Combinatorial and Algorithmic Aspects of Hyperbolic Polynomials

by

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Abstract

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We explore connections between hyperbolic polynomials and computer science problems involving optimization and counting. Specifically we investigate the following diverse set of topics, all of which are connected through the modern study of hyperbolic polynomials:

**Independent Sets.** To count independent sets of graphs, we study a multivariate generating polynomial for independence sets and prove generalizations of results of Chudnovsky and Seymour relating to real-rootedness of the polynomials.

**Differential Operators.** In the study of spectral graph theory and spectral discrepancy theory, the classical additive convolution is used to study the effect differential operators have on the roots of real rooted polynomials. We refine some existing tools for studying the movement of max root under differential operators and establish some conjectures for controlling the inner roots.

**Finite Difference Operators.** Lamprecht studies a $q-$extension of the classical multiplicative convolution which abstracts classical results. We extend the study of such generalizations by considering a finite-mesh preserving generalization of the classical additive convolution.

**Membership Algorithm.** We initiate a conversation around developing algorithms to solve fundamental tasks in this field by establishing an exact arithmetic polynomial time algorithm to determine whether a given bivariate polynomial is real-stable. By the characterization theorem of Brändén and Borcea this is equivalent to an algorithm for testing whether a given linear transformation $T : \mathbb{R}_d[x] \to \mathbb{R}[x]$ sends real rooted polynomials to real rooted polynomials.

**Convex Optimization Complexity.** Given a hyperbolic polynomial one can do convex optimization over its hyperbolicity cone. The exact algorithmic relationship between semidefinite programming and hyperbolic programming is an open question. In an attempt to further motivate complexity questions between these two primitives we give a random family of polynomials which require exponentially sized approximate semidefinite representations for their hyperbolicity cones.
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Chapter 1

Introduction and Background Material

In the early 20th century, the Riemann Hypothesis, Hermite, Laguerre, Jensen, Pólya, Schur, De Bruin and others began investigating linear operators reserving real-rootedness. Although their approach and what has followed has not made significant progress on this problem, the machinery has found a wide range of applications. Lee and Yang [1, 2] had some of the earli-est success with these tools, proving the partition function of the Ising model is non-vanishing when all the variables lie in the right half plane which implies the univariate restriction has all of its zeroes on the imaginary axis. Much of the modern interest in these fields is in deriving combinatorial results from this machinery such as Choe, Oxley, Sokal and Wagner’s [3] investigation of the relationship between the support of right half plane stable polynomi-als and matroids. Brändén [4] expanded these results by fully classifying multi-affine stable polynomials using a negative dependence condition known as the Strongly Rayleigh inequal-ities. This was further expanded by Borcea, Brändén and Liggett [5]. The theory of linear preservers of real rooted polynomials played a crucial role in Marcus, Spielman, Srivastava [6, 7, 7, 8] in proving existence of extremal expander graphs known as Ramanujan graphs and proving a decades old conjecture in operator algebras known as the Kadison-Singer Conjecture. Inspired by these new advances, Oveis Gharan, Saberi, and Singh [9] give an approximation algorithm for the asymmetric travelling salesman problem. In recent years there have been other exciting advances, such as the capacity program of Gurvits [10, 11, 12, 13, 14] which has led to new permanent approximation algorithms and a polynomial time algorithm for testing if a symbolic matrix in non-commuting variables is invertible or not.

The contributions of this thesis to the modern literature explore five questions:

**Multivariate Independence Polynomial** - The real-rootedness of the matching polynomial and the Heilmann-Lieb root bound are important results in the theory of undirected simple graphs. In particular, real-rootedness implies log-concavity and unimodality of the matchings of a graph, and recently in [6] the root bound was used to show the existence of Ramanujan graphs. Generalization of these results to the independence polynomial has
been partially successful. About a decade ago, Chudnovsky and Seymour \cite{15} established the real-rootedness of the independence polynomial for claw-free graphs. In Chapter 2 we prove a multivariate generalization which is completely self contained and implies the original Chudnovsky and Seymour result. By using a multivariate framework to directly prove the more general result, we obtain a simple inductive proof which we believe better captures the underlying structure. Our more general approach also yields a satisfactory explanation of the importance of claw-freeness in the graphs. In the second part of the chapter, we then extend the Heilmann-Lieb root bound to the multivariate independence polynomial by generalizing some of Godsil’s work on the matching polynomial.

**Submodularity of the Finite Free Convolution** - In Chapter 3 we expand upon the results of Marcus-Spielman-Srivastava and provide more general results about how all roots of a given polynomial are affected by finite free convolutions. We begin by noting how their framework can be generalized to submodularity of the max root of polynomials under the additive convolution. Further, we utilize the theory of hyperbolic polynomials to give more interesting root bounds on interior roots (other roots besides the largest). With these result in hand, we state a number of conjectures (and some counterexamples) in the direction of stronger univariate results on interior roots and of analogous multivariate results. Proving similar multivariate results seems to be a hard problem in general which would have important consequences relating to the Paving Conjecture.

**Mesh Generalizations of the Finite Free Convolution** - In \cite{16}, Lamprecht generalizes most of the classical results about the multiplicative convolution to a $q$-generalization of the convolution. Among his results, he proves the $q$-multiplicative convolution preserves the space of positive rooted polynomials in which the minimal ratio of two roots is bounded below by $q$. In \cite{17}, Brändén, Krasikov, and Shapiro pose a conjecture about an analogous generalization of the additive convolution, which we settle in Chapter 4. We give two proofs, one which uses the machinery that Lamprecht develops and the other which develops a novel connection between the $q$-multiplicative convolution and the $b$-additive convolution, allowing one to transfer multiplicative results to additive results.

**Real Stability Testing** - Algorithmic questions about counting, identifying, and quantifying real roots have been well studied. If one wants to study linear operators which preserve real rooted polynomials of bounded degree, the classification theorem of Brändén and Borcea shows it is equivalent to studying bivariate real stable polynomials. In this context it is natural to seek an exact arithmetic algorithm which answers the following membership question: Given a bivariate polynomial $p \in \mathbb{R}[x, y]$ is $p(x, y)$ a real stable polynomial? In Chapter 5 we provide a strongly polynomial time $O(n^5)$ algorithm using classical tools such as subresultants, Sturm sequences, Musser’s algorithm, and basic convexity properties of hyperbolicity.

**Complexity Lower Bounds of Hyperbolicity Cones** - Semidefinite programming can be encoded as a specific case of a more general convex programming primitive known as hyperbolic programming. To begin to quantify the relationship between the two optimization schemes, the algebraic Lax Conjectures explored the relationship between the parameterizations of these primitives, determinantal polynomials and hyperbolic polynomials. With these disproved by Brändén \cite{18}, the Geometric Lax Conjecture instead posits the convex sets the
primitives optimize over are equivalent. Regardless of the validity of this conjecture, the relative algorithmic complexity of the two primitives over the same convex body is unclear. In an attempt to understand the relationship between the different complexities, in Chapter 6 we construct an infinite family of hyperbolic polynomials in which any approximate encodings as SDPs must have exponential size in the degree.

This chapter will develop the context necessary to frame my main results. First we detail both classical and modern results about univariate real-rooted polynomials, including the classical convolutions, and mesh generalizations. Then we establish the fundamentals for the multivariate polynomial approaches, including both stability theory and hyperbolicity theory.

### Bibliographic Remarks

The main results presented in this thesis are all results of joint work. Chapters 3, 2, and 4 are all based on joint work [19, 20, 21] with Jonathan Leake. Chapter 5 is based on joint work [22] with Nikhil Srivastava and Prasad Raghavendra. Chapter 6 is based on joint work [23] with Nikhil Srivastava, Prasad Raghavendra, and Benjamin Weitz.

### Notation

In what follows, let $\mathcal{H}_+$ denote the open upper half-plane of $\mathbb{C}$, let $\mathbb{R}_{>0}$ denote the nonnegative real numbers, and let $\mathbb{K}$ denote a field, either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{K}[z_1, \ldots, z_n]$ denote the space of multivariate polynomials in variables $z_1, \ldots, z_n$ and let $\mathbb{K}_d[x]$ denote the space of univariate polynomials of degree at most $d$ with coefficients in $\mathbb{K}$. We will use different indeterminates throughout, but often $t, x, \text{ or } z$. For multivariate polynomials, we use $\mathbb{K}^\mu[z_1, \ldots, z_n]$ to denote the finite dimensional space of polynomials whose degree is bounded above by the tuple $\mu \in \mathbb{N}^n$. For $p \in \mathbb{K}[z_1, \ldots, z_n]$ and $t = (t_1, \ldots, t_n) \in \mathbb{K}^n$, define $p(tz) := p(t_1 z, t_2 z, \ldots, t_n z)$, which is a univariate polynomial. For $t = (t_1, \ldots, t_n) \in \mathbb{K}^n$ and $k \in [n] := \{1, \ldots, n\}$, let $(t_1, \ldots, \hat{t}_k, \ldots, t_n)$ denote the vector in $\mathbb{K}^{n-1}$ which is the vector $t$ with the $k^{th}$ element removed. Also, for all $k$ we use the shorthand $\partial_{z_k} := \frac{\partial}{\partial z_k}$. When it is clear from context which variable we are taking derivative with respect to we will use $D$ instead. Given a tuple $(\mu_1, \ldots, \mu_n) = \mu \in \mathbb{N}^n$, we use the following short hand to denote the iterated derivative $\partial^\mu = \partial_{z_1}^{\mu_1} \circ \ldots \circ \partial_{z_n}^{\mu_n}$. Finally, given a real-rooted univariate polynomial $p(x)$ we enumerate its roots $\lambda_n(p) \leq \lambda_{n-1}(p) \leq \ldots \leq \lambda_1(p) = \lambda_{\text{max}}(p)$.

### 1.1 Pólya-Schur Program

In the early 20th century Pólya-Schur set out a program to understand which linear operators preserve real-rootedness of univariate polynomials. Central to this program were the following two questions:
1. Which linear operators $T : \mathbb{R}[x] \to \mathbb{R}[x]$ send real rooted polynomials to real rooted polynomials?

2. Which linear operators $T : \mathbb{R}[x] \to \mathbb{R}[x]$ send real rooted polynomials of degree at most $d$ to real rooted polynomials of degree at most $d$?

There are two natural families of linear operators that were considered as sub-problems: Diagonal operators and differential operators. Diagonal operators are linear transformations which have the monomials as eigenvectors, namely $T(x^k) = c_k x^k$ for any $c_k$. Differential operators are linear transformations which are polynomial in the derivative operator, $T = \sum a_k D^k$ where $D = \frac{\partial}{\partial x}$. For both class of operators the unbounded situation is straightforward:

**Theorem 1.1.1.** A differential operator $T = \sum a_k D^k = p(D)$ preserves all real rooted polynomials if and only if $p$ is a real rooted polynomial. A diagonal operator $T[x^k] = c_k x^k$ preserves all real rooted polynomials if and only if $T[(1+x)^n] = \sum \binom{n}{k} c_k x^k$ is real rooted for all $n$.

To resolve the program in the bounded degree case we need new tools, namely the additive and multiplicative convolutions.

**Classical Convolutions**

The Walsh additive ([24]) and Grace-Szegö multiplicative ([25]) polynomial convolutions on $f, g \in \mathbb{C}_n[x]$ have been denoted $\boxplus^n$ and $\boxtimes^n$ respectively (e.g., in [26]):

$$f \boxplus^n g := \frac{1}{n!} \sum_{k=0}^{n} \frac{\partial^k f \cdot (\partial^{n-k}_x g)(0)}{k!}$$

$$f \boxtimes^n g := \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k g_k x^k$$

Notice we get a differential operator if we fix a polynomial $q$ and view the additive convolution as a linear operator $p \mapsto p \boxplus^n q$, and we can obtain all differential operators in this fashion. Similarly we get all diagonal operators on bounded degree polynomials using the multiplicative convolution.

This notation is suggestive, as these convolutions can be thought of producing polynomials whose roots are contained in the (Minkowski) sum and product of complex discs containing the roots of the input polynomials. When the inputs have real roots (additive) or non-negative roots (multiplicative), this fact also holds in terms of real intervals containing the roots.

These convolutions play a central role in analyzing linear operators which preserve bounded degree real rooted polynomials from certain classes. The following theorems illustrate this:
Theorem 1.1.2. A differential operator $T = \sum a_k D^k$ preserves the space of real rooted degree at most $n$ polynomials if and only if $T = p \boxplus^n \cdot$ for some fixed real rooted polynomial $p$. A diagonal operator $T[x^k] = c_k x^k$ preserves the space of real rooted polynomials of degree at most $n$ if and only if $T = p \boxstar^n \cdot$ for some fixed positive rooted polynomial $p$.

Some well known properties of the additive convolution are given as follows, where we let $\lambda(p)$ denote the non-increasing vector of roots of $p$ counting multiplicities.

Proposition 1.1.3. Let $p, q \in \mathbb{R}^n[t]$ be real-rooted polynomials of degree at most $n$. We have the following:

1. (Symmetry) $p \boxplus^n q = q \boxplus^n p$

2. (Shift-invariance) $(p(t + a) \boxplus^n q)(t) = (p \boxplus^n q)(t + a) = (p \boxplus^n q(t + a))(t)$ for $a \in \mathbb{R}^n$

3. (Scale-invariance) $(p(at) \boxplus^n q(at)) = a^n \cdot (p \boxplus^n q)(at)$ for $a \in \mathbb{R}$

4. (Derivative-invariance) $(Dp) \boxplus^n q = D(p \boxplus^n q) = p \boxplus^n (Dq)$ for all $k \in [n]$

5. (Stability-preserving) $p \boxplus^n q$ is real rooted

6. (Triangle inequality) $\lambda_1(p \boxplus^n q) \leq \lambda_1(p) + \lambda_1(q)$

Analogous properties hold in the case of the multiplicative convolution.

1.2 Submodularity

Recently, there has been interest in understanding how certain differential operators preserving real-rootedness affect the roots of the input polynomial. Much of this interest derives from the notion of interlacing families, heavily studied by Marcus, Spielman, and Srivastava in their collection of papers ([6],[7],[8]) containing their celebrated resolution of Kadison-Singer. Most uses of interlacing families share the same loose goal: to study spectral properties of random combinatorial objects. To do this, one equates random combinatorial operations on the objects to differential operators on associated characteristic polynomials. Then, understanding the spectrum of the random objects is reduced to understanding how the roots of certain polynomials are affected by differential operators preserving real-rootedness.

The most robust way to study the effects of a differential operator on roots comes from framework of Marcus, Spielman, and Srivastava. They associate an $R$-transform to polynomials, inspired from free probability theory, which gives tight bounds on the movement of the largest root via the additive convolution mentioned above. This framework was used in particular in [6] to prove the existence of Ramanujan bipartite graphs. The strength of their framework is that it gives tight largest root bounds for a general class of differential operator preserving real-rootedness, replacing many of the ad hoc methods used before to study specific desired operators.
The triangle inequality falls short often in analyzing the max root under a fixed linear operator: Let \( q \) be any polynomial whose max root is zero, then the triangle inequality says \( \lambda_{\text{max}}(p \boxplus^n q) \leq \lambda_{\text{max}}(p) \). It is desirable to come up with a more descriptive bound which takes into account more information about \( p \). To accomplish this, in [26], the authors consider the effects of a certain class of differential operators on the largest root of a given real-rooted polynomial:

\[
U_\alpha := 1 - \alpha D
\]

This differential operator is inspired by the Cauchy transform, via the following equivalence:

\[
U_\alpha p(x) = 0 \iff p(x) - \alpha p'(x) = 0 \iff \frac{p'(x)}{p(x)} = \frac{1}{\alpha} =: \omega
\]

Restricting to points larger than the largest root of \( p \), we have that \( \frac{d}{dx} \) is a bijection between \((\lambda_1(p), \infty)\) and \((0, \infty)\). Let \( K_\omega(p) \) denote the inverse of \( \omega \). Note that as \( \omega \to 0 \) our inverse tends to infinity, while as \( \omega \to \infty \) our inverse tends to \( \lambda_1(p) \). Furthermore, \( \lambda_1(U_\alpha p) = K_\omega(p) \).

This definition is inspired by similar objects from free probability, as discussed in [26]. The main result from [26] regarding these \( U_\alpha \) is given as follows:

**Theorem 1.2.1** ([26]). Let \( p, q \in \mathbb{R}^n[t] \) be real-rooted polynomials of degree \( n \). For any \( \alpha > 0 \) we have:

\[
\lambda_1(U_\alpha(p \boxplus^n q)) + n\alpha \leq \lambda_1(U_\alpha(p)) + \lambda_1(U_\alpha(q))
\]

As discussed above, every differential operator on polynomials in \( \mathbb{R}^n[t] \) can be represented as \( T(p) = p \boxplus^n q \) for some polynomial \( q \in \mathbb{R}^n[t] \). In particular we can represent \( U_\alpha \) via

\[
U_\alpha(p) = p \boxplus u_\alpha
\]

where \( u_\alpha(t) := t^n - n\alpha \cdot t^{n-1} \). Notice that here we have \( \lambda_1(u_\alpha) = n\alpha \), which means that the above result can be restated as follows:

\[
\lambda_1(p \boxplus^n q \boxplus^n u_\alpha) + \lambda_1(u_\alpha) \leq \lambda_1(p \boxplus^n u_\alpha) + \lambda_1(q \boxplus^n u_\alpha)
\]

This is a submodularity relation for the additive convolution. Further, by rearranging this result, it can also be seen as a diminishing returns property of the convolution:

\[
\lambda_1(p \boxplus^n q \boxplus^n u_\alpha) - \lambda_1(q \boxplus^n u_\alpha) \leq \lambda_1(p \boxplus^n u_\alpha) - \lambda_1(u_\alpha)
\]

The operation \( p \mapsto p \boxplus^n q \) can be interpreted as spreading out the roots of \( p \) (see the discussion at the beginning of §3.1). The above expression then says that, as the roots of a polynomial become more spread out, the operation of convolving by \( p \) has less of an effect on the largest root.

The natural next question is:

**Question 1.2.2.** Can \( u_\alpha \) be replaced by a larger class of real-rooted polynomials in the above expression?

This question and related topics are explored in Chapter 3. The main result of that chapter is an affirmative answer to the question encapsulated in Theorem 3.0.1.
CHAPTER 1. INTRODUCTION AND BACKGROUND MATERIAL

1.3 Mesh Properties

Another analytic property of roots that is studied is the minimal distance between two roots, known as mesh.

Definition 1.3.1. Fix $p \in \mathbb{R}[x]$ with all non-negative roots. We say that $p$ is $q$-log mesh if the minimum ratio (greater than 1) between any pair of non-zero roots of $p$ is at least $q$. We also say that $p$ is strictly $q$-log mesh if the minimum ratio is greater than $q$. In these situations, we write $\text{lmesh}(p) \geq q$ and $\text{lmesh}(p) > q$ respectively.

