A Contemporary Study in Gauge Theory and Mathematical Physics: Holomorphic Anomaly in Gauge Theory on ALE Space & Freudenthal Gauge Theory

by

Sheng-Yu Darren Shih

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Committee in charge:

Professor Bruno Zumino, Chair
Professor Ori J. Ganor
Professor Mariusz Wodzicki

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Abstract

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This thesis covers two distinct parts: Holomorphic Anomaly in Gauge Theory on ALE Space and Freudenthal Gauge Theory.

In part I, I presented a concise review of the Seiberg-Witten curve, Nekrasov's Ω-background, geometric engineering and the holomorphic anomaly equation followed by my published work: Holomorphic Anomaly in Gauge Theory on ALE Space, where an Ω-deformed $\mathcal{N}=2$ $SU(2)$ gauge theory on A_1 space and its five dimension lift is studied. We find that the partition functions can be reproduced via special geometry and the holomorphic anomaly equation. Schwinger type integral expressions for the boundary conditions at the monopole/dyon point in moduli space are inferred. The interpretation of the five-dimensional partition function as the partition function of a refined topological string on $A_1\times(\text{local } \mathbb{P}^1\times\mathbb{P}^1)$ is suggested.

In part II, I give a comprehensive review of the *Freudenthal Triple System*, including Freudenthal's original construction from *Jordan Triple Systems* and its relation to Lie algebra, Yang-Baxter equation, and 4d $\mathcal{N}=2$ BPS black holes, where the novel *Freudenthal-dual* was discovered. I also present my published work on the Freudenthal Gauge Theory, where we construct the most generic gauge theory admitting *F-dual*, and prove a no-go theorem that forbids coupling of a *F-dual* invariant gauge theory to supersymmetry.

I dedicate this work to my advisor Prof. Bruno Zumino. It is really my greatest honor to work with one of the most insightful physicist and finest gentleman in the world.

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Chapter 1

Introduction

This thesis is composed of two parts: Holomorphic Anomaly in Gauge Theory on ALE Space and Freudenthal Gauge Theory. They summarize my two distinct research topics during my graduate study at UC Berkeley.

In part I, we focus on the holomorphic anomaly in gauge theory in Asymptotically Locally Euclidean (ALE) spaces. Holomorphic anomaly equation [11, 12] is long known to be a powerful tool in the study of topological string moduli space and the gauge theories descended from it. It admits a generalization, called extended holomorphic anomaly equation [54, 55], which is applicable to the mysterious Ω-background originally introduced by N. Nekrasov to regularize the divergent moduli space measure of the gauge theory in $\mathbb{R}^4$ [71]. In this study, we take a step further towards a non-trivial spacetime; We consider four-dimensional Ω-deformed $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory on A_1 space and its lift to five dimensions. We find that the partition functions can be reproduced via special geometry and the holomorphic anomaly equation, whose boundary conditions at the monopole/dyon point in moduli space can be expressed by Schwinger type integrals. We also find the five-dimensional partition function bears an interpretation as the partition function of a refined topological string on $A_1\times$(local $\mathbb{P}^1 \times \mathbb{P}^1$). The result is presented in our published paper [53].

In part II, we move the topic to an exotic algebraic system called *Freudenthal Triple System*, which was originally introduced by mathematician Hans Freudenthal in his study of exceptional Lie algebras [29, 30]. Such an algebraic system distinguished by a non-symmetric ternary product plays important role in both pure math and theoretical physics research. From the mathematics side, besides its original appearance in the study of exceptional Lie algebras, it's also shown to lead to solutions to the renowned Yang-Baxter equation [74, 75], which paves the way to new insights in the study of braiding groups, and integrable systems. From the physics side, it's known to classify the scalar manifold of the 4d $\mathcal{N} = 2$ Maxwell-Einstein supergravity, and governs the black hole entropy in the theory. Later, it was discoverd that such black holes are related by a non-linear duality called *Freudenthal-dual* (*F-dual* for short). Being a Leibniz algebra, a *Freudenthal Triple System* also naturally defines a gauge theory similar to the famous BLG theory for the supersymmetric M2-branes [8, 38]. In this study we analyze the mathematical structure of *F-dual*, and it's possible application towards a new gauge theory. We gave a mathematical proof of the *F-dual* invarance of the black hole entropy and presented a no-go theorem forbidding coupling of a *F-dual* invariant gauge theory to supersymmetry. Our result is published in [60].

The organization of the thesis is as follows: in part I, we give a short review in chapter 2, which covers the related background knowledge to out work of "HHolomorphic Anomaly in Gauge Theory on ALE Space" [53], which is presented in chapter 3. The review is meant to be concise, with only the minimal exposure to the vast knowledge of topological string and supersymmetric Yang-Mills theory. We focus, instead, on presenting one coherent logic line linking various aspects of the theory we studied in detail in chapter 3. In part II, a more detailed account of the *Freudenthal Triple System* and its relation to the 4d $\mathcal{N} = 2$ Maxwell-Eienstein supergravity and the Yang-Baxter equation is presented in chapter 4. And chapter 5 is adapted from an un-published version of our paper "Freudenthal Gauge Theory" [60]. We try to cover as many aspects of the *FTS* as possible to moderate detail, hoping by doing so will familiarize the readrer with the relatively un-familar *Freudenthal Triple System*. Appendix A is the Lie bracket table for a Lie algebra constructed from a *FTS*. In appendix B and C, we collect the review of *special geometry* from string theory construction and spin chain approach of an integrable systerm, which supplement other aspects of the topics covered in the main context.

Part I

Holomorphic Anomaly in Gauge Theory on ALE Space

Chapter 2

Background Knowledge

2.1 Review of $\mathcal{N} = 2$ $SU(2)$ gauge theory

The objective of this part is the holomorphic anomaly of gauge theory in ALE spaces. We shall start by reviewing the pure $\mathcal{N} = 2$ $SU(2)$ gauge theory in $\mathbb{R}^4$ spacetime following [78] here. It has a $\mathcal{N} = 2$ vector multiplet $\mathcal{A}$, containing a gauge field A_μ, two Weyl fermions λ, ψ, and a scalar ϕ, all in the adjoint representation of the gauge group.

The classical potential of the pure $\mathcal{N} = 2$ gauge theory is

$$V(\phi) = \frac{1}{g^2} tr[\phi, \phi^\dagger]^2. \tag{2.1}$$

This potential vanishes when ϕ and $\phi^\dagger$ commute; therefore the classical theory has a family of vacuum states labeled by its field configuration at infinity $\phi_\infty \equiv \phi(x \to \infty)$, which is generated by the Cartan subalgebra of the gauge group. In our $SU(2)$ case, one may choose by gauge transformation the asymptotic field configuration to be $\phi_\infty = \frac{1}{2} a\sigma^3$, where σ^i are the standard Pauli matrices and a is a complex number called Coulomb parameter that labels the vacua. The Weyl group of $SU(2)$ acts by $a \leftrightarrow -a$, so the gauge invariant quantity parameterizing the space of classical vacua is $u \equiv tr\phi_\infty^2 = \frac{1}{2}a^2$. Note, classically, there is a singularity at $u = 0$. where the full $SU(2)$ gauge symmetry is restored; on the other hand at a generic point point in the classic moduli space, the $SU(2)$ breaks into $U(1)$ and some of the fields acquire mass via Higgs mechanism, this is called the Coulomb branch of the theory.

2.1.1 Low Energy Effective Theory and Prepotential

The low energy effective action of the light fields for a generic $\langle\phi\rangle$ is constrained by $\mathcal{N}=2$ supersymmetry. In terms of $\mathcal{N}=1$ fields, the lagrangian is [1]

$$\frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta\, tr\Big(\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi}\overline{\Phi}\Big)+\int d^2\theta\,\frac{1}{2}tr\Big(\frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^2}W_\alpha^2\Big)\right], \tag{2.2}$$

where $\Phi=(\psi,\phi)$ and $W_\alpha=(A_\mu,\lambda)$ are the $\mathcal{N}=1$ chiral and vector multiplet in the $\mathcal{N}=2$ vector multiplet $\mathcal{A}$. Because of the $\mathcal{N}=2$ supersymmetry, the super potential is not lifted by quantum corrections, as a result the quantum theory has a non-trivial moduli space $\mathcal{M}$. In fact $\mathcal{M}$ is a one complex dimensional Kähler manifold. The Kähler metric can be written in terms of the effective low energy prepotential $\mathcal{F}$ as

$$(ds)^2=\mathrm{Im}\left(\frac{\partial^2\mathcal{F}(a)}{\partial a^2}\right)\,da d\bar{a}. \tag{2.3}$$

In the classical theory, $\mathcal{F}$ can be read off from the tree level lagrangian of the $SU(2)$ gauge theory and is $\mathcal{F}(a)=\frac{1}{2}\tau_{cl}\,a^2$ where $\tau_{cl}=\frac{\theta}{2\pi}+i\frac{4\pi}{g^2}$. In the quantum case, the θ parameter and gauge coupling g may be obtained from $\tau(a)\equiv\frac{\theta(a)}{2\pi}+i\frac{4\pi}{g^2(a)}$ as functions of a via:

$$\tau(a)=\frac{\partial^2\mathcal{F}}{\partial a^2}, \tag{2.4}$$

where the a dependence comes from quantum corrections to the prepotential $\mathcal{F}(a)$. The quantum corrections to $\mathcal{F}$ are studied in [80], it contains a perturbative part and a non-perturbative instanton contribution:

$$\begin{aligned}\mathcal{F}(a)&=\mathcal{F}_{pert}(a)+\mathcal{F}_{inst}(a;\mathfrak{q})\\&=i\frac{1}{2\pi}a^2\log\frac{a^2}{\Lambda^2}+a^2\sum_{k=1}^{\infty}\Big(\frac{\mathfrak{q}}{a^4}\Big)^k\alpha_k,\end{aligned} \tag{2.5}$$

with numerical coefficients α_k. Here Λ is a dynamically generated scale and $\mathfrak{q}\equiv\Lambda^4$ is the instanton counting parameter s.t. the $\mathfrak{q}^k$ term in the summation comes from k instanton contributions. Note, the perturbative part contains tree level and one-loop contributions and all the higher loop terms are absent.

2.1.2 Seiberg-Witten Curve

The one-loop part of the prepotential (2.5) implies that for large $|a|$,

$$\tau(a)\approx\frac{i}{\pi}\Big(\log\frac{a^2}{\Lambda^2}+3\Big), \tag{2.6}$$

[1] For a generic $\langle\phi\rangle$ labeled by non-zero a, the $SU(2)$ symmetry is broken to $U(1)$, i.e. we are in the Coulomb branch where the trace of lagrangian only gives an overall scaling.

whose imaginary part is single-valued and positive as required to be the Kähler metric $(ds)^2 = \mathrm{Im}\,\tau(a)\,da\,d\bar{a}$ on the moduli space $\mathcal{M}$. However, it cannot be globally defined since the harmonic function $\mathrm{Im}\,\tau(a)$ cannot have a minimum over the non-compact space $\mathcal{M}$. One thus concludes that the above description of metric is valid only locally.

To introduce another local coordinate, we define $a_D \equiv \partial\mathcal{F}/\partial a$, and re-organize the metric as

$$(ds)^2 = \mathrm{Im}\,da_D\,d\bar{a} = -\frac{i}{2}\big(da_D\,d\bar{a} - da\,d\bar{a}_D\big), \tag{2.7}$$

with the coupling $\tau(a) = \partial a_D/\partial a$. This form is symmetric in a and a_D, and if we choose a_D as our local parameter, the metric will be in the same general form with a different harmonic function replacing $\mathrm{Im}\,\tau$.

To have a global description of the moduli space $\mathcal{M}$, we take $u = tr\phi_\infty^2$ as the global coordinate on $\mathcal{M}$ and treat $(a_D(u), a(u)) \in X \cong \mathbb{C}^2$ as holomorphic coordinate map from $\mathcal{M}$ to X. The moduli space $\mathcal{M}$ actually has singularity and the holomorphic sections $a_D(u)$, $a(u)$ have non-trivial monodromy over such singular points. To see it explicitly, consider a circle of the u-plane at large $|u|$. Since $u = \frac{1}{2}a^2$, one has under $\log u \to \log u + 2\pi i : \log a \to \log a + \pi i$ or $a \to -a$. And from the one-loop correction formula $\mathcal{F}_{one-loop} = \frac{i}{2\pi}\,a^2 \log\big(a^2/\Lambda^2\big)$ for large $|a|$:

$$a_D \equiv \frac{\partial\mathcal{F}}{\partial a} \approx \frac{2ia}{\pi}\log\big(a/\Lambda\big) + \frac{ia}{\pi} \tag{2.8}$$

would acquire a monodromy $a_D \to -a_D + 2a$.

Moreover, the monodromy must form a complete representation of the fundamental group of the punctured moduli space $\mathcal{M}$ in $SL(2,\mathbb{Z})$. [2] This indicates the existence of two more singularities on u-plane, which are related by $\mathbb{Z}_2$ duality $u \leftrightarrow -u$. One of the singularity arises at the point u_0 of the moduli space where the magnetic monopole becomes massless, that is, where $a_D(u_0) = 0$. By choosing the dynamic scale, we may as well take this point as $u_0 = \Lambda^2$. Thus, near this singular point, a_D forms a good local coordinate

$$a_D \approx c_0(u - \Lambda^2) \tag{2.9}$$

with some constant c_0. And the magnetic coupling is $\tau_D \approx -\frac{i}{\pi}\log a_D$. From $\tau_D = -\frac{da(a_D)}{da_D}$, we may integrate τ_D and get

$$a(u) = -a(a_D(u)) \approx a_0 + \frac{i}{\pi}a_D\log a_D \approx a_0 + \frac{i}{\pi}c_0(u-\Lambda^2)\log(u-\Lambda^2), \tag{2.10}$$

where $a_0 = a|_{u=\Lambda^2}$ is a (non-zero) integration constant. The other singularity located at $u = -\Lambda^2$ is similarly obtained by taking electric-magnetic duality.

[2] Denote by $a^\alpha = (a_D, a)$, with $\alpha = 1, 2$ and $\epsilon_{\alpha\beta}$ the Levy-Civita tensor in 2-dim, the Kähler metric in the u coordinate is

$$(ds)^2 = -\frac{i}{2}\epsilon_{\alpha\beta}\frac{da^\alpha}{du}\frac{d\bar{a}^\beta}{d\bar{u}}du\,d\bar{u},$$

which has isometry group $Sp(2,\mathbb{R}) = SL(2,\mathbb{R})$. While a further investigation of the gauge fields and the electric and magnetic charges further reduces the isometry group to $SL(2,\mathbb{Z})$, one concludes that (a_D, a) must form a representation of $SL(2,\mathbb{Z})$.

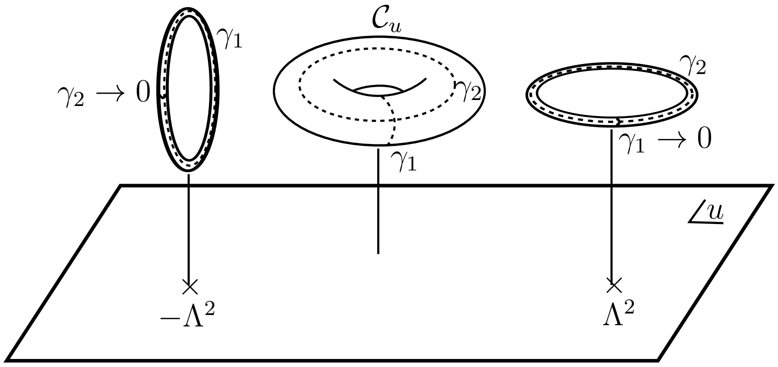

Figure 2.1: The dual 1-cycles (γ_1, γ_2) of $\mathcal{C}_u$ over the global coordinate $u \in \mathbb{C}^*$ of $\mathcal{M}$.

Thus the moduli space together with its coordinate maps $\mathcal{M} \to X \cong \mathbb{C}^2$ satisfying appropriate monodromy property is described by a family of genus one complex curve $\mathcal{C}_u$ fibration over the punctured complex u-plane, where

$$\mathcal{C}_u : y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u). \tag{2.11}$$

The genus one curve $\mathcal{C}_u$ so designed is singular exactly at $u = \pm\Lambda^2, \infty$. And the local coordinates $a(u)$ and $a_D(u)$ are obtained as the period of some properly chosen meromorphic one form λ_{SW} (called Seiberg-Witten 1-form) integrating over the two independent (dual) 1-cycles γ_1, γ_2 of $\mathcal{C}_u$ (see Fig. 2.1):

$$\begin{aligned} a_D(u) &= \oint_{\gamma_1} \lambda_{SW} \\ a(u) &= \oint_{\gamma_2} \lambda_{SW}, \end{aligned} \tag{2.12}$$

such that at singular points $u \to \pm\Lambda^2$, one of the cycles of $\mathcal{C}_u$ shrinks to zero and the corresponding period becomes a good local coordinate of the singular locus while its dual period assumes desired monodromy property.

2.2 Nekrasov's Localization Formula and the Ω Background

2.2.1 Nekrasov's Partition Function

Despite several attempts to calculate the instanton corrections to the prepotential (2.5) in indirect ways, the gauge theory approach didn't go further than two-instanton corrections. The difficulty was caused by the increasingly intricated structure of the instanton measure as the instanton charge k increases. The revolution came from Nekrasov's localization method

[71], which by properly regularizing the divergent instanton measure successfully gives a systematic way to calculate the instanton contribution to any instanton number k as

$$Z^{Nek}_{inst}(a;\epsilon_1,\epsilon_2;\mathfrak{q}) \\ =\sum_{k=0}^{\infty}\sum_{\substack{(Y_1,Y_2)\\|Y_1|+|Y_2|=k}}\prod_{m,n=1}^{2}\frac{\mathfrak{q}^k}{\prod_{s\in Y_n}E_s(\epsilon_1+\epsilon_2-a_{nm};Y_n,Y_m;-\vec{\epsilon})\prod_{t\in Y_m}E_t(a_{nm};Y_n,Y_m;\vec{\epsilon})}, \quad (2.13)$$

where (Y_1,Y_2) are a pair of Young-diagrams, $\vec{\epsilon}\equiv(\epsilon_1,\epsilon_2)$ are regularization parameters, and $a_{nm}\equiv a_n-a_m$ with $(a_1,a_2)=(a,-a)$ for our $SU(2)$ gauge theory. The function $E_s\big(x;Y_n,Y_m;(\epsilon_1,\epsilon_2)\big)$ is defined as

$$E_s\big(x;Y_n,Y_m;(\epsilon_1,\epsilon_2)\big)=x-\epsilon_1L_{Y_n}(s)+\epsilon_2\big(A_{Y_m}(s)+1\big), \quad (2.14)$$

where $L_Y(s)$ and $A_Y(s)$ denotes the leg- and arm-length function for the box $s=(i,j)$ of the Young-diagram Y.

Using the same regularization, Nekrasov also obtained the perturbative part of the partition function, which can be organized as a Schwinger integral:

$$Z^{Nek}_{pert}(a;\epsilon_1,\epsilon_2)=\exp\Big(\int_{\varepsilon}^{\infty}\frac{ds}{s}\frac{2\cosh(2sa)}{\sinh\big(\frac{s\epsilon_1}{2}\big)\sinh\big(\frac{s\epsilon_2}{2}\big)}\Big), \quad (2.15)$$

with the singularity in the UV cut-off ε dropped. Together, the full Nekrasov's partition function

$$Z^{Nek}(a;\epsilon_1,\epsilon_2;\mathfrak{q})=Z^{Nek}_{pert}(a;\epsilon_1,\epsilon_2)\times Z^{Nek}_{inst}(a;\epsilon_1,\epsilon_2;\mathfrak{q}) \quad (2.16)$$

is identified to the Seiberg-Witten prepotential (2.5) as [71]

$$\mathcal{F}(a;\mathfrak{q})=\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2\log Z^{Nek}(a;\epsilon_1,\epsilon_2;\mathfrak{q}). \quad (2.17)$$

2.2.2 The Ω Background

To understand the meaning of the regularization parameter (ϵ_1,ϵ_2), also called Nekrasov's Ω-background, we shall have a short review of the idea behind the localization method. Our approach here is adapted from [73]. Schematically, the partition function of the low energy lagrangian (2.2) after Wick rotation is of the form:

$$\int\mathcal{D}A_\mu\, exp\Big(-i\frac{\tau_{cl}}{8\pi}\int_{\mathbb{R}^4}trF*F\Big)\times\cdots, \quad (2.18)$$

where the integration is over distinct gauge orbits of connections and the dots stand for other terms involving fields other than the gauge field A_μ in the supermultiplet $\mathcal{A}$. While the supersymmetry invariance asserts that the gauge field must satisfy the anti-self-dual equation

$$F=-\big(*F\big), \quad (2.19)$$

the path integral is equivalent to a counting of gauge inequivalent configurations of the following topological invariants

$$k = \frac{-1}{8\pi^2} \int_{\mathbb{R}^4} tr\, F \wedge F \tag{2.20}$$

called instanton number. Therefore, the partition function (2.18) is schematically

$$Z_{pert} \times \sum_{k \geq 0} \Lambda^{4k} \int_{\overline{\mathcal{M}}_k} 1, \tag{2.21}$$

where $\overline{\mathcal{M}}_k$ is the smooth partial compactification of the k-instanton moduli space, and $\Lambda = \mu\, e^{i\pi\, \tau_{cl}/4}$ with μ the UV cut-off of the path integral.

The problem is; the k-instanton moduli space $\overline{\mathcal{M}}_k$ is not a compact manifold so that the integration over $\overline{\mathcal{M}}_k$ diverges. To understand this divergence, physically, one may regard an element in $\overline{\mathcal{M}}_k$ as an non-linear superposition of k instantons of charge one spread over the space time $\mathbb{R}^4$. The divergence arises because the instanton wave function may wander off to infinity and is not normalizable. To solve this problem, Nekrasov introduces a regularization scheme compatible to the $\mathcal{N} = 2$ supersymmetry, where the spacetime $\mathbb{R}^4$ is identified as $\mathbb{C}^2$ with its complex structure corresponding to the symplectic form $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$. In term of the complex coordinates $z_1 = x^1 + ix^2$, $z_2 = x^3 + ix^4$,

$$\omega = \frac{i}{2}\big(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2\big). \tag{2.22}$$

Then the divergent volume integral $\int_{\mathbb{R}^4} \omega \wedge \omega$ is regularized by the symplectic form

$$\omega_{\epsilon_1,\epsilon_2} \equiv e^{-\epsilon_1 \pi |z_1|^2}\, dz_1 \wedge d\bar{z}_1 + e^{-\epsilon_2 \pi |z_2|^2}\, dz_2 \wedge d\bar{z}_2, \tag{2.23}$$

such that

$$\int_{\mathbb{R}^4} \omega \wedge \omega \approx \int_{\mathbb{R}^4} \omega_{\epsilon_1,\epsilon_2} \wedge \omega_{\epsilon_1,\epsilon_2}\Big|_{\epsilon_1,\epsilon_2 \to 0} = \lim_{\epsilon_1,\epsilon_2 \to 0} \Big(\frac{1}{\epsilon_1 \epsilon_2}\Big). \tag{2.24}$$

Note that the Hamiltonian flow generated by $H_i = \frac{1}{2}|z_i|^2$ w.r.t. the symplectic form ω is a rotation about the origin $z_i = 0$ with velocity 1. Hence the regularization parameters (ϵ_1, ϵ_2) have a physical meaning as the generator of the maximal torus $\mathbb{T}^2_{\epsilon_1,\epsilon_2}$ of $SO(4) = SU(2) \times SU(2)$.

In this spirit, Nekrasov regularizes the partition function of $\mathcal{N} = 2$ pure $SU(2)$ gauge theory with boundary configuration $\phi \to \frac{1}{2} a\sigma^3$ as

$$Z^{Nek}_{inst}(a; \epsilon_1, \epsilon_2) = \Big\langle \exp\Big(\frac{-i\,\tau_{cl}}{8\pi} \int_{\mathbb{R}^4} e^{-2\pi(\epsilon_1 H_1 + \epsilon_2 H_2)}\, trF * F\Big) \times \cdots \Big\rangle_a, \tag{2.25}$$

where the subscript a indicates the vacuum expectation value is taken with boundary configuration a while the dots stand for terms involving interaction with other fields in the theory and establishes the expected identification to the Seiberg-Witten prepotential (2.17) [71].

2.3 Geometric Engineering of $\mathcal{N} = 2$ $SU(2)$ Pure Gauge Theory

Here we shall review how the $\mathcal{N} = 2$ $SU(2)$ pure gauge theory arises from a type IIA string theory construction called geometric engineering [51, 50], which reveals yet another way to understand the gauge theory moduli space. It will also serve as the link to the holomorphic anomaly, which we will discuss in the next section.

2.3.1 Generalities of Geometric Engineering

The idea behind geometric engineering is the following: if one consider a super string theory in an 10-dim ambient space of the form $\mathbb{R}^4 \times M$. Regarding the $\mathbb{R}^4$ as our (Euclidean) spacetime, then various field contents in a supersymmetric gauge theory may arise as the $\mathbb{R}^4$ projection of string theory objects wrapping on suitable non-trivial cycles of a properly engineered internal space M in certain limit. For instance, the requirement of having $\mathcal{N} = 2$ supersymmetry in 4D is equivalent to requiring the internal space M to be a Calabi-Yau manifold. To get a gauge theory with gauge group of ADE type in Cartan's classification, one needs a ADE type singularity in a $K3$ geometry (which has complex dimension 2), where the Cartan's matrix is identified as the intersection number of the 2-spheres as the blow-up of the singular points in $K3$. To get a gauge theory in 4D, one further take the $K3$ as a fibration over some complex dimension 1 base manifold, whose non-trivial cycles will lead to various matter content in the result gauge theory.

$SU(2)$ Case

To be explicit, let's consider the $\mathcal{N} = 2$ $SU(2)$ pure gauge theory here. It requires a A_1 singularity, which upon blow-up is a single 2-sphere $\mathbb{P}^1_f$ in $K3$. Since we only want to consider pure gauge theory, we may take a genus 0 surface, that is a 2-sphere denoted as $\mathbb{P}^1_b$ as our base manifold and get the desired fibration $\mathbb{P}^1_f \to \mathbb{P}^1_b$. [3] The $W^{\pm}$ boson in the $SU(2)$ gauge theory is then obtained as D_2-branes wrapping the fiber $\mathbb{P}^1_f$ with opposite orientation, and the $W^{\pm}$ mass is proportional to the Kähler moduli t_f of $\mathbb{P}^1_f$. A gauge instanton of charge k is identified with a string wrapping the base $\mathbb{P}^1_b$ k times located at spacetime point $x \in \mathbb{R}^4$. Finally, the bare gauge coupling g_0 at the string scale is related to the Kähler moduli t_b of the base $\mathbb{P}^1_b$ as $\frac{1}{g_0^2} \propto t_b$.

[3] The details of the fibration is governed by the requirement that as a whole M has to be a Calabi-Yau manifold. In our pure $SU(2)$ case, there exist a family of possible $\mathbb{P}^1_f \to \mathbb{P}^1_b$ fibrations categorized as Hirzebruch surfaces F_n. In the simplest case $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ describes the trivial bundle. It has been checked up to $n = 2$ that all these fibrations leads to the same $SU(2)$ gauge theory in appropriate limit [50].

2.3.2 Double Scaling Limit

To obtain the pure gauge theory, one has to decouple the theory from gravity/string effect by taking appropriate limit as discussed in [49]. First of all, the bare gauge coupling g_0 at string scale M_{Planck} should go to zero as $M_{Planck} \to \infty$, that is; the $SU(2)$ theory is asymptotically free. On the other hand, our $W^{\pm}$ should have finite mass m_W, which in the string unit is $\big(m_W/M_{Planck}\big) \propto t_f \to 0$ in this limit. Finally, in the weak coupling limit, the gauge coupling runs as

$$\frac{1}{g_0^2} \approx \log \frac{m_W}{M_{Planck}}. \tag{2.26}$$

Thus, we have the double scaling limit:

$$\begin{cases} t_b \to \infty \\ t_f \to 0 \end{cases}, \text{with} \quad t_b \approx -\text{const.} \log t_f. \tag{2.27}$$

This relation can be made more precise by recalling that the k instanton correction to the prepotential (2.5) has the form $\left(\frac{\Lambda^4}{a^4}\right)^k$, where a is the Coulomb parameter, which is proportional to t_f. [4] While in the string theory picture, the instanton is weighted by $\Lambda^4 \sim e^{-t_b}$. Thus we should take the double scaling limit to be

$$e^{-t_b} \sim \varepsilon^4 \Lambda^4, \quad t_f \sim \varepsilon\, a, \tag{2.28}$$

as $\varepsilon \to 0$, where $\Lambda \sim M_{Planck}$ is the dynamically generated scale. Note, the gauge theory obtained as the double scaling limit of the type IIA string theory construction essentially has only one parameter; the Coulomb parameter a which forms a local coordinate of the $SU(2)$ pure gauge theory moduli space $\mathcal{M}$.

