.6, we consider initial conditions set by the dominant eigenvector of the rate equations of eternal inflation. These initial conditions are selected by global duals of the local measures we consider; they describe the late-time attractor regime of the global solution. The distribution is dominated by the longest-lived metastable vacuum, and we find that the single-initial-vacuum approximation is reliable in this case. We show that the probabilities for other initial vacua are much smaller than the amplitude to transition dynamically to such vacua from the longest-lived vacuum. Thus, if an arrow of time is predicted in the single-initial-vacuum approximation, the same conclusion is obtained with the full eigenvector. This implies that global measures such as the light-cone time cutoff or scale factor time cutoff are in accord with the observed arrow of time subject to certain conditions on the vacuum structure.

### 4.2 Conventions and Approximations

#### Branching Ratios

A landscape has a collection of stable or metastable vacua labeled by the index \(i\), each with a cosmological constant \(\Lambda_i\). For simplicity we assume these vacua have been labeled in order of increasing cosmological constant,

\[
\Lambda_1 < \Lambda_2 < \ldots < \Lambda_N .
\]

When \(\Lambda_i\) is negative, the vacuum \(i\) is terminal, in the sense that a geodesic entering a pocket universe with \(\Lambda < 0\) will terminate on a future singularity, the big crunch.

Any vacuum \(j\) has a decay rate per unit four-volume \(\Gamma_{ij}\) (which could be extremely small) to a different vacuum \(i\). The total decay rate of vacuum \(j\) is defined as the sum of all possible decay rates from \(j\),

\[
\Gamma_j = \sum_i \Gamma_{ij} .
\]

It is often convenient to define a dimensionless decay rate \(\kappa_{ij}\),

\[
\kappa_{ij} = \frac{4\pi \Gamma_{ij}}{3H_j^4} ,
\]

\(^1\)We thank Don Page for a discussion that brought this possibility to our attention.
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where

\[ H_i = \left( \frac{\Lambda_i}{3} \right)^{1/2} \]  

(4.4)

is the expansion rate for non-terminal vacua at late times. The total dimensionless decay rate is defined as

\[ \kappa_j = \sum_i \kappa_{ij} \]  

(4.5)

The branching ratio from vacuum \( j \) to \( i \) is defined as the corresponding decay rate divided by the total decay rate of the parent vacuum \( j \),

\[ \beta_{ij} = \frac{\Gamma_{ij}}{\Gamma_j} = \frac{\kappa_{ij}}{\kappa_j} \]  

(4.6)

At a coarse-grained level, the expected number of times a geodesic starting in vacuum \( j \) enters vacuum \( i \) is given by a sum over products of branching ratios along all possible paths from \( j \) to \( i \) [15],

\[ e_{ij} = \sum_p \sum_{i_1, i_2, \ldots, i_{p-1}} \beta_{i_{p-1}i} \cdots \beta_{iij} \]  

(4.7)

The sum over \( p \) indicates a sum over paths of any length.

**Double Exponential Arithmetic**

In this section we review the arithmetic of double exponential numbers. An exponentially large (or small) number is one of the form \( e^x \) (or \( e^{-x} \)), where \( x \) is large. We denote this by a double inequality, \( e^x \gg 1 \). A double exponentially large (or small) number is one of the form \( e^x \) (or \( e^{-x} \)) with \( x \gg 1 \), which we denote by a triple inequality \( e^x \gg 1 \). For example, the Boltzmann brain production rate \( \Gamma_{BB} < e^{-S_{BB}} \), where \( S_{BB} \) is the minimum entropy required to create a Boltzmann brain, is a double exponentially small number. (The inequality here comes from the fact that a Boltzmann brain requires a minimum free energy of the order \( S_{BB} \), so its production is accordingly suppressed.)

If \( x \) and \( y \) are at least exponentially large and \( x > y \), then

\[ x \pm y \approx x \]  

(4.8)

Assuming \( x \) and \( y \) are double exponentially large, we can apply the previous rule to the exponent to obtain a product rule. Again assuming \( x > y \),

\[ xy \approx \frac{x}{y} \approx x \]  

(4.9)

Similar rules apply for exponentially or double exponentially small numbers, but with \( x \) replaced by \( 1/x \).
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We can apply this arithmetic to landscape decay rates to obtain useful approximate identities. For example,

$$\frac{\Gamma_{BB,i}}{\Gamma_i} \approx \begin{cases} \Gamma_{BB,i} < e^{-S_{BB}} & \text{if } \Gamma_{BB,i} < \Gamma_i, \\ \Gamma_i^{-1} > e^{S_{BB}} & \text{if } \Gamma_{BB,i} > \Gamma_i, \end{cases}$$

(4.10)

where \(\Gamma_{BB,i}\) is the Boltzmann brain production rate in vacuum \(i\).

Dominant History Method

In an eternally inflating space-time, the number of occurrences of different types are infinite. This means that the relative probability of event \(I\) compared to event \(J\),

$$\frac{P_I}{P_J} = \frac{\langle N_I \rangle}{\langle N_J \rangle},$$

(4.11)

where \(\langle N_I \rangle\) is the expected number of events of type \(I\), which is infinite, is ill-defined. To address this problem, we must introduce a cutoff procedure that regulates infinities by counting only a finite subset of the events \(I\).

In this paper, we use the causal patch cutoff [102]. Probabilities are defined by counting events in a single causally connected region of space-time (the “patch”), in a weighted average over initial conditions and decoherent histories of the patch. In a theory with long-lived vacua \(i\), the expected number of events \(\langle N_i \rangle\) of type \(I\) can be computed as

$$\langle N_I \rangle = \sum_{i,j} N_{II} e_{ij} P_j.$$  

(4.12)

Here \(P_j\) is the probability of starting in vacuum \(j\); \(e_{ij}\) is the probability that a geodesic that starts in \(j\) will enter vacuum \(i\), given by Eq. (4.7); and \(N_{II}\) is the number of events of type \(I\) in vacuum \(i\), within the patch.

Here the events of interest, \(I\), will be observations made by Boltzmann brains, \(BB\), versus those made by ordinary observers, \(OO\). The expected number of \(BB\)s produced in vacuum \(i\) is equal to the lifetime of vacuum \(i\) multiplied by the rate of \(BB\) production in \(i\), so \(N_{BB,i} = \Gamma_{BB,i}/\Gamma_i\). In the case \(\Gamma_{BB,i} < \Gamma_i\), where \(\Gamma_{BB,i}\) is a double-exponentially small number, this can be represented by a branching ratio, if decay channels are augmented by a “decay to Boltzmann brains”:

$$N_{BB,i} \approx \beta_{BB,i} \equiv \frac{\Gamma_{BB,i}}{\Gamma_i + \Gamma_{BB,i}},$$

(4.13)

By double exponential arithmetic, Eq. (4.10), we can also approximate \(N_{BB,i} \approx \Gamma_{BB,i}\). Numbers that are not double-exponentials, such as the precise number of ordinary observers within a patch that contains any, and the number of observations made by each, can be set to unity.
There is a convenient method for estimating probabilities using the causal patch cutoff, based on its branching-tree implementation [15, 111]. A path through the landscape is represented as a sequence of arrows between vacua along the path. The branching ratios for each individual process are written above these arrows, using the notation $1'$ to represent a branching ratio that is one up to a small correction.

In addition to labels denoting vacua (which denote generic states in these vacua, e.g., empty de Sitter space), we also include labels “$OO$” and “$BB$” to represent ordinary observers and Boltzmann brains. Whether such observers form depends not only on the vacuum, but also on how it is approached along the decay path. If ordinary observers are produced by normal dynamical evolution after the decay of a higher $\Lambda$ vacuum, then we denote the decay as branching to $OO$, with the branching ratio set by the decay rate of the parent vacuum. This is followed by an eventual decay to empty de Sitter space, with branching ratio $1'$. For Boltzmann brains, we use Eq. (4.13).

However, if no observers are produced in the approach to equilibrium in the new vacuum, then ordinary observer production requires an extra up-tunneling. It is then even more suppressed than Boltzmann brains, since the latter require a much smaller entropy decrease: $\Gamma_{OO} \ll \Gamma_{BB} < e^{-S_{BB}}$. This is denoted by decay to a de Sitter vacuum followed by another, highly suppressed “decay” to observers.

To account for general initial conditions, each path begins with the label “$I.C.$” followed by an arrow to some vacuum. The probability of starting in this vacuum will be written above the arrow.

The total amplitude for ordinary observers or Boltzmann brains is obtained by summing over all paths that include such observers. In realistic models, most branching ratios and initial probabilities will be double-exponentially small, and each sum will be dominated by a particular path or class of paths. It is then sufficient to compare only the two dominant paths to determine which class of observers wins.

As an example, consider a toy landscape with two de Sitter vacua $A$ and $B$. $B$ is connected to a terminal vacuum $T$ with negative cosmological constant. Suppose also that only $B$ contains observers of any kind, and that ordinary observers are produced after $A$ decays to $B$. Then the dominant path to Boltzmann brains is represented as

$$I.C. \xrightarrow{P_B} B \xrightarrow{\Gamma_{BB,B}} BB [\xrightarrow{1'} B \xrightarrow{1'} T \xrightarrow{\frac{1}{\Lambda}} \text{crunch}].$$

Here $1'$ above an arrow represents a branching ratio that is nearly unity.

Meanwhile, the dominant path to ordinary observers is

$$I.C. \xrightarrow{P_A} A \xrightarrow{1} OO [\xrightarrow{1'} B \xrightarrow{1'} T \xrightarrow{\frac{1}{\Lambda}} \text{crunch}].$$

For completeness, we show a probable completion of the path in square brackets; this completion does not contribute to the amplitudes.

The total branching ratio for each type of observer is obtained by multiplying the probabilities for the corresponding dominant path. One obtains $P_B \Gamma_{BB,B}$ for Boltzmann brains, and $P_A$ for ordinary observers. Thus, an arrow of time is predicted if and only if $P_A/P_B > \Gamma_{BB,B}$. 

4.3 Two Toy Models

Figure 4.1: (a) This toy landscape leads to an arrow of time even if initial conditions select the high entropy vacuum $C$. This result persists if Hartle-Hawking initial conditions are chosen. (b) With one extra terminal vacuum, an arrow of time is still predicted if initial conditions strictly select $C$. However, Boltzmann brains dominate in the Hartle-Hawking state, which adds a small amplitude to start in $B$.

In this section, we consider the two toy landscapes shown in Fig. 4.1. Both are one-dimensional, and only neighboring vacua are connected by decay. This is for simplicity, to illustrate the key physics, and general results will not depend on this assumption. The toy models differ from each other only through the number of terminal vacua. Each theory predicts an arrow of time if initial conditions are entirely concentrated in the de Sitter vacuum with highest entropy. However, the two models may behave very differently under a tiny admixture of initial probability in other de Sitter vacua.

A Landscape With One Terminal

Consider the effective potential depicted in Fig. 4.1a. In this toy landscape, there are three de Sitter vacua $A$, $B$, and $C$, and one terminal vacuum $T$. It is assumed that ordinary observers are produced if $A$ decays to $B$; that both ordinary observers and Boltzmann brains can exist only in vacuum $B$; and that vacuum $B$ decays faster than it produces Boltzmann brains:

$$\Gamma_B > \Gamma_{BB,B}.$$  \hspace{1cm} (4.16)

While simplistic, this toy model captures key features that are expected of the string landscape: for instance, there are large step sizes $|\Delta \Lambda| \gg S_{BB}^{-1}$ between vacua.
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Single-Vacuum Initial Conditions

Suppose now that initial conditions select the dominant vacuum $C$. From there the leading path to ordinary observer production is

$$C \xrightarrow{1} A \xrightarrow{1} OO \xrightarrow{\gamma} B \xrightarrow{\gamma} T \xrightarrow{1} \text{crunch}.$$  (4.17)

Naively, the second arrow has a branching ratio $\beta_{BA}$. However, $C$ is a dead-end vacuum; if $A$ decays back to $C$, all that can happen is a return to $A$. Strictly speaking one should sum over paths in which the entry to $B$ is preceded by any number of oscillations between $C$ and $A$, but the net effect is a branching ratio of 1 from $A$ to $B$.

The dominant path to Boltzmann brain production is

$$C \xrightarrow{1} A \xrightarrow{1} OO \xrightarrow{\gamma} B \xrightarrow{\gamma} BB \xrightarrow{\gamma B B} BB \xrightarrow{\gamma B B} T \xrightarrow{1} \text{crunch}.$$  (4.19)

The Boltzmann path is suppressed by the extra factor $\Gamma_{BB} < e^{-S_{BB}}$, so ordinary observers win and an arrow of time is predicted. (Note the importance of Eq. (4.16); with the opposite inequality, each path would contain of order $\Gamma_B$ additional Boltzmann brain production events before the vacuum $B$ decays to the terminal, and Boltzmann brains would win.)

Multi-Vacuum Initial Conditions

Now suppose each de Sitter vacuum $A, B, C$ has initial probabilities $P_A, P_B$ and $P_C$ respectively. For each initial vacuum, we identify the dominant path to ordinary observers and the dominant path to $BB$s as follows:

$I.C. \hspace{0.5cm} P_C \rightarrow C \xrightarrow{1} A \xrightarrow{1} OO \xrightarrow{\gamma} B \xrightarrow{\gamma} T \xrightarrow{1} \text{crunch}.$  (4.19)

$I.C. \hspace{0.5cm} P_C \rightarrow C \xrightarrow{1} A \xrightarrow{1} B \xrightarrow{\gamma_{BB,B}} BB \xrightarrow{\gamma B B} BB \xrightarrow{\gamma B B} T \xrightarrow{1} \text{crunch}.$  (4.20)

$I.C. \hspace{0.5cm} P_A \rightarrow A \xrightarrow{1} OO \xrightarrow{\gamma} B \xrightarrow{\gamma} T \xrightarrow{1} \text{crunch}.$  (4.21)

$I.C. \hspace{0.5cm} P_A \rightarrow A \xrightarrow{1} B \xrightarrow{\gamma_{BB,B}} BB \xrightarrow{\gamma B B} BB \xrightarrow{\gamma B B} T \xrightarrow{1} \text{crunch}.$  (4.22)

$I.C. \hspace{0.5cm} P_B \rightarrow B \xrightarrow{\gamma_{OO,B}} OO \xrightarrow{\gamma} B \xrightarrow{\gamma} T \xrightarrow{1} \text{crunch}.$  (4.23)

$I.C. \hspace{0.5cm} P_B \rightarrow B \xrightarrow{\gamma_{BB,B}} BB \xrightarrow{\gamma B B} BB \xrightarrow{\gamma B B} T \xrightarrow{1} \text{crunch}.$  (4.24)

We see that with initial conditions purely in $C$ or $A$, ordinary observers would win. But initial conditions in $B$ favor Boltzmann brains, since $\Gamma_{OO,B} \ll \Gamma_{BB,B}$. More generally, the outcome will, in principle, depend on the distribution of initial conditions.
Ordinary observers dominate if at least one of the following two conditions is satisfied:

\[
\frac{P_B}{P_C} < \frac{1}{\Gamma_{BB,B}}, \\
\frac{P_B}{P_A} < \frac{1}{\Gamma_{BB,B}}.
\]  

(4.25)  

(4.26)

Since \( \Gamma_{BB,B} < e^{-S_{BB}} \), we find that a sufficient but not necessary condition is

\[
\frac{P_B}{P_C} < e^{S_{BB}}.
\]  

(4.27)

Boltzmann brains dominate if both of the following conditions hold:

\[
\frac{P_B}{P_C} > \frac{1}{\Gamma_{BB,B}}, \\
\frac{P_B}{P_A} > \frac{1}{\Gamma_{BB,B}}.
\]  

(4.28)  

(4.29)

From Eq. (4.27), it is clear that ordinary observers will win for a large range of initial probabilities, including any case where initial conditions have the largest support in vacuum C.

A Landscape With Two Terminals

Now consider the landscape represented in Fig. 4.1b. The only difference is the extra terminal vacuum \( T' \), which can be reached by a decay from C. This adds another feature expected to hold in the string landscape: every de Sitter vacuum can decay to a vacuum with lower cosmological constant. Consequently the branching ratio from C to A is no longer 1. The up-tunneling rate to A contains a suppression factor \( e^{-S_C} \ll e^{-S_{BB}} \). Unless the decay rate to \( T' \) is similarly suppressed, the branching ratio from C to A will be double-exponentially small.

Single-Vacuum Initial Conditions

With initial conditions entirely in vacuum C, the extra terminal does not affect the conclusion that ordinary observers win. The smaller branching ratio affects paths to OOs and BBs equally. Paths from C to \( T' \) dominate overall but they are “sterile”: they do not contribute to paths to either type of observers.

Multi-Vacuum Initial Conditions

However, the extra terminal has large implications for the case with more general initial conditions. Suppose each de Sitter vacuum \( A, B, C \) has initial probabilities \( P_A, P_B \) and \( P_C \)
respectively. For each initial vacuum, let us identify the dominant path to OO's and the dominant path to BBs:

\[
I.C. \quad P_C \xrightarrow{\beta_{AC}} C \xrightarrow{\beta_{BA}} A \xrightarrow{\Gamma_{BB,B}} [V' \rightarrow B \rightarrow T \xrightarrow{1} \text{crunch}] . 
\]

\[
I.C. \quad P_C \xrightarrow{\beta_{AC}} C \xrightarrow{\beta_{BA}} B \xrightarrow{\Gamma_{BB,B}} [V' \rightarrow B \rightarrow T \xrightarrow{1} \text{crunch}] . 
\]

\[
I.C. \quad P_A \xrightarrow{\beta_{BA}} OO \xrightarrow{1'} B \rightarrow T \xrightarrow{1} \text{crunch} , 
\]

\[
I.C. \quad P_A \xrightarrow{\beta_{BA}} B \xrightarrow{\Gamma_{BB,B}} BB \xrightarrow{1'} B \rightarrow T \xrightarrow{1} \text{crunch} . 
\]

\[
I.C. \quad P_B \xrightarrow{\Gamma_{OO,B}} OO \xrightarrow{1'} B \rightarrow T \xrightarrow{1} \text{crunch} , 
\]

\[
I.C. \quad P_B \xrightarrow{\Gamma_{BB,B}} BB \xrightarrow{1'} B \rightarrow T \xrightarrow{1} \text{crunch} . 
\]

Again, with initial conditions purely in C or A, OO's win. But as before, initial conditions in B favor Boltzmann brains since \( \Gamma_{OO,B} \ll \Gamma_{BB,B} \). More generally, the outcome will depend on the distribution of initial conditions, and in this case we find this puts tighter restrictions on our initial conditions.

**Ordinary observers dominate** if initial conditions sufficiently disfavor B compared to at least one of A and C. That is, at least one of the following two conditions must be satisfied:

\[
\frac{P_B}{P_C} < \frac{\beta_{BA}\beta_{AC}}{\Gamma_{BB,B}} ,
\]

\[
\frac{P_B}{P_A} < \frac{\beta_{BA}}{\Gamma_{BB,B}} .
\]

These formulas have a very natural interpretation. Ordinary observers dominate if the amplitude for starting in B and then producing Boltzmann brains, \( P_B\Gamma_{BB,B} \) is smaller than the amplitude for starting in some other vacuum and tunneling over to B, for example \( P_C\beta_{BA}\beta_{AC} \). Note that in the latter case ordinary observers are automatically produced.

Since \( \Gamma_{BB,B} < 1 \), we find that a sufficient but not necessary condition is

\[
\frac{P_B}{P_C} < \beta_{BA}\beta_{AC} .
\]

As we will discuss in Sec. 4.6, with dominant eigenvector initial conditions this condition is satisfied and hence OO's are predicted.

**Boltzmann brains dominate** if both of the following conditions hold:

\[
\frac{P_B}{P_C} > \frac{\beta_{BA}\beta_{AC}}{\Gamma_{BB,B}} ,
\]

\[
\frac{P_B}{P_A} > \frac{\beta_{BA}}{\Gamma_{BB,B}} .
\]

As we will discuss in Sec. 4.5, these conditions can both be satisfied with Hartle-Hawking initial conditions, leading to BB domination.
4.4 A Large Landscape

Now we generalize to the case of the large landscape, and derive sufficient conditions on the initial probability distribution for the existence or absence of an arrow of time. We will make the following assumptions about the basic structure of the landscape:

- **No tuning.** The low energy physics necessary for complex phenomena such as observers is fine-tuned, so that only a tiny fraction of vacua have observers of any type.

- **Large step size.** Vacuum transitions generically change the cosmological constant by a large amount $|\Delta \Lambda| \gg \frac{1}{S_{BB}}$.

- **Not too large.** The effective number of vacua is less than $e^{S_{BB}}$.

- **Not effectively one-dimensional.** For any two de Sitter vacua $i, j$ with $\Lambda_i, \Lambda_j < \frac{1}{S_{BB}}$, there exists a semiclassical decay path from $i$ to $j$ that does not pass through any vacuum $k$ with $\Lambda_k < \frac{1}{S_{BB}}$.

These conditions are believed to be satisfied by the string landscape.

We make one additional assumption, that all vacua decay faster than they produce Boltzmann brains:

$$\Gamma_{BB,i} < \Gamma_i.$$ (4.41)

Whether this assumption holds in the landscape of string theory is not known though there is circumstantial evidence in its favor [112].

In the case of single-vacuum initial conditions, the above conditions are sufficient for the absence of Boltzmann brains. We will review this argument in Sec. 4.4; it relies on the no-tuning assumption, which ensures (generically) that the single initial vacuum cannot produce Boltzmann brains at all. In Sec. 4.4, we consider the general case, allowing an admixture of initial vacua some of which may have a nonzero rate of producing Boltzmann brains. The question is whether this spoils the prediction of an arrow of time.

### Single Initial Vacuum

Here we briefly review the case of single vacuum initial conditions in the string landscape; for further details, the original papers [98, 99, 100] should be consulted. We consider a slightly generalized setup where the initial vacuum is not necessarily the dominant vacuum, but the proof follows through exactly as before. Suppose the dominant path leading to both ordinary observers and Boltzmann brains starts in the de Sitter vacuum $i$, with probability 1. There are two cases, depending on the size of the cosmological constant of the initial vacuum.

The first case is small initial vacuum energy: $\Lambda_i < \frac{1}{S_{BB}}$. (In the toy model of the previous section, this corresponds to starting in vacuum $C$.) By the large-step-size property, vacua
with observers of any type cannot be accessed directly from $i$. Thus, all paths to observers of any kind begin with an up-tunneling from $i$ to a second vacuum $i_1$ with $\Lambda_{i_1} \gg S_{BB}^{-1}$, so
\[ e^{S_{i_1}} \ll e^{S_{BB}} . \] (4.42)

Detailed balance\footnote{Detailed balance is supported by explicit computation of Coleman de Lucia tunneling rates between de Sitter vacua, considered here. It does not apply to decays involving terminal vacua, which are irreversible at the semiclassical level.} relates the up-tunneling and down-tunneling rates between $i$ and $i_1$:
\[ \Gamma_{ii_1} e^{S_i} = \Gamma_{i_1i} e^{S_{i_1}} . \] (4.43)

We also know that the decay time of $i_1$ cannot be larger than the recurrence time or faster than the Planck time,
\[ 1 > \Gamma_{ii_1} > e^{-S_{i_1}} . \] (4.44)

Combining, we find
\[ e^{S_{BB}} \gg e^{S_{i_1}} > \Gamma_{ii_1} e^{S_i} > 1 \gg e^{-S_{BB}} . \] (4.45)

The second vacuum $i_{1,OO}$ in the dominant path to ordinary observers need not be the same as the second vacuum $i_{1,BB}$ in the dominant path to Boltzmann brains. However, the above inequalities imply that to an accuracy better than $e^{S_{BB}}$, the first up-tunneling suppresses paths to either Boltzmann brains or ordinary observers by the same amount:
\[ e^{-S_{BB}} \ll \frac{\beta_{i_{1,OO}}}{\beta_{i_{1,BB}}} \ll e^{S_{BB}} . \] (4.46)

This is sufficient accuracy to neglect the effects of the first up-tunneling, since we will find later that ordinary observers will be favored by a factor larger than $e^{S_{BB}}$. At this point the rest of the proof follows as in Case II, with $i_1$ now taken as the starting point.

The second case is large initial vacuum energy: $\Lambda_i > S_{BB}^{-1}$. (In the toy model of the previous section, this corresponds to starting in vacuum $A$.) By the assumed multi-dimensionality of the landscape, it is always possible to find a path through vacua that all have $\Lambda_{i_k} > S_{BB}^{-1}$ for all $k \geq 2$. The branching ratios through these paths will be less suppressed than $e^{-S_{BB}}$. By the assumption that the landscape has fewer than $e^{S_{BB}}$ vacua, the sum over paths does not introduce any large factors for either type of observer. Let $e_{OO,i}$ and $e_{BB,i}$ be the total branching ratios for producing either $OO$s or $BB$s starting from $i$ (note that $OO$s and $BB$s do not in general need to be produced in the same vacuum). Then, including the first up-tunneling, we have
\[ e_{OO,i} > X_{OO} e^{-S_{BB}} , \] (4.47)
\[ e_{BB,i} < X_{BB} e^{-S_{BB}} , \] (4.48)

where $X_{OO}$ and $X_{BB}$ differ by less than a factor of $e^{\pm S_{BB}}$. Thus to double-exponentially good accuracy,
\[ e_{BB,i} < e_{OO,i} , \] (4.49)

so $OO$s dominate over $BB$s given initial conditions with support only in $i$. 
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General Initial Conditions

Now we analyze the same large landscape, but with general initial conditions. The probability for each class of observers can be computed by considering each initial vacuum with nonzero probability \( P_i \), computing the expected number of \( OO \)s and \( BB \)s as in the single-initial-vacuum case, and then summing with weighting \( P_i \).

Naively, this means that \( OO \)s win. We showed in the previous subsection that they do so independently of the initial vacuum, so they should still win in a weighted average over initial vacua. However, there is a key difference: we can no longer assume that all vacua with nonzero initial probability are unable to produce Boltzmann brains. This contributes a new term to the expected number of Boltzmann brains, \( \sum_j P_j \Gamma_{BB,j} \). Therefore, ordinary observers win if and only if

\[
\sum_j P_j \Gamma_{BB,j} < \sum_i P_i e^{OO,i},
\]

(4.50)

where each sum runs over all de Sitter vacua.

Because the landscape has fewer than \( e^{S_{BB}} \) vacua, we expect that each sum is “dominated” by one path, in the following extremely weak sense: one can find a path such that dropping all other paths decreases the sum by less than a factor of \( e^{S_{BB}} \). This will be useful in some arguments below.

Note that the left hand side is always less than \( e^{-S_{BB}} \), so ordinary observers win if the right hand side is greater than this double-exponentially small quantity. For some theories of initial conditions, this is obviously the case: for example, if the probability is distributed evenly over all de Sitter vacua, or with the tunneling wavefunction \([109, 110]\). In these cases, an arrow of time is predicted. This is not surprising, since such initial conditions select for low initial entropy in any case. What made the result of Ref. \([98]\) interesting is that it was possible to obtain an arrow of time even after starting the universe in a state of arbitrarily large entropy.

Thus, we will focus here on theories of initial conditions which select (or might select) for such initial states, but which give nonzero probability to several different initial vacua. We will examine whether an arrow of time is still predicted when more than one initial vacuum is taken into account, i.e., whether Eq. (4.50) is satisfied or violated. In Sec. 4.5 we consider Hartle-Hawking initial conditions and find that they predict the absence of an arrow of time when the flow towards terminals is strong enough. In Sec. 4.6 we consider the initial conditions picked out by the global dual to the causal patch measure and closely related measures; we find that an arrow of time is predicted.

4.5 Hartle-Hawking Initial Conditions

We now consider several specific choices of initial conditions and their implications for the existence of an arrow of time. In this section, we first analyze the case of Hartle-Hawking initial conditions. We will assume throughout the condition necessary for an arrow of time
in the single vacuum case, i.e., that all vacua decay faster than they produce Boltzmann brains. Hence the branching ratio to Boltzmann brains is small in all vacua:

\[ \beta_{BB,j} = \frac{\Gamma_{BB,j}}{\Gamma_j} \ll 1 \]  

(4.51)

**Hartle-Hawking Proposal**

In the Hartle-Hawking no-boundary proposal \[113\], the wave function of the universe, \( \Psi[h_{ij}, \ldots] \), is given by a path integral over all compact Euclidean four-manifolds whose only boundary is a given three-dimensional spacelike surface with metric \( h_{ij} \). It would be nice to obtain probabilities directly in this framework, by computing the amplitude for specified field configurations, perhaps subject to additional conditions such as the presence of observers and some or all of their previous observations. However, the measure problem cannot be circumvented so easily. The data that can be conditioned on can only include what is available in the observer’s past light-cone. But the number of possible different quantum states in a past light-cone is finite \[114, 115, 116, 117\]. There will be infinitely many saddlepoint geometries that contain the specified patch, and the sum over saddlepoints diverges. Therefore, we will apply the Hartle-Hawking prescription only to obtain a probability distribution at an initial time. We will use the causal patch measure or its close relatives to obtain well-defined probabilities in the resulting ensemble of semiclassical geometries.