Definition 1.3.2. Fix $p \in \mathbb{R}[x]$ with all real roots. We say that $p$ is $b$-mesh if the minimum non-negative difference of any pair of roots of $p$ is at least $b$. We also say that $p$ is strictly $b$-mesh if this minimum difference is greater than $b$. In these situations, we write $\text{mesh}(p) \geq b$ and $\text{mesh}(p) > b$ respectively.

\textbf{Note}

The $q$-Multiplicative Convolution

In [16], Lamprecht proves logarithmic mesh preservation properties of $q$-multiplicative convolution. This convolution is defined as follows, where $p_k$ and $r_k$ are the coefficients of $p$ and $r$, respectively. (Note that as $q \to 1$ this limits to the classical multiplicative convolution.)

$$p \boxtimes_q^n r := \sum_{k=0}^{n} \binom{n}{k}_q^{-1} q^{-\binom{k}{2}} (-1)^k p_k r_k x^k$$

Here, $\binom{n}{k}_q$ denotes the $q$-binomial coefficients, defined as follows

Definition 1.3.3. For any $n \in \mathbb{Z}$, we have:

$$\binom{n}{q} := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}$$

We then extend this notation to $(n)_q! := (n)_q(n-1)_q \cdots (2)_q(1)_q$ and $\binom{n}{k}_q := \binom{(n)_q!}{(k)_q((n-k)_q)!}$.

Theorem 1.3.4 (Lamprecht). Let $p$ and $r$ be polynomials of degree at most $n$ such that $\text{lmesh}(p) \geq q$ and $\text{lmesh}(r) \geq q$, for some $q \in (1, \infty)$. Then, $\text{lmesh}(p \boxtimes_q^n r) \geq q$.

This result is actually an analogue of an earlier result of Suffridge [27] regarding polynomials with roots on the unit circle. In Suffridge’s result, $q$ is taken to be an element of the unit circle, and log mesh translates to mean that the roots are pairwise separated by at least the argument of $q$. Roughly speaking, he obtains the same result for the corresponding $q$-multiplicative convolution. Remarkably, the known proofs of his result (even a proof of Lamprecht) differ fairly substantially from Lamprecht’s proof of the above theorem.
Additionally, we note here that Lamprecht uses different notation and conventions in [16]. In particular, he uses \( q \in (0, 1) \), considers polynomials \( p \) with all non-positive roots, and his definition of \( \boxplus^n_q \) does not include the \((-1)^k\) factor. These differences are generally speaking unsubstantial, but it is worth noting that the arguments of §4.1 seem to require the \((-1)^k\) factor.

The \( b \)-Additive Convolution

In Chapter 4, we show the \( b \)-additive convolution (or, finite difference convolution) preserves the space of polynomials with root mesh at least \( b \). Brändén, Krasikov, and Shapiro define the \( b \)-additive convolution (only for \( b = 1 \)) in [17] as follows. (Note that as \( b \to 0 \) this limits to the classical additive convolution.)

\[
p \boxplus^n_b r := \frac{1}{n!} \sum_{k=0}^{n} \Delta_b^k p \cdot (\Delta_b^{n-k} r)(0)
\]

Here, \( \Delta_b \) is a finite \( b \)-difference operator, defined as:

\[
\Delta_b : p \mapsto \frac{p(x) - p(x-b)}{b}
\]

Our main result in Chapter 4 then solves the first (and second) conjecture stated in [17]. We state it formally here.

**Theorem 1.3.5 (4).** Let \( p \) and \( r \) be polynomials of degree at most \( n \) such that mesh(\( p \)) \( \geq b \) and mesh(\( r \)) \( \geq b \), for some \( b \in (0, \infty) \). Then, mesh(\( p \boxplus^n_b r \)) \( \geq b \).

As a note, Brändén, Krasikov, and Shapiro actually use the forward finite difference operator in their definition of the convolution. This is not a problem as our result then differs from their conjecture by a shift of the input polynomials.

**Remark 1.3.6.** Although the \( q \)-multiplicative and \( b \)-additive convolutions preserve \( q \)-log mesh and \( b \)-mesh respectively, they do not preserve real-rootedness.

To see this we calculate: \( x^2 \boxplus^2_1 x^2 = \frac{1}{2}(2x^2 - 2x + 1) \). This polynomial has discriminant \(-4\) and hence is not real rooted.

1.4 Interlacing

The notion of interlacing polynomials is intimately related to the theory of stable polynomials and mesh preservation. We now define this notion and state a few of its important properties.

**Definition 1.4.1.** Let \( p, q \in \mathbb{R}[z] \) be real-rooted polynomials given by \( p(z) = C_1 \prod_{k=1}^{n}(z - \lambda_k) \) and \( q(z) = C_2 \prod_{k=1}^{m}(z - \gamma_k) \), where \( n \) and \( m \) differ by at most 1 and \( m \leq n \). We write \( q \ll p \), or say \( q \) interlaces \( p \), if \( \lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \gamma_2 \geq \cdots \) and \( C_1 \cdot C_2 > 0 \). If the roots alternate in the same way but \( C_1 \cdot C_2 < 0 \), we swap the order of \( p \) and \( q \) in this relation.
We give a short proof of this now. If \( f \) such that \( f \) has all coefficients. We say that \( f \) and \( g \) have a common interlacing, if there exists \( f \in \mathbb{R}[z] \) such that \( f \ll p_k \) for all \( k \in [m] \).

Given \( f, g \in \mathbb{R}[x] \) with interlacing roots, we say \( f \ll g \) iff \( f'g - fg' \leq 0 \) iff \( \left( \frac{f'}{g} \right)' \leq 0 \) wherever defined. Further, we say \( f \ll g \) strictly if \( f'g - fg' < 0 \). Additionally, it is well known that \( f \) and \( g \) are real-rooted with interlacing roots then one of \( f \ll g \) or \( g \ll f \) holds. Further, \( f \) and \( g \) have strictly interlacing roots (no shared roots) iff \( f \ll g \) strictly or \( g \ll f \) strictly.

Let \( \lambda_f \) denote the largest root of \( f \). If \( f \) and \( g \) are monic, then \( f \ll g \) implies \( \lambda_f \leq \lambda_g \). We give a short proof of this now. If \( f \) has a double root at \( \lambda_f \), then interlacing implies \( g(\lambda_f) = 0 \), and therefore \( \lambda_f \leq \lambda_g \). Otherwise, consider that \( f \ll g \) implies \( f'(\lambda_f) \cdot g(\lambda_f) = (f'g - fg')(\lambda_f) \leq 0 \). Since \( f \) is monic, we have \( f'(\lambda_f) > 0 \) which in turn implies \( g(\lambda_f) \leq 0 \). Since \( g \) is monic, this implies the result. Note that this further implies that if \( f \) and \( g \) are monic and \( \deg(f) = \deg(g) - 1 \), then \( f \) and \( g \) have interlacing roots iff \( f \ll g \).

Another classical result allows us to combine interlacing relations. If \( f \ll g \) and \( f \ll h \), then \( f \ll ag + bh \) for any \( a, b \in \mathbb{R}_{>0} \). (This is well known whenever \( g, h \) have leading coefficients of the same sign, and we will prove it in full generality below.) A similar result holds if \( g \ll f \) and \( h \ll f \). Note also that \( f \ll g \) iff \( g \ll -f \), and that \( af \ll bf \) for all \( a, b \in \mathbb{R} \). Finally, the Hermite-Biehler theorem says \( af + bg \) is real-rooted for all \( a, b \in \mathbb{R} \) iff either \( f \ll g \) or \( g \ll f \).

Remark 1.4.3. A polynomial \( f \) with non-negative roots is \( q \)-log mesh if and only if \( f \ll f(q^{-1}x) \) and strictly \( q \)-log mesh if and only if \( f \ll f(q^{-1}x) \) strictly (for \( q > 1 \)). Similarly, a polynomial \( f \) with real roots is \( b \)-mesh if and only if \( f \ll f(x - b) \) and strictly \( b \)-mesh if and only if \( f \ll f(x - b) \) strictly (for \( b > 0 \)).

Now let \( f \) have \( n \) simple real roots, \( \alpha_1, ..., \alpha_n \), and let \( g \) be of degree at most \( n + 1 \). By partial fraction decomposition, we have:

\[
\frac{g(x)}{f(x)} = (dx + c) + \sum_{k=1}^{n} \frac{c_{\alpha_k}}{x - \alpha_k}
\]

Denoting \( f_{\alpha_k}(x) := \frac{f(x)}{x - \alpha_k} \), this implies:

\[
g(x) = (dx + c)f(x) + \sum_{k=1}^{n} c_{\alpha_k} f_{\alpha_k}(x)
\]

If \( g(\alpha_k) = 0 \), then \( c_{\alpha_k} = 0 \). Otherwise we compute:

\[
c_{\alpha_k} = \lim_{x \to \alpha_k} \frac{(x - \alpha_k)g(x)}{f(x)} = \left[ \frac{f'(\alpha_k)}{g(\alpha_k)} \right]^{-1} = \left[ \left( \frac{f}{g} \right)'(\alpha_k) \right]^{-1}
\]
For monic \( f \), we then further compute:

\[
d = \lim_{x \to \infty} \frac{f(x)g'(x) - f'(x)g(x)}{f(x)^2} = \lim_{x \to \infty} \left( \frac{g}{f} \right)'(x)
\]

This leads to the first result, which is a classical one.

**Proposition 1.4.4.** Fix \( f, g \in \mathbb{R}[x] \). Suppose \( f \) is monic and has \( n \) simple real roots, \( \alpha_1, \ldots, \alpha_n \), and suppose \( g \) is of degree at most \( n + 1 \). Consider the decomposition:

\[
g(x) = (dx + c)f(x) + \sum_{k=1}^{n} c_{\alpha_k} f_{\alpha_k}(x)
\]

Then, \( g \ll f \) iff \( d \leq 0 \) and \( c_{\alpha_k} \geq 0 \) for all \( k \), and \( f \ll g \) iff \( d \geq 0 \) and \( c_{\alpha_k} \leq 0 \) for all \( k \).

**Proof.** (\( \Rightarrow \)). If \( f \ll g \), then \( \left( \frac{f}{g} \right)' \leq 0 \) and \( \left( \frac{g}{f} \right)' \geq 0 \). This implies \( c_{\alpha_k} \leq 0 \) for all \( k \) and \( d \geq 0 \). If \( g \ll f \), then \( f \ll -g \). The same argument implies \( c_{\alpha_k} \geq 0 \) for all \( k \) and \( d \leq 0 \).

(\( \Leftarrow \)). Supposing \( d \geq 0 \) and \( c_k \leq 0 \) for all \( k \), we write:

\[
R(x) := \frac{g(x)}{f(x)} = (dx + c) + \sum_{k=1}^{n} \frac{c_{\alpha_k}}{x - \alpha_k}
\]

Note that this implies \( R(\alpha_k + \epsilon) < 0 \) and \( R(\alpha_k - \epsilon) > 0 \) for small enough \( \epsilon > 0 \) and for all \( k \). This implies that \( g \) has at least one root between each pair of adjacent asymptotes of \( f \). Note that this demonstrates interlacing up to a few missing roots of \( g \).

To show \( f \ll g \) we just need to prove that the remaining roots of \( g \) do not disrupt this interlacing property, and that the leading coefficient of \( g \) is the correct sign. Casework on the possible values of \( c, d \) then implies the result. Finally, the result for \( d \leq 0 \) and \( c_k \geq 0 \) for all \( k \) follows similarly.

There is actually another way to state this result, in terms of cones of polynomials. Let \( \text{cone}(f_1, \ldots, f_m) \) denote the closure of the positive cone generated by the polynomials \( f_1, \ldots, f_m \).

**Corollary 1.4.5.** Let \( f \in \mathbb{R}[x] \) be a monic polynomial with \( n \) distinct (not necessarily simple) roots, \( \alpha_1, \ldots, \alpha_n \). Then:

\[
\{ g \in \mathbb{R}[x] : g \ll f \} = \text{cone}(-xf, -f, f_{\alpha_1}, \ldots, f_{\alpha_n})
\]

\[
\{ g \in \mathbb{R}[x] : f \ll g \} = \text{cone}(xf, f, -f, -f_{\alpha_1}, \ldots, -f_{\alpha_n})
\]

Here we define \( f_{\alpha_k} := \frac{f(x)}{x - \alpha_k} \) even when \( \alpha_k \) is not a simple root of \( f \).
Proof. First suppose that \( f \) has degree exactly \( n \) (all roots simple). If the roots of \( f, g \) interlace, then \( g \) is of degree at most \( n + 1 \). Since any such \( g \) can be written as a linear combination of \( xf, f, f_{\alpha_1}, ..., f_{\alpha_n} \), the result follows from the previous proposition.

Now if \( f \) has a root \( \alpha_k \) of multiplicity \( m \geq 2 \), then any polynomial \( g \) for which \( f \ll g \) or \( g \ll f \) must have a root at \( \alpha_k \) of multiplicity at least \( m - 1 \). In this case we have:

\[
\{ g \in \mathbb{R}[x] : g \ll f \} = (x - \alpha_k)^{m-1} \cdot \left\{ h \in \mathbb{R}[x] : h \ll \frac{f(x)}{(x - \alpha_k)^{m-1}} \right\}
\]

Inducting on this idea then implies the result.

This immediately yields the following result concerning linear operators preserving certain interlacing relations. Notice that here we restrict to only considering polynomials \( g \) of degree at most \( n \), where \( n \) is the degree of \( f \).

**Definition 1.4.6.** Given a real linear operator \( T : \mathbb{R}_n[x] \to \mathbb{R}[x] \), and a real-rooted polynomial \( f \), we say that \( T \) preserves interlacing with respect to \( f \) if \( g \ll f \) implies \( T[g] \ll T[f] \) and \( f \ll g \) implies \( T[f] \ll T[g] \) for all \( g \in \mathbb{R}_n[x] \).

**Corollary 1.4.7.** Fix a real linear operator \( T : \mathbb{R}_n[x] \to \mathbb{R}[x] \), and fix \( f \in \mathbb{R}_n[x] \). Suppose \( f \) is monic with \( n \) simple roots, \( \alpha_1, ..., \alpha_n \), and that \( T[f_{\alpha_k}] \ll T[f] \) for all \( k \). Then, \( T \) preserves interlacing with respect to \( f \).

The next result gives a link between the concept of interlacing and the roots of linear combinations of polynomials. It is typically attributed to Obreshkoff, but can be viewed as a reformulation of the Hermite-Biehler theorem.

**Proposition 1.4.8 (Obreshkoff’s Theorem).** For \( p, q \in \mathbb{R}[z] \) with real roots, \( \alpha p + \beta q \) is real-rooted for all \( \alpha, \beta \in \mathbb{R} \) iff \( p \ll q \) or \( q \ll p \).

To generalize Obreshkoff’s Theorem to many polynomials, Chudnovsky and Seymour make the following definition and prove the following equivalence.

**Definition 1.4.9.** We say that \( p_1, ..., p_m \in \mathbb{R}[z] \) are compatible if all convex combinations are real rooted.

**Theorem 1.4.10 ([15]).** Let \( p_1, ..., p_k \in \mathbb{R}[z] \) be polynomials with positive leading coefficients. The following are equivalent.

1. \( p_i \) and \( p_j \) are compatible for all \( i \neq j \).
2. \( p_i \) and \( p_j \) have a common interlacing for all \( i \neq j \).
3. \( p_1, ..., p_k \) are compatible.
4. \( p_1, ..., p_k \) have a common interlacing.
1.5 Multivariate Polynomials

Real Stability

To develop a complete answer to the Pólya-Schur program, Brändén and Borcea used the theory of real stable polynomials in [28, 29]. First we recall their conclusion to the program, and then talk about the necessary background related to stable polynomials:

**Definition 1.5.1.** A polynomial \( p \in \mathbb{R}[x_1, \ldots, x_n] \) is called *real stable* if it is identically zero\(^1\) or if \( p(z_1, \ldots, z_n) \neq 0 \) whenever \( \text{Im}(z_i) > 0 \) for all \( i = 1, \ldots, n \). Equivalently, \( p \) is real stable if and only if the univariate restrictions

\[ t \mapsto p(te_1 + x_1, te_2 + x_2, \ldots, te_n + x_n) \]

are real rooted whenever \( e_1, \ldots, e_n > 0 \) and \( x_1, \ldots, x_n \in \mathbb{R} \).

The equivalence between the two formulations above is an easy exercise. Note that a univariate polynomial is real stable if and only if it is real rooted. Note that we consider the zero polynomial to be real-rooted.

**Theorem 1.5.2 ([28, 29]).** Given a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^n \), it sends real rooted polynomials to real rooted polynomials if and only if one of the following conditions holds:

1. \( T[(x + w)^n] = \sum_k \binom{n}{k} T[x^k]w^{n-k} \) is a real stable polynomial
2. \( T[(1 + xw)^n] = \sum_k \binom{n}{k} T[x^k]w^k \) is a real stable polynomial
3. The image of \( T \) is a two-dimensional subspace of real-rooted polynomials, equivalently the span of two interlacing polynomials

We will frequently use the elementary fact that a limit of real-rooted polynomials is real-rooted, which follows from Hurwitz’s theorem (see, e.g. [30, Sec. 2]), or from the argument principle.

To get a sense of how these polynomials behave, it’s helpful to look at a bunch of basic operations which preserve stability.

**Proposition 1.5.3 (Closure Properties).** Let \( p, q \in \mathbb{K}[z_1, \ldots, z_n] \) be stable (resp. real stable) polynomials, and fix \( k \in [n] \). Then the following are also stable (resp. real stable).

1. \( p \cdot q \) (product)
2. \( \partial_{z_k} p \) (differentiation)
3. \( z_k \partial_{z_k} p \) (degree-preserving differentiation)

---

\(^1\)Some works (e.g. [28]) consider only nonzero polynomials to be stable, while others [30] include the zero polynomial. We find the latter convention more convenient.
(iv) \( p(z_1, \ldots, z_{k-1}, r, z_{k+1}, \ldots, z_n) \), for \( r \in \mathbb{R} \) (real specialization)

(v) \( p(z_1, \ldots, z_{k-1}, z_1, z_{k+1}, \ldots, z_n) \) (projection)

(vi) \( z_k^{\text{deg}_k(p)} p(z_1, \ldots, z_{k-1}, -z_k^{-1}, z_{k+1}, \ldots, z_n) \) (inversion)

Here, \( \text{deg}_k(p) \) is the degree of \( z_k \) in \( p \).

The next result is a stability equivalence theorem of Borcea and Brändén, which is essentially a generalization of the Hermite-Biehler theorem.

