Special Cycles on GSpin Shimura Varieties

by

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Abstract

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In this thesis we prove that a certain generating function of special cycles on GSpin Shimura varieties is modular. More specifically, we consider the Shimura variety corresponding to the reductive group $\text{Res}_{F/Q} G$, where $G = \text{GSpin}(V)$ the GSpin group for $V$, a quadratic space over a totally real number field $F$, $[F : \mathbb{Q}] = d$ with certain conditions at the infinite places. We construct a generating function in the sense of Kudla and Millson and show that its image in cohomology is an automorphic form.
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To Nancy, Pam, Jessica, Artemis, and Apollo
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Chapter 1
Introduction

For Hilbert modular surfaces, Hirzebruch and Zagier show in [4] that a certain series with coefficients the Hirzebruch-Zagier divisors are modular forms of weight 1. Further inspired by this work, Gross, Kohnen and Zagier show in [1] that a generating series that has Heegner divisors as coefficients is modular of weight 3/2. This approach is unified by Borcherds in [2], where he shows more generally the modularity of generating series with Heegner divisor classes as coefficients in the Picard group over $\mathbb{Q}$.

Kudla and Millson further extend in [5], based on work from [8], [9], [7], the results to Shimura varieties of orthogonal type over a totally real number field and show the modularity in the cohomology group. This is further extended by Yuan, Zhang and Zhang in [11], who showed the modularity of the generating series in the Chow group.

In the current paper, inspired by the above work of Kudla and Millson, we construct special cycles on a different Shimura variety of orthogonal type over a totally real number field $F$ and show the modularity of Kudla’s generating series in the cohomology group.

We consider the Shimura variety corresponding to the reductive group $\text{Res}_{F/\mathbb{Q}} G$, where $G = \text{GSpin}(V)$ the GSpin group for $V$, a quadratic space over a totally real number field $F$, $[F : \mathbb{Q}] = d$. We suppose $V$ has signature $(n, 2)$ at $e$ real places and signature $(n + 2, 0)$ at the remaining $d - e$ places. Kudla, Millson and Yuan, Zhang, Zhang have treated the case of $e = 1$, while we allow $e \in \{1, \ldots, d\}$.

We present now the setting of the paper. For $F$ a totally real field with real embeddings $\sigma_1, \ldots, \sigma_d$, let $\mathbb{A} = \mathbb{A}_F$ the ring of adeles and let $V$ be a quadratic space over $F$ with signature $(n, 2)$ at infinite places $\sigma_1 \ldots \sigma_e$ and with signature $(n + 2, 0)$ elsewhere. Let $G$ denote the reductive group $\text{GSpin}(V)$ over $F$. We define the hermitian symmetric domain $D$ corresponding to $G$ to be:

$$D = D_1 \times D_2 \times \ldots \times D_e,$$

where $D_i$ is the Hermitian symmetric domain of oriented negative definite 2–planes in $V_{\sigma_i}$. Note that $D$ has complex dimension $en$. 
Theorem 1.0.1. (Res\(_{F/\mathbb{Q}} G, D\)) is a Shimura datum and for any open compact subgroup \(K\) of \(G(\mathbb{A}_f)\), this gives us the complex Shimura variety:

\[
M_K(\mathbb{C}) \cong G(F)\backslash D \times G(\mathbb{A}_f)/K.
\]

For \(i = 1, \ldots, e\) we let \(L_{D_i}\) be the complex line bundle corresponding to the points of \(D_i\) (each point of \(D_i\) is an oriented negative definite \(2\)-plane which can uniquely be given the structure of a complex line). We also have the projection maps \(p_i : D \rightarrow D_i\). Then \(p_i^* L_{D_i} \in \text{Pic}(D)\) descends to a line bundle \(L_{K,i} \in \text{Pic}(M_K) \otimes \mathbb{Q}\).

We will construct first the special cycles \(Z(x, g)_K\) for \(x \in V\) with \(q(x) \in F_+\), the totally positive units of \(F\) and \(g \in G(\mathbb{A}_f)\). We let \(V_x\) be the orthogonal complement of \(x\) in \(V\). We let \(G_x = \text{GSpin}(V_x)\) and \(D_x\) be the Hermitian symmetric domain associated to \(G_x\). We actually have natural identifications:

\[
G_x = \{ g \in G : \text{gx} = x \}, \quad D_x = \{(\tau_1, \ldots, \tau_e) \in D : \langle \tau, \tau_i \rangle = 0 \ \forall i \}.
\]

Then \((\text{Res}_{F/\mathbb{Q}} G_x, D_x)\) is a Shimura datum and we have an injection \((\text{Res}_{F/\mathbb{Q}} G_x, D_x) \rightarrow (\text{Res}_{F/\mathbb{Q}} G, D)\) of Shimura data. For \(K \subset G(\mathbb{A}_f)\) an open compact subgroup, we can then define the complex Shimura variety:

\[
M_{gKg^{-1}, x} = G_x(F)\backslash D_x \times G_x(\mathbb{A}_f)/(gKg^{-1} \cap G_x(\mathbb{A}_f)).
\]

Moreover, we have an injection of \(M_{gKg^{-1}, x}\) into \(M_K\) given by:

\[
M_{gKg^{-1}, x} \rightarrow M_K, \quad [\tau, h] \rightarrow [\tau, hg].
\]

We define the cycle \(Z(x, g)_K\) to be the image of the morphism above. Note that \(Z(x, g)_K\) is represented by the subset \(D_x \times G_x(\mathbb{A}_f)gK\) of \(D \times G(\mathbb{A}_f)\).

For \(x \in V\) with \(q(x) \notin F_+\), we naturally extend the definition as follows:

\[
Z(x, g)_K = \begin{cases} 
(-1)^e c_1(L_{K,1}) \ldots c_1(L_{K,e}) & \text{for } x = 0, \\
0 & \text{for } x \neq 0, q(x) \notin F_+.
\end{cases}
\]

Here \(c_1\) denotes the Chern class of a line bundle.

Now we define the generating function. For Schwartz-Bruhat functions \(\phi_f \in \mathcal{S}(V(\mathbb{A}_f))^K\) and \(g' \in \widetilde{\text{SL}}_2(\mathbb{A})\), we take:

\[
Z(g', \phi_f) = \sum_{x \in G(F)\backslash V} \sum_{g \in G_x(\mathbb{A}_f)G(\mathbb{A}_f)/K} r(g') \phi_f(g^{-1}x) W_{q(x)}(g'x) Z(x, g)_K
\]

Here \(r\) is the Weil representation of \(\widetilde{\text{SL}}_2(\mathbb{A})\) and \(W_{q(x)}\) is the standard Whittaker function.

The following is the main theorem of the paper.

**Theorem 1.0.1.** The series \([Z(g', \phi_f)]\) is an automorphic form, discrete of weight \(1 + \frac{g}{2}\) for \(g' \in \text{SL}_2(\mathbb{A})\) valued in \(H^{2e}(M_K, \mathbb{C})\).
Remark 1.0.2. As in the literature, the above is taken to mean that for any functional
\[ \ell : H^{2e}(M_K, \mathbb{C}) \to \mathbb{C} \]
such that \( \ell([Z(g', \phi_f)]) \) is convergent, we have that \( \ell([Z(g', \phi_f)]) \) is an automorphic form, discrete of weight \( 1 + \frac{n}{2} \) for \( g' \in \text{SL}_2(\mathbb{A}) \) with respect to the Weil representation on \( S(V) \).

The case \( e = 1 \) was proved by Kudla and Millson in [5], based on work from [8], [9], [7]. One can further conjecture that the series \( Z(g', \phi_f) \) is an automorphic form, discrete of weight \( 1 + \frac{n}{2} \) for \( g' \in \text{SL}_2(\mathbb{A}) \) valued in \( \text{CH}^c(M_K)_{dR} \). This is out of reach at the moment, but one can expect to extend the methods of Borcherds (see [2]) to show the modularity in the Chow group.

We will present now the ideas of the proof. We prove the cases \( e > 1 \) by extending the ideas of Kudla and Millson. For each cycle \( Z(x, g) \) we want to construct a Green current \( \eta(x, g) \) of \( Z(x, g) \) in \( M_K(\mathbb{C}) \). Via the isomorphism \( H^e_{dR}(X_K, \mathbb{C}) \simeq H_{2e(n-1)}(X_K, \mathbb{C}) \), where the former is deRham cohomology while the latter is singular homology, we have the identification of cohomology classes:
\[ [Z(x, g)] = [\omega(\eta(x, g))], \]
where \( \omega(\eta(x, g)) \) is the Chern form corresponding to the Green current \( \eta(x, g) \).

In order to construct the currents \( \eta(x, g) \), we take the currents defined by Kudla-Millson \( \eta_{0, i}(x, \tau_i) \) of \( D_{x,i} \) in \( D_i \). Taking the pullbacks via the projections \( p_i : D \to D_i \) and taking the \( * \)-product, we obtain a Green current of \( D_x \) in \( D \):
\[ \eta_2(x, g) = p_1^* \eta_0(x, \tau_1) \ast p_2^* \eta_0(x, \tau_2) \ast \cdots \ast p_e^* \eta_0(x, \tau_e). \]
Furthermore, by averaging the current on the lattice \( \Gamma = G_x(F) \backslash G(F) \cap G_x(\mathbb{A}_F)gK \), we get \( \eta_2(x, \tau; g, h) = \sum_{\gamma \in \Gamma} \eta_1(x, \gamma \tau) \) a Green current for \( G(\mathbb{Q})(D_x \times G_x(\mathbb{A}_F)gK)/K \) in \( G(\mathbb{Q})(D \times G(\mathbb{A}_F))/K \) that descends to a Green current \( \eta_3(x, \tau; g, h) \) of \( Z(x, g)_K \) in \( M_K \). Here \( (\tau, h) \in M_K \).