2.4 Review of Holomorphic Anomaly Equation

Although Nekrasov's formula (2.13) provides a systematic way to calculate the Seiberg-Witten prepotential (2.5) in the weak coupling limit $u \sim \frac{1}{2}a^2 \to \infty$, it has a apparent draw back; it doesn't easily generalize to other point u in the moduli space $\mathcal{M}$. The rescue comes from the holomorphic anomaly equation, which was discovered [11, 12] in a worldsheet analysis of a topologically twisted string theory known as the B-model. The holomorphic anomaly equation gives a recursive way to calculate the B-model free energy as a genus expansion of the string amplitudes starting from the tree level amplitudes together with appropriate boundary conditions at certain points of the topological string moduli space. From the previous section, we know that the $\mathcal{N} = 2$ gauge theory in $\mathbb{R}^4$ (and some of its generalization, for instance a ALE space) can be identified as the double scaling limit

[4] Recall the Coulomb parameter a is nothing but the expectation value of the Cartan generator of the $SU(2)$, which is identified to be the $\mathbb{P}^1_f$ in our type IIA string theory construction.

(2.28) of a string theory on some internal Calabi-Yau 3-fold via geometrical engineering; therefore, we may utilize the holomorphic anomaly equation to get a global description of the prepotential (2.5) over the whole moduli space $\mathcal{M}$.

2.4.1 Topological Twist

A superconformal $\mathcal{N}=2$ (string) theory has four supercharges and two $U(1)$ currents, which can be separated into the left moving part $(G^{\pm}, J)$, and the right moving part $(\overline{G}^{\pm}, \overline{J})$, where the $\pm$ sign over G's indicate their $U(1)$ charge with respect to the corresponding $U(1)$ current. Taking the linear combination [5]

$$Q = G^{+} + \overline{G}^{+}, \quad \overline{Q} = G^{-} + \overline{G}^{-}, \tag{2.29}$$

the $\mathcal{N}=2$ supersymmetry algebra gives:

$$\begin{aligned} Q^2 &= \overline{Q}^2 = 0, \\ \{Q, \overline{Q}\} &= 2(H_L + H_R) = 2H, \\ [Q, H] &= [\overline{Q}, H] = 0, \end{aligned} \tag{2.30}$$

where (H_L, H_R) denote the left/right-moving hamiltonian on the worldsheet. Equipped with these supercharges, the ground state of the theory is spanned by the fields in their cohomology. Denote by $\{\phi_i\}$, called chiral fields, a basis of the Q cohomology, they satisfy

$$[Q, \phi] = 0, \quad \phi \sim \phi + [Q, \Lambda], \tag{2.31}$$

and a similar equation for the basis $\{\overline{\phi_i}\}$ of $\overline{Q}$ cohomology, called anti-chiral fields. As the first component of a superfield, the chiral fields can perturb the super string action $S \to S + \delta S(t^i, \overline{t}^{\overline{i}})$ with the term:

$$\begin{aligned} \delta S(t^i, \overline{t}^{\overline{i}}) &= t^i \int d^2z d^2\theta^+ \phi_i + \overline{t}^{\overline{i}} \int d^2z d^2\theta^- \overline{\phi}_i \\ &= t^i \int d^2z \{G^-, [\overline{G}^-, \phi_i]\} + \overline{t}^{\overline{i}} \int d^2z \{G^+, [\overline{G}^+, \overline{\phi}_i]\}, \end{aligned} \tag{2.32}$$

where $(t^i, \overline{t}^{\overline{i}})$ are complex parameters that describe the deformation of the vacuum configuration of the string theory, i.e. they form a local coordinates of the string theory moduli space $\mathcal{M}$. [6]

---

[5] There are two different choice of linear combinations possible here $Q = G^+ + \overline{G}^+$ and $Q = G^+ + \overline{G}^-$. They are related by the choice of sign $(\overline{J} \to -\overline{J})$, and hence as far as the cohomology is concerned, they are equivalent. However, the ground state operators defined w.r.t. the two choices are not equivalent.

[6] For a unitary theory, one requires that $\overline{t}^{\overline{i}} = (t^i)^*$. However, we won't impose such condition in the following discussion.

However, with arbitrary string world sheet Σ_g of genus g, there is a problem: as seen from the 2-dim supersymmetry algebra (2.30), the anti-commutator of two global fermionic transformations generates a bosonic translation, i.e. a global holomorphic vector field on the worldsheet Σ_g, which can exist only if Σ_g has genus zero or one. The resolution is provided by topological twist [84, 85], which states that one may twist the worldsheet Lorentz group by the additional $U(1)$ symmetry so that Q has charge zero with respect to the new (twisted) Lorentz group. Then Q becomes a scalar field and its cohomology is now regarded as a BRST cohomology in the twisted theory that couples to the worldsheet gravity without further difficulties.

2.4.2 Holomorphic Anomaly Equation

In the topologically twisted theory, one expects the genus g string free energy $\mathcal{G}^{(g)}$ should be a holomorphic function on the genus g string moduli space $\mathcal{M}_g$ only, since an variation along an anti-holomorphic direction $\bar{t}^{i}$ corresponds to an insertion of the BRST exact operator $\{G^+, [\overline{G}^+, \overline{\phi}_i]\}$. It turns out the above argument is too naive. Consider the following anti-holomorphic derivative $\bar{\partial}_{\bar{i}}\mathcal{G}^{(g)}$ of the genus g free energy $\mathcal{G}^{(g)}$ for $g \geq 2$:

$$\begin{aligned}
\frac{\partial}{\partial \bar{t}^{\bar{i}}}\mathcal{G}^{(g)} &= \int_{\mathcal{M}_g}[dm]\int d^2z \left\langle \oint_{C_z} G^+ \oint_{C'_z} \overline{G}^+ \overline{\phi}_{\bar{i}}(z) \prod_{a,\bar{a}=1}^{3g-3}\int \mu_a G^- \int \overline{\mu}_{\bar{a}}\overline{G}^- \right\rangle_{\Sigma_g} \\
&= \int_{\mathcal{M}_g}[dm]\sum_{b,\bar{b}=1}^{3g-3}\left\langle \int \overline{\phi}_{\bar{i}}(z)\int 2\mu_b T \int 2\overline{\mu}_{\bar{b}}\overline{T}\prod_{a\neq b}\int \mu_a G^- \prod_{\bar{a}\neq\bar{b}}\int \overline{\mu}_{\bar{a}}\overline{G}^- \right\rangle_{\Sigma_g} \\
&= \int_{\mathcal{M}_g}[dm]\sum_{b,\bar{b}=1}^{3g-3} 4\frac{\partial^2}{\partial m_b \partial \overline{m}_{\bar{b}}}\left\langle \int \overline{\phi}_{\bar{i}}(z)\prod_{a\neq b}\int \mu_a G^- \prod_{\bar{a}\neq\bar{b}}\int \overline{\mu}_{\bar{a}}\overline{G}^- \right\rangle_{\Sigma_g}, \qquad (2.33)
\end{aligned}$$

where $[dm]$ is the measure of the module space $\mathcal{M}_g$, and $(\mu_a, \overline{\mu}_{\bar{a}})$ denote the $6g-6$ Beltrami differentials inserted at $(w_a, w_{\bar{a}})$ that saturate the complex structure moduli of the genus g worldsheet Σ_g. C_z, C'_z are two contours surrounding the $\overline{\phi}_{\bar{i}}$ insertion point z. When moving these contours around the Riemann surface Σ_g, the integral of OPE's picks up the commutators $\oint_{C_w} G^+ \cdot G^-(w) = 2T(w)$ and $\oint_{C_w} \overline{G}^+ \cdot \overline{G}^-(\overline{w}) = 2\overline{T}(\overline{w})$, where $T(w), \overline{T}(\overline{w})$ are the left/right moving components of the energy-momentum tensor on the worldsheet. Since energy-momentum tensor operator $T(w), \overline{T}(\overline{w})$ generates a variation of the worldsheet metric, the second line is equivalent to the derivatives with respect to the moduli $m, \overline{m}$ of Σ_g.

As seen from (2.33), $\bar{\partial}_{\bar{i}}\mathcal{G}^{(g)}$ is indeed a total derivative, but the integral is non-zero because the boundary of $\mathcal{M}_g$ has $([g/2]+1)$ irreducible components $\mathcal{D}_g^r$ $(r = 0, 1, \ldots, [g/2])$, each corresponds to surfaces $\widehat{\Sigma}_g^r$ with a singular node (Fig. 2.2). One can argue that insertion of $\overline{\phi}_{\bar{i}}(z)$ contributes only when it's located on the throat because otherwise one gets an $\overline{\phi}_{\bar{i}}$ inserted on regular punctured Riemann surface $\Sigma_{g-1}(z_j, z_k)$, which is zero by (2.33), see Fig.

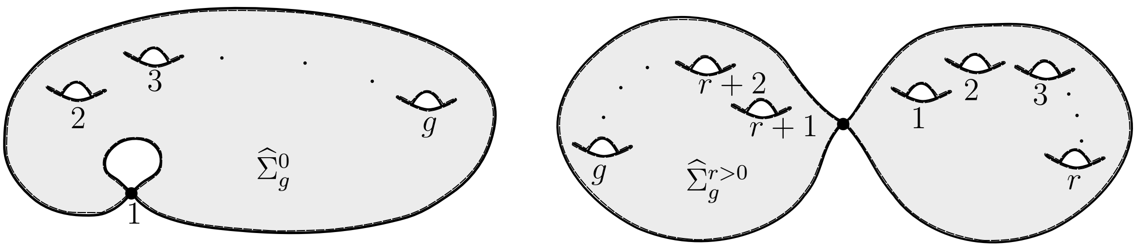

Figure 2.2: The nodal surfaces corresponding to $\mathcal{D}_g^0$ and $\mathcal{D}_g^{r>0}$.

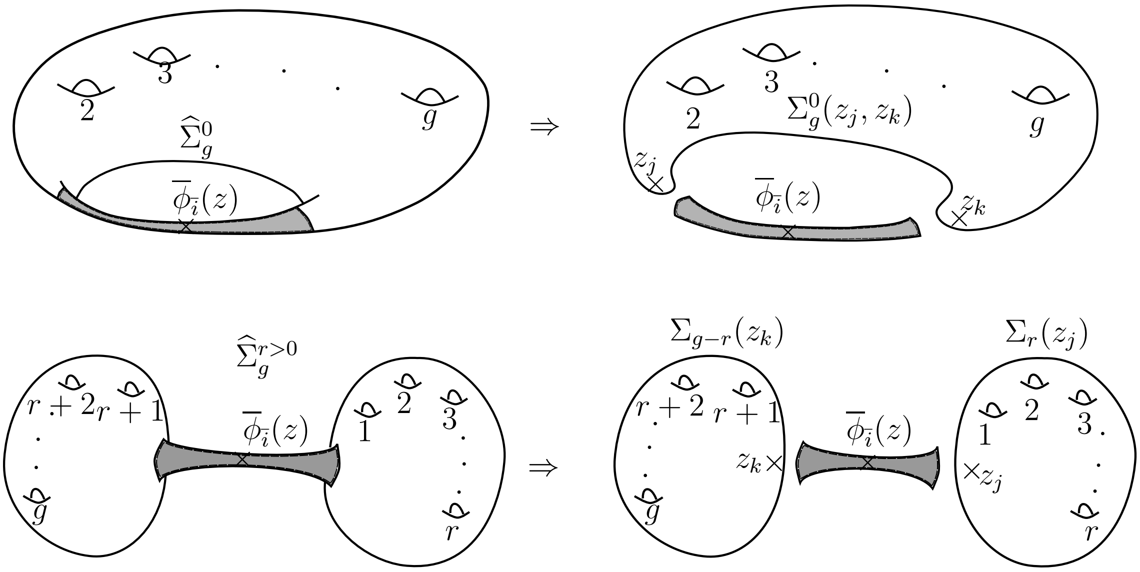

Figure 2.3: The nodal surface $\widehat{\Sigma}_g^r$ is conformally equivalent to regular surfaces with an infinitely long throat, which connects the two punctures (z_j, z_k) arising from removal of the node of $\widehat{\Sigma}_g^r$.

2.3. Therefore, the boundary term in (2.33) can be calculated as the three point function $\overline{C}_{\bar{i}}^{jk}$ from the throat area glued with the lower genus free energy from the punctured surface(s). We obtain the following holomorphic anomaly equation:

$$\bar{\partial}_{\bar{i}}\mathcal{G}^g = \frac{1}{2}\overline{C}_{\bar{i}}^{jk}\Big(D_jD_k\mathcal{G}^{g-1} + \sum_{r=1}^{g-1} D_j\mathcal{G}^r D_k\mathcal{G}^{g-r}\Big), \tag{2.34}$$

where D_j is the covariant derivative on $\mathcal{M}$. Supplemented with appropriate boundary condition at the singular point of the moduli space $\mathcal{M}$ that fixes the holomorphic part of $\mathcal{G}^g$, the holomorphic anomaly equation provides a systematic way to evaluate the string free energy $\mathcal{G}^g$ recursively.

2.4.3 Extended Holomorphic Anomaly Equation

For a generic Ω background, despite some success in refined topological vertex formalism [45] as a generalization of a weighted stable map counting from string worldsheet to a toric Calabi-Yau 3-fold, its string theory picture remains unclear. Therefore, a priori, we don't expect the existence of a simple recursive formula that generates the Nekrasov's partition function with generic ϵ_1, ϵ_2. However, it was observed that such an generalization called extended holomorphic anomaly equation does exist [54].

To set the stage, let's reparametrize ϵ_1, ϵ_2 as

$$\epsilon_1 = \beta^{1/2}\lambda, \quad \epsilon_2 = -\beta^{-1/2}\lambda.$$

Then, the Nekrasov's partition function (2.16) can be organized in the following form:

$$\begin{aligned} \mathcal{F}^{Nek}(a;\epsilon_1,\epsilon_2;\mathfrak{q}) &\equiv \log\, Z^{Nek}(a;\epsilon_1,\epsilon_2;\mathfrak{q}) \\ &= \sum_{i,j=0}^{\infty}(\epsilon_1+\epsilon_2)^i(\epsilon_1\epsilon_2)^{j-1}\mathcal{F}^{(\frac{i}{2},j)}(a;\mathfrak{q}) \\ &= \sum_{i,j=0}^{\infty}(-1)^j(\beta^{1/2}-\beta^{-1/2})^i\mathcal{F}^{(\frac{i}{2},j)}(a;\mathfrak{q})\,\lambda^{i+2j-2}, \end{aligned} \tag{2.35}$$

whose genus zero part $\mathcal{F}^{(0,0)}$ is the Seiberg-Witten prepotential (2.5). In the following, we shall suppress the instanton counting parameter $\mathfrak{q}$ to simplify the notation. While in this case all the $\mathcal{F}^{(\frac{i}{2},j)}$ with odd i actually vanishes, we may re-organize the free energy as

$$\mathcal{F}^{Nek}(a;\epsilon_1,\epsilon_2) = \sum_{g_1,g_2=0}^{\infty} \lambda^{2(g_1+g_2)-2}\,\mathcal{G}^{(g_1,g_2)}(a;\beta), \tag{2.36}$$

with λ now being regarded as a genus counting parameter and g_1+g_2 as the genus of the string worldsheet. [7] Here, each of the topological amplitude $\mathcal{G}^{(g_1,g_2)}(a;\beta)$ should be regarded as a function over the whole moduli space parametrized by the global coordinate u via the explicit u dependence of the period $a(u)$ (2.12). It turns out that $\mathcal{G}^{(g_1,g_2)}(u,\bar{u};\beta)$ for $g_1+g_2\geq 2$ satisfies a extended holomorphic anomaly equation:

$$\bar{\partial}_{\bar{u}}\,\mathcal{G}^{(g_1,g_2)} = \frac{1}{2}\,\overline{C}_{\bar{u}}^{uu}\Big(\mathcal{G}_{uu}^{(g_1,g_2-1)} + \sum_{\substack{r_1+r_2>0,\\ (r_1,r_2)\neq(g_1,g_2)}} \mathcal{G}_u^{(r_1,r_2)}\,\mathcal{G}_u^{(g_1-r_1,g_2-r_2)}\Big), \tag{2.37}$$

where $\mathcal{G}_u^{(g_1,g_2)}$ stands for the covariant derivative of $\mathcal{G}^{(g_1,g_2)}$ with respect to the global coordinate u and $\overline{C}_{\bar{u}}^{uu}$ is the Levi-Civita connection of the Weil-Petersson metric $g(u,\bar{u})\approx \mathrm{Im}\tau(u)$

[7] This picture arises from the type IIA construction reviewed in the previous section, where the prepotential is essentially counting the genus 0 worldsheet instantons. The higher genus corrections are argued to come from the R^2F^{2g-2} coupling, where R is the spacetime curvature and F is the graviphoton field strength [12].

(2.3):

$$g_{u\bar{u}} = \frac{i}{2}\left(\frac{da}{du}\frac{d\overline{a}_D}{d\bar{u}} - \frac{da_D}{du}\frac{d\overline{a}}{d\bar{u}}\right). \tag{2.38}$$

Together with appropriate boundary condition at various singular point of the moduli space $\mathcal{M}$ that fixes the holomorphic ambiguity of $\mathcal{G}^{(g_1,g_2)}$ at each order, the extended holomorphic equation (2.37) yields the desired Nekrasov's free energy $\mathcal{F}^{Nek}(u;\epsilon_1,\epsilon_2)$ for a generic Ω-background over the whole moduli space $\mathcal{M}$ iteratively.

Chapter 3

Holomorphic Anomaly in Gauge Theory in ALE Space

3.1 Introduction

Recently, there has been a renewed interest in gauge theory on Asymptotically Locally Euclidean (ALE) space. This has been mainly triggered by the findings of [10] (see also [16, 15, 86, 46]), hinting towards an extension of the proposed correspondence between instanton partition functions of $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions and conformal blocks of Liouville theory [4], to a correspondence between instanton partition functions of gauge theories on ALE space and conformal blocks in super Liouville theory.

The instanton partition function of four-dimensional $\mathcal{N}=2$ $U(N)$ gauge theory with and without matter can be efficiently calculated using localization on the moduli space of instantons by making use of the so-called Ω-deformation acting on the space-time $\mathbb{R}^4 \cong \mathbb{C}^2$ via the rotation

$$\Omega:\ (z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)\,, \tag{3.1}$$

as put forward in [71] (extending the earlier works [65, 57, 64]). Subsequently, this localization technique has been applied to gauge instantons on ALE space [34]. Their approach builds on the construction of self-dual gauge connections one ALE space of [56], utilizing that ALE spaces can be obtained from the minimal resolution of orbifolds of type $\mathbb{C}^2/\Gamma$ with Γ a discrete Kleinian subgroup of $SU(2)$. Due to the topological nature of the quantities under consideration it is sufficient to stick to the orbifolds, though one may also directly consider the resolved geometries (*c.f.*, [15]).

As shown in [54, 55], Ω-deformed gauge theory partition functions on $\mathbb{C}^2$ can be reproduced under the reparameterization

$$\epsilon_1 = \sqrt{\beta}\,\lambda\,, \quad \epsilon_2 = -\frac{1}{\sqrt{\beta}}\,\lambda\,, \tag{3.2}$$

in the limit $\lambda \to 0$ with β fixed via special geometry and the holomorphic anomaly equation of [11, 12]. The essential effect of the Ω-deformation being the change of boundary conditions coming from the dyon/monopole point in moduli space. The leading order in λ of the free energy, determined via special geometry, is just the prepotential of the gauge theory. Essentially, this is the celebrated Seiberg-Witten solution of $\mathcal{N} = 2$ gauge theory [78, 79]. Higher powers in λ, recursively determined via the holomorphic anomaly equation, supposedly correspond to a 1-parameter family of gravitational corrections.

One may ask if the "B-model" technique of resorting to special geometry and the holomorphic anomaly equation can be applied as well to the instanton partition functions of gauge theory on ALE space. Due to the linkage between modularity and the holomorphic anomaly [1], it is not obvious that this is the case, since already in the unrefined case the modular properties of these partition functions are (to the best of our knowledge) somewhat unclear, *c.f.*, [82, 14].

As one of our main results we will find that the partition functions of pure $SU(2) \subset U(2)$ gauge theory on the simplest ALE space, namely A_1, can be reproduced via the standard B-model approach, albeit supplemented with new boundary conditions. One may take this result as evidence that the applicability of the B-model approach is more general than anticipated. In particular, this indicates that the partition functions of the theories on ALE space still possess modular properties. We will also find that the gauge theory on A_1 exhibits a new feature. Namely, the symmetry between the monopole and dyon point may be broken under coupling to gravity.

In the realm of geometric engineering, the partition function of the undeformed $SU(2)$ gauge theory on $\mathbb{C}^2$ arises as the low energy effective limit of the A-model topological string partition function on local $\mathbb{P}^1 \times \mathbb{P}^1$ [51]. For the string theory interpretation of the Ω-deformed gauge theory partition function one usually invokes the M-theory lift. While in the undeformed case the partition function counts BPS states of spinning M2-branes with respect to their left spin under the $SU(2)_L \times SU(2)_R$ rotation group, the Ω-deformation corresponds to a refinement in the sense of counting with respect to left and right spin [41, 45]. In particular, the so defined refined topological string partition function equals for geometric engineering geometries the partition function of the corresponding Ω-deformed gauge theory in five dimensions compactified on a circle [41, 45].

Not very surprisingly, the B-model approach to the Ω-deformation can be applied to the refined topological string partition function, with similar boundary conditions as for the Ω-deformed gauge theory. In particular, the boundary conditions are determined by the $c = 1$ string at $R = \beta$ [54]. This is perhaps the strongest hint so far available in favor of the existence of a worldsheet formulation of refinement (for some proposals, see [5, 69]).

Since it is natural to lift the four-dimensional gauge theory partition function on ALE space to five dimensions, in similar fashion as the gauge theory on $\mathbb{C}^2$, it is tempting to speculate about a (refined) topological string interpretation with ALE space-time. In this note we will find some hints towards this direction. Namely, we will find that the natural five-dimensional lift of the Ω-deformed gauge theory on A_1 can be reproduced via the B-model techniques applied to the mirror geometry of local $\mathbb{P}^1 \times \mathbb{P}^1$. This suggests that it makes sense to take the five-dimensional gauge theory as a definition of a (refined) topological string on $A_1 \times$(local $\mathbb{P}^1 \times \mathbb{P}^1$).

The outline is as follows. In the next section we will briefly recall some basics about instanton counting via localization in four dimensions and its generalization to ALE space. The partition function obtained in this manner will be the benchmark to fix the holomorphic ambiguities in the B-model approach, which we will follow in section 3.3 for $SU(2) \subset U(2)$ gauge theory on A_1. The lift of the instanton calculus via localization to five dimensions will be discussed in section 3.5. In particular, we will present the natural five dimensional partition functions in the $U(1)$ case, supposedly corresponding to the partition functions of a refined topological string on the resolved conifold with A_1 space-time, and for $SU(2)$, which is supposed to correspond to a refined topological string on $A_1\times$(local $\mathbb{P}^1 \times \mathbb{P}^1$). The partition functions will be confirmed in section 3.5 via the B-model approach applied to the mirror geometry of local $\mathbb{P}^1 \times \mathbb{P}^1$. We present some concluding words in section 3.6.

3.2 Instanton counting via localization

In this section we briefly recall the calculation of the partition function of $\mathcal{N} = 2$ supersymmetric gauge theory on $\mathbb{C}^2$ via localization and its generalization to A_1 space-time, mainly following [71, 14, 16], to there we refer for a more detailed treatment. Though the localization calculation in the general case of $U(N)$ with matter is clear (in the orbifold formalism also with general A_n space-time), for brevity we stick to pure $U(2)$ on A_1, as this is the theory of main concern in this note.

3.2.1 Generalities

According to [71], the instanton partition function of $U(2)$ gauge theory on $\mathbb{R}^4 \simeq \mathbb{C}^2$ can be calculated via localization with respect to the $\mathbb{T}^2_{a_1,a_2}\times\mathbb{T}^2_{\epsilon_1,\epsilon_2}$ group action on the (compactified) moduli space $\mathcal{M}_k$ of k-instantons of $U(2)$ gauge theory. Here $\mathbb{T}^2_{a_1,a_2}$ stands for the maximal torus of the $U(2)$ gauge group with generators a_i being the Coulomb parameters, while $\mathbb{T}^2_{\epsilon_1,\epsilon_2}$ refers to the maximal torus of the $SO(4)$ space-time rotation group with generators ϵ_i as in (3.1). Localization reduces the calculation of the partition function to a weighted summation over the fixed points under the group action $\mathbb{T}^2_{a_1,a_2} \times \mathbb{T}^2_{\epsilon_1,\epsilon_2}$. Denoting the set of fixed-points in $\mathcal{M}_k$ as $\Sigma_k \subset \mathcal{M}_k$ and the weights as $\omega_k(f;\vec{a};\epsilon_1,\epsilon_2)$, the partition function reads

$$Z^{\mathbb{C}^2}_{inst}(\vec{a};\epsilon_1,\epsilon_2;\mathfrak{q}) = \sum_k \sum_{f\in\Sigma_k} \omega_k[f](\vec{a};\epsilon_1,\epsilon_2)\, \mathfrak{q}^k\,, \tag{3.3}$$

Here, $\mathfrak{q}$ stands for the instanton counting parameter, which we will later identify with the dynamical scale of the gauge theory as $\mathfrak{q} = \Lambda^4$. The set of fixed-points Σ_k can be conveniently encoded in pairs of Young diagrams $\vec{Y} = (Y_1, Y_2)$ with $|Y_1| + |Y_2| = k$ [71]. Furthermore, utilizing the parameterization of the fixed-points via Young diagrams, the weights can be expressed simply as [71, 20, 66]

$$\omega_k[\vec{Y}] = \prod_{n,m=1}^{2} \frac{1}{\prod_{s\in Y_n} E_s(\epsilon_1+\epsilon_2-a_{nm};Y_n,Y_m;-\epsilon_1,-\epsilon_2)\prod_{t\in Y_m} E_t(a_{nm};Y_n,Y_m;\epsilon_1,\epsilon_2)}\,, \tag{3.4}$$

with $a_{nm} := a_n - a_m$ and where

$$E_s(a,Y_n,Y_m;\epsilon_1,\epsilon_2) = a - \epsilon_1 L_{Y_n}(s) + \epsilon_2(A_{Y_m}(s)+1)\,.$$

Here, $L_Y(s)$ and $A_Y(s)$ denote the usual leg-, respectively, arm-length functions for the box $s=(i,j)$ of the partition Y. Note that the partition function $Z^{\mathbb{C}^2}_{inst}$ has to be supplemented by hand by a proper perturbative part $Z^{\mathbb{C}^2}_{pert}$, *i.e.*,

$$Z^{\mathbb{C}^2}(\vec{a};\epsilon_1,\epsilon_2;\mathfrak{q}) = Z^{\mathbb{C}^2}_{pert}(\vec{a};\epsilon_1,\epsilon_2)\, Z^{\mathbb{C}^2}_{inst}(\vec{a};\epsilon_1,\epsilon_2;\mathfrak{q})\,.$$

Explicit closed formulas for $Z^{\mathbb{C}^2}_{pert}$ can be found in [70, 68]. Note also that later we will mainly restrict to $SU(2)\subset U(2)$ via setting $a=a_1=-a_2$.