**Theorem 1.5.4** ([28], Lemma 1.8). For \( p, q \in \mathbb{R}[z_1, \ldots, z_n] \), \( p + z_{n+1}q \) is real stable iff for every \( t \in \mathbb{R}_{>0}^n \) and every \( y \in \mathbb{R}^n \), we have that \( q(tz + y) \) interlaces \( p(tz + y) \).

Finally, we give an equivalent condition for real stability of multi-affine polynomials. This will be a useful result for demonstrating counterexamples to real-rootedness.

**Definition 1.5.5.** A polynomial \( p \in \mathbb{K}[z_1, \ldots, z_n] \) is said to be **multi-affine** if it is of degree at most one in each variable.

**Proposition 1.5.6** ([4]). A multi-affine polynomial \( p \in \mathbb{R}[z_1, \ldots, z_n] \) is real stable iff \( p \) is strongly Rayleigh. That is, iff for every \( j \neq k \in [n] \) and every \( x \in \mathbb{R}^n \), we have the following.

\[
(\partial_{z_j} p)(x) \cdot (\partial_{z_k} p)(x) \geq (\partial_{z_j} \partial_{z_k} p)(x) \cdot p(x)
\]

**1.6 Graph Polynomials**

In this section, we discuss the multivariate analogues of the independence and matching polynomials. Though somewhat counter-intuitive, considering the multivariate versions of these polynomials actually simplifies the situation. In the multivariate world, one can directly manipulate how particular vertices and edges influence the polynomial by manipulating the associated variable. And further, these polynomials are multiaffine: important operations like differentiation and evaluation at 0 have intuitive interpretations.

Notions like real-rootedness and root bounds become trickier in the multivariate world, but real stability and similar notions can often play the analogous parts. This is true for the multivariate matching polynomial and somewhat true for the multivariate independence polynomial, as we will see below. But first, let’s set up some notation.

**Graph Notation**

Let \( G = (V_G, E_G) \) be an undirected graph, which is simple unless otherwise specified. As usual, \( V_G \) is the set of vertices and \( E_G \) is the set of edges. We employ standard notation surrounding these first objects:

- \( \{u, v\} \in E_G \) iff there is an edge between vertices \( u \) and \( v \).
• $u \in e$ for $e \in E_G$ iff $u$ is a vertex of the edge $e$

• $N_G[v]$ (resp. $N_G(v)$) denotes the closed (resp. open) neighborhood of $v$

• $H \subseteq G$ (resp. $H \leq G$) iff $H$ is a subgraph (resp. induced subgraph) of $G$

As usual, we will leave off the subscript $G$ when unambiguous. We also generalize the definition of “claw” in the following standard way. As usual, let $K_{m,n}$ denote the complete bipartite graph with $m+n$ vertices. So, we refer to $K_{1,3}$ as a claw or as a $3$-star. Generalizing, we refer to $K_{1,n}$ as an $n$-star. For any graph $H$, we say that $G$ is $H$-free if it does not contain $H$ as an induced subgraph.

Finally, we denote the line graph of $G$ by $L(G)$. This is the graph formed by considering the edges of $G$ to be the vertices of $L(G)$, with adjacency in $L(G)$ determined by whether or not the corresponding edges of $G$ share a vertex in $G$.

**The Matching Polynomial**

The univariate and multivariate matching polynomials have been well studied. In 1972, Heilmann and Lieb proved that for any graph the multivariate matching polynomial is real-stable. This implies the real-rootedness of the univariate matching polynomial, and in fact Heilmann and Lieb gave bounds on its largest root. More recently, Choe, Oxley, Sokal, and Wagner [3] gave a simpler proof of this fact using a special linear operator on polynomials, called the “multi-affine part”. We their proof below.

First though, we define and discuss a few multivariate matching polynomials. The reader should be aware that our notation will be slightly different from that which is standard; we do this to emphasize the connection between the matching and independence polynomials. We give examples of all these polynomials in Figure 2.

Given any graph $G$, we define the multi-affine *vertex matching polynomial* of $G$ as follows.

$$
\mu_V(G) \equiv \mu_V(G)(x) := \sum_{M \subseteq E \text{ matching}} \prod_{\{u,v\} \in M} -x_u x_v
$$

Notice that the univariate restriction of $\mu_V(G)$ is the univariate matching polynomial used by Godsil and Heilmann-Lieb, but with the degrees inverted. So, for instance, Heilmann and Lieb’s upper bound on the absolute value of the roots of the matching polynomial would translate to a bound away from zero for this inverted polynomial. We will discuss this further later. We also define the multiaffine *edge matching polynomial* of $G$ as follows.

$$
\mu_E(G) \equiv \mu_E(G)(x) := \sum_{M \subseteq E \text{ matching}} \prod_{e \in E} x_e
$$

We now give the proof of real stability of the vertex matching polynomial, and show its connection to the edge matching polynomial.
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Theorem 1.6.1 ([31], [3], [29]). For any graph $G$, the vertex matching polynomial $\mu_V(G)$ is real stable.

Proof. Let MAP (“Multi-Affine Part”) denote the linear operator on multivariate polynomials which removes any terms which are not multi-affine. By [29], this operator preserves real stability. We then have the following.

$$\mu_V(G)(x) \overset{\text{MAP}}{=} \prod_{\{u,v\} \in E} (1 - x_u x_v)$$

Since $(1 - x_u x_v)$ is real stable and the product of real stable polynomials is real stable, this implies the result.

This then implies real-rootedness of the univariate matching polynomial via univariate restriction. As for the edge matching polynomial, we don’t quite have real stability. However, we do have same-phase stability, which still implies real-rootedness of the univariate restriction.

Corollary 1.6.2. For any graph $G$, the edge matching polynomial $\mu_E(G)$ is same-phase stable.

Proof. Let $\Pi^\downarrow$ be the projection operator, which sends all variables $x_v$ to a single variable $x$. Fixing $(t_e)_{e \in E} \in \mathbb{R}^{\lvert E \rvert}_+$, we have the following.

$$\mu_E(G)(-tx^2) = \sum_{M \subseteq E, M \text{ matching}} \prod_{e \in M} -t_e x^2 = (\Pi^\downarrow \circ \text{MAP}) \left( \prod_{\{u,v\} \in E} (1 - t_e x_u x_v) \right)$$

By closure properties of real stability and the fact that $t_e > 0$ implies $(1 - t_e x_u x_v)$ is real stable, the right-hand side of the above equation is real-rooted. So, $\mu_E(G)(-tx^2)$ is real-rooted, which implies $\mu_E(G)(tx)$ is real-rooted. (In fact, it has all its roots on the negative part of the real line.) Since $t$ was arbitrary, this implies the result.

It’s well-known that matchings of graphs are related to independent sets of line graphs. This connection is made particularly clear by considering the (multivariate) edge matching polynomial, as we will see in the next section.

The univariate independence polynomial of a graph is another well-studied graph polynomial. However, consideration of its roots has proven a bit more difficult. For example, the independence polynomial of a graph is not real-rooted in general, and it has only been about a decade since the first proof of real-rootedness for claw-free graphs was published in [15]. Since then a number of proofs of real-rootedness have appeared, along with interesting results about location and modulus of certain roots ([32], [33], [34], [35]). In Chapter 2, we give another proof of real-rootedness for claw-free graphs by proving something stronger:
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namely, that the multivariate independence polynomial of a graph is same-phase stable if and only if the graph is claw-free. In their original proof, Chudnovsky and Seymour show real-rootedness using an intricate recursion based on a combinatorial structure known as a “simplicial clique”. By encoding the recursive compatibility using our notion of same-phase stability, we are able to avoid the introduction of simplicial cliques and use simpler graph structures in the recursion. Same-phase stability of the edge matching polynomial then serves as the base case.

1.7 Hyperbolic Polynomials

Real Stable polynomials are closely related to the following more general class of polynomials.

Definition 1.7.1. A homogeneous polynomial \( p \in \mathbb{R}[x_1, \ldots, x_n] \) is called hyperbolic with respect to a point \( e = (e_1, \ldots, e_n) \in \mathbb{R}^n \) if \( p(e) > 0 \) and the univariate restrictions

\[
t \mapsto p(te + x)
\]

are real rooted for all \( x \in \mathbb{R}^n \). The connected component of \( \{ x \in \mathbb{R}^n : p(x) \neq 0 \} \) containing \( e \) is called the hyperbolicity cone of \( p \) with respect to \( e \), and will be denoted \( K(p, e) \).

Real stable polynomials and hyperbolic polynomials are related by the following:

Proposition 1.7.2 ([28]). Given a polynomial \( p \in \mathbb{R}[x_1, \ldots, x_n] \) of degree \( d \), we can homogenize the polynomial to obtain \( p_H = z^d p(x_1/z, \ldots, x_n/z) \). Then \( p \) is real stable if and only if \( p_H \) is homogeneous with respect to all vectors in \( \mathbb{R}_{>0}^n \times \{0\} = \{(x_1, \ldots, x_n, 0) | x_i > 0 \} \).

The most important theorem regarding hyperbolic polynomials says that hyperbolicity cones are always convex, and that hyperbolicity at one point in the cone implies hyperbolicity at every other point.

Theorem 1.7.3 (Garding [36]). If \( p \in \mathbb{R}[x_1, \ldots, x_n] \) is hyperbolic with respect to \( e \in \mathbb{R}^n \) then:

1. \( K(p, e) \) is an open convex cone.
2. \( p \) is hyperbolic with respect to every point \( y \in K(p, e) \).

Perhaps the most familiar example of a hyperbolic polynomial is the determinant of a symmetric matrix:

\[
X \mapsto \det(X)
\]

for real symmetric \( X \), which is hyperbolic with respect to the identity matrix since the characteristic polynomial of a symmetric matrix is always real rooted. The corresponding hyperbolicity cone is the cone of positive semidefinite matrices. By taking certain restrictions of this polynomial we can end up with polynomials of the form \( (x_1, \ldots, x_n) \mapsto \det(\text{sum} x_i A_i) \).
where $A_i$ is symmetric. In this case the hyperbolicity cone with respect to the identity matrix is the \textit{spectrahedral cone} $\{x \mid \sum x_i A_i \succ 0\}$. These are exactly linear slices of the cone of positive definite matrices which show up in the theory of optimization using semidefinite programming. From this point of view hyperbolic polynomials give us an interesting collection of cones which potentially could generalize semidefinite programming. To explore this generalization we need to understand which aspects of semidefinite programming abstract to the hyperbolic setting.

From the standpoint of convex optimization, hyperbolicity cones yield a rich family of convex sets that one can efficiently optimize over — in particular, interior point methods can be used to efficiently optimize over the hyperbolicity cone $K_p$ for a polynomial $p$, given an oracle to evaluate $p$ and its gradient and Hessian [37, 38]. This optimization primitive, referred to as hyperbolic programming, captures linear and semidefinite programming as special cases.

Given that hyperbolic programming yields a common generalization of linear and semidefinite programs, it is natural to ask whether hyperbolic programming as an algorithmic primitive is strictly more powerful than semidefinite programming? Conversely, can optimization over hyperbolicity cones be always \textit{simulated} by semidefinite programming? The analogous question of comparing the power of semidefinite programs to linear programs has been satisfactorily resolved by the work on LP extended formulations [39]. Specifically, an LP extended formulation of a convex set $C \in \mathbb{R}^m$, is a polytope $P \in \mathbb{R}^n$ for $n > m$ along with a projection $\Pi : \mathbb{R}^n \to \mathbb{R}^m$ such that $\Pi \circ P = C$. The size of the extended formulation is the number of constraints representing $P$. By showing a lower bound on the size of the extended formulation of a convex set $C$, one is proving that the set $K$ is not efficiently representible as \textit{projections of polytopes}, thus optimization over $C$ cannot be efficiently simulated by linear programming. Notice that in an extended formulation one is allowed to introduce new auxiliary variables and new constraints (i.e., $C$ is a \textit{projection} of $P$), in order to obtain a compact representation of the set $C$.

To study the relationship between hyperbolic programming and semidefinite programming a series of conjectures have appeared. The Lax conjecture in its original stronger algebraic form asked whether every polynomial in three variables hyperbolic with respect to the direction $(1, 0, 0)$, could be written as $\det(xI + yB + zC)$ for some symmetric matrices $B, C$ (sometimes called a \textit{definite determinantal representation}). This immediately implies that all hyperbolicity cones in three dimensions are spectrahedral, and was proved by Helton and Vinnikov and Lewis, Parrilo, and Ramana [40, 41]. The algebraic conjecture is easily seen to be false for $n > 3$ by a count of parameters, since the set of hyperbolic polynomials is known to have nonempty interior [42] and is of dimension $n^d$, whereas the set of $n-$tuples of $d \times d$ matrices has dimension $n\binom{d}{2}$. This led to the weaker conjecture that for every hyperbolic $p(x)$ there is an integer $k$ such that $p(x)^k$ admits a definite determinantal representation, which was disproved by Bränden in [18]. The Generalized Lax conjecture, which is a geometric statement, is equivalent to the yet weaker algebraic statement that for every hyperbolic $p(x)$ there is a hyperbolic $q(x)$ such that $K_p \subseteq K_q$ and $p(x)q(x)$ admits a definite determinantal representation.
Conjecture 1.7.4. (Generalized Lax Conjecture) Every hyperbolicity cone is a spectrahedral cone, i.e., a linear section of the cone of positive semidefinite matrices in some dimension.

An SDP extended formulation of a cone $C$ would be a representation of $C = \pi \circ K$ for a projection $\pi : \mathbb{R}^n \to \mathbb{R}^m$ and a spectrahedral cone $K$. The size of the SDP extended formulation is the dimension $B$ of the psd cone of which $K$ is a linear section. A spectrahedral representation is a special case of an SDP extended formulation wherein $\pi = Id$, i.e., no additional auxiliary variables are introduced in the formulation. With this terminology, the question of power of hyperbolic vs semidefinite programming can be rephrased as asking whether there exists hyperbolicity cones with no small SDP extended formulations.

Even more basic questions about SDP extended formulations of hyperbolicity cones remain open. For instance, does every hyperbolicity cone admit an SDP extended formulation of some finite size? It is entirely possible that every hyperbolicity cone admits a spectrahedral representation. This is known as the Generalized Lax conjecture that has received significant attention in recent work.

For several special classes of hyperbolic polynomials, the corresponding hyperbolicity cones are known to be spectrahedral. The elementary symmetric polynomial of degree $d$ in $n$ variables is given by $e_d(x) = \sum_{S \subseteq [n], |S| = d} \prod_{i \in S} x_i$. Bränden [43] showed that the hyperbolicity cones of elementary symmetric polynomials are spectrahedral with matrices of dimension $O(n^d)$. If a polynomial $p$ is hyperbolic with respect to a direction $e \in \mathbb{R}^n$, then its directional derivatives along $e$ are hyperbolic too [44]. Directional derivatives of the polynomial $p(x) = \prod_{i \in [n]} x_i$ [45, 46, 43, 3] and the first derivatives of the determinant polynomial [47] are also known to satisfy the generalized Lax conjecture. Amini [48] has shown that certain multivariate matching polynomials are hyperbolic and that their cones are spectrahedral, of dimension $n!$.

Given the above examples, it is natural to wonder whether exponential blowup in dimension is an essential feature of passing from hyperbolicity cones to spectrahedral representations, and even assuming the generalized Lax conjecture to be true, the size of the spectrahedral representation of a hyperbolicity cone is interesting from a complexity standpoint. Can one find a hyperbolic polynomial which can be evaluated quickly but has no polynomial sized spectrahedral representation? Even stronger, can we find such a polynomial for which all spectrahedral representations also must be exponentially sized? Can one prove that a sufficiently random hyperbolic polynomial satisfies the above? In an attempt to begin understanding questions like these, in Chapter 6 we produce an explicit random family of polynomials such that all approximate spectrahedral representations are exponentially sized.

1.8 Stability Testing

Algorithmic questions surrounding real rootedness have been well-studied. Given a univariate polynomial, $p(x) \in \mathbb{R}[x]$ with real coefficients, the following are all classically studied
questions: Is $p(x)$ nonnegative, i.e. is $p(x) \geq 0$ for all $x \in \mathbb{R}$? How many roots of $p(x)$ are real? Given some error threshold, $\epsilon > 0$, can we find intervals $I_k$ of radius at most $\epsilon$ which contain the $k$th real root of $p(x)$?

The fundamental difference between questions about real roots and complex roots lie in the ability to get accurate counts of real roots in a given domain. Given a univariate polynomial $p(z)$, if we want to count the number of zeroes in a simply connected domain $D \subset \mathbb{C}$ we can use the residue theorem by integrating along the boundary $\partial D$. However, the integral becomes poorly behaved as the boundary approaches zeroes of the polynomial, which leads to issues with bit complexity. Problems involving real roots avoid this mainly by using Sturm sequences [49], given a polynomial $p$:

$p_0 = p$

$p_1 = p'$

$p_{i+1} = -\text{rem}(p_{i-1}, p_i)$

Given a point $\chi \in \mathbb{R}$, define $V(\chi)$ to be the total number of sign changes (ignoring zeroes) along the Sturm sequence, $p_0(\chi), p_1(\chi), \ldots$.

**Theorem 1.8.1** (Sturm’s Theorem[49]). *Given a square-free polynomial $p(x)$, the number of distinct zeroes in the interval $(a, b]$ is*

$V(a) - V(b)$

To handle polynomials which are not square-free we first define a square-free decomposition

**Definition 1.8.2.** Given a polynomial $p$ we call $a_1 \ldots a_k$ a *square-free decomposition* of $p$ if $p = a_1 \ldots a_k$ and $a_i$ are all square-free polynomials and coprime to each other. Note that $a_i$ is the product of the irreducible factors, $r$, of $p$ such that $r^i$ divides $p$ and $r^{i+1}$ doesn’t.

There are many algorithms, such as Yun’s algorithm [50], for producing such decompositions. Note the combination of Sturm’s sequence and Yun’s algorithm solve all three of our problems since we have the following reductions: $p$ is real rooted if and only if all of the square-free polynomials in a square-free decomposition are real rooted. In order for $p$ to be nonnegative it must have no real roots of odd multiplicity, since this would imply locally $p$ is both negative and positive. We conclude $p$ is a nonnegative polynomial if and only if the odd degree terms in the square-free decomposition have no real roots and $p$ has positive leading coefficient. Finally, in order to locate the roots, given an oracle which counts the number of real roots in an interval we can use bisection method to obtain membership intervals of a specified radius.