The probability to start in a de Sitter vacuum \( i \), with cosmological constant \( \Lambda_i \), is set by the action of the corresponding Euclidean de Sitter instanton. It is proportional to the number of quantum states associated with empty de Sitter space \[118\]. In this section, we will find it convenient to work with unnormalized probabilities, i.e., we set

\[ P_i = e^{S_i} = \exp(3\pi/\Lambda_i) \]  

(4.52)

Since we compute a relative probability for ordinary observers vs. Boltzmann brains, the overall normalization drops out in any case. The probability to start in a state with \( \Lambda \leq 0 \) is assumed to vanish.

As a theory of initial conditions, the Hartle-Hawking proposal is problematic because as we will see, it exponentially favors initial conditions with large entropy. (See Ref. \[119\] for a detailed discussion and further references.) In the toy model of Sec. 4.3, the proposal is nevertheless in accord with observation \[98\]. However, we will show that the addition of another terminal vacuum destroys the prediction of an arrow of time if the initial vacuum can decay into it fast enough. We will also examine more generally the conditions on a large landscape under which the Hartle-Hawking proposal is viable.

**Toy Model**

Hartle-Hawking initial conditions can lead to \( BB \) domination in the toy model with three vacua and an extra terminal depicted in Fig. 4.1, as we will now show. We may neglect paths that start in \( A \).
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Boltzmann brains have unnormalized probability

\[ P_{BB} = e^{S_B} \beta_{BB,B} = \frac{e^{S_B} \Gamma_{BB,B}}{\Gamma_B} = \frac{N_{BB,B}}{\Gamma_B}, \]  

(4.53)

where we have introduced the quantity

\[ N_{BB,B} \equiv e^{S_B} \Gamma_{BB,B}. \]  

(4.54)

Since \( e^{S_B} \) is the total number of quantum states associated with de Sitter space, \( N_{BB,B} \) can be interpreted as the number of states that contain Boltzmann brains. This number is always greater than \( e^{S_{BB}} \), where \( S_{BB} \) is the minimum coarse grained entropy of a Boltzmann brain. But it may be much greater, since all other systems, and particularly the horizon, contribute to and typically dominate the entropy.

By detailed balance,

\[ \Gamma_{AC} e^{S_C} = \Gamma_{CA} e^{S_A}, \]  

(4.55)

so the unnormalized probability for ordinary observers is

\[ P_{OO} = \frac{\beta_{BA} \Gamma_{CA} e^{S_A}}{\Gamma_{T'C}}. \]  

(4.56)

where we have assumed that \( \Gamma_{T'C} \gg \Gamma_{AC} \). Boltzmann brains win if and only if

\[ \Gamma_{T'C} > \frac{\Gamma_B \beta_{BA} \Gamma_{CA} e^{S_A}}{N_{BB,B}}. \]  

(4.57)

Since \( \Lambda_A > S_{BB}^{-1} \), the quantities \( \beta_{BA}, \Gamma_{CA}, \) and \( e^{-S_A} \) are all large compared to \( e^{-S_{BB}} \), whereas \( N_{BB,B}^{-1} \ll e^{-S_{BB}} \). By double-exponential arithmetic, Boltzmann brains win if and only if

\[ \Gamma_{T'C} > \frac{\Gamma_B}{N_{BB,B}}. \]  

(4.58)

The right hand side is very small, so for a large range of parameters, the Hartle-Hawking proposal will not predict an arrow of time in this model. However, note that Eq. (4.58) does not involve the factor \( e^{-S_C} \) associated with up-tunneling from the vacuum \( C \). Thus, it is not necessary to interpose a de Sitter vacuum between \( C \) and \( T' \) to keep the Hartle-Hawking viable. It suffices to make the down-tunneling rate from \( C \) to \( T' \) smaller than the down-tunneling rate of \( B \) (which could be quite large), divided by the number of Boltzmann states in vacuum \( B \).

Large Landscape

Our analysis of the toy model extends easily to the case of the large landscape considered in Sec. 4.4. With our choice of normalization, the amplitude for Boltzmann brains is closely related to the number of “Boltzmann states,” summed over all vacua:

\[ P_{BB} = \sum_j e^{S_j} \beta_{BB,j} = \sum_j \frac{N_{BB,j}}{\Gamma_j} \equiv \mathcal{N}. \]  

(4.59)
Here \( \mathcal{N}_{BB,j} = e^{S_j} \Gamma_{BB,j} \) is the number of quantum states, in the de Sitter vacuum \( j \), that contain at least one Boltzmann brain. This number is greater than \( e^{S_{BB}} \), where the exponentially large number \( S_{BB} \) is the coarse-grained entropy of a minimal Boltzmann brain. It may be much greater since horizon entropy and the entropy of all other matter in the patch contributes to \( \mathcal{N}_{BB,B} \).

The general amplitude for ordinary observers is given by

\[
P_{OO} = \sum_i e^{S_i} e^{S_{OO,i}}.
\] (4.60)

We can restrict the sum to vacua with \( \Lambda_i \lesssim (\log \mathcal{N})^{-1} < S_{BB}^{-1} \). Vacua with larger cosmological constant have \( P_i < \mathcal{N} \); they are too suppressed by the Hartle-Hawking initial conditions to be able to compete with Eq. (4.59). By arguments analogous to those in Sec. 4.4, we may further restrict the sum to a single initial vacuum and path, which dominates in the following weak sense: that dropping all other terms changes the sum by a factor less than \( \mathcal{N} \). Again up to factors double-exponentially smaller than \( \mathcal{N} \), one then finds that the amplitude for ordinary observers can be estimated from the initial portion of this dominant path, keeping only the initial probability and the branching ratio for the first up-tunneling to the vacuum \( i_1 \):

\[
P_{OO} \approx e^{S_i} e^{S_{OO,i}} \approx \beta_{i_1,i} e^{S_i} = \frac{\Gamma_{ii_1} e^{S_{i_1}}}{\Gamma_i}.
\] (4.61)

By the large step size property, \( \Lambda_{i_1} > S_{BB}^{-1} \). Thus \( e^{S_{i_1}} \) is larger than one and smaller than \( \mathcal{N} \), since \( 1 < e^{S_{i_1}} < e^{S_{BB}} < \mathcal{N} \). Similarly \( \Gamma_{ii_1} \) is smaller than one and larger than \( \mathcal{N}^{-1} \), since \( 1 > \Gamma_{ii_1} > e^{-S_{i_1}} > \mathcal{N}^{-1} \). By double exponential arithmetic, it follows that ordinary observers win if and only if

\[
\Gamma_i < P_{BB}^{-1}.
\] (4.62)

If the sum in Eq. (4.59) is dominated by a single term \( j \) (which is plausible), the necessary and sufficient condition for an arrow of time becomes

\[
\Gamma_i < \frac{\Gamma_j}{\mathcal{N}_{BB,j}}.
\] (4.63)

That is, the Hartle-Hawking proposal remains viable in a large landscape, if and only if ordinary observers are mainly produced along a path starting in a de Sitter vacuum \( i \) whose lifetime exceeds the lifetime of the most Boltzmann-friendly vacuum \( j \) by a factor of the total number of Boltzmann states in \( j \).

It is not known whether or not the string theory landscape satisfies Eq. (4.62). Note that it is possible for a landscape to satisfy this condition, and to simultaneously satisfy the usual assumption that all vacua decay faster than they produce Boltzmann brains.

### 4.6 Dominant Eigenvector Initial Conditions

Several of the most attractive measure proposals are related by global-local duality [13]. For instance, the scale-factor, light-cone time and CAH+ cutoffs are dual to the local “fat
geodesic”, causal patch, and Hubbletube cutoffs, respectively [12, 99, 13]. This duality holds for a very specific choice of initial conditions for the local measures, with probabilities given by the dominant eigenvector of the rate equation for eternal inflation.

We will review the rate equation, the dominant eigenvector, and a method for computing the dominant eigenvector. We will then argue that OO will always win with these initial conditions.

**Rate Equation**

Eternal inflation is the process by which a landscape of vacua is populated through exponential expansion and vacuum decay. Consider a family of geodesics orthogonal to some fiducial hypersurface. Let $f_j(t)$ be the fraction of the comoving volume occupied by vacuum $j$ at time $t$. The rate equation describing the volume distribution is \[ df_j \over dt = \sum_i (-\kappa_{ij} f_j + \kappa_{ji} f_i) \] (4.64)

This equation is appropriate to the scale factor time cutoff, which defines initial conditions for the fat geodesic by duality. Essentially the same rate equation holds for the light-cone time cutoff, which is dual to the causal patch [13].

We are interested only in the asymptotic distribution of de Sitter vacua, which is governed by the matrix equation

\[ {df_i \over dt} = \sum_j R_{ij} f_j , \] (4.65)

where

\[ R_{ij} = \kappa_{ij} - \kappa_i \delta_{ij} , \] (4.66)

and indices $i, j$ are now restricted to run over vacua with positive cosmological constant. The total decay rates $\kappa_i$ will in general contain some contributions from decays to terminals.

At late times, generic solutions to this equation evolve to an attractor distribution,

\[ f_i(t) = s_i e^{-qt} + ... , \] (4.67)

given by the dominant eigenvector $s_i$ of the transition matrix $R_{ij}$, that is, the eigenvector with the eigenvalue of smallest magnitude, $-q$ [121]:

\[ \sum_j R_{ij} s_j = -qs_i . \] (4.68)

Generically, the eigenvector will be dominated by the longest lived de Sitter vacuum, $*$:

\[ s_i \approx \delta_{i*} . \] (4.69)

In previous work this approximation was assumed in the analysis of the Boltzmann brain problem. Here we will go beyond the zeroth-order approximation and show that this does not change the conclusion.
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Toy Model

Let us revisit the toy model depicted in Fig. 4.1b, with three de Sitter vacua and two terminal vacua. We assume that $C$ is the longest-lived de Sitter vacuum. We will derive the initial probability distribution by estimating the corrections to Eq. (4.69).

The transition matrix is

$$
R = \begin{pmatrix}
-\kappa_C & 0 & \kappa_{CA} \\
0 & -\kappa_B & \kappa_{BA} \\
\kappa_C \epsilon & \kappa_{AB} & -\kappa_A \\
\end{pmatrix}.
$$

We assume that

$$
\epsilon \equiv \frac{\kappa_{AC}}{\kappa_C} \ll 1.
$$

This is natural since tunneling from $C$ to $A$ is “up-tunneling”; it increases the vacuum energy and decreases the entropy. Note that $\kappa_A = \kappa_{BA} + \kappa_{CA}$. Expanding to first order in $\epsilon$, the eigenvalue $-q$ is written as

$$
q = q^{(0)} + \epsilon q^{(1)} + \ldots,
$$

and the eigenvector is

$$
s = \begin{pmatrix}
1 \\
\frac{1}{s_B} + \epsilon s_B^{(1)} + \ldots \\
\frac{1}{s_A} + \epsilon s_A^{(1)} + \ldots
\end{pmatrix}.
$$

We have chosen a normalization in which the longest-lived vacuum has unit weight, for simplicity. At leading order in $\epsilon$, the subleading entries will still be correctly normalized.

The eigenvalues are the roots of the characteristic polynomial, a cubic equation in $q$:

$$
\text{Det}(R + qI) = \epsilon \kappa_C \kappa_{CA} (\kappa_B - q) \\
- (\kappa_C - q) [-\kappa_{AB} \kappa_{BA} + (\kappa_B - q)(\kappa_A - q)].
$$

At zeroth order in $\epsilon$, the correct choice is the smallest-magnitude root, $q^{(0)} = \kappa_C$. The first order correction is

$$
\epsilon q^{(1)} = \frac{(\kappa_B - \kappa_C)\kappa_{CA}\kappa_{AC}}{(\kappa_B - \kappa_C)(\kappa_A - \kappa_C) - \kappa_{AB}\kappa_{BA}}.
$$

Substituting Eq. (4.73) into the rate equation Eq. (4.68) with the matrix Eq. (4.70), and using our results for $q^{(0)}$ and $q^{(1)}$, we recover Eq. (4.69) at zeroth order:

$$
\begin{align*}
s_B^{(0)} &= 0, \\
s_A^{(0)} &= 0.
\end{align*}
$$
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At first order, we find

\[ \epsilon_s^{(1)} B = \kappa_B \kappa_A \left( \kappa_B - \kappa_C \right) \left( \kappa_A - \kappa_C \right) - \kappa_{AB} \kappa_{BA}, \]  

(4.78)

\[ \epsilon_s^{(1)} A = \left( \kappa_B - \kappa_C \right) \kappa_A \left( \kappa_B - \kappa_C \right) \left( \kappa_A - \kappa_C \right) - \kappa_{AB} \kappa_{BA}. \]  

(4.79)

As decay rates can be assumed exponentially small, and as \( C \) is the longest-lived vacuum by assumption, we may set

\[ \kappa_A - \kappa_C \approx \kappa_A, \quad \kappa_B - \kappa_C \approx \kappa_B. \]  

(4.80)

Thus, to first order in the up-tunneling branching ratio \( \epsilon = \beta_{AC} \), we find that the initial probability for vacuum \( B \) is given by

\[ P_B \approx s_B \approx \frac{\kappa_{BA} \kappa_{AC}}{\kappa_B \kappa_A - \kappa_{AB} \kappa_{BA}} + O(\epsilon^2). \]  

(4.81)

Recall from Eq. (4.36) that a sufficient condition for ordinary observers to win in this toy model is

\[ \frac{P_B}{P_C} < \frac{\beta_{BA} \beta_{AC}}{\Gamma_{BB,B}}. \]  

(4.82)

With \( P_C \approx 1 \) and Eq. (4.81), this condition reduces to

\[ \beta_{AB} \beta_{BA} + \frac{\Gamma_{BB,B}}{\kappa_B} \kappa_C < 1, \]  

(4.83)

This condition is easily satisfied, since Eq. (4.16) implies that both terms on the left hand side are double-exponentially small. Recall that Eq. (4.16) [or more generally Eq. (4.41)] must be assumed; if it did not hold, Boltzmann brains would dominate already at zeroth order.

To see this in detail, we recall that \( \Gamma_{BB,B} \) is double-exponentially small, whereas the powers of \( \Lambda_B \) by which \( \kappa_B \) differs from \( \Gamma_B \) can be assumed to be at most exponentially small. Thus, the second term in Eq. (4.83) can be approximated as \( \frac{\Gamma_{BB,B}}{\kappa_B} \kappa_C \ll \kappa_C \ll 1 \). Note also that vacuum \( A \) must have more free energy than ordinary observers in vacuum \( B \), which in turn have more free energy than Boltzmann brains in vacuum \( B \). This implies that \( BB \) production in \( B \) is enhanced compared to up-tunneling: \( \Gamma_{AB} \ll \Gamma_{BB,B} \). Together with Eq. (4.16), this implies \( \beta_{AB} \ll \Gamma_{BB,B} \ll 1 \). Since \( \beta_{BA} < 1 \), we conclude that the first term on the left hand side of Eq. (4.83), too, is very small.

Thus if ordinary observers win with initial conditions entirely in the longest-lived vacuum \( C \), then they will still win when initial conditions are refined to reflect the dominant eigenvector at first order in the up-tunneling branching ratio from \( C \), including the support in vacua other than \( C \). Furthermore, they win easily: The probability to start in \( B \) has a structure similar to the branching ratio along paths from the dominant vacuum \( C \) to \( B \). For \( OO \)s to win this must only be bounded by that exact branching ratio multiplied by the huge
factor $\Gamma^{-1}_{BB,B}$. By comparison, recall that a sufficient condition Eq. 4.58 for no arrow in the same toy model with Hartle-Hawking initial conditions was also easily satisfied when the decay from $C$ to the terminal $T'$ was large compared to the double exponentially suppressed quantity $\Gamma_B N^{-1}_{BB,B}$.

**Large Landscape**

Consider the general landscape of Sec. 4.4, subject to the assumptions stated there. These assumptions ensure that ordinary observers win with dominant eigenvector initial conditions, in the approximation that initial conditions have support entirely in the longest-lived de Sitter vacuum. We argue that this conclusion survives corrections that take into account that the dominant eigenvector has small support in other vacua as well.

In a terminal landscape, up-tunnelings will generally be suppressed compared to down-tunnelings. Thus it is appropriate to consider a general method of perturbation theory in up-tunneling branching ratios, which is discussed in detail in App. E. At leading order, the correction to single-vacuum initial conditions is given by

$$s_i^{(n_0)} = \sum_p \sum^{(n_0)}_{i_1, \ldots, i_{p-1}} \frac{\kappa_{ii_{p-1}}}{D_i - D_*} \cdots \frac{\kappa_{i_1*}}{D_{i_1} - D_*} .$$

(4.84)

The superscript $(n_0)$ indicates summation over paths with exactly $n_0$ up-tunnelings and an arbitrary number of down-tunnelings. Approximating down-tunneling rates by total decay rates, and neglecting $\kappa_*$ compared to $\kappa_i$ for all $i$, we obtain

$$s_i^{(n_0)} = \sum_p \sum^{(n_0)}_{i_1, \ldots, i_{p-1}} \frac{\kappa_s}{\kappa_i} \beta_{ii_{p-1}} \cdots \beta_{i_1*} .$$

(4.85)

Notice that this result involves precisely the product of branching ratios that determine $e_{i*}$ at the first nonzero order in the number of up-tunnelings. Assuming both $e_i$ and $s_i$ are well approximated at this order, we may conclude that

$$s_i \approx \frac{\kappa_s}{\kappa_i} e_{i*} .$$

(4.86)

Thus, the probability to start in vacuum $i$, $P_i \approx s_i$, is much smaller than the amplitude to decay to $i$ from $*$, $e_{i*}$. Multiplying each by the production rate of Boltzmann brains in vacuum $i$, the former yields the correction we seek while the latter, by the results of the previous section, is the zeroth order amplitude for Boltzmann brains. Since $BB$s lose at zeroth order, this shows that the corrections cannot change the outcome.

To see this in detail, recall that we have already established from Eq. 4.49 that

$$e_{BB,*} < e_{OO,*} .$$

(4.87)
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This inequality implies the dominance of ordinary observers at zeroth order, i.e., with initial conditions purely in *. Moreover, by Eqs. (4.12), (4.13), and (4.41), the number of Boltzmann brains produced along paths that start in the * vacuum is very small:

\[ e_{BB,*} = \sum_j \frac{\kappa_{BB,j}}{\kappa_j} e_{j,*}. \]  

(4.88)

We will now combine these results with Eq. (4.86) to show that ordinary observers will still win with initial conditions given by the full dominant eigenvector.

Technically, we must demonstrate that Eq. (4.50) is satisfied. Since \( s_i \) is small for \( i \neq * \), we can set \( P_i \approx s_i \) and use Eq. (4.86) to bound the left hand side of Eq. (4.50):

\[ \sum_j P_j \Gamma_{BB,j} \approx \kappa_* \sum_j \frac{\Gamma_{BB,j}}{\kappa_j} e_{j,*} < \kappa_* e_{BB,*}. \]  

(4.89)

The sum is taken over all \( j \) since \( \Gamma_{BB,*} = 0 \), so the contribution for \( j = * \) vanishes. Also, in the second step we have used Eq. (4.88) and the fact that \( \Gamma_{BB,j} < \kappa_{BB,j} \) in Planck units. With Eq. (4.87), it follows that

\[ \sum_j P_j \Gamma_{BB,j} < \kappa_* e_{OO,*} \ll e_{OO,*} \approx \sum_i P_i e_{OO,i}, \]  

(4.90)

so that Eq. (4.50) is indeed satisfied. Ordinary observers win.
Chapter 5

Anthropic Origin of the Neutrino Mass from Cooling Failure

5.1 Introduction

In a theory with a large multidimensional potential landscape [122], the smallness of the cosmological constant can be anthropically explained [17]. The lack of a viable alternative explanation for a small or vanishing cosmological constant, the increasing evidence for a fine-tuned weak scale, and several other complexity-favoring coincidences and tunings in cosmology and the Standard Model, all motivate us to consider landscape models seriously, and to extract further pre- or post-dictions from them.

A large landscape can also explain an aspect of the Standard Model that has long remained mysterious: the origin of the masses and mixing angles of the quarks and leptons. Plausible landscape models allow for some of the first generation quark and lepton masses to be anthropically determined, while the remaining parameters are set purely by the statistical distribution of the Yukawa matrices. Results are consistent with the observed hierarchical, generation, and pairing structures [149, 150, 151, 152, 153, 154, 155, 156]. In such analyses, the overall mass scale of neutrinos may be held fixed and ascribed, e.g., to a seesaw mechanism. But ultimately, one expects that the mass scale will vary, no matter what the dominant origin of neutrino masses is in the landscape. For Dirac neutrinos, Yukawa couplings can vary; in the seesaw, a coupling or the right-handed neutrino mass scale can vary.

Thus we may ask whether anthropic constraints play a role in determining the overall scale, or sum, of the standard model neutrino masses,

\[ m_\nu \equiv m(\nu_e) + m(\nu_\mu) + m(\nu_\tau) \]  \hspace{1cm} (5.1)

1It cannot be explained in a one-dimensional landscape, no matter how large [123, 124, 125], because an empty universe is produced. The string theory landscape [122, 126, 127] is an example of a multidimensional landscape in which the cosmological constant scans densely and our vacuum can be produced with sufficient free energy. Related early work includes [128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141]. Reviews with varying ranges of detail and technicality are available, for example [142, 143, 144, 145, 146, 147, 148].
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Current observational bounds imply

$$58 \text{meV} \leq m_\nu \lesssim 0.23 \text{eV}.$$ (5.2)

The lower bound comes from the mass splittings observed via solar and atmospheric neutrino oscillations [157]. The upper bound comes from cosmological observations that have excluded the effects that more massive neutrinos would have had on the cosmic microwave background and on large scale structure [158, 5]. The proximity of the lower to the upper bound gives us confidence that cosmological experiments in the coming decade will detect $m_\nu$, and that they may determine its value with a precision approaching the $10^{-2}$eV level [159].

An anthropic origin of the neutrino mass scale is suggested by the remarkable coincidence that neutrinos have affected cosmology just enough for their effects to be noticeable, but not
enough to significantly diminish the abundance of galaxies. A priori, $m_\nu$ could range over dozens of orders of magnitude. If $m_\nu$ was only two orders of magnitude smaller than the observed value, its effects on cosmology would be hard to discern at all. If $m_\nu$ was slightly larger, fewer galaxies would form, and hence fewer observers like us. The goal of this paper is to assess this question quantitatively.

The basic framework for computing probabilities in a large landscape of vacua is reviewed in Sec. 5.2, and the probability distribution $dP/d\log m_\nu$ is computed in Sec. 5.3. In the remainder of this introduction, we will describe the key physical effects that enter into the analysis, and we will present our main results.

**Summary:** There are two competing effects that determine the neutrino mass sum. We assume that the statistical distribution of neutrino masses among the vacua of the landscape favors a large neutrino mass sum, with a force of order unity or less (see Sec. 5.2 for the definition of the multiverse force).

If the anthropic approach is successful, we must demonstrate a compensating effect: that neutrino masses much greater than the observed value are not frequently observed. That is, we must multiply the prior probability for some value of $m_\nu$ by the number of observers that will be produced in regions where $m_\nu$ takes this value. Observers are usually represented by some proxy such as galaxies. We consider two models for observations: at any given time, their rate is proportional to the number of Milky Way-like galaxies, or proportional to the growth rate of this galaxy population (see Sec. 5.2). We sum this rate over a spacetime region called the causal patch [15] (a standard regulator for the divergent spacetime that results from a positive cosmological constant; see Sec. 5.2). The product of prior distribution and the abundance of galaxies yields a predicted probability distribution. As usual, if the observed value lies some number of standard deviations from the mean of the predicted distribution, we reject the model (in this case the anthropic approach to $m_\nu$) at the corresponding level of confidence.

The neutrino mass spectrum—the individual distribution of masses among the three active neutrinos—has a noticeable effect on structure formation. We consider two extreme cases. In the *normal hierarchy*, one neutrino contributes dominantly to the mass sum $m_\nu$; here we approximate the remaining two as massless. In this case the observed mass splittings require $m_\nu \geq 58$ meV. In the *degenerate hierarchy*, each mass is of order $m_\nu/3$ (and here we approximate them as exactly equal). This case will soon be tested by cosmological observations, since the observed mass splittings would imply $m_\nu \gtrsim 150$ meV, near the present upper limit. We do not explicitly consider the intermediate case of an inverted hierarchy, with two nearly degenerate massive neutrinos and one light or massless neutrino.

The main challenge lies in estimating the galaxy abundance as a function of $m_\nu$. The effects of one or more massive neutrinos on structure formation are somewhat complex; hence, we compute the linear evolution of density perturbations numerically using Boltzmann codes CAMB [160] and CLASS [161], wherever possible. We will now summarize the key physical effects. A more extensive summary and analytic approximations are given in Sec. 5.3 and Appendix G; we recommend Refs. [162, 163] for detailed study.


Figure 5.2: No cooling boundary: if one assumed that observers trace dark matter halos of mass $10^{12} M_\odot$ or greater, one would find a bimodal probability distribution over the neutrino mass sum $m_\nu$. This distribution is shown here for a normal hierarchy (orange/upper curve) and degenerate hierarchy (green curve). The range of values $m_\nu$ consistent with observation ($58$ meV $< m_\nu < 0.23$ eV, shaded in red) is greatly disfavored, ruling out this model. — By contrast, we shall assume here that observers trace galaxies. Crucially, we shall argue that for $m_\nu \gtrsim 10$ eV, galaxies do not form even though halos do. This novel catastrophic boundary excludes the mass range above 10 eV, leading to a successful anthropic explanation of the neutrino mass (see Fig. 5.3).

After becoming nonrelativistic, neutrinos contribute approximately as pressureless matter to the Friedmann equation. However, they contribute very differently from cold dark matter (CDM) to the growth of perturbations, because neutrinos are light and move fast. This introduces a new physical scale into the problem of structure formation: the free-streaming scale is set by the distance over which neutrinos travel until becoming nonrelativistic. It is roughly given by the horizon scale when they become nonrelativistic, with comoving wavenumber $k_{nr}$ (see Appendix G for more details). On this and smaller scales, $k \gtrsim k_{nr}$, neutrinos wipe out their own density perturbations. More importantly, as a nonclustering matter component they change the rate at which CDM perturbations grow, from linear growth in the scale factor ($\delta \propto a$) on large scales, to sub-linear growth on smaller scales $k \gtrsim k_{nr}$. This suppresses the CDM power spectrum at small scales, see Fig. 5.1a.

The linear quantity most closely related to the abundance of dark matter halos on the galactic scale $R$ is not the dimensionless CDM power spectrum $k^3P_{cc}(k)$. Rather, halo abundance is controlled by the smoothed density contrast $\sigma_R$, which is approximately given by the integrated power,

$$\sigma_R \sim \int_0^{1/R} \frac{dk}{k} k^3 P_{cc}(k) ,$$

up to the wavenumber corresponding to the relevant scale. (A more precise formula is given in the main text, where we also describe in detail how halo abundance is computed from
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\( \sigma_R \) using the Press-Schechter formalism.) This distinction turns out to be crucial for large neutrino masses.

We see from Fig. 5.1a that for small neutrino masses \( m_\nu \lesssim 10 \text{ eV} \), the integrand \( k^3 P_{cc}(k) \) increases monotonically. Hence the integral for \( \sigma_R \) is dominated by its upper limit, i.e., by the power on the scale \( k_{gal} \sim 1/R \). This yields the “bottom-up” scenario of hierarchical structure formation familiar from our own universe: small halos typically form first, and more massive halos virialize later.

However, for \( m_\nu \gtrsim 10 \text{ eV} \), the small scale power becomes so suppressed that the dimensionless power spectrum develops a maximum at the free-streaming scale \( k_{nr} < k_{gal} \). In this regime, the smoothed density contrast \( \sigma_R \) on galactic scales \( R \) is no longer dominated by the power at wavenumber \( \sim 1/R \). Instead, the power at larger scales than \( R \) contributes dominantly to \( \sigma_R \). This results in a top-down scenario, where halos first form on cluster scales, nearly simultaneously with galactic-scale halos.

The transition from bottom-up to top-down structure formation around \( m_\nu \approx 10 \text{ eV} \) has not (to our knowledge) been noted in the context of anthropic explanations of the neutrino mass sum. We find here that it is crucial to the analysis, for two reasons. First, it implies that the scales that dominantly contribute to \( \sigma_R \) are unaffected by free-streaming for \( m_\nu \gtrsim 10 \text{ eV} \). Therefore, increasing \( m_\nu \) beyond \( 10 \text{ eV} \) does not suppress CDM structure. In fact, we find that \( \sigma_R \) increases in this range (Fig. 5.1b). The second implication works in the opposite direction: in the top-down scenario that arises for \( m_\nu \gtrsim 10 \text{ eV} \), galaxies will not form inside halos at an abundance comparable to our universe.

Let us discuss each of these implications in turn. We begin by pretending that the stellar mass per halo mass is unaffected by \( m_\nu \); in particular, let us suppose that there is no dramatic suppression of star formation in the top-down regime, \( m_\nu \gtrsim 10 \text{ eV} \). If so, we would be justified in regarding halos as a fair proxy for observers. Here we consider \( 10^{12} M_\odot \) halos [164]. From Fig. 5.1b, we see that halo abundance decreases with \( m_\nu \) up to \( m_\nu \sim 10 \text{ eV} \); then it begins to increase. Combining this with the assumed prior distribution that favors large \( m_\nu \), we would find the probability distribution over \( m_\nu \) is bimodal (Fig. 5.2). The first peak is at \( m_\nu \approx 1 \text{ eV} \), followed by a minimum near \( 10 \text{ eV} \) and a second peak at much greater mass.\(^2\) Therefore, if observers traced dark matter halos with \( M \gtrsim 10^{12} M_\odot \), one should conclude that small neutrino masses are greatly disfavored. Such a result would be in significant tension with the current upper bound of \( 0.23 \text{ eV} \), and it would seem to render an anthropic origin of the neutrino mass sum implausible.