3.2.2 A_1 via orbifold

As already mentioned in the introduction, A_1 space-time can be obtained as the minimal resolution of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, where the orbifold, whose action we denote as $\mathcal{I}$, acts on $(z_1,z_2)\in\mathbb{C}^2$ as

$$\mathcal{I}:(z_1,z_2)\to(e^{\pi i}z_1,e^{-\pi i}z_2)\,. \tag{3.5}$$

In particular, the fixed points in the instanton moduli space under the action of Ω are automatically invariant under the orbifold action. Due to the topological nature of the instanton partition function, one can utilize the orbifold projection to infer the instanton partition function on A_1 space-time. Explicitly, this means to restrict the summation in (3.3) to fixed points which are invariant under $\mathcal{I}$, combined with a projection of the weights. For that, note that $\mathcal{I}$ shifts the ϵ_i parameters of the Ω deformation (*c.f.*, (3.1)), as

$$\epsilon_1\to\epsilon_1+\pi\,,\quad \epsilon_2\to\epsilon_2-\pi\,. \tag{3.6}$$

Because of the orbifold singularity, in general the $U(2)$ gauge field can have non-trivial holonomy when circling around the non-shrinkable cycle. The holonomy can be labeled by a pair of charges (q_1,q_2) associated with the two Cartan generators of the $U(2)$. Therefore, the Coulomb parameters a_i transform under $\mathcal{I}$ as

$$a_i\to a_i+\pi q_i\,. \tag{3.7}$$

There are four possible holonomy sectors: $(0,0)$, $(1,1)$, those partition functions can be reproduced from the Neveau-Schwarz sectors in the associated super Liouville CFT [10], and $(0,1)$, $(1,0)$ which are related to the Rammond sector of the CFT [46]. All other fields, being covariantly constant with respect to the gauge field, acquire a phase shift induced from the twisted boundary condition of the gauge field when going around the non-trivial cycle.

As discussed in detail in [34], the projection on the fixed-points can be implemented by associating to each box (i,j) in a Young diagram Y_n a charge $q_n + i - j$ and counting the total number of boxes with zero charge, k_0, and of charge one, k_1, for each pair of Young diagrams (corresponding to a fixed-point). The topological classification of the instanton solution on $\mathbb{C}^2/\mathbb{Z}_2$ according to its first and second Chern class [56] then yields selection rules on the allowed values of (k_0, k_1) for given holonomy. Explicitly, we take as in [46] for the four different sectors of $U(2)$ on $\mathbb{C}^2/\mathbb{Z}_2$:

$$(0,0),(0,1),(1,0) : k_0 - k_1 = 0\,, \quad (1,1) : k_0 - k_1 = 1\,.$$

Note that it may seem that this corresponds only to a small subset of possible instanton solutions. However, we will see in the next section that this subset in fact contains all non-trivial information.

It remains to project the weights. The projection is simply given by keeping only the factors in (3.4) which are even under $\mathcal{I}$, acting via (3.6) and (3.7) on each factor. Following this prescription, it is straight-forward to calculate the partition function $Z^{(q_1,q_2)}$ for each holonomy sector.

We reparameterize the partition functions in the variables (3.2), restrict to $SU(2) \subset U(2)$ and expand the partition functions in λ, keeping β fixed. The expansion goes into even powers of λ only, *i.e.*,

$$\mathcal{F}(a;\epsilon_1,\epsilon_2;\mathfrak{q}) := \log Z(a;\epsilon_1,\epsilon_2;\mathfrak{q}) \sim \sum_{n=0}^{\infty} \mathcal{G}^{(n)}(a;\beta;\mathfrak{q})\,\lambda^{2n-2}\,. \tag{3.8}$$

For the applicability of the usual Seiberg-Witten approach, we would like to have that

$$\mathcal{G}^{(0)} \sim \mathcal{F}^{(0)}\,, \tag{3.9}$$

(up to some rescaling of λ) where $\mathcal{F}^{(0)}$ is the ordinary Seiberg-Witten prepotential of $SU(2)$ gauge theory on $\mathbb{C}^2$. In physical language, this means that we would like to see a difference only under coupling to gravity, which is responsible for the higher order terms in λ in (3.8). Extracting explicitly the prepotential $\mathcal{G}^{(0)}$ for each sector $Z^{(q_1,q_2)}$, however, shows that this is generally not the case. Rather, one has to take the combinations of holonomy sectors

$$\begin{aligned} Z_2^+ &:= Z^{(0,1)} + Z^{(1,0)}\,,\\ Z_2^- &:= Z^{(0,0)} + Z^{(1,1)}\,, \end{aligned} \tag{3.10}$$

in order that (3.9) holds (strictly speaking, taking the combination leading to Z_2^+ is not necessary since $Z^{(0,1)} = Z^{(1,0)}$). That is, one should combine sectors with same first Chern class. Note that while Z_2^+ obtains contributions only from regular instantons, Z_2^- includes contributions from regular and fractional instantons.

Hence, $SU(2) \subset U(2)$ gauge theory on A_1 possesses two distinguished sectors under coupling to gravity, with partition functions as defined via (3.10). (Sometimes we will also refer to Z_2^+ as coming from the even sector and Z_2^- as coming from the odd sector.) These are the two combinations of sectors which are well-behaved (in the sense of having modular properties), and on which we therefore focus on in this work.

3.2.3 A_1 via blowup

While the orbifold approach has the advantage that it is rather simple, efficient to explicitly compute and straight-forward to generalize, it has one obvious drawback. Namely, it is not immediately clear how to project the perturbative part of the partition function. However, calculating the partition function instead via a more algebraic geometrical approach one can in fact obtain a prediction. The price to pay is that this approach is somewhat more complicated and less efficient to explicitly compute. Therefore, we will mainly use the orbifold approach for explicit calculations and use the algebraic approach only to gain some additional insight. The construction goes (roughly) as follows [16] (see also [19] and references therein).

The compactification of the instanton moduli space of $U(2)$ gauge theory on the minimal resolution $\mathcal{O}(-2) \to \mathbb{P}^1$ of the A_1 singularity can be described in terms of the moduli space $\mathcal{M}(2, k, n)$ of a rank two framed torsion free coherent sheave $\mathcal{E}$ on a stacky compactification of $\mathcal{O}(-2) \to \mathbb{P}^1$. The moduli space is characterized by the first Chern class $c_1(\mathcal{E}) = kE$ (with E the exceptional divisor resolving the singularity) and the discriminant $\Delta(\mathcal{E}) = c_2(\mathcal{E}) - \frac{1}{4}c_1^2(\mathcal{E}) = n$. It is important to keep in mind that due to the stacky compactification k can be integer and half-integer (*c.f.*, [19]). As usual, the instanton partition function is obtained via localization with respect to a $\mathbb{T}^2_{a_1,a_2} \times \mathbb{T}^2_{\epsilon_1,\epsilon_2}$ action on the moduli space, as a weighted sum over the fixed-points. The fixed-points under the torus action are given by (twisted) ideal sheaves characterized by a pair $\vec{k} = (k_1, k_2)$ with $k = k_1 + k_2$ and a pair of Young diagrams (Y_1, Y_2) with $|Y_1| + |Y_2| = n + \frac{1}{2}(k_1 - k_2)^2$. Though it is not hard to infer the weights, they are too lengthy to explicitly recall here. The details of the localization calculation are anyway not of our main concern in this note. Therefore, we just state the final result for the instanton partition function, which takes the form [16]

$$Z^{A_1}_{inst} = \sum_k \mathfrak{q}^{|k|^2/2} Z^{(k)}(\vec{a}; \epsilon_1, \epsilon_2; \mathfrak{q}) \,, \tag{3.11}$$

with

$$\begin{aligned} Z^{(k)}(\vec{a}; \epsilon_1; \epsilon_2; \mathfrak{q}) = \sum_{k_1+k_2=k} \mathfrak{q}^{|k_1-k_2|^2/2} \prod_{\alpha,\beta=1}^{2} \ell(\vec{a}; k_\alpha, k_\beta; \epsilon_1, \epsilon_2) \\ \times Z^{\mathbb{C}^2}_{inst}(\vec{a} - 2\epsilon_1\vec{k}; 2\epsilon_1, \epsilon_2 - \epsilon_1; \mathfrak{q}) Z^{\mathbb{C}^2}_{inst}(\vec{a} - 2\epsilon_2\vec{k}; \epsilon_2 - \epsilon_1, 2\epsilon_2; \mathfrak{q}) \,, \end{aligned} \tag{3.12}$$

and

$$\ell(\vec{a}; k_\alpha, k_\beta; \epsilon_1, \epsilon_2) = \begin{cases} \prod_{\sigma_+} \frac{(-1)}{i\epsilon_1 + j\epsilon_2} & k_{\alpha\beta} > 0 \\ \prod_{\sigma_-} \frac{1}{(i+1)\epsilon_1 + (j+1)\epsilon_2} & k_{\alpha\beta} < 0 \\ 1 & k_{\alpha\beta} = 0 \end{cases} ,$$

where we defined $k_{\alpha\beta} = k_\alpha - k_\beta$ and $Z_{inst}^{\mathbb{C}^2}$ is as defined via (3.3). The selection rules on the products are $\sigma_+ := \{i, j \geq 0 : i + j \leq 2(k_{\alpha\beta} - 1) \wedge i + j - 2k_{\alpha\beta} = 0 \mod 2\}$ and $\sigma_- := \{i, j \geq 0 : i + j \leq -2(k_{\alpha\beta} + 1) \wedge i + j - 2k_{\alpha\beta} = 0 \mod 2\}$.

Some remarks are in order. The so defined partition function (3.11) is a summation over all topological sectors. In order to compare to the orbifold partition functions of the previous section, one has to identify the corresponding values of k. It is not hard to infer that

$$Z^{(0)}(\vec{a}; \epsilon_1; \epsilon_2; \mathfrak{q}) = Z_2^- , \quad Z^{(1/2)}(\vec{a}; \epsilon_1; \epsilon_2; \mathfrak{q}) = \mathfrak{q}\, Z_2^+ , \tag{3.13}$$

with $Z_2^\pm$ as defined in (3.10). This is as expected since fractional k corresponds to non-trivial holonomy (*c.f.*, [19]). Furthermore, one can in fact observe that the sum over the integer and half-integer sectors of (3.12) factorize separately, *i.e.*,

$$Z_{inst}^{A_1} = \vartheta_3(0; \mathfrak{q}^{1/2})\, Z_2^- + \mathfrak{q}^{-1/8} \vartheta_2(0; \mathfrak{q}^{1/2})\, Z_2^+ ,$$

where $\vartheta_n(z; q)$ denote the standard auxiliary theta functions. A related observation has been made for the $\mathcal{N} = 2^*$ theory in [16], and one may also obtain the above factorization via taking an appropriate limit thereof. This explains the statement of the previous section that only the topological sectors leading to Z_2^+ and Z_2^- yield non-trivial information and are thereof of relevance to us.

Finally, we also briefly recall from [16] that the form of the partition function (3.11) can be used to predict the perturbative contribution. The main idea behind is that if one requires that (3.12) is expressed in a similar fashion in terms of the full partition function $Z^{\mathbb{C}^2}$ (*i.e.*, that a similar blowup equation as in [66] holds), the perturbative contribution is essentially fixed and can be extracted.

3.3 B-model approach to $SU(2)$ gauge theory in 4d

In this section we are going to reproduce the partition functions of the gauge theory on A_1 space of the previous section using special geometry and the holomorphic anomaly equation. In particular, we will infer the behavior at the dyon/monopole point in moduli space. Also, this approach definitely fixes the perturbative contribution, for which we will find closed and simple expressions.

3.3.1 Generalities

We consider $\mathcal{N} = 2$ $SU(2)$ gauge theory in four dimensions and denote by u a global coordinate on the quantum moduli space of vacua $\mathcal{M}$, which can be identified with the base space of a family of complex curves $\mathcal{C}_u$. For instance the hyperelliptic curve,

$$\mathcal{C}_u: \ y^2 = (x^2 - \Lambda^4)(x - u)\,,$$

where Λ denotes the dynamical scale of the gauge theory. The quantum moduli space has three special points. Namely, the weak coupling region $u = \infty$, where the gauge boson (vector-multiplet) become massless, the strongly coupled monopole point $u = \Lambda^2$ with a massless monopole (hyper-multiplet) and the strongly coupled dyon point $u = -\Lambda^2$ with a massless dyon (hyper-multiplet). Note that the monopole and dyon points are related by a $\mathbb{Z}_2$ symmetry. For latter reference, we parameterize the strongly coupled singular points via the "discriminant" $\Delta = 0$ with

$$\Delta = \Delta_+ \Delta_-\,, \tag{3.14}$$

and

$$\Delta_\pm = u \pm \Lambda^2\,. \tag{3.15}$$

The family of curves is equipped with a meromorphic one-form λ_{SW}, such that

$$a = \oint_A \lambda_{SW}\,, \quad a_D = \oint_{A_D} \lambda_{SW}\,,$$

and

$$a_D = \frac{\partial \mathcal{F}^{(0)}}{\partial a}\,,$$

for appropriately chosen 1-cycles A and A_D, and under elimination of u, via the so-called mirror map $u(a)$ obtainable by inverting the period $a(u)$. In order to obtain the periods a and a_D in different corners of moduli space, it is convenient to resort to the Picard-Fuchs equation,

$$\mathcal{L}\,\omega(u) = 0\,,$$

satisfied by all the periods of λ_{SW}. Taking a as local (flat) coordinate around $u \to \infty$, the differential operator $\mathcal{L}$ takes the form

$$\mathcal{L} = \partial_a \frac{1}{C_{aaa}} \partial_a^2\,,$$

where $C_{aaa} := \partial_a^3 \mathcal{F}^{(0)}$ is referred to as Yukawa coupling.

We now apply the holomorphic anomaly equations of [11, 12]. That is, we consider the amplitudes $\mathcal{G}^{(2n-2)}(a;\beta;\mathfrak{q})$ (defined in (3.8)) to be given by the holomorphic limit $\bar{a} \to \infty$ of non-holomorphic, but globally defined objects $\mathcal{G}^{(n)}(u,\bar{u})$ over $\mathcal{M}$.

For $g > 1$, the $\mathcal{G}^{(g)}(u,\bar{u})$ satisfy the holomorphic anomaly equation

$$\bar{\partial}_{\bar{u}} \mathcal{G}^{(g)} = \frac{1}{2} \sum_{r=1}^{g-1} \bar{C}_{\bar{u}}^{uu} \mathcal{G}_u^{(g-r)} \mathcal{G}_u^{(r)} + \frac{1}{2} \bar{C}_{\bar{u}}^{uu} \mathcal{G}_{uu}^{(g-1)}\,, \tag{3.16}$$

where $\mathcal{G}^{(g)}_{uu} = D_u \mathcal{G}^{(g)}_u = D^2_u \mathcal{G}^{(g)}$ and D_u is the covariant derivative over $\mathcal{M}$. In particular, the connection takes the form

$$\lim_{\bar{a}\to\infty} \Gamma^u_{uu} = \partial_u \log \frac{\partial a(u)}{\partial u} \,.$$

As usual, indices are raised and lowered using the Weil-Petersson metric. The one-loop amplitude satisfies

$$\bar{\partial}_{\bar{u}} \partial_u \mathcal{G}^{(1)} = \frac{1}{2} \bar{C}^{uu}_{\bar{u}} C_{uuu} \,. \tag{3.17}$$

There are several techniques on the market to solve the holomorphic anomaly equation (3.16). Hence, we will not dwell here into the details on how to actually solve (3.16). For that, we refer to the (extensive) literature on this topic.

Clearly, (3.16) and (3.17) only capture the non-holomorphic part of the amplitude and hence have to be supplemented by an appropriately chosen holomorphic function. The determination of the holomorphic function is the main difficulty in the holomorphic anomaly approach. Fortunately, the holomorphic ambiguities can be fixed by taking boundary conditions from other points in moduli space into account [43, 44]. In particular, for $SU(2)$ gauge theory in the Ω background, expansion of the amplitudes at the monopole or dyon point is sufficient, since the amplitudes feature at this point in moduli space a gap with a specific singular leading term, yielding enough boundary conditions. In detail, the leading terms are determined by the free energy of the $c = 1$ string at $R = \beta$ [54].

Let us pause for a moment to comment on a related approach in the literature. For the same purpose of reproducing refined partition functions via holomorphic anomaly equations the authors of [42] promote a "generalized" version of the holomorphic anomaly equation. The reason behind lies in their two parameter expansion of the free energy $\mathcal{F}(a; \epsilon_1, \epsilon_2; \mathfrak{q})$ (as defined in (3.8)) , *i.e.*,

$$\mathcal{F}(a; \epsilon_1, \epsilon_2; \mathfrak{q}) = \sum_{n_1,n_2=0}^{\infty} \mathcal{F}^{(n_1,n_2)}(a; \mathfrak{q})(\epsilon_1\epsilon_2)^{n_1-1}(\epsilon_1+\epsilon_2)^{2n_2} \,.$$

However, one can infer that the relation to our 1-parameter expansion discussed above is just a resummation,

$$\mathcal{G}^{(g)}(a; \beta; \mathfrak{q}) = \sum_{(n_1,n_2)\subset\sigma(g,2)} \left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)^{2n_2} \mathcal{F}^{(n_1,n_2)}(a; \mathfrak{q}) \,,$$

where $\sigma(g, 2)$ denotes the set of integer (including zero) partitions of g of length 2. In particular, their "generalized" holomorphic anomaly equation follows from the above relation. Hence, the refinement is generally captured by the ordinary holomorphic anomaly equation equipped with new boundary conditions, as put forward in [54]. For completeness, we also mention that one can also consider the extended holomorphic anomaly equation of [83] in this context in order to capture certain shift degree of freedoms, as also discussed in [54, 55].

3.3.2 Z_2^+

Let us start the discussion of the gauge theory on A_1 space with the sector with partition function Z_2^+. As already stated in section 3.2.2, via explicit expansion of the instanton partition function as in (3.8), one infers that

$$\mathcal{G}^{(0)} = \frac{1}{2}\mathcal{F}^{(0)} ,$$

and hence we can work with the usual special geometry of $SU(2)$ gauge theory (under a rescaling of λ). Furthermore, we observe that the 1-loop sector is reproduced via the usual solution of the 1-loop holomorphic anomaly equation (3.17), given by

$$\mathcal{G}^{(1)} = \frac{1}{2}\log\left(\partial_u a(u)\right) + a_{\mathcal{G}}^{(1)}(u;\beta) , \tag{3.18}$$

with holomorphic ambiguity fixed to

$$a_{\mathcal{G}}^{(1)}(u;\beta) = \frac{1}{48}\left(\beta + \frac{1}{\beta}\right)\log\Delta + \frac{1}{8}\log\Delta .$$

It is interesting to note that this holomorphic ambiguity seems also to be related to the 1-loop partition function of a specific two dimensional gravity model, *c.f.*, [13], similar as is the case for the ordinary Ω-deformed 1-loop amplitude.

For higher genus, we parameterize the holomorphic ambiguities as usual via the functions

$$a_{\mathcal{G}}^{(g>1)}(u;\beta) = u^{3-g}\sum_{n=1}^{2g-2} a_n^{(g)}(\beta)\frac{\Lambda^{4n-4}}{\Delta^n} ,$$

with constants $a_n(\beta)$ to be determined.

Via explicit expansion of the higher $\mathcal{G}^{(g)}$ at the weak coupling and monopole/dyon points, and comparing with the expected results from the localization calculation outlined in section 3.2.2, we are able to fix the holomorphic ambiguities, that is, the $a_n^{(g)}(\beta)$. In particular, we observe that the genus g amplitudes expanded at the monopole/dyon points have the structure

$$\mathcal{G}^{(1)}(a_D;\beta) = \Psi_2^{(1)}(\beta)\,\log(a_D) + \ldots , \quad \mathcal{G}^{(g>1)}(a_D;\beta) = \Psi_2^{(g)}(\beta)\,a_D^{2-2g} + \mathcal{O}(a_D^0) , \tag{3.19}$$

i.e., feature a gap and a distinguished leading singular term parametrized as $\Psi_2^{(g)}$. We find for the first few of the coefficients

$$\begin{aligned}
\Psi_2^{(1)}(\beta) &= \frac{1}{48}\left(\beta + \frac{1}{\beta}\right) + \frac{1}{8} , \\
\Psi_2^{(2)}(\beta) &= -\frac{7 + 180\beta + 10\beta^2 + 180\beta^3 + 7\beta^4}{11520\beta^2} , \\
\Psi_2^{(3)}(\beta) &= \frac{31 + 3150\beta + 49\beta^2 + 3780\beta^3 + 49\beta^4 + 3150\beta^5 + 31\beta^6}{322560\beta^3} , \\
&\;\;\vdots
\end{aligned} \tag{3.20}$$

We can also extract the perturbative contribution to the partition function, which we parameterize as

$$\log Z_2^{pert} \sim 2\,\Phi_2^{(1)}(\beta)\log a + \sum_{n>1} 2\,\Phi_2^{(n)}(\beta)\,\frac{\lambda^{2n-2}}{(2a)^{2n-2}}\,.$$

We find

$$\begin{aligned}
\Phi_2^{(1)}(\beta) &= -\frac{1}{24}\left(\beta + \frac{1}{\beta}\right)\,,\\
\Phi_2^{(2)}(\beta) &= \frac{1+40\beta^2+\beta^4}{1440\beta^2}\,,\\
\Phi_2^{(3)}(\beta) &= -\frac{1+154\beta^2+154\beta^4+\beta^6}{10080\beta^3}\,,\\
\Phi_2^{(4)}(\beta) &= \frac{3+1880\beta^2+1568\beta^4+1880\beta^6+3\beta^8}{60480\beta^4}\,,\\
&\vdots
\end{aligned} \tag{3.21}$$

Note that we can reproduce both the $\Psi_2^{(g)}(\beta)$ and $\Phi_2^{(g)}(\beta)$ via the Schwinger type integrals [1]

$$\begin{aligned}
\frac{1}{2}\int \frac{ds}{s} e^{-s\mu} \frac{\cosh\left(\frac{s(\epsilon_1+\epsilon_2)}{2}\right)}{\sinh(s\epsilon_1)\sinh(s\epsilon_2)} &\sim \cdots + \sum_{n>0} \Psi_2^{(n)}(\beta)\,\frac{\lambda^{2n-2}}{\mu^{2n-2}}\,,\\
\frac{1}{2}\int \frac{ds}{s} e^{-s\mu} \frac{e^{\frac{s}{2}(\epsilon_2-\epsilon_1)}\cosh\left(\frac{s(\epsilon_1+\epsilon_2)}{2}\right)}{\sinh(s\epsilon_1)\sinh(s\epsilon_2)} &\sim \cdots + \sum_{n>2} \Phi_2^{(n/2)}(\beta)\,\frac{\lambda^{n-2}}{\mu^{n-2}}\,,
\end{aligned} \tag{3.22}$$

where the relation between the first and second integral is just a shift of μ,

$$\mu \to \mu + \frac{1}{2}(\epsilon_1 - \epsilon_2)\,. \tag{3.23}$$

One should note that the Schwinger integral for Φ_2 presented in equation (3.22) has an additional sector in odd powers of λ and is further not symmetric in the exchange of the ϵ_i. Both problems can be evaded if one considers instead the integral

$$\frac{1}{4}\int \frac{ds}{s} e^{-s\mu} \frac{\cosh\left(\frac{s(\epsilon_1+\epsilon_2)}{2}\right)\cosh\left(\frac{s(\epsilon_1-\epsilon_2)}{2}\right)}{\sinh(s\epsilon_1)\sinh(s\epsilon_2)}\,. \tag{3.24}$$

The odd sector disappears while the even sector is unchanged. Hence, one should interpret the first integral in (3.22) and the integral in (3.24) as due to integrating out a massive

[1] The reader should not be confused about sinh versus sin, respectively cosh versus cos in comparison to other Schwinger type integrals in the literature. What occurs depends on the definition/convention used.

hyper-, respectively, vector-multiplet in the Ω-deformed A_1 background. In particular, one can see the vector-multiplet contribution (3.24) as the combination of two hyper-multiplets shifted with the shift (3.23) with opposite signs, as it should be since we specialized to $SU(2) \subset U(2)$. Note that in contrast to gauge theory in ordinary space-time (*c.f.*, [55]), the hyper- and vector-multiplets give already for $\epsilon_1 = -\epsilon_2$ different contributions.

Since we have at genus g exactly $2g-2$ unknowns $a_n^{(g)}$, the gap provides $2g-1$ boundary condition and we have a closed expression for the $\Psi_2^{(2)}$, we can (at least theoretically) solve for the partition function to any desired order in the B-model formalism.

Let us also comment on the Nekrasov-Shatashvilli limit (for short NS limit) [72] of the gauge theory partition function in the A_1 case. Since the partition function is entirely determined by the Schwinger integrals above, it is sufficient to consider the NS limit thereof. We have

$$\lim_{\epsilon_2 \to 0} \epsilon_2 \int \frac{ds}{s} e^{-s\mu} \frac{\cosh\left(\frac{s(\epsilon_1+\epsilon_2)}{2}\right)}{\sinh(s\epsilon_1)\sinh(s\epsilon_2)} = \frac{1}{2}\int \frac{ds}{s^2} \frac{e^{-\mu s}}{\sinh\left(\frac{s\epsilon_1}{2}\right)} .$$

Comparison with [3] shows that in the NS limit the boundary conditions, hence also the full partition function, of the A_1 case becomes identical to the ordinary case (up to an overall factor of 1/2). In particular, this implies that the quantum integrable system one can associate in the NS limit with the Ω-deformed gauge theory, following [72], is as for the gauge theory on $\mathbb{C}^2$.

3.3.3 Z_2^-

Let us now discuss the other sector. Similar as before, we have that

$$\mathcal{G}^{(0)} = \frac{1}{2}\mathcal{F}^{(0)} ,$$

and the usual special geometry applies. However, in contrast to above, we find that the 1-loop amplitude is reproduced via the ambiguity in (3.18) fixed to

$$a_{\mathcal{G}}^{(1)} = \frac{1}{48}\left(\beta + \frac{1}{\beta}\right)\log\Delta + \frac{1}{8}\log\Delta_+ - \frac{1}{8}\log\Delta_- . \tag{3.25}$$

In particular, this shows that the $\mathbb{Z}_2$ symmetry between the monopole and dyon point in moduli space at $u = \pm\Lambda^2$ is broken at 1-loop. This is similar as in the shifted two massless flavor case considered in [54]. Correspondingly, in order to obtain sufficient boundary conditions we have to expand the higher genus amplitudes separately around both points in moduli space. We take as Ansatz for the holomorphic ambiguities

$$a_{\mathcal{G}}^{(g)}(u;\beta) = u^{3-g}\sum_{n=1}^{2g-2}\frac{1}{\Delta^n}\left(a_n^{(g)}(\beta)\Lambda^{4n-4} + \tilde{a}_n^{(g)}(\beta)u^{2n+1}\Lambda^2\right) ,$$

where we assumed twice as much unknowns, $a_n^{(g)}$ and $\tilde{a}_n^{(g)}$, as we had before because of the breaking of the $\mathbb{Z}_2$ symmetry. Fixing the unknown coefficients via comparison with the localization results and analytic continuation of the amplitudes to the monopole and dyon point shows that the amplitudes still feature a gap structure, *i.e.*,

$$\mathcal{G}_{\pm}^{(1)}(a_D;\beta) = \Psi_2^{(1)}(\pm\beta)\,\log(a_D) + \dots\,, \quad \mathcal{G}_{\pm}^{(g>1)}(a_D;\beta) = \Psi_2^{(g)}(\pm\beta)\,a_D^{2-2g} + \mathcal{O}(a_D^0)\,, \quad (3.26)$$

where the $\pm$ distinguishes between expansion at the monopole, respectively, dyon point in moduli space. We observe that the coefficients Ψ_2 are as in (3.20). The sole difference between the expansions at the two points is a flip of sign of β.