Taking the Chern forms, the \( * \)-product turns into wedge product and the averages, as well as the pullbacks are preserved. We get the Chern form \( \omega_3(x, \tau; g, h) \) corresponding to the divisor \( Z(x, g)_K \) that is the pullback under the projection map \( G(\mathbb{Q})(D \times G(\mathbb{A}_F))/K \to M_K \) of the Chern form
\[ \omega_2(x, \tau; g, h) = \sum_{\gamma \in \Gamma} \omega_1(x, \gamma \tau), \]
of the Green function \( \eta_2(x, \tau; g, h) \), and \( \omega_1(x, \tau) = p_1^* \omega_0(x, \tau_1) \wedge p_2^* \omega_0(x, \tau_2) \wedge \cdots \wedge p_e^* \omega_0(x, \tau_e) \) are the Chern forms of \( \eta_1(x, \tau) \). Here \( \omega_0(x, \tau_i) = \omega(\eta_0(x, \tau_i)) \) is the Chern form of \( \eta_0(x, \tau_i) \) that is defined by Kudla-Millson in [5], based on work from [8], [9], [7].

Plugging in \([\omega_3(x, \tau; g, h)]\) for the cohomology class of \([Z(x, g)]\), after unwinding the sums we get:
\[ [Z(g', \phi)] = \sum_{x \in V_F} r(g'_f) \phi_f(x) W_q(x)(g'_x)[\omega_1(x, \tau)]. \]
Finally, using the properties of the Kudla-Millson form on the weight of each individual $\omega_0(x, \tau)$, we get:

$$[Z(g', \phi)] = \sum_{x \in V_F} r(g'_f) \phi_f(x) r(g'_x) [e^{-2\pi q(x)} \omega_0(x, \tau)],$$

and this is a theta function of weight $(n + 2)/2$ with values in $H^{2e}(M_K, \mathbb{C})$, thus it is automorphic.
Chapter 2

Background

2.1 Notation

We use the same conventions as in [12], which we quickly review.

Let $k$ be a local field of a totally real number field. If $k = \mathbb{R}$ then we use the standard absolute value $|\cdot|$ on $k$, and use the additive character $\psi: k \to \mathbb{C}^\times$ given by $\psi(x) = e^{2\pi i x}$.

Otherwise $k/\mathbb{Q}_p$ is non-archimedean, and we define $|\cdot|$ by sending the uniformizer to $N^{-1}$ where $N$ is the size of the residue field. Then we define the additive character $\psi = \psi_{\mathbb{Q}_p} \circ \text{tr}_{k/\mathbb{Q}_p}$. Here $\psi_{\mathbb{Q}_p}$ is defined by $\psi_{\mathbb{Q}_p}(x) = e^{-2\pi i(x)}$ where $\iota: \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$ is the natural embedding.

We take the measure $dx$ on $k$ to be the unique Haar measure on $k$ self-dual with respect to $\psi$ in the sense that the Fourier transform:

$$\hat{\Phi}(y) := \int_k \Phi(x)\psi(xy)dx$$

satisfies $\hat{\Phi}(x) = \Phi(-x)$.

Similarly for $(V,q)$ a quadratic space over $k$, we take the unique self-dual measure on $V$ with respect to $(V,q)$ and $\psi$ such that

$$\hat{\Phi}(y) := \int_V \Phi(x)\psi(\langle x, y \rangle)dx$$

satisfies $\hat{\Phi}(x) = \Phi(-x)$. Here $\langle x, y \rangle := q(x+y) - q(x) - q(y)$ is the bilinear pairing corresponding to $q$.

To work globally, fix $F$ totally real and let $\mathbb{A} = \mathbb{A}_F$. By taking tensor products of the above, we obtain an absolute value $|\cdot|_{\mathbb{A}}$, an additive character $\psi$, and a self-dual measure $dx$ satisfying the inversion formula. Moreover, $\psi$ is actually $F$-invariant.
2.2 Quadratic Spaces over Totally Real Fields

Let $F$ be a totally real field of degree $d$ and $(V, q)$ a nondegenerate quadratic space of dimension $n + 2$. That is, $V$ is an $F$–vector space with a function

$$q : V \to F$$

such that

$$q(cv) = c^2 q(v)$$

for $c \in F$ and $v \in V$ and such that

$$\langle x, y \rangle := q(x + y) - q(x) - q(y)$$

is a nondegenerate symmetric bilinear form. If we choose a basis of $V$, then we can have

$$q(v) = v^t A v$$

for $A \in M_{n,n}(F)$ symmetric.

Now let $\sigma_1 \ldots \sigma_d$ be the infinite places of $F$. Then for each $\sigma_i$ we have a quadratic space $(V_i, q_i)$ over $\mathbb{R}$. One concrete way to realize this is to use the same basis as above, then the quadratic form $q_i$ can be defined as $q_i(v) = v^t \sigma_i(A)v$.

Recall also that $\sigma_i(A)$ can be diagonalized with nonzero diagonal entries (by nondegeneracy). If the diagonal has $a$ positive entries and $b$ negative entries then we say $(V, q_i)$ has signature $(a, b)$. For shorthand, we say that $V$ has signature $(a, b)$ at the place $\sigma_i$.

**Example 2.2.1.** We’re interested in producing examples which satisfy the hypotheses of the main theorem. Additionally suppose $F/\mathbb{Q}$ is Galois. Then any two real embeddings differ by an automorphism $\tau \in \text{Gal}(F/\mathbb{Q})$. Thus for a primitive element $\alpha \in F$, we have $\sigma_i(\alpha) \neq \sigma_j(\alpha)$ for $i \neq j$. Thus the set

$$\{\sigma_1(\alpha), \ldots \sigma_d(\alpha)\}$$

is a set of $d$ distinct real numbers. For any $0 \leq e \leq d$, we can choose $q_e \in \mathbb{Q}$ so that

$$\{\sigma_1(\alpha + q_e), \ldots \sigma_d(\alpha + q_e)\}$$

has $d - e$ positive elements and $e$ negative elements. Then we can let $V = F^{n+2}$ and for $x = (x_1, \ldots x_{n+2})$ we have the quadratic form

$$q(x) = x_1^2 + \ldots x_n^2 + (\alpha + q_e)x_{n+1}^2 + (\alpha + q_e)x_{n+2}^2$$

over $F$. By construction, $V$ has signature $(n, 2)$ at $e$ distinct infinite places and signature $(n + 2, 0)$ otherwise.
2.3 The Weil Representation and Theta Functions

Let $k$ be a non-archimedean local field and $(V, q)$ a quadratic space over $k$. Let $O(V, q)$ be the orthogonal group of $(V, q)$. That is

$$O(V, q) := \{ A \in \text{GL}(V) \mid q(Av) = q(v) \ \forall v \}.$$

Let $\mathcal{S}(V)$ be the space of locally constant and compactly supported complex-valued functions (Schwartz functions) on $V$. We wish to describe an action of the group $\widetilde{\text{SL}_2(k)} \times O(k)$ on $\mathcal{S}(V)$. Recall that $\widetilde{\text{SL}_2(k)}$ is a central extension of $\text{SL}_2(k)$ by $\{ \pm 1 \}$. We can write $\widetilde{\text{SL}_2(k)} = \text{SL}_2(k) \times \{ \pm 1 \}$ with the group law given by

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2\beta(g_1, g_2))$$

where $\beta : \text{SL}_2(k) \times \text{SL}_2(k) \to \{ \pm 1 \}$ is a cocycle in $H^2(\text{SL}_2(k), \{ \pm 1 \})$. We first fix some notation. Consider the following elements of $\text{SL}_2(k)$ for $a \in k^\times$ and $b \in k$:

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We will also use the notation $m(a)$ and $n(b)$ to denote the elements $((m(a), 1)$ and $(n(b), 1)$ in $\widetilde{\text{SL}_2(k)}$. Also let $h \in O(k)$. Then we describe the Weil representation $r$ of $\widetilde{\text{SL}_2(k)} \times O(k)$ on $\mathcal{S}(V)$ as follows. For $\Phi \in \mathcal{S}(V)$:

- $r(h)\Phi(x) = \Phi(h^{-1}x)$
- $r(m(a))\Phi(x) = \chi_{(V, q)}(a)|a|^{\frac{n(V)}{2}} \Phi(ax)$
- $r(n(b))\Phi(x) = \psi(bq(x))\Phi(x)$
- $r(w, \epsilon)\Phi = e^{\epsilon^{\frac{n(V)}{2}}}\gamma(V, q)\hat{\Phi}.$

The character $\chi_{(V, q)}$ is given by

$$\psi_{(V, q)}(a) = (a, (-1)^{\frac{n(V)}{2}} \det(V, q))$$

where the right hand side is the Hilbert symbol. The Weil index $\gamma(V, q)$ is an eighth root of unity. Finally

$$\hat{\Phi}(x) := \int_V \Phi(y)\psi(\langle x, y \rangle)dy$$

is the Fourier transform and where

$$\langle x, y \rangle := q(x + y) - q(x) - q(y)$$
is the corresponding inner product. This extends easily to the case where \( k = \mathbb{R} \), where we demand that \( \Phi \) decays rapidly.