However, our fundamental assumption is that observers trace galaxies, not halos. In some cosmologies including our own, galaxies in turn trace halos; if they do, halos are an equally good proxy. But the change of regime from bottom-up to top-down structure formation for \( m_\nu \gtrsim 10 \text{ eV} \) is catastrophic for galaxy formation.

\(^2\)Fig. 5.2 does not show the entire peak since CAMB gives results only for \( m_\nu \lesssim 40 \text{ eV} \). Absent the earlier catastrophic boundary at \( 10 \text{ eV} \) that we will assert, a robust effect that would eventually suppress the probability at large neutrino mass is the smallness of the baryon fraction for \( m_\nu \gtrsim 100 \text{ eV} \). This would suppress the number of baryons (and hence, observers) in the causal patch [165]. It would also impose dynamical obstructions to star formation [166].
Figure 5.3: Our main result: the probability distribution over the neutrino mass sum $m_\nu$ for (a) a normal hierarchy and (b) a degenerate hierarchy, assuming that observers require galaxies. The plot is the same as Fig. 5.2, but the mass range is cut off at 7.7 eV (10.8 eV) in the normal (degenerate) case. For greater masses, the first halos form late and are of cluster size; we argue that galaxies do not form efficiently in such halos. We use the second observer model described in Sec. 5.2; results look nearly identical with the first model. We assume a flat prior over $m_\nu$ (see Fig. 5.4 for other priors).—The central $1\sigma$ and $2\sigma$ regions are shaded. Vertical red lines indicate the lowest possible values for the neutrino mass consistent with available data: $m_\text{obs} \approx 58$ meV for a normal hierarchy, and $m_\text{obs} \approx 150$ meV for a degenerate hierarchy. We find that these values are within $2\sigma$ of the median. The agreement would further improve with a less conservative treatment of the detrimental effects of neutrinos on gas cooling in halos, and/or the cosmological detection of a neutrino mass sum larger than the minimal value.

From observation, we know that stars do not form efficiently in bound structures that are much larger than the mass scale of our own galactic halo, $10^{12} M_\odot$. Heuristically, this can be explained by noting that in halos of this size, the cooling timescale for the baryonic gas is greater than the age of the universe [167, 168, 169, 170, 166]. In our universe there are galaxies because galactic halos, which produce stars efficiently, formed earlier than these larger halos, which do not. Clusters inherit galaxies that formed in smaller halos, but they do not have significant star formation themselves.

In a top-down scenario due to large neutrino mass, however, galactic halos would form much later. They would typically be embedded in larger halos that virialize roughly at the same time, with masses characteristic of galaxy groups clusters—but without many galaxies to inherit. The virial temperature and dynamical timescale relevant for baryon cooling will be set by the largest of the nested halos. (See Sec. 5.3 and Appendix I for details.) Therefore, cooling will not be efficient: the top-down scenario produces star-poor dark matter clumps, with most baryons remaining in hot gas.

As a first approximation for this cooling boundary, we cut off the probability distribution at a value $m_\nu \sim 10$ eV that corresponds to the onset of the top-down regime. This overestimates the amount of galaxies just below the cutoff and underestimates it just above. In future
work, we plan to include explicit models for successful galaxy cooling beyond the crude top-down vs. bottom-up criterion. This should replace the sharp cutoff by a smooth decay of the probability.

We believe that our argument for a cooling catastrophe is robust, because the transition to a top-down scenario is a drastic change of regime. However, the underlying physics is complicated, involving shocks, complicated cooling functions, fragmentation, and feedback from stars, black holes, and supernovae. Suppose therefore that we are wrong. That is, suppose that at $m_\nu \gtrsim 10\text{ eV}$, some unanticipated combination of processes lead to a stellar mass inside the causal patch that is not much less than in our universe. Then one would find that large neutrino masses are unsuppressed (Fig. 5.2), and the observed value of $m_\nu$ cannot be explained anthropically. In this sense, the cooling catastrophe we assert can be regarded as a prediction of the anthropic approach to the neutrino mass. To test this prediction, it will be important to investigate galaxy formation for $m_\nu \gtrsim 10\text{ eV}$ using simulations that give an adequate treatment of cooling flows and feedback.

**Results:** Our main results, with the cooling cutoff $m_\nu \lesssim 10\text{ eV}$ imposed, are shown in Fig. 5.3. We find that the currently allowed range of values for $m_\nu$ is entirely consistent with an anthropic explanation, at better than $2\sigma$. Fig. 5.4 shows that our approach succeeds for a wide range of prior distributions $dP_{\text{vac}}/dm_\nu \propto m_\nu^{-1}$: assuming a normal (degenerate) hierarchy, $m_{\text{obs}}$ lies within $2\sigma$ of the median if $0.09 < n \leq 1.0$ ($0.09 < n < 1.4$).

Our chief conclusion is that the neutrino mass sum can be anthropically explained, but only if detrimental effects of neutrinos on galaxy and star formation (rather than halo formation) already become significant at or below $m_\nu \approx 10\text{ eV}$.

Our results favor larger $m_\nu$ than the minimum values allowed by the observed mass splittings, and in particular they favor a degenerate over a normal hierarchy. Since the observed range is consistent within $2\sigma$ in either case, these are mild preferences rather than sharp predictions.

There are however two additional reasons why a degenerate hierarchy appears more natural in the context of the anthropic approach. First, with a normal hierarchy one might expect that each neutrino mass scans separately with prior $n_i$. Each prior would have to be assumed positive and $O(1)$. The prior for $m_\nu$ would then be $n = \sum n_i$, and it becomes less plausible that $n$ should be small enough to render the anthropic prediction compatible with observation. A degenerate hierarchy, on the other hand, may be the result of some flavor symmetry that links the masses of the individual neutrinos, leaving only a single scanning parameter. Then it is more plausible that $n$ is small enough to include the observed $m_\nu$.

The second reason to prefer a degenerate hierarchy is that it eliminates a viable anthropic window where two neutrinos are extremely massive. If each neutrino has mass of order MeV or greater, neutrons would be stable, leading to a (catastrophic) helium-dominated universe [171]. But neutrons will be unstable and the catastrophe is averted, if one neutrino remains light and only the other two become very heavy. With a normal or inverted hierarchy, one has to explain why the one or two heavy neutrinos did not end up in the extremely large mass range above the MeV scale. This can be resolved by assuming that the prior
Figure 5.4: The prior distribution of cosmologically produced vacua is assumed to favor large neutrino mass and to have no special feature near the observed magnitude: \(dP_{\text{vac}}/d\log m_\nu \propto m_\nu^n, n > 0\). One then expects \(n \sim O(1)\), and the previous two figures all show the case \(n = 1\). This figure shows that the same conclusions obtain for a considerable range of \(n\). (a) The median of the probability distribution as a function of the multiverse force \(n\). (b) The standard deviation of the worst case observed value from the median, as a function of \(n\). The 2\(\sigma\) region is shaded. Orange (the upper curve at large \(n\) in each plot) is for a normal hierarchy; green is for a degenerate hierarchy.

Distributions for the individual neutrino masses do have a feature between the eV and the MeV scale, such that the much larger scale is disfavored. With a degenerate hierarchy, this problem does not arise in the first place, since either all neutrinos are light or all are heavy.

**Relation to earlier work:** Our analysis builds on the pioneering work of Tegmark, Vilenkin and Pogosian \[171, 172\] (see also \[173, 174\]), who were the first to argue that the neutrino mass admits an anthropic explanation. We agree with their conclusion, but we claim here that the nature of the relevant catastrophic boundary was not correctly identified.

Ref. \[171\] does not justify its restriction to the region \(m_\nu \lesssim 10\) eV. Moreover, it employs an analytic approximation to \(\sigma_R\) that greatly underestimates the halo abundance for \(m_\nu \gtrsim 5 - 10\) eV. With this approximation, the probability distribution appears to vanish near 10 eV due to a paucity of CDM structure; see Fig. 5.5. Thus, suppression of CDM structure due to massive neutrinos—rather than the obstruction to cooling at \(m \gtrsim 10\) eV—would appear to provide the relevant catastrophic boundary underlying the anthropic explanation of the neutrino mass sum.

Here we go further in two respects: our numerical computations show that CDM structure becomes unsuppressed for \(m_\nu \gtrsim 10\) eV. Hence, if neutrino masses have an anthropic origin, a different catastrophic boundary is relevant. And we identify a specific physical effect, the transition to a top-down regime, which had not been noted and which supplies a suitable boundary by suppressing galaxy formation.

The analytic approximation in question is Eq. (5) in Ref. \[171\]. It assumes that massive neutrinos suppress the smoothed density contrast \(\sigma_R\) on galactic scales by the same factor by
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FAILURE

Figure 5.5: The dashed black line shows the probability distribution found by Tegmark et al. [171] for a flat prior over $m_\nu$. This result would seem to remain compatible at about $2\sigma$ with current observational constraints (red shaded region). However, the analytic fitting function for the density contrast $\sigma_R$ used in [171, 172] underestimates $\sigma_R$ above a few eV. The solid curves show the probability distribution that results when $\sigma_R$ is computed numerically from Boltzmann codes: orange/upper=normal hierarchy; green/lower=degenerate hierarchy. They differ slightly from Fig. 5.2 because Ref. [171] used a different measure and observer model. Either way, a successful anthropic explanation of $m_\nu$ requires the identification of a catastrophic boundary at or below 10 eV.

which they suppress the matter power on galactic scales. This is accurate for small neutrino masses, because in a bottom-up scenario the shortest scales dominate the integral for $\sigma_R$. The approximation underestimates the abundance of dark matter halos for $m_\nu \gtrsim 10$ eV, because in this regime $\sigma_R$ is dominated by power at larger scales, which is relatively unsuppressed by free-streaming. More details can be found after Eq. (5.3) and in Sec. 5.3.

The discrepancy is revealed by explicit numerical computation of the smoothed density contrast $\sigma_R$ on galactic scales from Boltzmann codes (see Fig. 5.1). One also finds significantly different results for a normal versus degenerate hierarchy, a distinction that was suppressed in the analytical approximation of Ref. [171].

When the halo abundance is correctly computed, the need for a novel catastrophic boundary at or before 10 eV becomes evident (Fig. 5.5). Without it, the probability distribution would strongly disfavor small neutrino masses. It would be in significant tension with the an upper bound of 0.23 eV or even 1 eV, and it would seem to render an anthropic origin of the neutrino mass sum implausible.

Our computation of $\sigma_R$ from Boltzmann codes, and our compensating identification of a novel catastrophic boundary at 10 eV are the main differences to Ref. [171]. Another difference is that we use the causal patch measure [15] to regulate the infinities of eternal inflation. Refs. [171, 172] used a different measure that is no longer viable; see Sec. 5.2 for details. This has a visible but comparatively small effect on the probability distribution: by comparing Fig. 5.2 with Fig. 5.5, one sees that the causal patch is somewhat more favorable
to an anthropic explanation of $m_\nu$. The causal patch also renders more robust our conjecture that star formation is ineffective for $m_\nu \gtrsim 10$ eV, as discussed in more detail later.

We also build on the seminal investigation of catastrophic boundaries in cosmology by Tegmark et al. [166] (see also Ref. [175]), who emphasized the crucial role of cooling. We believe that our present work is the first to associate catastrophic cooling failure to a top-down structure formation scenario. Ref. [171] points out a number of distinct catastrophes at very large neutrino mass: For example, neutrinos act as cold dark matter for $m_\nu \gg 100$ eV, which also may be detrimental to star formation. (However, this does not counteract the abundance of CDM structure we find at $m_\nu \gtrsim 10$ eV. Fig. 5.5 illustrates that a cutoff at any scale larger than 10 eV, say at $m_\nu \approx 30$ eV, would make small neutrino masses too improbable for an anthropic explanation to work.)

5.2 Predictions in a Large Landscape

If a theory has a large number of metastable vacua, most predictions will be statistical in nature. We are usually interested in understanding the magnitude of a particular parameter $x$, such as the cosmological constant or in the present case, the neutrino mass; hence we wish to compute a probability distribution $dP/d\log x$.

Fundamentally, the probability $dP$ is proportional to the number of observations $dN_{\text{obs}}$ that find the parameter to lie in the range $(\log x, \log x + d\log x)$. Thus, our task is to compute $dN_{\text{obs}}$. This can be done by weighting a prior probability distribution $f(x)$, which comes from the underlying theory, by the number $w(x)$ of observations that will be made in a vacuum where $x$ takes on a particular value:

$$
\frac{dP}{d\log x} = w(x)f(x). \quad (5.4)
$$

We will discuss each factor in turn. Our presentation in each subsection will be general at first, before specializing to the case of the neutrino mass, $x = m_\nu$.

Prior as a Multiverse Probability Force

The prior is defined by

$$
f(x) = \frac{dN_{\text{vac}}}{d\log x}, \quad (5.5)
$$

Here $x$ will be a parameter in the effective theory at low energies whose scale $\log x$ one would like to predict or explain; $dN_{\text{vac}} = f(x)d\log x$ is the number of long-lived metastable vacua\(^3\) in which the parameter takes on values in the range $(\log x, \log x + d\log x)$.

\(^3\)Strictly, what matters is not the abundance of such vacua in the effective potential but in the multiverse: cosmological dynamics could favor the production of some vacua over others. For most low-energy parameters one expects that such selection effects are uncorrelated with $x$ in the range of interest. In any case, we shall take the prior $f$ to be an effective distribution that incorporates cosmological dynamics.
With the notable exception of the cosmological constant, the prior distribution for most parameters is not well known. This is a technical problem: in the string landscape, $f(x)$ should in principle be computable. In practice, it is difficult to derive phenomena far below the fundamental scale (the Planck or string scale) directly. However, this need not be an obstruction to progress, any more than the fact that we cannot derive the Standard Model from a more fundamental theory prevented us from discovering it.

Consider an arbitrary low-energy parameter $x$. In any large landscape the prior distribution $f(x)$ should admit an effective description [149, 176, 175]. To avoid putting in the answer, one may assume that $f(x)$ has no special features (such as a maximum) in a wide logarithmic range of values. This range should include but be much larger than the range compatible with observation. One can then parametrize the prior distribution by a “statistical pressure” or “multiverse probability force” towards large or smaller values,

$$n \equiv \frac{d \log f}{d \log x},$$

where $n$ is approximately constant. For example, a flat prior distribution over $\log x$ corresponds to $n = 0$. If the prior is flat over $x$, $dN_{\text{vac}}/dx = \text{const.}$, then $n = 1$.

Suppose that there is a regime change sufficiently near the observed value $\log x_0$, such that the number of observers (or at least, of observers like us) $w(x)$ drops dramatically above or below a critical value $\log x_c$. Suppose for definiteness that $x_0 \lesssim x_c$. If the probability force favors large values of $x$, but not too strongly $[n > 0, n \sim O(1)]$, then the observed value can be explained. Similarly, with a negative probability force, one can explain the proximity of $x$ to nearby catastrophic boundary at some smaller $x_c \lesssim x_0$.

Recent successful examples of this approach include an explanation of the coincidence that dark and baryonic densities are comparable [165], the fine-tuning of the weak scale [177, 165, 178], and the comparability of several large, a priori unrelated timescales in cosmology [175]. In each case, the required assumption about the probability force is weak and qualitative: $n \sim \pm O(1)$. Thus phenomenological models of the landscape have significant explanatory value, while constraining the underlying prior distributions through the sign (and roughly the strength) of the probability force $n$. It is particularly instructive to keep track of the combination of (and possible conflict between) forces $n_i$ needed to simultaneously explain multiple parameters $x_i$ [175, 178, 179].

Now let us turn to the prior for the total neutrino mass, $P_{\text{vac}}(m_\nu)$. We know of no physical reason why a minimum neutrino mass should be necessary for observers. Hence, to obtain a normalizable probability distribution $f$, we must assume that the effective prior distribution favors large $m_\nu$:

$$\frac{dP_{\text{vac}}}{d \log m_\nu} \propto m_\nu^n, \quad n > 0,$$

in some large logarithmic neighborhood of the observed value, $\sim 0.1$ eV. A natural and simple choice is $n = 1$, and we will use this value for definiteness in most plots. More generally, we will find that a comfortable range of values $0 < n \approx O(1)$ is consistent with an anthropic explanation of the neutrino mass, but not a value much greater than 1 (see Fig. 5.4).
Anthropic Weighting

The probability distribution over $\log x$ relevant for comparing the theory with observation is obtained by conditioning $\tilde{p}$ on the presence of observers. More quantitatively, one weights by the number of observations

$$w(x) = \frac{dN_{\text{obs}}}{dN_{\text{vac}}}$$

that are made in a vacuum where $x$ takes on a specific value. Generically, $w(x)$ will be unsuppressed in a large region either above or below the observed value, or both. Thus, the anthropic factor is not doing all the work; the prior distribution is crucial for comparing the theory to observation.

In this paper we will consider two different models for the number of observations $w(x)$ that are performed in the universe. Both are based on the assumption that observers require galaxies, say of halo mass comparable to the Milky Way’s, $10^{12}$ solar masses. The first model assumes that the rate at which observations occur in a given spatial region per unit proper time, $\dot{w}(x)$, is proportional to the total mass $M_{\text{gal}}$ of such galaxies, at every instant; hence

$$w(x) = \int dt \ M_{\text{gal}}(t) \quad \text{(Observer Model 1)}. \quad (5.9)$$

The second model (which reduces to the choice made in [171]) assumes instead that the rate of observation is proportional to the rate $\dot{M}$ at which the above total galaxy mass grows:

$$w(x) = \int dt \ \dot{M}_{\text{gal}}(t) \quad \text{(Observer Model 2)}. \quad (5.10)$$

The two models can be thought of as two different approximations taken to an extreme. In the first, observations would be made continuously in the galaxy, at fixed rate per unit stellar mass, no matter how old the stars become. In the second model, observations would occur instantaneously as baryons cool and form stars; no observations would be assigned to a galaxy that is not growing. (The second model was used in Ref. [171]; note that in the context of the measure used there the integral over time is trivial, yielding the collapse fraction $F_R$.) The truth is likely somewhere in between the two models. However, we will find that our results depend only weakly on the model, so we expect our conclusions to be robust.

Measure

A cosmology with at least one long-lived de Sitter vacuum gives rise to eternal inflation: the universe will grow without bound and remain at finite temperature in arbitrarily large volumes at late times. Hence, all possible events will occur infinitely many times. This applies in particular to observations. Thus a regulator or “measure” must be introduced to obtain a finite anthropic factor $w(x)$. For this problem to exist, it is not necessary that the theory predict a large landscape; one de Sitter vacuum (such as, apparently, ours [6, 7]) is
enough. But the measure problem becomes particularly glaring in the landscape context: globally, every type of vacuum bubble is produced infinitely many times, and each bubble universe contains an infinite comoving volume.

Existing analyses of the anthropic origin of neutrino masses preceded a period of significant progress on the measure problem of eternal inflation. Following Weinberg [17], Refs. [171, 172] regulate the divergences of the cosmological dynamics by estimating the number of observers \textit{per baryon}. This measure can no longer be considered viable [10, 180]. Note, however, that our choice of measure is \textit{not} responsible for the main differences between our results and those of [171], as described at the end of Sec. 5.1.

In this paper, we will use the causal patch measure [15], which regulates eternal inflation by considering a single causally connected region and averaging over its possible histories. This proposal is very generally defined, requiring only causal structure. It is also well motivated: it merely applies to cosmology an existing restriction that was already needed for the unitary evaporation of black holes [14]. Though proposed on formal grounds, the causal patch has met with phenomenological success; two examples are described in Appendix F. We take this as evidence that it approximates the correct measure well (at least in regions with positive cosmological constant [181]).

A potential landscape is consistent with the observed cosmological history only if it is multi-dimensional with large energy differences between neighboring vacua [147].\footnote{The decay of our parent vacuum must release enough energy to heat our universe at least to the temperature of big bang nucleosynthesis, which requires \Delta \Lambda \gg 1 (MeV)^4. This is the reason why a multidimensional landscape is essential. One-dimensional “washboard” landscapes [123, 124, 125] are ruled out, because they must have \Delta \Lambda \lesssim 10^{-35} (MeV)^4 so as to naturally include at least one vacuum like ours.} String theory gives rise to such a structure upon compactification to three spatial dimensions [122], with \Delta \Lambda not much below unity.

The causal patch will contain a particular decay chain through de Sitter vacua in the landscape, ending with a big crunch in a vacuum with negative cosmological constant; each such chain is weighted by its probability, i.e., by the product of branching ratios [15]. For a typical decay chain, none of the vacua will have anomalously small cosmological constant \Lambda \ll \Delta \Lambda. Thus, after conditioning on observers, there will be one vacuum with small cosmological constant in the causal patch, and we need only be concerned with how the causal patch regulates the volume of the corresponding bubble universe.

Here we focus on the variation of the neutrino mass only, so we shall take this vacuum to be otherwise like ours. In particular we set the cosmological constant to the observed value, \Lambda \sim 10^{-123}, and we take the spatial geometry to be flat. The metric is of the Friedman-Robertson-Walker (FRW) type:

\[ ds^2 = -dt^2 + a(t)^2 (dr^2 + r^2 d\Omega^2), \quad (5.11) \]

where \( a \) is the scale factor, \( r \) is the comoving radius, \( t \) is proper time, and \( d\Omega^2 \) is the metric on the unit two-sphere.

By definition, the causal patch is the causal past of the future endpoint of a geodesic; thus its boundary consists of the past light-cone of such a point. We are interested in
the boundaries of the causal patch during the time when a long-lived de Sitter vacuum still contains matter. A future decay has an exponentially small effect on the location of the patch boundary at much earlier times, so the patch can be computed by treating the vacuum as completely stable. The patch boundary is thus the cosmological event horizon. Its comoving radius at FRW time $t$ is obtained by tracing a light-ray back from future de Sitter infinity:

$$r_{\text{patch}}(t) = \int_{t}^{\infty} \frac{dt'}{a(t')} .$$

(5.12)

The physical volume of the patch is

$$V_{\text{phys}}(t) = \frac{4\pi}{3} a(t)^3 r_{\text{patch}}(t)^3 .$$

(5.13)

As described in the previous subsection, we estimate the rate of observations per unit time as proportional to the total mass of all galaxies in the physical volume of the patch (for observer model 1), or to the rate of increase of this mass (for observer model 2). We can write this quantity as

$$M_{\text{gal}}(t) = \rho_{bc}(t) V_{\text{phys}}(t) F_R(t) G_R(t) .$$

(5.14)

The first two factors give the total mass $M_{bc}$ of baryons and cold dark matter in the patch at the time $t$. The collapse fraction $F_R$ is the fraction of this mass that is contained in halos of mass greater than $10^{12} M_\odot$, corresponding to a comoving distance scale $R$: $M_{\text{halo}} = M_{bc} F_R$. The galaxy fraction $G_R$ is the fraction of this latter mass that represents baryons in galaxies, $M_{\text{gal}} = M_{\text{halo}} G_R$.

Combining this with Eqs. (5.9), (5.7), and (5.4), the (unnormalized) probability distribution over the neutrino mass is given by

$$\frac{dp}{d \log m_\nu} \propto m_\nu^n \int dt (r_{\text{patch}} a)^3 \rho_{bc} F_R G_R .$$

(5.15)

in the first observer model; we replace $F_R G_R$ by $\frac{d}{dt} (F_R G_R)$ for the second observer model.\footnote{Note that the time derivative should not be taken of the entire integrand, for this model. The loss of mass across the horizon due to the shrinking comoving volume of the patch does not produce “negative galaxies” inside the patch. At some cost in readability, we could have made this more explicit by defining the integrand in Eq. (5.10) as the causal patch volume times the rate of change of the average physical density contributed by galaxies.}

Factors in the integrand may in general depend on both $m_\nu$ and $t$.

### 5.3 Calculation of $dP/d \log m_\nu$

#### Fixed, Variable, and Time-dependent Parameters

We will consider a one-parameter family\footnote{It would clearly be of interest to compute the probability distribution over several parameters including the neutrino mass; for examples of multivariate probability distributions in the landscape, see e.g. [172].} of cosmologies, differing from our universe only in the total mass of active neutrinos. More precisely, we consider two such families, since
we treat the cases of normal and degenerate neutrino hierarchy separately. Thus, we hold fixed all fundamental parameters other than $m_\nu$. In particular, we fix the vacuum energy density, $\rho_\Lambda = \Lambda / 8\pi G$, and the spatially flat geometry of the universe (imposed, presumably, by a mechanism like inflation that is uncorrelated with $m_\nu$). We also hold fixed $\chi_b \equiv \rho_b / n_\gamma$ and $\chi_c \equiv \rho_c / n_\gamma$, the masses per photon of baryons and CDM. These quantities remain invariant under changes of $m_\nu$, since we hold fixed the fundamental processes that produced the observed baryon and CDM abundances.

For the actual values of these parameters, we use the Planck TT+lowP+lensing+ext best fit cosmological parameters \cite{Ade:2015xua}; see Table 5.1. The best fit assumes a neutrino mass of about 0.06 eV \cite{Ade:2015xua}, whereas strictly, one should use a best fit marginalized over $m_\nu$ for the purposes of our paper. However, this has virtually no effect on the fixed cosmological parameters such as $\rho_\Lambda$, $\chi_b$, and $\chi_c$, because neutrinos are already constrained to contribute a very small fraction to the total density. For example, the best-fit for the Hubble parameter\footnote{Unless otherwise specified, we quote the Hubble parameter in units km s$^{-1}$ Mpc$^{-1}$ throughout.} (Planck TT+lowP+lensing+ext \cite{Ade:2015xua}) shifts from 67.9 $\pm$ 0.55 ($m_\nu \approx$ 0.06 eV) to 67.7 $\pm$ 0.6 (marginalized over $m_\nu$). This difference is negligible compared to current error bars and the discrepancies between different cosmological datasets.

When considering entire cosmological histories, as we do, it is best to specify each cosmology in terms of time-independent parameters such as $\Lambda$, $\chi_b$, $\chi_c$, and $m_\nu$. However, we use Boltzmann codes such as CAMB and CLASS to compute power spectra wherever possible (i.e., for $z \geq 0$). These codes expect input parameters that specify the cosmological model in terms of their present values, at redshift $z = 0$. It is not clear what one would mean by the “present” time in an alternate cosmology, but for the purposes of CAMB and CLASS, $z = 0$ is defined to be the time at which the CMB temperature takes the observed value, $T_{\text{CMB}} \approx 2.7$ K.

Thus we must derive the values of various time-dependent quantities at the time when the universe reaches this temperature, as a function of $m_\nu$, with other time-independent parameters fixed as described above. One finds for the Hubble parameter and the density parameters

\begin{align}
H(m_\nu; z = 0) &= H_0 \left( \frac{\chi_b \Lambda \nu}{\chi_b \Lambda \nu_0} \right)^{1/2} , \\
\Omega_X(m_\nu; z = 0) &= \frac{\chi_X}{\chi_b \Lambda \nu} , \quad X \in \{ b, c, \Lambda, \nu \} .
\end{align}

Here multiple indices imply summation, for example $\chi_{bc} \equiv \chi_b + \chi_c$. The fixed parameters $\chi_b$ and $\chi_c$ were defined above. The fixed parameter $\chi_\Lambda \equiv \rho_\Lambda / n_\gamma (z = 0)$ is defined for notational convenience as the observed vacuum energy per photon at the present observed
Table 5.1: The cosmological parameters used in our calculation, as well as the resulting mass per photon of baryons and CDM, $\chi_b$ and $\chi_c$. $T_{\text{CMB}}$ is a Planck TT+lowP+BAO fit, while all others are from Planck TT+lowP+lensing+ext best fit values. We take $k_{\text{pivot}} = 0.05$ Mpc$^{-1}$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{\text{CMB}}$</td>
<td>2.722 K</td>
</tr>
<tr>
<td>$H_0$</td>
<td>67.90</td>
</tr>
<tr>
<td>$\Omega_m$</td>
<td>0.3065</td>
</tr>
<tr>
<td>$\Omega_A$</td>
<td>0.6935</td>
</tr>
<tr>
<td>$\Omega_b h^2$</td>
<td>0.02227</td>
</tr>
<tr>
<td>$\Omega_c h^2$</td>
<td>0.1184</td>
</tr>
<tr>
<td>$10^9 A_s$</td>
<td>2.143</td>
</tr>
<tr>
<td>$n_s$</td>
<td>0.9681</td>
</tr>
<tr>
<td>$\chi_b$</td>
<td>0.5745 eV</td>
</tr>
<tr>
<td>$\chi_c$</td>
<td>3.054 eV</td>
</tr>
</tbody>
</table>

CMB temperature. The $m_\nu$-dependent parameter $\chi_\nu(m_\nu) = \frac{3}{11} m_\nu$ is the neutrino mass per photon. $H_0 \equiv H(0.06 \text{eV}; z = 0)$ and $\chi_\nu(0.06 \text{eV})$ are observed values, corresponding to the Planck best fit baseline model.