Although this gives an efficient algorithm for determining whether a given polynomial is real rooted, other contexts for determining whether a polynomial is real rooted require different tools. In §5.2 we introduce new machinery in order to handle such a situation.
In Chapter 5 we explore an extension of this question to the bivariate scenario: Given a bivariate polynomial $p(x, y) \in \mathbb{R}[x, y]$, is there an exact arithmetic polynomial time algorithm which determines whether or not $p(x, y)$ is a real-stable polynomial? By a generic argument using quantifier elimination such an algorithm exists. Our main result of Chapter 5, Theorem 5.0.1, constructs an explicit algorithm using elementary techniques which runs in $O(n^5)$ time.
Chapter 2

Generalizations of the Matching Polynomial to the Multivariate Independence Polynomials

Given a graph \( G = (V, E) \), the matching polynomial of \( G \) and the independence polynomial of \( G \) are defined as follows.

\[
\mu(G) := \sum_{M \subseteq E, M, \text{matching}} (-x^2)^{|M|}
\]

\[
I(G) := \sum_{S \subseteq V, S, \text{independent}} x^{|S|}
\]

The real-rootedness of the matching polynomial and the Heilmann-Lieb root bound are important results in the theory of undirected simple graphs. In particular, real-rootedness implies log-concavity and unimodality of the matchings of a graph, and recently in \([6]\) the root bound was used to show the existence of Ramanujan graphs. Additionally, it is well-known that the matching polynomial of a graph \( G \) is equal to the independence polynomial of the line graph of \( G \). With this, one obtains the same results for the independence polynomials of line graphs. This then leads to a natural question: what properties extend to the independence polynomials of all graphs?

Generalization of these results to the independence polynomial has been partially successful. About a decade ago, Chudnovsky and Seymour \([15]\) established the real-rootedness of the independence polynomial for claw-free graphs. (The independence polynomial of the claw is not real-rooted.) A general root bound for the independence polynomial was also given by \([51]\), though it is weaker than that of Heilmann and Lieb. As with the original results, these generalizations are proven using univariate polynomial techniques.

More recently, Borcea and Brändén used their characterization of stability-preserving operators \([28]\), \([29]\) to give a simple and intuitive proof of the real-rootedness of the matching polynomial by showing that the multivariate matching polynomial is real stable. Beyond its surprising simplicity, their proof also suggests that the multivariate approach may be the more natural one. That said, the first part of this chapter is a partial generalization
of this stability result to the multivariate independence polynomial of claw-free graphs. In particular, we prove a result related to the real-rootedness of certain weighted independence polynomials. This result was originally proven by Engström in [52] by bootstrapping the Chudnovsky and Seymour result for rational weights and using density arguments. The proof we give here is completely self-contained and implies both the original Chudnovsky and Seymour result as well as the weighted generalization. By using a multivariate framework to directly prove the more general result, we obtain a simple inductive proof which we believe better captures the underlying structure.

In addition, the full importance of the claw (3-star) graph is not immediately clear from the univariate framework. Since the result of Chudnovsky and Seymour, there have been attempts to explain more conceptually why the claw-free premise is needed for real-rootedness. In particular, some graphs containing claws actually have real-rooted independence polynomials, disproving the converse to the univariate result. On the other hand, the stronger stability-like property we use here turns out to be equivalent to claw-freeness, yielding a satisfactory converse.

In the second part of this chapter, we then extend the Heilmann–Lieb root bound by generalizing some of Godsil’s work on the matching polynomial. In [53], Godsil demonstrated the real-rootedness of the matching polynomial of a graph by showing that it divides the matching polynomial of a related tree. (For a tree, root properties are more easily derived.) We prove a similar result for the multivariate matching polynomial, and then we determine conditions for which these divisibility results extend to the multivariate independence polynomial. Further, we prove the Heilmann–Lieb root bound for the independence polynomial of a certain subclass of claw-free graphs. By considering a particular graph called the Schläfli graph, we demonstrate that this root bound does not hold for all claw-free graphs and provide a weaker bound in the general claw-free case.

**Same-phase Stability**

We now introduce a new notion of stability. Notice that the connection between the following conditions is similar to that which is given by Proposition 1.5.1.

**Definition 2.0.1.** A polynomial \( p \in \mathbb{R}[z_1, ..., z_n] \) is said to be same-phase stable if one of the following equivalent conditions is satisfied.

1. For every \( t \in \mathbb{R}^n_+ \), the univariate restriction \( p(tz) \) is stable (and therefore real rooted).
2. If \( \arg(z_1) = \arg(z_2) = \cdots = \arg(z_n) \), then \( p(z_1, ..., z_n) = 0 \) implies \( z_k \not\in \mathcal{H}_+ \) for some \( k \).

We will primarily make use of condition (i).

This notion is strictly weaker than that of “stable”, and it will serve as the basic concept in what follows (as stability and real stability did in the previous section). Next, we define a notion of compatibility for real same-phase stable polynomials, which is similar to that of Chudnovsky and Seymour in [15].
Definition 2.0.2. Polynomials $p_1, \ldots, p_m \in \mathbb{R}_+ [z_1, \ldots, z_n]$ with nonnegative coefficients are said to be same-phase compatible if $p_k$ is same-phase stable for all $k$, and the polynomials \{\(p_k(tz)\)\}_{k=1}^m$ are compatible for each $t \in \mathbb{R}_+^n$. Note that by Theorem 1.4.10, we could instead require \{\(p_k(tz)\)\}_{k=1}^m$ have a common interlacing for each $t \in \mathbb{R}_+^n$.

Remark 2.0.3. In order to utilize the theory of interlacing and compatible polynomials, we need to assume that the polynomials we are using have nonnegative coefficients. This is because results like Theorem 1.4.10 no longer hold if negative or complex coefficients are allowed. That said, this restriction is not required to define same-phase stable polynomials, and many other properties also hold without it.

We now can apply Chudnovsky and Seymour’s equivalence result (Theorem 1.4.10) to get the following:

Corollary 2.0.4. Let $p_1, \ldots, p_k \in \mathbb{R}_+ [z_1, \ldots, z_n]$ be polynomials with nonnegative coefficients. The following are equivalent.

1. $p_i$ and $p_j$ are same-phase compatible for all $i \neq j$.
2. $p_1, \ldots, p_k$ are same-phase compatible.

Same-phase Stability for Multi-affine Polynomials

We now begin to develop a general theory of same-phase stability for multi-affine real polynomials. This class of polynomials is of particular importance here, as most multivariate graph polynomials are real and multi-affine. We start by giving some basic closure properties.

Proposition 2.0.5 (Closure Properties). Let $p \in \mathbb{R}[z_1, \ldots, z_n]$ and $q \in \mathbb{R}[w_1, \ldots, w_m]$ be multi-affine same-phase stable polynomials, and fix $k \in [n]$. Then the following are also multi-affine same-phase stable. Note that if in addition $p$ and $q$ have nonnegative coefficients, then the following do as well.

(i) $p \cdot q$ (disjoint product)
(ii) $\partial_{z_k} p$ (differentiation)
(iii) $z_k \partial_{z_k} p$ (variable selection)
(iv) $p(z_1, \ldots, z_{k-1}, 0, z_{k+1}, \ldots, z_n)$ (variable deselection)
(v) $z_1 z_2 \cdots z_n p(z_1^{-1}, \ldots, z_n^{-1})$ (selection inversion)

Proof. (i) Straightforward.
(ii) Fix $t \in \mathbb{R}_+^n$, letting $t_k$ vary. Also, define $t_0 := (t_1, \ldots, t_k, \ldots, t_n)$. So, $p(tz)$ is real-rooted for any $t_k \in \mathbb{R}_+$. By Hurwitz’s theorem,

$$(\partial_{z_k} p)(t_0 z) = \lim_{t_k \to \infty} t_k^{-1} p(tz).$$
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is also real-rooted. So, \( \partial_{z_k} p \) is same-phase stable.

(iii) This follows from (i), since \((z_k \partial_{z_k} p)(tz) = t_k z(t \partial_{z_k} p)(t_0 z)\) is real-rooted iff \((\partial_{z_k} p)(t_0 z)\) is.

(iv) For any \( t \in \mathbb{R}^n_+ \) with \( t_k = 0 \), we have that \( p(t_1 z, \ldots, t_{k-1} z, 0, t_{k+1} z, \ldots, t_n z) = p(t z) \) is real-rooted by definition of same-phase stability.

(v) Given \( t \in \mathbb{R}^n_+ \) with strictly positive entries, we have that \( p(t^{-1} z) \) has real roots, say at \( \gamma_1, \ldots, \gamma_m \). So, \( z^m p(t^{-1} z^{-1}) = t_1 z \ldots t_n z \cdot p((t_1 z)^{-1}, \ldots, (t_n z)^{-1}) \) has real roots at \( \gamma_1^{-1}, \ldots, \gamma_m^{-1} \). Of course, some of these inverse zeros may be missing when some \( \gamma_j = 0 \), and there may be extra zeros at \( z = 0 \). However, this will not affect the real-rootedness of the inverted polynomial. Hurwitz’s theorem then allows us to limit to all \( t \in \mathbb{R}^n_+ \).

The names given to some of the closure properties are specific to multi-affine polynomials. In particular, “variable selection” (resp. “variable deselection”) refers to the fact that the associated actions will pick out the terms of \( p \) which contain (resp. do not contain) a particular variable. Then, “selection inversion” inverts which terms contain which variables. The idea here is to give a combinatorial interpretation to these actions. For example, if the variables correspond to vertices on some graph, then variable deselection might correspond to removal of some vertex.

The next definition is inspired by \( p + z_{n+1} q \) used in Theorem 1.5.4. The proposition that follows then relates this definition to multi-affine polynomials.

**Definition 2.0.6.** Let \( p, f_0, f_1, \ldots, f_m \in \mathbb{R}[z_1, \ldots, z_n] \) be polynomials, not necessarily multi-affine, such that

\[
p = f_0 + z_{i_1} f_1 + \cdots + z_{i_m} f_m.
\]

We call such an expression a proper splitting of \( p \) (with respect to \( \{z_{i_j}\}_j \)) if none of the \( f_k \)'s depend on any of the \( z_{i_j} \)'s. We also say that \( \{z_{i_j}\}_{j=1}^m \) splits \( p \).

**Proposition 2.0.7.** Let \( p \in \mathbb{K}[z_1, \ldots, z_n] \) be a multi-affine polynomial, and suppose \( \{z_{i_j}\}_{j=1}^m \) splits \( p \). Then \( p \) has a unique proper splitting with respect to \( \{z_{i_j}\}_j \), expressed as

\[
p = p_0 + \sum_{j=1}^m z_{i_j} \partial_{z_{i_j}} p,
\]

where \( p_0 \) is the polynomial \( p \) with the variables \( \{z_{i_j}\}_j \) evaluated at 0.

Another way to think about this proposition is as follows. For a multi-affine polynomial \( p \in \mathbb{K}[z_1, \ldots, z_n] \), we have that \( \{z_{i_j}\}_{j=1}^m \) splits \( p \) iff no term of \( p \) contains more than one variable from \( \{z_{i_j}\}_{j=1}^m \). This naturally leads to the use of “variable selection” \( (z_{i_j} \partial_{z_{i_j}} p) \) and “variable deselection” \( (p_0) \) in the decomposition of \( p \) into the above sum of polynomials.

We now reach the main theorem of this section. As mentioned before, this can be seen as a loose analogue of the stability equivalence theorem (1.5.4) of Borcea and Brändén.
Theorem 2.0.8. Let $p \in \mathbb{R}_+[z_1, \ldots, z_n]$ be a multi-affine polynomial with nonnegative coefficients. The following are equivalent.

(i) The polynomial $p$ is same-phase stable.

(ii) Given any proper splitting

$$p = f_0 + \sum_{j=1}^{m} z_i f_j$$

we have that $f_0, z_i f_1, \ldots,$ and $z_m f_m$ are same-phase compatible.

(iii) There exists some proper splitting

$$p = f_0 + \sum_{j=1}^{m} z_i f_j$$

such that $f_0, z_i f_1, \ldots,$ and $z_m f_m$ are same-phase compatible.

Proof. $(i) \Rightarrow (ii)$ Let $p = f_0 + \sum_{j=1}^{m} z_i f_j$ be a proper splitting of $p$. By uniqueness of the proper splitting, $f_0$ is the polynomial $p$ with variables $\{z_i\}$ evaluated at 0, and $z_i f_j = z_i \partial_{z_i} p$. So, by closure properties, each of $f_0, z_i f_1, \ldots,$ and $z_m f_m$ is same-phase stable. Now, fix $t \in \mathbb{R}_n$ and $\lambda \in \mathbb{R}_m$, and let $\lambda t$ be defined as:

$$(\lambda t)_i := \begin{cases} \lambda_j t_i, & i = i_j \\ t_i, & i \notin \{i_j\}_{j=1}^{m} \end{cases}$$

That is, $\lambda t$ is obtained by multiplying the $i_j$’th entry of $t$ by $\lambda_j$ for all $j \in [m]$. With this, same-phase stability of $p$ implies

$$(\lambda t) \cdot p = f_0(tz) + \sum_j \lambda_j [t_i z f_j(tz)]$$

is real-rooted for every choice of $\lambda$, which means every convex combination of $f_0(tz), t_i z f_1(tz), \ldots,$ and $t_m z f_m(tz)$ is real-rooted. So, $f_0(tz), t_i z f_1(tz), \ldots,$ and $t_m z f_m(tz)$ have a common interlacing. Since $t$ was arbitrary, this implies $f_0, z_i f_1, \ldots,$ and $z_m f_m$ are same-phase compatible.

$(ii) \Rightarrow (iii)$ This is trivial, given the existence of some proper splitting. In particular, $p = p(0, z_2, \ldots, z_n) + z_1 \partial_{z_1} p$ is always a proper splitting for multi-affine $p$.

$(iii) \Rightarrow (i)$ Fix $t \in \mathbb{R}_n$. Same-phase compatibility of $f_0, z_i f_1, \ldots,$ and $z_m f_m$ implies $f_0(tz), z f_1(tz), \ldots,$ and $z f_m(tz)$ have a common interlacing. So,

$$(1 + \sum_j t_{ij})^{-1} p(tz) = \frac{f_0(tz) + \sum_j t_{ij} z f_j(tz)}{1 + \sum_j t_{ij}}$$

is real-rooted. Since $t$ was arbitrary, this implies $p$ is same-phase stable. \qed
The power of this statement comes from the fact that same-phase compatibility of any particular splitting implies same-phase compatibility of every possible splitting. We will use this to our advantage in an inductive argument to follow.

2.1 Same-phase Stability of the Multivariate Independence Polynomial

Here, we give another proof of real-rootedness for claw-free graphs by proving something stronger: namely, that the multivariate independence polynomial of a graph is same-phase stable if and only if the graph is claw-free. In their original proof, Chudnovsky and Seymour show real-rootedness using an intricate recursion based on a combinatorial structure known as a “simplicial clique”. By encoding the recursive compatibility using our notion of same-phase stability, we are able to avoid the introduction of simplicial cliques and use simpler graph structures in the recursion. Same-phase stability of the edge matching polynomial then serves as the base case.

Before giving this proof, we need to set up the relevant notation. Given any graph $G$, we define the multi-affine independence polynomial of $G$ as follows.

$$ I(G) \equiv I(G)(x) := \sum_{S \subset V, S \text{ independent}} \prod_{v \in S} x_v $$

Stability properties of the multivariate independence polynomial have been previously studied by Scott and Sokal. In [54], they observe this polynomial as a specific case of a more general statistical-mechanical partition function, and generic lower bounds on the modulus of the roots are studied. In particular, the Lovász local lemma is used to give a universal lower bound of $\frac{1}{e^{2 \Delta}}$, where $\Delta$ is the maximum degree of $G$.

As discussed in the notation above, for a given graph $G$ we denote the line graph of $G$ by $L(G)$. Since line graphs are claw-free, we have the following first step toward the desired result.

**Corollary 2.1.1.** For any graph $G$, the independence polynomial $I(L(G))$ of the line graph of $G$ is same-phase stable.

**Proof.** By considering the fact that the operator $L$ maps edges to vertices and shared vertices to edges, we actually have the following identity.

$$ \mu_E(G) = I(L(G)) $$

The previous corollary gives the desired result.

Of course, this is quite far from the claim that all claw-free graphs are same-phase stable. However, as it turns out, line graphs will serve a base case in our induction on general claw-free graphs. To illustrate this, we first give the following lemma.
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\[
\mu_E(C_6, x) = 1 + x_{ab} + x_{bc} + x_{cd} + x_{de} + x_{ef} + x_{fa} + x_{ab}x_{cd} + x_{ab}x_{de} + x_{bc}x_{ef} + x_{bc}x_{fa} + x_{cd}x_{ef} + x_{cd}x_{fa} + x_{de}x_{fa} + x_{ab}x_{cd}x_{ef} + x_{bc}x_{de}x_{fa}
\]

\[
\mu_V(C_6, x) = 1 - x_a x_b - x_b x_c - x_c x_d - x_d x_e - x_e x_f - x_f x_a + x_a x_b x_c x_d + x_a x_b x_d x_e + x_a x_b x_c x_f + x_b x_c x_d x_e + x_b x_c x_e x_f + x_b x_c x_f x_a + x_c x_d x_e x_f + x_c x_d x_f x_a + x_d x_e x_f x_a - x_a x_b x_c x_d x_e x_f - x_b x_c x_d x_e x_f x_a
\]

\[
I(C_6, x) = 1 + x_a + x_b + x_c + x_d + x_e + x_f + x_a x_c + x_a x_d + x_b x_d + x_b x_e + x_b x_f + x_c x_e + x_c x_f + x_d x_f + x_a x_c x_e + x_b x_d x_f
\]

Figure 2.1: A small graph $C_6$ with associated independence polynomial, vertex/edge matching polynomials.

**Lemma 2.1.2.** Let $G$ be a connected claw-free graph which is also triangle-free. Then, $G$ is either a path or a cycle. In particular, $G$ is a line graph.

**Proof.** Given a vertex $v \in G$, if the degree of $v$ is greater than 2 then we get either a claw with $v$ as the base or a triangle. We conclude that a graph which is connected, claw-free, and triangle-free is equivalent to being connected and triangle-free with all vertices degree 1 or 2.

With this, we now give the proof of same-phase stability for claw-free graphs, using the theory of same-phase compatibility developed above. As mentioned in the introduction, this result is a reformulation of a theorem of Engström given in [52].