Now suppose that \( F \) is a number field and \((V, q)\) is a quadratic space over \( F \). Choose an \( \mathcal{O}_F \)-lattice \( V_{\mathcal{O}_F} \). Let \( V_k = V \otimes_F \mathbb{A} \) and recall our choice \( \psi : \mathbb{A}/F \to \mathbb{C} \). We can define an action of \( \overline{SL_2(\mathbb{A})} \times O(V_k) \) on the space \( \mathcal{S}(V_k) \) which is the restricted tensor product of the local spaces with spherical element given by the characteristic function of \( V_{\mathcal{O}_F} \) by taking the product of the local actions. The representation depends on the choice of \( \psi \).

The Weil representation allows us to define theta functions. We define the theta series as a function on \( \overline{SL_2(\mathbb{A})} \times O(V_A) \). Let \( \Phi \in \mathcal{S}(V_k) \), then

\[
\theta(g, h, \Phi) = \sum_{x \in V} r(g, h)\Phi(x).
\]

The following is well-known in the literature.

**Lemma 2.3.1.** The theta function

\[
\theta(g, h, \Phi) = \sum_{x \in V} r(g, h)\Phi(x).
\]

is invariant under \( \overline{SL_2(F)} \times O(V) \).

### 2.4 The Shimura Variety

First we recall the definition of \( G\text{Spin}(V) \). Let \((V, q)\) be a quadratic space over \( \mathbb{R} \) and \( C(V, q) = (\oplus_k V^\otimes k) / I \) be the Clifford algebra of \((V, q)\), where we are taking the quotient by the ideal \( I = \{ q(v) - v \otimes v \mid v \in V \} \).

Then \( C(V, q) \) has dimension \( 2^{\dim(V)} \) and we have a \( \mathbb{Z} \)-grading on \( T(V) = \bigoplus_k V^\otimes k \) that descends to a \( \mathbb{Z}/2\mathbb{Z} \)-grading on \( C(V, q) \). We write \( C(V, q) = C_0(V, q) \oplus C_1(V, q) \). We naturally have \( V \subset C_1(V, q) \). Then we can define the \( G\text{Spin} \) group of \( V \):

\[
G\text{Spin}(V) = \{ g \in C_0(V, q)^\times \mid gVg^{-1} = V \}.
\]

**Example 2.4.1.** Let \( V = \mathbb{R}^2 \) with \( q(x) = -x_1^2 - x_2^2 \). Let \( \{e_1, e_2\} \) be the standard basis. Then

\[
-2 = (e_1 + e_2) \otimes (e_1 + e_2) = e_1 \otimes e_1 + e_2 \otimes e_2 + (e_1 \otimes e_2 + e_2 \otimes e_1) = -2 + (e_1 \otimes e_2 + e_2 \otimes e_1)
\]

so that \( e_1 \otimes e_2 = -e_2 \otimes e_1 \). Then \( C(V, q) \) has basis \( \{1, e_1, e_2, e_1 \otimes e_2\} \) and it is easy to see that \( C(V, q) \cong \mathbb{H} \) as \( \mathbb{R} \)-algebras. We also have

\[
C_0(V, q)^\times = \{ a + b(e_1 \otimes e_2) \mid (a, b) \neq (0, 0) \}.
\]
In this case we have
\[ G\text{Spin}(V) = C_0(V, q) \cong \mathbb{C}^\times. \]

Let \( G' = \text{Res}_{F/Q}(G\text{Spin}(V)) \). We have \( G'(\mathbb{R}) = \prod_{v=1}^e G\text{Spin}(V_v) \). Thus to give a morphism
\[ h_0 : S_{\mathbb{R}} \rightarrow G'(\mathbb{R}) \]
is to give morphisms
\[ h_v : S_{\mathbb{R}} \rightarrow G\text{Spin}(V_v). \]
We let \( h_v \) be trivial for \( v > d \), corresponding to the places with signature \((n + 2, 0)\). For \( v \leq d \), consider the even Clifford algebra \( E_v \) of \( V_v \). Since the signature of \( V_v \) is \((n, 2)\) we can choose orthogonal vectors \( e_1, e_2 \) such that \( q(e_1) = q(e_2) = -1 \). Then we are interested in the element \( j_v = e_1 e_2 \in E_v \), where we omit the tensor product for readability. By the definition of the Clifford algebra, we have
\[ (e_1 + e_2)^2 = q(e_1 + e_2) = -2. \]
However we can alternatively expand as
\[ e_1^2 + e_1 e_2 + e_2 e_1 + e_2^2. \]
This implies \( e_1 e_2 = -e_2 e_1 \). Now we compute
\[ j_v^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -(-1)(-1) = -1. \]
We claim that \( a + b j_v \in G\text{Spin}V_v \) as long as \( (a, b) \neq (0, 0) \). Recall that
\[ G\text{Spin}V_v = \{ g \in E_v^\times | gV g^{-1} = V \}. \]
First, it is clear that such \( a + b j_v \in E^\times \) as
\[ (a + b j_v) \cdot \frac{a - b j_v}{a^2 + b^2} = 1. \]
Now to finish, we extend our set \( \{e_1, e_2\} \) to an orthonormal basis \( \{e_1, e_2, \ldots, e_{n+2}\} \) of \( V \). By the same computation as before, we have \( e_i e_j = -e_j e_i \) for all \( i \neq j \). We consider the injective linear map
\[ (a + b j_v) : V \rightarrow E, \ v \rightarrow (a + b j_v) v (a - b j_v). \]
If we show that the image is contained in \( V \), then the image must be \( V \) and thus \( a + b j_v \in G\text{Spin}V \). First we compute
\[ (a + b j_v) e_1 (a - b j_v) = (a + b e_1 e_2) e_1 (a - b e_1 e_2) \]
\[ = a^2 e_1 - ab e_1 e_1 e_2 + ab e_1 e_2 e_1 - b^2 e_1 e_2 e_1 e_2 \]
\[ = a^2 e_1 + 2ab e_2 - b^2 e_1 \in V. \]
The calculation for \( e_2 \) is similar. Now we must consider the case \( e_i \) where \( i > 2 \). We have
\[
(a + be_1e_2)e_i(a - be_1e_2) = a^2e_i - abc_i e_1e_2 + abc_i e_2e_3 - b^2e_1e_2e_3e_1e_2.
\]
Now, since we have to apply two transpositions both of which change the sign, \( e_3e_1e_2 = e_1e_2e_3 \). Thus the problematic inner terms cancel out. For the last term, \( e_1e_2e_3e_1e_2 = e_1e_1e_2e_2 = -e_1e_1e_2 \). Thus \((a + be_1e_2)e_i(a - be_1e_2) \in V \). Therefore the image of \((a + bj_0)\) is contained in \( V \), and is thus equal to \( V \). Dividing this map by the norm \( a^2 + b^2 \) shows that the conjugation map is an isomorphism and thus that \( a + bj_0 \in \text{GSpin}(V) \).

It can be checked that the pair \((G', D)\) is a Shimura datum. For additional details see [10].

### 2.5 Complex Geometry

We will recall now some background from complex geometry. Let \( X \) be a connected compact complex manifold of dimension \( m \). Suppose \( Y \) is a closed compact complex submanifold of codimension \( d \). Then \( Y \) has no boundary and is thus a \( 2(m-d) \) chain in \( X \). We can take the class of \( Y \) to be \([Y] \in H_{2(m-d)}(X, \mathbb{C})\). Now recall the perfect pairing:
\[
H_{2(m-d)}(X, \mathbb{C}) \times H^{2(m-d)}_{dR}(X, \mathbb{C}) \to \mathbb{C},
\]
given by \((Y, \eta) \to \int_Y \eta\). Thus \( H_{2(m-d)}(X, \mathbb{C}) \cong H^{2(m-d)}_{dR}(X, \mathbb{C})^\vee \). Also recall the perfect pairing:
\[
H^{2(m-d)}_{dR}(X, \mathbb{C}) \times H^{2d}_{dR}(X, \mathbb{C}) \to \mathbb{C},
\]
given by \((\eta, \omega) \to \int_X \eta \wedge \omega\). Thus \( H^{2(m-d)}_{dR}(X, \mathbb{C}) \cong H^{2d}_{dR}(X, \mathbb{C}) \). We can compose these isomorphisms to get:
\[
H_{2(m-d)}(X, \mathbb{C}) \cong H^{2d}_{dR}(X, \mathbb{C}). \tag{2.1}
\]
In light of the above isomorphism, a closed \( 2d \)-form \( \omega \) on \( X \) represents \([Y]\) iff
\[
\int_Y \eta = \int_X \omega \wedge \eta
\]
for any closed \( 2(m-d) \) form \( \eta \) on \( X \).