**Homogeneous Evolution**

For computing the volume of the causal patch, the factor $(r_{\text{patch}} a)^3$ in Eq. (5.15), we will need to know the scale factor. Unless structure is present, the integrand will be suppressed by the Press-Schechter factor $F_R$; hence it suffices to use an analytic solution valid to excellent approximation in the matter and vacuum eras:

$$a(m_\nu; t) = \left[ \cot \lambda \sinh \left( \frac{3 \sin \lambda}{2} H_0 t \right) \right]^{2/3},$$

(5.18)

The solution depends on $m_\nu$ through $\Omega_\Lambda(m_\nu; z = 0) \equiv \sin^2 \lambda$.

Since $\chi_{\text{bc}}$ does not depend on $m_\nu$ and $\rho_{\text{bc}} = n_{\gamma} \chi_{\text{bc}}$, $\rho_{\text{bc}}(z = 0)$ does not depend on $m_\nu$. Moreover, since the scale factor in Eq. (5.18) is normalized so that $a = 1$ at $z = 0$, we have $\rho_{\text{bc}}(t) = \rho_{\text{bc}}(z = 0)/a(t)^3$ for all values of $m_\nu$. Thus Eq. (5.15) simplifies to

$$\frac{d\mathcal{P}}{d \log m_\nu} \propto m_\nu^n \int dt r_{\text{patch}}^3 F_R G_R,$$

(5.19)

where $r_{\text{patch}}$ is given by Eq. (5.12), and we are dropping $m_\nu$-independent normalization factors as usual.
Figure 5.6: The comoving volume of the causal patch for $m_\nu = 0$ (black), $m_\nu = 4$ (blue), and $m_\nu = 8$ (red).

The comoving volume of the causal patch is shown in Fig. 5.6. We note that already at the homogeneous level, a nonzero neutrino mass is slightly disfavored because it decreases the size of the causal patch. We also note that the patch size is maximal at early times and decreases rapidly. Hence galaxies that form very late effectively fail to contribute to the probability for a given parameter value.

Halo Formation

The next factor in Eq. (5.19) is the collapse fraction $F_R(m_\nu, t)$. It captures the effects of neutrinos on structure formation: recall that $F_R$ is defined as the fraction of baryonic and cold dark matter that is contained in halos of mass $10^{12} M_\odot$ or greater. It captures the effects of neutrinos on structure formation. Recall that $F_R$ is defined as the fraction of baryonic and cold dark matter in virialized halos of mass scale $10^{12} M_\odot$ or greater. This corresponds to a comoving distance scale $R \sim 1.8$ Mpc.\(^8\)

The collapse fraction can be determined using the Press-Schechter formalism [182]. Before nonlinearities are important, the density contrast\(^9\) $\delta(x, t)$ smoothed on a scale $R$ has a Gaussian distribution,

$$\mathcal{P}(\delta, t) d\delta \sim \exp \left( -\frac{\delta^2}{2\sigma_R^2} \right) d\delta ,$$

with standard deviation $\sigma_R(t)$. Fluctuations that exceed a certain threshold $\delta_* \sim O(1)$ in

\(^8\)The comoving scale $R$ is independent of $m_\nu$ because $\rho_{bc}(z = 0)$ is. However, when expressed in units of Mpc/$h$ it depends on $m_\nu$ through Eq. (5.16).

\(^9\)We use the CDM density contrast and power spectrum to compute the Press-Schechter factor $F$. This matches $N$-body simulations better than using the full matter density contrast including neutrinos [183]. It is also a conservative choice, since the total matter power spectrum is further suppressed at large $m_\nu$, by a factor $(1 - f_\nu)^2$ below the free streaming scale.
the linear analysis will have become gravitationally bound. Hence,
\[ F_R(t) = \int_{\delta_*}^{\infty} P(\delta, t)d\delta = \text{erfc} \left( \frac{\delta_*}{\sqrt{2}\sigma_R(t)} \right). \] (5.21)

We use the canonical value \( \delta_* = 1.69 \), which is obtained by comparing the linear perturbation to a spherical collapse model.\(^{10}\)

The standard deviation of the smoothed density contrast is given by [185]
\[ \sigma^2_R \equiv \langle \delta^2_R(x) \rangle, \] (5.22)

with
\[ \delta_R \equiv \int d^3x' \delta(x) W_R(|x - x'|), \] (5.23)
where \( \delta(x) = \delta\rho_c/\rho_c \) is the fractional overdensity of cold dark matter. We use the *top hat* window function, \( W_R(x) = 1 \) for \( |x| \leq R \) and \( W_R(x) = 0 \) otherwise.

Equivalently, the smoothed density contrast can be computed from Fourier-transformed quantities:
\[ \sigma^2_R = \int_0^\infty \frac{dk}{k} \frac{k^3 P_{cc}(k)}{2\pi^2} |W_R(k)|^2, \] (5.24)

where \( W_R(k) = \frac{3}{(kR)^3}(\sin kR - kR \cos kR) \). The CDM power spectrum is defined by
\[ \langle \delta(k)\delta(k') \rangle = (2\pi)^3 P_{cc}(k)\delta^3(k - k'), \] (5.25)
where \( \delta(k) \) is the Fourier transform of \( \delta(x) \) and \( \delta^3 \) is the Dirac delta function.

To evaluate \( \sigma_R(m_\nu, t) \), we use the CAMB code [160] to compute the CDM power spectrum \( P_{cc}(k) \) as a function of time, in models with different neutrino mass. We evaluate the integral in Eq. (5.24) numerically. We have also checked our results using the CLASS code [161].

We noticed a small discrepancy in the output of \( k^3P_{cc} \) at the largest neutrino masses we consider, \( m_\nu \sim 10 \text{ eV} \), where CLASS gives a slightly larger amplitude for the free-streaming peak. By lowering the cutoff on \( m_\nu \) described in Sec. 5.3, the CLASS output would only strengthen the anthropic explanation of the observed neutrino mass range.

Available Boltzmann codes do not return power spectra for negative redshifts, that is, for times when the CMB temperature is below 2.7 K. *In this regime only*, we estimate \( \sigma_R \) by extrapolating our numerical results to negative redshifts semi-analytically as described in Appendix H. This regime is not a dominant contributor to the overall probability distribution, due to the smallness of the causal patch at late times, and since vacuum domination terminates structure formation in any case.

We compute \( F_R \) and \( \dot{F}_R \) from Eq. (5.21); the results are shown in Fig. 5.7.

\(^{10}\)For structure that forms in the vacuum era, the collapse threshold is slightly lowered [171], whereas in the presence of an appreciable neutrino fraction \( \delta_* \) should be slightly increased [184]. If we adapted \( \delta_* \) accordingly, the net effect would be to further suppress structure at large \( m_\nu \), in favor of an anthropic origin of the neutrino mass. However, appropriate values of \( \delta_* \) have so far been estimated only for rather small neutrino masses. Ultimately, it would be preferable to sidestep the Press-Schechter approximation altogether. Our analysis could be dramatically improved by using proper N-body simulations to compute structure formation, including an adequate treatment of baryonic physics.
CHAPTER 5. ANTHROPOIC ORIGIN OF THE NEUTRINO MASS FROM COOLING FAILURE

Figure 5.7: The Press-Schechter factor (solid lines) and its derivative (dashed lines) at the galaxy scale, $10^{12} M_\odot$, for a normal hierarchy with $m_\nu = 0$ eV (black), $m_\nu = 2$ eV (brown), $m_\nu = 4$ eV (blue), $m_\nu = 6$ eV (purple) and $m_\nu = 8$ eV (red). Each is used to define either of the two observer models of Sec. 5.2. Note that massive neutrinos suppress structure at all times, but much more so at early times [186, 187, 188, 189, 190, 191].

Galaxy Formation: Neutrino-Induced Cooling Catastrophe

The final factor $G_R(m_\nu, t)$ in Eq. (5.19) is the fraction of the halo mass in baryons within galaxies. To approximate this, we must first investigate the effect of a top-down structure scenario (present at $m_\nu \gtrsim 8 - 10$ eV, as discussed in Sec. 5.1) on galaxy formation.

In our universe galaxies form in halos with masses between $10^7 M_\odot$ and $10^{12} M_\odot$. Larger halos can inherit galaxies from mergers, resulting in galaxy groups and clusters, with masses ranging from $10^{13} M_\odot$ to $10^{15} M_\odot$. However, halos in the latter mass range do not themselves produce a significant amount of stars, relative to their total mass.

This fact can be understood as a consequence of the ability, or failure, of baryons to cool rapidly inside newly formed dark matter halos. (For more detail, see Appendix I and references given there.) Baryons are shock-heated to a virial temperature $T_{\text{vir}}$ when they fall into a large dark matter halo. In order to condense into a galaxy at the center of the halo, the baryons must first shed their thermal energy. Cooling can occur by bremsstrahlung at temperatures large enough to ionize hydrogen, or by atomic and molecular line cooling at the lower temperatures attained in smaller halos.

Analytically, one can estimate the time it takes baryons to cool, $t_{\text{cool}}$. The cooling time grows with the mass of the halo (for large masses), and with the time of its formation. It is also easy to compute the gravitational timescale of the halo, $t_{\text{grav}}$, which is somewhat shorter than the time of its formation.

A good match to observation is obtained by the following criterion. If $t_{\text{cool}} < t_{\text{grav}}$, then cooling is efficient. A significant fraction of baryons (up to 10%) is converted into stars. This process occurs rapidly, on a timescale that can be treated as instantaneous compared to the age of the universe when the halo formed.

On the other hand, if $t_{\text{cool}} > t_{\text{grav}}$, then star formation is limited by the cooling time. In
this regime, one would still expect a certain amount of rapid star formation at the dense core of the halo, but this is not seen in observations. (This is known as the cooling flow problem.) Observations do not constrain the possibility that a significant portion of baryons will form stars in the distant future, on a timescale much greater than the age of the universe. This time would greatly exceed $t_A$. Since the causal patch is of a fixed physical size of order the de Sitter horizon scale, there will be exponentially few halos left in it at late times. Thus, star formation at very late times does not contribute to the probability of a particular universe. (This sensitivity to the matter content inside the cosmological horizon is a key feature distinguishing the causal patch from other interesting measures, such as the fat geodesic or scale factor time cutoff [99], and it is responsible for several of the chief successes of the causal patch, e.g. [192, 193, 194, 175, 196, 197, 165].) Thus, we may take $t_{cool} < t_{grav}$ as a robust condition for galaxy formation to occur in a newly formed halo.

The cooling function that determines the rate of heat dissipation has a complicated form in the relevant halo mass range (see [166] and references therein). Appendix I describes two different approximations to $t_{cool}$ and $t_{grav}$ that capture different cooling regimes that halos in our analysis might explore. One finds in either regime that at late times, cooling is inefficient for halo masses above the scale of the Milky Way halo:

$$M_{\text{vir}} > 10^{12} M_\odot, \quad t_{\text{vir}} \gtrsim O(\text{Gyr}) \implies \text{No Galaxy} \quad (5.26)$$

Importantly, the boundary is consistent with the observation that in our universe, there are no galaxies much larger than the Milky Way.

It would be interesting to implement a more precise version of the above boundary as a cutoff on the time until galaxy formation is efficient, at any value of $m_\nu$. Massive neutrinos delay structure formation more dramatically than they suppress it (Fig. 5.7), so such a cutoff would exclude an appreciable fraction of halos from contributing to galaxy formation even at rather small $m_\nu$. Thus it would lead to a greater suppression of intermediate neutrino masses between 1 and 10 eV, and thus would favor the anthropic approach. Instead, we will argue more conservatively for a cooling cutoff on $m_\nu$ around 10 eV. We will now identify a change of regime for $m_\nu \gtrsim 10$ eV. As we shall see, this transition places the dominant halo population so far into the regime of inefficient cooling, that the above rough estimate suffices to conclude that galaxy formation is highly suppressed.

For $m_\nu \lesssim 8 - 10$ eV, recall that the dimensionless matter power spectrum $k^3 P_{cc}(k)$ increases monotonically with $k$ (see Fig. 5.1a), and the integral for the smoothed density contrast $\sigma_R$ in Eq. (5.24) is dominated by the power at the small galactic scale $R$. In this range, the power spectrum preserves the standard hierarchical structure formation we see in our universe, where low mass halos generally form earlier than more massive ones. Thus, it is not likely for a $10^{12} M_\odot$ halo to be nested inside a more massive overdensity that collapses at the same time.

Above $m_\nu \approx 8 - 10$ eV, neutrinos suppress small scale power so much that the dimensionless power spectrum $k^3 P_{cc}(k)$ develops a maximum near the scale associated with free streaming $k_{nr}$ (Fig. 5.1a). This corresponds to a mass of order $5 - 100$ times the scale of
the Milky Way halo, roughly the scale of galaxy clusters.\footnote{The peak (the free streaming scale) moves to smaller scales as \( m_\nu \) is increased. Eventually it crosses the galaxy scale: for \( m_\nu \gtrsim 100 \text{ eV} \) neutrinos act as cold dark matter. But this does not yield an anthropically allowed region, because the dark matter to baryon density ratio \( \zeta \) will be too large. This may be detrimental to disk fragmentation \cite{[171,166]}. If the causal patch is used, \( \zeta \gg 1 \) is robustly suppressed independently of any effects on galaxy and star formation, because the total mass of baryons (and thus of observers) in the patch scales like \( (1 + \zeta)^{-1} \) \cite{[165]}.} It implies that the smoothed density contrast on small scales such as \( 10^{12} M_\odot \) is no longer dominated by the power at the corresponding wavenumber \( k \). Instead, the integral in Eq. (5.24) is dominated by the maximum of the integrand, near \( k_{\text{nr}} \).

This implies that \( 10^{12} M_\odot \) overdensities become gravitationally bound at the same time as overdensities on larger scales: a top-down scenario. The virial temperature and cooling time will be set by the largest scale that the \( 10^{12} M_\odot \) overdensity is embedded in, \( M_{\text{vir}} \gg 10^{12} M_\odot \). Moreover, for such large halos virialization will occur quite late (see Fig. 5.7), \( t_{\text{vir}} \gg 5 \) Gyr. Hence, for \( m_\nu \gtrsim 8 - 10 \text{ eV} \), the cooling condition in Eq. (5.26) becomes violated, by a substantial margin.

Note that this conclusion is insensitive to the halo mass scale we associate with observers. Whether we require \( 10^{10} M_\odot \) or \( 10^{12} M_\odot \) halos: if the power spectrum peaks at larger scales, the putative galactic halos will be embedded in and virialize together with perturbations on a mass scale well above \( 10^{12} M_\odot \), leading to a cooling problem.

Let us summarize these considerations and formulate our cooling cutoff on the neutrino mass. If there exists some large scale \( k_* < k_{\text{gal}} \) such that \( k_*^3 P_{\text{cc}}(k_*) > k_{\text{gal}}^3 P_{\text{cc}}(k_{\text{gal}}) \), we interpret this as indicating top-down structure formation. Let \( m_\nu^{\text{max}} \) be the greatest neutrino mass sum for which this criterion is not met, i.e., the largest neutrino mass compatible with bottom-up structure formation. From Boltzmann codes we find \( m_\nu^{\text{max}} = 7.7 \) eV for the normal hierarchy and \( m_\nu^{\text{max}} = 10.8 \) eV for the degenerate hierarchy. We have argued that cooling fails substantially in the top-down regime, because the first virialized halos are large and form late. Hence, we treat \( m_\nu^{\text{max}} \) as a sharp catastrophic boundary. We approximate \( G_R \) as a step function that vanishes past this critical mass:

\[
G_R(m_\nu, t) = \begin{cases} 1, & m_\nu < m_\nu^{\text{max}} \\ 0, & m_\nu \geq m_\nu^{\text{max}} \end{cases}. \tag{5.27}
\]

We evaluate the integral in Eq. (5.19) numerically using Mathematica. The integration is started before structure begins to form, at redshift \( z = 12 \), when \( F_R \) is negligible. The integration is terminated deep in the vacuum era when \( r_{\text{patch}} \) becomes exponentially small. Our final result is described in Sec. 5.1; see Figures 5.3 and 5.4.
Chapter 6

Null Geodesics, Local CFT Operators and AdS/CFT for Subregions

6.1 Introduction

The AdS/CFT correspondence [198, 199] provides an important tool for obtaining insight into quantum gravity. Yet even today, the seemingly basic question of how bulk locality is encoded in the boundary theory—in other words, which CFT degrees of freedom describe a given geometrical region in the bulk—has resisted a simple, precise answer.

In this paper, we investigate the related question of AdS/CFT subregion dualities. That is, we consider the possibility that a CFT restricted to a subset of the full AdS boundary is dual to a geometric subset of the AdS bulk. There is no obvious reason that a geometric region on the boundary has to correspond to a geometric region in the bulk, but there are strong arguments for such a subregion duality in certain simple cases [200], and intriguing hints [201, 202, 203] that it may be true more generally.

The problem of precisely what bulk region should be associated with a given boundary region is complicated and has been explored recently by [204, 205, 206]. We will not propose or adopt a rule for constructing such an association. Instead, we focus on one nice feature of the global AdS/CFT duality which does not generalize to arbitrary subregions, namely the ability to reconstruct the bulk using local CFT operators in the classical limit [207, 208, 209, 210]. Specifically, we will emphasize the role of continuity of the bulk reconstruction, and propose a simple geometric diagnostic testing whether continuous reconstruction holds for a given subregion (see also Ref. [211] for related work).

To motivate our investigation, first consider the full global AdS/CFT duality. We will work in Lorentzian signature and fix the Hamiltonian of the CFT, which corresponds to fixing all the non-normalizable modes in the bulk. Now take the $G_N \to 0$ limit in the bulk; the bulk theory reduces to solving classical field equations in a fixed background. The non-normalizable modes are fully determined and non-dynamical, but there are still many allowed solutions because of the normalizable modes. CFT data on the boundary should be
sufficient to specify a particular bulk solution. Normalizable modes in the bulk approach zero at the boundary, but a nonzero boundary value can be defined by stripping off a decaying factor,

$$\phi(b) \equiv \lim_{z \to 0} z^{-\Delta} \Phi(b, z),$$

(6.1)

where $z$ is the usual coordinate that approaches zero at the boundary, $b$ stands for the boundary coordinates, and $\Phi$ is a bulk field. We will also use the notation $B = (b, z)$ where convenient. By the “extrapolate” version of the AdS/CFT dictionary [212], these boundary values are dual to expectation values of local operators,

$$\phi(b) = \langle O(b) \rangle.$$

(6.2)

We can now ask a classical bulk question: do the boundary values $\phi$ determine the bulk solution everywhere? This is a nonstandard type of Cauchy problem, because we are specifying data on a surface that includes time.

In a simple toy model where the bulk contains only a single free field with arbitrary mass, Hamilton et al. [207, 208] showed explicitly that this boundary data does specify the bulk solution completely in global AdS. The fact that the boundary data specifies the bulk solution can be considered the classical, non-gravitational limit of AdS/CFT. It is a nontrivial fact that expectation values of local CFT operators are sufficient to reconstruct the bulk field in this case.

A proposed subregion duality must pass the same test. Is the CFT data in a boundary subregion sufficient to reconstruct the bulk solution within the corresponding bulk subregion? In principle, the CFT data is quite complicated. The simplification that occurred in global AdS/CFT, that expectation values of local boundary operators were sufficient, may or may not carry over to other cases, and our task is to properly account for when it does. This is a problem in the theory of classical differential equations which we can hope to solve. Simple examples show that the problem is subtle, however, and to properly capture the physics of the problem we need to differentiate between bulk reconstruction and continuous bulk reconstruction.

The simplest illustration comes from AdS-Rindler space, which can be described as follows. In the global duality, the CFT is formulated on a sphere cross time and the associated bulk is global AdS. Let us divide the boundary sphere at some time across the equator. In the bulk, the extremal surface ending on the boundary equator is a hyperboloid, and we can use Rindler-type coordinates in AdS so that this extremal surface is a Rindler horizon. The northern hemisphere on the boundary extends naturally into a small causal diamond on the boundary, namely the region determined by time evolution of the data in the northern hemisphere. The corresponding bulk region is a Rindler wedge, shown in Fig. 6.1, which we will call AdS-Rindler space.

Does the global boundary data, restricted to the small boundary diamond, determine the bulk solution in the corresponding AdS-Rindler wedge? Hamilton et al. [208] also addressed this question. They determined that a particular analytic continuation of the boundary data was necessary to reconstruct the bulk. Here we provide a different answer that does not rely
null geodesics, local CFT operators and AdS/CFT for subregions

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Figure 6.1: Here we show the AdS-Rindler wedge inside of global AdS, which can be defined as the intersection of the past of point $A$ with the future of point $B$. The asymptotic boundary is the small causal diamond defined by points $A$ and $B$. The past lightcone of $A$ and the future lightcone of $B$ intersect along the dashed line, which is a codimension-2 hyperboloid in the bulk. There is a second AdS-Rindler wedge, defined by the points antipodal to $A$ and $B$, that is bounded by the same hyperboloid in the bulk. We refer to such a pair as the “right” and “left” AdS-Rindler wedges.

on analytic continuation of the boundary data. We claim that there is a direct map from the boundary data to the bulk field, but that the map is not continuous. This leaves the physical interpretation open to doubt.

There are two reasons to focus on the question of continuity. First, if the subregion duality is correct, we would expect that measuring boundary data to finite precision should determine the bulk data to a corresponding precision. This is only true if the bulk solution depends continuously on the boundary data. Second, the question of continuous reconstruction seems to be mathematically robust; we will be able to make heuristic contact with nice mathematical theorems about when continuous reconstruction is possible.

Continuous reconstruction fails because there are finite excitations in the bulk Rindler wedge with an arbitrarily small imprint on the boundary data. The physics of these exci-
tations is simple: there exist null geodesics that pass through the bulk Rindler wedge, but avoid the boundary diamond. One can construct solutions where geometric optics is an arbitrarily good approximation and the energy is concentrated along such a null geodesic. In this way, we can construct solutions that are finite in the bulk but have arbitrarily small boundary data in the Rindler wedge.

We can also ask a slightly different mathematical question, which is closely related to bulk reconstruction from the boundary data but simpler to analyze: the question of unique continuation. Suppose we are given the bulk solution in some region near the boundary, and we want to continue the solution further into the bulk. In the AdS context, evolution inward is roughly dual to RG flow in the CFT. This question is closely related to the previous one, and again can be diagnosed with null geodesics [213]. In the case of the bulk Rindler wedge we find that unique continuation fails as well. We cannot evolve the solution radially inward in this case.

Given the connection to continuity and local reconstruction, as well as geometrical simplicity, we are motivated to propose a diagnostic for continuous bulk reconstruction from local CFT operators:

**Does every null geodesic in the bulk subregion have an endpoint on the corresponding boundary subregion?**

Despite the failure of this diagnostic for AdS-Rindler, there are good reasons to think that this particular subregion duality actually holds. The Rindler wedge can be thought of as an eternal black hole with a hyperbolic horizon. This suggests that a duality holds, by analogy with the ordinary eternal black hole: the CFT in the Hartle-Hawking state may be restricted to one boundary component, and the resulting thermal state is dual to one of the two exterior region of an eternal AdS-Schwarzschild black hole [200].

Since continuous reconstruction from CFT one-point functions fails for this subregion, we learn that nonlocal boundary operators must play an important role in the duality even in the classical limit. Generalizing this result, we learn that nonlocal CFT operators [214, 215] are important when subregions are small enough that the boundary region no longer captures all null rays passing through the bulk.

The remainder of the paper is organized as follows. In Section 6.2 we review the general procedure for reconstructing the bulk solution from boundary data which was employed by Hamilton et al. in their work. We also show how to determine continuity of the reconstruction map. The general method is applied to global AdS, AdS-Rindler space, the Poincare patch, and Poincare-Milne space. In Section 6.3 we formulate the geometric diagnostic of capturing null geodesics and relate it to continuity of the reconstruction map, making contact with results in the mathematics literature. We apply the diagnostic to the black hole geometries, as well, without finding an explicit reconstruction map. In Section 6.4, we exhibit arguments that a subregion duality does exist for AdS-Rindler space. In Section 6.5, we note that this can be reconciled with the failure of continuous reconstruction from local fields if the duality involves nonlocal boundary operators in an essential way.
CHAPTER 6. NULL GEODESICS, LOCAL CFT OPERATORS AND ADS/CFT FOR SUBREGIONS

6.2 The Reconstruction Map

General Formulas

We begin this section by reviewing the procedure for obtaining a bulk solution from boundary data using eigenmodes of the wave equation, generalizing the approach of Ref. [208]. A classical, free bulk field $\Phi$ can be expanded in terms of orthonormal modes $F_k$ which depend on a collection of conserved quantities $k$:

$$\Phi(B) = \int dk \, a_k F_k(B) + \text{c.c.} \quad (6.3)$$

Near the boundary, the modes $F_k$ have the asymptotic form $F_k(B) \sim r^{-\Delta} f_k(b)$. Thus we find the boundary field $\phi = \lim_{r \to \infty} r^\Delta \Phi$ has the expansion

$$\phi(b) = \int dk \, a_k f_k(b) + \text{c.c.} \quad (6.4)$$

Given $\phi(b)$, we can ask whether it is possible to reconstruct $\Phi(B)$. Recall that $\phi(b)$ is dual to a one-point function in the CFT, hence this is equivalent to asking whether the bulk field is determined by CFT one-point functions. This is possible when the $a_k$ can be extracted from $\phi$ through an inner product of the form

$$a_k = W_k \int db \, f_k^*(b) \phi(b), \quad (6.5)$$

where $W_k$ is a weighting factor. Equivalently, the boundary mode functions should satisfy the orthogonality relation

$$\int db \, f_k^*(b) f_{k'}(b) = W_k^{-1} \delta_{k,k'} \quad (6.6)$$

There is no guarantee that a relation such as (6.6) will hold in general. We will see both possibilities in the examples below.

Given (6.5), it is a simple matter to solve for $\Phi(B)$:

$$\Phi(B) = \int dk \, \left[ W_k \int db \, f_k^*(b) \phi(b) \right] F_k(B) + \text{c.c.} \quad (6.7)$$

We emphasize that at this stage 6.7 is, in principle, a recipe for computing the bulk field in terms of the boundary field.

However, there is an important simplification when the order of integration over $k$ and $b$ can be exchanged. Then we have

$$\Phi(B) = \int db \, K(B|b) \phi(b), \quad (6.8)$$
where

\[ K(B|b) = \int dk \, W_k f_k^*(b) F_k(B) + \text{c.c.} \]  

(6.9)

This is a nontrivial simplification which does not occur in all cases. We will see below that when the order of integration is illegitimately exchanged, as in the example of the AdS-Rindler wedge, the integral over \( k \) in (6.9) does not converge \(^1\).

Non-convergence of the integral in (6.9) is due to growth of the eigenmodes at large \( k \). The large \( k \)-behavior of the modes is closely related to the question of continuity of the reconstruction map, \( \phi(b) \mapsto \Phi(B) \). To examine continuity, we need to adopt definitions for the bulk and boundary norms. On the boundary, we will follow Ref. [216] and use the norm

\[ ||\phi||_b^2 = \int db \left| \nabla_b \phi \right|^2 + |\phi|^2 . \]  

(6.10)

Here \( |\nabla_b \phi|^2 \) is positive-definite, not Lorentzian, even though we are in a Lorentzian space-time. In other words, the norm looks like an integral of an energy density (over both space and time), not an action. We will leave its exact form unspecified here, but will be explicit in the examples below. The correct norm to choose is an open question, and a different choice may affect the answer. Our choice is motivated by related results in the mathematics literature, but it may not be a natural choice for this problem. For now, this norm will serve to illustrate the possible answers to the continuity question. Because of (6.6), we will find that \( ||\phi||_b^2 \propto \int p(k) W_k^{-1} |a_k|^2 \), where \( p(k) \) is a quadratic polynomial in the conserved momenta.

In the bulk, a convenient and natural norm is given by the energy of the solution. Adopting the standard Klein-Gordon normalization for the modes \( F_k(B) \), the energy is given by

\[ ||\Phi||_B^2 = E[\Phi] = \int dk \, |\omega(k)||a_k|^2 , \]  

(6.11)

where \( \omega(k) \) is the frequency written as a function of the conserved quantities (one of which may be the frequency itself). The reconstruction map is continuous if and only if there is a constant \( C > 0 \) such that

\[ ||\Phi||_B^2 \leq C ||\phi||_b^2 . \]  

(6.12)

That is, a bulk solution of fixed energy cannot have arbitrarily small imprint on the boundary. Equivalently, by going to momentum space, the product \( \omega(k)W_k/p(k) \) must be bounded from above. In the remainder of this section we apply these general formulas to several specific cases to find smearing functions and check continuity. We restrict ourselves to a 2+1-dimensional bulk for simplicity.