**Theorem 2.1.3** (Engström). For any claw-free graph $G$, the independence polynomial $I(G)$ is same-phase stable.

**Proof.** We induct on the number of vertices. If $G$ is disconnected, then its independence polynomial is the product of the independence polynomials of its connected components. The inductive hypothesis on components of $G$ (along with the disjoint product closure property for same-phase stable polynomials) then implies the result for $G$. If $G$ is connected and contains no 3-cliques (triangles), then $G$ is a line graph by the previous lemma. The line graph corollary then implies the result for $G$. If neither of these conditions is satisfied, then $G$ is a connected graph with at least one 3-clique. Let $u, v, w$ denote the vertices of this 3-clique.
In the independence polynomial $I(G)$, let the variables $z_u, z_v, z_w$ represent the vertices $u, v, w$, respectively. Consider the following equivalent expressions of $I(G)$.

$$I(G) = I(G)_{u=v=w=0} + z_u \partial_{z_u} I(G) + z_v \partial_{z_v} I(G) + z_w \partial_{z_w} I(G)$$

$$= I(G \setminus \{u, v, w\}) + z_u I(G \setminus N[u]) + z_v I(G \setminus N[v]) + z_w I(G \setminus N[w])$$

$$= [I((G \setminus \{u\}) \setminus \{v, w\}) + z_u I((G \setminus \{u\}) \setminus N[v]) + z_w I((G \setminus \{u\}) \setminus N[w])] + z_u I(G \setminus N[u])$$

The square-bracketed sections of the last three expressions are proper splittings of $I(G \setminus \{u\})$, $I(G \setminus \{v\})$, and $I(G \setminus \{w\})$, respectively. By the inductive hypothesis and the same-phase stability theorem, these proper splittings have terms which are same-phase compatible. So, the terms of the first expression of $I(G)$ are pairwise same-phase compatible. By Corollary 2.0.4, we have that all the terms of the first expression are same-phase compatible. These terms give a proper splitting of $I(G)$, and so Theorem 2.0.8 implies $I(G)$ is same-phase stable.

An interesting feature of the above proof is the fact that the inductive step did not use the fact that $G$ is claw-free. This suggests that perhaps the theorem can be extended to certain clawed graphs. However, the following corollary shows that this is not the case.

**Corollary 2.1.4.** For any graph $G$, the independence polynomial $I(G)$ is same-phase stable if and only if $G$ is claw-free (3-star-free).

**Proof.** By the above theorem, we only need to show that the independence polynomial of a graph with a claw is not same-phase stable. To get a contradiction, let $G$ be a graph such that the vertices $u, v, w, x$ form a claw, and yet $I(G)$ is same-phase stable. Let $p(z_u, z_v, z_w, z_x)$ be the polynomial obtained by evaluating $I(G)$ at zero for all other variables (besides $z_u$, $z_v$, $z_w$, and $z_x$). By closure properties, $p$ is also same-phase stable. With this we compute $p(tz)$ for $t = (1, 1, 1, 1)$:

$$p(z, z, z, z) = 1 + 4z + 3z^2 + z^3$$

This polynomial is not real-rooted, which gives the desired contradiction.

With this equivalence in mind, one might wonder for what smaller class of graphs the independence polynomial is actually real stable. A somewhat surprising result is the following.

**Proposition 2.1.5.** For any connected graph $G$, the independence polynomial $I(G)$ is real stable if and only if $G$ is complete (2-star-free).

**Proof.** If $G$ is a complete graph, then the independence polynomial of $G$ is $1 + \sum_{v \in V} x_v$, which is real stable. On the other hand, suppose $G$ is some connected incomplete graph such that $I(G)$ is real stable. By incompleteness and connectedness, $G$ contains an induced path
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P of length at least 2. (E.g., consider the shortest path between two non-adjacent vertices.)
In fact, we can assume P is of length exactly 2 by removing all but 3 consecutive vertices.
Notice that P is now an induced 2-star. Evaluating I(G) at 0 the variables x_v for which
v ∈ P, we obtain I(P), the independence polynomial of P. Closure properties imply I(P)
is real stable.

Labeling the vertices of P as u, v, w, we then have

\[ I(P)(x) = 1 + x_u + x_v + x_w + x_u x_w, \]

which, for \(x_0 = (-1, 1, -1)\), gives

\[ \partial_{x_u} I(P)(x_0) \cdot \partial_{x_w} I(P)(x_0) = 0 < 1 = \partial_{x_u x_w} I(P)(x_0) \cdot I(P)(x_0). \]

That is, I(P) is not strongly Rayleigh. So, I(P) is not real stable, which is the desired
contradiction.

\[ \square \]

2.2 Root Bounds

In addition to proving real rootedness of the matching polynomial, Heilmann and Lieb
established bounds on the modulus of roots of the matching polynomial. Since we use the
inverted matching polynomial, this result bounds the roots, \(\lambda\), of \(\mu_V(G)\) away from zero:

\[ |\lambda| \geq \frac{1}{2\sqrt{\Delta - 1}} \]

Since \(\mu_V(G)(x) = I(L(G))(-x^2)\) this result can be stated equivalently as a bound on the
root closest to zero, \(\lambda_1\), for the independence polynomial of line graphs. To do this note that
the maximum degree, \(\Delta\), of a graph is equal to the clique size, \(\omega\), of its line graph.

\[ \lambda_1(I(L(G))) \leq \frac{1}{4(\omega - 1)} \]

Since all line graphs are claw-free graphs, we can seek out similar bounds for the inde-
pendence polynomial of claw-free graphs. In what follows, we adapt the methods of Godsil
to determine such root bounds for a certain subclass of claw-free graphs, namely those which
contain a simplicial clique. (Although we were able to avoid simplicial cliques in the proof of
real-rootedness, they turn out to be crucial to generalizing the Heilmann-Lieb root bound.)
We then discuss how the bound does not extend to all claw-free graphs.

To this end, we first discuss Godsil’s original divisibility result which was key to his proof
of the Heilmann-Lieb root bound. We do this in the multivariate world, though, so as to
provide context for the later results on the independence polynomial.
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Path Trees

A basic element of Godsil’s proof of the root bound is the notion of a path tree of a graph. We now define this notion as he did, and subsequently discuss what needs to be altered in order to apply it to the multivariate matching polynomial.

Definition 2.2.1. Given a graph $G$ and vertex $v$, we define the (labeled) path tree $T_v(G)$ of $G$ with respect to $v$ recursively as follows. If $G$ is a tree, we define $T_v(G) = G$, and we say that $v$ is the root of $T_v(G)$. We also label the vertices of $T_v(G)$ using the vertices of $G$. (In the recursive step, we will continue to label using vertices of $G$.)

For an arbitrary graph $G$, we first consider the forest which is the disjoint union of the labeled trees $T_w(G \setminus \{v\})$ for each $w \in N(v)$. We then define $T_v(G)$ by appending a vertex (the root) labeled $v$ and connecting it to the roots of each of these trees.

Remark 2.2.2. Figure 2.2 gives an example of a path tree. Note that it is defined in such a way that the paths stemming from $v$ in $G$ and from the root, $v$ in $T_v(G)$, are in order preserving bijection (where the order on paths is the subpath ordering).

In Godsil’s proof of the root bound for the matching polynomial, he shows that the univariate vertex matching polynomial of $G$ divides that of $T_v(G)$ for any $v$. In the multivariate world, this divisibility relation won’t be possible, a priori, since there are potentially far more vertices (and hence, variables) in $T_v(G)$ than in $G$. However, using the labeling of the vertices described above, we can in fact extend this divisibility result. We now formalize this notion of labeling, so as to easily generalize it to all relevant multivariate graph polynomials.

Let $G, H$ be two graphs, and let $\phi : G \to H$ be a graph homomorphism. We call this homomorphism a labeling of $G$ by $H$. For a graph $G$, we define the relative vertex matching polynomial (with respect to $\phi$) as follows.

$$\mu^\phi_v(G) \equiv \mu^\phi_v(G)(x) := \sum_{M \subset E(G) \atop M, \text{matching}} \prod_{\{u,v\} \in M} -x_{\phi(u)}x_{\phi(v)}$$

We define the relative edge matching polynomial and the relative independence polynomial (with respect to $\phi$) analogously. When unambiguous, we will remove the $\phi$ superscript from the notation. Notice that the univariate specialization of each of the normal matching and independence polynomials is the same as that of the relative matching and independence polynomials, for any $\phi$. This notion then gives us a way to compare multivariate matching and independence polynomials from different graphs without destroying any univariate information.

Now, consider the labeling of vertices described in the construction of $T_v(G)$ above. This can extended to a graph homomorphism, $\phi_v : T_v(G) \to G$ in a unique way. Specifically, the vertices of $T_v(G)$ are mapped to the vertices of $G$ via the labeling given above (e.g., the root of $T_v(G)$ maps to $v \in G$, the neighbors of the root are mapped to the neighbors of $v \in G$, etc.). An edge $\{u, w\}$ of $T_v(G)$ is then mapped to the edge $\{T_v(u), T_v(w)\}$ in $G$, which exists by the inductive construction given above.
In what follows, we will consider the graph polynomials $\mu_v^G(T_v(G))$ and $\mu_E^G(T_v(G))$. For simplicity of notation, we will from now on denote these polynomials $\mu_V(T_v(G))$ and $\mu_E(T_v(G))$, respectively. That is, reference to $\phi_v$ will be dropped.

With this, we now state the generalization of Godsil’s divisibility theorem for the vertex matching polynomial. We omit the proof, as this theorem turns out to be a corollary of a more general result related to independence polynomials.

**Theorem 2.2.3** (Godsil). Let $v$ be a vertex of the graph $G = (V, E)$, and let $T \equiv T_v(G)$ be the path tree of $G$ with respect to $v$. Further, let $\mu_V(T) \equiv \mu_v^G(T)$ denote the relative vertex matching polynomial. We then have the following.

$$\frac{\mu_V(G)}{\mu_V(G \setminus v)} = \frac{\mu_V(T)}{\mu_V(T \setminus v)}$$

Further, $\mu_V(G)$ divides $\mu_V(T)$.

By univariate specialization, this gives us the first step toward the well-known Heilmann and Lieb root bound (up to inversion of the input variable). We now attempt to generalize this divisibility to independence polynomials. First, however, we will need to develop some path tree analogues.

**Path Tree Analogues**

**Induced Path Trees**

Given a graph $G$ and a vertex $v$, the *induced path tree* $T_v^\angle(G)$ of $G$ with respect to $v$ is intuitively defined as follows: it is the path tree that is constructed when only *induced* paths are considered. That is, we use the recursive process of creating the usual path tree, only we forbid traversal of vertices which are neighbors of previously traversed vertices. So, another name that could be used for this tree is the “neighbor-avoiding” path tree.

We now give an explicit definition of the induced path tree. The crucial difference between this definition and the definition of the path tree given above is that neighbors of a vertex are excluded in the recursive step.

**Definition 2.2.4.** Given a graph $G$ and vertex $v$, we define the *induced path tree* $T_v^\angle(G)$ of $G$ with respect to $v$ recursively as follows. If $G$ is a tree, we define $T_v^\angle(G) = G$, and we say that $v$ is the root of $T_v^\angle(G)$.

For an arbitrary graph $G$, we first consider the forest which is the disjoint union of the trees $T_w^\angle(G \setminus N[v] \cup \{w\})$ for each $w \in N(v)$. We then define $T_v^\angle(G)$ by appending a vertex corresponding to $v$ (the root) and connecting it to the roots of each of these trees.

We also define a slightly different version of the induced path tree. As will be seen, this adjusted definition is more appropriate for our purposes.
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Definition 2.2.5. Given a graph $G$ and a clique $K$, the induced path tree $T^*_K(G)$ of $G$ with respect to $K$ is defined as follows. Construct a new graph $G^*$ by attaching a new vertex $*$ to $G$, with the property that $\{*, u\} \in E(G^*)$ iff $u \in K$. Then, define $T^*_K(G) := T^*_\{\} (G^*)$.

Remark 2.2.6. As with the path tree, we can label the vertices of the induced path tree in a natural way. This gives rise to graph homomorphisms $\phi_v : T^*_v(G) \to G$ and $\phi_K : T^*_K(G) \to G^*$.

Simplicial Clique Trees

We need two graph theoretic concepts before defining our final path tree analogue. Given a graph $G$, let $K \leq G$ be an induced clique. Then, $K$ is called a simplicial clique if for all $u \in K$, $N[u] \cap (G \setminus K)$ is a clique as an induced subgraph of $G$ (or equivalently, as an induced subgraph of $G \setminus K$). Intuitively, this means that neighborhoods of each $u \in K$ are two cliques joined at $u$: one is $K$ itself, and the other consists of the remaining neighbors of $u$. Simplicial cliques have been studied frequently in relation to the independence polynomial of a graph, and in particular, they were used in Chudnovsky and Seymour’s original proof of real-rootedness for claw-free graphs.

We further say that a graph $G$ is simplicial if it is claw-free and contains a simplicial clique. It may at first seem strange as to why “claw-free” is included in this definition. The main reason is the useful recursive structure that can be extracted from the following lemma.

Lemma 2.2.7 ([15]). Let $G$ be claw-free, and let $K \leq G$ be a simplicial clique in $G$. For any $u \in K$, $N[u] \cap (G \setminus K)$ is a simplicial clique in $G \setminus K$.

Remark 2.2.8. One can easily check that our definition of a simplicial graph is equivalent to having a recursive structure of simplicial cliques as indicated in the previous lemma.

A block graph (or clique tree) is a graph in which every maximal 2-connected subgraph is a clique [55]. As it turns out, block graphs are precisely the line graphs of trees. From this observation we note that there is a natural tree-like recursive structure on block graphs. Specifically, let $B$ be a block graph, and let $K$ be a clique in $B$. Then, $B \setminus K$ is a “forest of block graphs”. That is, if we refer to $K$ as the “root clique” in $B$, then each “root clique” in the forest $B \setminus K$ is connected to some vertex of $K$ in $B$.

We now define a special kind of clique tree. Notice that while the term “tree” is used, the graphs defined here are not actually trees in the usual sense.

Definition 2.2.9. Given a simplicial graph $G$ and simplicial clique $K \leq G$, we define the (simplicial) clique tree $T^K_{\langle\rangle}(G)$ of $G$ with respect to $K$ recursively as follows. If $G = K$, we define $T^K_{\langle\rangle}(G) = G$, and we say that $K$ is the “root clique” of $T^K_{\langle\rangle}(G)$.

For an arbitrary graph $G$, we first consider the “forest of simplicial clique trees” which is the disjoint union of $T^K_{\langle\rangle}(G \setminus K)$ for each $u \in K$. (Here, we define $J_u := N[u] \cap (G \setminus K)$.) Note that this is valid, since the previous lemma implies $J_u$ is a simplicial clique for all $u \in K$. We
then define $T^\otimes_K(G)$ by appending the clique $K$ (the root clique) and connecting each vertex $u \in K$ to each vertex of the root clique of $T^\otimes_{d_u}(G \setminus K)$.

**Remark 2.2.10.** We can label the vertices of the (simplicial) clique tree in the usual way, and this gives rise to a natural graph homomorphism $\phi_K : T^\otimes_K(G) \to G$.

For examples of the induced path tree and the simplicial clique tree, see Figures 2.2 and 2.3.

### Divisibility Relations

Given the above definitions, the main goal of this section is to demonstrate the following theorem. Here, for $v \in G$ we define $K_v \leq L(G)$ via $K_v := L(\{e \in E(G) : v \in e\})$. That is, $K_v$ can be thought of as “the clique in $L(G)$ associated to $N[v]$”.

**Theorem 2.2.11.** Let $L$ be the line graph operator, $T_v$ the path tree operator with respect to $v$, $T^\perp_K$ the induced path tree operator with respect to $K$, and $T^\otimes_K$ the clique tree operator with respect to $K$. Then the following diagram commutes up to isomorphism.

\[
\begin{array}{ccc}
\{\text{graphs}\} & \xrightarrow{T_v} & \{\text{trees}\} \\
\downarrow L & & \downarrow L \\
\{\text{simplicial graphs}\} & \xrightarrow{T^\otimes_K} & \{\text{simpl. block graphs}\}
\end{array}
\]

In the upper left triangle, commutativity is achieved for $K = K_v$.

This can be broken down into a few results, which we give now.

**Lemma 2.2.12.** For any graph $G$ and any $v \in G$, $K_v$ is a simplicial clique of $L(G)$. In particular, $L(G)$ is simplicial.

**Proof.** It is easy to see that $K_v$ is a clique. If we consider $w \in K_v$, this corresponds to an edge $e_w \in E(G)$ that has $v$ as an endpoint. Then given any two neighbors of $w$ that are not in $K_v$, we know they correspond to two edges which share an endpoint with $e_w$ but do not have $v$ as an endpoint. Hence they both share the other endpoint of $e_w$ and are therefore connected in the line graph. This shows that $N[w] \setminus K_v$ is a clique, so $K_v$ is a simplicial clique.

It is well known that line graphs are claw-free, so all line graphs are simplicial.

**Proposition 2.2.13.** For any (nonempty) graph $G$ and any $v \in V$, the induced path tree of $L(G)$ with respect to $K_v$ is isomorphic to the path tree of $G$ with respect to $v$. That is, $T^\perp_{K_v}(L(G)) \cong T_v(G)$. 


Proof. First, let $G$ be the graph with one vertex, $v$. Then, $L(G)$ is the empty graph and $T_{K_v}^\circ L(G)$ is also the graph with one vertex (recall that the operator $T_{K_v}^\circ$ adds an extra vertex to the input graph). On the other hand, $T_v(G)$ is the graph with one vertex, and the result holds in this case.

Now, let $G$ be a connected graph consisting of two or more vertices, and let $v$ be some vertex of $G$. (We can assume WLOG that $G$ is connected, since $T_v$ and $T_{K_v}^\circ$ only deal with connected components of $v$ and $K_v$, respectively.) We proceed inductively, adopting the convention that $K_v\leq L(G)$ and $K_v^\prime\leq L(G\setminus\{v\})$ are the cliques associated to $N[u]$ in the respective line graphs.

We first consider $T_v(G)$. For each $u\in N(v)$, we have that $T_u(G\setminus\{v\})$ is naturally a subtree of $T_v(G)$. In fact, $T_v(G)$ can be viewed as the disjoint union of $T_u(G\setminus\{v\})$ for all $u\in N(v)$, connected to a single vertex corresponding to $v$.

We next consider $T_{K_v}^\circ L(G)$. Notice that $L(G\setminus\{v\})\cong L(G)\setminus K_v$. For any $u\in N(v)$, this implies $T_{K_v}^\circ L(G\setminus\{v\})\cong T_{K_u}^\circ L(\overline{G}\setminus K_v)$, where $J_u:=K_u\cap (L(G\setminus K_v))$. Recall that the $T_{K_v}^\circ$ operator adds an extra vertex attached to each vertex of $K$. So, we can view $T_{K_v}^\circ L(G)$ as the disjoint union of $T_{K_u}^\circ (L(G)\setminus K_v)$ for all $u\in N(v)$, along with an extra vertex connected to each of the added extra vertices in the disjoint union.

By the induction hypothesis, we have $T_u(G\setminus\{v\})\cong T_{K_u}^\circ L(G\setminus\{v\})$ for all $u\in N(v)$. This implies that the two descriptions given above of $T_v(G)$ and $T_{K_v}^\circ L(G)$, respectively, are equivalent. Therefore, $T_v(G)\cong T_{K_v}^\circ L(G)$.