If \( X \) is noncompact, as it will be in what follows, we must make some modifications. Our chain of isomorphisms becomes
\[
H^{BR}_{2(m-d)}(X, \mathbb{C}) \cong H^{2(m-d)}_{dR, c}(X, \mathbb{C})^\vee \cong H^{2d}_{dR}(X, \mathbb{C}).
\]
Here \( H^{BR}_{2(m-d)}(X, \mathbb{C}) \) denotes the Borel-Moore homology, which allows infinite linear combinations of simplices, and \( H^{2(m-d)}_{dR, c}(X, \mathbb{C}) \) is de-Rham cohomology with compact support. Then
we say a closed differential 2d-form $\omega$ on $X$ represents $[Y]$ if

$$\int_Y \eta = \int_X \omega \wedge \eta$$

for any closed and compactly-supported differential 2$(m - d)$ form $\eta$ on $X$.

### 2.6 Green’s currents and Chern classes

We recall some background on Green currents, following mainly [SABK].

Let $A^{p,q}(X)$ and $A^c_{p,q}(X)$ denote the space of $(p, q)$-differential forms, and, respectively, $(p, q)$-differential forms with compact support. Let $D_{p,q}(X) = A^c_{p,q}(X)^*$ be the space of functionals that are continuous in the sense of deRham [SABK]. That is, for a sequence $\{\omega_i\} \in A^{p,q}(X)$ with support contained in a compact set $K \subset X$ and for $T \in D_{p,q}(X)$, we must have $T(\omega_i) \to 0$ if $\omega_i \to 0$, meaning that the coefficients of $\omega_i$ and finitely many of their derivatives tend uniformly to 0.

We also recall the differential operators:

$$d = \partial + \bar{\partial}, \quad d^c = \frac{1}{4\pi i}(\partial - \bar{\partial}), \quad dd^c = \frac{i}{2\pi} \bar{\partial} \partial.$$

We let $D^{p,q} = D_{m-p,m-q}$. Then we have an inclusion:

$$A^{p,q}(X) \to D^{p,q}(X), \quad \omega \to [\omega]$$

where we define the current $[\omega](\alpha) = \int_X \omega \wedge \alpha$ for any $\alpha \in A^{m-p,m-q}(X)$. For $Y \subset X$, let $\iota : Y \hookrightarrow X$ be the natural inclusion and we also define a current $\delta_Y \in D^{p,p}(X)$ by:

$$\delta_Y(\alpha) = \int_Y \iota^* \alpha,$$

for any $\alpha \in A^{d-p,d-p}$.

**Definition 2.6.1.** A Green current for a codimension $p$ analytic subvariety $Y \subset X$ is a current $g \in D^{p-1,p-1}(X)$ such that

$$dd^c g + \delta_Y = [\omega_Y]$$

for some smooth form $\omega_Y \in A^{p,p}(X)$.

One natural way to obtain Green currents is from Green functions. For $Y \subset X$ a closed compact submanifold of codimension 1, a Green’s function of $Y$ is a smooth function

$$g : X \setminus Y \to \mathbb{R}$$
which has a logarithmic singularity along $Y$. This means that for any pair $(U, f_U)$ with $U \subset X$ open and $f_U : U \to \mathbb{C}$ a holomorphic function such that $Y \cap U$ is defined by $f_U = 0$, then the function

$$g + \log |f_U|^2 : U \setminus (Y \cap U) \to \mathbb{R}$$

extends uniquely to a smooth function on $U$.

Now let $g$ be a Green function for $Y \subset X$, and for $U \subset X$ and $f_U = 0$ the local defining equation of $U \cap Y$, we define locally:

$$\omega_U = dd^c(g + \log |f_U|^2)$$

By gluing together all $\omega_U$ we get a differentiable form $\omega = \omega(g)$ over $X$. We call this the Chern form associated to the Green’s function $g$. Then we have further an important result:

**Theorem.** (The Poincare-Lelong formula) Let $g$ be a Green function for $Y \subset X$. For any $\eta \in A^{m-1,m-1}(X)$ we have the formula:

$$\int_X gdd^c(\eta) = \int_X \omega_Y \wedge \eta - \int_Y \eta.$$ 

Note that this implies that for $g$ a Green’s function, $[g]$ is a Green current of $Y$ in $X$.

This is true also for currents $g \in D^{n,p}(X)$ by definition, and it implies that for a closed form $\eta$ the LHS equals 0, and thus $\int_X \omega \wedge \eta = \int_Y \eta$. Thus for $g$ a Green’s function of $Y$ in $X$, we have as cohomology classes in the isomorphism (2.1):

$$[Y] = [\omega_Y].$$

Another natural way to get more Green currents is by taking their $\star$-product. We recall the $\star$-product for Green currents following [SABK]. For $Y, Z$ closed irreducible subvarieties of $X$ such that $Y$ and $Z$ intersect properly, let $g_Y, g_Z$ Green currents, respectively. We define the $\star$-product:

$$g_Y \star g_Z = g_Y \wedge \delta_Z + [\omega_Y] \wedge g_Z.$$ 

Moreover, from [SABK] (Theorem 4, page 50), when $Y$ and $Z$ have the Serre intersection multiplicity 1, then $g_Y \star g_Z$ is a Green current for $Y \cap Z$. 

Chapter 3

Construction of Green currents and Chern forms

In this chapter we construct a Green’s current of $Z(x,g)_K$ in $M_K$. In the following we consider the case of $x \in V(F)$ with $q(x) \in F_+$. 

3.1 Green functions of $D_x$ in $D$

We first recall how to construct a Green’s function of $D_x$ in $D$. Let $V$ be as in the previous sections. For a place $\sigma_i$, $1 \leq i \leq e$, the real vector space $V_{\sigma_i}$ has signature $(n,2)$. Let $\tau \in D_i$. It corresponds to a negative definite 2--plane $W$ in $V$ and we can write any $x \in V$ as $x = x_\tau + x_{\tau\perp}$ where $x_\tau \in W$ and $x_{\tau\perp} \in W^\perp$. We define:

$$R(x,\tau) = -2q(x_\tau), \quad q_\tau(x) = q(x) + R(x,\tau).$$

Note that this implies $R(x,\tau) = 0$ if and only if $\tau \in D_{i,x}$. We have the following crucial lemma:

**Lemma 3.1.1.** For $g \in \text{GSpin}(V)$, we have $R(gx,g\tau) = R(x,\tau)$.

**Proof.** Given $\tau$ we have a negative definite 2–plane $W$ with basis $\{\alpha, \beta\}$. Then $g\tau$ gives the subspace $gW$ with basis $\{g\alpha, g\beta\}$. Then

$$\langle gx, g\alpha \rangle_{g\tau} = \frac{\langle gx, g\alpha \rangle}{\langle g\alpha, g\alpha \rangle} g\alpha + \frac{\langle gx, g\beta \rangle}{\langle g\beta, g\beta \rangle} g\beta = \frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} g\alpha + \frac{\langle x, \beta \rangle}{\langle \beta, \beta \rangle} g\beta = g(x_\tau).$$

Thus

$$R(gx, g\tau) = -2q((gx)_{g\tau}) = -2q(g(x_\tau)) = -2q(x_\tau) = R(x,\tau).$$

\[ \square \]
In terms of an orthogonal basis we can write \( \tau = \alpha + i\beta \) such that \( \langle \alpha, \beta \rangle = 0 \) and \( \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle < 0 \). Then we have:

\[
R(x, \tau) = -\frac{\langle x, \alpha \rangle^2}{\langle \alpha, \alpha \rangle} - \frac{\langle x, \beta \rangle^2}{\langle \beta, \beta \rangle}
\]

Moreover, we show below that \(- \log(R(x, \tau)) \) is a Green’s function for \( D_{i,x} \) in \( D_i \):

**Lemma 3.1.2.** For fixed \( x \neq 0 \) and \( \tau \in D \setminus D_x \), the function \(- \log(R(x, \tau)) \) is a Green’s function for \( D_{i,x} \) in \( D_i \).

**Proof.** For \( \tau \in D_i \), we have a negative definite 2–plane \( W_\tau \subset V \) with an orientation. Choose \( \alpha, \beta \) an orthogonal basis of \( W \) with \( q(\alpha) = q(\beta) = -1 \). Then since we have an orientation we can give \( W_\tau \) a complex structure via \( J : W \to W, J(\alpha) = \beta, J(\beta) = -\alpha \).

Then we have the tautological complex line bundle \( L_{D_i} \) over \( D_i \) with fiber \( L_\tau = W_\tau \). We have a map:

\[
s_x(\tau) : L_\tau \to \mathbb{C}, \ v \mapsto \langle x, v \rangle.
\]

This defines an element \( s_x(\tau) \in L_\tau^\vee \). As \( \tau \) varies, we get a map

\[
s_x : D_i \to L_{D_i}^\vee, \ \tau \mapsto s_x(\tau).
\]

Then \( s_x \) is a holomorphic section of the line bundle \( L_{D_i}^\vee \). This section has a hermitian metric

\[
\|s_x(\tau)\|^2 = \frac{|\langle x, v \rangle|^2}{|\langle v, v \rangle|},
\]

where \( v \in L_\tau \) is any nonzero vector. In terms of an orthogonal basis we can write \( v = \alpha + i\beta \) such that \( \langle \alpha, \beta \rangle = 0 \) and \( \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle < 0 \). Then

\[
\|s_x(\tau)\|^2 = \frac{\langle x, \alpha \rangle^2 + \langle x, \beta \rangle^2}{\langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle} = -\frac{\langle x, \alpha \rangle^2}{2\langle \alpha, \alpha \rangle} - \frac{\langle x, \beta \rangle^2}{2\langle \beta, \beta \rangle}
\]

and also \( x_\tau = \frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha + \frac{\langle x, \beta \rangle}{\langle \beta, \beta \rangle} \beta \).