We can also read of the perturbative contribution. The first few terms read

$$\begin{aligned} \tilde{\Phi}_2^{(1)}(\beta) &= -\frac{1}{24}\left(\beta + \frac{1}{\beta}\right) + \frac{1}{4}\,, \\ \tilde{\Phi}_2^{(2)}(\beta) &= \frac{1 - 50\beta^2 + \beta^4}{1440\beta^2}\,, \\ \tilde{\Phi}_2^{(3)}(\beta) &= -\frac{1 - 161\beta^2 - 161\beta^4 + \beta^6}{10080\beta^3}\,, \\ &\vdots \end{aligned} \quad (3.27)$$

It is instructive to write down the corresponding Schwinger integrals, which are closely related to the ones given in (3.22). We have

$$\begin{aligned} \frac{1}{2}\int \frac{ds}{s} e^{-\mu s} \frac{\cosh\left(\frac{s(\epsilon_1 \pm \epsilon_2)}{2}\right)}{\sinh(s\epsilon_1)\sinh(s\epsilon_2)} &\sim \dots + \sum_{n>0} \Psi_2^{(n)}(\pm\beta)\,\frac{\lambda^{2n-2}}{\mu^{2n-2}}\,, \\ \frac{1}{2}\int \frac{ds}{s} e^{-\mu s} \frac{e^{\frac{1}{2}s(\epsilon_1+\epsilon_2)}\cosh\left(\frac{s(\epsilon_1 + \epsilon_2)}{2}\right)}{\sinh(s\epsilon_1)\sinh(s\epsilon_2)} &\sim \dots + \sum_{n>0} \tilde{\Phi}_2^{(n/2)}(\beta)\,\frac{\lambda^{n}}{\mu^{n}}\,. \end{aligned} \quad (3.28)$$

Note that the second integral in (3.28) is related to the first integral with plus sign via a shift

$$\mu \to \mu - \frac{1}{2}(\epsilon_1 + \epsilon_2)\,. \quad (3.29)$$

As in the previous subsection, one should get rid off the odd sector in the integral for $\tilde{\Phi}_2$ via considering instead the integral

$$\frac{1}{4}\int \frac{ds}{s} e^{-\mu s} \frac{\cosh^2\left(\frac{s(\epsilon_1 + \epsilon_2)}{2}\right)}{\sinh(s\epsilon_1)\sinh(s\epsilon_2)}\,.$$

Again, this can be seen as the combination of two hyper-multiplets oppositely shifted via (3.29).

Since we have at genus g in total $4g-4$ unknowns $a_n^{(g)}$ and $\tilde{a}_n^{(g)}$, the two gaps provide $4g-2$ boundary conditions, and the Schwinger integrals in (3.28) provide 3 additional boundary conditions, we have again enough boundary information available to calculate the partition function in the B-model formalism to any desired order.

We also remark that trivially the $\Psi_2^{(n)}(\pm\beta)$ have the same $\epsilon_2 \to 0$ limit for both signs. Hence, the statement of the previous section regarding the NS limit still applies.

Finally, let us note that due to the identity

$$\frac{\cosh\left(\frac{s(\epsilon_1+\epsilon_2)}{2}\right)+\cosh\left(\frac{s(\epsilon_1-\epsilon_2)}{2}\right)}{\sinh(s\epsilon_1)\sinh(s\epsilon_2)}=\frac{1}{2\sinh(\frac{s\epsilon_1}{2})\sinh(\frac{s\epsilon_2}{2})}\,, \tag{3.30}$$

we have that

$$\Psi_1^{(g)}(\beta)=\Psi_2^{(g)}(\beta)+\Psi_2^{(g)}(-\beta)\,, \tag{3.31}$$

where $\Psi_1^{(g)}(\beta)$ denote the usual boundary conditions of $SU(2)$ gauge theory on Ω-deformed $\mathbb{C}^2$ space-time at the monopole/dyon point [54].

The relation (3.31) neatly illustrates the essential point of the B-model discussion performed in this section. Namely, the boundary conditions from the dyon/monopole points are projected to either $\Psi_2^{(g)}(+\beta)$ or $\Psi_2^{(g)}(-\beta)$, separately for each point. This freedom incorporates the two different sectors of the gauge theory on A_1 space-time outlined in section 3.2.2 into the B-model formalism.

3.4 Instanton partition functions in 5d

In this section we are going to consider the natural lift of the localization calculation for four-dimensional gauge theory of section 3.2 to five dimensions. In particular, we take the so-obtained partition functions as definition of a refined topological string with A_1 space-time.

3.4.1 Generalities

The partition function of five-dimensional supersymmetric gauge theory with eight supercharges compactified on a circle of radius R_c can be obtained via a certain deformation of the instanton calculation for the same gauge theory in four dimensions [71, 70, 67]. In detail, the deformation is given by a simple change of weights

$$E_s(a;Y_n,Y_m;\epsilon_1,\epsilon_2)\to 1-\exp\left(R_c\,E_s(a;Y_n,Y_m;\epsilon_1,\epsilon_2)\right)\,, \tag{3.32}$$

in the localization formula (3.4). We can absorb R_c in a simultaneous rescaling of the Coulomb-parameters a_i and the parameters ϵ_i. Therefore we just set R_c to one in (3.32).

By definition, the partition function of the five-dimensional gauge theory can be identified with the partition function of a refined topological string on a geometric engineering geometry which yields in the effective field-theory limit the four-dimensional gauge theory partition function [41, 45].

It seems natural to conjecture that the "lift" (3.32) applied to the gauge theory on ALE space also corresponds to the partition function of a five dimensional gauge theory, presumably on ALE$\times S^1$ (at least locally), which, one may take as the definition of a refined topological string with ALE space-time.

3.4.2 $U(1)$

Let us consider first the $U(1)$ gauge theory in five dimensions, those partition function defines the refined topological string on the conifold, that is, $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$ [41]. Since we are considering $U(1)$ gauge theory, we only have to sum over a single partition R. Before orbifolding, one has [66, 67]

$$Z_{U(1)} = \sum_R \mathfrak{q}^{|R|} C_R C_{R^t}\,,$$

with

$$C_R := \prod_{(i,j)\in R} \left(1-e^{\epsilon_1 L_R(i,j)-\epsilon_2(A_R(i,j)+1)}\right)^{-1}\,, \tag{3.33}$$

where as before $L_R(i,j)$ denote the leg-length, $A_R(i,j)$ the arm-length of the box $(i,j)\in R$ and $\mathfrak{q}$ the instanton counting parameter, which is identified with the Kähler parameter t of the geometry as

$$\mathfrak{q} = e^{\frac{1}{2}(\epsilon_1+\epsilon_2)} Q\,,$$

where we also defined $Q := e^{-t}$. Note that we performed an additional shift of t in order to obtain an expansion of the corresponding free energy into even powers of λ only. In particular, under this identification one can show that

$$Z = \exp\left(\sum_{k=1}^{\infty} \frac{Q^k}{4k\sinh(\frac{k\epsilon_1}{2})\sinh(\frac{k\epsilon_2}{2})}\right) = \prod_{i,j}^{\infty}(1-Qe^{\epsilon_1(i-1/2)-\epsilon_2(j-1/2)})\,, \tag{3.34}$$

and this is the refined partition function we expect from the M-theory spin state counting point of view, following [41]. Note also that the partition function of $\mathcal{O}(-2)\oplus\mathcal{O}(0)\to\mathbb{P}^1$ differs from this only by $Z\to Z^{-1}$.

Let us now consider the orbifold projection. Following the implementation of the projection into the localization scheme, as described in section 3.2.2, we infer that we obtain as projected partition function $Z^+_{U(1)}$,

$$Z^+_{U(1)} = \sum_{R_+} \mathfrak{q}^{|R_+|/2} \tilde{C}_{R_+} \tilde{C}_{R^t_+}\,, \tag{3.35}$$

where R_+ denotes the projected set of partitions with an even number of boxes, and $\tilde{C}_R$ is the projection of (3.33), that is, only boxes in R are taken to contribute for which

$$L_R(i,j) + A_R(i,j) + 1 = 0 \mod 2\,,$$

holds.

In order to find a similar expression as (3.34) for the orbifold, a natural guess for an Ansatz would be to directly apply the orbifold projection to the infinite product occurring in (3.34). Let us assume that the Kähler modulus t is charged under the orbifold action $\mathcal{I}$ given in (3.5) with charge $q^c\pi$. The product is invariant under $\mathcal{I}$ if

$$(i+j-1)+q^c = 0 \mod 2\,,$$

holds. For q^c even, the projected partition function is given by

$$Z^+_{U(1)} = \prod_{i_1,j_2\,\text{even}}^{\infty}\ \prod_{j_1,i_2\,\text{odd}}^{\infty} (1-Qe^{\epsilon_1(i_1-1/2)-\epsilon_2(j_1-1/2)})(1-Qe^{\epsilon_1(i_2-1/2)-\epsilon_2(j_2-1/2)})\,,$$

while for q^c odd by

$$\tilde{Z}^-_{U(1)} = \prod_{i_1,j_1\,\text{even}}^{\infty}\ \prod_{j_2,i_2\,\text{odd}}^{\infty} (1-Qe^{\epsilon_1(i_1-1/2)-\epsilon_2(j_1-1/2)})(1-Qe^{\epsilon_1(i_2-1/2)-\epsilon_2(j_2-1/2)})\,.$$

We can rewrite the first partition function as

$$Z^+_{U(1)} = \exp\left(\sum_{k=1}^{\infty} \frac{e^{\frac{1}{2}k(\epsilon_1+\epsilon_2)}(1+e^{k(\epsilon_1+\epsilon_2)})}{k(e^{2k\epsilon_1}-1)(e^{2k\epsilon_2}-1)} Q^k\right) = \exp\left(\sum_{k=1}^{\infty} \frac{\cosh\left(\frac{k(\epsilon_1+\epsilon_2)}{2}\right)}{2k\sinh(k\epsilon_1)\sinh(k\epsilon_2)} Q^k\right)\,, \tag{3.36}$$

and observe that for even charge (+) the summand is the same as the first integrand in (3.22). This is similar as in the ordinary refined case. Furthermore, explicit expansion of the corresponding free energies (parameterized via (3.2)) for small Q and λ shows that (3.35) equals (3.36). For odd charge (−) we infer instead that

$$\tilde{Z}^-_{U(1)} = \exp\left(\sum_{k=1}^{\infty} \frac{e^{\frac{1}{2}k(\epsilon_1+\epsilon_2)}(e^{k\epsilon_1}+e^{k\epsilon_2})}{k(e^{2k\epsilon_1}-1)(e^{2k\epsilon_2}-1)} Q^k\right) = \exp\left(\sum_{k=1}^{\infty} \frac{e^{\frac{k}{2}(\epsilon_2-\epsilon_1)}\cosh\left(\frac{k(\epsilon_1-\epsilon_2)}{2}\right)}{2k\sinh(k\epsilon_1)\sinh(k\epsilon_2)} Q^k\right)\,, \tag{3.37}$$

and this is the same as the first integral in (3.28) with negative sign under an additional shift of Q. It is more natural to define

$$Z^-_{U(1)}(\epsilon_1,\epsilon_2;Q) := \tilde{Z}^-_{U(1)}(\epsilon_1,\epsilon_2;Qe^{-\frac{1}{2}(\epsilon_2-\epsilon_1)})\,.$$

In particular, under this definition the relation between $Z^+_{U(1)}$ and $Z^-_{U(1)}$ is just a flip of sign of one of the ϵ_i combined with an overall sign change, *i.e.*,

$$Z^+_{U(1)}(\epsilon_1,\epsilon_2;Q) = -Z^-_{U(1)}(\epsilon_1,-\epsilon_2;Q) = -Z^-_{U(1)}(-\epsilon_1,\epsilon_2;Q)\,. \tag{3.38}$$

Also note that

$$Z_{U(1)}(\epsilon_1,\epsilon_2;Q) = Z^+_{U(1)}(\epsilon_1,\epsilon_2;Q)\times Z^-_{U(1)}(\epsilon_1,\epsilon_2;Q)\,,$$

or, in terms of the corresponding free energies $\mathcal{F} = \log(Z)$,

$$\mathcal{F}(\epsilon_1, \epsilon_2; Q) = \mathcal{F}^+(\epsilon_1, \epsilon_2; Q) + \mathcal{F}^-(\epsilon_1, \epsilon_2; Q)\,,$$

due to relation (3.30).

Hence, it is suggested to identify $Z^+_{U(1)}$ and $Z^-_{U(1)}$ as possible partition functions of the refined topological string on $A_1 \times (\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1)$. The partition functions on $A_1 \times (\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathbb{P}^1)$ follow as usual via $Z^\pm_{U(1)} \to \left(Z^\pm_{U(1)}\right)^{-1}$. Of course, due to (3.38) there is no essential difference between $Z^+_{U(1)}$ and $Z^-_{U(1)}$. However, as we will see in the next subsection, it is useful to explicitly distinguish between the two cases.

3.4.3 $U(2)$

Let us now discuss the pure $U(2)$ gauge theory in five dimensions, since this relates to the refined topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$, *i.e.*, on $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$. We denote the two Kähler parameters of the geometry, which correspond to the sizes of the two $\mathbb{P}^1$, as t_1 and t_2. We also define $Q_i := e^{-t_i}$.

As for $U(1)$, the essential lift from four to five dimensions is implemented via the simple change of weight (3.32). From [41, 45] one can infer that the identification between the $SU(2) \subset U(2)$ gauge theory partition function in five dimensions, $Z_{SU(2)}(\epsilon_1, \epsilon_2; a; \mathfrak{q})$, and the refined topological string partition function (defined via M-theory left-right spin counting) on local $\mathbb{P}^1 \times \mathbb{P}^1$ goes as

$$Z_{\mathbb{P}^1\times\mathbb{P}^1}(\epsilon_1, \epsilon_2; Q_1, Q_2) = Z_{U(1)}(\epsilon_1, \epsilon_2; Q_1 e^{\frac{\epsilon_1+\epsilon_2}{2}}) Z_{U(1)}(\epsilon_1, \epsilon_2; Q_1 e^{-\frac{\epsilon_1+\epsilon_2}{2}}) \times Z_{SU(2)}(\epsilon_1, \epsilon_2; a; \mathfrak{q})\,,$$

where $Z_{U(1)}(\epsilon_1, \epsilon_2; Q)$ is as in (3.34), and with identification of parameters

$$a = -\frac{1}{2}\log Q_1\,, \quad \mathfrak{q} = \frac{Q_2}{Q_1} e^{\frac{1}{2}(\epsilon_1+\epsilon_2)}\,.$$

(Usually, the two $Z_{U(1)}$ factors are referred to as perturbative contribution.) One can also interchange the role of Q_1 and Q_2, due to the symmetry of the geometry. It is interesting to note that while the full partition function $Z_{\mathbb{P}^1\times\mathbb{P}^1}$ has an expansion into even powers of λ only, the equivariant limit to either of the $\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathbb{P}^1$ parts of the geometry does not (only under additional shifts of the Q). That means that the cancellation of the odd sector in λ is a global phenomenon.

Using the results of the previous sections, it is now easy to suggest the partition function of the refined topological string on $A_1 \times (\text{local } \mathbb{P}^1 \times \mathbb{P}^1)$. Namely,

$$Z^\pm_{\mathbb{P}^1\times\mathbb{P}^1}(\epsilon_1, \epsilon_2; Q_1, Q_2) = Z^\pm_{U(1)}(\epsilon_1, \epsilon_2; Q_1 e^{\frac{\epsilon_1+\epsilon_2}{2}}) Z^\pm_{U(1)}(\epsilon_1, \epsilon_2; Q_1 e^{-\frac{\epsilon_1+\epsilon_2}{2}}) \times Z^\pm_{SU(2)}(\epsilon_1, \epsilon_2; a; \mathfrak{q})\,, \tag{3.39}$$

In particular the $\pm$ in $Z^\pm_{SU(2)}$ stand for the even, respectively odd, projection discussed in section 3.2, while $Z^\pm_{U(1)}$ is given in (3.36), respectively (3.37). One should note that only for the same pairing of $\pm$ between the $U(1)$ and $SU(2)$ part one obtains a partition function which is symmetric under exchange of Q_1 and Q_2. Note also that while in $Z^+_{\mathbb{P}^1\times\mathbb{P}^1}$ only integer powers of Q_i appear, $Z^-_{\mathbb{P}^1\times\mathbb{P}^1}$ possesses also half-integer powers, reflecting the presence of "fractional" instantons in the effective field theory limit. In the following section, we will give strong support in favor of the interpretation of (3.39) as a (refined) topological string partition function.

3.5 B-model approach to $U(2)$ gauge theory in 5d

In this section we are going to reproduce the five dimensional partition functions of the previous section via the B-model approach applied to the mirror geometry of local $\mathbb{P}^1 \times \mathbb{P}^1$. This hints towards the interpretation of the five-dimensional partition functions as the partition function of a refined topological string on $A_1\times$(local $\mathbb{P}^1 \times \mathbb{P}^1$).

3.5.1 Generalities

The B-model formalism for the topological string on this geometry has been exhaustively discussed in the literature, hence we will be brief and only recall the necessities.

Via mirror symmetry, the (large volume) tree-level data of the topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$, can be obtained by solving the system of Picard-Fuchs equations [23]

$$\begin{aligned} \mathcal{L}_1 &= \theta_1^2 - 2z_1(\theta_1+\theta_2)(1+2\theta_1+2\theta_2)\,, \\ \mathcal{L}_2 &= \theta_2^2 - 2z_2(\theta_1+\theta_2)(1+2\theta_1+2\theta_2)\,, \end{aligned} \tag{3.40}$$

with $\theta_i := z_i\partial_{z_i}$ and where z_i denote the two complex structure parameters mirror to the two Kähler parameters. The moduli space of the mirror geometry has a rich structure, see [2]. However, for our purposes only the large volume point and the conifold locus are of relevance. [2] The former to compare to the five dimensional gauge theory results, and the latter to fix the holomorphic ambiguities. In particular, since we are only interested in these points, we do not need to bother about resolving singularities in moduli space. Hence, working solely with the (singular) moduli space parameterized by the discriminant

$$\Delta\tilde{\Delta} = 0\,,$$

with

$$\Delta = (1 - 8(z_1+z_2) + 16(z_1-z_2)^2)\,, \tag{3.41}$$

and

$$\tilde{\Delta} = \tilde{\Delta}_1\tilde{\Delta}_2\,; \quad \tilde{\Delta}_i = z_i\,,$$

[2] One could also use the point in moduli space corresponding to the geometric engineering limit to compare directly to the 4d gauge theory. We performed this analysis and obtained the expected results. However, we omit this discussion in this note for brevity, since it does not provide any major new insights.

is sufficient for our purposes.

Solving the set of Picard-Fuchs equations (3.40) yields the mirror maps and the prepotential $\mathcal{F}^{(0)}$ for the large volume point ($z_1 = z_2 = 0$). For the conifold point, we can for instance choose $z_1 = z_2 = \frac{1}{16}$, as this point lies only on $\Delta = 0$, and is non-singular. A possible choice of coordinates around this point is (see for example [39])

$$z_1^c = 1 - \frac{z_1}{z_2}\,, \quad z_2^c = 1 - \frac{z_2}{\frac{1}{8} - z_1}\,. \tag{3.42}$$

Solving the Picard-Fuchs system (3.40) in the new coordinates z_i^c gives the mirror maps and the prepotential at the conifold locus.

Having the tree-level data at the large volume and a conifold point in moduli space at hand, one can solve for the higher genus amplitudes recursively via the holomorphic anomaly equations, both, in the ordinary and refined case. Since the local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry effectively behaves like a 1-parameter geometry, the 1-parameter holomorphic anomaly equations introduced at hand of $SU(2)$ gauge theory in section 3.3 are sufficient. Note also that the effect of refinement is a pure change of boundary conditions at the conifold point, as anticipated in [54] and explicitly confirmed in [42].

3.5.2 $Z^+_{SU(2)}$

Via explicit comparison to the localization results of section 3.4.3, we find that the 1-loop free energy of the lift of the gauge theory on A_1 to five dimensions can be reproduced in the B-model formalism via the usual solution of the 1-loop holomorphic anomaly equation

$$\mathcal{G}^{(1)} = -\frac{1}{2}\log\left(\det G\right) + a_{\mathcal{G}}^{(1)}(z_1, z_2; \beta)\,, \tag{3.43}$$

with $G_{ij} := \partial_{Q_i} z_j$, holomorphic ambiguity $a_{\mathcal{G}}^{(1)}$ parameterized as

$$a_{\mathcal{G}}^{(1)}(z_1, z_2; \beta) = \kappa_1 \log \Delta + \kappa_2 \log \tilde{\Delta}\,, \tag{3.44}$$

albeit under fixing

$$\kappa_1 = -\frac{1}{48}\left(\beta + \frac{1}{\beta} + 6\right)\,, \quad \kappa_2 = -\frac{1}{48}\left(24 - \beta - \frac{1}{\beta}\right)\,.$$

Note that the coefficient κ_1 of the "conifold term" has the expected universal structure, *i.e.*, is proportional to $\Psi_2^{(1)}$ given in (3.20).

Having the tree-level and 1-loop data at hand, we can try to solve for the higher genus amplitudes using the holomorphic anomaly equations supplemented with the expected boundary conditions (3.20) for the conifold point in moduli space. We parameterize the holomorphic ambiguity as usual, *i.e.*, via

$$a_{\mathcal{G}}^{(g>1)}(z_1, z_2; \beta) = \frac{1}{\Delta^{2g-2}} \sum_{n_i \leq 4(g-1)} a_{n_1,n_2}^{(g)}(\beta)\, z_1^{n_1} z_2^{n_2}\,.$$

Similar as in the original case, the symmetry of the partition function under exchange of z_1 and z_2, the known constant map contribution and conifold behavior yield enough constraints to fix the coefficients $a^{(g)}_{n_1,n_2}$ for arbitrary g.

In this manner, the amplitudes can be calculated to high degree in Q_i. Clearly, the explicit results are too lengthy to be explicitly shown here. We just give the leading terms of the 2-loop amplitude for illustration purposes, *i.e.*,

$$\mathcal{G}^{(2)} = \frac{1+40\beta^2+\beta^4}{720\beta^2}(Q_1+Q_2) - \frac{59+1440\beta-2950\beta^2+1440\beta^3+59\beta^4}{360\beta^2}Q_1Q_2 + \ldots .$$

Finally, we also note that we have found an all integer BPS state type expansion of the five dimensional partition function. We will come back to this elsewhere.

3.5.3 $Z^-_{SU(2)}$

Let us now consider the other sector. Since in the underlying four-dimensional gauge theory the $\mathbb{Z}_2$ symmetry between the monopole and dyon point is broken at 1-loop, we expect also something new to happen for the refined topological B-model on $A_1\times$ (local $\mathbb{P}^1\times\mathbb{P}^1$). Indeed, the genus expansion of the free energy obtained from the localization calculation for the lifted gauge theory shows that the parameters Q_i also occur with half-integer powers at 1-loop and beyond, which is not immediately clear how to be reproduced by the B-model since the ordinary 1-loop amplitude (3.43) for general ambiguity (3.44) clearly has an expansion into integer powers of Q_i only. However, similar as for the gauge theory in four dimensions, the solution lies in a "refinement" of the conifold locus in moduli space at 1-loop and beyond. For that, note that the parameterization of the conifold locus ($\Delta = 0$ with Δ as in (3.41)), can actually be factorized as

$$\Delta = \Delta_1\Delta_2\Delta_3\Delta_4 = 0\,,$$

with

$$\begin{aligned} \Delta_1 &= -1+2(\sqrt{z_1}-\sqrt{z_2})\,, \quad \Delta_2 = 1+2(\sqrt{z_1}-\sqrt{z_2})\,, \\ \Delta_3 &= -1+2(\sqrt{z_1}+\sqrt{z_2})\,, \quad \Delta_4 = 1+2(\sqrt{z_1}+\sqrt{z_2})\,. \end{aligned} \tag{3.45}$$

As it will turn out, for our purposes it is enough to consider the two combinations $\Delta_{12} = \Delta_1\Delta_2$ and $\Delta_{34} = \Delta_3\Delta_4$. We observe that we can reproduce the localization result at 1-loop if we fix the holomorphic ambiguity of the amplitude (3.43) to

$$a^{(1)}_{\mathcal{G}}(z_1,z_2;\beta) = -\frac{1}{48}\left(\beta+\frac{1}{\beta}\right)\log\Delta - \frac{1}{8}\left(\log\Delta_{12}-\log\Delta_{34}\right) - \frac{1}{48}\left(30-\beta-\frac{1}{\beta}\right)\log\tilde{\Delta}\,.$$

Note the qualitative similarity of the first two terms to (3.25). In particular, the breaking of the $\mathbb{Z}_2$ symmetry between the monopole and dyon point in the four dimensional gauge theory translates to a breaking of the degeneration of the conifold locus in moduli space. Hence, in order to obtain enough boundary conditions for the higher genus amplitudes, we have to expand the amplitudes around two different conifold points in moduli space, *i.e.*, the conifold locus breaks into two components at 1-loop and beyond. The point corresponding to the choice of coordinates (3.42) lies on $\Delta_3 = 0$, thus we take for instance as the other point $(z_1, z_2) = (\frac{1}{4}, 1)$, which lies on $\Delta_2 = 0$. We can take as coordinates around this point

$$\tilde{z}_1^c = 1 - \frac{4z_1}{z_2}\,, \quad \tilde{z}_2^c = 1 - \frac{z_2}{\frac{1}{2} + 2z_1}\,.$$

Transforming the Picard-Fuchs equations (3.40) to these coordinates, one obtains as solution the corresponding mirror maps and prepotential. With the tree-level and 1-loop data at hand, we solve for the higher genus amplitudes using the holomorphic anomaly equation and expand around the large volume and the two conifold points. We observe that the gauge theory results of section 3.4.3 can be reproduced via parameterizing the holomorphic ambiguity via

$$a_{\mathcal{G}}^{(g>1)}(z_1, z_2; \beta) = \frac{1}{\Delta^{2g-2}} \left(\sum_{n_i \leq 4(g-1)} a_{n_1,n_2}^{(g)}(\beta)\, z_1^{n_1} z_2^{n_2} + \sum_{\substack{n_i < 8(g-1) \\ n_i \text{ odd}}} \tilde{a}_{n_1,n_2}^{(g)}(\beta) z_1^{n_1/2} z_2^{n_2/2} \right) .$$

Further, the amplitudes around the two conifold points feature two independent gaps with coefficient of the leading (singular) term $\sim \Psi_2(+\beta)$, respectively, $\sim \Psi_2(-\beta)$. This yields sufficiently many boundary conditions to solve the amplitudes to any desired order. Again, the explicit amplitudes are too lengthy to be displayed here. However, for the readers convenience we give the leading terms of the 2-loop amplitude:

$$\begin{aligned} \mathcal{G}^{(2)} = &\frac{(\beta-1)^2}{4} Q_1^{1/2} Q_2^{1/2} + \frac{1 - 50\beta^2 + \beta^4}{720\beta^2}(Q_1 + Q_2) \\ &- \frac{59 - 540\beta + 1010\beta^2 - 540\beta + 59}{360\beta^2} Q_1 Q_2 + \frac{5(\beta-1)^2}{2\beta^2}(Q_1^{3/2} Q_2^{1/2} + Q_1^{1/2} Q_2^{3/2}) + \ldots\,. \end{aligned} \tag{3.46}$$

3.6 Conclusion

In this work we have initiated the study of gauge theory on ALE space and its five-dimensional lift from a special geometry and holomorphic anomaly point of view.

Besides having explicitly shown that the partition function of pure $SU(2) \subset U(2)$ gauge theory on the simplest example of an ALE space, namely A_1, can be reproduced using the "B-model" approach of invoking special geometry and the holomorphic anomaly equation, we also showed that the partition function resulting from the naive lift to five dimensions of the gauge theory on A_1 still enjoys this property. In particular, the corresponding tree-level geometry is local $\mathbb{P}^1 \times \mathbb{P}^1$, hinting towards that the lifted gauge theory can be identified with a sort of refined topological string on $A_1 \times$(local $\mathbb{P}^1 \times \mathbb{P}^1$).

We see room for further investigations in various directions. Perhaps most interesting would be the generalization to include matter, other internal backgrounds and more general ALE space-times, in order to clarify if the $SU(2) \subset U(2)$ on A_1 case discussed in this note is a mere mathematical curiosity, or, if there is a general structure behind. For that, it would be useful to construct an orbifolded version of the refined topological vertex ("A-model" approach). This appears to be relatively straight-forward, *i.e.*, the only missing essential ingredient being the proper projection of the framing factor. The application of the B-model approach seems in general to be a bit more tricky due to the observed "refinement" of moduli space under coupling to gravity, which is expected to be a general feature for theories on ALE space-time. It appears that one should simultaneously pursue the A- and B-model approach in order to be able to definitely fix both. It would also be beneficial to properly understand the resummation of the free energies into a generating function counting (projected) BPS-states. We have found indications that such a resummation exists. We hope to come back to these and related thematics in more detail elsewhere.