\(^1\)With certain extra assumptions on the fields, however, [208] is able to construct a complexified smearing function.
CHAPTER 6. NULL GEODESICS, LOCAL CFT OPERATORS AND ADS/CFT FOR SUBREGIONS

Global AdS

The AdS\(_{2+1}\) metric in global coordinates is
\[
\text{ds}^2 = -\frac{1}{\cos^2 \rho} \text{dt}^2 + \frac{1}{\cos^2 \rho} \text{d}\rho^2 + \tan^2 \rho \, \text{d}\theta^2. \tag{6.13}
\]
The Klein-Gordon equation in these coordinates reads
\[
-\cos^2 \rho \partial_t^2 \Phi + \frac{\cos^2 \rho}{\tan \rho} \partial_\rho (\tan \rho \partial_\rho \Phi) + \frac{1}{\tan^2 \rho} \partial_\theta^2 \Phi = m^2 \Phi. \tag{6.14}
\]
The normalizable solutions are
\[
F_{nl} = N_{nl} e^{-i\omega t} e^{i l \theta} \cos^\Delta \rho \sin |l| \rho \mathcal{F}_{nl}(\rho). \tag{6.15}
\]
where
\[
N_{nl} = \sqrt{\frac{\Gamma(n + |l| + 1) \Gamma(\Delta + n + |l|)}{n! \Gamma^2(|l| + 1) \Gamma(\Delta + n)}}, \tag{6.16}
\]
\[
\mathcal{F}_{nl}(\rho) = 2 F_1(-n, \Delta + n + |l|, |l| + 1, \sin^2 \rho), \tag{6.17}
\]
and the frequency is \(\omega = \Delta + 2n + |l|\). The boundary modes are
\[
f_{nl} = \lim_{\rho \to \pi/2} \cos(\rho)^{-\Delta} F_{nl} = (-1)^n e^{i l \theta - i \omega t} \sqrt{\frac{\Gamma(\Delta + n + |l|) \Gamma(\Delta + n)}{n! \Gamma^2(\Delta) \Gamma(n + |l| + 1)}}. \tag{6.18}
\]
Following the general procedure outlined above, we can compute the smearing function
\[
K(\theta, t, \rho | \theta', t') = \sum_{n,l} \frac{1}{4\pi^2} \frac{\Gamma(\Delta) \Gamma(n + |l| + 1)}{\Gamma(\Delta + n)} (-1)^n e^{-i \omega (t - t')} e^{i l (\theta - \theta')} \cos^\Delta \rho \sin |l| \rho \mathcal{F}_{nl}(\rho) + \text{c.c.} \tag{6.19}
\]
This can be summed to obtain the result of Ref. [208].

The boundary norm in this case is given by\(^2\)
\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \int \text{d}\theta \left( (\partial_t \phi)^2 + (\partial_\theta \phi)^2 + \phi^2 \right) = 4\pi^2 \sum_{nl} \left( \omega^2 + l^2 + 1 \right) \frac{\Gamma(\Delta + n + |l|) \Gamma(\Delta + n)}{n! \Gamma^2(\Delta) \Gamma(n + |l| + 1)} |a_{nl}|^2. \tag{6.20}
\]
The reconstruction map is continuous if and only if the following quantity is bounded:
\[
\frac{\omega W_{nl}}{1 + \omega^2 + l^2} = \frac{\omega}{4\pi^2(1 + \omega^2 + l^2)} \frac{n! \Gamma^2(\Delta) \Gamma(n + |l| + 1)}{\Gamma(\Delta + n + |l|) \Gamma(\Delta + n)}. \tag{6.21}
\]
This ratio clearly remains finite for all values of \(n\) and \(l\), thus proving continuity.

\(^2\)In global coordinates, the norm in position space is properly defined as an average over time. This is related to the fact that the frequencies are discrete.
CHAPTER 6. NULL GEODESICS, LOCAL CFT OPERATORS AND ADS/CFT FOR SUBREGIONS

AdS-Rindler

We now turn to the AdS-Rindler wedge, which in 2+1 dimensions has the metric

$$ds^2 = \frac{1}{z^2} \left[ - \left( 1 - \frac{z^2}{z_0^2} \right) dt^2 + \frac{dz^2}{1 - \frac{z^2}{z_0^2}} + dx^2 \right].$$

(6.22)

The Rindler horizon is located at $$z = z_0$$, while the AdS boundary is at $$z = 0$$. The Klein-Gordon equation is

$$- \frac{z^2}{1 - z^2/z_0^2} \partial_t^2 \Phi + z^2 \partial_z \left( \frac{1}{z} \left( 1 - \frac{z^2}{z_0^2} \right) \partial_z \Phi \right) + z^2 \partial_x^2 \Phi = m^2 \Phi.$$  

(6.23)

The normalizable solutions are

$$F_{\omega k} = N_{\omega k} e^{-i\omega t} e^{ikx} z^\Delta \left( 1 - \frac{z^2}{z_0^2} \right)^{-i\hat{\omega}/2} {}_2F_1 \left( \frac{\Delta - i\hat{\omega} - ik}{2}, \frac{\Delta - i\hat{\omega} + ik}{2}, \Delta; \frac{z^2}{z_0^2} \right).$$

(6.24)

where $$\hat{\omega} = \omega z_0$$, $$\hat{k} = k z_0$$, and

$$N_{\omega k} = \frac{1}{\sqrt{8\pi^2 |\omega|}} \left| \frac{\Gamma(\Delta+i\hat{\omega}+ik) \Gamma(\Delta+i\hat{\omega}-ik)}{\Gamma(\Delta) \Gamma(i\hat{\omega})} \right|.$$  

(6.25)

The boundary modes are then

$$f_{\omega k} = \lim_{z \to 0} z^{-\Delta} F_{\omega k} = N_{\omega k} e^{ikx - i\omega t}.$$  

(6.26)

We can attempt to construct the smearing function following Eq. 6.9, but, as discussed below that equation, we will find that the integral over $$k$$ does not converge:

$$K(x, t, z | x', t') =$$

$$\frac{1}{4\pi^2} \int dk \omega \ e^{i k (x - x')} e^{-i \omega (t - t')} z^\Delta \left( 1 - \frac{z^2}{z_0^2} \right)^{-i\hat{\omega}/2} {}_2F_1 \left( \frac{\Delta - i\hat{\omega} - ik}{2}, \frac{\Delta - i\hat{\omega} + ik}{2}, \Delta; \frac{z^2}{z_0^2} \right),$$  

(6.27)

$$= \infty.$$  

(6.28)

This divergence is due to the exponential growth in $$k$$ of the hypergeometric function when $$k \gg \omega$$ [208],

$$2F_1 \left( \frac{\Delta - i\hat{\omega} - ik}{2}, \frac{\Delta - i\hat{\omega} + ik}{2}, \Delta; \frac{z^2}{z_0^2} \right) \sim \exp[\hat{k} \sin^{-1}(z/z_0)].$$  

(6.30)
The boundary norm is given by

\[
\int dtdx \left( (\partial_t \phi)^2 + (\partial_x \phi)^2 + \phi^2 \right) = \int d\omega dk\ 4\pi^2 N_{\omega k}^2 (1 + \omega^2 + k^2)|a_{\omega k}|^2 .
\]

We see that the ratio which must be bounded in order that continuity hold is

\[
\frac{\omega_{W_{nl}}}{1 + \omega^2 + k^2} = \frac{2\omega^2}{1 + \omega^2 + k^2} \left| \frac{\Gamma(\Delta)\Gamma(i\omega)}{\Gamma(\Delta + i\omega + ik)\Gamma(\Delta + i\omega - ik)} \right|^2 .
\]

This ratio remains bounded for fixed \(k\), but when \(k \gg \omega\) it grows like \(\exp(\pi k)\). So we find both that the smearing function does not exist and that continuity fails.

**Physical Interpretation**  In this case, the problem with reconstructing the bulk solution occurs regardless of the bulk point we are interested in. The discontinuity can be understood physically. At first, it is surprising that modes with \(\omega < k\) are even allowed; in the Poincaré patch, obtained as the \(z_0 \to \infty\) limit of AdS-Rindler, they are not.\(^3\) Near the Rindler horizon frequency is redshifted relative to its value at infinity, while momentum is unaffected. So a local excitation with proper frequency comparable to its proper momentum appears at infinity as a mode with \(\omega < k\). The modes with \(\omega < k\) are confined by a potential barrier that keeps them away from the boundary; for large \(k\) the height of the barrier is proportional to \(k^2\). This causes the boundary data to be suppressed relative to the bulk by a WKB factor \(\exp(-\int \sqrt{V}) \sim \exp(-\pi k)\).

We have seen that there is no smearing function in this case because a divergence at large momentum prevents us from exchanging the order of integration. To understand the physical meaning of this divergence, we can ask about computing a more physical quantity, which will regulate the divergence. Instead of trying to find an expression for the bulk field at a specified bulk point, consider instead a bulk field smeared with a Gaussian function of some width \(\sigma\) in the transverse direction,

\[
\Phi_{\sigma}(t, x, z) \equiv \int dx' \exp\left(-\frac{(x' - x)^2}{\sigma^2}\right) \Phi(t, x', z) .
\]

We only smear in the \(x\) direction because the only divergence is in \(k\), and we drop various numerical factors and polynomial prefactors that will be unimportant for our conclusion. We will also set \(z_0 = 1\) (which is always possible by an appropriate scaling of coordinates) for the remainder of this section.

The smeared field has a perfectly fine expression in terms of local boundary fields. We can use symmetries to place the bulk point at \(t = x = 0\); then

\[
\Phi_{\sigma}(0, 0, z) = \int dt' dx' K_{\sigma}(0, 0, z|x', t')\phi(x', t')
\]

\(^3\)In Ref. [217], in the context of the BTZ black hole, it is suggested that these modes are connected with finite temperature effects, and the associated exponential factors are interpreted as Boltzmann weights. We consider this to be very suggestive, but have not found a concrete connection to this work.
with

$$K_\sigma(0, 0, z|x', t') = \int d\omega dk e^{i\omega t'} - kx' - k^2 \sigma^2 (1 - z^2)^{-i\omega/2} \, _2F_1 \left( \frac{\Delta - i\omega - ik}{2}, \frac{\Delta - i\omega + ik}{2}, \Delta, z^2 \right).$$

(6.35)

The important question is the large \( k \) behavior of this function. To get a feeling for it, replace the hypergeometric function by its large \( k \) limit,

$$\approx g(\omega, \Delta, z)k^{\Delta - 1} \cosh(2k\theta) \quad (6.36)$$

where \( \theta \) depends on the distance from the boundary, \( \sin \theta = z \), and \( g \) a function that does not depend on \( k \). We ignore the polynomial prefactor and focus on the exponential dependence. Performing the integral, we get

$$K_\sigma(0, 0, z|x', t') \approx g(t, \Delta, z) \exp \left( \frac{\theta^2}{\sigma^2} - \frac{x'^2}{\sigma^2} - 2i\frac{\theta}{\sigma^2}x' \right).$$

(6.37)

where the quotation marks indicate that this is only a cartoon of the correct answer that captures the large momentum behavior of the smearing function. Now we can write the smeared bulk field in terms of the boundary values,

$$\Phi_\sigma(0, 0, z) = \int dx'dt' K_\sigma(0, 0, z|x', t') \phi(x', t').$$

(6.38)

What is the behavior of this function as we localize the bulk field by taking the width small, \( \sigma \to 0 \)? \( K_\sigma \) is strongly dependent on \( \sigma \): the maximum value of \( K_\sigma \) is exponentially large at small \( \sigma \), \( K_\sigma^{\text{max}} = \exp(\theta^2/\sigma^2) \), where again \( \theta \) is related to the distance from the horizon, ranging from \( \theta = \pi/2 \) at the horizon to \( \theta = 0 \) at the boundary. It varies rapidly, with characteristic wavenumber \( \theta/\sigma^2 \), and has a width set by \( \sigma \).

The physical length over which the bulk point is smeared is \( \sigma_{\text{phys}} = \sigma/z \), and up to an order-one factor we can approximate \( \theta \approx z \). Restoring factors of the AdS radius \( L \), we find that the smeared smearing function \( K_\sigma \) is a rapidly oscillating function with maximum value

$$K_\sigma^{\text{max}} \sim \exp \left( \frac{L^2}{\sigma_{\text{phys}}^2} \right).$$

(6.39)

Note that the dependence on the radial location has disappeared upon writing things in terms of the physical size. Attempting to measure the bulk field at scales smaller than the AdS radius requires exponential precision in the boundary measurement, because we are trying to compute an order-one answer (the bulk field value) by integrating an exponentially large, rapidly oscillating function multiplied by the boundary field value.
We note here an interesting technical feature of this construction. We chose to compute a bulk operator smeared with a Gaussian profile in the transverse direction. Normally, the exact form of a smeared operator is not physically relevant. In particular, we can ask whether it is possible to construct an analogous function $K$ for smeared bulk operators which have smooth but compact support in the transverse direction. Unfortunately this is impossible. In order to overcome the exponential divergence at large $k$ in the mode functions, we had to smear against a bulk profile which dies off at least exponentially fast at large $k$. Such a function is necessarily analytic in $x$, and hence will not have compact support. Therefore we cannot truly localize our smeared bulk operators in the above construction; some residual leaking to infinity is required.

Poincare Patch

The Poincare patch is the canonical example of a subregion duality that works. With our chosen norms, we will find that continuity actually fails in the Poincare patch, even though a smearing function exists. This suggests that the Poincare patch may already reveal subtleties that we claim exist in the AdS-Rindler case. However, we will see that the nature of the discontinuity is very different from that of the AdS-Rindler wedge. Later, in Section 6.3, we will argue that this discontinuity may be a harmless relic of our choice of norm, and that a more reliable answer is given by the geometric criterion presented there.

The metric of the Poincare patch is

$$ds^2 = \frac{dz^2 - dt^2 + dx^2}{z^2},$$

and the Klein-Gordon equation in these coordinates reads

$$-z^2 \partial_t^2 \Phi + z^3 \partial_z \left( \frac{1}{z} \partial_z \Phi \right) + z^2 \partial_x^2 \Phi = m^2 \Phi .$$

In this case we label the eigenmodes by $k$ and $q$, with $q > 0$. The frequency is given by $\omega = \sqrt{q^2 + k^2}$. Properly normalized, the modes are $F_{qk} = (4\pi \omega)^{-1/2} e^{ikx} z \sqrt{q} J_\nu(qz)$. We have introduced the notation $\nu = \Delta - 1 = \sqrt{1 + m^2}$. Then the boundary modes are

$$f_{qk} = \lim_{z \to 0} z^{-\Delta} F_{qk} = \frac{q^{\nu+1}}{2^{\nu} \Gamma(\Delta)} e^{i(kx-\omega t)} \sqrt{4\pi \omega} .$$

The smearing function can easily be computed,

$$K(x, t, z|x', t') = \int dqdk \frac{2^{\nu} \Gamma(\Delta)}{4\pi^2 q^{\nu-1} \omega} e^{ik(x-x')} e^{-i\omega(t-t')} z J_\nu(qz) + \text{c.c.},$$

and this matches with the result of Ref. [208].
The boundary norm is
\[ \int dt dx \left( (\partial_t \phi)^2 + (\partial_x \phi)^2 + \phi^2 \right) = \int dq dk \frac{\pi q^{2\nu}}{4^{\nu} \Gamma^2(\Delta)} (1 + \omega^2 + k^2) |a_{qk}|^2. \] (6.44)

The ratio which must remain bounded for continuity to hold is
\[ \frac{\omega W_{qk}}{1 + \omega^2 + k^2} = \frac{4^{\nu} \Gamma^2(\Delta) \omega}{\pi q^{2\nu} (1 + \omega^2 + k^2)}. \] (6.45)

For large \( q, k \) this remains bounded, but as \( q \to 0 \) it does not. The physics of the problem is the following. Starting with any solution, we can perform a conformal transformation that takes
\[ z \to \lambda z, \quad x \to \lambda x, \quad t \to \lambda t. \] (6.46)

For large \( \lambda \), this moves the bulk solution towards the Poincare horizon and away from the Poincare boundary, resulting in a small boundary imprint. Under this scaling, \( q \to \lambda^{-1} q \), so it is exactly the small \( q \) behavior above that allows for such an “invisible” solution.

As stated above, we believe that this discontinuity may merely be a problem of the choice of norm. In particular, this is an “infrared” discontinuity, and the difficulties of the AdS-Rindler wedge were “ultraviolet” in character. The smearing function seems to be sensitive only to the ultraviolet discontinuities, which suggests that those are more troublesome. Furthermore, in Section 6.3 we will see that the Poincare patch (marginally) passes the geometric test of continuity while the AdS-Rindler wedge clearly fails. A remaining problem for future work to provide a more concrete connection between “ultraviolet” and “infrared” discontinuities and the existence or non-existence of a smearing function.

**Poincare-Milne**

Poincare-Milne space is the union of the collection of Milne spaces at each value of \( z \) in the Poincare patch. It is useful to contrast the Poincare-Milne case with the AdS-Rindler case considered above. The reason is that the conformal boundary of Poincare-Milne space is identical to that of the AdS-Rindler wedge, but the Poincare-Milne bulk is larger, as shown in Fig. 6.2.\(^4\) We expect that the boundary theory of the AdS-Rindler boundary is dual to the AdS-Rindler bulk space and not more [204, 205, 206], and so it is an important check on our methods that they do not provide false evidence for a Poincare-Milne subregion duality. While we have no proof that the free theory constructions we have considered so far cannot be extended to Poincare-Milne, we can show that the most obvious construction breaks down in a very curious way.

The metric of Poincare-Milne space is
\[ ds^2 = \frac{dz^2 - dt^2 + t^2 dx^2}{z^2}, \] (6.47)\(^4\)For definiteness we discard the future light cone of the point \( E \) in the figure, so that the boundary is exactly AdS-Rindler, with no extra null cone.
where we restrict to $t > 0$. The Klein-Gordon equation in these coordinates reads

$$-z^2 t^{-1} \partial_t(t \partial_t \Phi) + z^3 \partial_z \left( \frac{1}{z} \partial_z \Phi \right) + z^2 t^{-2} \partial_x^2 \Phi = m^2 \Phi . \tag{6.48}$$

The $z$-dependence and $x$-dependence of the normalizable eigenmodes are identical to the Poincare patch case, and the $t$-dependence comes from solving the equation

$$-t^{-1} \partial_t(t \partial_t \psi) - t^{-2} k^2 \psi = q^2 \psi . \tag{6.49}$$

The general solution to this equation is a linear combination of Hankel functions, $\psi = AH^{(1)}_{ik}(qt) + Be^{\pi k} H^{(2)}_{ik}(qt) = AH^{(1)}_{ik}(qt) + B[H^{(1)}_{ik}(qt)]^*$. 

Figure 6.2: Here we depict Poincare-Milne space, together with an AdS-Rindler space that it contains. The bulk of Poincare-Milne can be defined as the intersection of the past of point $A$ with the future of line $BE$. Clearly this region contains the AdS-Rindler space which is the intersection of the past of $A$ and the future of $B$. Furthermore, the asymptotic boundary of the Poincare-Milne space and the AdS-Rindler space is identical, being the causal diamond defined by $A$ and $B$ on the boundary.
As we will demonstrate, no equation like 6.6 can hold for solutions to this equation. To see this, it is convenient to define $\tilde{\psi} = (qt)^{1/2}\psi$. Then we have

$$-\partial^2_t \tilde{\psi} - \frac{k^2 + 1/4}{t^2} \tilde{\psi} = q^2 \tilde{\psi}. \quad (6.50)$$

This is a Schrödinger equation for a scattering state in an attractive $1/t^2$ potential. To simplify the calculation, we will normalize the solutions so that $A = 1$ always. The standard expectation from quantum mechanics is that $B$ is then completely determined as a function of $q$, and in particular we will only have a single linearly independent solution for a given value of $q$. However, from the bulk point of view there should always be two solutions for any $q$, corresponding to the positive and negative frequency modes. Indeed, the coefficient $B$ is usually determined by the boundary condition $\tilde{\psi}(0) = 0$, but here that is trivially satisfied for all $B$. Hence $B$ is a free parameter. We will now demonstrate another strange fact about this potential, that eigenmodes with different values of $q$ are not orthogonal, which shows that Eq. 6.6 does not hold.

To see this, consider two solutions $\tilde{\psi}_1$ and $\tilde{\psi}_2$ corresponding to $q_1$ and $q_2$. We have

$$(q_1^2 - q_2^2) \int_0^\infty dt \, \tilde{\psi}_1^* \tilde{\psi}_2 = \tilde{\psi}_1^* \partial_t \tilde{\psi}_2 - \partial_t \tilde{\psi}_1^* \tilde{\psi}_2 \bigg|_0^\infty. \quad (6.51)$$

We can compute the inner product once we know the asymptotic behavior of the solutions near $t = \infty$ and $t = 0$.

First, we use the large argument asymptotic form of the Hankel function,

$$H^{(1)}_{ik}(qt) \approx \sqrt{\frac{2}{\pi qt}} e^{i(qt-\pi/4)} e^{k\pi/2}, \quad (6.52)$$

so that

$$\tilde{\psi}_i \approx \sqrt{\frac{2}{\pi}} e^{k\pi/2} \left( e^{i(q_it-\pi/4)} + B_i e^{-i(q_it-\pi/4)} \right). \quad (6.53)$$

Then we find

$$\lim_{t \to \infty} \frac{1}{q_1^2 - q_2^2} \left( \tilde{\psi}_1^* \partial_t \tilde{\psi}_2 - \partial_t \tilde{\psi}_1^* \tilde{\psi}_2 \right) = 2e^{\pi k} (1 + B_1^* B_2) \delta(q_1 - q_2), \quad (6.54)$$

where we have used the fact that $\lim_{x \to \infty} e^{-iqx}/q = \pi \delta(q)$ and $\delta(q_1 + q_2) = 0$ when $q_1$ and $q_2$ are both positive. The result is proportional to a $\delta$-function, as it had to be. For large $t$ the solution approaches a plane wave, and plane waves of different frequencies are orthogonal.

Near $t = 0$ we use the small argument expansion

$$H^{(1)}_{ik}(qt) \approx \frac{1 + \coth \pi k}{\Gamma(1 + ik)} \left( \frac{qt}{2} \right)^{ik} \frac{\Gamma(1 + ik)}{\pi k} \left( \frac{qt}{2} \right)^{-ik}, \quad (6.55)$$

so that

$$\tilde{\psi}_i \approx C_i t^{ik+1/2} + D_i t^{-ik+1/2}, \quad (6.56)$$
where \( C_i \) and \( D_i \) are determined in terms of \( B_i \) and \( q_i \). Then we have
\[
\lim_{t \to 0} \tilde{\psi}_1^* \partial_t \tilde{\psi}_2 - \partial_t \tilde{\psi}_1^* \tilde{\psi}_2 = 2ik (C_1^* C_2 - D_1^* D_2) .
\]
(6.57)

In order to ensure orthogonality, this combination has to vanish for arbitrary choices of the parameters. This is clearly not the case. We note in passing that imposing an extra constraint of the form \( D = e^{i\delta} C \), with \( \delta \) a new independent parameter, will make the wavefunctions orthogonal. Tracing through the definitions, one can see that this also fully determines \( B \) in terms of \( q \) and \( \delta \), and that \(|B| = 1\) as expected by unitarity. The choice of \( \delta \) corresponds to a choice of self-adjoint extension, necessary to make the quantum mechanics well-defined (for additional discussion of this point see Ref. [218], and see references therein for more on the \( 1/t^2 \) potential in quantum mechanics). As we pointed out above, however, such a prescription is not relevant for our current task, as it would eliminate a bulk degree of freedom.

**AdS-Rindler Revisited**

We would like to emphasize that the above analysis of Poincare-Milne space is not a no-go theorem. As an example, we now show that AdS-Rindler space, analyzed in a certain coordinate system, suffers from the same pathologies. By a change of coordinates, one can show that the AdS-Rindler wedge can be written in a way that is precisely analogous to Poincare-Milne:
\[
\frac{ds^2}{z^2} = \frac{dz^2 - x^2 dt^2 + dx^2}{z^2},
\]
(6.58)

where we restrict to the region \( x > 0 \). Using this coordinate system, and following the usual procedure, we encounter problems very similar to those of Poincare-Milne space discussed above. The \( x \)-dependence of the eigenmodes is found by solving
\[
-x^{-1} \partial_x (x \partial_x \psi) - x^{-2} \omega^2 \psi^2 = -q^2 \psi .
\]
(6.59)

This is equivalent to a Schrödinger equation in the same potential as before, except now we are finding bound states instead of scattering states. The analysis is completely analogous to the Poincare-Milne case. There is a continuous spectrum of bound states (unusual for quantum mechanics!), and they are not generically orthogonal. Thus we cannot carry out the program of mapping boundary data to bulk solutions. In this scenario, the choice of a self-adjoint extension would involve quantizing the allowed values of \( q \), and by restricting \( q \) correctly we can find a set of orthogonal states. While that is appropriate for quantum mechanics, here the bulk physics is well-defined without such a restriction.
6.3 A Simple, General Criterion for Continuous Classical Reconstruction: Capturing Null Geodesics

In this section, we propose a general, geometric criterion for classical reconstruction of the bulk from the boundary. To our knowledge, the case of AdS has not been analyzed explicitly. However, mathematicians such as Bardos et al. [216] have analyzed the analogous situation in flat spacetime: Consider a field that solves the classical wave equation in some region $\Omega$ of Minkowski space with a timelike boundary $\partial \Omega$, with Neumann boundary conditions everywhere on the boundary. Now suppose the boundary value of the field is given in some region $R \subset \partial \Omega$ of the boundary. When is this sufficient to determine the bulk field everywhere in $\Omega$?

The central result is that every null geodesic in $\Omega$ should intersect $R$ in order for continuous reconstruction to be possible. The basic intuition is that if there is some null geodesic that does not hit $R$, then by going to the geometric optics limit we can construct solutions that are arbitrarily well localized along that geodesic. These solutions are “invisible” to the boundary observer who only can observe $\phi$ in the region $R$, in the sense that the boundary imprint can be made arbitrarily small while keeping the energy fixed.

It is not surprising that capturing every null geodesic is a necessary condition for continuous reconstruction, and this will be the important point for us. In many situations, however, the null geodesic criterion is actually sufficient. As long as every null geodesic hits $R$, the entire bulk solution can be reconstructed. (The theorems are quite a bit more general than we have described here, applying to general second-order hyperbolic partial differential equations, and generalizing to nonlinear problems.)

We propose to extrapolate this condition to AdS and its asymptotic boundary, and subregions thereof. The statement is that continuous reconstruction of a bulk subregion is only possible if every null geodesic in that subregion reaches the asymptotic boundary of that subregion. Applying this to a small diamond on the boundary, we conclude that there is no bulk region for which boundary data on the small diamond can be continuously mapped to a bulk field. As shown in Fig. 6.3, it is possible to find a null geodesic through any bulk point that does not intersect a small diamond on the AdS boundary.

A rigorous generalization of the null geodesic criterion to the case of AdS is desirable. In global AdS, at least for the special case of the conformally coupled scalar field, the theorems of Ref. [216] are already strong enough in their current form to ensure continuity. That is because the problem is equivalent to a particular wave equation in a (spatially) compact region with boundary, i.e., the Penrose diagram. And indeed, there we found that the reconstruction was continuous in the way predicted by the theorems.

As stated above, a subtlety arises for the Poincare patch. From the point of view of null geodesics, the Poincare patch is a marginal case. In the Penrose diagram, the boundary of the Poincare patch seems to be just barely large enough to capture all null geodesics passing through the bulk. Why, then, did we find that the reconstruction map is discontinuous, in
apparent violation of the theorems of Ref. [216]? In fact, the Poincare patch just barely fails the criterion because the boundary region is not an open set, as required by the theorem that guarantees continuous reconstruction. We believe this may explain the “infrared” discontinuity we found, and we also believe that a different choice of norm could cure the problem. The existence of an explicit smearing function shows that the problems of the Poincare patch are not fatal.

AdS-Rindler space is of an entirely different character. As we mentioned above, it is clear that there are null geodesics which pass through the bulk and do not intersect even the closure of the boundary. We believe that this is why the discontinuity is in the “ultraviolet,” and also why the smearing function does not exist.

Unique Continuation, Null Geodesics, and RG Flow

There is another important physical question which brings null geodesics to the fore, and it is less subtle than continuity. The trouble with continuity, as we have seen, is that precise statements depend on a choice of boundary norm, and we have been unable to specify a natural choice for this problem. However, even without a boundary norm, we can ask the bulk question of unique continuation of a solution in the radial direction. In AdS/CFT, the radial evolution of the fields is related to a renormalization group flow of the CFT [219, 220,
Let $r$ be a radial coordinate such that $r = \infty$ is the boundary, which represents the UV of the CFT. In the CFT, the IR physics is determined by the UV physics, which suggests that a bulk field configuration near $r = \infty$ can be radially evolved inward and determine the field configuration for all $r$. This intuition can be checked for any given proposed subregion duality.

It is a well-studied problem in mathematics to take a classical field, which solves some wave equation, specified in the region $r > r_*$ and ask if it can be uniquely continued to the region $r < r_*$. If we ask the question locally, meaning that we only ask to continue in a neighborhood of $r = r_*$, then the answer is simple and apparently very robust: the continuation is unique if and only if all null geodesics that intersect the surface $r = r_*$ enter the known region $r > r_*$. (This is usually stated by saying that the extrinsic curvature tensor of the surface, when contracted with any null vector, should have a certain sign.) The intuition here is the same as with continuous reconstruction: if a null geodesic grazes the surface but does not enter the region where we are given the solution, then we can construct geometric optics type solutions that are zero in the known region, but nonzero inside [213].