Proposition 2.2.14. For any simplicial graph $G$ and any simplicial clique $K\leq G$, the line graph of the induced path tree of $G$ with respect to $K$ is isomorphic to the clique tree of $G$ with respect to $K$. That is, $L(T_K^\circ(G))\cong T_K^\circ(G)$.

Proof. There is a natural grading on the edges of $T_K^\circ(G)$, where the edges from $*$ to vertices in $K$ have grading 1, and edges from vertices $v\in K$ to vertices in $N[v]\setminus K$ have grading 2, and so forth. Then under the line graph operation we get a grading on the vertices of $L\circ T_K^\circ(G)$.

Similarly $T_K^\circ(G)$ has a natural grading on the vertices by grading $K$ as grade 1, and for every vertex $v\in K$, grading the clique $N[v]\setminus K$ as grade 2, and so forth.

Now we can induct on the number of vertices in $G$. The result is obviously true for the graph with one vertex. It is then clear that the first grades of $L\circ T_K^\circ(G)$ and $T_K^\circ(G)$ are isomorphic: they are both cliques of size $K$. We then label the vertices of the first grade in $L\circ T_K^\circ(G)$ by vertices in $K$ as follows. Each vertex of the first grade comes from an edge in $T_K^\circ(G)$ of the form $\{*,v\}$, for some $v\in K$. So, we label this first-grade vertex in $L\circ T_K^\circ(G)$ by "$v$".

In $L\circ T_K^\circ(G)$, this vertex labeled "$v$" connects to edges in $G$ from $v$ to vertices in $N[v]\setminus K$ in $T_K^\circ(G)$. In this way we see viewing the sub-clique tree (obtained by looking at $v$ and all of the grades below it) rooted at the vertex labeled $v$ in $L\circ T_K^\circ(G)$ is $L\circ T_N^\circ(v\setminus K)(G)$. Likewise by looking at the vertex labeled $v$ in $T_K^\circ(G)$ we see the sub-clique tree obtained by looking
at \( v \) and all grades below it is exactly \( T_N^{\otimes}(G) \), by definition of the simplicial clique tree. By induction our claim is proved.

There are two comments to be made about this diagram. First, we can consider the induced path tree operator as some sort of “inverse” or “adjoint” to the line graph operator. In fact, for \( G \in \{\text{trees}\} \) (resp. \( G \in \{\text{simpl. block graphs}\} \)) we have that \( T_K^{\angle} \) is the left (resp. right) inverse of \( L \).

Second, consider the outer rectangle of the diagram. We see that the line graph operator “passes” the path tree operator to the clique tree operator. So, if Godsil’s divisibility relation can be shown to hold between a simplicial graph and its clique tree, we will be able to derive the same relation between a graph and its path tree as a corollary. (The corollary will actually be for the edge matching polynomial. A simple argument then gives the result for the vertex matching polynomial, as we will see below.)

We now generalize Godsil’s theorem.

**Theorem 2.2.15.** Let \( K \) be a simplicial clique of the simplicial graph \( G = (V, E) \), and let \( T \equiv T^K_{\eta}(G) \) be the clique tree of \( G \) with respect to \( K \). Further, let \( I(T) \equiv I^{\phi_K}(T) \) denote the relative independence polynomial. We then have the following.

\[
\frac{I(G)}{I(G \setminus K)} = \frac{I(T)}{I(T \setminus K)}
\]
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Figure 2.3: An example of a graph, its induced path tree and simplicial clique tree. $W_6$ is not a line graph.

Proof. We induct on $|V(G)|$. Note that if $G$ is a simplicial block graph, then $T = G$, and so the result is true.

For the general case we get:

$$\frac{I(G)}{I(G \setminus K)} = \frac{I(G \setminus K) + \sum_{v \in K} x_v I(G \setminus N[v])}{I(G \setminus K)}$$

$$= 1 + \sum_{v \in K} \frac{x_v I(T_{\infty}^{\geq}(G \setminus K) \setminus N[v])}{I(T_{\infty}^{\geq}(G \setminus K))}$$

$$= 1 + \sum_{v \in K} \frac{x_v I(T_{\infty}^{\geq}(G \setminus K) \setminus N[v]) \prod_{w \in K, w \neq v} I(T_{w}^{\geq}(G \setminus K))}{I(T_{\infty}^{\geq}(G) \setminus K)}$$

$$= 1 + \sum_{v \in K} \frac{x_v I(T_{\infty}^{\geq}(G) \setminus N[v])}{I(T_{\infty}^{\geq}(G) \setminus K)}$$

$$= \frac{I(T_{\infty}^{\geq}(G) \setminus K) + \sum_{v \in K} x_v I(T_{\infty}^{\geq}(G) \setminus N[v])}{I(T_{\infty}^{\geq}(G) \setminus K)}$$

$$= \frac{I(T_{\infty}^{\geq}(G))}{I(T_{\infty}^{\geq}(G) \setminus K)}$$
In the above we use the recursion formula for the independence polynomial expanding at a clique and the fact that $N[v]$ is a simplicial clique in $G \setminus K$ when $K$ is a simplicial clique. Notice also that the relative independence polynomial $I \equiv I^K$ is needed in order for the last equality to hold.

**Remark 2.2.16.** We compute the independence polynomials of the appropriate graphs from Figure 2.2 to illustrate the divisibility relations proved in the preceding theorem:

$I(L(P), x) = 1 + x_s + x_y + x_z + x_w = 1 + x_s + x_y + x_z + x_w + x_s x_w, \quad I(T^\otimes_1(L(P))) = (1 + x_s + x_y + x_z + x_w + x_s x_w) \cdot (1 + x_w) = I(L(P), x) \cdot (1 + x_w)$

The proof we gave for the previous theorem is essentially the one Godsil gives for his original theorem, except that we deal with simplicial cliques rather than vertices. The previous theorem now yields the following corollaries.

**Corollary 2.2.17.** $I(G)$ divides $I(T^\otimes_K(G))$ for any simplicial graph $G$ with simplicial clique $K$.

**Proof.** We have seen that $G \setminus K$ is a simplicial graph. The previous theorem can be written as:

$$
\frac{I(T^\otimes_K(G))}{I(G)} = \frac{I(T^\otimes_K(G) \setminus K)}{I(G \setminus K)} = \prod_{v \in K} \frac{I(T^\otimes_{N[v]\setminus K}(G \setminus K))}{I(G \setminus K)}
$$

Then since $N[v] \setminus K$ is a simplicial clique in $G \setminus K$, by induction we have the denominator divides any term in the numerator, so the right hand side is a polynomial, as desired.

**Corollary 2.2.18.** Given a simplicial graph $G$, we have that $\lambda_1(G) \leq \frac{-1}{4(\omega - 1)}$.

**Proof.** By the previous corollary we have $\lambda_1(G) \leq \lambda_1(T^\otimes_K(G))$. Then by the commutativity of the diagram, we have seen $T^\otimes_K(G) = L(T^\otimes_K(G))$. Hence we have $\lambda_1(T^\otimes_K(G)) \leq \frac{-1}{4(\omega - 1)}$ is equivalent to the identical root bound on $\mu_E(T^\otimes_K(G))$. Godsil provides bounds on this root by relating the matching polynomial of a tree to its characteristic polynomial, and then bounding the roots of the characteristic polynomial by its maximal degree $\Delta$. Since the maximum degree of the vertices in $T^\otimes_K(G)$ is $\omega$, we get our desired bound.

**Remark 2.2.19.** In their original paper, Heilmann and Lieb prove a root bound for weighted matching polynomials, where one puts weights on the vertices. Since the previous corollary works in the multivariate case, one could use this framework to derive similar results for weighted independence polynomials.

**Other Bound on $\lambda_1$**

Briefly we mention some easy lower bounds on $\lambda_1(G)$. In what follows we let $G$ be any graph. First we note how modifying our graph by removing edges or removing vertices affects $\lambda_1(G)$.
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Proposition 2.2.20. Let $G$ be any graph, $v$ a vertex in that graph, and $e = \{u, w\}$ an edge in the graph.

1. $\lambda_1(G \setminus v) \leq \lambda_1(G)$
2. $\lambda_1(G \setminus e) \leq \lambda_1(G)$

Proof. To prove these we need the following recurrences:

$$I(G) = I(G \setminus v) + xI(G \setminus N[v])$$

$$I(G) = I(G \setminus e) - x^2I(G \setminus (N[u] \cup N[w]))$$

To prove the first statement we prove the following statement by induction: Given any $H \subset V(G)$, we have $I(G \setminus H)$ is nonnegative on the interval $[\lambda_1(G), \infty)$. If $G \setminus H$ is not the empty graph, then $I(G \setminus H)$ is not the zero polynomial so this implies that $\lambda_1(G \setminus H) \leq \lambda_1(G)$. If $G \setminus H$ is the empty graph it is trivially true. For $|V(G)| = 1$, it is easily checked to be true. Assuming this to be true for $|V(G)| \leq n-1$, let $G$ be a graph with $|V(G)| = n$. Then if $H = G$, we noted this is trivially true. Then it suffices to show that $\lambda_1(G \setminus v) \leq \lambda_1(G)$. By induction we know $I(G \setminus N[v])$ is nonnegative on $[\lambda_1(G \setminus v), \infty)$. Then we know $xI(G \setminus N[v])$ is nonpositive on $[\lambda_1(G \setminus v), 0)$ (all the roots of independence polynomials are negative). By the recurrence relation, $I(G)$ at $\lambda_1(G \setminus v)$ is nonpositive, so by the intermediate value theorem $I(G)$ has a root in $[\lambda_1(G \setminus v), 0)$, as desired.

To prove the second claim, since $G \setminus (N[u] \cup N[w])$ is a induced subgraph of $G \setminus e$, we have $I(G \setminus (N[u] \cup N[w]))$ is nonnegative on $[\lambda_1(G \setminus e), \infty)$. By the recurrence, we have that $I(G)$ evaluated at $\lambda_1(G \setminus e)$ is nonpositive, and so by the intermediate value theorem we see $\lambda_1(G \setminus e) \leq \lambda_1(G)$. $\Box$

Using this we can get the following simple lower bound on $\lambda_1$:

Proposition 2.2.21. $\frac{-1}{\omega} \leq \lambda_1(G)$

Proof. Let $K_\omega \leq G$ be the largest clique in $G$. Then by our previous proposition we have $\lambda_1(K_\omega) \leq \lambda_1(G)$. We have $I(K_\omega) = 1 + \omega x$, so $\lambda_1(K_\omega) = \frac{-1}{\omega}$. $\Box$

These results hold for all graphs, but combining these with our previous results for simplicial graphs $G$, we see:

$$\frac{-1}{\omega} \leq \lambda_1(G) \leq \frac{-1}{4(\omega - 1)}$$
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2.3 Failure of the Root Bounds

Recall we have the following inclusions of types of graphs:

{Line Graphs} ⊂ {Simplicial Graphs} ⊂ {Claw-Free Graphs}

The root bounds for the matching polynomial carry over to the independence polynomial for line graphs. And by extending the proof method of Godsil, we demonstrated the equivalent root bounds for simplicial graphs. The natural next question is: how general can the graphs get before the root bound fails?

In what follows we provide a claw-free graph (which is not simplicial) for which the root bound fails. We then provide a much weaker root bound for claw-free graphs. It is unknown whether this weaker root bound is tight due to our lack of examples of claw-free graphs which are not simplicial.

Schläfli Graph

The Schläfli graph is the unique strongly regular graph with parameters 27, 16, 10, 8. It is the complement of the Clebsch graph, the intersection graph of the 27 lines on a cubic surface. The Clebsch graph is triangle free, and hence the Schläfli graph is claw-free. We refer the reader to [56] for a comprehensive reference on the Schläfli graph and related graphs.

Keeping in mind that our root bound is equivalent to the statement

$$\lambda_1(G) \cdot 4 \cdot (\omega - 1) \leq -1,$$

we calculate the following.

Lemma 2.3.1. We have the following:

(i) The independence polynomial of the Schläfli graph is

$$45t^3 + 135t^2 + 27t + 1.$$

(ii) The clique size of the Schläfli graph is 6.

(iii) $$\lambda_1(\text{Schläfli graph}) \cdot 4 \cdot (\omega - 1) > -1$$

Proof. One can calculate the independence polynomial and clique size using any computer algebra system; we used Sage.

To show our graph breaks the root bound it suffices to show that $$I(G)(t/20)$$ has a root in $$(-1, 0)$$. In fact we can easily calculate that $$I(G)(-1/20) = -29/1600$$ while $$I(G)(0) = 1$$, so there is a root in $$(-1, 0)$$.

Weaker Root Bounds for Claw-free Graphs

Given any claw-free graph $$G$$, we can introduce a simplicial clique by modifying the graph as follows:

Lemma 2.3.2. Let $$G$$ be a claw-free graph. Given any vertex $$v \in G$$, we can form a new graph $$S_v(G)$$ by connecting all of $$N[v]$$ together to form a clique. Then, $$S_v(G)$$ is claw-free and $$\{v\}$$ is a simplicial clique in $$S_v(G)$$. 

CHAPTER 2. GENERALIZATIONS OF THE MATCHING POLYNOMIAL TO THE MULTIVARIATE INDEPENDENCE POLYNOMIALS

Proof. It is clear that \( \{v\} \) will be a simplicial clique in \( S_v(G) \). To see that \( S_v(G) \) is claw-free, suppose one of the added edges creates a claw. Then we have \( u, w \in N(v) \) and a claw with some \( u \) as the internal node and \( w \) as a leaf. Since we have connected all of the neighbors of \( v \) together, we must have the other two leaves of the claw outside of \( N[v] \). However these two vertices therefore are not connected to \( v \) or each other, and hence form a claw with \( u \) as the internal node and \( v \) as the other leaf. This provides a contradiction since \( G \) is claw-free.

When analyzing the clique tree of \( S_v(G) \) starting at the newly formed simplicial clique \( \{v\} \), we notice that the first rung of the clique tree is \( \{v\} \), the second rung is \( N(v) \), and beyond that are clique trees that live in \( G \setminus N[v] \). This observation immediately yields the following:

**Proposition 2.3.3.** Given any claw-free graph \( G \) and a vertex \( v \in G \), we have:

\[
\lambda_1(S_v(G)) \leq -\frac{1}{4 \cdot \max\{\omega - 1, \deg(v)\}}
\]

This yields the following root bound for \( G \):

\[
\lambda_1(G) \leq -\frac{1}{4 \cdot \max\{\omega - 1, \delta\}}
\]

Proof. By Proposition 2.2.20, we have \( \lambda_1(G) \leq \lambda_1(S_v(G)) \). To optimize the bound we pick the vertex \( v \) which has minimal degree in the graph, \( \delta \).

In the Schlafli graph we have a large gap between the clique size of 6 and minimal degree of 16. We think that other non-simplicial claw-free graphs with a large gap between clique size and minimal degree may provide good candidates for studying this root bound. Further, finding a family of graphs which require this looser bound could assist in showing how optimal this bound is for non-simplicial claw-free graphs.

### 2.4 Other Remarks

Above, we presented independence polynomials analogues to the real-rootedness (subsequently real stability) and the root bounds of the matching polynomial. We expect other results about the matching polynomial to be generalizable to the independence polynomial. In what follows we list a few examples and comment on these.

In [51], Fisher and Solow remark that \( I(G)^{-1} \) can be viewed as a generating function which enumerates the number of \( n \) letter words, where the letters are the vertices of the graph and two letters commute iff they have an edge between them on the graph. Similarly in [53], Godsil shows that \( \frac{ x^{(2n+\mu V(G,x^{-1})^\mu)} }{ x^{2n+\mu V(G,x^{-1})^\mu} } \) is a generating function in \( x^{-1} \) for closed tree-like walks in \( G \). We believe that there is a multivariate generalization of Fisher and Solow’s remark by working in the ring \( \mathbb{Z}[x_1, \ldots, x_n] \) where variables commute if and only if they
correspond to vertices in the graph $G$ which share an edge. Godsil’s tree-like result should be a combinatorial consequence of the more general Fisher and Solow result.

In a previous paper of Bencs, Christoffel-Darboux like identities are established for the independence polynomial [35]. One can similarly establish multivariate generalizations of these identities. By generalizing in this way, one can give a single identity that implies all the others through simple multivariate operations.

Another area of interest is studying independent sets in hypergraphs. One can naturally define the multivariate independence polynomial of a hypergraph. Namely given a hypergraph $G = (V, E)$ a set $S \subset V$ is independent if $e \not\subset S$ for all edges $e \in E$. If two edges are comparable in $G (e \subset f)$, then we note that by removing $f$ from the edge set we do not change the independent sets of $G$. If $G$ contains any edges of size one, then that vertex never shows up in the independence polynomial so we can further reduce $G$ by removing that vertex. Thus we can do this to obtain the reduction, $\tilde{G}$, of $G$ which has the same multivariate independence polynomial and has no comparable edges and no edges of size 1.

**Proposition 2.4.1.** Given a hypergraph $G$, $I(G, x)$ is same-phase stable if and only if $\tilde{G}$ is a 2-uniform claw-free graph.

**Proof.** As noted, $I(G, x) = I(\tilde{G}, x)$, so if $\tilde{G}$ is 2-uniform and claw-free we see $I(G, x)$ is same-phase stable by previous results. If $\tilde{G}$ is not 2-uniform, then we have some edge $e$ with $|e| > 2$. If $I(\tilde{G}, x)$ were same-phase stable then we could restrict to the subgraph of vertices in $e$ and obtain a same-phase stable independence polynomial. Since no other edges are comparable to $e$ by construction of $\tilde{G}$, we have this subgraph only contains the edge $e$. Then we can diagonalize to get the independence polynomial $(1 + x)^n - x^n$. If $I(\tilde{G}, x)$ were same-phase stable, this polynomial would be real rooted. However this would imply that its derivatives were real rooted, namely $(1 + x)^3 - x^3 = 1 + 3x + 3x^2$ would be real rooted, a contradiction. \qed
Chapter 3

Further Structure of Finite Free Convolutions

We expand upon the previous results of Marcus-Spielman-Srivastava and provide more general results about how all roots are of a given polynomial are affected by finite free convolutions. To do this, we first expand their bound on the movement of the largest root to all differential operators preserving real-rootedness. Further, we utilize the theory of hyperbolic polynomials to give more interesting root bounds on interior roots (other roots besides the largest).

With these result in hand, we state a number of conjectures (and some counterexamples) in the direction of stronger univariate results on interior roots and of analogous multivariate results. Proving similar multivariate results seems to be a hard problem in general. But it is the authors’ hope that by better fleshing out the details of the additive convolution in the univariate case, one can better abstract to the multivariate case to handle problems such as Kadison-Singer, the Paving conjecture, and Heilman-Lieb root bounds.