Part II

Freudenthal Gauge Theory

Chapter 4

Background Knowledge

The *Freudenthal Triple System* (*FTS* for short) was originally constructed from *Jordan Triple System*'s (*JTS*'s for short) by mathematician Hans Freudenthal in mid 1950's in his famous series study on exceptional Lie algebra [29, 30, 32, 31, 33], which culminated in the discovery of *Freudenthal's magic square* (Table 4.3) that bears his name. Besides its importance in the representation theory of exceptional Lie algebra, *FTS* is also closely related to study of integrable systems; it is shown that given a *FTS*, one can construct a solution to the Yang-Baxter equation, which in turn defines a formal integrable system [74, 75]. It was also realized that both *JTS* and *FTS* plays a fundamental role in $\mathcal{N} = 2$ supergravity in 4d and 5d [35, 36]. More explicitly, it was found that when couple n_V Maxwell super-multiplet to $\mathcal{N} = 2$ supergravity in five dimensional spacetime, the scalar manifold, which describes the background configuration, is completely classified by elements of a *JTS* $\mathfrak{J}$ and upon dimensional reduction to 4d, the background configuration is classified by the corresponding *FTS* $\mathfrak{K}(\mathfrak{J})$. Recently, *FTS* gains great interest from the study of 4-dim $\mathcal{N} = 2$ supersymmetric black holes by Dull et al. [17], where they discovered a non-linear duality called *F-dual* that relates supersymmetric black holes with the same entropy.

In the following, we present a short review of the mathematical construction of a *Freudenthal Triple System* from a *Jordan algebra of cubic form* following Freudenthal's original construction. We shall also review Okubo's solution [74, 75] to the Yang-Baxter equation and the role it plays in integrable systems. Then, we will discuss the role played by *JTS*'s and *FTS*'s in $\mathcal{N} = 2$ supergravity.

4.1 Freudenthal Triple System

Classically, a *Freudenthal Triple System* was constructed from a *Jordan Triple System* [29, 30, 87]. Let's start with a review of the *Jordan Algebras of Cubic Forms*.

4.1.1 Review of Jordan Algebras of Cubic Forms

A *Jordan Algebras of Cubic Forms* $\mathfrak{J}$ is a special kind of Jordan algebras, which includes all of the five Jordan algebras $\mathfrak{J}_{magic}$ of dimensions $1, 6, 9, 15, 27$ of the form $\mathfrak{J}_{magic} = \mathbb{R}$, $H_3(\mathfrak{A})$ that will lead to the five exceptional Lie algebras in the *Freudentahl's magic square* (Table 4.3) [61]. Here, $H_3(\mathfrak{A})$ denotes the Jordan algebra of 3×3 hermitian matrices with elements in one of the four *normed division algebras* $\mathfrak{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{Q}$, where the *Jordan product* $X \circ Y$ is defined as $\frac{1}{2}(XY + YX)$ via matrix multiplications.

A *Jordan algebra of Cubic Form* $\mathfrak{J}$ is constructed from a suitable triple $(N_{\mathfrak{J}}, \#, 1_{\mathfrak{J}})$, consisting of a totally symmetric *cubic norm* [1] $N_{\mathfrak{J}} : \mathfrak{J} \otimes \mathfrak{J} \otimes \mathfrak{J} \to \mathbb{R}$ with a *base point* [2] $1_{\mathfrak{J}} \in \mathfrak{J}$ s.t. $N_{\mathfrak{J}}(1_{\mathfrak{J}}) = 1$, and a quadratic *sharp mapping* $\#$.

From these, one can introduce the following forms:

$$\begin{aligned} Tr_{\mathfrak{J}}(X) &\equiv 3\, N_{\mathfrak{J}}(X, 1_{\mathfrak{J}}, 1_{\mathfrak{J}}) && (\textit{Linear Trace Form}) \\ S_{\mathfrak{J}}(X,Y) &\equiv 6\, N_{\mathfrak{J}}(X, Y, 1_{\mathfrak{J}}) && (\textit{Bilinear Spur Form}) \\ Tr_{\mathfrak{J}}(X,Y) &\equiv Tr_{\mathfrak{J}}(X)\, Tr_{\mathfrak{J}}(Y) - S_{\mathfrak{J}}(X,Y) && (\textit{Bilinear Trace Form}) \end{aligned} \tag{4.1}$$

Given a *cubic norm* $N_{\mathfrak{J}}$ with the *base point* $1_{\mathfrak{J}}$, the *sharp map* is a quadratic map $\# : \mathfrak{J} \to \mathfrak{J}$ strictly satisfying three conditions:

$$\begin{aligned} Tr_{\mathfrak{J}}(X^{\#}, Y) &= N_{\mathfrak{J}}(X, X, Y) && (\textit{Trace-Sharp Formula}) \\ (X^{\#})^{\#} &= N_{\mathfrak{J}}(X)\, X && (\textit{Adjoint Identity}) \\ X \# 1_{\mathfrak{J}} &= Tr_{\mathfrak{J}}(X)\, 1_{\mathfrak{J}} - X && (1_{\mathfrak{J}}\textit{-Sharp Identity}), \end{aligned} \tag{4.2}$$

where the symmetric bilinear *sharp product* $X \# Y$ is defined as

$$X \# Y \equiv (X+Y)^{\#} - X^{\#} - Y^{\#}. \tag{4.3}$$

Let's remark here that a *Jordan algebra of cubic form* $\mathfrak{J}$ is entirely fixed by the *cubic norm* $N_{\mathfrak{J}}$ via the identity:

$$X \circ Y = \frac{1}{2}\Big\{ X \# Y + Tr_{\mathfrak{J}}(X)\, Y + Tr_{\mathfrak{J}}(Y)\, X - S_{\mathfrak{J}}(X,Y)\, 1_{\mathfrak{J}} \Big\}, \tag{4.4}$$

and we shall take the *cubic norm* $N_{\mathfrak{J}}$ as the fundamental entity in the discussion in following sections.

[1] Strictly speaking, the $N_{\mathfrak{J}} : \mathfrak{J} \otimes \mathfrak{J} \otimes \mathfrak{J} \to \mathbb{R}$ defined here is the *full linearization* of the *cubic norm* $N_{\mathfrak{J}} : \mathfrak{J} \to \mathbb{R}$, satisfying $N_{\mathfrak{J}}(\lambda X) = \lambda^3 N_{\mathfrak{J}}(X), \forall\, (X, \lambda) \in \mathfrak{J} \times \mathbb{R}$. Explicitly, $\forall\, X, Y, Z \in \mathfrak{J}$

$$N_{\mathfrak{J}}(X,Y,Z) \equiv \frac{1}{6}\Big\{ N_{\mathfrak{J}}(X+Y+Z) - N_{\mathfrak{J}}(X+Y) - N_{\mathfrak{J}}(Y+Z) - N_{\mathfrak{J}}(Z+X) + N_{\mathfrak{J}}(X) + N_{\mathfrak{J}}(Y) + N_{\mathfrak{J}}(Z) \Big\},$$

with the normalization such that $N_{\mathfrak{J}}(X) = N_{\mathfrak{J}}(X,X,X)$.

[2] The *base point* $1_{\mathfrak{J}}$ is sometimes also called the *multiplicative identity*, which is the unity of the *Jordan product*.

4.1.2 Freudenthal's construction

After reviewing the *Jordan Triple System*, we are ready to construct a family of *Symplectic Triple System* (*STS* for short), called *Freudenthal Triple System*, which is out main focus in this half of the thesis. We will follow Freudenthal's original construction based on Jordan algebras [29, 30, 87].

Let $\mathfrak{J}$ be a *Jordan algebra of Cubic Form*, for any pair $X, Y \in \mathfrak{J}$, one can define a linear transformation $\mathcal{L}_{\mathfrak{J}}(X,Y) \in \mathfrak{gl}(\mathfrak{J})$ by

$$\mathcal{L}_{\mathfrak{J}}(X,Y)\,Z \equiv 2Y\#(X\#\,Z) - \frac{1}{2}Tr_{\mathfrak{J}}(Y,Z)\,X - \frac{1}{6}Tr_{\mathfrak{J}}(X,Y)\,Z, \quad \forall Z \in \mathfrak{J}. \tag{4.5}$$

Note, $\mathcal{L}_{\mathfrak{J}}(X,Y)$ is neither symmetric nor anti-symmetric in X and Y.

Denote by $\mathfrak{H}$ the subspace spanned by $\{\mathcal{L}_{\mathfrak{J}}(X,Y)\,|X,Y \in \mathfrak{J}\}$ in $\mathfrak{gl}(\mathfrak{J})$, one can construct the following two vector spaces (as direct sum of vector spaces):

$$\mathfrak{K} = \mathfrak{J} \oplus \mathfrak{J} \oplus \mathbb{R} \oplus \mathbb{R}, \quad \text{with elements of the form } x = (X, Y, \xi, \omega)\text{; and}$$

$$\mathfrak{L} = \mathfrak{H} \oplus \mathbb{R} \oplus \mathfrak{J} \oplus \mathfrak{J}, \quad \text{with elements of the form } \Theta = (\textstyle\sum_i \mathcal{L}_{\mathfrak{J}}(X_i, Y_i), \rho, A, B),$$

where in the last line, the sum is over some pairs $X_i, Y_i \in \mathfrak{J}$. For any two elements $x_i = (X_i, Y_i, \xi_i, \omega_i)_{i=1,2}$ in $\mathfrak{K}$, one can further define a symplectic form [3] $\langle x_1, x_2\rangle$ and an element $x_1 \times x_2 \in \mathfrak{L}$ as follows:

$$\begin{aligned}\langle x_1, x_2\rangle &\equiv Tr_{\mathfrak{J}}(X_1,Y_2) - Tr_{\mathfrak{J}}(X_2,Y_1) + \xi_1\omega_2 - \xi_2\omega_1,\\ x_1 \times x_2 &\equiv \tfrac{1}{2}\begin{pmatrix} \mathcal{L}_{\mathfrak{J}}(X_1,Y_2) + \mathcal{L}_{\mathfrak{J}}(X_2,Y_1) \\ \tfrac{3}{4}\big(\xi_1\omega_2 + \xi_2\omega_1\big) - \tfrac{1}{4}\Big(Tr_{\mathfrak{J}}(X_1,Y_2) + Tr_{\mathfrak{J}}(X_2,Y_1)\Big) \\ -Y_1\#\,Y_2 + \tfrac{1}{2}\big(\xi_1 X_2 + \xi_2 X_1\big) \\ X_1\#\,X_2 - \tfrac{1}{2}\big(\omega_1 Y_2 + \omega_2 Y_1\big)\end{pmatrix}.\end{aligned} \tag{4.6}$$

Geometrically, each element $\Theta = (\sum_i \mathcal{L}_{\mathfrak{J}}(X_i,Y_i), \rho, A, B) \in \mathfrak{L}$ corresponds to a linear transformation of the vector space $\mathfrak{K}$ given by:

$$\Theta\,x = \begin{pmatrix} \Big(\sum_i \mathcal{L}_{\mathfrak{J}}(X_i,Y_i) + \tfrac{1}{3}\rho\Big)X + 2B\#\,Y + \omega A \\ -\Big(\sum_i \mathcal{L}_{\mathfrak{J}}(Y_i,X_i) + \tfrac{1}{3}\rho\Big)Y + 2A\#\,X + \xi B \\ Tr_{\mathfrak{J}}(A,Y) - \rho\xi \\ Tr_{\mathfrak{J}}(B,X) + \rho\omega \end{pmatrix}, \quad \forall\, x = (X,Y,\xi,\omega) \in \mathfrak{K}. \tag{4.7}$$

[3]This symplectic form $\langle x_i, x_j\rangle$ is non-degenerate provided the *Bilinear Trace Form* is non-degenerate, which is always true in our case.

Note the different ordering of X_i, Y_i of the $\sum_i \mathcal{L}_{\mathfrak{J}}(X_i, Y_i)$ term in the first line and the second line. Restricting to the subspace spanned by $\mathfrak{L}_{\mathfrak{K}} \equiv \{\mathcal{L}_{x_i x_j} \equiv x_i \times x_j \,|\, x_i, x_j \in \mathfrak{K}\} \subset \mathfrak{L}$, this linear transformation induces a triple product $x_1x_2x_3 : \mathfrak{K} \otimes \mathfrak{K} \otimes \mathfrak{K} \to \mathfrak{K}$ defined by:

$$x_1x_2x_3 \equiv (x_1 \times x_2)\, x_3, \tag{4.8}$$

which is clearly symmetric in the first two variables but not symmetric with respect to any other exchange of indices. This triple product satisfies the following relations:

$$\begin{aligned} x_1x_2x_3 - x_1x_3x_2 &= 2\lambda \langle x_2, x_3\rangle\, x_1 - \lambda \langle x_1, x_2\rangle\, x_3 - \lambda \langle x_3, x_1\rangle\, x_2 \\ x_1x_2(x_3x_4x_5) - x_3x_4(x_1x_2x_5) &= (x_1x_2x_3)x_4x_5 + x_3(x_1x_2x_4)x_5 \\ \langle x_1x_2x_3, x_4\rangle &= -\langle x_3, x_1x_2x_4\rangle \end{aligned} \quad , \tag{4.9}$$

where λ is a numerical constant that equals $\frac{1}{8}$ in our explicit construction here. We shall call a triple of $\mathfrak{K}$, the ternary product (4.8), and symplectic form as a whole satisfying (4.9) with non-zero λ a *Freudenthal Triple System* (*FTS*).

4.2 Freudenthal Triple System and Lie Algebra

There is a 1-1 correspondence between a *FTS* $\mathfrak{K}$ and a simple Lie algebra $\mathfrak{g}$. In the special case, when the *FTS* is constructed from a magic *Jordan Triple System* $\mathfrak{J}_{magic}$ as reviewed in the previous section, Freudenthal proved that the Lie-algebra so obtained are the exceptional Lie algebras, which is summarized in the *Freudenthal's magic square* Table 4.3. On the other hand, the backward construction provides us rich examples of *FTS*'s which might not have a *JTS* construction available.

We will review both directions of the construction in the following two subsections and present the Lie algebra table in the appendix A.1 for reader's convenience.

4.2.1 From FTS to (Exceptional) Lie Algebra

Given a *Freudenthal Triple System* $\mathfrak{K}$ (4.9), one can construct a Lie algebra $\mathfrak{g}(\mathfrak{K})$ [6, 75]. We will follow the approach by H. Asano here [6]. To start with, let's consider the vector space $\mathfrak{g}(\mathfrak{K}) = \mathfrak{L}_{\mathfrak{K}} \oplus \mathfrak{su}(2) \oplus \mathfrak{K} \oplus \mathfrak{K}$, where $\mathfrak{L}_{\mathfrak{K}} \equiv \{\mathcal{L}_{x_i x_j} \equiv x_i \times x_j \,|\, x_i, x_j \in \mathfrak{K}\} \subset \mathfrak{L} \subset End(\mathfrak{K})$ defined in (4.7). Label the two copies of the *FTS* $\mathfrak{K}$ as $(\mathfrak{K}_1, \mathfrak{K}_2)$, and group the pair $(x_1, x_2) \in \mathfrak{K}_1 \oplus \mathfrak{K}_2$ as a column vector $\vec{x}$ such that the standard $\mathfrak{su}(2)$ generators $\{\sigma_i\}_{i=1,2,3}$ act as matrix multiplication. Then, one can check that for any $u, v, z, w \in \mathfrak{K}$ satisfying (4.9) the following

subspace	elements
$\mathfrak{g}^2$	σ^+
$\mathfrak{g}^1$	$x \in \mathfrak{K}_1$
$\mathfrak{g}^0$	$\sigma_z, \mathcal{L}_{uv}$
$\mathfrak{g}^{-1}$	$y \in \mathfrak{K}_2$
$\mathfrak{g}^{-2}$	σ^-

Table 4.1: Grading of the Lie algebra $\mathfrak{g}(\mathfrak{K})$

commutators:

$$\begin{aligned}
[\mathcal{L}_{uv}, \vec{x}\,] &\equiv \mathcal{L}_{uv}\vec{x}, \quad \text{element-wise operation,} \\
[\mathcal{L}_{uv}, \mathcal{L}_{zw}] &\equiv \mathcal{L}_{(uvz)w} + \mathcal{L}_{z(uvw)}, \\
[\,\sigma_i, \mathcal{L}_{uv}] &\equiv 0, \\
[\,\sigma_i, \vec{x}] &\equiv \sigma_i\vec{x}, \quad \text{matrix multiplication,} \\
[\,\vec{x}, \vec{y}\,] &\equiv \lambda\big(\langle y_1, x_2\rangle - \langle x_1, y_2\rangle\big)\,\sigma_z + 2\lambda\langle x_1, y_1\rangle\,\sigma^+ - 2\lambda\langle x_2, y_2\rangle\,\sigma^- \\
&\quad + \mathcal{L}_{x_1y_2} - \mathcal{L}_{y_1x_2},
\end{aligned} \tag{4.10}$$

together with the standard commutator of $(\sigma^{\pm}, \sigma_z)$ make $\mathfrak{g}(\mathfrak{K})$ a Lie algebra, which is simple if and only if the symplectic form $\langle x, y\rangle$ is non-degenerate [87, 26]. In the special case when the *Freudenthal Triple System* was constructed from the magic *Jordan Triple System* $\mathfrak{K}(\mathfrak{J}_{magic})$, Freudenthal proves that the Lie algebra $\mathfrak{g}(\mathfrak{K})$ so constructed are the exceptional Lie algebras [29, 30, 33, 62].

4.2.2 From Lie Algebra to FTS

In this subsection, we will review the construction of a *FTS* $\mathfrak{K}(\mathfrak{g})$ from a Lie algebra $\mathfrak{g}$ following closely to [81]. We first observe the Lie algebra $\mathfrak{g}(\mathfrak{K})$ constructed in the previous section admits a grading and can be decomposed as $\mathfrak{g}(\mathfrak{K}) = \mathfrak{g}^2 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$, such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ for all $\{i, j, (i+j)\}$ integer between -2 to 2. The elements in the Lie algebra is then grouped as Table 4.1.

To construct a *FTS* $\mathfrak{K}(\mathfrak{g})$, let's consider a simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{e}$ of rank r with $\mathfrak{h}$ its Cartan subalgebra. Denote by Δ the root lattice of $\mathfrak{g}$ and $\Delta^o \subset \Delta$ its simple roots, we shall introduce the Chevalley basis, which is a special choice of the basis $\{H_i\}_{i=1,\dots,r} \in \mathfrak{h}$ and

subspace	elements
$\mathfrak{g}^2$	$E_{\vec{\rho}}$
$\mathfrak{g}^1$	$\{E_{\vec{\alpha}} \in \mathfrak{e} \mid \vec{\alpha}\cdot\vec{\rho} = 1\}$
$\mathfrak{g}^0$	$\{E_{\vec{\alpha}} \in \mathfrak{e} \mid \vec{\alpha}\cdot\vec{\rho} = 0\} \oplus \mathfrak{h}$
$\mathfrak{g}^{-1}$	$\{E_{\vec{\alpha}} \in \mathfrak{e} \mid \vec{\alpha}\cdot\vec{\rho} = -1\}$
$\mathfrak{g}^{-2}$	$E_{-\vec{\rho}}$

Table 4.2: Grading of the Lie algebra $\mathfrak{g}$ from its root system

$\{E_{\vec{\alpha}}\} \in \mathfrak{e}$, such that:

$$\begin{aligned}[H_i, H_j] &= 0,\\ [H_i, E_{\vec{\alpha}_j}] &= \alpha_{ij}\, E_{\vec{\alpha}_j},\\ [E_{\vec{\alpha}_i}, E_{-\vec{\alpha}_i}] &= H_i,\end{aligned} \tag{4.11}$$

for all simple roots $\vec{\alpha}_{j=1,\ldots,r} \in \Delta^o$; and for any $\vec{\alpha}, \vec{\beta} \in \Delta$, one has

$$[E_{\vec{\alpha}}, E_{\vec{\beta}}] = \begin{cases} \pm(q+1)\, E_{\vec{\alpha}+\vec{\beta}}\,, & \text{if } \vec{\alpha}+\vec{\beta} \in \Delta \\ 0, & \text{otherwise} \end{cases}\,, \tag{4.12}$$

where $\alpha_{ij} \in M(r, \mathbb{Z})$ is the Cartan matrix of $\mathfrak{g}$ and q is the largest integer satisfying $\vec{\beta} - q\vec{\alpha} \in \Delta$.

We may rescale the root lattice so that the highest roots have length square 2. [4] Picking one generator $E_{\vec{\rho}} \in \Delta$ of highest root $\vec{\rho}$ and define $H \equiv [E_{\vec{\rho}}, E_{-\vec{\rho}}] \in \mathfrak{h}$, then $(E_{\vec{\rho}}, E_{-\vec{\rho}}, H)$ forms an $\mathfrak{su}(2)$ sub-algebra in $\mathfrak{g}$, which will be identified as $(\sigma^+, \sigma^-, \sigma_z)$ in the last subsection. With respect to $E_{\vec{\rho}}$, we can introduce a -2 to 2 grading to all the elements $E_{\vec{\alpha}} \in \mathfrak{e}$ according to the inner product $\vec{\alpha}\cdot\vec{\rho}$. The result is listed in Table 4.2.

Then, $\mathfrak{g}^1$ equipped with a symplectic two form $\langle x, y\rangle : \mathfrak{g}^1 \otimes \mathfrak{g}^1 \to \mathbb{R}$ and a ternary product $xyz : \mathfrak{g}^1 \otimes \mathfrak{g}^1 \otimes \mathfrak{g}^1 \to \mathfrak{g}^1$ for all $x, y, z \in \mathfrak{g}^1$ defined as:

$$\begin{aligned}\langle x, y\rangle E_{\vec{\rho}} &= \frac{1}{2\lambda}[x, y]\\ xyz &= \frac{1}{2}\big([z, [y, [x, E_{-\vec{\rho}}]]] + [z, [x, [y, E_{-\vec{\rho}}]]]\big),\end{aligned} \tag{4.13}$$

will satisfy all the conditions in (4.9), which establishes the claim: $\mathfrak{g}^1 \cong \mathfrak{K}(\mathfrak{g})$.

[4] For a simple Lie algebra, it can be shown that there are at most only two possible length of roots called long roots and short roots. We can normalize the long roots to be of length square 2.

4.3 Yang-Baxter Equation and Okubo's solution

4.3.1 Overview of Yang-Baxter Equation

Formally, the Yang-Baxter equation is an algebraic equation of the form:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{4.14}$$

which is understood as an equation over $V \otimes V \otimes V$. Here $\mathcal{R}_{ij}$ signifies the operator on $V \otimes V \otimes V$, which acts as a matrix $R \in End_{\mathbb{C}}(V \otimes V)$ on the ith and the jth components and as identity on the last component; for instance

$$\mathcal{R}_{12} = \mathcal{R} \otimes 1, \quad \mathcal{R}_{23} = 1 \otimes R. \tag{4.15}$$

Introducing a basis for V, one may write down the Yang-Baxter equation explicitly as a matrix equation (see appendix C.2 for more details):

$$R^{b'a'}_{a_1b_1}(\theta)R^{c'a_2}_{a'c_1}(\theta')R^{c_2b_2}_{b'c'}(\theta'') = R^{n'm'}_{b_1c_1}(\theta'')R^{c_2l'}_{a_1n'}(\theta')R^{b_2a_2}_{l'm'}(\theta), \quad \text{with } \theta' = \theta + \theta'', \tag{4.16}$$

where $\theta, \theta', \theta'' \in \mathbb{C}$ are called spectral parameters of the *R-matrix* $R^{cd}_{ab}(\theta)$. It is the core object of interest in the research of both theoretical physics and applied math ranging from statistical physics [48], integrable systems [48, 21], the braid group[88, 52], to the quantum group [48, 21, 59]. An interested reader may find more information in appendix C.

4.3.2 Okubo's solution

A systematic approach to its solution starts by seeking a solution of the form $r(\theta) : \mathbb{C} \to \mathfrak{g}\otimes\mathfrak{g}$ for some Lie algebra $\mathfrak{g}$ that solves the Yang-Baxter equation in the classical limit, and then a solution $R(\theta)$ is obtained by appropriate quantization of $r(\theta)$ [47]. This approach pioneered by V. Drinfel'd opened up the research of quantum groups and Yangian algebra [24, 9, 77, 58].

However, there also exist exotic solutions constructed by S. Okubo from a *Freudenthal Triple System* [74, 75]. Starting from a *FTS* $\mathfrak{K}$ of real dimension M, one may introduce a totally symmetric triple product [5] $[x_1, x_2, x_3] : \mathfrak{K} \otimes \mathfrak{K} \otimes \mathfrak{K} \to \mathfrak{K}$ by symmetrizing the non-symmetric triple product $x_1x_2x_3$. Denoting $\{e_1, \ldots, e_M\}$ a basis of the vector space $\mathfrak{K}$, one introduces the *R-matrix*:

$$R^{ab}_{cd}(\theta) \equiv \langle e^a, [e^b, e_c, e_d]_\theta \rangle,$$

and seeks for a solution to the (quantum) Yang-Baxter Equation (YBE for short) of the form:

$$[x_1, x_2, x_3]_\theta = P(\theta)[x_1, x_2, x_3] + A(\theta)\langle x_1, x_2\rangle\, x_3 + B(\theta)\langle x_3, x_1\rangle\, x_2 + C(\theta)\langle x_2, x_3\rangle\, x_1,$$

[5] The totally symmetric triple product $[x_1, x_2, x_3] : \mathfrak{K}\otimes\mathfrak{K}\otimes\mathfrak{K} \to \mathfrak{K}$ here is identical to the triple product $T(x_1, x_2, x_3)$ used in the following sections. We adhere to Okubo's convention here for notation simplicity and easy reference to his original paper [74, 75].

where the indices had been raised using the symplectic metric $g_{ab} \equiv \langle e_a, e_b \rangle$ over $\mathfrak{K}$. [6]

This is solved by Okubo in [74, 75], with

$$A(\theta) = \frac{M + 12\, b\, \theta + 1}{4(M + 24\, b\, \theta - 2)}\, P(\theta),$$
$$B(\theta) = \frac{M + 48\, b\, \theta - 2}{48}\, P(\theta),$$
$$C(\theta) = -\frac{M + 24\, b\, \theta + 4}{192\, b\, \theta}\, P(\theta),$$

provided

$$x_1 \cdot x_2 \cdot x_3 \equiv (x_1 x_2 e^j) e_j x_3 + \frac{M + 16}{24} x_1 x_2 x_3 = 0$$

is satisfied for all $x_1, x_2, x_3 \in \mathfrak{K}$. Here, b is an arbitrary constant and $P(\theta)$ can be any arbitrary function.

Remark: From this explicit solution to the YBE, the θ-dependent triple product $[x_1, x_2, x_3]_\theta$ doesn't have any symmetry; while the *R-matrix* $R^{ab}_{cd}(\theta)$ defined from it enjoys the symmetry $R^{ab}_{cd}(\theta) = R^{ba}_{dc}(\theta)$ as required in Okubo's construction.

Okubo further analyzed the condition $x_1 \cdot x_2 \cdot x_3 = 0$ under the assumption that the space $\mathfrak{K}$ is an irreducible $\mathfrak{g}$-module of some simple Lie algebra $\mathfrak{g}$. It turns out that only five solutions exist with the simple Lie algebras being $\mathfrak{g} = A_1, C_3, A_5, D_6$, and E_7 and the dimensions $M = 4, 14, 20, 32$, and 56 respectively. Observing that the above simple Lie algebras are actually the maximal simple subalgebras of the five exceptional Lie algebras $\mathfrak{g}_0 = G_2, F_4, E_6, E_7$, and E_8, one finds the five solutions can be neatly summarized in the *Freudenthal's magic square* Table 4.3, where one finds the five exceptional Lie algebras $\mathfrak{g}_0$ and their maximal simple subalgebra $\mathfrak{g}$ sit in the last two rows of the *magic square*. Moreover, taking the *Jordan algebras* on the top to be our $\mathfrak{J}$ in $\mathfrak{K} \equiv \mathfrak{J} \oplus \mathfrak{J} \oplus \mathbb{R} \oplus \mathbb{R}$, one sees that the vector spaces $\mathfrak{K}$ do have the right dimension required to give a solutions to the YBE.

4.4 5d/4d $\mathcal{N} = 2$ Supergravity and F-dual

Jordan Triple System and its corresponding *Freudenthal Triple System* $\mathfrak{K}$ has long known to play fundamental role in 5-dim and 4-dim $\mathcal{N} = 2$ supergravity since the pioneering work by Günaydin, Sierra, and Townsend in mid 1980's [35, 36]. It describes the scalar manifold, i.e. the moduli space, of the theory. It, therefore, also governs the black holes in the theory. In particular, in the 4d $\mathcal{N} = 2$ Maxwell-Einstein supergravity with $n_V + 1$ vector multiplet, a BPS black hole has $2n_V + 2$ electric and magnetic charges, which can be organized as an element $x \in \mathfrak{K}$ of real dimension $2n_V + 2$. Its entropy is given by the formula (4.30):

$$S^{(4)}_{BH} = \pi \sqrt{|\Delta(x)|},$$

[6] We adapt the NW-SE convention when raising/lowering the indices.