By this same reasoning, one might conclude that reconstruction from the boundary is not unique when there are null geodesics which avoid the boundary, as opposed to the reconstruction being merely discontinuous as stated previously. The resolution has to do with the technical definitions behind the phrasing, which differ slightly between the two questions. In the present context, the non-uniqueness of the solution comes from going all the way to the geometric optics limit along some geodesic which does not enter $r > r_*$. But this is a singular limit, and one might wish to exclude such configurations from being solutions to the equation. That is the choice we implicitly made in previous sections when we talked about continuity. Continuity is broken because of the same type of geometric optics solutions with a singular limit, but we do not have to include the limiting case itself; continuity only depends on the approach to the limit. So the null geodesic criterion, and the reasoning behind it, is the same even though certain technical aspects of the description change based on convenience for the particular question being asked. The point of discussing unique continuation at all is that the boundary is not involved in the question, and so a boundary norm need not be chosen.

For the case of the AdS-Rindler wedge, the same analysis of null geodesics as above indicates that unique continuation fails as well. Knowing the solution for $r > r_*$ does not determine the solution for smaller $r$. Furthermore, the Poincare patch is again a marginal case for this question. Using the standard $z$ coordinate, then for any $z_*$ there are null geodesics which do not deviate from $z = z_*$.

The Diagnostic in Other Situations

To get a sense for how seriously to take our diagnostic, we can apply it to a variety of familiar situations to test its implications.
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AdS black hole formed in a collapse Suppose we begin at early times with matter near the AdS boundary, and then at some later time it collapses to make a large black hole. In this case, every null geodesic reaches the boundary. For a given geodesic, just follow it back in time: at early times there is no black hole and no singularity, and we know that all null geodesics in AdS hit the boundary. So for a black hole formed in a collapse, every null geodesic is captured by the boundary, and it is likely that continuous reconstruction of the bulk is possible, both inside and outside the horizon.

Eternal AdS black holes and black branes In the case of an eternal black hole, there are some null geodesics that never reach the boundary; they go from the past singularity to the future singularity. The bulk can be continuously reconstructed from the boundary data only outside \( r = 3G_N M \). (3 is the correct numerical factor in 3+1 dimensions. More generally, the bulk can be reconstructed down to the location of the unstable circular orbit.)

We will show this explicitly, focusing initially on a spherical black hole in 3+1 dimensions. The metric is

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2_2
\]

with \( f(r) = 1 + r^2/L^2 - 2G_N M/r \). The null geodesics are extrema of the action

\[
S = \int d\lambda g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \int d\lambda \left( E^2 f - \dot{r}^2 - r^2 \dot{\Omega}^2 \right).
\]

Identifying the conserved quantities \( E = f\dot{t} \) and \( l = r^2 \dot{\Omega} \), the equation of motion can be read off from the condition that the worldline is null:

\[
0 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{E^2}{f} - \frac{\dot{r}^2}{f} - \frac{l^2}{r^2}.
\]

This derivation leads to a simple equation for null geodesics,

\[
\dot{r}^2 + V_{\text{eff}}(r) = E^2 \quad \text{with} \quad V_{\text{eff}} = \frac{fl^2}{r^2}.
\]

The effective potential has a maximum at \( r = 3G_N M \), independent of \( l \) (see Fig. 6.4). So null geodesics that begin outside this radius will inevitably reach the boundary, either in the past or the future. But there are null geodesics that exit the past horizon, bounce off the potential barrier, and enter the future horizon. Because of these, it will be impossible to reconstruct the bulk region near the horizon.

In this case, rather than conclude that there is anything wrong with the correspondence, the natural interpretation is that our classical analysis is breaking down. The “lost” null geodesics are being lost because they fall into the singularity. To recover this information, we will need to go beyond the classical approximation and resolve the singularity.
We can also ask about unique continuation. Starting with the data at large $r$, we can try to integrate in to find the solution at smaller $r$. This process will work fine down to $r = 3G_N M$. However, trying to continue the solution across $3G_N M$ will be impossible.

In the case of a black brane with a planar horizon in AdS$_D$, the effective potential for the null geodesics becomes

$$V_{\text{eff}} = a - \frac{b}{r^{D-1}}$$

where $a$ and $b$ are positive constants. Unlike the spherical black hole, there is no local maximum in the effective potential. For every value of $r$, there are null geodesics which exit the past horizon, travel to that value of $r$, then exit the future horizon. So there is no bulk region that can be continuously reconstructed from the boundary data.

**General conclusion about black hole reconstruction** In the cases of eternal black holes and black branes, the presence of singularities led to the existence of null geodesics which did not reach the boundary, and consequently regions of the bulk which could not be reconstructed from the boundary data. This is not a sign that AdS/CFT is breaking down, but rather an indication that our classical reconstruction procedure is not valid. We know that classical physics breaks down in the neighborhood of the singularity, but the null geodesic criterion suggests that there is a problem even in low-curvature regions. Since the problematic null geodesics begin and end on singularities, it is possible that the physics of singularities needs to be resolved before this question can be answered. A second possibility...
is that nonlocal boundary operators in the CFT encode the physics of the missing bulk regions. As we emphasize in Section 6.4, this latter possibility is the expected outcome for AdS-Rindler space, where we believe there is an exact duality between particular bulk and boundary subregions.

6.4 Arguments for an AdS-Rindler Subregion Duality

In this section we exhibit several arguments in favor of a subregion duality for AdS-Rindler space, despite the failure of continuous reconstruction from local boundary fields.

Probing the Bulk

In the previous section, we asked whether we could classically reconstruct the bulk field $\Phi(B)$ from data on the boundary $\phi(b)$. In essence, we restricted ourselves to considering only one point functions $\langle O(b) \rangle$ on the boundary, and sought to reconstruct bulk fields from integrals of these local boundary operators. However, from an operational standpoint, there is no reason to expect this to be the most efficient way of reconstructing the bulk in general. The boundary theory is equipped with many inherently nonlocal operators. For instance, higher point correlation functions such as $\langle O(b_1)O(b_2) \rangle$ could provide a much better probe of the bulk than one point functions.\textsuperscript{5}

From a physical standpoint, basic properties of AdS/CFT and causality \cite{201, 223} are enough to argue that the theory on the boundary diamond should be capable of reconstructing, at the very least, the AdS-Rindler bulk \cite{204, 205, 206}. Consider a bulk observer Bob who lives near the boundary. The boundary theory should be able to describe Bob, and thus it would be inconsistent for Bob to have information about the bulk which the boundary theory does not. Since Bob can send and receive probes into regions of the bulk which are in the intersection of the causal future and causal past of his worldline, he can probe the entire bulk diamond. Thus, it should be the case that the entire bulk diamond can reconstructed from data on the boundary.

The question of classical reconstruction—restricting to one-point functions on the boundary—amounts to only allowing Bob to make measurements of the field value at his location. If the value of the field decays rapidly near the boundary, Bob would need extremely high resolution to resolve the field. Allowing higher point functions on the boundary amounts to allowing Bob to send and receive probes into the bulk which directly measure the field away from the boundary. This could potentially be a far more efficient way of reconstructing the bulk.

\textsuperscript{5}It may be the case that higher point functions, which can be obtained by solving classical bulk equations of motion with quantum sources, encounter similar obstructions in the classical limit. However the boundary theory also contains many additional nonlocal operators, such as Wilson loops, which we expect to behave differently in this regime.
Figure 6.5: The geometry defining the Hartle-Hawking state for AdS-Rindler. Half of the Lorentzian geometry, containing the $t > 0$ portion of both the left and right AdS-Rindler spaces, is glued to half of the Euclidean geometry. The left and right sides are linked by the Euclidean geometry, and the result is that the state at $t = 0$ is entangled between the two halves.

**Hyperbolic Black Holes**

A CFT dual for AdS-Rindler arises as a special case of the AdS/CFT duality for hyperbolic black holes. The conformal boundary of AdS-Rindler can be viewed as the Rindler patch of Minkowski space, by Eq. 6.58. The CFT vacuum, when restricted to the Rindler patch, appears as a thermal Unruh state, indicating the presence of a thermal object in the bulk. Indeed, the AdS-Rindler metric in Eq. 6.22 with the replacement $z = 1/r$ is exactly the $\mu = 0$ case of the metric of the hyperbolic black hole studied in Ref. [224]:

$$ds^2 = -\left(\frac{r^2}{L^2} - 1 - \frac{\mu}{r^{d-2}}\right)dt^2 + \left(\frac{r^2}{L^2} - 1 - \frac{\mu}{r^{d-2}}\right)^{-1}dr^2 + r^2dH_{d-1}^2,$$

(6.65)

where the spatial hyperbolic plane has the metric

$$dH_{d-1}^2 = \frac{d\xi^2 + dx_i^2}{\xi^2}.$$

(6.66)

A CFT dual for hyperbolic black holes follows from an adaptation of Maldacena’s analysis of the eternal AdS black hole [225], which generalizes easily to the hyperbolic case. In
particular, hyperbolic black holes have a bifurcate Killing horizon, allowing for a definition of a Hartle-Hawking state from a Euclidean path integral [226] (Fig. 6.5). The boundary consists of two disconnected copies of $\mathbb{R} \times H^{d-1}$ (the boundary diamonds). The boundary Hartle-Hawking state is defined through a Euclidean path integral performed on $I_{\beta/2} \times H^{d-1}$, where $I_{\beta/2}$ is an interval of length $\beta/2$ and $\beta$ is the inverse Rindler temperature. (Of course, the Hartle-Hawking state for AdS-Rindler is equivalent to the global vacuum. This follows since $I_{\beta/2} \times H^{d-1}$ is conformal to a hemisphere, which is half of the boundary of Euclidean AdS.) The right and left wedges, regions I and IV in Fig. 6.6, are entangled:

$$|\psi\rangle = \sum_n e^{-\beta E_n/2} |E_n\rangle_R |E_n\rangle_L .$$

(6.67)

Restricting to only region I or IV therefore yields a thermal density matrix.

Excitations above the Hartle-Hawking vacuum can be constructed through operator insertions in the Euclidean geometry. In these states, all particles that enter and leave region I through the Rindler/hyperbolic black hole horizon will be entangled with particles in region IV. One may therefore question to what extent region I can be reconstructed without access to region IV. Small excitations above the Hartle-Hawking vacuum, with energy below the temperature $1/2\pi L$, will appear as an indiscernible fluctuation in the thermal noise when restricted to region I. More energetic states, however, are Boltzmann-suppressed. The density matrix in I will, to a good approximation, accurately register the presence of particles.

Figure 6.6: The Penrose diagram for a hyperbolic black hole. In the $\mu = 0$ case, regions I and IV become the right and left AdS-Rindler wedges. In this case, the singularity is only a coordinate singularity, so the spacetime can be extended to global AdS.
with energy above $1/2\pi L$. Hence, the boundary theory of region I, i.e., its density matrix, should encode at least the high-energy states in the bulk region I [227].

### 6.5 Discussion

If an AdS/CFT duality to is to make sense physically, it should be the case that a physicist with a large but finite computer can simulate the CFT and learn something about the bulk. Knowing particular boundary observables to some accuracy should determine the bulk to a corresponding accuracy. In the case of global AdS/CFT, Hamilton et al. [208] found simple boundary observables—local, gauge invariant operators—which are sufficient to reconstruct the bulk. We have shown that this reconstruction is continuous, meaning that it is a physical duality in the above sense.

In the case of the proposed AdS-Rindler subregion duality, we have seen that these operators are not sufficient to perform the same task. We have shown that, given Eq. 6.10 as our choice of boundary norm, the classical reconstruction map in AdS-Rindler is not continuous. This indicates that we must specify the boundary theory to arbitrary precision to learn anything about the bulk, signaling a breakdown in the physicality of the correspondence.

It is true that our argument for the breakdown depends on the specific boundary norm we choose. We are always free to pick a different norm, for instance one which better respects the symmetries of the boundary theory, and it may be useful to investigate this possibility further. However, the null geodesic criterion gives a simple and intuitive picture of the failure of classical reconstruction, and we would find it surprising if a natural choice of norm could cure the difficulties.

The failure of our diagnostic does not necessarily signal the death of a AdS-Rindler subregion duality. The crucial point is that besides taking the classical limit, we additionally assumed that bulk operators could only be expressed as integrals of local boundary quantities. By removing this extra assumption, a full duality may be recovered—and it would seem surprising if, in general, local boundary quantities were always sufficient for classical reconstruction in all situations. The CFT contains many nonlocal operators, such as complicated superpositions of Wilson loops [214, 215], in addition to local ones. Our results suggest that these additional operators are necessary to see locality in the bulk, even in the classical limit.
Appendix A

Hodge Eigenvalue Decomposition for Forms

In this Appendix, we review some facts about the mathematics of \( p \)-forms and the Hodge decomposition, which we will need to decompose the higher dimensional fields in terms of components that are convenient for the Kaluza-Klein reduction. For a more extensive introduction, see for example [228].

**Review of \( p \)-forms**

For (in general complex-valued) \( p \)-forms \( \omega \) and \( \eta \) on an \( N \)-dimensional manifold \( \mathcal{N} \) with metric \( \gamma \) and covariant derivative \( \nabla \), we define the positive definite inner product

\[
(\omega, \eta) = \int d^N y \sqrt{\gamma} \frac{1}{p!} \omega^*_{n_1...n_p} \eta^{n_1...n_p}.
\]  

(A.1)

The standard exterior derivative taking \( p \)-forms to \((p+1)\)-forms is,

\[
(d\omega)_{n_1...n_{p+1}} = (p+1) \nabla_{n_1} \omega_{n_2...n_{p+1}}.
\]  

(A.2)

A \( p \)-form \( \omega \) is closed if \( d\omega = 0 \), and the space of closed \( p \)-form fields is denoted \( \Lambda^p_{\text{closed}}(\mathcal{N}) \). A \( p \)-form \( \omega \) is exact if \( \omega = d\eta \), for some \((p-1)\)-form \( \eta \). The space of exact \( p \)-form fields is denoted by \( \Lambda^p_{\text{exact}}(\mathcal{N}) \). An exact form is closed, since \( d^2 = 0 \), so \( \Lambda^p_{\text{exact}}(\mathcal{N}) \) is a subspace of \( \Lambda^p_{\text{closed}}(\mathcal{N}) \).

The co-exterior derivative taking \( p \)-forms to \((p-1)\)-forms is defined by

\[
(d^\dagger \omega)_{n_1...n_{p-1}} = -\nabla^m \omega_{mn_1...n_{p-1}}.
\]  

(A.3)

The operator \( d^\dagger \) is the adjoint of \( d \) with respect to the inner product Eq. (A.1), i.e. \( (d\alpha, \omega) = (\alpha, d^\dagger \omega) \), for \( p \)-form \( \omega \) and \((p-1)\)-form \( \alpha \). A \( p \)-form \( \omega \) is co-closed if \( d^\dagger \omega = 0 \), and the space of co-closed \( p \)-form fields is denoted \( \Lambda^p_{\text{co-closed}}(\mathcal{N}) \). A \( p \)-form \( \omega \) is co-exact if \( \omega = d^\dagger \eta \), for some
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$(p+1)$-form $\eta$. The space of co-exact $p$-form fields is denoted by $\Lambda_p^{\text{co-exact}}(\mathcal{N})$. A co-exact form is co-closed, since $(d^\dagger)^2 = 0$, so $\Lambda_p^{\text{co-exact}}(\mathcal{N})$ is a subspace of $\Lambda_p^{\text{co-closed}}(\mathcal{N})$.

The Hodge Laplacian is defined by

$$\Delta = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d \ , \quad (A.4)$$

or in terms of components for a $p$-form $\omega$,

$$(\Delta \omega)_{n_1...n_p} = -\nabla^2 \omega_{n_1...n_p} + \sum_{i=1}^p [\nabla^m_{n_i}, \nabla_{n_i}] \omega_{n_1,...,n_{i-1},m,n_{i+1}...n_p}$$

$$= -\nabla^2 \omega_{n_1...n_p} + \sum_{i=1}^p R^{m}_{n_i} \omega_{n_1,...,n_{i-1},m,n_{i+1}...n_p}$$

$$- 2 \sum_{i,j=1 \ (i<j)}^p R^{m_1}_{n_i} R^{m_2}_{n_j} \omega_{n_1,...,n_{i-1},m_1,n_{i+1}...n_j-1,m_2,n_{j+1}...n_p} \ , \quad (A.5)$$

where $R^{m}_{npq}$ is the Riemann curvature. A $p$-form $\omega$ is called harmonic if $\Delta \omega = 0$. A form is harmonic if and only if it is closed and co-closed. The vector space of harmonic $p$-forms is denoted $\Lambda_p^{\text{harm}}(\mathcal{N})$.

The Laplacian is self-adjoint in the scalar product Eq. (A.1),

$$(\omega, \Delta \eta) = (\Delta \omega, \eta) \ . \quad (A.6)$$

We can decompose the space of $p$-forms into eigenspaces of the Laplacian,

$$\Lambda^p(\mathcal{N}) = \bigoplus_{\lambda} E^p_{\lambda}(\mathcal{N}) \ . \quad (A.7)$$

where $E^p_{\lambda}(\mathcal{N})$ are the subspaces $\{\omega \in \Lambda^p(\mathcal{N}) | \Delta \omega = \lambda \omega\}$. Each subspace is finite dimensional. We only consider those $\lambda$’s such that the subspaces are non-trivial, and this forms the spectrum of the Laplacian. Due to self-adjointness and non-negativity of the Laplacian, the spectrum is real and non-negative, $\lambda \geq 0$, and the eigenspaces for different $\lambda$ are orthogonal with respect to the inner product (A.1). The zero eigenspace is the same as the space of harmonic forms $E^p_{\lambda=0}(\mathcal{N}) = \Lambda_p^{\text{harm}}(\mathcal{N})$.

We have the Hodge decomposition theorem, giving the following orthogonal (under the inner product Eq. (A.1)) direct sum decomposition

$$\Lambda^p(\mathcal{N}) = \Lambda^p_{\text{exact}}(\mathcal{N}) \oplus \Lambda^p_{\text{co-exact}}(\mathcal{N}) \oplus \Lambda_p^{\text{harm}}(\mathcal{N}) \ . \quad (A.8)$$

This means that any $p$-form $\omega$ can be written uniquely as a sum of an exact form, a co-exact form, and a harmonic form. We also have $\Lambda^p_{\text{closed}}(\mathcal{N}) = \Lambda^p_{\text{exact}}(\mathcal{N}) \oplus \Lambda_p^{\text{harm}}(\mathcal{N})$, and $\Lambda^p_{\text{co-closed}}(\mathcal{N}) = \Lambda^p_{\text{co-exact}}(\mathcal{N}) \oplus \Lambda_p^{\text{harm}}(\mathcal{N})$. The maps

$$\Lambda^p_{\text{co-exact}}(\mathcal{N}) \xrightarrow{d} \Lambda^{p+1}_{\text{exact}}(\mathcal{N}) \ , \quad \Lambda^p_{\text{co-exact}}(\mathcal{N}) \xleftarrow{d^\dagger} \Lambda^{p+1}_{\text{exact}}(\mathcal{N}) \ . \quad (A.9)$$
are bijections.

The differential and co-differential commute with the Laplacian,

\[ d\Delta = \Delta d, \quad d^\dagger \Delta = \Delta d^\dagger. \]  

(A.10)

Thus \( \Lambda^p_{\text{exact}}(\mathcal{N}) \) and \( \Lambda^p_{\text{co-exact}}(\mathcal{N}) \) are invariant subspaces of \( \Delta \). In fact, The Hodge decomposition commutes with the eigenvalue decomposition. Thus we can do a Hodge decomposition on each eigenspace separately for \( \lambda \neq 0 \) (the \( \lambda = 0 \) case is already done because \( E^0_0(\mathcal{N}) = \Lambda^p_{\text{harm}}(\mathcal{N}) \)),

\[ E^p_\lambda(\mathcal{N}) = E^p_{\lambda,\text{exact}}(\mathcal{N}) \oplus E^p_{\lambda,\text{co-exact}}(\mathcal{N}), \]  

(A.11)

and the spaces \( \Lambda^p_{\text{exact}}(\mathcal{N}) \) and \( \Lambda^p_{\text{co-exact}}(\mathcal{N}) \) can be decomposed into eigenspaces

\[ \Lambda^p_{\text{exact}}(\mathcal{N}) = \sum_{\oplus \lambda \neq 0} E^p_{\lambda,\text{exact}}(\mathcal{N}), \]  

(A.12)

\[ \Lambda^p_{\text{co-exact}}(\mathcal{N}) = \sum_{\oplus \lambda \neq 0} E^p_{\lambda,\text{co-exact}}(\mathcal{N}). \]  

(A.13)

The following sequences are exact for \( \lambda > 0 \),

\[ \cdots \rightarrow E^p_\lambda(\mathcal{N}) \xrightarrow{d} E^p_{\lambda+1}(\mathcal{N}) \xrightarrow{d} \cdots \]  

(A.14)

\[ \cdots \leftarrow E^p_\lambda(\mathcal{N}) \xleftarrow{d^\dagger} E^p_{\lambda+1}(\mathcal{N}) \xleftarrow{d^\dagger} \cdots \]  

(A.15)

The kernel of \( d \) on \( E^p_\lambda(\mathcal{N}) \) is \( E^p_{\lambda,\text{exact}}(\mathcal{N}) \), and the kernel of \( d^\dagger \) on \( E^p_\lambda(\mathcal{N}) \) is \( E^p_{\lambda,\text{co-exact}}(\mathcal{N}) \).

**The Kaluza-Klein Ansatz**

The result of all this is that any \( p \)-form \( \omega \) can be written as

\[ \omega_{n_1 \cdots n_p} = \sum_{i_p} a^{\lambda}_p Y_{i_p,n_1 \cdots n_p} + \sum_{i_{p-1}} b^{\lambda}_{p-1} \left( d Y_{i_{p-1}} \right)_{n_1 \cdots n_p} + \sum_{\alpha_p} c^{\lambda}_p Y_{\alpha_p,n_1 \cdots n_p}. \]  

(A.16)

Here, \( Y_{i_p} \) is a basis of co-closed \( p \)-forms,

\[ \nabla^m Y_{i_p,mn_2 \cdots n_p} = 0, \]  

(A.17)

which are also eigenvalues of the Laplacian with positive eigenvalues

\[ \Delta Y_{i_p,n_1 \cdots n_p} = \lambda_{i_p} Y_{i_p,n_1 \cdots n_p}, \quad \lambda_{i_p} > 0. \]  

(A.18)

The index \( i_p \) runs over the basis, including multiplicities for the eigenspaces. The \( Y_{\alpha_p} \) are a basis of harmonic \( p \)-forms,

\[ \Delta Y_{\alpha_p,n_1 \cdots n_p} = 0, \]  

(A.19)
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(which, like all harmonic forms, are closed and co-closed, $\nabla^m Y_{\alpha_p, mn_2 \cdots n_p} = \nabla [n_1 Y_{\alpha_p, n_2 \cdots n_{p+1}}] = 0$). The $a^{i_p}$'s, $b^{i_p-1}$'s and $c^{\alpha_p}$ in Eq. (A.16) are constant coefficients which are unique for a given form $\omega$.

We also choose the bases above to be orthonormal in the product Eq. (A.1). For the co-exact forms we have
\[
\frac{1}{p!} \int d^N y \sqrt{\gamma} Y^*_{i_p, n_1 \cdots n_p} Y^{n_1 \cdots n_p}_{j_p} = \delta_{i_p j_p} ,
\]
and for the harmonic forms we have
\[
\frac{1}{p!} \int d^N y \sqrt{\gamma} Y^*_{\alpha_p, n_1 \cdots n_p} Y^{n_1 \cdots n_p}_{\beta_p} = \delta_{\alpha_p \beta_p} .
\]
The three parts of the decomposition Eq. (A.16) are all orthogonal to each other. The expression Eq. (A.16) is an expansion of a $p$-form over a complete orthonormal basis of forms, a generalization of a Fourier expansion, one which is well suited for Kaluza-Klein reductions.

Finally, we will impose that the bases have the following properties under complex conjugation,
\[
Y^*_{i_p, n_1 \cdots n_p} = Y_{i_p, n_1 \cdots n_p} , \quad Y^*_{\alpha_p, n_1 \cdots n_p} = Y_{\alpha_p, n_1 \cdots n_p} .
\]
Here $\overline{i_p}$ and $\overline{\alpha_p}$ represent some involution on the set of indices $i_p, \alpha_p$ respectively.\footnote{As an example, if we were considering the space of scalar functions $\phi$ on the manifold $\mathcal{N}$, there is always only one harmonic function, a constant, and given the normalization condition Eq. (A.21) this constant is $\frac{1}{\sqrt{\mathcal{V}_N}}$, where $\mathcal{V}_N$ is the volume of the manifold. The co-exact zero-forms are the functions orthogonal to the constant function, and to label them we will generally use the index $a$ in place of the index $i_0$. There are no exact zero-forms. Our decomposition Eq. (A.16) for functions then reads
\[
\phi = \sum_a a^a \psi_a + \frac{1}{\sqrt{\mathcal{V}_N}} c^0 ,
\]
where $\psi^a$ is a complete orthonormal set of eigenvectors of the scalar Laplacian with positive eigenvalues,
\[
\Box \psi^a = -\lambda_a \psi^a , \quad \lambda_a > 0 ,
\]
for the case $p = 0$, where we are considering scalar functions $\phi$ on the manifold $\mathcal{N}$, there is always only one harmonic function, a constant, and given the normalization condition Eq. (A.21) this constant is $\frac{1}{\sqrt{\mathcal{V}_N}}$, where $\mathcal{V}_N$ is the volume of the manifold. The co-exact zero-forms are the functions orthogonal to the constant function, and to label them we will generally use the index $a$ in place of the index $i_0$. There are no exact zero-forms. Our decomposition Eq. (A.16) for functions then reads
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\]}

The expression Eq. (A.16) is an expansion of a $p$-form over a complete orthonormal basis of forms, a generalization of a Fourier expansion, one which is well suited for Kaluza-Klein reductions.

For the case $p = 0$, where we are considering scalar functions $\phi$ on the manifold $\mathcal{N}$, there is always only one harmonic function, a constant, and given the normalization condition Eq. (A.21) this constant is $\frac{1}{\sqrt{\mathcal{V}_N}}$, where $\mathcal{V}_N$ is the volume of the manifold. The co-exact zero-forms are the functions orthogonal to the constant function, and to label them we will generally use the index $a$ in place of the index $i_0$. There are no exact zero-forms. Our decomposition Eq. (A.16) for functions then reads
\[
\phi = \sum_a a^a \psi_a + \frac{1}{\sqrt{\mathcal{V}_N}} c^0 ,
\]
where $\psi^a$ is a complete orthonormal set of eigenvectors of the scalar Laplacian with positive eigenvalues,
\[
\Box \psi^a = -\lambda_a \psi^a , \quad \lambda_a > 0 ,
\]
\[ \int d^N y \sqrt{\gamma} \psi^*_a \psi_b = \delta_{ab} , \quad (A.26) \]

For the case \( p = 1 \), where we are considering covariant vectors \( V_m \) on the manifold \( N \), we will often use \( i \) in place of \( i_1 \) for the index labeling the co-exact vectors, and the label \( \alpha \) instead of \( \alpha_1 \) for the harmonic vectors. The decomposition Eq. (A.16) then becomes

\[ V_m = \sum_i b^i Y_{i,n} + \sum_a b^a \partial_n \psi_a + \sum_{\alpha} c^\alpha Y_{\alpha,n} , \quad (A.27) \]

where \( \psi_a \) is the orthonormal basis of positive eigenvalue eigenmodes of the scalar Laplacian (the same basis as appears in Eq. (A.24)), and \( Y_{i,n} \) is a basis of orthonormal co-exact eigenvalues of the vector Laplacian,

\[ \Delta Y_{i,n} = -\Box Y_{i,n} + R^m Y_{i,m} = \lambda_i Y_{i,n} , \quad \lambda_i > 0 , \quad (A.28) \]

\[ \nabla^n Y_{i,n} = 0 \quad , \quad (A.29) \]

\[ \int d^N y \sqrt{\gamma} Y^*_i m Y^*_j m = \delta_{ij} \quad . \quad (A.30) \]

The \( Y_{\alpha,n} \) are the orthonormal harmonic forms,

\[ \Delta Y_{\alpha,n} = 0 \quad , \quad (A.31) \]

\[ \int d^N y \sqrt{\gamma} Y^*_\alpha,m Y^*_\beta,m = \delta_{\alpha\beta} \quad . \quad (A.32) \]

As we will review in Appendix B, in the case where the metric \( \gamma \) is Einstein (the only case of interest in the Kaluza-Klein decompositions of gravity), harmonic vectors can only exist when the scalar curvature is non-positive, \( R(N) \leq 0 \). In this case the set indexed by \( \alpha \) is empty when \( R(N) > 0 \).

**Hodge Star**

The Hodge star, \( * \), maps \( p \)-forms to \( (N-p) \)-forms,

\[ (*\omega)_{n_{p+1} \cdots n_N} = \frac{1}{p!} \epsilon^{n_{p} \cdots n_N}_{n_{p+1} \cdots n_N} \omega_{n_{p+1} \cdots n_N} \quad , \quad (A.33) \]

where \( \epsilon_{n_1 \cdots n_N} = \sqrt{\tilde{\epsilon}_{n_1 \cdots n_N}} \) is the volume form and \( \tilde{\epsilon}_{n_1 \cdots n_N} \) is the anti-symmetric symbol with \( \tilde{\epsilon}_{12 \cdots N} = 1 \). The volume form satisfies \( \epsilon^{p_{1} \cdots p_{p} n_{1} \cdots n_{d-p} \cdots p_{m_{1} \cdots m_{d-p}}} = p! (d-p)! \delta^{p_{1} \cdots p_{p} n_{1} \cdots n_{d-p} \cdots p}_{m_{1} \cdots m_{d-p}} \).