Computing directly gives us \( R(x, \tau) = 2\|s_x(\tau)\|^2 \). It follows by a theorem of Poincare-Lelong (see [SABK] II.2) that \(- \log(R(x, \tau)) \) is a Green’s function for \( D_{i,x} \) in \( D_i \). \qed

We take the Green function defined by Kudla-Millson (see [5]):

\[
\eta(x, \tau) = f(2\pi R(x, \tau)),
\]

where

\[
f(t) = -Ei(-t) = \int_t^\infty \frac{e^{-x}}{x} \, dx
\]
is the exponential integral. Note that we have
\[ f(t) = -\log(t) - \gamma - \int_0^t \frac{e^{-x} - 1}{x} \, dx. \]
Here \( \gamma \) is the Euler-Mascheroni constant. The function \( f(t) \) is smooth on \((0, \infty)\), \( f(t) + \log(t) \) is infinitely differentiable on \([0, \infty)\), and \( f(t) \) decays rapidly as \( t \to \infty \). Using Lemma 3.1.2, we easily see that \( f(R(x, \tau)) \) is a Green’s function of \( D_{i,x} \) in \( D_i \).

Let \( p_i : D \to D_i \) be the natural projections. Then \( p_i^* f(R(x, \tau)) \) is a Green’s function of \( p_i^* D_{i,x} \) in \( D \). By taking the \(*\)-product, we define for \( \tau = (\tau_1, \ldots, \tau_e) \in D \setminus D_x \):
\[ \eta_1(x, \tau) = p_1^* f(R(x, \tau_1)) \ast \cdots \ast p_e^* f(R(x, \tau_e)). \]
This is a Green current of \( D_x \) in \( D \). This follows from [SABK] (Theorem 4, page 50), as the divisors \( D_{x,i} \) intersect transversally in \( D \), thus have Serre’s multiplicity 1.

To obtain a Green’s function for \( Z(x, g) \) in \( M_K \), we take the average:
\[ \eta_2(x, \tau, g, h) = \sum_{\gamma \in G_x(F) \backslash G(F)} \eta_1(x, \gamma \tau) 1_{G_x(\mathbb{A}_f)gK}(\gamma h) \]
Note that this can be rewritten as \( \eta_2(x, \tau) = \sum_{\gamma \in \Gamma} \eta_1(\gamma^{-1} x, \tau) \), where \( \Gamma = G_x(F) \backslash G(F) \cap G_x(\mathbb{A}_f)gK^{-1} \) is a lattice in \( G(\mathbb{A}_f) \). Note that it is not obvious that this function converges. We are going to show below that actually \( \eta_2(x, \tau) \) is a Green’s function of \( G(F)(D_x \times G_x(\mathbb{A}_f)gK/K) \) in \( G(F)(D \times G(\mathbb{A}_f)/K) \) in Proposition 3.2.1.

Finally, as \( \eta_2 \) is invariant under the action of \( G(F) \), it descends to a Green’s function:
\[ \eta_3(x, \tau, g, h), \]
which is a Green’s function of \( Z(x, g)_K \) in \( M_K \).

### 3.2 Kudla-Millson function \( \varphi_{KM} \)

We will now recall some results from Kudla (see [5]), based on previous work of Kudla and Millson (see [8], [9] and [7]). Let \( V \) be a quadratic space over \( \mathbb{R} \) with signature \((n, 2)\). Let \( G = G\text{Spin}(V) \) and \( D \) be the space of oriented negative 2–planes in \( V \). Let \( z_0 \in D \) and let \( K = \text{Stab}(z_0) \). Then
\[ D \cong G/K \cong SO(n, 2)/(SO(n) \times SO(2)). \]
Let \( g_0 = \text{Lie}(G) \) be the Lie algebra of \( G \) and \( f_0 \) that of \( K \). We denote the complexification of these Lie algebras by \( g \) and \( f \). Then we have a Harish-Chandra decomposition
\[ g = f + p_+ + p_- \]
and the space of differential forms of type \((a,b)\) on \(D\) is
\[
\Omega^{a,b}(D) \cong [C^\infty(G) \otimes \wedge^{a,b}(\mathfrak{p}^*)]^K.
\]

**Theorem.** There exists an element \(\varphi^0_{KM}(x, \tau) \in (\mathcal{S}(V) \otimes \Omega^{1,1}(D))^G\) with the following properties:

1. \(r(k')\varphi^0_{KM} = (A + Bi)^{(n+2)/2}\varphi^0_{KM}\), for \(k' \in K', K' = \iota^{-1}\left\{\left(\begin{array}{cc} A & B \\ -B & A \end{array}\right), A + iB \in U(1)\right\}\)

via the map \(\iota : \widetilde{Sp}_2(\mathbb{R}) \to Sp_2(\mathbb{R})\), \(\det(k') = \det(\iota(k')) = A + iB\) for \(\iota(k') = k = \left(\begin{array}{cc} A & B \\ -B & A \end{array}\right)\).

2. \(d\varphi^0_{KM} = 0\) i.e. for any \(x \in V\), the form \(\varphi^0_{KM}(x, \cdot)\) is a closed \((1,1)\)-form on \(D\) which is \(G_x\)-invariant.

Note that \(\varphi^0_{KM}(gx, g\tau) = \varphi^0_{KM}(x, \tau)\), for \(g \in G\) from the \(G\)-invariance in the definition.

We write below \(\varphi^0_{KM}\) explicitly following [5]. First, note that we have an isomorphism \([\mathcal{S}(V) \otimes \Omega^{1,1}(D)]^G \cong [\mathcal{S}(V) \otimes \wedge^{1,1}\mathfrak{p}^*]_G\) given by evaluating at \(z_0\). Then we can identify \(\mathfrak{p}_0 \subset \mathfrak{g}_0 = \text{Lie}(G)\) as
\[
\mathfrak{p}_0 = \left\{\left(\begin{array}{cc} 0 & B \\ B^T & 0 \end{array}\right) : B \in M_{n \times 2}(\mathbb{R})\right\} \cong M_{n \times 2}(\mathbb{R}).
\]

Moreover, we can give \(\mathfrak{p}_0\) a complex structure using \(J = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \in \text{GL}_2(\mathbb{R})\) acting as multiplication on the right. Then we have the differential forms \(\omega_{ij} \in \Omega^1(D) = \Omega^{1,0}(D) \oplus \Omega^{0,1}(D), 1 \leq i \leq n, 1 \leq j \leq 2\), defined by the function in \(\omega_{ij} \in \mathfrak{p}_0\), \(\omega_{ij} : \mathfrak{p}_0 \cong M_{n \times 2}(\mathbb{R}) \to \mathbb{R}\) given by the map \(u = (u_{st})_{1 \leq s \leq n, 1 \leq t \leq 2} \to u_{ij}\). We have
\[
\varphi^0_{KM}(x) = e^{-\pi q_{z_0}(x)} \left(\sum_{i,j=1}^{n} 2x_i x_j \omega_{i1} \wedge \omega_{j2} - \frac{1}{2\pi} \sum_{i=1}^{n} \omega_{i1} \wedge \omega_{i2}\right).
\]

Finally, \(\varphi_{KM}\) is defined as:
\[
\varphi_{KM}(x) = e^{\pi q(x)} \varphi^0_{KM}(x) = e^{-2\pi R(x, z_0)} \left(\sum_{i,j=1}^{n} 2x_i x_j \omega_{i1} \wedge \omega_{j2} - \frac{1}{2\pi} \sum_{i=1}^{n} \omega_{i1} \wedge \omega_{i2}\right) \quad (3.1)
\]

We finish by recalling the crucial property \(\varphi_{KM}\) has, proven in ([6], Proposition 4.10):
\[
\ddbar^c \eta_{0,i}(x) + \delta_{D_{x,i}} = [\varphi_{KM}(x)]. \quad (3.2)
\]

Thus \(\varphi_{KM}(x)\) represents \(D_{x,i}\).
Convergence of $\eta_2(x, \tau)$

Now we are ready to show the convergence of $\eta_2(x, \tau)$. Moreover, we are going to show:

**Proposition 3.2.1.** For $x \in V(F)$ with $q(x) \in F_+$, $\eta_2(x, \tau)$ is a Green’s function of $G(F)(D_x \times G_x(A_f) gK/K)$ in $G(F)(D \times G(A_f)/K)$.

Recall that for $\tau = (\tau_1, \ldots, \tau_e)$, we let $f_i(x, \tau) = p_i^*(f(R(x, \tau_i)))$, where $p_i : D \to D_i$ is the projection on the $i^{th}$ component of $D$. Equation (3.2) is preserved under pullbacks, thus we have:

$$dd^c[f_i(x, \tau)] + \delta_{D_i,x} = [\omega_1(x, \tau)],$$

where $\omega_1(x, \tau) = p_i^* \omega_{KM}$.