The main theorem of this Chapter is the following:

**Theorem 3.0.1.** Let \( p, q, r \in \mathbb{R}^n[t] \) be real-rooted polynomials of degree \( n \). We have:

\[
\lambda_1(p \boxplus_n q \boxplus_n r) + \lambda_1(r) \leq \lambda_1(p \boxplus_n r) + \lambda_1(q \boxplus_n r)
\]

To prove this, we adapt and simplify the proof of the original MSS result above. We leave this proof to §3.4, where we actually prove slightly more general results.

### 3.1 Strengthening MSS and Associated Conjectures

The triangle inequality above gives the most basic bound on the largest root of the convolution to two polynomials. The first main collection of interior root bounds can be stated in terms of *majorization*. The majorization order is a partial order on vectors in \( \mathbb{R}^n \) which can be thought of morally as saying that the coordinates of one vector are more spread out
than the coordinates of the other. Formally, majorization is defined as follows. We refer the reader to [57] for more discussion on the following equivalent definitions.

**Definition 3.1.1.** Given $x, y \in \mathbb{R}^n$, we say that $x$ majorizes $y$ and write $y \prec x$ if one of the following equivalent conditions holds. We let $x^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow)$ denote the ordering of the entries $x$ in non-increasing order.

1. $\sum_{i=1}^k y_i^\downarrow \leq \sum_{i=1}^k x_i^\downarrow$ for all $k$, with equality for $k = n$.
2. $y$ is contained in the convex hull of $\{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \mid \sigma \in S_n\} \subset \mathbb{R}^n$.
3. There exists a doubly stochastic matrix $D$ (each row and column sum is 1) such that $Dx = y$.
4. There is a sequence of pinches, of the form $x \mapsto (x_1, \ldots, x_j + \alpha, \ldots, x_k - \alpha, \ldots, x_n)$ such that the $j^{th}$ and $k^{th}$ coordinates are getting closer together (without crossing), which takes $x$ to $y$.

This makes $\prec$ a partial order on $\mathbb{R}^n$ for all $n$.

Note that condition (1) applied to the vectors of roots of two polynomials can be interpreted as a root bound involving interior roots. What we need then is some way to prove majorization-type results about the additive convolution. One way to do this is via hyperbolic polynomials, which enables us to convert inequalities regarding matrix eigenvalues into inequalities regarding roots of polynomials.

**Theorem 3.1.2 ([58], [59]).** Fix a hyperbolic polynomial $p$ with respect to $e$. For $v, w \in \mathbb{R}^n$, Horn’s inequalities hold for $\lambda(v + w)$, $\lambda(v)$, and $\lambda(w)$. In particular, the following majorization relation holds:

$$\lambda(v + w) \prec \lambda(v) + \lambda(w)$$

To apply this result, we need to view the additive convolution as a hyperbolic polynomial. We do this in the following.

**Proposition 3.1.3.** Consider the following, where $\boxplus$ only acts on the $x$ variables.

$$p(x, a_1, \ldots, a_n, b_1, \ldots, b_n) := \left( \prod_{k=1}^n (x - a_k) \right) \boxplus^n \left( \prod_{k=1}^n (x - b_k) \right)$$

Then $p$ is hyperbolic with respect to $e = (1, 0, \ldots, 0)$.

**Proof.** For $q(t) := \prod_k (t - a_k)$, $r(t) = \prod_k (t - b_k)$, and $y = (c, a_1, \ldots, a_n, b_1, \ldots, b_n)$ we compute:

$$p(et + y) = p(t + c, a_1, \ldots, a_n, b_1, \ldots, b_n) = (q \boxplus^n r)(t + 2c)$$

So $p(et + y)$ is real-rooted since $q$ and $r$ are and $\boxplus$ preserves real-rootedness. Also, $p(e) = 1 > 0$. 

$\square$
This fact allows us to immediately apply the previous theorem to the additive convolution. In what follows, we let $\lambda(p)$ denote the vector of roots of $p$ in nonincreasing order.

**Corollary 3.1.4.** Let $p, q \in \mathbb{R}^n[x]$ be real-rooted and of degree exactly $n$. Then:

$$\lambda(p \boxplus^n q) \prec \lambda(p) + \lambda(q)$$

**Proof.** Let $v = (0, a_1, ..., a_n, 0, ..., 0)$ and $w = (0, 0, ..., 0, b_1, ..., b_n)$, where the $a_k$ and $b_k$ are the roots of $p$ and $q$, respectively. Then $\lambda(v) = \lambda(p)$ and $\lambda(w) = \lambda(q)$. Further, $\lambda(v + w) = \lambda(p \boxplus^n q)$. The result then follows from the previous theorem. \qed

Note that this immediately gives us interior root inequalities of the following form:

$$\sum_{i=1}^{k} \lambda_i(p \boxplus^n q) \leq \sum_{i=1}^{k} \lambda_i(p) + \sum_{i=1}^{k} \lambda_i(q)$$

Using the same proof as in the corollary, we can similarly obtain all of Horn’s inequalities for the roots of $p$, $q$, and $p \boxplus^n q$. For instance, we obtain the Weyl inequalities (for all $i, j$) which more directly bound the interior roots:

$$\lambda_{i+j-1}(p \boxplus^n q) \leq \lambda_i(p) + \lambda_j(q)$$

Whenever $i = j = 1$, this boils down to the triangle inequality:

$$\lambda_1(p \boxplus^n q) \leq \lambda_1(p) + \lambda_1(q)$$

As it turns out, Theorem 3.1.2 above also yields an important majorization preservation result regarding the additive convolution. In [60], Borcea and Brändén give a complete characterization of linear operators which preserve majorization of roots. Roughly speaking, the result says that a linear operator $T$ (with certain degree restrictions) which preserves real-rootedness has the following property:

$$\lambda(p) \prec \lambda(q) \implies \lambda(T(p)) \prec \lambda(T(q))$$

Their result then immediately applies to the operator $T_q(p) := p \boxplus^n q$ for any fixed real-rooted $q$. This result also has a nice proof via hyperbolicity, and we demonstrate this now. As a note, the following proof immediately generalizes to any degree-preserving linear operator preserving real-rootedness. It is likely that one could generalize it further to the full Borcea-Brändén result, using some of the results regarding polynomial degree from [60].

**Corollary 3.1.5.** Let $p, q, r \in \mathbb{R}^n[x]$ be real-rooted polynomials of degree exactly $n$ such that $\lambda(p) \prec \lambda(q)$. Then:

$$\lambda(p \boxplus^n r) \prec \lambda(q \boxplus^n r)$$
Proof. Let $a_k, b_k, c_k$ be the roots of $p, q,$ and $r,$ respectively. By Definition 3.1.1 and the fact that $\lambda(p) \prec \lambda(q),$ we have that $(a_k)$ is in the convex hull of the permutations of $(b_k).$ That is,

$$(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} \beta_\sigma \cdot (b_{\sigma(1)}, \ldots, b_{\sigma(n)})$$

where $\beta_\sigma \geq 0$ and $\sum \beta_\sigma = 1.$ With this, we use the following notation:

$$v := (0, a_1, \ldots, a_n, c_1, \ldots, c_n) \quad w_\sigma := (0, b_{\sigma(1)}, \ldots, b_{\sigma(n)}, c_1, \ldots, c_n)$$

And so we also have that $v = \sum_{\sigma \in S_n} \beta_\sigma w_\sigma.$

Since $\prec$ is a partial order, we can induct on the majorization relation of Theorem 3.1.2, using the hyperbolic polynomial from Proposition 3.1.3:

$$\lambda(v) = \lambda \left( \sum_{\sigma \in S_n} \beta_\sigma w_\sigma \right) \prec \sum_{\sigma \in S_n} \lambda(\beta_\sigma w_\sigma)$$

By the scale-invariance property of $\boxplus$ (see Proposition 1.1.3), we have that $\lambda(\beta_\sigma w_\sigma) = \beta_\sigma \cdot \lambda(w_\sigma).$ This implies:

$$\lambda(p \boxplus^n r) = \lambda(v) \prec \sum_{\sigma \in S_n} \beta_\sigma \cdot \lambda(w_\sigma) = \sum_{\sigma \in S_n} \beta_\sigma \cdot \lambda(q \boxplus^n r) = \lambda(q \boxplus^n r)$$

As we saw in §1.2, their result can be extended to 3 polynomials in the form of our main result (Theorem 1.2.1):

$$\lambda_1(p \boxplus^n q \boxplus^n r) + \lambda_1(r) \leq \lambda_1(p \boxplus^n r) + \lambda_1(q \boxplus^n r)$$

A natural next question becomes: what other inequalities on roots can we achieve in the 3 polynomial case. With this, we arrive at our main collection of conjectures. To simplify the notation, we first make the following definitions.

**Definition 3.1.6.** Fix $n \in \mathbb{N}$ and let $I, J, K \subset [n].$ We call $(I, J, K)$ a Horn’s triple if for all Hermitian $n \times n$ matrices $A, B$ we have:

$$\sum_{i \in I} \lambda_i(A + B) \leq \sum_{j \in J} \lambda_j(A) + \sum_{k \in K} \lambda_k(B)$$

That is, if $(I, J, K)$ give rise to one of Horn’s inequalities.

**Definition 3.1.7.** Fix $n \in \mathbb{N}$ and let $I, L, J, K \subset [n].$ We call $(I, L, J, K)$ a valid 4-tuple if for all real-rooted $p, q, r$ of degree $n$ we have:

$$\sum_{i \in I} \lambda_i(p \boxplus^n q \boxplus^n r) + \sum_{i \in L} \lambda_i(r) \leq \sum_{j \in J} \lambda_j(p \boxplus^n r) + \sum_{k \in K} \lambda_k(q \boxplus^n r)$$
We want to determine all of the valid 4-tuples. It is worth noting that the method of hyperbolic polynomials (which worked for inequalities relating the roots of 2 polynomials) does not work for determining valid 4-tuples. In fact we have the following, even for diagonal matrices:

$$\lambda_1(A + B + C) + \lambda_1(C) \not\leq \lambda_1(A + C) + \lambda_1(B + C)$$

For example, let $A = B = \text{diag}(2, 0)$ and $C = \text{diag}(0, 2)$.

With these notions in hand, we can now succinctly state our main conjectures. The first is a natural generalization of Horn’s inequalities for two polynomials.

**Conjecture 3.1.8.** Let $p, q, r \in \mathbb{R}[x]$ be real-rooted and of degree exactly $n$, and let $(I, J, K)$ be a Horn’s triple. Then $(I, I, J, K)$ is a valid 4-tuple.

Note that the indices of the left-hand side of the inequality are the same for both polynomials. But perhaps this does not have to be the case here? That is, can we pick $L \neq I$ such that the inequality for $(I, L, J, K)$ is stronger than the inequality for $(I, I, J, K)$, and yet it is still a valid 4-tuple?

This turns out to be a difficult question in general. However, we state a few conjectures in this direction. The first would be a negative result if true.

**Conjecture 3.1.9.** Let $p, q, r \in \mathbb{R}[x]$ be real-rooted and of degree exactly $n$, and let $(I, L, J, K)$ be a valid 4-tuple. Then $(I, J, K)$ is a Horn’s triple.

Of course you can make the set $L$ a “weaker” set of indices than $I$ (meaning that the inequality for $(I, L, J, K)$ is logically weaker than the inequality for $(I, I, J, K)$) to get a new valid 4-tuple. Since such inequalities follow from the conjecture given above, we will ignore these 4-tuples. That said, the only question left is just how much “stronger” the set $L$ can be. We give yet another conjecture regarding this question, albeit only in the case where $|I| = |L| = |J| = |K| = 1$. To ease notation, we say that $(i, j, k)$ and $(i, l, j, k)$ are a Horn’s triple and a valid 4-tuple, respectively (replace singleton sets with the single index).

**Conjecture 3.1.10.** Let $p, q, r \in \mathbb{R}[x]$ be real-rooted and of degree exactly $n$, and let $(i, j, k)$ be a Horn’s triple. Note that this is equivalent to $i \geq j + k - 1$ (see the Weyl inequalities above, which are strongest Horn’s triples of this form). Then $(i, \max(j, k), j, k)$ and $(i, n + 1 - \max(j, k), j, k)$ are valid 4-tuples.

Notice that for small $j, k$ the first 4-tuple given in the above conjecture is stronger, and for large $j, k$ the second 4-tuple given in above conjecture is stronger.

### 3.2 The Multivariate Case

All of the root bounds and conjectures discussed in this chapter thus far have been for univariate polynomials. However, in their resolution of Kadison-Singer, Marcus-Spielman-Srivastava give bounds on how the points above the roots of a given multivariate polynomial
change under the action of differential operators. This prompts an obvious question: are there multivariate generalizations of the root bounds discussed in this chapter?

To attempt to answer this, we will give the natural multivariate generalization of the additive convolution, along with some basic analogous results. But first we define the points above the roots of a polynomial, and state a few of its properties. This notion should be interpreted as a multivariate generalization of the largest root of a polynomial.

Note that in the univariate case, \( \text{Ab}(p) \) is the interval \([\max\text{root}(p), \infty)\).

**Proposition 3.2.1.** Let \( p \in \mathbb{R}[x_1, \ldots, x_n] \) be real stable. Then \( \text{Ab}(p) \) is convex and is the closure of a connected component of the non-vanishing set of \( p \).

**Proof.** Follows from the theory of hyperbolic polynomials. E.g., see [38].

With the notion of \( \text{Ab}(p) \) in hand, we now define and discuss the multivariate version of the additive convolution including a multivariate generalization of the triangle inequality.

**Definition 3.2.2.** For \( p, q \in \mathbb{R}^\gamma[x_1, \ldots, x_n] \) we define the bilinear function:

\[
(p \boxplus q)(x) := \sum_{0 \leq \mu \leq \gamma} \partial_x^\mu p(x) \cdot \partial_x^{-\mu} q(0)
\]

**Proposition 3.2.3.** Let \( p, q \in \mathbb{R}^\gamma[x_1, \ldots, x_n] \) be real stable polynomials. We have the following:

1. (Symmetry) \( p \boxplus q = q \boxplus p \)
2. (Shift-invariance) \( (p(x + a) \boxplus q)(x) = (p \boxplus q)(x + a) = (p \boxplus q(x + a))(x) \) for \( a \in \mathbb{R}^n \)
3. (Scale-invariance) \( (p(ax) \boxplus q(ax))(x) = a^\gamma \cdot (p \boxplus q)(ax) \) for \( a \in \mathbb{R}^n \)
4. (Derivative-invariance) \( (\partial_{x_k} p) \boxplus q = \partial_{x_k} (p \boxplus q) = p \boxplus (\partial_{x_k} q) \) for all \( k \in [n] \)
5. (Stability-preserving) \( p \boxplus q \) is real stable
6. (Triangle inequality) \( \text{Ab}(p \boxplus q) \supseteq \text{Ab}(p) + \text{Ab}(q) \), where \( + \) is Minkowski sum

**Proof.** (1), (2), (3) and (4) are straightforward. To prove (5), one can consider the Borcea-Brändén symbol (see [28]) of the operator

\[
\boxplus^\gamma : \mathbb{R}^{(\gamma, \gamma)}[x_1, \ldots, x_n, z_1, \ldots, z_n] \to \mathbb{R}^{\gamma}[x_1, \ldots, x_n]
\]

which is defined on products of polynomials (i.e., simple tensors) via

\[
\boxplus^\gamma(p(x)q(z)) := (p \boxplus q)(x)
\]

and linearly extended. Note that if \( \boxplus^\gamma \) preserves stability, then (5) follows as a corollary. That said, the symbol of \( \boxplus^\gamma \) takes on a very nice form, using property (2):

\[
\text{Symb}_{BB}(\boxplus^\gamma) = \boxplus^\gamma((x + y)^\gamma(z + w)^\gamma) = (x + y)^\gamma \boxplus^\gamma (x + w)^\gamma = (x + y + w)^\gamma
\]
This polynomial is obviously real stable, and (5) follows.

To prove (6), we first assume $0 \in \text{Ab}(p) \cap \text{Ab}(q)$ by shifting, since $\text{Ab}(p(x + a)) = \text{Ab}(p) + \{-a\}$. Note also that $0 \in \text{Ab}(p)$ if and only if $p$ has coefficients all of the same sign. (One direction is easy, the other follows by induction and the fact that $\text{Ab}(p) \subseteq \text{Ab}(\partial_x p)$ by a standard argument.) In this case, $p$ and $q$ have coefficients all of the same sign, and therefore so does $p \boxplus q$. That is, in this case $0 \in \text{Ab}(p \boxplus q)$.

To complete the proof, we utilize this case to show that $a \in \text{Ab}(p)$ and $b \in \text{Ab}(q)$ implies $a + b \in \text{Ab}(p \boxplus q)$. Note that by shifting we have that $0 \in \text{Ab}(p(x + a)) \cap \text{Ab}(q(x + b))$, which implies $0 \in \text{Ab}((p \boxplus q)(x + a + b))$ by the previous paragraph. This in turn implies $a + b \in \text{Ab}(p \boxplus q)$. $\square$

Again, in the univariate case $\text{Ab}(p)$ is literally the interval $[\maxroot(p), \infty)$. The triangle inequality stated above then is equivalent to the classical version: $\maxroot(p \boxplus q) \leq \maxroot(p) + \maxroot(q)$. This is what justifies our calling it “the triangle inequality”.

The upshot of the previous proposition is that many of the nice classical properties of the univariate convolution are shared with the multivariate additive convolution. That said, it becomes natural to ask a similar question for the stronger results discussed in this chapter; that is: what more can we say about how the multivariate additive convolution relates to points above the roots?

Our first conjecture in this direction is a combining of our main theorem (1.2.1) and the multivariate triangle inequality.

**Conjecture 3.2.4.** Let $p, q, r \in \mathbb{R}[x_1, \ldots, x_n]$ be real stable. Then:

$$\text{Ab}(p \boxplus q \boxplus r) + \text{Ab}(r) \supseteq \text{Ab}(p \boxplus r) + \text{Ab}(q \boxplus r)$$

In a (as of yet unpublished) paper of Brändén and Marcus, a multivariate analogue of the Marcus-Spielman-Srivastava root bound is given. We believe that this result should follow from the previous conjecture, but it is currently unclear whether or not the methods of Brändén-Marcus can be adapted to prove the conjecture itself.

### 3.3 A Natural (But False) Conjecture

It can be shown that the previous conjecture is not enough to prove optimal bounds for the paving conjecture. For this we need something a bit more refined, which we give in the following. This conjecture represents the most natural generalization of the univariate root bound, and the fact that it precisely implies optimal paving bounds only increases its importance. In addition it has been considered independently of the authors by Mohan Ravichandran (personal correspondence; also see [61]) in attempt to prove optimal paving bounds, and this even further suggests its centrality.