$\mathfrak{A}\diagdown\mathfrak{J}$	$\mathbb{R}$	$H_3(\mathbb{R})$	$H_3(\mathbb{C})$	$H_3(\mathbb{H})$	$H_3(\mathbb{O})$
$\mathbb{R}$	0	A_1	A_2	C_3	F_4
$\mathbb{C}$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	A_1	C_3	A_5	D_6	E_7
$\mathbb{O}$	G_2	F_4	E_6	E_7	E_8
$dim_{\mathbb{R}}\mathfrak{J}$	1	6	9	15	27
$dim_{\mathbb{R}}\mathfrak{K}$	4	14	20	32	56

Table 4.3: Freudenthal's magic square

spin	**2**	**3/2**	**1**	**1/2**	**0**	**indices**
1 gravity	$e_{\hat{\mu}}^{m}$	$\psi_{\hat{\mu}\dot{a}}$	$A_{\hat{\mu}}^{0}$			$\dot{a} = 1, 2 \in Sp(2)$
n_V $U(1)$ vector			$A_{\hat{\mu}}^{A}$	$\lambda_{\dot{a}}^{A}$	ϕ^A	$A = 1, \ldots, n_V$
vector/scalar content			$A_{\hat{\mu}}^{I}$		ϕ^A	$I = (0, A)$

Table 4.4: 5-dim $\mathcal{N} = 2$ SUGRA field contents

where $\Delta(x) : \mathfrak{K} \to \mathbb{R}$ is a quartic form, which we shall review later in this section, see (4.31). Since then, different aspects of the 5d/4d $\mathcal{N} = 2$ supergravity and its black holes have been intensively studied [37], including various string/M-theory constructions, see [63, 76]. However, not until very recently an unexpected non-linear duality, called F-dual, among such BPS black holes is discovered [17]. This discovery revives the interest in the study of *FTS*'s, in particular its mathematical structure, which is the main focus of the second half of this thesis.

In the following subsections, we will present a short review on 5d/4d $\mathcal{N} = 2$ supergravity focusing on the role played by *JTS*'s and the corresponding *FTS*'s.

4.4.1 5-dim $\mathcal{N} = 2$ SUGRA

Here we will review the 5dim $\mathcal{N} = 2$ supergravity; in particular, we want to see the roles played by the *JTS* in this theory. In the minimal setup, we consider a graviton multiplet, n_V $U(1)$ vector multiplets and one hyper multiplet, which doesn't play a essential role in the following. We list the field content in Table 4.4.

We are interested in the part of the 5*dim* action that couples the vector multiplets to the gravity, which is collected in the following:

$$S^{(5)} = \int d^5x \sqrt{-g_{(5)}} \left(R_{(5)} - G_{AB}\partial_{\hat{\mu}}\phi^A\partial^{\hat{\mu}}\phi^B\right) \\ - \int \mathring{a}_{IJ}\, F^I *_5 F^J + \frac{1}{24}\int C_{IJK}A^I \wedge F^J \wedge F^K. \tag{4.17}$$

Here, the real scalar fields $\{\phi^A\}$ take values on a hypersurface

$$\mathcal{M}_5 = \Big\{\xi^I \in \mathbb{R}^{n_V+1} \,\Big|\, N(\xi) \equiv \frac{1}{6}C_{IJK}\xi^I\xi^J\xi^K = 1\Big\}, \tag{4.18}$$

and are called *special (local) coordinates* of $\mathcal{M}_5$, distinguished by their relation to the $\mathcal{N} = 2$ multiplet structure.

In the action, the "couplings" G_{AB} and $\mathring{a}_{IJ}$ are functions of the scalar fields $\{\phi^A\}$, while the structure constant C_{IJK} of the *Chern-Simons* like term $A \wedge F \wedge F$ is totally symmetric and a constant as required by gauge invariance of the action. To be explicit, let's present the pullback metric G_{AB} on $\mathcal{M}_5$ and the gauge coupling $\mathring{a}_{IJ}$ here:

$$G_{AB} = -\frac{1}{2}\frac{\partial\xi^I}{\partial\phi^A}\frac{\partial\xi^J}{\partial\phi^B}\,\partial_{\xi^I}\partial_{\xi^J}N(\xi)\Big|_{\xi\in\mathcal{M}_5} \\ a_{IJ} = -\frac{1}{2}\,\partial_{\xi^I}\partial_{\xi^J}N(\xi), \quad \mathring{a}_{IJ} \equiv a_{IJ}(\xi)\big|_{\xi\in\mathcal{M}_5} = a_{IJ}\big(\xi(\phi^A)\big). \tag{4.19}$$

Based on the requirement of $\mathcal{N} = 2$ supersymmetry and the assumption that the manifold $\mathcal{M}_5$ is a symmetric space, Günaydin et al. studied the algebraic constraints that elements $\{\xi^I\}$ must satisfy. It turns out that the conditions so obtained match exactly with the defining conditions of a *JTS* with the cubic form (see section 4.1.1), and $N(\xi)$ is identified with the cubic norm $N_{\mathfrak{J}}(\xi)$ of the *JTS* $\mathfrak{J}$ [36].

4.4.2 Dimension Reduction to 4-dim

When taking the Kaluza-Klein reduction from 5d to 4d, each 5d vector field $A^I_{\hat{\mu}}$ splits into a 4d vector field together with one scalar field:

$$A^I_{\hat{\mu}} = (A^I_\mu, a^I)\ , \tag{4.20}$$

where $\mu = 0, 1, 2, 3$ is the four dimensional vector index. On the other hand, the 5d algebraic constraint $N_{\mathfrak{J}}(\xi) = 1$ is relaxed to $N_{\mathfrak{J}}(\xi) = e^{3\sigma}$, where σ is the KK scalar (i.e. dilaton) of the original 5d metric. Thus, upon absorbing the dilaton degree of freedom, the now unconstrained fields $\{\xi^I\}$ can combine with a^I to form a set of complex fields

$$\mathcal{X}^I \equiv a^I + i\xi^I \in \mathbb{C}^{n_V+1}, \tag{4.21}$$

which are the homogeneous holomorphic coordinates of a projective *special Kähler manifold* $\mathcal{M}_4$, which will also be called $\mathcal{M}_V$ in the following, of real dimension $2n_V$ with prepotential [7]

$$\mathcal{F} = N_{\mathfrak{J}}(\mathcal{X})/\mathcal{X}^0. \tag{4.22}$$

Denote by $\mathcal{F}_I = \partial_I \mathcal{F} \equiv \frac{\partial \mathcal{F}}{\partial \mathcal{X}^I}$, and $\mathcal{F}_{IJ} = \partial_I \partial_J \mathcal{F}$, the Kähler potential of $\mathcal{M}_V$ is:

$$\mathcal{K}(\mathcal{X}, \overline{\mathcal{X}}) = -\log\Big(i\,\overline{\mathcal{X}}^I \mathcal{F}_I - i\,\mathcal{X}^I \overline{\mathcal{F}}_I\Big), \tag{4.23}$$

which gives the Kähler metric in the (projective) *special coordinates* $\mathcal{Z}^A \equiv \mathcal{X}^A/\mathcal{X}^0$ as:

$$\frac{\partial^2 \mathcal{K}}{\partial \mathcal{Z}^A \partial \bar{\mathcal{Z}}^B} = \frac{3}{2}\, e^{-2\sigma} \mathring{a}_{AB}. \tag{4.24}$$

The gauge coupling can be re-organized by introducing the so-called *period matrix*:

$$\mathcal{N}_{IJ} = \overline{\mathcal{F}}_{IJ} + 2i \frac{\mathrm{Im}(\mathcal{F}_{IL})\mathrm{Im}(\mathcal{F}_{JM})\mathcal{X}^L \mathcal{X}^M}{\mathrm{Im}(\mathcal{F}_{PQ})\mathcal{X}^P \mathcal{X}^Q}. \tag{4.25}$$

Together, the $S^{(5)}$ now descends to a 4d action:

$$\begin{aligned} S^{(4)} = & \int d^4x\, \sqrt{-g_4}\, \Big(R_4 - \frac{3}{2}\, e^{-2\sigma} \mathring{a}_{AB}\, \partial_\mu \mathcal{Z}^A \partial^\mu \mathcal{Z}^B\Big) \\ & - \frac{1}{4}\int \mathrm{Im}(\mathcal{N}_{IJ}) F^I *_4 F^J + \frac{1}{8}\int \mathrm{Re}(\mathcal{N}_{IJ}) F^I \wedge F^J, \end{aligned} \tag{4.26}$$

and the algebraic constraint over ξ^I in 5d leading to the defining property of a *JTS* $\mathfrak{J}$ now descends to the defining property of a *FTS*, which states the homogeneous coordinates (4.21) $\mathcal{X}^I \in \mathbb{C}^{n_V+1}$ of $\mathcal{M}_V$ is identified as an element of a *FTS* $\mathfrak{K}(\mathfrak{J})$ of real dimension $2n_V + 2$.

4.4.3 The Freudenthal Triple System and $4D$ Black Holes

Given the 4-dim action $S^{(4)} = \int d^4x\, \mathcal{L}^{(4)}$ (4.26), the black hole entropy is given by

$$S^{(4)}_{BH} = 2\pi \int_\Sigma \frac{\partial \mathcal{L}^{(4)}}{\partial R_{\mu\nu\rho\sigma}} \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} \sqrt{-g_4}\, d\Omega, \tag{4.27}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor, and the integral is over the event horizon Σ of the black hole. It can be evaluated in terms of the magnetic/electric charges $(p^I, q_J) \in \mathbb{Z}^{2n_V+2}$ of the black hole via solving the attractor flow equation. In our *special coordinates* it can be simplified to the algebraic equation:

$$\left[\overline{Z} \begin{pmatrix} \mathcal{X}^I \\ \mathcal{F}_J \end{pmatrix} - Z \begin{pmatrix} \overline{\mathcal{X}}^I \\ \overline{\mathcal{F}}_J \end{pmatrix}\right]_\Sigma = i \begin{pmatrix} p^I \\ q_J \end{pmatrix},$$

[7] The terminology *prepotential* here and *period matrix* later are originated from string theory approach of the $\mathcal{N} = 2$ supergravity discussed here. Interested reader may find a short review in appendix B.

where

$$Z \equiv e^{\mathcal{K}/2}(p^I \mathcal{F}_I - q_I \mathcal{X}^I), \tag{4.28}$$

is called the central charge function, which is a spacetime field that asymptotes to the central charge of the energy-momentum tensor at infinity. [8]

It turns out that the black hole entropy (4.27) so evaluated can be organized neatlyby grouping the charges (p^0, p^A, q_0, q_A) as an element x in FTS: [9]

$$x = \left(p^I \hat{X}_I, q_I \hat{X}^I, p^0, q_0\right) \equiv \left(X, Y, \xi, \omega\right) \in \mathfrak{K}(\mathfrak{J}), \tag{4.29}$$

where $\left\{\hat{X}_I\right\}_{I=1,\ldots,n_V}$ is a basis of the JTS $\mathfrak{J}$. Then, the entropy is given by

$$S_{BH}^{(4)} = \pi\sqrt{|\Delta(x)|}, \tag{4.30}$$

with the quartic form $\Delta(x) : \mathfrak{K} \to \mathbb{R}$ defined as

$$\Delta(x) = -4\left\{\xi N_{\mathfrak{J}}(X) + \omega N_{\mathfrak{J}}(Y) + \frac{1}{4}\left(\xi\omega - Tr_{\mathfrak{J}}(X,Y)\right)^2 - Tr_{\mathfrak{J}}(X^{\#}, Y^{\#})\right\}. \tag{4.31}$$

4.4.4 F-dual

A novel observation by Duff et al. in [17] was that by introducing a nondegenerate symplectic form $\langle x_1, x_2 \rangle : \mathfrak{K} \otimes \mathfrak{K} \to \mathbb{R}$ as

$$\langle x_1, x_2 \rangle \equiv \xi_1\omega_2 - \xi_2\omega_1 + Tr_{\mathfrak{J}}(X_1, Y_2) - Tr_{\mathfrak{J}}(X_2, Y_1),$$

one can define a totally symmetric *Freudenthal triple product* $T(x_1, x_2, x_3) : \mathfrak{K}^{\otimes 3} \to \mathfrak{K}$ from the equation:

$$\langle\, T(x_1, x_2, x_3), x_4 \,\rangle = 2\,\Delta(x_1, x_2, x_3, x_4),$$

where $\Delta(x_1, x_2, x_3, x_4)$ is the *full linearization* of the quartic form $\Delta(x)$ defined above. And the black hole entropy is invariant under the *Freudenthal dual* (dubbed as *F-dual*):

$$x \to \tilde{x} \equiv T(x)|\Delta(x)|^{-1/2},$$

with $T(x) \equiv T(x, x, x)$. The proof of this statement can be found in the appendix of the paper presented in the next chapter, see section 5.6.1.

Remark: The confusing terminology *Freudenthal Triple Product* is of historical origin. It is related to the ternary product $xyz : \mathfrak{K}^{\otimes 3} \to \mathfrak{K}$ introduced in section 4.1.2 via symmetrization: $T(x_1, x_2, x_3) = \frac{1}{3!}\sum_{\sigma \in S_3} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$.

[8] The string theory aspects of the central charge function may be find in (B.6)

[9] Here, we adapt the convention in subsection 4.1.2, where an element $x \in \mathfrak{K}(\mathfrak{J})$ will be represented as (X, Y, ξ, ω) with $X, Y \in \mathfrak{J}$ and $\xi, \omega \in \mathbb{R}$.

We collect the explicit formula of $T(x_1, x_2, x_3)$ and $\Delta(x_1, x_2, x_3, x_4) = \langle T(x_1, x_2, x_3), x_4\rangle$ below for reader's reference.

$$T(x_1, x_2, x_3) \equiv \sum_{\sigma\in S_3} \begin{pmatrix} \frac{1}{2}Y_1\#\big(Y_2\# Y_3\big) - \frac{1}{8}\Big(Tr_{\mathfrak{J}}(X_2, Y_3) - \xi_2\omega_3\Big)X_1 - \frac{1}{4}\,\omega_1\, Y_2\# Y_3 \\ -\frac{1}{2}X_1\#\big(X_2\# X_3\big) + \frac{1}{8}\Big(Tr_{\mathfrak{J}}(Y_2, X_3) - \omega_2\xi_3\Big)Y_1 + \frac{1}{4}\,\xi_1\, X_2\# X_3 \\ -\frac{1}{12}N_{\mathfrak{J}}(Y_1, Y_2, Y_3) + \frac{1}{8}\Big(Tr_{\mathfrak{J}}(X_2, Y_3)\Big)\xi_1 - \frac{1}{8}\,\omega_1\xi_2\xi_3 \\ \frac{1}{12}N_{\mathfrak{J}}(X_1, X_2, X_3) - \frac{1}{8}\Big(Tr_{\mathfrak{J}}(Y_2, X_3)\Big)\omega_1 + \frac{1}{8}\,\xi_1\omega_2\omega_3 \end{pmatrix}_{\sigma}, \tag{4.32}$$

and

$$\langle\, T(x_1, x_2, x_3), x_4\,\rangle = \sum_{\sigma\in S_4} \left\{ \begin{matrix} \frac{1}{4}Tr_{\mathfrak{J}}(X_1\# X_2, Y_3\# Y_4) - \frac{1}{12}\Big(N_{\mathfrak{J}}(X_1, X_2, X_3)\,\xi_4 + N_{\mathfrak{J}}(Y_1, Y_2, Y_3)\,\omega_4\Big) \\ -\frac{1}{16}\Big(Tr_{\mathfrak{J}}(X_1, Y_2) - \xi_1\omega_2\Big)\Big(Tr_{\mathfrak{J}}(X_3, Y_4) - \xi_3\omega_4\Big) \end{matrix} \right\}_{\sigma}. \tag{4.33}$$

Here, $\sum_{\sigma\in S_n}(\ldots)_\sigma$ means summing over all permutation of the indices $1, 2, \ldots, n$.

Chapter 5

Freudenthal Gauge Theory

5.1 Introduction

The idea that a ternary algebra might be an essential structure governing a physical system has a long history. It can be traced back to the early 70's, when Y. Nambu proposed a generalized Hamiltonian system based on a ternary product: the Nambu-Poisson bracket. Since then, physicists have tried to apply ternary algebras to various physical systems without much success. Despite some partial results [22] the quantization of the Nambu-Poisson bracket remains to be a long term puzzle. Almost four decades latter, the ternary algebra re-appeared in the study of M-theory by J. Bagger, N. Lambert [8] and separately by A. Gustavsson [38], where a ternary Lie-3 algebra is proposed as the underlying gauge symmetry structure on a stack of supersymmetric M2-branes. This is the famous BLG theory. When taking the Nambu-Poisson bracket as an infinite-dimensional generalization of the Lie-3 bracket, one gets from the BLG theory a novel six-dimensional field theory, which can be interpreted as a non-commutative version of the M5-brane theory [40].

In this note, we will propose a new gauge field theory based on another ternary algebra: the *Freudenthal Triple System* (*FTS*). [1] In the minimal setup, our model contains a bosonic scalar field $\phi(x)$ valued in the *FTS* $\mathfrak{K}$ together with a gauge field $A_\mu(x)$ taking values in $\mathfrak{K} \otimes \mathfrak{K}$. Similar to the BLG theory, the gauge transformation in our theory is constructed from a triple product defined over the *FTS* $\mathfrak{K}$. However, unlike the totally anti-symmetric Lie-3 bracket used in the BLG theory, this triple product doesn't have simple symmetry

[1] Historically, there are several different notions of *Freudenthal Triple System*, which differ by the symmetry structure of their triple product. They were introduced in the mathematics literature addressing different algebraic properties of the triple system. Although simply related, those different definitions of *FTS* has different properties, which of course can be translated from one to another. In the physics literature, the *FTS* we focus on in this paper are sometimes also called the *Generalized Freudenthal Triple System*, which makes the derivation property more transparent. Since there is no general agreement on the standard notion of the *FTS*, we will simple denote the triple system in this paper by *Freudenthal Triple System* (*FTS*), to which, the *FTS* introduced in the $\mathcal{N} = 2$ Maxwell-Einstein supergravity literature can be regarded as a special case of it.

structure with respect to exchanging a pair of its arguments. Nevertheless, one can still show the gauge invariance of this model is guaranteed by the algebraic properties of the *FTS*.

Besides the gauge symmetry, our model also possesses a novel global symmetry; the *Freudenthal duality* (*F-dual* for short), which is a non-linear mapping from $\mathfrak{K}$ to $\mathfrak{K}$. Mathematically, it was a long observed non-linear identity over *FTS*, which (to the author's best knowledge) can be traced back from the early days of the *FTS* [18] In physics literature, the *F-dual* was first observed in $\mathcal{N}=2$ Maxwell-Einstein supergravity as a duality of black hole charges that keeps the black hole entropy invariant [17]; where the electric and magnetic charges of a supersymmetric black hole for the n $U(1)$ vector multiplets $(\vec{q}_e, \vec{q}_m)$ is regarded as an element in the *FTS* $\mathfrak{K}$. Various attempts to understand the *F-dual* from other aspects of the $\mathcal{N}=2$ Maxwell-Einstein supergravity have been made recently, see for instance [28]. While the relation (if any) between our model and the supergravity is unclear at this moment, our model can be regarded as the minimal setup that admits the *F-duality*.

This paper is organized as follows: In section 2, we present the definition of a *Freudenthal Triple System* and give a short summary on the properties of the *FTS* that is relevant to our model. In section 3, the gauge transformation constructed from the triple product is defined. In the same section, we propose a minimal action that possesses both gauge symmetry and *F-dual* and then give a detailed proof of its invariance under both of the symmetries. Section 4 is devoted to the generalization of the minimal action introduced in section 3. There, we couple the *FTS* $\mathfrak{K}$ to a most general algebraic system and study the mathematical structure that is required to define the *F-dual*. The results is summarized as a No-Go theorem in section 4.2. We summarize the construction of the *Freudenthal Gauge Theory* and remark on its potential link to other fields of physics in section 5.

5.2 Freudenthal Triple Systems

Historically, *Freudenthal Triple System* (*FTS*) $\mathfrak{K}$ was first introduced by H. Freudenthal in his study of exceptional Lie-algebras. In the classical construction, $\mathfrak{K}$ is taken to be the direct sum of two copies of a *Jordan Triple System* (*JTS*) $\mathfrak{J}$ and two copies of real numbers $\mathbb{R}$: $\mathfrak{K}(\mathfrak{J})=\mathfrak{J}\oplus\mathfrak{J}\oplus\mathbb{R}\oplus\mathbb{R}$. Over the vector space $\mathfrak{K}$ a symplectic form and a triple product can be defined via the totally symmetric tri-linear form (also known as the *cubic norm*) of the *JTS*. Such a triple product can be re-interpreted as a linear map $\mathcal{L}_{\phi_I\phi_J}$ over $\mathfrak{K}$ parametrized by a pair of elements $\phi_I, \phi_J \in \mathfrak{K}$, which we will review momentarily. To get the five exceptional Lie-algebra, one first restrict the *JTS* $\mathfrak{J}$ to one of the following five cases(denoted by $\mathfrak{J}_{magic}$): $\mathfrak{J}=\mathbb{R}$ or $\mathfrak{J}=H_3(\mathfrak{A})$, where $H_3(\mathfrak{A})$ is the algebra hermitian 3×3 matrices with entries taking values in one of the four *normed division algebras* $\mathfrak{A}=\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. Denote by $\mathfrak{M}$ a hypersurface in $\mathfrak{K}(\mathfrak{J}_{magic})$ defined by $\mathfrak{M}=\big\{\phi_I\in\mathfrak{K}(\mathfrak{J}_{magic})\,|\,\mathcal{L}_{\phi_I\phi_I}\phi_K=0,\ \forall\,\phi_K\in\mathfrak{K}(\mathfrak{J}_{magic})\big\}$, then the five exceptional Lie-algebras $\mathfrak{g}$ arise as the the direct sum of the sub-algebra $\mathrm{Inv}(\mathfrak{M})$ of $\mathfrak{psl}\big(\mathfrak{K}(\mathfrak{J}_{magic})\big)$ that keeps the surface $\mathfrak{M}$ invariant together with a copy of $\mathfrak{su}(2)$ and two copies of $\mathfrak{K}(\mathfrak{J}_{magic})$ [29, 30, 87]: $\mathfrak{g}=\mathrm{Inv}(\mathfrak{M})\oplus\mathfrak{su}(2)\oplus\mathfrak{K}(\mathfrak{J})\oplus\mathfrak{K}(\mathfrak{J})|_{\mathfrak{J}=\mathfrak{J}_{magic}}$. In the mathematics literature, there are several different notions of *FTS*, which differ by the symmetric structure of the triple product, for instance: [18, 26, 27]. All of these "*FTS*'s" are closely related by simple redefinitions; however, because they have different symmetric properties, some algebraic properties of the *FTS* will be transparent only in a specific setup.

5.2.1 Definition of a Freudenthal Triple System

To our purpose, a *FTS* is a *Symplectic Triple System*, which is a symplectic vector space $\mathfrak{K}$ equipped with a triple product $\phi_I\phi_J\phi_K : \mathfrak{K}\otimes\mathfrak{K}\otimes\mathfrak{K} \to \mathfrak{K}$, defined for any three elements $\phi_I, \phi_J, \phi_K \in \mathfrak{K}$. The triple product satisfies the following axioms:mplectic vector space $\mathfrak{K}$ equipped with a triple product $\phi_I\phi_J\phi_K : \mathfrak{K}\otimes\mathfrak{K}\otimes\mathfrak{K} \to \mathfrak{K}$, defined for any three elements $\phi_I, \phi_J, \phi_K \in \mathfrak{K}$. The triple product satisfies the following axioms:

(*i*) $\phi_I\phi_J\phi_K = \phi_J\phi_I\phi_K$

(*ii*) $\phi_I\phi_J\phi_K = \phi_I\phi_K\phi_J + 2\lambda\langle\phi_J,\phi_K\rangle\phi_I + \lambda\langle\phi_I,\phi_K\rangle\phi_J - \lambda\langle\phi_I,\phi_J\rangle\phi_K$

(*iii*) $\phi_L\phi_M(\phi_I\phi_J\phi_K) = (\phi_L\phi_M\phi_I)\phi_J\phi_K + \phi_I(\phi_L\phi_M\phi_J)\phi_K + \phi_I\phi_J(\phi_L\phi_M\phi_K)$

(*iv*) $\langle\phi_L\phi_M\phi_I,\phi_J\rangle + \langle\phi_I,\phi_L\phi_M\phi_J\rangle=0$,

where $\langle\phi_I,\phi_J\rangle : \mathfrak{K}\otimes\mathfrak{K} \to \mathbb{R}$ is the symplectic form defined in $\mathfrak{K}$, and λ is an arbitrary constant. Note that the third axiom above allows one to introduce for any pair of $\phi_L, \phi_M \in \mathfrak{K}$ a linear operator $\mathcal{L}_{\phi_L\phi_M} \in \mathfrak{gl}(\mathfrak{K})$ acting on $\phi_K \in \mathfrak{K}$ as:

$$\mathcal{L}_{\phi_L\phi_M}\,\phi_K \equiv \phi_L\phi_M\phi_K, \tag{5.1}$$

which is a derivation with respect to the triple product. And then the fourth axiom states that the symplectic form is invariant under the variation generated by $\mathcal{L}_{\phi_L\phi_M}$. That is, we can reformulate the last two axioms as:

(*iii′*) $\mathcal{L}_{\phi_L\phi_M}(\phi_I\phi_J\phi_K) = (\mathcal{L}_{\phi_L\phi_M}\,\phi_I)\phi_J\phi_K + \phi_I(\mathcal{L}_{\phi_L\phi_M}\,\phi_J)\phi_K + \phi_I\phi_J(\mathcal{L}_{\phi_L\phi_M}\,\phi_K)$

(*iv′*) $\mathcal{L}_{\phi_L\phi_M}\langle\phi_I,\phi_J\rangle = \langle\mathcal{L}_{\phi_L\phi_M}\,\phi_I,\phi_J\rangle + \langle\phi_I,\mathcal{L}_{\phi_L\phi_M}\,\phi_J\rangle = 0.$

Let us remark here that when $\lambda \neq 0$ the fourth axiom can actually be derived from the previous three. Mathematically, whenever $\lambda \neq 0$ the second axiom is a very strong compatibility condition that constrains the structure of the triple product and the symplectic form defined over the vector space $\mathfrak{K}$, and hence the non-trivial algebraic structure of the *FTS*. On the other hand, when $\lambda = 0$, the first three axioms reduce to the defining properties of a Lie-3 algebra defined over Grassmanian numbers, which in general is not a *FTS*. And hence, one has to further impose the fourth axiom as a compatibility condition between the (now totally symmetric) triple product and the symplectic form and hence restores the algebraic structure of the *FTS* $\mathfrak{K}$. In any rate, we include the fourth condition as part of the axioms so that the most generic situation will be included.

It will be convenient for our discussion later to introduce a basis $\{e_a\}$ of $\mathfrak{K}$ that allows one to write down the symplectic metric and the *triple product structure constant* (or simply *structure constant* for short) as:

$$\begin{aligned} \langle e_a, e_b \rangle &= \omega_{ab} \\ e_a e_b e_c &= f_{abc}{}^d e_d. \end{aligned} \tag{5.2}$$

When the symplectic metric ω_{ab} is non-degenerate, which we will always assume to be true in this paper, one has an isomorphism between the vector space $\mathfrak{K}$ and its dual space, and hence can lower the last index of the *structure constant* [2] : $f_{abcd} \equiv f_{abc}{}^e \omega_{ed}$. In terms of these coefficients, the defining axioms can be re-written as:

(i) $f_{abcd} = f_{bacd}$

(ii) $f_{abcd} = f_{acbd} + 2\lambda\omega_{ad}\omega_{bc} - \lambda\omega_{ca}\omega_{bd} - \lambda\omega_{ab}\omega_{cd}$

(iii) $f_{abc}{}^d f_{efd}{}^g = f_{efa}{}^d f_{dbc}{}^g + f_{efb}{}^d f_{adc}{}^g + f_{efc}{}^d f_{abd}{}^g$

(iv) $f_{abcd} = f_{abdc}$.