Inversely,

\[ \omega_{n_{p+1} \cdots n_N} = \frac{1}{(N-p)!} \epsilon_{n_{p+1} \cdots n_N} \omega_{n_{p+1} \cdots n_N} (\omega)_{n_{p+1} \cdots n_N} \quad . \quad (A.34) \]

We have \( **\omega = (-1)^{p(N-p)} \omega \), and the Hodge star is an isomorphism between the space of \( p \)-forms and the space of \( (N-p) \)-forms.
We can write the co-exterior derivative Eq. (A.3) as 
\[ d^\dagger = (-1)^{N(p+1)+1} \ast d \ast, \] 
and so the Hodge star intertwines \( d \) and \( d^\dagger \) and commutes with the Laplacian,
\[ d \ast \omega = (-1)^{p} \ast d^\dagger \omega, \quad d^\dagger \ast \omega = (-1)^{p+1} \ast d \omega, \quad \ast \Delta = \Delta \ast, \] \tag{A.35}
for a \( p \)-form \( \omega \). In addition, it preserves the inner product,
\[ (\ast \omega, \ast \eta) = (\omega, \eta), \quad (\ast \alpha, \omega) = (-1)^{p(N-p)}(\alpha, \ast \omega), \] \tag{A.36}
for \( p \)-forms \( \omega \) and \( \eta \) and an \((N-p)\)-form \( \alpha \).

The Hodge star is an isomorphism between the space of harmonic \( p \)-forms and the space of harmonic \((N-p)\)-forms
\[ \Lambda^p_{\text{harm}}(M) \leftrightarrow \Lambda^{N-p}_{\text{harm}}(M). \] \tag{A.37}
This means that the index \( \alpha_p \) takes values over the same set as the index \( \alpha_{N-p} \). We can therefore choose our basis of harmonic \((N-p)\)-forms to be the dual of the basis of \( p \)-forms,
\[ Y_{\alpha_{N-p}} = \ast Y_{\alpha_p}, \quad 0 \leq p < N/2. \] \tag{A.38}

In the case of even \( N \), the space of harmonic \( N/2 \)-forms is mapped to itself under the Hodge star. For \( N = 4 \mod 4 \), we have \( \ast^2 = 1 \), so the space of harmonic \( N/2 \)-forms splits into two eigenspaces of \( \ast \) with eigenvalues \( \pm 1 \), the self-dual and anti-self dual forms, and we can choose our basis of \( Y_{\alpha_{N/2}} \) to line up with this split. We split the index set \( \alpha_{N/2} \) into two sets, \( \alpha_+ \) and \( \alpha_- \) such that \( \ast Y_{\alpha_+} = \pm Y_{\alpha_-} \). For \( N = 2 \mod 4 \), we have \( \ast^2 = -1 \) and so the space of \( N/2 \)-forms splits into two eigenspaces of \( \ast \) with eigenvalues \( \pm i \), the imaginary self-dual and imaginary anti-self dual forms, and we choose our basis to transform as \( \ast Y_{\alpha_+} = \pm i Y_{\alpha_-} \). In this imaginary case we have \( Y_{\alpha_+} = Y_{\alpha_-} \). This case is useful for dealing with flux compactifications in \( N = 2 \).

For any given eigenvalue \( \lambda > 0 \), the Hodge star is an isomorphism between the space of exact \( p \)-forms of eigenvalue \( \lambda \) and the space of co-exact \((N-p)\)-forms of eigenvalue \( \lambda \).
\[ E^p_{\lambda,\text{exact}}(M) \leftrightarrow E^{N-p}_{\lambda,\text{co-exact}}(M), \quad \lambda > 0. \] \tag{A.39}
This means the index \( i_p \) ranges over the same set as the index \( i_{N-1-p} \). We may choose our basis of forms so that
\[ Y_{i_{N-1-p}} = \frac{1}{\sqrt{\lambda_{i_p}}} \ast dY_{i_p}, \quad 0 \leq p \leq [N/2]. \] \tag{A.40}

The normalization on the right hand side ensures that the left hand side is normalized properly under the scalar product Eq. (A.1).

These duality relations are important in the case of the flux compactifications of Section 2.6. For example, there can be harmonic one-forms in the expansion of the off-diagonal components of the higher dimensional graviton, corresponding to vectors in the uncompactified \( d \)-dimensional space. In this case there are also harmonic \((N-1)\)-forms coming from
the expansion of the \((N - 1)\)-form potential with all indices in the compactified dimensions. These harmonic \((N - 1)\)-forms correspond to scalars in the \(d\) dimensions, but because of Hodge duality these scalars carry the same index, \(\alpha\), as the harmonic one-forms from the graviton. These scalars mix with the vectors coming from the graviton. Similarly, there are co-exact vectors in the expansion of the off-diagonal components of the higher dimensional graviton, again corresponding to vectors in the uncompactified \(d\)-dimensional space with index \(i\). There are co-exact \((N - 2)\)-forms in the expansion of the \((N - 1)\)-form flux with one index in the \(d\)-dimensional space, corresponding to vectors in \(d\) dimensions. These co-exact \((N - 2)\)-forms are dual to the exact two-forms, and hence, because of the isomorphism Eq. (A.9), carry the same index, \(i\), as the co-exact one-forms. These vectors mix with the vectors of the graviton.
Appendix B

Killing Vectors on Closed Einstein Manifolds

In the Kaluza-Klein decomposition of gravity, a special role is played by Killing vectors on the internal manifold, which end up corresponding to massless vector fields in the uncompactified space. In this Appendix we review some needed facts about Killing vectors on closed Einstein manifolds.

We are considering a closed Riemannian $N$-manifold which is Einstein, i.e. the metric satisfies

\[ R_{mn} = \frac{R_{(N)}}{N} \gamma_{mn} , \]  

(B.1)

with $R_{(N)}$ constant.

A Killing vector $K^m$ is a vector along which the metric has an isometry,

\[ \mathcal{L}_K \gamma_{mn} \equiv \nabla_m K_n + \nabla_n K_m = 0 . \]  

(B.2)

Taking the trace and divergence of Eq. (B.2), and using the Einstein condition Eq. (B.1) we find that Killing vectors satisfy

\[ \nabla_m K^m = 0 , \quad \Box K^m + \frac{R_{(N)}}{N} K^m = 0 . \]  

(B.3)

First, we prove some vanishing theorems which will tell us when Killing vectors can be present. Consider the following identity for a general vector field $V_m$.

\[ \nabla_m (V^n \nabla_n V^m - V^m \nabla_n V^n) = \nabla_m V_n \nabla^n V^m - (\nabla_m V^m)^2 + \frac{R_{(N)}}{N} V^2 . \]  

(B.4)

Upon integrating both sides, the left hand side is zero because it is a total derivative, so from the right hand side we have

\[ \int_{\mathcal{N}} \sqrt{\gamma} \left( \nabla_m V_n \nabla^n V^m - (\nabla_m V^m)^2 + \frac{R_{(N)}}{N} V^2 \right) = 0 . \]  

(B.5)
In the case where $V^m$ is a Killing vector $K^m$, we have, using the properties $\nabla_m K_n + \nabla_n K_m = 0$ and $\nabla_m K^m = 0$ from Eqs. (B.2) and (B.3),

$$\int_N \sqrt{\gamma} \left( -\nabla_m K_n \nabla^n K^m + \frac{R(N)}{N} K^2 \right) = 0 .$$

(B.6)

Suppose $R(N)$ is negative. Then the integrand is a sum of squares and the integral cannot be zero unless $K^m = 0$.

- If $R(N) < 0$, there can be no non-trivial Killing vectors.

This same trick allows us to prove a vanishing theorem for harmonic forms. Consider Eq. (B.5) in the case where $V_m$ is a harmonic one-form $H_m$. We have, using the properties that a harmonic form is closed and co-closed, $\nabla_m H_n = \nabla_n H_m$ and $\nabla_m H^m = 0$,

$$\int_N \sqrt{\gamma} \left( \nabla_m H_n \nabla^n H^m + \frac{R(N)}{N} H^2 \right) = 0 .$$

(B.7)

Suppose $R(N)$ is positive. Then the integrand is a sum of squares and the integral cannot be zero unless $H_m = 0$.

- If $R(N) > 0$, there can be no non-trivial harmonic forms.

In this sense, Killing vectors and harmonic forms are complementary. They only co-exist in the case where $R(N) = 0$. In this case, for the integral to be zero we see that both the Killing vectors and harmonic forms must satisfy $\nabla_m V_n = 0$. When $R(N) = 0$, the space of Killing vectors and the space of harmonic forms are identical.

The following integral is always greater than or equal to zero, and is equal to zero if and only if $V^m$ is a Killing vector,

$$0 \leq \int_N \sqrt{\gamma} \left( \nabla_m V_n + \nabla_n V_m \right) \left( \nabla^m V^n + \nabla^m V^m \right)$$

$$= 2 \int_N \sqrt{\gamma} \left[ -V_m \Box V^m - V^n \nabla_n (\nabla_m V^m) - \frac{R(N)}{N} V^2 \right]$$

$$= 2 \int_N \sqrt{\gamma} V_m (\Delta_K V)^m ,$$

(B.8)

(B.9)

(B.10)

where we have defined the following operator acting on the space of vector fields,

$$\left( \Delta_K V \right)^m = -\Box V^m - \nabla^m \left( \nabla_n V^n \right) - \frac{R(N)}{N} V^m .$$

(B.11)

It is easy to check that this operator is elliptic, self-adjoint, and, because of Eq. (B.8), positive definite. The Killing vectors are precisely the space of zero eigenvalues of this operator. Combining with Eq. (B.3), we have shown
$K^m$ is a Killing vector if and only if

\[ \Box K^m + \frac{R_{(N)}}{N} K^m = 0 \text{ and } \nabla_m K^m = 0. \]  \hspace{1cm} (B.12)

Recalling the Hodge Laplacian acting on one-form fields, $\Delta V_n = -\Box V_n + R_{nm} V_m$, and using Eq. (B.1), we have

- For $R_{(N)} > 0$, the space of Killing vectors is precisely the space of co-exact eigenvectors of the vector Laplacian with eigenvalue $\lambda = \frac{2R_{(N)}}{N}$,

\[ \Delta K^n = \left( \frac{2R_{(N)}}{N} \right) K^n, \quad \nabla_n K^n = 0. \]  \hspace{1cm} (B.13)

Recall that for $R_{(N)} = 0$, the space of Killing vectors is precisely the space of harmonic vectors, and by the vanishing theorem above, there are no Killing vectors when $R_{(N)} < 0$.

From Eq. (B.8), positivity of the operator $-\Box - \frac{R_{(N)}}{N}$, we see that $\lambda = \frac{2R_{(N)}}{N}$ is in fact a lower bound on the possible eigenvalues of $\Delta$ on the space of co-exact forms. It is saturated only for Killing vectors.
Appendix C

Conformal Killing Vectors on Closed Riemannian Manifolds

In the Kaluza-Klein decomposition of gravity, a special role is played by the so-called conformal scalars on the internal manifold. These end up accounting for absent modes from the point of view of the un-compactified space, so it is important to understand when they are present. In this Appendix we review some needed facts about conformal scalars on closed Einstein manifolds.

A conformal Killing vector $C^m$ is a vector along which the metric changes by an overall conformal factor,

$$\mathcal{L}_C \gamma_{mn} = \nabla_m C_n + \nabla_n C_m = f \gamma_{mn} .$$

(C.1)

where $f(y)$ is a scalar function. Taking the trace of Eq. (C.1), we find

$$f = \frac{2}{N} \nabla_mC^m ,$$

(C.2)

so the conformal Killing equations Eq. (C.1) can be written in the equivalent form

$$\nabla_mC_n + \nabla_n C_m - \frac{2}{N} (\nabla_p C^p) \gamma_{mn} = 0 .$$

(C.3)

The Killing vectors form a subspace of the conformal Killing vectors; they are precisely those for which $\nabla_mC^m = 0$.

In the following, we assume $N \geq 2$. The case $N = 1$ is trivial: every vector is a conformal Killing vector.

We start by proving some vanishing theorems. The following integral is always greater
APPENDIX C. CONFORMAL KILLING VECTORS ON CLOSED RIEMANNIAN
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than or equal to zero, and is equal to zero if and only if $V^m$ is a conformal Killing vector,

$$0 \leq \int_N \sqrt{\gamma} \left( \nabla_m V_n + \nabla_n V_m - \frac{2}{N} (\nabla_p V^p) \gamma_{mn} \right) \left( \nabla^m V^n + \nabla^n V^m - \frac{2}{N} (\nabla_p V^p) \gamma^{mn} \right)$$

$$= 2 \int_N \sqrt{\gamma} \left[ \nabla_n V_m \nabla^m V^n + \frac{N-2}{N} (\nabla_m V^m)^2 - \frac{R(N)}{N} V^2 \right]$$

$$= 2 \int_N \sqrt{\gamma} V_m (\Delta C V)^m , \quad \text{(C.4)}$$

where we have defined the following operator acting on the space of vector fields,

$$(\Delta C V)^m = -\Box V^m - \frac{N-2}{N} \nabla^m (\nabla_n V^n) - \frac{R(N)}{N} V^m . \quad \text{(C.5)}$$

It is easy to check that this operator is elliptic, self-adjoint, and, by Eq. (C.4), positive. The conformal Killing vectors are precisely the space of zero eigenvalues of this operator.

Suppose $R(N) \leq 0$, and $V^m$ is a conformal Killing vector. Looking at the middle line Eq. (C.4), we see that the integrand is a sum of squares and the integral cannot be zero unless $V^m = 0$. If $R(N) = 0$, then we have $\nabla_m V_n = 0$. We thus have the following vanishing theorems:

- If $R(N) < 0$, there are no non-trivial conformal Killing vectors.
- If $R(N) = 0$, there are no conformal Killing vectors which are not also Killing vectors.

Since there are no non-Killing conformal Killing vectors $R(N) \leq 0$, we turn to the case $R(N) > 0$ for which they may exist. Taking the divergence of Eq. (C.3), we have

$$\Box C^m + \frac{R(N)}{N} C^m + \frac{N-2}{N} \nabla^m (\nabla_n C^n) = 0 , \quad \text{(C.6)}$$

which is the same as the operator Eq. (C.5), so we may write $(\Delta C C)^m = 0$. Taking another divergence gives

$$\Delta (\nabla_m C^m) = \frac{R(N)}{N-1} (\nabla_m C^m) , \quad \text{(C.7)}$$

where $\Delta = -\Box$ is the scalar Laplacian. Thus $\nabla_m C^m$ is an eigenfunction of the Laplacian with eigenvalue $\lambda = \frac{R(N)}{N-1}$. In terms of this value of $\lambda$, we can rearrange derivatives in Eq. (C.7) to give

$$\nabla^m (\Delta C_m - \lambda C_m) = 0 , \quad \text{(C.8)}$$

where $\Delta$ is the Laplacian on vectors. This implies

$$\Delta C_m = \lambda C_m + V_{m}^{\text{co-closed}} , \quad \text{(C.9)}$$

where $V_{m}^{\text{co-closed}}$ is an arbitrary co-closed form, $\nabla^m V_{m}^{\text{co-closed}} = 0$. Now, decompose $C_m$ into its co-closed and exact parts,

$$C_m = \nabla_m C + C_{m}^{\text{co-closed}} , \quad \text{(C.10)}$$
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where $C$ is a scalar with no constant component, and $\nabla^m C_{m}^{\text{co-closed}} = 0$. Plugging Eq. (C.10) into Eq. (C.9), and reading off the co-closed and exact parts, we have

$$\nabla_m (\Delta C - \lambda C) = 0, \quad (\Delta - \lambda) C_{m}^{\text{co-closed}} = V_{m}^{\text{co-closed}}.$$  \hfill (C.11)

The second equation in Eq. (C.11) can be inverted for $C_{m}^{\text{co-closed}}$ given $V_{m}^{\text{co-closed}}$. The first equation tells us that $C$ is an eigenvalue of the Laplacian up to a constant $C_0$, $\Delta C = \lambda C + C_0$, but since $C$ was assumed to have no constant piece, we must have $C_0 = 0$. In summary, we have

$$C_m = \nabla_m C + C_{m}^{\text{co-closed}}, \quad \Delta C = \lambda C,$$  \hfill (C.12)

where $C_{m}^{\text{co-closed}}$ is an arbitrary form satisfying $\nabla_m C_{m}^{\text{co-closed}} = 0$.

Now, plugging Eq. (C.12) into Eq. (C.6), we find $\Box C_{m}^{\text{co-closed}} + R(N) C_{m}^{\text{co-closed}} = 0$, which combined with the fact that $C_{n}^{\text{co-closed}}$ is co-closed and the result Eq. (B.12), tells us that $C_{n}^{\text{co-closed}}$ is a Killing vector: $\nabla_m C_{n}^{\text{co-closed}} + \nabla_n C_{m}^{\text{co-closed}} = 0$. Therefore, the space of conformal Killing vectors is all vectors of the form

$$\nabla^m C + K^m, \quad \Delta C = \frac{R(N)}{N-1} C,$$  \hfill (C.13)

where $K^m$ is an arbitrary Killing vector.

- The subspace of conformal Killing vectors orthogonal to the subspace of Killing vectors is $\nabla^m C$, $\Delta C = \frac{R(N)}{N-1} C$, i.e. the image under the gradient of the eigenspace of the scalar Laplacian with eigenvalue $\lambda = \frac{R(N)}{N-1}$.

These scalars, spanning the eigenspace of the scalar Laplacian with eigenvalue $\lambda = \frac{R(N)}{N-1}$, whose gradients give the non-Killing conformal Killing vectors, are called conformal scalars. They must be treated specially in the decomposition of the graviton.

**Lichnerowicz Bound**

A theorem of Lichnerowicz [229] says that given a closed $N$-dimensional Riemannian manifold $N \geq 2$, if the Ricci scalar $R(N)$ satisfies $R(N) \geq k$ for some constant $k > 0$, then the first nonzero eigenvalue $\lambda$ of the scalar Laplacian satisfies $\lambda \geq \frac{k}{N-1}$. Our interest is in Einstein spaces, for which $R(N)$ is constant, so we have the bound

$$\lambda \geq \frac{R(N)}{N-1}.$$  \hfill (C.14)

Thus, the conformal Killing vectors of a closed Einstein manifold come precisely from the gradients of the lowest possible non-zero eigenfunctions of the scalar Laplacian, those that saturate the Lichnerowicz bound.

By a theorem of Obata [230], the equality in Eq. (C.14) is obtained if and only if the manifold is isometric to the sphere, so conformal scalars (and hence non-Killing conformal
Killing vectors exist only for the sphere. The eigenvalues of the scalar Laplacian on an $N$ sphere of radius $R$ are given by $l(l + N - 1)/R^2$, where $l = 0, 1, 2, \ldots$, or in terms of the curvature $R_{(N)} = N(N - 1)/R^2$,

$$\lambda_l = \frac{l(l + N - 1)}{N(N - 1)} R_{(N)} . \quad (C.15)$$

We see that the Lichnerowicz bound Eq. (C.14) is saturated by the eigenmodes with $l = 1$. In summary, conformal scalars exist only on the sphere, and are precisely the $l = 1$ spherical harmonics.

Turning to the case $N = 1$, we note that all scalars orthogonal to the constant scalar are conformal scalars. In this case $R_{(N)} = 0$, and the Lichnerowicz bound does not apply.
Appendix D

Hodge Eigenvalue Decomposition for Symmetric Tensors

In this Appendix we describe the analog of the Hodge decomposition for symmetric tensors. This is needed to decompose the extra-dimensional components of the graviton. The Killing vectors described in Appendix B and the conformal scalars described in Appendix C play a role in this expansion and must be treated with care.

Consider the space of (complex) symmetric tensors $h_{mn}$ on an Einstein space $N$. This space is denoted $S^2(N)$. We define the positive definite inner product

$$
(h, h') = \int d^N y \sqrt{\gamma} h^*_{mn} h'^{mn}.
$$

The natural Laplacian on this space is the Lichnerowicz operator [231], defined by

$$
\Delta_L h_{mn} = -\nabla^2 h_{mn} + \frac{2R(N)}{N} h_{mn} - 2R_{mpnq} h^{pq}.
$$

The Lichnerowicz operator is self-adjoint with respect to the inner product Eq. (D.1), $(h, \Delta_L h') = (\Delta_L h, h')$, thus we can decompose the space of symmetric tensors into eigenspaces of the Lichnerowicz operator,

$$
S^2(N) = \sum_{\lambda \in \Lambda} S^2_{\lambda}(N).
$$

where $E^p_{\lambda}(N)$ are the subspaces $\{\omega \in \Lambda^p(N) | \Delta \omega = \lambda \omega \}$. Each subspace is finite dimensional. We only consider those $\lambda$'s such that the subspaces are non-trivial, and this forms the spectrum of the Lichnerowicz Laplacian. Due to self-adjointness, the $\lambda$'s are all real and the eigenspaces for different $\lambda$ are orthogonal with respect to the inner product Eq. (D.1).

The space of symmetric tensors can also be decomposed as the orthogonal sum of the space of traverse tensors $S^2_T(N)$, i.e. those tensors $h^T_{mn}$ satisfying $\nabla^m h^T_{mn} = 0$, and all tensors of the form $\nabla_{(m} V_{n)}$ for some vector $V_m$ [232, 233]

$$
S^2(N) = S^2_T(N) \oplus \text{Im} \left( \nabla_{(m} \cdot n) \right).
$$

(D.4)
APPENDIX D. HODGE EIGENVALUE DECOMPOSITION FOR SYMMETRIC TENSORS

The vector can in turn be split into transverse and longitudinal parts according to the standard Hodge decomposition Eq. (A.8): \( V_m = \partial_m \phi + V^T_m, \nabla^m V^T_m = 0 \), where \( \phi \) is some scalar. Finally we extract the trace, with the result that we can write any symmetric tensor as

\[
 h_{mn} = h_{TT}^{mn} + 2 \nabla_m V^T_n + 2 \left( \nabla_m \nabla_n \phi - \frac{1}{N} \nabla^2 \phi \right) + \frac{1}{N} \bar{\phi} \gamma_{mn},
\]  

(D.5)

where \( h_{TT}^{mn} \) is transverse and traceless: \( \nabla_m h_{TT}^{mn} = 0, \gamma_{mn} h_{TT}^{mn} = 0 \), and \( \bar{\phi} \) is another scalar, which carries the trace: \( \bar{\phi} = h \). This is the Hodge decomposition for symmetric tensors.

The four parts of this decomposition are all orthogonal with respect to the inner product Eq. (D.1).

The Lichnerowicz operator commutes with traces, divergences and symmetrized derivatives,

\[
 \Delta_L (\phi \gamma_{mn}) = \gamma_{mn} \Delta \phi, \quad \Delta_L (\nabla_m V_n) = \nabla_m (\Delta V_n), \quad \nabla^m \Delta_L h_{mn} = \Delta (\nabla^m h_{mn}),
\]  

(D.6) (D.7) (D.8)

where \( V_m \) and \( \phi \) are any one-form and scalar, and \( \Delta \) is the Hodge Laplacian Eq. (A.5) (we also have \( \nabla^m \Delta V_m = \Delta (\nabla^m V_m) \)).

In light of these relations, the decomposition Eq. (D.3) according to Lichnerowicz eigenvalues commutes with the tensor Hodge decomposition Eq. (D.5). This means we can write an arbitrary symmetric tensor \( h_{mn} \) as

\[
 H_{mn} = \sum_I c^T h_{mn,I} + \sum_{i \neq \text{Killing}} c^i (\nabla_m \xi_{m,i} + \nabla_n \xi_{n,i}) + \sum_{a \neq \text{conformal}} c^a \left( \nabla_m \nabla_a \psi_a - \frac{1}{N} \nabla^2 \psi_a \gamma_{mn} \right) + \sum_{a} \frac{1}{N} c^a \psi_a \gamma_{mn} + \frac{1}{N} c^0 \gamma_{mn}.
\]  

(D.9)

The various \( c \)'s are all coefficients, the generalized Fourier coefficients; they are uniquely determined by and determine \( h_{mn} \). The different parts of this expression are as follows: \( h_{mn,I}^T \) are a complete orthonormal basis, indexed by \( I \), of transverse traceless tensors, chosen so that they are eigenvalues of the Lichnerowicz operator with eigenvalues \( \lambda_I \),

\[
 \Delta_L h_{mn,I}^T = \lambda_I h_{mn,I}^T, \quad \nabla^m h_{mn,I}^T = \gamma_{mn} h_{mn,I}^T = 0,
\]  

(D.10)

\[
 \int d^N y \sqrt{\gamma} (h_{TTmn,I}^*)^* h_{mn,J}^T = \delta^T_I. \quad \int d^N y \sqrt{\gamma} (\xi^m)_{i}^* \xi_{m,j} = \delta^j_i.
\]  

(D.11) (D.12) (D.13)
In the sum over \( i \) in Eq. (D.9), we omit those with eigenvalue \( \lambda = \frac{2R(N)}{N} \), because these are precisely those that are Killing vectors (see Appendix B), i.e. \( \nabla_m \xi_n + \nabla_n \xi_m = 0 \) (and hence these do not get their own independent Fourier coefficient \( \phi^i \)). The \( \psi^a \) are a complete orthonormal basis of the scalar Laplacian, as in Eqs. (A.24), (A.25), and (A.26). In the sum \( \sum_{a \neq \text{conf.}} \), we have indicated that we should leave out the conformal scalars (see Appendix C), those scalars with eigenvalue \( \lambda = \frac{R(N)}{N-1} \), because these are precisely those whose gradients are conformal Killing vectors, for which \( \nabla_m \nabla_n \psi - \frac{1}{N} \nabla^2 \psi \gamma_{mn} = 0 \), and hence should not get their own Fourier coefficient \( c^a \).

### Moduli Space of Einstein Structures

In the decomposition of the graviton, massless scalars appear corresponding to directions in the moduli space of Einstein structures of the internal manifold. The moduli space of Einstein structures on a manifold is the space of all possible Einstein metrics which can be put on the manifold, modulo diffeomorphisms, and with the condition that the total volume be fixed [232].

Considering small variations of the metric \( \delta \gamma_{mn} = h_{mn} \), we see from Eq. (D.4) that the condition that the variation not be a diffeomorphism is the condition that \( h_{mn} \) be transverse, \( \nabla^m h_{mn} = 0 \). The condition that the volume be fixed is the condition \( \delta \int d^N y \sqrt{\gamma} = \frac{1}{2} \int d^N y \sqrt{\gamma} \gamma_{mn} h_{mn} = 0 \), i.e. that \( h_{mn} \) be traceless, \( h \equiv \gamma^{mn} h_{mn} = 0 \).

The condition that a space be Einstein can be written \( R_{mn} - \frac{R(N)}{N} \gamma_{mn} = 0 \). The variation of the Einstein condition reads

\[
2 \delta \left( R_{mn} - \frac{R(N)}{N} \gamma_{mn} \right) = -\nabla^2 h_{mn} - 2R_{mpnq} h^{pq}.
\]

Thus, comparing to Eq. (D.2), we see that the tangent space to the moduli space, at a given Einstein metric, is spanned by transverse traceless tensors which are eigentensors of the Lichnerowicz operator with eigenvalue \( \frac{2R(N)}{N} \),

\[
\Delta_L h^{TT}_{mn} = \frac{2R(N)}{N} h^{TT}_{mn}.
\]

These give rise to massless scalar states in the Kaluza-Klein reduction of the graviton.
Appendix E

Perturbation theory

The dominant eigenvector can be computed perturbatively in up-tunneling branching ratios [234]. The rate equation matrix is separated into an upper triangular matrix $R^{(0)}$ consisting of down-tunnelings, and a lower triangular matrix containing up-tunneling rates, $R^{(1)}$:

$$R = R^{(0)} + R^{(1)}.$$  \hspace{1cm} (E.1)

The entries on the diagonal separate into total down-tunnelings,

$$D_i = \sum_{j<i} \kappa_{ji},$$  \hspace{1cm} (E.2)

which are included in $R^{(0)}$, and total up-tunnelings,

$$U_i = \sum_{j>i} \kappa_{ji},$$  \hspace{1cm} (E.3)

which are included in $R^{(1)}$.

Due to the high suppression of up-tunneling, the eigenvector and corresponding eigenvalue can be computed perturbatively in $R^{(1)}$: \footnote{The perturbative approach breaks down if up- and down-tunneling branching ratios become comparable. In a realistic landscape, this might be expected only for vacua near the Planck scale. For the purposes of the present analysis, however, no generality is lost in considering all states with $\Lambda > S_{BB}^{-1}$ as a single metastable de Sitter vacuum. Then up-tunneling can occur only from vacua with a very small cosmological constant and will be extremely suppressed.}

$$(R^{(0)} + \lambda R^{(1)})(s^{(0)} + \lambda s^{(1)} + \lambda^2 s^{(2)} + \ldots)$$

$$= -(q^{(0)} + \lambda q^{(1)} + \ldots)(s^{(0)} + \lambda s^{(1)} + \lambda^2 s^{(2)} + \ldots).$$  \hspace{1cm} (E.4)

Equating the coefficients at each order in $\lambda$ gives a collection of matrix equations that can be iteratively solved for the eigenvector at each order in perturbation theory,
\[ R^{(0)} s^{(0)} = -q^{(0)} s^{(0)} , \]  
\[ (R^{(0)} + q^{(0)} I) s^{(1)} = -(R^{(1)} + q^{(1)} I) s^{(0)} , \]  
\[ (R^{(0)} + q^{(0)} I) s^{(2)} = -q^{(2)} s^{(0)} - (R^{(1)} + q^{(1)} I) s^{(1)} , \]  
\[ \vdots \]  
\[ (R^{(0)} + q^{(0)} I) s^{(n)} = -\sum_{k=0}^{n-2} q^{(n-k)} s^{(k)} - (R^{(1)} + q^{(1)} I) s^{(n-1)} . \]  

At zeroth order, we posit that the unperturbed solution is a delta function with support only in the longest-lived de Sitter vacuum, which we denote \( * \),  
\[ s^{(0)}_i = \delta_{i*} . \]  

This selects the smallest magnitude eigenvector. (Note that this will only satisfy the zeroth order equation if there are no down-tunnelings from the dominant vacuum to non-terminal vacua [234].)