Before we continue, we mention two short lemmas that tell us about the behavior of $R(x, \tau)$ when $\tau$ varies in a compact set and $x$ varies in a lattice. The first lemma tells us that the quadratic forms $q_\tau$ bound each other:

**Lemma 3.2.2.** Let $K \subset D_i$ be a compact set. Fix $\tau_0 \in K$. Then there exist $c, d > 0$ such that

$$cq_{\tau_0}(x) \leq q_\tau(x) \leq dq_{\tau_0}(x)$$

for all $\tau \in K$.

**Proof.** Consider the function $\psi : K \times \{x \in V \mid q_{\tau_0}(x) = 1\} \to \mathbb{R}$, $\psi(\tau, x) = q_\tau(x)$. Since $q_{\tau_0}$ is positive definite, the set of vectors of norm 1 is a sphere and thus compact. Thus the domain is compact and thus the image is compact, and thus bounded. Since $x \neq 0$, it must also be bounded away from 0. Now let $\tau \in K$ and $x \in V$, $x \neq 0$. By the above we have

$$c \leq q_\tau \left( \frac{x}{q_{\tau_0}(x)} \right) \leq d$$

and thus $cq_{\tau_0}(x) \leq q_\tau(x) \leq dq_{\tau_0}(x)$ as desired. \hfill \Box

The second lemma tells us that how $R(x, \tau)$ increases when $x$ varies in a lattice:

**Lemma 3.2.3.** For a compact set $K \subset D$, there are only finitely many $\gamma \in \Gamma$ such that $R(\gamma^{-1}x, \tau_i) \leq \varepsilon$ for any $\tau = (\tau_1, \ldots, \tau_e) \in K$. More precisely, we have at most $O(\varepsilon^{n/2+1})$ such $\gamma \in \Gamma$.

**Proof.** Fix some $\tau_0 \in K \cap D_i$. If for $y \in \Gamma x$ we have $R(y, \tau_i) = \frac{q_{\tau_0}(y) - a}{2} < \varepsilon$, then from the previous lemma this implies that there exists $c > 0$ such that $q_{\tau_0}(y) < \frac{a + 2\varepsilon}{c}$. Thus $y$ lies in a $n + 2$ dimensional sphere in $V$ of radius $\sqrt{\frac{a + 2\varepsilon}{c}}$. The result follows. \hfill \Box
Now we want to write the individual terms of:

$$\eta_2(x, g; \tau, h) = \sum_{\gamma \in \Gamma} f_1(\gamma^{-1}x, \tau) \ast f_2(\gamma^{-1}x, \tau) \ast \cdots \ast f_e(\gamma^{-1}x, \tau)$$

where $\gamma = G_x(F) \backslash G(F) \cap G_x(F)gK^{-1}$.

We denote:

$$\varphi_i = \rho_i^x \varphi_{KM,i}$$

for $1 \leq i \leq e$. Then we can compute the formula for the $e$-star product:

**Lemma 3.2.4.** We have the $\ast$-product of $e$-terms:

$$f_1(x, \tau) \ast f_2(x, \tau) \ast \cdots \ast f_e(x, \tau) = \sum_{k=1}^{e} \varphi_1 \wedge \cdots \wedge \varphi_{k-1} \wedge f_k \wedge \delta_{D_{k+1}x} \wedge \cdots \wedge \delta_{D_{e}x}.$$

**Proof:** We denote $\delta_i = \delta_{D_{2i}}$ for $1 \leq i \leq e$ and $\delta_{i,j} = \delta_i \wedge \delta_{i+1} \cdots \wedge \delta_j$, $\varphi_{i,j} = \varphi_i \wedge \cdots \wedge \varphi_j$ for $i \leq j$ and we take $\delta_{i,j} = \varphi_{i,j} = 1$ for $i > j$. We show the result by induction. For $n = 2$, we have $f_1(x, \tau_1) \ast f_2(x, \tau_2) = f_1 \wedge \delta_2 + \varphi_1 \wedge f_2$. Assume the result is true for $n$. Then we have:

$$f_2 \ast f_3 \ast \cdots \ast f_{n+1} = \sum_{k=2}^{n+1} \varphi_{2,k-1} \wedge f_k \wedge \delta_{k+1,n+1}.$$

By definition, we have

$$f_1 \ast (f_2 \ast f_3 \ast \cdots \ast f_{n+1}) = f_1 \wedge (\delta_2 \wedge \cdots \wedge \delta_{n+1}) + \varphi_1 \wedge (f_2 \ast f_3 \ast \cdots \ast f_{n+1})$$

$$= f_1 \wedge (\delta_{2,n+1}) + \sum_{k=2}^{n+1} \varphi_1 \wedge \varphi_{2,k-1} \wedge f_k \wedge \delta_{k+1,n+1}$$

$$= \sum_{k=1}^{n+1} \varphi_{1,k-1} \wedge f_k \wedge \delta_{k+1,n+1}.$$

**Proof of Proposition 3.2.1:** To show the convergence of $\eta_2$, we are interested in averaging the terms $f_i(y, \tau)$ for $\tau$ inside a compact set $K \subset D$, where the average is taken over $y \in \Gamma x$. For the terms containing at least one $\delta_i$, the terms

$$\varphi_1(\gamma^{-1}x, \tau_1) \wedge \cdots \wedge \varphi_{k-1}(\gamma^{-1}x, \tau_{k-1}) \wedge f_k \wedge \delta_{D_{\gamma^{-1}x,k+1}}(\tau_{k+1}) \wedge \cdots \wedge \delta_{D_{\gamma^{-1}x,e}}(\tau_e)$$

are nonzero only for $\tau_i \in D_{i,\gamma^{-1}x}$. However, this implies $R(\gamma^{-1}x, \tau_i) = 0$ and this only happens for finitely many $\gamma \in \Gamma$ when $\tau_i \in K$ inside a compact from Lemma 3.2.3. Thus the sum:

$$F_1(x, \tau) = \sum_{k=1}^{e-1} \sum_{\gamma \in \Gamma} \varphi_1(\gamma^{-1}x, \tau_1) \wedge \cdots \wedge \varphi_{k-1}(\gamma^{-1}x, \tau_{k-1}) \wedge f_k \wedge \delta_{D_{\gamma^{-1}x,k+1}}(\tau_{k+1}) \wedge \cdots \wedge \delta_{D_{\gamma^{-1}x,e}}(\tau_e).$$
is finite. This leaves the last term:

\[ F_2(x, \tau) = \sum_{\gamma \in \Gamma} \varphi_1(\gamma^{-1}x, \tau_1) \wedge \cdots \wedge \varphi_{e-1}(\gamma^{-1}x, \tau_{e-1}) \wedge f_e, \]

which we treat below in Lemma 3.2.5. We show that the sum \( F_2(x, \tau) \) converges uniformly on compacts to a smooth function. This finishes the proof of the convergence in Proposition 3.2.1. Note that \( F_1(x, \tau) \) is a finite sum of currents, while \( F_2(x, \tau) \) is the average of wedge products of smooth functions which converges to a smooth function. One can show that the Green current condition is still met by \( \eta_2(x, \tau, g, h) \), which finishes the proof of Proposition 3.2.1.

As promised, we show the convergence of \( F_2(x, \tau) \) below:

**Lemma 3.2.5.** The average \( \sum_{y \in \Gamma \times} \varphi_1(y, \tau_1) \wedge \cdots \wedge \varphi_{e-1}(y, \tau_{e-1}) \wedge f_e(y, \tau_e) \) converges uniformly on compacts to a smooth function.

**Proof.** We are free to discard finitely many terms from our average of the star product without effecting convergence, so we discard terms where \( f_e(y, \tau_e) = 0 \) on \( K \). Using the results of Kudla-Millson, for \( y = (y_1, \ldots, y_{n+2}) \) coordinates determined by the point \( z_{0,k} \) in \( D_{k, x} \), we recall the explicit definition of \( \varphi_k(y) = p_k^* \varphi_{KM,k} \):

\[
\varphi_k(y) = \sum_{1 \leq i, j \leq n} y_i y_j p_k^*(\omega_{1i} \wedge \omega_{2j}) - \frac{1}{\pi} \sum_{1 \leq i \leq n} p_k^*(\omega_{1i} \wedge \omega_{t_2}).
\]

Thus in the average, all the terms are of the form:

\[
f_e(y, \tau_e) \left( \bigwedge_{r=1}^{e-1} (y_i y_j)^r \omega_{k,i1} \wedge \omega_{k,j2}(\tau_e) \right).
\]

The forms \( \omega_{k,rs} \) are smooth on \( K \) and the values of the smooth functions representing them in local coordinates are bounded. Now we sum over \( y \in \Gamma \times \) to get the coefficient

\[
\sum_{y \in \Gamma \times} f_e(y, \tau_e) P(y).
\]

Here \( P(y) \) is a polynomial of degree \( d \) in the entries of \( y \). We have \( f_e(y, \tau_e) \leq e^{-R(y, \tau_e)} \leq e^{-R(y, \tau_e)} \) for \( R(y, \tau_e) \geq 1 \), which happens for all except finitely many \( y \)'s. Furthermore, since there are \( O(z^{n+2}) \) vectors \( y \) in our sum with \( R(y, \tau_e) \leq z \), ignoring terms with \( y_1^2 + \ldots + y_{n+2}^2 < 1 \), we are reduced to the convergence of

\[
\sum_{z=1}^{\infty} e^{-z} z^{d(n+2)}
\]
which converges using the integral test. To compute the partial derivatives in \( \tau_1, \ldots, \tau_{e-1}, \) we get terms of the form:

\[
(y_i y_j)^{\epsilon} \frac{\partial}{\partial^{\epsilon} \tau_k \partial^{\epsilon} \tau_l} \omega_{1i} \wedge \omega_{2j}(\tau_k),
\]

where \( \epsilon \in \{0, 1\} \) and \( 1 \leq i, j \leq n \). Since \( \omega_{1i} \wedge \omega_{2j} \) are smooth on compacts, we can bound the terms \( \frac{\partial}{\partial^{\epsilon} \tau_k \partial^{\epsilon} \tau_l} \omega_{1i} \wedge \omega_{2j}(\tau_k) \) by a constant \( M_k \).