Note that property (i), (ii), and (iv) imply the *structure constants* are invariant under switching of the first pair and last pair of its indices, that is: $f_{abcd} = f_{cdab}$. This will be important in our construction of a Chern-Simons action for the gauge fields later.

5.2.2 Freudenthal Duality

Because of the fourth axiom, one may define a quartic form $\Delta(\phi)$ for any $\phi = \phi^a e_a \in \mathfrak{K}$ from the triple product and symplectic form as follows:

$$\Delta(\phi) = \frac{1}{2}\langle \phi\phi\phi, \phi \rangle = \frac{1}{2} f_{abcd}\phi^a\phi^b\phi^c\phi^d. \tag{5.3}$$

In the following, we will set $T(\phi) = \phi\phi\phi$ to simplify the notation. The quartic form $\Delta(\phi)$ has appeared in various places, most notably in the formula for the entropy of a supersymmetric $\mathcal{N} = 2$ black hole in 4-dimensional spacetime.

From the axioms of the *FTS* one can show that $\Delta(\phi)$ is invariant under the following transformation $F : \mathfrak{K} \to \mathfrak{K}$

$$F : \phi \mapsto \tilde{\phi} = \frac{T(\phi)}{\sqrt{6\lambda\,\Delta(\phi)}}. \tag{5.4}$$

That is $\Delta(\phi) = \Delta(\tilde{\phi})$, whose proof can be found in the appendix A by the end of this paper. In the physics literature, the map F is called "*Freudenthal Duality*" (or *F-duality* for short), which was first observed in [17] as a symmetry of the black hole entropy formula.

Several remarks are ready here:

[2] In this note, we will adopt the NE-WS convention when raising or lowering the indices using the symplectic metric.

- The *F-dual* F, if defined, squares to negative identity, i.e. $F \circ F = -1 \in \mathfrak{gl}(\mathfrak{K})$. This happens exactly when ϕ is an element in $\mathfrak{M}^c$, the complement of the hypersurface $\mathfrak{M} = \{\phi \in \mathfrak{K} \,|\, \mathcal{L}_{\phi\phi} = 0\} \subset \mathfrak{K}$ defined in the introduction of this section. Recall that for $\lambda \neq 0$ and for any $\phi \in \mathfrak{K}$, the *F-dual* $\tilde{\phi}$ is defined if and only if $\Delta(\phi) \neq 0$. Then this assertion is established by the identity [3] :

$$\langle \phi\phi\phi, \phi \rangle \equiv \langle \mathcal{L}_{\phi\phi}\phi, \phi \rangle = -\frac{1}{3} Tr\big(\mathcal{L}_{\phi\phi}\,\mathcal{L}_{\phi\phi}\big), \tag{5.5}$$

  where on the right hand side, one regards $\mathcal{L}_{\phi\phi}$ as hermitian matrices in $\mathfrak{gl}(\mathfrak{K})$; and the fact that the trace of the square of a hermitian matrix is zero if and only if the matrix is zero. Geometrically, one can understand the *F-duality* as follows: given any ϕ in the *Lagrangian submanifold* L of $\mathfrak{K}$ as a symplectic vector space, the *F-duality* F gives up to a sign an one-to-one pairing from ϕ to another element $\tilde{\phi}$ in the complement *Lagrangian submanifold* L^c of $\mathfrak{K}$ because of the property: $\tilde{\tilde{\phi}} = -\phi$. Thus, from the discrete orbit of the *F-duality*, one gets a four-component decomposition of the space $\mathfrak{K}$.

- The *F-dual* is not defined when $\lambda = 0$. Recall that when $\lambda = 0$, one has a totally symmetrized triple product. Thus, naively this statement seems to infer that no *F-dual* is defined when one has a totally symmetric triple product, which contradicts the observed *F-dual* in the black hole theory. While this is indeed true when λ is taken to be zero strictly, the *F-dual* can still be defined if one resolves the singularity by regarding $\lambda = 0$ as the limit of $\lambda \to 0$. Detailed analysis of Freudenthal's original construction shows that, as remarked in the previous section, there are indeed subtle λ dependence in the definition of the symplectic form and the triple product. In particular, if one takes $\lambda \to 0$ while keeping the symplectic form finite, then one gets a trivial triple product (except some very trivial examples in lower dimensions). Thus, in order to keep a non-trivial triple product when $\lambda \to 0$, the symplectic form has to diverge at the same time, s.t. the combination $\lambda\,\Delta(\phi)$ remains finite. Note, in this limit, one doesn't get a totally symmetric triple product when $\lambda \to 0$. Then the totally symmetric triple product used by the SUGRA society is really the non-symmetric triple product studied in this paper (with $\lambda = \frac{1}{6}$) whose anti-symmetric part being projected our when one only inserts symmetric arguments. Indeed, one can check that the axiom (iv) needed to define the four-form $\Delta(\phi)$ remains to be true if one instead consider a totally symmetric triple product $[\phi_I, \phi_J, \phi_K] = \phi_I\phi_J\phi_K - \lambda\,\langle\phi_J, \phi_K\rangle\phi_I - \lambda\,\langle\phi_I, \phi_K\rangle\phi_J = \frac{1}{3!}\sum_{\sigma\in S_3}\phi_{\sigma(I)}\phi_{\sigma(J)}\phi_{\sigma(K)}$.

We shall end this section with the following comment: *F-duality* is a non-linear map over $\mathfrak{K}$, and it is not a derivation with respect to the triple product over $\mathfrak{K}$. Thus this mathematical structure cannot be used to define an infinitesimal transformation consistently. This means

[3] This specific coefficient is valid for the case when the *FTS* $\mathfrak{K}$ is $\mathfrak{K}(H_3(\mathbb{O}))$. In other cases, one might get different, but non-zero coefficient, s.t. the argument remains to be valid.

the invariance observed above couldn't be a continuous symmetry; it is a duality of the system.

5.3 Freudenthal Gauge Theory

In this section, we will introduce the gauge transformation of our theory based on the *FTS* reviewed in previous sections. Our construction is parallel to the one in the BLG theory; however we present here a detailed analysis for the completeness of this paper, also we shall make several remarks during the construction to address the differences between our theory and BLG theory.

5.3.1 Global Symmetry

In the minimal setup, we consider a scalar field $\phi(x)$ valued in a *FTS* $\mathfrak{K}$ and we want to construct a Lagrangian density $\mathcal{L}[\phi(x)]$, which will have the desired symmetry. Since the Lagrangian density is necessarily a scalar quantity, all the terms in the Lagrangian density $\mathcal{L}[\phi(x)]$ must be schematically of the form $\alpha(\phi)\,\langle f(\phi), g(\phi)\rangle$, for some (potentially non-linear) scalar function $\alpha : \phi(x) \mapsto \mathbb{R}$ and functions $f, g : \phi(x) \mapsto \mathfrak{K}$. More precisely, at each point x in spacetime, $f\big(\phi(x)\big)$ and $g\big(\phi(x)\big)$ are elements of the subalgebra $\mathfrak{K}_{\phi(x)} \subset \mathfrak{K}$ generated by the element $\phi(x) \in \mathfrak{K}$. Explicitly, elements of $\mathfrak{K}_{\phi(x)}$ are polynomials in $\phi(x)$ with the multiplication defined by the non-associative triple product over $\mathfrak{K}$. Then the third axiom of the *FTS* allows one to define consistently an infinitesimal transformation $\mathcal{L}_\Lambda \in \mathfrak{gl}(\mathfrak{K})$, s.t.

$$\Big(f\big((1+\mathcal{L}_\Lambda)\phi(x)\big) - f\big(\phi(x)\big)\Big)_{linear\ order} = \mathcal{L}_\Lambda f\big(\phi(x)\big), \tag{5.6}$$

where $\Lambda \in \mathfrak{K}\otimes_S\mathfrak{K}$ is an infinitesimal parameter that generators the transformation. Note that only elements in the symmetric tensor product $\mathfrak{K}\otimes_S\mathfrak{K}$ can generate a transformation, since any elements in $\mathfrak{K}\otimes_A\mathfrak{K}$ will be projected out due to the symmetry property of exchanging the first two entries of the triple product. Now the fourth axiom states: for any $f(\phi), g(\phi) \in \mathfrak{K}$, $\langle f(\phi), g(\phi)\rangle$ transforms as

$$\mathcal{L}_\Lambda\langle f(\phi), g(\phi)\rangle = \langle \mathcal{L}_\Lambda f(\phi), g(\phi)\rangle + \langle f(\phi), \mathcal{L}_\Lambda g(\phi)\rangle = 0. \tag{5.7}$$

While, by the same argument, all the scalar functions $\alpha(\phi)$ are necessarily of this form, one concludes that any Lagrangian density $\mathcal{L}[\phi(x)]$ will be invariant under the infinitesimal transformation, i.e. by the four axioms, any Lagrangian $\mathcal{L}[\phi(x)]$ will be guaranteed to be invariant under the global symmetry generated by $\mathcal{L}_\Lambda$.

Before closing this subsection, let's remark here that the dual fields $\tilde{\phi}(x)$ are also elements in $\mathfrak{K}_{\phi(x)}$, which is easy to see from the explicit definition (5.4). And hence, it transforms in the same way as $\phi(x)$ under the global symmetry $\mathcal{L}_\Lambda$.

5.3.2 Gauge Symmetry

Now we will try to gauge the symmetry by promoting the infinitesimal generator $\Lambda \in \mathfrak{K}\otimes_S\mathfrak{K}$ to be a function $\Lambda(x)$ over spacetime. Adopting a basis $\{e_a\}$ for $\mathfrak{K}$, one can write down the gauge transformation of a field $\phi(x) = \phi^a(x)e_a$ in the following form:

$$\mathcal{L}_\Lambda\phi(x) = \Lambda^{ab}(x)\mathcal{L}_{e_ae_b}\phi(x) = f_{abc}{}^d\Lambda^{ab}(x)\phi^c(x)e_d, \tag{5.8}$$

where $\Lambda^{ab}(x)$ is an infinitesimal symmetric rank two tensor that generates the gauge transformation. In the following, we will often drop the explicit x-dependence of $\Lambda(x)$ to simplify the notation whenever confusion is unlikely to occur. It's convenient to define the linear operator $\hat{\Lambda} \in \mathfrak{gl}(\mathfrak{K})$ as

$$\widehat{\Lambda}_b{}^a = f_{cdb}{}^a\Lambda^{cd}. \tag{5.9}$$

In terms of this linear operator $\hat{\Lambda}$, the gauge symmetry transformation of a field $\phi(x)$ reduces to simple matrix multiplication $\mathcal{L}_\Lambda\phi^a = \widehat{\Lambda}_b{}^a\phi^b$. Note, as discussed in the previous section, the gauge transformation of the dual field $\tilde{\phi}(x)$ is automatically fixed to be $\mathcal{L}_\Lambda\tilde{\phi}^a = \widehat{\Lambda}_b{}^a\tilde{\phi}^b$ by construction.

Our next task is to introduce a gauge field $A_\mu^{ab}(x)$, and define the gauge covariant derivative D_μ valued in $\mathfrak{gl}(\mathfrak{K})$, which acts on the scalar field $\phi^a(x)$ as:

$$D_\mu\phi^a(x) \equiv \partial_\mu\phi^a(x) + (\widehat{A}_\mu)_b{}^a(x)\phi^b(x), \tag{5.10}$$

where to the gauge field $A_\mu(x) \equiv A_\mu^{ab}(x)\, e_a \otimes e_b$ valued in $\mathfrak{K}\otimes_S\mathfrak{K}$, $(\widehat{A}_\mu)_b{}^a(x) \equiv f_{cdb}{}^a A_\mu^{cd}(x)$ is the corresponding linear operator in $\mathfrak{gl}(\mathfrak{K})$. We then require $D_\mu\phi(x)$ transforms in the same way as $\phi(x)$:

$$\mathcal{L}_\Lambda\left(D_\mu\phi^a(x)\right) = (\mathcal{L}_\Lambda D_\mu)\phi^a(x) + D_\mu(\mathcal{L}_\Lambda\phi)^a(x) = \widehat{\Lambda}_b{}^a(x)(D_\mu\phi)^b(x), \tag{5.11}$$

which fixes the gauge transformation $\widehat{A}_\mu(x)$ to be:

$$\mathcal{L}_\Lambda\widehat{A}_\mu(x) = \partial_\mu\widehat{\Lambda}(x) + \left[\widehat{A}_\mu(x), \widehat{\Lambda}(x)\right] \equiv D_\mu\widehat{\Lambda}(x), \tag{5.12}$$

i.e. $\widehat{A}_\mu(x)$ transforms as an one-form valued in the adjoint representation of $\mathfrak{gl}(\mathfrak{K})$. Therefore, the *hat-map*: $\mathfrak{K}\otimes_S\mathfrak{K} \to \mathfrak{gl}(\mathfrak{K})$, which allows one to evaluate the gauge transformation defined via the triple product as standard matrix multiplications, gives an embedding of the our gauge transformation $\mathcal{L}_\Lambda$ into the Lie-algebra $\mathfrak{gl}(\mathfrak{K})$ as a sub-algebra. Note, this embedding ensures that the "trace" of the field strength $\widehat{F}_{\mu\nu}(x)$ defined as $\widehat{F}_{\mu\nu} \equiv [D_\mu, D_\nu] \in \mathfrak{gl}(\mathfrak{K})$ will automatically be gauge invariant. For later convenience, let's list the explicit form the the field strength $\widehat{F}_{\mu\nu}$ and its gauge transformation here:

$$\begin{aligned} \widehat{F}_{\mu\nu} &= \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu + [\widehat{A}_\mu, \widehat{A}_\nu] \\ \mathcal{L}_\Lambda\widehat{F}_{\mu\nu} &= [\widehat{F}_{\mu\nu}, \widehat{\Lambda}]. \end{aligned} \tag{5.13}$$

5.3.3 The Minimal Action

Given the construction in the previous sections, it is now easy to formulate the minimal action. First of all, for a generic term $\alpha(\phi)\langle f(\phi), g(\phi)\rangle$ in the Lagrangian, the anti-symmetry of the symplectic form requires $f(\phi)$ and $g(\phi) \in \mathfrak{K}_{\phi(x)}$ live in complement Lagrangian submanifolds of $\mathfrak{K}$. To the lowest order in $\phi(x)$ one gets the quadratic term: [4]

$$\langle \phi, \tilde{\phi} \rangle, \tag{5.14}$$

which will be used to construct the kinetic term of our theory. Since by construction, the gauge covariant derivative of $\phi(x)$ and $\tilde{\phi}(x)$ transform as vectors under gauge transformation, we propose a kinetic term of the form $\frac{1}{2}\langle D_\mu \phi, D^\mu \tilde{\phi}\rangle$, whose gauge invariance is guaranteed by construction.

By the embedding of our gauge transformation $\mathcal{L}_\Lambda$ into a sub-algebra of $\mathfrak{gl}(\mathfrak{K})$ discussed in the last section, in general dimensions, one can introduce a standard Maxwell term as the gauge invariant kinetic term for the gauge field $\hat{A}_\mu(x)$. Explicitly, we propose for a generic dimension bigger than three the following kinetic term: $\frac{1}{4} tr \widehat{F}^2$, with

$$tr\widehat{F}^2 = \big(\widehat{F}_{\mu\nu}\big)_a{}^b \big(\widehat{F}^{\mu\nu}\big)_b{}^a = \eta^{\mu\alpha}\eta^{\nu\beta} f_{cda}{}^b f_{efb}{}^a F^{cd}_{\mu\nu} F^{ef}_{\alpha\beta}, \tag{5.15}$$

whose gauge invariance follows simply from

$$\mathcal{L}_\Lambda\Big(\frac{1}{4} tr\widehat{F}^2\Big) = \frac{1}{2} tr\Big([\widehat{F}, \widehat{\Lambda}]\widehat{F}\Big) = 0. \tag{5.16}$$

Where in the above formula, we use the spacetime metric $\eta^{\mu\nu}$ to raise the indices of the two-form $\widehat{F}_{\mu\nu}$.

While, we also know that for any function $V(t) \in \mathcal{C}^\infty(\mathbb{R})$, $V\big(\Delta(\phi)\big)$ will be a gauge invariant potential term. We propose here the minimal gauge invariant action:

$$\mathcal{L} = \frac{1}{2}\langle D_\mu\phi, D^\mu\tilde{\phi}\rangle - V\big(\Delta(\phi)\big) + \frac{1}{4} tr\widehat{F}^2. \tag{5.17}$$

To check the invariance under the *F-duality* $F: \mathfrak{K} \to \mathfrak{K}$, one simply resorts to the fact that under the *F-dual*:

$$\phi \to \tilde{\phi}, \quad \tilde{\phi} \to -\phi,$$

together with the anti-symmetry of the symplectic form used to construct the scalar quantity. For example, the kinetic term transforms as:

$$F\left(\eta^{\mu\nu}\langle D_\mu\phi, D_\nu\tilde{\phi}\rangle\right) = \eta^{\mu\nu}\langle D_\mu\tilde{\phi}, D_\nu(-\phi)\rangle = \eta^{\mu\nu}\langle D_\nu\phi, D_\mu\tilde{\phi}\rangle, \tag{5.18}$$

[4] Recall that the dual-field $\tilde{\phi}(x)$ is defined as

$$\tilde{\phi} \equiv \frac{1}{\sqrt{6\lambda}} \frac{\phi\phi\phi}{\sqrt{\Delta(\phi)}},$$

which is a degree 1 rational function of ϕ living in the complement Lagrangian submanifold.

and the symmetry of the spacetime metric $\eta^{\mu\nu}$ establishes the stated invariance.

At this point, we should remark here that in the above construction, one doesn't need to specify the spacetime dimension. In fact, any dimension where the Maxwell action makes sense (i.e. four or above) will work, but even this is not a real restriction. In four dimensions, we can introduce a topological term, $tr\, \widehat{F} \wedge \widehat{F}$, which would not affect any of our conclusions. Likewise, in three dimensions, we should be able to include a Chern-Simons term for the action of the gauge field, which has the same form as in the BLG paper. Explicitly, one has:

$$\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\Big(f_{abcd}\, A^{ab}_{\mu}\, \partial_\nu A^{cd}_{\lambda} + \frac{2}{3} f_{cda}{}^{g} f_{efgb}\, A^{ab}_{\mu} A^{cd}_{\nu} A^{ef}_{\lambda}\Big), \tag{5.19}$$

whose gauge invariance can be shown easily given the property $f_{abcd} = f_{cdab}$ remarked previously.

5.4 Generalization?

Given the simple construction of the most general action for a spacetime scalar field $\phi(x)$ valued in $\mathfrak{K}$ admitting both gauge symmetry and the *F-dual* considered in the previous section. It is natural to seek for a similar action for some vector fields or spinor fields that is again invariant under both of the symmetries. Such kind of generalization is of interest to the physicists since it potentially might define a *sigma-model* type theory if one takes the spacetime in this paper as the *worldvolume* of some extended objects (for instance, $M2$-branes) and regarding the vector field as the image of the *worldvolume* in some target space. However, we shall prove in this section that such a generalization is not possible.

5.4.1 Couple to a Vector Space

To start the analysis, let's couple our *FTS* $\mathfrak{K}$ to a generic vector space $\mathfrak{V}$, over which we introduce various algebraic structures and make it into an algebra. For instance, spinors can be regarded as vectors with an anti-symmetric binary product that gives the fermionic statistics. In this way, our discussion for the formal algebraic system $\mathfrak{V}$ will cover the most generic space that couples to $\mathfrak{K}$. Thus, here we are considering a big vector space $\mathfrak{N} = \mathfrak{K} \otimes \mathfrak{V}$, whose element, denoted by Φ, is the tensor product of an element $\phi \in \mathfrak{K}$ and an element $v \in \mathfrak{V}$, i.e. $\Phi = \phi \otimes v$.

To be able to construct a Lagrangian density $\mathcal{L}[\Phi(x)]$ for the fields $\Phi(x) \in \mathfrak{N}$ obtained from promoting an element Φ in $\mathfrak{N}$ to a spacetime field $\Phi(x)$ taking values in $\mathfrak{N}$, we have to introduce a bilinear form (the metric) $\langle \Phi_I, \Phi_J \rangle \in \mathbb{R}$ for any two $\Phi_{I,J} = \phi_{I,J} \otimes v_{I,J}$ in $\mathfrak{N}$. This induces a metric on $\mathfrak{V}$ via direct evaluation:

$$\langle \Phi_I, \Phi_J \rangle = \langle \phi_I \otimes v_I, \phi_J \otimes v_J \rangle = \langle \phi_I, \phi_J \rangle \times (v_I, v_J)_{\mathfrak{V}}, \quad \forall\, \Phi_I, \Phi_J \in \mathfrak{N}, \tag{5.20}$$

where $(v_I, v_J)_{\mathfrak{V}} : \mathfrak{V} \otimes \mathfrak{V} \to \mathbb{R}$ is the induced metric over $\mathfrak{V}$, whose symmetry property is going to be determined by the required symmetry property of $\langle \Phi_I, \Phi_J \rangle$ of $\mathfrak{N}$.

Further more, in order to define the *F-dual* of this generalized theory, we need to introduce a triple product $\Phi_I\Phi_J\Phi_K : \mathfrak{N}\otimes\mathfrak{N}\otimes\mathfrak{N}\to\mathfrak{N}$ defined for any three elements $\Phi_I, \Phi_J, \Phi_K \in \mathfrak{N}$, which would then induce a trilinear triple product $[v_I, v_J, v_K]_{\mathfrak{V}} : \mathfrak{V}\otimes\mathfrak{V}\otimes\mathfrak{V}\to\mathfrak{V}$.

Here, we make a plausible conjecture that the *F-dual* is defined only for algebraic systems satisfying the four axioms of a *FTS* introduced in section 2. Thus, we require the metric $\langle\Phi_I, \Phi_J\rangle$ to be an anti-symmetric bilinear from (and append this as axiom (o) to the four axioms), and get the following five axioms for the algebra $\mathfrak{N}$:

(o) $\langle\Phi_I, \Phi_J\rangle = -\langle\Phi_J, \Phi_I\rangle$

(i) $\Phi_I\Phi_J\Phi_K = \Phi_J\Phi_I\Phi_K$

(ii) $\Phi_I\Phi_J\Phi_K = \Phi_I\Phi_K\Phi_J + 2\mu\,\langle\Phi_J, \Phi_K\rangle\,\Phi_I + \mu\,\langle\Phi_I, \Phi_K\rangle\,\Phi_J - \mu\,\langle\Phi_I, \Phi_J\rangle\,\Phi_K$

(iii) $\Phi_L\Phi_M(\Phi_I\Phi_J\Phi_K) = (\Phi_L\Phi_M\Phi_I)\Phi_J\Phi_K + \Phi_I(\Phi_L\Phi_M\Phi_J)\Phi_K + \Phi_I\Phi_J(\Phi_L\Phi_M\Phi_K)$

(iv) $\langle\Phi_L\Phi_M\Phi_I, \Phi_J\rangle + \langle\Phi_I, \Phi_L\Phi_M\Phi_J\rangle = 0,$

where, μ plays the role of the free parameter λ previously for the *FTS* $\mathfrak{N}$. Then, repeating the same construction discussed in the previous section to the *FTS* $\mathfrak{N}$, one gets the most general action invariant under the two desired symmetries.

5.4.2 A No-Go Theorem

These axioms of $\mathfrak{N}$ induce a set of corresponding axioms, in addition to the ones introduced already for other physical reasons, to the triple product $[v_I, v_J, v_K]_{\mathfrak{V}}$ and the metric $(v_I, v_J)_{\mathfrak{V}}$ defined on $\mathfrak{V}$. The reader can find the full set of the corresponding axioms for $\mathfrak{V}$ in appendix B. Among them, axiom $(B.\,iii)$ induced from the derivation property of $\mathfrak{N}$ leads to a particularly strong constraint. To see this, let's restrict to a subalgebra $\mathfrak{N}_\phi \equiv \mathfrak{K}_\phi\otimes\mathfrak{V}$ of $\mathfrak{N}$, where $\mathfrak{K}_\phi$ is the subalgebra in $\mathfrak{K}$ generated by a single generator $\phi\in\mathfrak{K}$. Over this subalgebra $\mathfrak{N}_\phi$, we shall get a weaker conditions on the algebraic structure of $\mathfrak{V}$. Now, if one takes five elements of the form $\Phi_{L,M,I,J,K} = \phi\otimes v_{L,M,I,J,K}$ in $\mathfrak{N}_\phi$, and inserts them into axiom $(B.\,iii)$, one gets a simplified (weaker) condition:

$$\phi\phi\,T(\phi)\otimes\Big(\big[v_L, v_M, [v_I, v_J, v_K]_{\mathfrak{V}}\big]_{\mathfrak{V}} - \big[v_I, v_J, [v_L, v_M, v_K]_{\mathfrak{V}}\big]_{\mathfrak{V}}\Big) = 0, \tag{5.21}$$

where the simplification comes from the fact that over the subalgebra $\mathfrak{K}_\phi$, $\mathcal{L}_{\phi T(\phi)}$ and $\mathcal{L}_{T(\phi)\phi}$ act as annihilation operators, whose proof can be found in appendix A. While given any two elements v_L, v_M in $\mathfrak{V}$ one gets a linear operator in $\mathfrak{gl}(\mathfrak{V})$ denoted by $\mathcal{L}_{v_L v_M}$, whose action is evaluated by the triple product as: $\mathcal{L}_{v_L v_M}\,v_I = [v_L, v_M, v_I]_{\mathfrak{V}}$ for any $v_I\in\mathfrak{V}$. Then, the weaker form of the axiom $(B.\,iii)$ states:

$$\big[\mathcal{L}_{v_L v_M}, \mathcal{L}_{v_I v_J}\big] = 0, \tag{5.22}$$

as matrix commutator in $\mathfrak{gl}(\mathfrak{V})$ for any $v_{I,J,L,M}\in\mathfrak{V}$.

This is in general possible only when $dim\mathfrak{V} = 1$, i.e. $\mathfrak{N} = \mathfrak{K} \otimes \mathbb{R}$, which is the scalar case discussed in the previous sections. Otherwise, one concludes that the subset $\{\mathcal{L}_{v_I v_J} \in \mathfrak{gl}(\mathfrak{V}) \,|\, v_I, v_J \in \mathfrak{V}\} \subset \mathfrak{gl}(\mathfrak{V})$ is a subset of the Cartan subalgebra of $\mathfrak{gl}(\mathfrak{V})$, i.e. [5] $\mathcal{L}_{v_I v_J} v_K = [v_I, v_J, v_K]_{\mathfrak{V}} = (v_I, v_J)_{\mathfrak{V}} \times v_K$, where "×" is multiplication by a scalar factor. The triple product so defined satisfies the strong form of axiom $(B.\,iii)$ and most of other axioms but is again refuted by axiom $(B.\,ii)$ whenever $\mathfrak{K}$ is bigger then a single generator algebra $\mathfrak{K}_\phi$.

This completes the proof of the No-Go theorem:

> Assuming the *F-dual* is defined only for an algebraic system $\mathfrak{N}$ satisfying all the four axioms introduced in section 2, then it is not possible to construct a Lagrangian $\mathcal{L}[\Phi(x)]$ for a vector/spinor field $\Phi(x)$ valued in $\mathfrak{K}$ that admits both gauge symmetry and *F-dual*.

5.5 Concluding Remarks

In this paper, we have introduced a family of gauge theories, which are invariant under two symmetries: a local gauge symmetry constructed from a *Freudenthal Triple System* (*FTS*) $\mathfrak{K}$, and a discrete symmetry called *Freudenthal Duality* (*F-duality*) F. In the minimal construction, which contains a single scalar field $\phi(x)$ valued in $\mathfrak{K}$ and a gauge field $A_\mu^{ab}(x) \in \mathfrak{K} \otimes_S \mathfrak{K}$, we presented the most general action invariant under the two symmetries discussed above. The algebraic structure of the *FTS* ensures that such a gauge theory is well-defined and has the requisite properties. F-duality, which is a highly non-linear duality, can be understood geometrically as a map from an element ϕ in the Lagrangian submanifold of $\mathfrak{K}$ to a dual-element $\tilde{\phi}$ in the complement Lagrangian submanifold of $\mathfrak{K}$. More over, the property $F \circ F = -1$ gives up to a sign an one-to-one pairing of elements in $\mathfrak{K}$, which is almost as strong as invertability of F.