The entry of \( R^{(0)} + q^{(0)} I \) corresponding to the dominant vacuum has a zero eigenvalue, rendering the matrix noninvertible. This means the system of equations is linearly dependent and has an infinite number of solutions; given Eq. (E.9), one can still satisfy the \( n \)th order equation by adding to a solution \( s^{(n)} \) any constant times \( s^{(0)} \).

We fix this ambiguity by requiring orthogonality,  
\[ s^{(0)} \cdot s^{(n)} = 0 \]  
for all \( n > 0 \). A different choice of constraint would correspond to re-normalizing and re-arranging the perturbative series. Our choice normalizes the dominant vacuum entry of the eigenvector to one,  
\[ s_* = 1 , \]  
so the eigenvector will no longer be normalized to 1 when higher order effects are taken into account. However, the criteria for \( OO \) or \( BB \) domination depend only on ratios of probabilities, and are thus independent of normalization.

In general, the leading-order correction to Eq. (E.9) may arise beyond first order, if it requires \( n_0 \geq 1 \) up-tunnelings to reach a given de Sitter vacuum \( i \) from the \( * \) vacuum. Before solving for this leading-order correction, we first consider the general solutions order-by-order up to \( n \)th order.

**Zeroth order**

We have required that the zeroth order solution to the eigenvector is given by Eq. (E.9),  
\[ s^{(0)}_i = \delta_{i*} . \]
From Eq. (E.5) we find the corresponding correction to the eigenvalue,

\[ q^{(0)} = D_\ast \, . \] (E.13)

First order

To solve for the first order correction, we invert Eq. (E.6), which is easy since the matrix is upper triangular. Each non-dominant entry gives an equation that can be solved iteratively:

\[ s_i^{(1)} = \hat{\beta}_{i\ast} + \sum_{j>i} \hat{\beta}_{ij} s_j^{(1)} , \] (E.14)

where

\[ \hat{\beta}_{ij} \equiv \frac{\kappa_{ij}}{D_i - D_\ast} . \] (E.15)

We claim that the entries \( s_i^{(1)} \) of the solution to this equation (except for the entry *, which is zero) are sums of products of the factors \( \hat{\beta} \) along paths from * to the corresponding vacuum, which begin with one up-tunneling and have no further up-tunnelings, and pass through only non-dominant vacua at intermediate steps:

\[ s_i^{(1)} = \sum_p \sum_{i_1, \ldots, i_{p-1} \neq \ast} \hat{\beta}_{ii_{p-1}} \cdots \hat{\beta}_{i_1 \ast} , \] (E.16)

where the superscript (1) indicates that we are summing over paths with only one up-tunneling that start with an up-tunneling. \( p \) labels path length, so the sum over \( p \) indicates that we are summing over paths of all length that are consistent with this requirement on up-tunnelings.

We can prove this using induction. First, for the vacuum \( N \) with the largest cosmological constant, Eq. (E.14) gives

\[ s_N^{(1)} = \hat{\beta}_{i\ast} , \] (E.17)

which satisfies Eq. (E.16).

Now suppose that Eq. (E.16) is satisfied by all entries \( i > k \), and assume that the \( k \)th entry does not correspond to *. Then by Eq. (E.14),

\[ s_k^{(1)} = \hat{\beta}_{k\ast} + \sum_{j>k} \hat{\beta}_{kj} \sum_p \sum_{i_1, \ldots, i_{p-1} \neq \ast} \hat{\beta}_{ii_{p-1}} \cdots \hat{\beta}_{i_1 \ast} \]

\[ = \sum_p \sum_{i_1, \ldots, i_{p-1} \neq \ast} \hat{\beta}_{ki_{p-1}} \cdots \hat{\beta}_{i_1 \ast} , \] (E.18)

so Eq. (E.16) is also satisfied by the \( k \)th entry.

The final case occurs when \( k = \ast \). In this case we replace the row with the constraint Eq. (E.10), which gives \( s_\ast^{(1)} = 0 \). Then proceeding to lower values of the cosmological
constant, a similar argument shows that the \((k-1)\)th entry satisfies Eq. (E.16). This proves our claim.

Finally, substituting our result back into Eq. (E.6), we can solve the row \(\ast\) for the corresponding correction to the eigenvalue,

\[
q^{(1)} = - \sum_j \kappa_{sj}^{(1)} + U_\ast .
\] (E.19)

**n**th order

For the \(n\)th order case with \(n > 1\), extra terms are generated that naively seem to deviate from this simple pattern. To see this, we first invert Eq. (E.8) as before. In terms of the compact notation

\[
\Delta D_i = D_i - D_* , \quad \Delta U_i = U_i - U_* ,
\] (E.20)

this gives the following iterative equation,

\[
s_i^{(n)} = \sum_{j<i} \hat{\beta}_{ij} s_j^{(n-1)} + \sum_{j>i} \hat{\beta}_{ij} s_j^{(n)} - \frac{U_i}{\Delta D_i} s_i^{(n-1)} + \sum_{k=0}^{n-1} q^{(n-k)} \frac{\Delta U_i}{\Delta D_i} s_i^{(k)} .
\] (E.21)

The third term comes from fact that for \(n > 1\) the vector \(s^{(n-1)}\) has components in non-dominant vacua, which multiply contributing diagonal entries of the matrix \(R^{(1)} - q^{(1)} I\).

It is possible to check that the last term generates paths that return through the dominant vacuum; hence it does not contribute to the leading order result and so we will discard it here. It suffices to consider only the following simplified version of the full iterative equation:

\[
s_i^{(n)} \approx \sum_{j<i} \hat{\beta}_{ij} s_j^{(n-1)} + \sum_{j>i} \hat{\beta}_{ij} s_j^{(n)} - \frac{\Delta U_i}{\Delta D_i} s_i^{(n-1)} .
\] (E.22)

We have included the up-tunneling factor \(U_\ast\) contained in \(q^{(1)}\) but set all other contributions from the last term in Eq. (E.21) to zero.

First, we can check—using an iterative argument that is very similar to the one we gave at first order—that the first two terms in this equation generate in the entries \(s_i^{(n)}\) all sums of products of the factors \(\hat{\beta}\) along paths from \(\ast\) to the corresponding vacuum, which begin with an up-tunneling and have a total of \(n\) up-tunnelings, and pass through only non-dominant vacua at intermediate steps:

\[
s_i^{(n)} \supset \sum_p \sum_{i_1, \ldots, i_{p-1}} (\hat{\beta}_{i_{p-1}i_p} \cdots \hat{\beta}_{i_1\ast}) ,
\] (E.23)

where as before the superscript \((n)\) indicates that we are summing over paths with only \(n\) up-tunnelings that start with an up-tunneling.

The effect of the third term in Eq. (E.22) is to add any number of factors \(-\Delta U_{ij}/\Delta D_{ij}\) at intermediate or final vacua \(i_j\) during the iteration. These factors appear non-trivially in front
of terms in the path sum where the number of up-tunnelings in the path does not saturate the order $n$ in perturbation theory. If the entries vanish up to some order $n_0$, the leading order, this first nonzero order will not contain these factors since they multiply paths that would have appeared at lower order if they existed. Thus the leading order in up-tunnelings result is

$$S_i^{(n_0)} = \sum_p \sum_{i_1, \ldots, i_{p-1}}^{(n_0)} \frac{\kappa_{ii_{p-1}}}{D_i - D_x} \cdots \frac{\kappa_{i_1*}}{D_{i_1} - D_x},$$

(E.24)

a sum over all possible paths with exactly $n_0$ up-tunnelings, which in general can have an arbitrary number of down-tunnelings. (A similar but inequivalent expression appears in Ref. [235].)
Appendix F

Cosmological Constant and the Causal Patch

The cosmological constant offers a nice example of the predictive power of a large landscape, and it also illustrates the advantages of the causal patch measure over competing proposals. In this appendix we review Weinberg’s 1987 prediction of a positive cosmological constant [17], which has since been confirmed by observation [6, 7]. We then turn to the more recent success of the causal patch measure in improving the quantitative agreement with the observed magnitude of $\Lambda > 0$ (particularly in settings where the primordial density contrast is also allowed to vary), while eliminating specific anthropic assumptions. The goal is to make contact between an example many readers will be familiar with, and the more general formalism for making predictions in the landscape described in Sec. 5.2.

Weinberg’s Prediction: $\Lambda \sim t_{\text{vir}}^{-2}$

Because $\Lambda = 0$ is not a special value from the point of view of particle physics, the prior distribution over the cosmological constant $\Lambda$ should have no sharp feature near $\Lambda = 0$; hence to leading order in a Taylor expansion, $dN_{\text{vac}}/d\Lambda \approx \text{const.}$ for $|\Lambda| \ll 1$.\(^1\) Hence we have

$$P_{\text{vac}}(\Lambda) \propto \Lambda = \exp(\log \Lambda) : \quad (F.1)$$

the prior favors large magnitude of the cosmological constant. So far, this is just a restatement of the cosmological constant problem in a landscape setting: among many (nonsupersymmetric) vacua, most will tend to have large $\Lambda$, since precise cancellations between the positive and negative contributions to $\Lambda$ are unlikely.

For $\Lambda > 0$, structure formation would be severely diminished if $\Lambda$ was large enough to dominate over the matter density of the universe before the time $t_{\text{gal}}$ when density perturbations on the scale of galactic haloes would otherwise become nonlinear. (For negative $\Lambda$ of sufficient magnitude, the universe recollapses too soon.) Crudely, the weighting factor $w(x)$

\(^1\)In this Appendix we work in Planck units, $G = \hbar = 1$. 
may be approximated as vanishing for $\Lambda > \rho_{\text{NL}}$ and constant for $\Lambda < \rho_{\text{NL}}$, where $\rho_{\text{vir}} \sim t_{\text{gal}}^{-2}$ is the energy density at that time [17]. A refinement [164] models $w(x)$ as the fraction of baryons that enter structure of a specified minimum mass. Thus, the resulting distribution $P(\log \Lambda) = wf$ peaks around $x \sim -2 \log t_{\text{gal}}$. $P$ is suppressed at larger values of $x$ due to the anthropic factor $w$, and at smaller values of $x$ because the prior probability $f$ is low. The model, proposed by Weinberg in 1987, thus predicted a nonzero cosmological constant not much smaller than $\rho_{\text{NL}}$. Just such a value has since been discovered [6, 7]. The model could have been ruled out at any level of confidence if, instead of a detection, the observational upper bound on $\Lambda$ had continued to improve, moving ever deeper into the region suppressed by the prior.

Weinberg’s argument had a few shortcomings, which we list here. First, the approach actually favors a somewhat larger value of $\Lambda$; the observed value is small at $2 - 3\sigma$ depending on the assumptions made about the size of galaxies required by observers. More concerning, the approach would not appear to be robust against variations of the initial density contrast $Q$. It strongly favors vacua in which both $Q$ and $\Lambda$ are larger than the observed values, unless the prior for $Q$ favors a small magnitude, or unless there is a catastrophic boundary very close to the observed values of $Q$. Neither of these arguments are easy to make.

Causal Patch Prediction: $\Lambda \sim t_{\text{obs}}^{-2}$

In much of the older literature, the divergences of eternal inflation were regulated by computing the number of observers per baryon. (See the beginning of Sec. 5.2 for a brief discussion of the measure problem, and Ref. [236] for a review.) This was a reasonable first guess, particularly in the context of a landscape where only the cosmological constant varies. However, it is no longer viable in light of more recent insights [10, 180].

The ratio is not well-defined in a landscape where some vacua may not contain any baryons. Worse, it does not actually regulate all infinities, since a long-lived metastable vacuum with positive cosmological constant (such as ours) will have infinite four-volume in any comoving volume; hence, an infinite number of observers “per baryon” will be produced by thermal fluctuations at late times. The number of measures that are well-defined and not clearly ruled out is surprisingly small, and the causal patch measure has had the greatest quantitative success so far (at least [181] when we are interested in relative probabilities for events in vacua with positive cosmological constant, as we are here). Here we give two examples.

First let us recompute the probability distribution over the cosmological constant, $dP/d \log \Lambda$ with $\Lambda > 0$ using the causal patch. We consider a class of observers that live at the (arbitrary but fixed) time $t_{\text{obs}}$; for comparing with out observations, we will choose $t_{\text{obs}} = 13.8 \text{ Gyr}$. But the causal patch at late times coincides with the interior of the cosmological horizon. Because of the exponential expansion, the average density decreases like $e^{-3t/t_{\Lambda}}$. If $t_{\text{obs}} \gg t_{\Lambda} \sim \Lambda^{-1/2} \sim O(10) \text{ Gyr}$, no observers will be present in the patch, no matter whether or not galaxies form. This is a much more stringent cutoff than the suppression of galaxy formation which only sets in for a larger value of $\Lambda$, such that $t_{\text{gal}} \gg t_{\Lambda}$. It agrees very
well with the observed value of $\Lambda$, resolving the mild $(2 - 3\sigma)$ tension with Weinberg’s estimate. It is unaffected by any increase in the primordial density contrast, since $t_{\text{obs}}$ contains Gyr time scales that are not shortened by hastening structure formation. It solves the “Why Now” problem directly. And it does all this without making any specific assumptions about the nature of observers, except that they are made of stuff that redshifts faster than vacuum energy. (However, in the present paper we do assume that observers require galaxies.)

The causal patch can also explain why dark and baryonic matter have comparable abundances: the “Why Comparable” coincidence. One makes the qualitative assumption that the dark-to-baryonic density ratio $\zeta$ favors large values. But when $\zeta \gg 1$, the causal patch suppresses baryonic observers by a factor $1/(1 - \zeta)$, which counteracts the prior distribution, leading to the prediction that $\zeta \sim O(1)$ [165].
Appendix G

Structure Formation with Neutrinos

Our calculation was done almost entirely using Boltzmann codes, not analytic approximations. However, for completeness we summarize here the physical origin of the effects of neutrinos on structure formation. In the final subsection H, we explain the semi-analytic extrapolation formula we have used to extend the code output to negative redshifts. For excellent in-depth treatments of neutrino cosmology, see Refs. [162, 163].

Neutrino Cosmology

Around a second after the big bang at the time of decoupling, neutrinos are frozen out with a Fermi-Dirac distribution whose temperature is set by the primordial plasma. Due to $e^\pm$ annihilations that heat up the plasma soon after neutrino decoupling, this temperature differs from the temperature of the CMB, which decouples from the plasma much later: $T_{\nu,0} = (4/11)^{1/3}T_{\text{CMB}} = 1.95\,\text{K}$.

The energy density and pressure of a single neutrino with mass $m$ at a fixed time since decoupling is thus approximately given by

$$\rho_{\nu} = 2 \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2 + m^2}}{e^{p/T_{\nu}(z)} + 1},\quad (G.1)$$

$$P_{\nu} = 2 \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\sqrt{p^2 + m^2}} \frac{1}{e^{p/T_{\nu}(z)} + 1},\quad (G.2)$$

where $T_{\nu}(z) = T_{\nu,0}(1 + z)$ is the neutrino temperature as it redshifts from the value set at decoupling.

At early times, neutrinos contribute as radiation and add to the total radiation density as

$$\rho_R = \left[ 1 + \frac{7}{8} \left( \frac{4}{11} \right)^{4/3} N_{\text{eff}} \right] \rho_\gamma,\quad (G.3)$$

where

$$\rho_\gamma = \frac{\pi^2}{15} T_{\text{CMB}}^4,\quad (G.4)$$
and where \( N_{\text{eff}} = 3.046 \) is the effective number of neutrino species, with a slight deviation from 3 due to non-thermal spectral distortions from the \( e^\pm \) annihilations.

Similarly, the number density of neutrinos per species is set by the CMB number density:

\[
n_\nu = \frac{3}{11} n_\gamma ,
\]

where

\[
n_\gamma = \frac{2\zeta(3)}{\pi^2} T_{\text{CMB}}^3.
\]

Neutrinos become approximately non-relativistic once their thermal energy drops below the relativistic kinetic energy, \( 3T_\nu(z) < m_\nu \), which occurs at a redshift \( z_{nr} \) of\(^1\)

\[
1 + z_{nr} = 1991 \left( \frac{m_\nu}{1 \text{ eV}} \right) .
\]

Well after this transition, the density of non-relativistic neutrinos asymptotes to

\[
\rho_\nu = m_\nu n_\nu ,
\]

where \( m_\nu \) is the sum of masses of all non-relativistic neutrino species. In terms of this, the neutrino density parameter counting only massive neutrinos is

\[
\Omega_\nu = \frac{\rho_\nu}{\rho_\ast} ,
\]

where \( \rho_\ast \) is the critical density defined by \( H^2 = 8\pi G \rho_\ast / 3 \), which gives

\[
\Omega_\nu h^2 = \left( \frac{m_\nu}{94.5 \text{ eV}} \right) .
\]

The neutrino free streaming scale is set by the typical distance neutrinos travel thermally up to a given time. Roughly, it is given by the horizon scale at early times and stops growing soon after the neutrinos become nonrelativistic; hence it can be crudely approximated by the horizon scale at the nonrelativistic transition, \( k_{nr} \).

On small scales, there are two effects by which neutrinos suppress structure. The most obvious is that density perturbations will be washed out. Thus, free streaming eliminates the contribution of neutrinos to structure, and thus suppresses the total matter power by a factor \( \sim (1 - f_\nu)^2 \), where

\[
f_\nu = \frac{\Omega_\nu}{\Omega_m} .
\]

defines the massive neutrino fraction. Conversely, on larger scales neutrinos will remain confined to the over-dense regions and will behave like cold dark matter.

\(^1\)The non-relativistic transition is far from sudden. The neutrino pressure Eq. (G.2) has a non-negligible tail long after the redshift Eq. (G.7), which smears out the transition. We thank J. Lesgourgues for explaining this point to us.
Figure G.1: The growth factor Eq. (H.1) (solid line), which behaves like $x^{1/3}$ (dashed line) during the matter era, and asymptotes to a constant value well above $x(t_\Lambda) = 1$.

A secondary but more important effect is that the density of massive neutrinos contribute via the Friedmann equation to the Hubble parameter, which controls the friction term in the growth of matter perturbations. But on short scales, they do not contribute to the source term (the density contrast). Therefore, CDM perturbations grow more slowly in the presence of a nonclustering matter component on short scales [237]:

$$
\delta_c \propto a , \quad k \lesssim k_{nr} , \\
\delta_c \propto a^p , \quad k > k_{nr} ,
$$

(G.12)

where

$$
p = \frac{-1 + \sqrt{1 + 24(1 - f_\nu)}}{4} \approx 1 - \frac{3}{5} f_\nu < 1 ,
$$

(G.13)

with the last approximation valid in the limit of small neutrino masses.
Appendix H

Late-Time Extrapolation of Numerical Results

Available Boltzmann codes do not offer output for negative redshifts. In order to estimate the smoothed density contrast $\sigma_R$ in this regime, we extrapolate our numerical results for $\sigma_R(z)$ from positive to negative $z$, i.e., from $a < 1$ to $a > 1$. The most straightforward approach would be a linear extrapolation in some time variable, fitting both the value and the derivative of $\sigma_R$ at $z = 0$. However, there is a physical effect that we must incorporate analytically: vacuum domination turns off structure growth on all scales. This effect is not strong enough at $z = 0$ to have a significant imprint on the value or time derivative of $\sigma_R$. However, the effect is also rather simple, and thus easy to incorporate analytically.

In a universe with negligible neutrino mass, the CDM density contrast grows as [164, 238]

$$\delta \propto G_A(x) \equiv \frac{5}{6} \sqrt{1 + \frac{1}{x} \int_0^x \frac{dy}{y^{1/6}(1 + y)^{3/2}}} ,$$

(H.1)

where

$$x \equiv \frac{\rho_A}{\rho_m} = \frac{\Omega_A}{\Omega_m} \bigg|_{z=0} (1 + z)^{-3} .$$

(H.2)

As seen in Fig. G.1, density perturbations grow like the scale factor during the matter dominated era; they asymptote to a constant value at times $t > t_A$.

With non-zero neutrino mass, a reasonable approximation is obtained by combining the analytic result for the matter era, Eq. (G.12), with the $m_\nu = 0$ transition to the vacuum dominated era:

$$\delta \propto G_A(x) \quad (k < k_{nr}) ,$$

(H.3)

$$\delta \propto G_A(x)^p \quad (k > k_{nr}) .$$

(H.4)

Recall that $P_{cc}(k) \propto \delta_c(k)^2$ by Eq. (5.25).

In order to improve on this result, we can incorporate the information gained from the use of Boltzmann codes. Instead of computing $p$ and $k_{nr}$ analytically as described in the
Figure H.1: The parameters $C_{\text{eff}}$ and $p_{\text{eff}}$, for normal (orange, top) and degenerate (green, bottom) hierarchies, obtained by fitting Eq. (H.9) to CAMB output for $\sigma_R$ and its derivative at $z = 0$. The resulting fitting function for $\sigma_R(z)$ is used to compute the Press-Schechter factor at negative redshift only. Note that $p_{\text{eff}} \approx 1$ throughout. This may seem surprising, but it is consistent with our earlier finding that at large neutrino masses, the scales whose power contributes dominantly to $\sigma_R$ are precisely the ones on which free-streaming is not effective. This is closely related to the discrepancy we find with Ref. [171], whose estimate $p_{\text{eff}} \approx p(k_{\text{gal}}) \approx 1 - 8f_\nu$ would yield a monotonically decreasing curve in (b).

In practice, it is cumbersome to extrapolate the power at each wave number only to integrate over scales to obtain the smoothed density contrast. By the late time corresponding to $z = 0$, for any neutrino mass, we expect that the integral in Eq. (5.24) is dominated by the power at some scale $k$ and will remain dominated by the same scale in the future ($z < 0$). For small neutrino masses, this scale will be set by the galaxy scale; for large $m_\nu$, it will be the scale of the peak of the spectrum $k^3P(k)$. We incorporate this by matching the analytic growth for $z < 0$ directly to the numerical results for $\sigma_R(x)$ at $z = 0$. For every $m_\nu$, we compute

$$p(k) = \frac{1}{2} \frac{d \log P_{cc}(x)}{d \log G_\Lambda(x)} .$$  \hspace{1cm} (H.5)

We can also fix the constant of proportionality $C$ by matching the magnitude of $P_{cc}$ obtained from CAMB at $z = 0$. This yields a semi-analytic power spectrum as a function of time, for any fixed $k$ and fixed neutrino mass:

$$P_{cc}(x) = CG_\Lambda(x)^{2p(k)} .$$  \hspace{1cm} (H.6)

previous subsection, we can read off a slope $p(k)$ from the numerical output near $z = 0$:

$$p(k) = \frac{1}{2} \frac{d \log P_{cc}(x)}{d \log G_\Lambda(x)} .$$  \hspace{1cm} (H.5)

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$$p_{\text{eff}} = \frac{d \log \sigma_R(x)}{d \log G_\Lambda(x)} \bigg|_{z=0} ,$$  \hspace{1cm} (H.7)

$$C_{\text{eff}} = \frac{\sigma_R}{G_\Lambda^{p_{\text{eff}}}} \bigg|_{z=0} .$$  \hspace{1cm} (H.8)
from the CAMB output for small nonnegative redshifts. The results are shown in Fig. H.1.

As our semi-analytic approximation entering the Press-Schechter factor $F$ for $z < 0$ we use

$$\sigma_R(z) = C_{\text{eff}} G_\Lambda(x(z))^{\rho_{\text{eff}}} \quad \text{[used for } z < 0 \text{ only]}$$  \hspace{1cm} (H.9)

with $G_\Lambda$ given by Eq. (H.1). We have checked that the same formula provides an excellent fit to the numerical results at $z > 0$, as one would expect. However, we stress again that we use the output from the CAMB code in this regime, not the fitting function. Moreover, the regime $z > 0$ dominates in our calculation because the comoving volume of the causal patch decreases rapidly below $z = 2$. 
Appendix I

Cooling and Galaxy Formation

In this Appendix, we review the basic time scales that are believed to control cooling flows in dark matter halos. Our discussion closely follows Ref. [175], where further details and references can be found.

Baryonic gas will fall into the gravitational well of newly formed dark matter halos. The baryons are thus shock-heated to high temperatures. In order for stars to form, the baryonic gas must cool and condense. The initial temperature of the baryons is called the virial temperature. By the virial theorem,

\[
\frac{GM_{\text{vir}} \mu}{5 R_{\text{vir}}} = T_{\text{vir}},
\]  

where \( M_{\text{vir}} \) is the mass of the halo and \( R_{\text{vir}} \) is its virial radius. In the regime of interest for us, \( T_{\text{vir}} \) is large enough to ionize hydrogen. Then one can take the average molecular mass \( \mu \) to be \( m_p/2 \), where \( m_p \) is the mass of the proton. With \( M_{\text{vir}} = \frac{4\pi}{3} \rho_{\text{vir}} R_{\text{vir}}^3 \) one finds

\[
T_{\text{vir}} \propto M_{\text{vir}}^{2/3} \rho_{\text{vir}}^{1/3},
\]  

where the “constants” of proportionality depend negligibly on \( M_{\text{vir}} \).

The timescale for cooling by bremsstrahlung is

\[
t_{\text{brem}} \propto \frac{T_{\text{vir}}^{1/2}}{\rho_{\text{vir}}} \propto \frac{M_{\text{vir}}^{1/3}}{\rho_{\text{vir}}^{5/6}}.
\]  

We will be interested in how this timescale compares to the age of the universe when the halo virializes,

\[
t_{\text{vir}} \propto \rho_{\text{vir}}^{-1/2}.
\]

If \( t_{\text{brem}} \lesssim t_{\text{vir}} \), then galaxy formation can be treated as instantaneous, i.e., as occurring nearly simultaneously with halo formation. Keeping track of all constants [175], one finds that this case corresponds to

\[
M_{\text{vir}} t_{\text{vir}}^2 \lesssim (10^{12} M_\odot)(2.2\text{Gyr})^2.
\]  

(I.5)
In the opposite case, $M_{\text{vir}} t_{\text{vir}}^2 \gg (10^{12} M_\odot)(2.2 \text{Gyr})^2$, we have $t_{\text{brems}} \gg t_{\text{vir}}$. In halos with these mass and virialization time combinations, galaxy formation cannot be treated as instantaneous. Instead, it takes a much greater time $t_{\text{brems}} \gg t_{\text{vir}}$ to convert a comparable fraction of baryons into stars. (If feedback or major mergers disrupt the cooling flow, the contrast would be even more drastic, but we will not assume this here.)

The above analysis assumed cooling of unbound charged particles by bremsstrahlung. This approximation is best for virial temperatures above $10^7$ K. At lower temperatures the cooling function is quite complicated, but one can get an estimate by treating it as independent of $T_{\text{vir}}$ in some range [239]. With this approximation, one obtains that the cooling condition is satisfied for

$$M_{\text{vir}}^2 t_{\text{vir}} < (10^{12} M_\odot)^2 (5.3 \text{Gyr}) .$$

With either scaling, one finds again that cooling is inefficient if $M_{\text{vir}} > 10^{12} M_\odot$, particularly for late virialization $t_{\text{vir}} \gtrsim 10$ Gyr.

So far, we have neglected the effects of the cosmological constant. For halos that form deep in the vacuum dominated era, one should use $\rho_{\text{vir}} \sim \rho_\Lambda$ instead of Eq. (I.4). But such halos contribute negligibly in the causal patch because they will be exponentially dilute.

We have also neglected neutrinos. However, Eq. (I.5) is sufficiently general to capture their main effect, which is to change the relation between $M_{\text{vir}}$ and $t_{\text{vir}}$. In a universe with $m_\nu \ll 8$ eV, $t_{\text{vir}}$ grows logarithmically with $M_{\text{vir}}$ for overdensities of a fixed relative amplitude. For $10^{12} M_\odot$ halos forming from $1\sigma$ ($2\sigma$) overdensities, $t_{\text{vir}} \approx 3.6 \text{Gyr}$ ($t_{\text{vir}} \approx 1.3 \text{Gyr}$) and by Eq. (I.5), cooling fails (succeeds).

In a universe with $m_\nu \gtrsim 8$ eV, however, small scale power is so suppressed that structure formation proceeds in a top-down manner. (This is shown in detail in the main text.) Then structure on all scales forms much later than 2.4 Gyr. Moreover, smaller structure is embedded in larger halos, which set the virial mass that enters Eq. (I.5). Hence, the timescale for a significant fraction of baryons to form stars is at least $t_{\text{brems}} \gg t_{\text{vir}} \gg O(\text{Gyr})$. 
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