For the partial derivatives in \( \tau_e \), note first that we have:

\[
\frac{\partial}{\partial \tau_e} f_e(R(y, \tau_e)) = \frac{e^{-R(y, \tau_e)}}{R(y, \tau_e)} \frac{\partial}{\partial \tau} R(y, \tau_e)
\]

\[
\frac{\partial}{\partial \tau_e} f_e(R(y, \tau_e)) = \frac{e^{-R(y, \tau_e)}}{R(y, \tau_e)} \frac{\partial}{\partial \tau_e} R(y, \tau_e)
\]

We get in general terms of the form:

\[
\frac{\partial}{\partial^{\alpha} \tau_e \partial^{\beta} \tau_e} f_e(R(y, \tau_e)) = e^{-R(y, \tau_e)} \sum_i \frac{e^{-c_i R(y, \tau_e)}}{R(y, \tau_e)} M_i R_y \cdot P_i(\partial R, y)
\]

where \( P_i(\partial R, y) \) is a polynomial in \( \frac{\partial}{\partial^{\alpha} \tau_e \partial^{\beta} \tau_e} R(y, \tau_e) \). This can be easily shown by induction. Excluding the terms for which \( R(y, \tau_e) \leq 1 \), and using the fact that \( \frac{\partial}{\partial^{\alpha} \tau_e \partial^{\beta} \tau_e} R(y, \tau_e) = -\sum_{j=1}^{n+2} y_j^2 \frac{\partial}{\partial^{n+2} \tau_e} R(e_j, \tau_e) \), a polynomial in \( y \) whose coefficients are smooth functions, we can bound the entire expression on our compact by

\[
\frac{\partial}{\partial^{\alpha} \tau_e \partial^{\beta} \tau_e} f_e(R(y, \tau_e)) \leq e^{-R(y, \tau_e)} Q(y)
\]

where \( Q(y) \) is a polynomial in \( (y_1, \ldots, y_{n+2}) \).

Taking the bounds on all the terms, we get as bounds terms of the form:

\[
\sum_{y \in \Gamma x} \bar{Q}(y) e^{-c \pi (y_1^2 + \ldots + y_{n+2}^2)} e^{-R(y, \tau_e)}
\]

and finally \( q_{\tau_e}(y) \geq c q_{\tau_0}(y) \), so we get as upper bound the sum independent of \( \tau \)’s:

\[
e^{2 \pi q(x)} \sum_{y \in \Gamma x} \bar{Q}(y) e^{-c \pi (q_{\tau_0}(y))} e^{-c \pi q_{\tau_0}(y)}
\]

To see that this converges, we sum over vectors \( y \) such that \( q_{\tau_0}(y) < C^2 \). There are \( O(C^{n+2}) \) such vectors \( y \). Suppose \( Q \) has degree \( d \). Then \( Q(y) \) is \( O(C^d) \). So the sum converges by the integral test. Thus all partial derivatives converge uniformly on compacts. \( \square \)
Averaging of Chern forms.

For $1 \leq i \leq e$, $\tau_i \in D_i$ we defined $\eta_{0,i}(x, \tau_i) = f(2\pi R(x, \tau_i))$ and we have the Chern forms $\varphi_{i,KM}(x)$ that satisfy the equation:

$$dd^c[\eta_{0,i}(x)] + \delta_{D_{x,i}} = [\varphi_{i,KM}(x)].$$

This is preserved under pullback, so we have for $\varphi_i = p^* \varphi_{i,KM}(x)$:

$$dd^c[f_i(x)] + \delta_{D_{x,i}} = [\varphi_i(x)].$$

We further defined $\eta_1(x, \tau) = f_1(x, \tau_1) \cdots f_e(x, \tau_e)$ a Green current of $D_x$ in $D$. As the star product turns into $\wedge$ product when we take the Chern forms ([SABK]), the Chern form associated to $\eta_1(x, \tau)$ is going to be:

$$\omega_1(x, \tau) = \varphi_1(x, \tau_1) \wedge \cdots \wedge \varphi_e(x, \tau_e),$$

and it satisfies:

$$dd^c[\eta_1(x)] + \delta_{D_x} = [\omega_1(x)].$$

We averaged further the Green currents:

$$\eta_1(x, g; \tau, h) = \sum_{\gamma \in G_x(F) \setminus G(F)} \eta_0(x, \gamma \tau) 1_{G_x(F) \beta K}(\gamma h)$$

to get a Green current of $Z(x, g)_K$ in $M_K$.

To get the Chern form we apply $dd^c$ locally and glue all the local forms. We can do that as the partial sums $F_1(x, \tau)$ and $F_2(x, \tau)$ converge to smooth functions. Then $\eta_2$ will descend to the Chern form:

$$\omega_2(x, g; \tau, h) = \sum_{\gamma \in \Gamma} \omega_1(\gamma^{-1}x, \tau).$$

Finally, as $\eta_2(x, \tau)$ is invariant under the action of $G(F)$ and it descends to $\eta_3(x, \tau)$, a Green’s function of $Z(x, g)_K$ in $M_K$, then the Chern form $\omega_3(x, \tau)$ of $\eta_3(x, \tau)$ is the pullback of $\omega_2(x, \tau)$ under the projection map $p : D \times G(\mathbb{A}_f)/K \rightarrow M_K$:

$$\omega_3(x, \tau) = p^* \omega_2(x, \tau).$$

Chern forms for $x = 0$

For $x = 0$, we define similarly $\omega_1(x, \tau) = c_1(L^{\tau_1}_{K,e}) \wedge \cdots \wedge c_1(L^{\tau_e}_{K,e})$. When we average, we actually have $\omega_1 = \omega_2 = c_1(L^{\tau_1}_{K,e}) \wedge \cdots \wedge c_1(L^{\tau_e}_{K,e})$. Finally we descend to $\omega_3$ a Chern form on $M_K$. From [6] we actually have:

**Lemma 3.2.6.** $\varphi_{KM,r}(0, \tau_r) = c_1(L^{\tau_r}_{K,e})$.

From this lemma we have as before $\omega_3(0, \tau) = p^* \omega_2(0, \tau)$ and $\omega_2(0, \tau) = \omega_1(0, \tau) = \varphi_1(0, \tau) \wedge \cdots \wedge \varphi_e(0, \tau)$, where $\varphi_i(0, \tau)$ is the pullback of the Kudla-Millson form $\varphi_{KM,r}(0, \tau_r) = -\frac{1}{2\pi} \sum_{i=1}^e \omega_{ii} \wedge \omega_{ij}$. Recall that $\varphi_{KM,r}(x, \tau_r)$ was defined in (3.1).
Chapter 4

Modularity of $Z(g', \phi)$

We recall the Weil representation $r$ of $\widetilde{SL}_2(\mathbb{A}_F)$ acting on the space $\mathcal{S}(V)$ of Schwartz-Bruhat functions on $V_{\mathbb{A}_F}$. Using the standard notation:

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the Weil representation for $\widetilde{SL}_2(\mathbb{A}_F) \times O(V_{\mathbb{A}_F})$ is defined by for $x \in V$ by:

- $r(m(a)) \phi(x) = \chi_V(a)|a|^{1/2} \phi(ax)$
- $r(n(b)) \phi(x) = \psi(bx^2) \phi(x)$
- $r(w) \Phi(x) = \gamma \hat{\phi}(x)$,
- $r(h) \Phi(x) = q(h) \Phi(h^{-1}x), \ h \in O(V)$

where $\psi$ is the standard additive character for $F$ and $\chi_V$ is the quadratic character corresponding to the quadratic space $V$ and $\gamma$ is the Weil constant that is an 8th root of unity.

For $g' \in SL_2(F_{\infty}) = \prod_{\sigma : F \rightarrow \mathbb{R}} SL_2(\mathbb{R}_\sigma)$, we have:

$$W_{q(x)}(g_x) = \prod_{\sigma : F \rightarrow \mathbb{R}} r(g_r) \phi_0(x_{\sigma_r}),$$

where $x_{\sigma_r}$ is the image of $x$ at the place $r$ via the embedding $\sigma_r : F \hookrightarrow \mathbb{R}$. We fix a basis given by the identity $z_{0,r}$ and define the standard Gaussian $\phi_0(x_1, \ldots, x_{n+2}) = e^{-\pi(x_1^2 + \cdots + x_{n+2}^2)}$. We denote $q_r(x) = x_1^2 + \cdots + x_{n+2}^2$ and note that in this notation we have $\phi_0(x_{\sigma_r}) = e^{-\pi q_r(x_r)}$.