We also analyzed the possibility of generalizing the minimal action by coupling to spacetime vector or spinor fields, which is usually a simple task when constructing gauge theories. It turns out that requiring *F-duality* imposes a very strong algebraic constraint that can be summarized as a No-Go theorem and concludes that our minimal action constructed in section 3 is actually the most general action admitting both the gauge symmetry and the *F-duality.* Bear this in mind, it is possible to couple the *FTS* to spacetime vector or spinor fields if one is willing to give up the *F-duality.* In such cases, there is, a priori, no restriction on the spacetime dimension. In particular, in three dimension, one has a gauge theory containing both bosonic and fermionic degrees of freedom together with the Chern-Simons term, which is very similar to the BLG action. However, we remark here that the algebraic structure of our action differs from the BLG action; for instance, our triple product has

[5] In general, one may propose $\mathcal{L}_{v_I v_J} v_K = h\big((v_I, v_J)_{\mathfrak{V}}\big) \times v_K$, for any function $h : \mathbb{R} \to \mathbb{R}$. However, the trilinearity of the triple product $[v_I, v_J, v_K]_{\mathfrak{V}}$ requires the function h to be linear. While the constant term of the linear function h leads to a trivial triple product and be easily refuted by other axioms, we conclude that up to an overall factor, $\mathcal{L}_{v_I v_J} v_K = (v_I, v_J)_{\mathfrak{V}} \times v_K$ is the most generic possibility.

more complicated symmetry structure than the Lie-3 algebra used in the BLG model. For this reason, there's no restriction on the dimension of our internal space $\mathfrak{K}$; on the other hand, this complicated symmetry structure makes supersymmetrization of our model a hard problem, which we hope to come back in the future.

We should also point out that given the close relation between the *FTS*'s and exceptional Lie-algebras $\mathfrak{g}$, one might try to make link from our model to a $\mathfrak{g}$ Yang-Mills theory. This is certainly possible, but recall that the exceptional Lie-groups cannot be embedded into standard matrix groups, the Yang-Mills theory so obtained won't have the standard Maxwell terms constructed from trace over matrices. Geometrically, a better way to understand this model is by realizing that the exceptional Lie-groups can be embedded as matrix groups over octonion numbers [7], thus one in principle can formulate dual to our model a standard Yang-Mills theory over octonion numbers. It is also worth a while to point out that the *FTS* has another geometrical interpretation over an eccentric geometry called *metasymplectic geometry* introduced by H. Freudenthal [32, 31], where two points can define instead of a line passing through them as in the standard geometry two more relations, called *interwoven* and *hinged*; and to each set of points, there corresponds a set of dual geometrical objects called *symplecta* satisfying relations dual to the three among the points discussed above. In this bizarre geometrical setup, the axioms stated in this note gains natural geometrical interpretation and their relation to the exceptional Lie-algebra becomes more transparent. We postpone the possible physical interpretation of such geometry as a future project.

5.6 Appendix

5.6.1 A Freudenthal duality

Here we present the proof that the four form $\Delta(\phi) \equiv \frac{1}{2}\langle T(\phi), \phi\rangle$ is invariant under the *Freudenthal dual* $F : \phi \to \tilde{\phi} = \frac{T(\phi)}{\sqrt{6\lambda\,\Delta(\phi)}}$.

Let's start with the following observation: by axiom (i) and (iii), one has

$$\begin{aligned}\phi_I\phi_J(\phi_I\phi_J\phi_K) &= (\phi_I\phi_J\phi_I)\phi_J\phi_K + \phi_I(\phi_I\phi_J\phi_J)\phi_K + \phi_I\phi_J(\phi_I\phi_J\phi_K)\\ &= \mathcal{L}_{(\phi_I\phi_J\phi_I)\phi_J}\,\phi_K + \mathcal{L}_{\phi_I(\phi_J\phi_I\phi_J)}\,\phi_K + \phi_I\phi_J(\phi_I\phi_J\phi_K),\end{aligned} \tag{5.23}$$

i.e. $\big(\mathcal{L}_{(\phi_I\phi_J\phi_I)\phi_J} + \mathcal{L}_{\phi_I(\phi_J\phi_I\phi_J)}\big)\phi_K = 0$, for any $\phi_{I,J,K} \in \mathfrak{K}$. This means, by restricting to the case $\phi_I = \phi_J = \phi$ and again the axiom (iii), $\mathcal{L}_{T(\phi)\phi}$ and $\mathcal{L}_{\phi T(\phi)}$ act like annihilation operators on $\mathfrak{K}$. While by axiom (ii) and the definition of $\Delta(\phi)$ one gets:

$$\begin{aligned}\mathcal{L}_{T(\phi)T(\phi)}\phi &= T(\phi)T(\phi)\phi\\ &= T(\phi)\phi\,T(\phi) + 2\lambda\,\langle T(\phi), \phi\rangle\,T(\phi) + \lambda\,\langle T(\phi), \phi\rangle\,T(\phi) - \lambda\,\langle T(\phi), T(\phi)\rangle\,\phi\\ &= 6\lambda\,\Delta(\phi)\,T(\phi),\end{aligned} \tag{5.24}$$

and similarly, $\phi\phi\, T(\phi) = -6\lambda\, \Delta(\phi)\phi$. Then, direct evaluation of $T(T(\phi))$ leads to:

$$\begin{aligned} T(T(\phi)) &= \mathcal{L}_{T(\phi)T(\phi)} T(\phi) = 6\lambda\, \Delta(\phi)\Big(T(\phi)\phi\phi + \phi\, T(\phi)\phi + \phi\phi\, T(\phi)\Big) \\ &= -\big(6\lambda\, \Delta(\phi)\big)^2 \phi, \end{aligned} \tag{5.25}$$

from which, for $6\lambda\, \Delta(\phi) \neq 0$, one can check the following two statements are satisfied at the same time:

- The *F-dual* $F : \phi \to \tilde{\phi} = \frac{T(\phi)}{\sqrt{6\lambda\, \Delta(\phi)}}$ squares to negative identity, i.e. $F \circ F = -1 \in \mathfrak{gl}(\mathfrak{K})$,
- The four-form $\Delta(\phi) = \frac{1}{2}\langle T(\phi), \phi\rangle$ is invariant under *F-dual*, i.e. $\Delta(\phi) = \Delta(\tilde{\phi})$,

which then completes the proof.

5.6.2 B Axioms of $\mathfrak{V}$

$(B.\,o)$ $(v_I, v_J)_{\mathfrak{V}} = (v_J, v_I)_{\mathfrak{V}}$

$(B.\,i)$ $[v_I, v_J, v_K]_{\mathfrak{V}} = [v_J, v_I, v_K]_{\mathfrak{V}}$

$(B.\,ii)$
$$\begin{aligned} &(\phi_I\phi_J\phi_K) \otimes \Big([v_I, v_J, v_K]_{\mathfrak{V}} - [v_I, v_K, v_J]_{\mathfrak{V}}\Big) \\ &\quad = \langle\phi_J, \phi_K\rangle\, \phi_I \otimes \Big(2\mu\, (v_J, v_K)_{\mathfrak{V}} \times v_I - 2\lambda\, [v_I, v_J, v_K]_{\mathfrak{V}}\Big) \\ &\quad\quad + \langle\phi_I, \phi_K\rangle\, \phi_J \otimes \Big(\mu\, (v_I, v_K)_{\mathfrak{V}} \times v_J - \lambda\, [v_I, v_J, v_K]_{\mathfrak{V}}\Big) \\ &\quad\quad - \langle\phi_I, \phi_J\rangle\, \phi_K \otimes \Big(\mu (v_I, v_J)_{\mathfrak{V}} \times v_K - \lambda\, [v_I, v_J, v_K]_{\mathfrak{V}}\Big) \end{aligned}$$

$(B.\,iii)$
$$\begin{aligned} 0 &= (\phi_L\phi_M\phi_I)\phi_J\phi_K \otimes \Big(\big[v_L, v_M, [v_I, v_J, v_K]_{\mathfrak{V}}\big]_{\mathfrak{V}} - \big[[v_L, v_M, v_I]_{\mathfrak{V}}, v_J, v_K\big]_{\mathfrak{V}}\Big) \\ &\quad + \phi_I(\phi_L\phi_M\phi_J)\phi_K \otimes \Big(\big[v_L, v_M, [v_I, v_J, v_K]_{\mathfrak{V}}\big]_{\mathfrak{V}} - \big[v_I, [v_L, v_M, v_J]_{\mathfrak{V}}, v_K\big]_{\mathfrak{V}}\Big) \\ &\quad + \phi_I\phi_J(\phi_L\phi_M\phi_K) \otimes \Big(\big[v_L, v_M, [v_I, v_J, v_K]_{\mathfrak{V}}\big]_{\mathfrak{V}} - \big[v_I, v_J, [v_L, v_M, v_K]_{\mathfrak{V}}\big]_{\mathfrak{V}}\Big) \end{aligned}$$

$(B.\,iv)$ $\Big([v_L, v_M, v_I]_{\mathfrak{V}}, v_J\Big)_{\mathfrak{V}} + \Big(v_I, [v_L, v_M, v_J]_{\mathfrak{V}}\Big)_{\mathfrak{V}} = 0$

Appendix A

Lie Algebra from FTS

For any $\alpha, \beta \in \mathfrak{g}(\mathfrak{K}) = \mathfrak{L}_{\mathfrak{K}} \oplus \mathfrak{su}(2) \oplus \mathfrak{K} \oplus \mathfrak{K}$, we list their Lie bracket $[\alpha, \beta] \in \mathfrak{g}(\mathfrak{K})$ in Table. A.1. Here, x, y, z, u, v, w are elements in $\mathfrak{K}$ and $\mathcal{L}_{xy}$ is an element in $\mathfrak{L}_{\mathfrak{K}} \subset End(\mathfrak{K})$. The standard $\mathfrak{su}(2)$ generators $\{\sigma_i\}_{i=1,2,3}$ will be regrouped as creation/annihilation operators $\sigma^{\pm}$ and σ_z in the table. To distinguish the elements in the different copies of $\mathfrak{K}$ in $\mathfrak{g}$, we add to $x \in \mathfrak{K}$ a subscript $x_{(1)}$ or $x_{(2)}$ indicating the specific copy of $\mathfrak{K}$ it is in. One may check, with the help of relations listed in (4.9), the Lie bracket defined here is indeed closed and satisfies Jacobi identity.

$[\boldsymbol{\alpha}, \boldsymbol{\beta}]$	$\boldsymbol{\sigma}^+$	$\mathbf{y}_{(1)}$	$\boldsymbol{\sigma_z}$	$\mathcal{L}_{\mathbf{zw}}$	$\mathbf{y}_{(2)}$	$\boldsymbol{\sigma}^-$
$\boldsymbol{\sigma}^+$	0	0	$-2\sigma_z$	0	$y_{(1)}$	σ_z
$\mathbf{x}_{(1)}$	0	$2\lambda\langle x, y\rangle\, \sigma^+$	$x_{(1)}$	$-(zwx)_{(1)}$	$\lambda\langle y, x\rangle\, \sigma_z$	$-x_{(2)}$
$\boldsymbol{\sigma_z}$	$2\sigma^+$	$y_{(1)}$	0	0	$-y_{(2)}$	$-2\sigma^-$
$\mathcal{L}_{\mathbf{uv}}$	0	$(uvy)_{(1)}$	0	$\mathcal{L}_{(uvz)w} + \mathcal{L}_{z(uvw)}$	$(uvy)_{(2)}$	0
$\mathbf{x}_{(2)}$	$-x_{(1)}$	$\lambda\langle y, x\rangle\, \sigma_z$	$x_{(2)}$	$-(zwx)_{(2)}$	$-2\lambda\langle x, y\rangle\, \sigma^-$	0
$\boldsymbol{\sigma}^-$	$-\sigma_z$	$y_{(2)}$	$2\sigma^-$	0	0	0

Table A.1: Product rule for Lie bracket.

Appendix B

Review of Special Geometry

Over the ambient space of $\mathcal{M}_V$ one can consider a $Sp(2n_V+2)$ vector bundle $\mathcal{E}_V$. Denoting a section Ω by its coordinates $(\mathcal{X}^I, \mathcal{F}_I)$, the symplectic form

$$\langle \Omega, \Omega' \rangle = \mathcal{X}^I \mathcal{F}'_I - \mathcal{X}'^I \mathcal{F}_I \tag{B.1}$$

endows the fibers with a phase space structure, derived from the symplectic form $\langle d\Omega, d\Omega \rangle = d\mathcal{X}^I \wedge d\mathcal{F}_I$ over $\mathcal{E}_V$.

The geometry of the scalar manifold $\mathcal{M}_V$ (with an intrinsic local coordinate $z^i \in \mathbb{C}^{n_V}$) is completely determined by a choice of a holomorphic section $\Omega(z) = \big(\mathcal{X}^I(z), \mathcal{F}_I(z)\big)$ taking value in a Lagrangian cone, i.e. a projective subspace satisfying $d\mathcal{X}^I \wedge d\mathcal{F}_I = 0$. Note, this is equivalent to saying the 1-form $\mathcal{F}_I \, d\mathcal{X}^I$ when restricting to $\mathcal{M}_V$ is closed and hence allows one to introduce (locally) the *prepotential* $\mathcal{F}(\mathcal{X}^I)$, s.t.

$$\mathcal{F}_I \, d\mathcal{X}^I = d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \mathcal{X}^I} \, d\mathcal{X}^I. \tag{B.2}$$

At generic points, the sections $\{\mathcal{X}^I(z)\}_{I=0,\ldots,n_V}$ may be taken as the homogeneous holomorphic coordinates on $\mathcal{M}_V$ and their ratios $\{\mathcal{Z}^A(z) = \mathcal{X}^A/\mathcal{X}^0\}_{A=1,\ldots,n_V}$ is called (projective) *special coordinates*.

Once the holomorphic section $\Omega(z)$ is given, the metric on $\mathcal{M}_V$ is obtained from the Kähler potential

$$\mathcal{K}(z,\bar{z}) = -log\Big[\, i \left\langle \bar{\Omega}(\bar{z}), \Omega(z) \right\rangle \Big]. \tag{B.3}$$

B.1 String Theory Construction

The $\mathcal{N}=2$ Maxwell-Einstein supergravity with n_V vector-multiplets and n_H hyper-multiplets can be obtained from both type IIA and type IIB string theory with appropriate internal Calabi-Yau (CY) three-folds $\widetilde{Y}, Y$. In this construction, various spacetime fields are realized as string theory objects wrapping around appropriate non-trivial cycles in the internal

$String/CY$	$h_{3,0}(CY)$	$h_{2,1}(CY)$	$h_{1,1}(CY)$	$h_{2,2}(CY)$
IIB/Y	1	n_V	$n_H - 1$	$n_H - 1$
	$A^I, B_I \in H_3(Y, \mathbb{R})$, symplectic basis. $\Omega_Y \in H^{3,0}(Y, \mathbb{R})$.		$\mathcal{M}_V \to$ complex structure of Y. $\mathcal{M}_H \to$ field configuration & complex Kähler structure of Y.	
IIA/$\widetilde{Y}$	1	$n_H - 1$	n_V	n_V
	$\mathcal{M}_V \to$ cpx. Kähler structure of $\widetilde{Y}$. $\mathcal{M}_H \to$ field configuration & complex structure of $\widetilde{Y}$		$J = B_{NS} + i\omega_K$, complex Kähler form of $\widetilde{Y}$. $\gamma^A \in H_{1,1}(\widetilde{Y}, \mathbb{Z}),\ \gamma_A \in H_{2,2}(\widetilde{Y}, \mathbb{Z})$.	

Table B.1: Mirror pair $(\widetilde{Y}, Y)$ in type IIA/IIB construction.

Calabi-Yau manifolds. [1] For instance, in the type IIA construction, the vector multiplet is obtained as a $D2$-brane wrapping a non-trivial $(1,1)$ cycle of the Calabi-Yau three-fold $\widetilde{Y}$. In the type IIB construction, the vector multiplet comes from $D3$-brane wrapping non-trivial $(2,1)$ cycles of the Calabi-Yau three-fold Y. In both case, the electro-magnetic duality $\vec{q}_e \leftrightarrow \vec{q}_m$ is realized as the Poincare dual in the internal CY. Since type IIA and type IIB string theory are related by T-duality, to achieve the same supregravity theory, the internal Calabi-Yau manifolds $\widetilde{Y}$ and Y are related by mirror symmetry, which dictates the relation between various algebraic topology contents of the mirror pair $(\widetilde{Y}, Y)$. For example, the scalar manifold $\mathcal{M}_V$, which plays the role as the moduli space of complex Kähler structure of $\widetilde{Y}$, will be the complex structure moduli space for the mirror Calabi-Yau Y. In Table B.1, we list the relevant geometrical objects and their role on the two sides of the mirror pair $(\widetilde{Y}, Y)$ used in our type II/$\widetilde{Y}$ and type IIB/Y discussion later.

B.1.1 IIB/Y

In type IIB/Y compactification, the holomorphic section $\Omega(z) = \big(\mathcal{X}^I(z), \mathcal{F}_I(z)\big)$ at a point $z \in \mathcal{M}_V$ is calculated by

$$\mathcal{X}^I(z) = \int_{A^I(z)} \Omega_Y(z), \quad \mathcal{F}_I(z) = \int_{B_I(z)} \Omega_Y(z). \tag{B.4}$$

[1] We have encountered one example in the review of geometric engineering, where pure $SU(2)$ gauge theory was considered, see section 2.3.

Here the 3-cycles $A^I, B_I \in H_3(Y(z), \mathbb{Z})$ are regarded as sections over $\mathcal{M}_V$ since a point $z \in \mathcal{M}_V$ determines a complex structure over Y. The Kähler potential $\mathcal{K}$ on the moduli of complex structure is:

$$\mathcal{K}(z, \bar{z}) = -\log \left[i \int_{Y(z,\bar{z})} \Omega_Y(z) \wedge \bar{\Omega}_Y(\bar{z}) \right]. \tag{B.5}$$

The central charge of a state with electric-magnetic charges p^I, q_I may be rewritten as:

$$Z(\mathcal{X}^I, \mathcal{F}_I) = \frac{\int_\gamma \Omega_Y}{\sqrt{i \int_Y \Omega_Y \wedge \bar{\Omega}_Y}}, \tag{B.6}$$

where $\gamma = q_I A^I - p^I B_I$, and is recognized as the mass of a D3-brane wrapped on a special Lagrangian 3-cycle $\gamma \in H_3(Y, \mathbb{X})$.

B.1.2 IIA/$\widetilde{\mathbf{Y}}$

The 4d *special coordinates* $\mathcal{Z}^A = \mathcal{X}^A/\mathcal{X}^0$ (recall $\mathcal{Z}^0 = \mathcal{X}^0/\mathcal{X}^0 = 1$) of $\mathcal{M}_V$ can also be calculated in the type IIA/$\widetilde{Y}$ compactification as:

$$\mathcal{Z}^A = \int_{\gamma^A} J, \quad \mathcal{F}_A/\mathcal{X}^0 = \int_{\gamma_A} J \wedge J. \tag{B.7}$$

Note, the special coordinates $\{\mathcal{Z}^A\}_{A=1,\ldots,n_V}$ are non-generic holomorphic coordinates, singled out by there relation to the $\mathcal{N} = 2$ vector-multiplets, while $\{z^i\}_{i=1,\ldots,n_V}$ are (generic) intrinsic local coordinates of $\mathcal{M}_4$, i.e. $\mathcal{Z}^A = \mathcal{Z}^A(z^i)$ holomorphic.

In the large volume limit, the Kähler potential (in the gauge $\mathcal{X}^0(z) = 1, \mathcal{X}^A(z) = z^A$) is given by the volume in string units,

$$\mathcal{K}(z, \bar{z}) = -\log \int_{\widetilde{Y}(z,\bar{z})} J \wedge J \wedge J, \tag{B.8}$$

originated from the cubic prepotential

$$\mathcal{F} = -\frac{1}{6} C_{ABC} \frac{\mathcal{X}^A \mathcal{X}^B \mathcal{X}^C}{\mathcal{X}^0} + \mathcal{O}(vol^{-1})_{\text{(instanton corrections)}}. \tag{B.9}$$

Here, C_{ABC} are the triple intersection numbers of 4-cycles $\gamma_{A,B,C} \in H_{2,2}(\widetilde{Y}, \mathbb{Z})$.

Appendix C

The Yang-Baxter Equation and the Spin Chain

Integrable system by now is a big industry in physics and mathematics. Throughout the years, people have developed several (in the end equivalent?) ways of viewing it; some geometrical, some algebraic, and some remains "mysterious." In different formulations the role played by the YBE can have very different interpretation. Here we review the spin chain approach and how an integrable system arises from a solution to the YBE following L. D. Faddeev's review [25].

C.1 Spin Chain and Integrable Systems

In the spin chain theory, one studies the eigenvalue problem of a closed chain of N lattice sites periodically identified. Let h_n, $n = 1, \ldots, N$ be the Hilbert space of states at the nth site and [1] $L = dim_{\mathbb{C}}\, h_n$ the complex dimension of the local quantum space h_n, one introduces the total Hilbert space: $\mathcal{H}_N \equiv \otimes_{n=1}^{N} h_n$. The goal is to find a family of N mutually commuting operators $\{Q_l\}$ over the L^N dimensional Hilbert space and their eigenvalues. The trick to solve this problem is to introduce one more vector space V, usually called the auxiliary space, of the same dimension as h_n, and construct the Lax operator $L_{n,a}(\lambda)$, which is a function of one complex variable λ taking values in $End_{\mathbb{C}}(h_n \otimes V)$. In the literature, λ is usually called the spectral parameter, where the name is rooted from its role as an eigenvalue in the original Lax operator.

Now, consider two examplers of Lax operators $L_{n,a_1}(\lambda)$ and $L_{n,a_2}(\mu)$ with the same quantum space h_n and V_1 and V_2 as the corresponding auxiliary spaces. Regarded as operators acting on the triple tensor product $h_n \otimes V_1 \otimes V_2$, the *ordered* operator products $L_{n,a_1}(\lambda)L_{n,a_2}(\mu)$ and $L_{n,a_2}(\mu)L_{n,a_1}(\lambda)$ make sense, and are claimed to be related by similarity transformations

[1] Here, we will assume all the lattice sites have the same Hilbert space of complex dimension L.

with the *intertwiner* acting only in $V_1 \otimes V_2$ and so not containing quantum operators. [2] In other words, there exists an invertible matrix $R_{a_1,a_2}(\lambda-\mu) \in End_{\mathbb{C}}(V_1 \otimes V_2)$, s.t.

$$R_{a_1,a_2}(\lambda-\mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu). \tag{C.1}$$

This relation, which is usually called the (local) Fundamental Commutation Relation (aka, FCR) in the literature, is the key to making contact to the YBE and various other approaches of the integrable system as we will see later.

The Lax operator $L_{n,a}(\lambda)$ has a "natural" geometric interpretation as a *connection* along the chain, defining the transport between the sites n and $n+1$ via the Lax equation:

$$\vec{\psi}_{n+1} = L_{n,a}(\lambda)\,\vec{\psi}_n, \tag{C.2}$$

where, $\vec{\psi}_n$ is a vector in the auxiliary space V taking values in h_n (naturally embedded in $\mathcal{H}$). Then, one can consider the "monodromy" $T_{N,a}(\lambda)$ around the chain:

$$T_{N,a}(\lambda) \equiv L_{N,a}(\lambda)\ldots L_{1,a}(\lambda), \tag{C.3}$$

which is regarded as a matrix in $End_{\mathbb{C}}(V)$ with entries being elements in $End_{\mathbb{C}}(\mathcal{H})$. The monodromy matrices $T_{N,a_1}(\lambda)$ and $T_{N,a_2}(\mu)$ satisfy the same FCR,

$$R_{a_1,a_2}(\lambda-\mu)T_{N,a_1}(\lambda)T_{N,a_2}(\mu) = T_{N,a_2}(\mu)T_{N,a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu) \tag{C.4}$$

which follows from the local FCR at each lattice sites and the commutative property of the Lax operator between different sites.

Taking the natural embedding of $T_{N,a_1}(\lambda)$ and $T_{N,a_2}(\mu)$ as $T_{N,a_1}(\lambda)\otimes 1_{V_2}$ and $1_{V_1}\otimes T_{N,a_2}(\mu)$ respectively in $End_{\mathbb{C}}(V_1 \otimes V_2)$, and regarding the above FCR as a commutation relation of $End_{\mathbb{C}}(V_1 \otimes V_2)$ matrices, one easily sees

$$Tr_{V_1\otimes V_2}\Big(T_{N,a_1}(\lambda)T_{N,a_2}(\mu) - T_{N,a_2}(\mu)T_{N,a_1}(\lambda)\Big) = 0, \tag{C.5}$$

from the cyclic symmetry of the trace and the invariance of the *R-matrix* $R_{a_1,a_2}(\lambda-\mu)$ under a diagonal shift of $(\lambda,\mu) \in \mathbb{C}^2$. Note, this is not a trivial equation, since $T_{N,a_1}(\lambda)\,T_{N,a_2}(\mu)$ and $T_{N,a_2}(\mu)\,T_{N,a_1}(\lambda)$ are still ordered operator products of elements in $End_{\mathbb{C}}(\mathcal{H})$. Now one can proceed to evaluate the trace and gets:

$$[F(\lambda), F(\mu)] = 0, \tag{C.6}$$

where $[F(\lambda), F(\mu)]$ is the commutator of elements in $End_{\mathbb{C}}(\mathcal{H})$, and $F(\lambda) \equiv Tr_{V_1}\,T_{N,a_1}(\lambda) = Tr_{V_2}\,T_{N,a_2}(\lambda)$ (recall the auxiliary spaces V_1 and V_2 are isomorphic as complex vector spaces).

[2] Note, the operator products $L_{n,a_1}(\lambda)L_{n,a_2}(\mu)$ and $L_{n,a_2}(\mu)L_{n,a_1}(\lambda)$, beings elements in $End_{\mathbb{C}}(h_n \otimes V_1 \otimes V_2)$, are regarded as elements in $End_{\mathbb{C}}(h_n) \otimes End_{\mathbb{C}}(V_1 \otimes V_2)$, which is isomorphic to $End_{\mathbb{C}}(h_n \otimes V_1 \otimes V_2)$.

Taking the Laurent expansion of $F(\lambda)$:

$$F(\lambda) = \sum_{l-\infty}^{\infty} \mathcal{Q}_l \, \lambda^l, \tag{C.7}$$

where $l - \infty$ means that the series starts from a finite negative integer (and we will always assume our operator $F(\lambda)$ admits such an expansion), we get the following main result of this subsection:

> The FCR asserts the existence of a family of mutually commuting operators $\{\mathcal{Q}_l \,|\, \mathcal{Q}_l \in End_{\mathbb{C}}(\mathcal{H})\}$ constructed from the monodromy operator $T_{N,a}(\lambda)$.

C.2 From YBE To FCR

The relation between the FCR and the YBE comes from a more general interpretation of the FCR due to V. Drinfeld. In his approach, the generating object for the Lax operators is a universal *R-matrix* $\mathcal{R}$ defined as an element in $\mathcal{A} \otimes \mathcal{A}$ for some algebra $\mathcal{A}$, satisfying the abstract YBE:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{C.8}$$

which is understood as an equation over $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$. Here $\mathcal{R}_{ij}$ signifies the operator on $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, which acts as the matrix $\mathcal{R}$ on the ith and the jth components and as identity on the other component; for instance

$$\mathcal{R}_{12} = \mathcal{R} \otimes 1, \quad \mathcal{R}_{23} = 1 \otimes \mathcal{R}. \tag{C.9}$$

The algebra $\mathcal{A}$, called Yangian by Drinfeld, must have a family of representations $\rho(\lambda, a)$, parametrized by a continuous parameter $\lambda \in \mathbb{C}$ and a discrete label a, which parametrizes different representations of $\mathcal{A}$.

The concrete Lax operators are obtained via the evaluation of the universal $\mathcal{R}$-*matrix* $\mathcal{R}$ with respect to the representations, i.e.

$$L_{n,a}(\lambda - \mu) = \Big(\rho(a, \lambda) \otimes \rho(n, \mu)\Big)\mathcal{R} = R_{a,n}(\lambda - \mu). \tag{C.10}$$

Now, the local FCR is obtained if one applies the representation $\rho(a_1, \lambda) \otimes \rho(a_2, \mu) \otimes \rho(n, \sigma)$ to the abstract YBE and set $\sigma = 0$ with a_1, a_2 and n all refer to the same representation. From here, the construction in the previous subsection carries over, and one can proceed to study the integrable system corresponding to a solution to the YBE.

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