Using the Iwasawa decomposition $\widetilde{SL}_2(\mathbb{R})$, we can write each $g_r$ in the form:

$$g_r = \begin{pmatrix} 1 & u_r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_r^{1/2} & 0 \\ 0 & v_r^{-1/2} \end{pmatrix} k'_0,$$
where \( k'_\theta \) has the image \( \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \) under the map \( \hat{\text{SL}}_2(\mathbb{R}) \rightarrow \text{SL}_2(\mathbb{R}) \). Then we have from the definition of the standard Whittaker function:

\[
W_q(x)(g_{\infty}) = \prod_{\sigma : F \rightarrow \mathbb{R}} u^\frac{n+2}{2} e^{\frac{n+2}{2} \theta r} e^{-\pi q_r(x_r)}.
\]

For \( g' \in \hat{\text{SL}}_2(\mathbb{A}_F) \), \( \phi \in (\mathcal{S}(V_{\mathbb{A}_F}))^K \), we defined the generating series:

\[
[Z(g', \phi)] = \sum_{x \in G(F) \backslash V_F} \sum_{g \in G_x(\mathbb{A}_F,f) \backslash G(\mathbb{A}_F,f)/K} r(g,g')\phi_f(x)W_q(x)(g_{\infty}')[Z(x,g)_K]. \tag{4.1}
\]

We will show:

**Theorem 4.0.1.** The function \( Z(g', \phi) \) is an automorphic form of weight \( 1 + n/2 \) for \( g' \in \hat{\text{SL}}_2(\mathbb{A}_F) \), \( \phi \in \mathcal{S}(V_{\mathbb{A}_F}) \) with values in \( H^{2n}(X, \mathbb{C}) \).

Recall that in \( H^{2n}(X, \mathbb{C}) \) we have \( [Z(x,g)] = [\omega_3(x,g,\tau,h)] \) as cohomology classes. Thus we can replace in the sum (4.1) the cohomology class of the special cycle \( Z(x,g) \) with the cohomology class of \( \omega_3(x,g,\tau,h) \). We are going to show first

**Lemma 4.0.2.** The cohomology class \( [Z(g', \phi)] \) descends on \( D \times G(\mathbb{A}_f)/K \) to the cohomology class:

\[
[Z(g', \phi)] = \sum_{x \in V_F} r(g',\phi_f(h^{-1}x))W_q(x)(g_{\infty}')[\omega_1(x, \tau)].
\]

**Proof.** The pullback to \( D \times G(\mathbb{A}_f)/K \) of \( \omega_3(x,\tau,h) \) is \( \omega_2(x,\tau) \). Then by taking the pullback to \( D \times G(\mathbb{A}_f)/K \) in 4.1, we have:

\[
[Z(g', \phi)] = \sum_{x \in G(F) \backslash V_F} \sum_{g \in G_x(\mathbb{A}_F,f) \backslash G(\mathbb{A}_F,f)/K} r(g,g')\phi_f(x)W_q(x)(g_{\infty}')\omega_2(x,g,\tau,h).
\]

By plugging in the definition \( \omega_2(x,g,\tau,h) = \sum_{\gamma \in G_x(F) \backslash G(F)} \omega_1(x,\gamma\tau)1_{G_x(F)gK}(\gamma h) \) in (4.1), we get \([Z(g', \phi)]\) equal to the cohomology class of:

\[
\sum_{x \in G(F) \backslash V_F} \sum_{g \in G_x(\mathbb{A}_F,f) \backslash G(\mathbb{A}_F,f)/K} r(g,g')\phi_f(x)W_q(x)(g_{\infty}') \sum_{\gamma \in G_x(F) \backslash G(F)} \omega_1(x,\gamma\tau)1_{G_x(F)gK}(\gamma h).
\]

We will unwind the sum below to get the result of the lemma. We interchange the summations to get:

\[
\sum_{x \in G(F) \backslash V_F} \sum_{\gamma \in G_x(F) \backslash G(F)} \sum_{g \in G_x(\mathbb{A}_F,f) \backslash G(\mathbb{A}_F,f)/K} r(g,g')\phi_f(x)W_q(x)(g_{\infty}')\omega_1(x,\gamma\tau)1_{G_x(F)gK}(\gamma h).
\]

Note that \( 1_{G_x(F)gK}(\gamma h) \neq 0 \) iff \( \gamma h \in G_x(F)gK \), or equivalently if \( g \in G_x(F)\gamma hK \), and since we are summing for \( g \in G_x(\mathbb{A}_Q,f) \backslash G(\mathbb{A}_Q,f)/K \), we can replace \( g \) by \( \gamma h \) everywhere and get:

\[
[Z(g', \phi)] = \sum_{x \in G(F) \backslash V_F} \sum_{\gamma \in G_x(F) \backslash G(F)} r(\gamma h,g')\phi_f(x)W_q(x)(g_{\infty}')\omega_1(x, \gamma\tau).
\]
CHAPTER 4. MODULARITY OF $Z(g', \phi)$

Since the action of $G(\mathbb{A}_{Q,f})$ on $\phi$ is given by $r(\gamma h, g')\phi_f(x) = r(g')\phi_f(h^{-1}\gamma^{-1}x)$ and $\omega_1(x, \gamma \tau) = \omega_1(\gamma^{-1}x, \tau)$, then we have:

$$[Z(g', \phi)] = \sum_{x \in V_F} r(g')\phi_f(h^{-1}x)W_{q(x)}(g'_x)\omega_1(x, \tau),$$

which gives us the result of the lemma. □

Now we are ready to show Theorem 4.0.1. We will first recall the definition of the standard Whittaker function. Recall that we have $\omega_1(x, \tau) = p_1^* \varphi_{KM,1}(x, \tau_1) \wedge \cdots \wedge p_\epsilon^* \varphi_{KM,\epsilon}(x, \tau_\epsilon)$. From the Theorem of Kudla-Millson, we have $r(g'_r)\varphi_{KM,\epsilon}(x, \tau_r) = e^{-2\pi q_r(x)}\varphi_{KM,\epsilon}(x, \tau_r)$. When we take the pullback, the Schwartz-Bruhat function is preserved, thus we also have:

$$r(g'_r)\varphi_r(x, \tau_r) = e^{-2\pi q_r(x)}\varphi_r(x, \tau_r).$$

Denote $\varphi^0_r(x, \tau_r) = e^{-\pi q_r(x)}\varphi_r(x, \tau_r)$ and then we have from the above property $W_{q(x)}(g_r)\varphi_r(x, \tau_r) = r(g_r)\varphi^0_r(x, \tau_r)$ and thus:

$$W_{q(x)}(g_r)\varphi_r(x, \tau_r) = r(g_r)\varphi^0_r(x, \tau_r).$$

Note that we can write:

$$W_{q(x)}(g_x)\varphi_1(x, \tau_1) \wedge \cdots \wedge \varphi_e(x, \tau_e) =$$

$$(W_{q_1}(g_1)\varphi_1(x, \tau_1) \wedge \cdots \wedge W_{q_\epsilon}(g_\epsilon)\varphi_\epsilon(x, \tau_\epsilon)) \prod_{r=e+1}^d W_{q_r}(g_r),$$

and using the property above this is:

$$r(g_1)\varphi^0_{KM}(x_1, \tau_1) \wedge \cdots \wedge r(g_\epsilon)\varphi^0_{KM,\epsilon}(x_\epsilon, \tau_\epsilon) = r(g_x)\phi^0(x, \tau),$$

where $\phi^0(x, \tau) = \varphi^0_{KM}(x_1, \tau_1) \wedge \cdots \varphi^0_{KM,\epsilon}(x_\epsilon, \tau_\epsilon) \prod_{r=e+1}^d \phi_0(x_r)$. Recall that for $r \geq e + 1$, $W_{q(x)}(g_r) = r(g_r)\phi_0(x_r)$, where $\phi_0(x_r)$ is the standard Gaussian.

Going back to the sum, we get:

$$[Z(g', \phi)] = \sum_{x \in V_F} r(g')\phi_f(h^{-1}x)r(g'_x)\phi^0(x, \tau),$$

and this is a theta function of weight $(n + 2)/2$ with values in the cohomology group $H^{2n}(X, \mathbb{C})$. Thus when we take the pullback to $M_K$, we get a theta function as well. This means that for any functional acting on the cohomology part of $\phi^0(x, \tau)$, we still get a theta function in $g'$, and this is indeed being preserved by the pullback.

We can also easily check the weight of the theta function by computing $r(k_0)\phi^0(x, \tau) = r(k_1)\varphi^0_{KM}(x_1, \tau_1) \wedge \cdots \varphi^0_{KM,\epsilon}r(k_\epsilon)(x_\epsilon, \tau_\epsilon) \prod_{r=e+1}^d r(k_r)\phi_0(x_r)$ which gives us the factor $e^{i\frac{n+2}{2}\theta_r}$ at each place $r.$
Bibliography