Unfortunately though, the conjecture is false in general. We will state it in two equivalent forms, and provide a counterexample.
To do this, we first must relate the notion of $\text{Ab}(p)$ to the notion of potential in the multiaffine case. Potential was used by Marcus-Spielman-Srivastava to delicately keep track of root bounds, and so this connection comes at no surprise. We will use the standard definition of potential in what follows:

$$\Phi_i^p(a) := \frac{\partial x_i p}{p}(a)$$

**Corollary 3.3.1.** Let $p \in \mathbb{R}^{(1^n)}[x_1, \ldots, x_n]$ be real stable and multiaffine with $p(0) > 0$ and $0 \in \text{Ab}(p)$. Then:

$$\Phi_i^p(0) \leq 1 \iff -e_i \in \text{Ab}(p)$$

**Proof.** Since $p(x) > 0$ for $x \in \mathbb{R}_+^n$ and $p$ is multiaffine we have:

$$\Phi_i^p(c \cdot e_i) < 1 \iff 0 < p(c \cdot e_i) - \partial x_i p(c \cdot e_i) = p(0) + (c - 1) \partial x_i p(0) = p((c - 1) \cdot e_i)$$

It is straightforward that $\Phi_i^p(c \cdot e_i)$ is strictly decreasing in $c$ (or else identically zero) for $c \geq 0$, and therefore:

$$\Phi_i^p(0) \leq 1 \iff \Phi_i^p(c \cdot e_i) < 1 \text{ for all } c > 0$$

Combining these gives:

$$\Phi_i^p(0) \leq 1 \iff p((c - 1) \cdot e_i) > 0 \text{ for all } c > 0$$

Note now that $p((c - 1) \cdot e_i)$ is linear in $c$, and that $(c - 1) \cdot e_i = 0 \in \text{Ab}(p)$ for $c = 1$. Therefore Proposition 3.2.1 implies $\Phi_i^p(0) \leq 1$ iff $-e_i \in \text{Ab}(p)$. □

We now state the false conjecture, once in terms of potential and once in terms of points above the roots.

**Conjecture 3.3.2** (Strong conjecture, first form (see [61])). Let $p, q \in \mathbb{R}^{(1^n)}[x_1, \ldots, x_n]$ be real stable multiaffine polynomials, and let $a$ and $b$ be above the roots of $p$ and $q$ respectively. Suppose for some $\varphi_1, \ldots, \varphi_n \in \mathbb{R}_{++}$, we have the following for all $i \in [n]$:

$$\Phi_i^p(a) \leq \varphi_i, \quad \Phi_i^q(b) \leq \varphi_i$$

Then for all $i \in [n]$ we have:

$$\Phi_i^{p \boxplus_q} \left( a + b - \frac{1}{\varphi_i} \right) \leq \varphi_i$$

**Conjecture 3.3.3** (Strong conjecture, second form). Let $p, q \in \mathbb{R}^{(1^n)}[x_1, \ldots, x_n]$ be real stable multiaffine polynomials. Suppose for all $i \in [n]$ we have:

$$-e_i \in \text{Ab}(p), \quad -e_i \in \text{Ab}(q)$$

Then for all $i \in [n]$ we have:

$$-1 - e_i \in \text{Ab}(p \boxplus_q)$$
**Proof of equivalence.** By the previous corollary \(-e_i \in \text{Ab}(p)\) is equivalent to \(\Phi^i_p(0) \leq 1\). The conclusion of the above conjecture is that \(\Phi^i_{p \boxplus q}(-1) \leq 1\). Again by the previous corollary, this is equivalent to \(-e_i \in \text{Ab}((p \boxplus q)(x-1))\). This in turn is equivalent to \(-1 - e_i \in \text{Ab}(p \boxplus q)\).

As a final note, we can restrict to this seemingly less general case (i.e., \(\Phi^i_p(0) \leq 1\) instead of \(\Phi^i_p(a) \leq \varphi_i\)) via shifting and scaling, which completes the proof.

To disprove this, we give a counterexample to the second formulation. The key idea is to use a polynomial which is extremal with respect to the strongly Rayleigh conditions. These conditions are nice convexity-type properties which are equivalent to real stability for multiaffine polynomials. We recall them now.

**Proposition 3.3.4 ([4]).** Fix multiaffine \(p \in \mathbb{R}^{(1^n)}[x_1, ..., x_n]\). We have that \(p\) is real stable iff for all \(x \in \mathbb{R}^n\) and all \(i, j \in [n]\) we have:

\[
\partial_{x_i}p(x) \cdot \partial_{x_j}p(x) - p(x) \cdot \partial_{x_i} \partial_{x_j}p(x) \geq 0
\]

It is also of interest to note that the polynomial in the above inequality does not depend on \(x_i\) or \(x_j\). One can see this by taking the partial derivative of the above expression with respect to \(x_i\) or \(x_j\), recalling that \(p\) is multiaffine (this expression will be 0). This makes it relatively easy to determine whether or not 3-variable multiaffine polynomials are real stable, as in the following example.

**Counterexample 3.3.5.** The polynomial

\[
p = q = \frac{8}{21}x_1x_2x_3 + \frac{80}{21}x_1x_2 + \frac{27}{7}x_1x_3 + x_2x_3 + 4x_1 + 4x_2 + 4x_3 + 4
\]

provides a counterexample to the above conjectures.

**Proof.** First we prove that \(p = q\) is real stable. By the above comment, we obtain simple expressions for the strongly Rayleigh conditions:

\[
\partial_{x_1}p(x) \cdot \partial_{x_2}p(x) - p(x) \cdot \partial_{x_1} \partial_{x_2}p(x) = \frac{1}{21}(7x_3 + 4)^2
\]

\[
\partial_{x_1}p(x) \cdot \partial_{x_3}p(x) - p(x) \cdot \partial_{x_1} \partial_{x_3}p(x) = \frac{4}{7}(2x_2 + 1)^2
\]

\[
\partial_{x_2}p(x) \cdot \partial_{x_3}p(x) - p(x) \cdot \partial_{x_2} \partial_{x_3}p(x) = \frac{4}{147}(22x_1 + 21)^2
\]

Notice that all of these expressions are nonnegative for all \(x\), which means that \(p = q\) is real stable. Also, notice that these polynomials are on the boundary of the set of nonnegative polynomials, and so in some sense \(p = q\) is on the boundary of the set of real stable polynomials. Note that this polynomial has 0 above its roots (with \(p(0) > 0\)), and it is easy to see that \(-e_i \in \text{Ab}(p)\) for all \(i \in [n]\).
We now compute $p \boxplus q = p \boxplus p$ as follows:

\[
p \boxplus q = \frac{64}{441} x_1 x_2 x_3 + \frac{1280}{441} x_1 x_2 + \frac{144}{49} x_1 x_3 + \frac{16}{21} x_2 x_3 + \frac{4768}{147} x_1 + \frac{32}{3} x_2 + \frac{226}{21} x_3 + \frac{1520}{21}
\]

Since $\boxplus$ preserves real stability, this polynomial is real stable. Further, we have $0 \in \text{Ab}(p \boxplus q)$ and $(p \boxplus q)(0) > 0$, and so $(p \boxplus q)(x) \geq 0$ for all $x \in \text{Ab}(p \boxplus q)$. With this, we show that $(p \boxplus q)(-1 - e_1) < 0$ which contradicts the above conjecture:

\[
(p \boxplus q)(-1 - e_1) = -\frac{1450}{441}
\]

\[
\Box
\]

### 3.4 Proof of the Main Result

We now set out to prove our main result, Theorem 1.2.1. Recall the following from the theory of interlacing polynomials:

**Lemma 3.4.1.** Fix real-rooted $p, q, r \in \mathbb{R}^n[x]$ with positive leading coefficients. If $q \ll p$, then:

\[
q \boxplus^n r \ll p \boxplus^n r
\]

Now we introduce the notation used in [26]. Given a monic polynomial $p$ of degree $n$ with at least 2 distinct roots, we write:

\[
p(x) = \prod_{i=1}^{n} (x - \lambda_i)
\]

Order the roots $\lambda_1 \geq \cdots \geq \lambda_n$, and let $k$ be minimal such that $\lambda_1 \neq \lambda_k$. Define $\mu_0 := \frac{\lambda_1 + \lambda_k}{2}$ and $\mu_1 := \lambda_1$. Further, for $\mu \in [\mu_0, \mu_1]$ we define:

\[
\tilde{p}_\mu(x) := (x - \mu)^2 \prod_{i \neq 1, k} (x - \lambda_i)
\]

We then define:

\[
\hat{p}_\mu(x) := p(x) - \tilde{p}_\mu(x) = ((2\mu - (\lambda_1 + \lambda_k))x - (\mu^2 - \lambda_1 \lambda_k)) \prod_{i \neq 1, k} (x - \lambda_i)
\]

For $\mu > \mu_0$, we have that $\hat{p}_\mu$ is of degree $n - 1$ with positive leading coefficient and the extra root is at least $\lambda_1$. (Note that when $\mu = \mu_0$, we have that $\hat{p}_\mu$ is of degree $n - 2$ with negative leading coefficient.) To see this, notice:

\[
\rho := \frac{\mu^2 - \lambda_1 \lambda_k}{2\mu - (\lambda_1 + \lambda_k)} \geq \lambda_1 \iff \mu^2 - 2\mu \lambda_1 + \lambda_1^2 = (\mu - \lambda_1)^2 \geq 0
\]
Figure 3.1: Above is an illustration of larger roots of the described polynomials. The labels of the roots are below the number line, while their respective multiplicity is above. \(*\) is the multiplicity of \(\lambda_k\).

This then implies that for \(f_\mu(x) := (x - \mu)\prod_{i \neq 1,k} (x - \lambda_i)\), we have \(f_\mu \ll \tilde{p}_\mu\), \(f_\mu \ll \hat{p}_\mu\), and \(f_\mu \ll p\). In Figure 3.1, we illustrate one possibility for the largest roots of these polynomials.

In what follows, we additionally fix a real-rooted \(r \in \mathbb{R}[x]\) of degree \(n\).

**Lemma 3.4.2.** Fix any \(\mu, \mu'\) with \(\mu_0 \leq \mu \leq \mu_1\) where \(\mu_0, \mu_1\) are defined as above. We have:

\[
\lambda_1(\tilde{p}_{\mu_0} \boxplus^n r) \leq \lambda_1(p \boxplus^n r) \leq \lambda_1(\tilde{p}_{\mu_1} \boxplus^n r)
\]

\[
\lambda_1(\hat{p}_{\mu_0} \boxplus^n r) \leq \lambda_1(p \boxplus^n r)
\]

**Proof.** The first inequality of the first line follows from the fact that the roots of \(\tilde{p}_{\mu_0}\) are majorized by that of \(p\) (this is because \(\tilde{p}_{\mu_0}\) can obtained via a “pinch” of the roots of \(p\); see property 4 of Definition 3.1.1). The second inequality of the first line follows from Lemma 3.4.1 and the fact that \(p \ll \tilde{p}_{\mu_1}\). The second line follows from Lemma 3.4.1 and the fact that \(\hat{p}_\mu \ll g \ll \hat{p}_{\mu'}\) for \(g(x) := (x - \mu)(x - \mu')\prod_{i \neq 1,k} (x - \lambda_i)\).

**Corollary 3.4.3.** There exists \(\mu \in [\mu_0, \mu_1]\) such that \(\lambda_1(\tilde{p}_\mu \boxplus r) = \lambda_1(p \boxplus r)\).

**Proof.** The above lemma and continuity.

Now, let \(\mu_*\) denote the maximal \(\mu \in [\mu_0, \mu_1]\) such that the previous corollary holds. For simplicity, we will denote \(\tilde{p} := \tilde{p}_{\mu_*}\) and \(\hat{p} := \hat{p}_{\mu_*}\).

**Proposition 3.4.4.** For \(\mu_*\) defined as above, we have that \(\mu_* > \mu_0\) and:

\[
\lambda_1(\hat{p} \boxplus^n r) = \lambda_1(p \boxplus^n r) = \lambda_1(\tilde{p} \boxplus^n r)
\]
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Proof. The second equality follows from the definition of $\mu_*$. So we only need to prove the first equality. By linearity $\hat{p} \boxplus r$ has a root at $\lambda_1(p \boxplus r)$, and so $\lambda_1(\hat{p} \boxplus r) \geq \lambda_1(p \boxplus r)$. So in fact we only need to show that $\lambda_1(\hat{p} \boxplus r) \leq \lambda_1(p \boxplus r)$.

If $\mu_* = \mu_1$, then $\lambda_1(\hat{p}) = \lambda_1(p)$ and $\hat{p} \ll p$. This implies $\lambda_1(\hat{p} \boxplus r) \leq \lambda_1(p \boxplus r)$. Otherwise $\mu_0 \leq \mu_* < \mu_1$. Then for $\mu > \mu_*$, we have $\lambda_1(\hat{p}_\mu \boxplus r) > \lambda_1(p \boxplus r)$ by Lemma 3.4.2 which implies $\hat{p}_\mu \boxplus r > 0$ at $\lambda_1(\hat{p}_\mu \boxplus r)$. Recalling the definition of $f_\mu$ above, $f_\mu \ll \hat{p}_\mu$ implies $\lambda_1(f_\mu \boxplus r) \leq \lambda_1(\hat{p}_\mu \boxplus r)$, and $f_\mu \ll \hat{p}_\mu$ implies $\hat{p}_\mu \boxplus r$ has at most one root greater than $\lambda_1(f_\mu \boxplus r)$. Combining all this with the fact that $\hat{p}_\mu \boxplus r$ has positive leading coefficient gives $\lambda_1(\hat{p}_\mu \boxplus r) < \lambda_1(\hat{p}_\mu \boxplus r)$. Limiting $\mu \to \mu_*$ from above then implies $\lambda_1(\hat{p} \boxplus r) \leq \lambda_1(\hat{p} \boxplus r) = \lambda_1(p \boxplus r)$.

Now suppose that $\mu_* = \mu_0$, so as to get a contradiction. As $\mu \to \mu_*$ from above, $\hat{p}_\mu \boxplus r$ has positive leading coefficient limiting to zero. So $\hat{p} \boxplus r$ then has one less root, and has negative leading coefficient as discussed above. However, since $\lambda_1(\hat{p}_\mu \boxplus r) < \lambda_1(\hat{p}_\mu \boxplus r) \leq \lambda_1(\hat{p}_\mu \boxplus r)$ for all $\mu > \mu_*$ (as noted earlier in this proof), $\hat{p}_\mu \boxplus r$ must have a root limiting to $-\infty$ as $\mu \to \mu_*$. Therefore the second-from-leading coefficient of $\hat{p}_\mu \boxplus r$ (the sum of negated roots scaled by the leading coefficient) is eventually non-negative as $\mu \to \mu_*$. This contradicts the fact that $\hat{p} \boxplus r$ has negative leading coefficient. (Note that this crucially uses the fact that $\mu_*$ is maximal.) \qed

The next lemma provides the base case to a more streamlined induction for the proof. In fact, it may even lead to a proof of some sort of majorization relation.

Definition 3.4.5. For real-rooted $p \in \mathbb{R}_n[x]$ not necessarily of degree $n$, let $\lambda^n(p) \in \mathbb{R}^n$ be the list of roots of $p$, padded with the mean of the roots, and then ordered in non-increasing order.

Lemma 3.4.6. Fix real-rooted $p, q, r \in \mathbb{R}_n[x]$ such that $\deg(q) = \deg(r) = n$ and $\deg(p) = 1$. Then:

$$\lambda^n(p \boxplus q \boxplus^n r) + \lambda^n(r) \prec \lambda^n(p \boxplus^n r) + \lambda^n(q \boxplus^n r)$$

Proof. By shifting, we may assume WLOG that $p, q, r$ all have have roots which sum to 0. Since $\deg(p) = 1$, the result is then equivalent to the following:

$$\lambda^n(D^{n-1}(q \boxplus^n r)) + \lambda^n(r) \prec \lambda^n(D^{n-1}r) + \lambda^n(q \boxplus^n r)$$

Since $\boxplus$ preserves the set of polynomials whose roots sum to 0, this is equivalent to:

$$\lambda^n(r) \prec \lambda^n(q \boxplus^n r)$$

Since $r = x^n \boxplus^n r$ and $\lambda(x^n) \prec \lambda(q)$, the result follows from Corollary 3.1.5. \qed

The following is an immediate corollary of the previous lemma.

Corollary 3.4.7. Fix real-rooted $p, q, r \in \mathbb{R}_n[x]$ such that $\deg(q) = \deg(r) = n$ and $\deg(p) = 1$. Then:

$$\lambda_1(p \boxplus q \boxplus^n r) + \lambda_1(r) \leq \lambda_1(p \boxplus^n r) + \lambda_1(q \boxplus^n r)$$
We now prove the main result.

**Theorem 3.4.8.** Fix real-rooted $p,q,r \in \mathbb{R}[x]$ such that $\deg(q) = \deg(r) = n$ and $\deg(p) = k \leq n$. Then:
$$
\lambda_1(p \boxplus^n q \boxplus^n r) + \lambda_1(r) \leq \lambda_1(p \boxplus^n r) + \lambda_1(q \boxplus^n r)
$$

**Proof.** We induct on $k$, using the previous corollary as the base case. Let $p$ be a polynomial of degree $k$ with roots in $[-R,R]$ (for any fixed $R$) which maximizes (by compactness):
$$
\beta(p) := \lambda_1(p \boxplus^n q \boxplus^n r) + \lambda_1(r) - \lambda_1(p \boxplus^n r) - \lambda_1(q \boxplus^n r)
$$

To get a contradiction, we assume $\beta(p) > 0$. In particular this implies $p$ has at least 2 distinct roots, allowing us to apply the above discussion, notation, and results. By induction we have $\beta(\hat{p}) \leq 0$, which implies:
$$
\lambda_1(\hat{p} \boxplus^n q \boxplus^n r) \leq \lambda_1(\hat{p} \boxplus^n r) + \lambda_1(q \boxplus^n r) - \lambda_1(r)
$$
$$
= \lambda_1(p \boxplus^n r) + \lambda_1(q \boxplus^n r) - \lambda_1(r)
$$
$$
= \lambda_1(p \boxplus^n q \boxplus^n r) - \beta(p)
$$

Since $\mu_* > \mu_0$ by the previous proposition, $\hat{p}$ has positive leading coefficient. This implies $\hat{p} \boxplus^n q \boxplus^n r = (p - \hat{p}) \boxplus^n q \boxplus^n r < 0$ when evaluated at $\lambda_1(p \boxplus^n q \boxplus^n r)$. Since $\hat{p}$ has positive leading coefficient, this gives:
$$
\beta(\hat{p}) - \beta(p) = \lambda_1(\hat{p} \boxplus^n q \boxplus^n r) - \lambda_1(p \boxplus^n q \boxplus^n r) > 0
$$

This contradicts the maximality of $\beta(p)$, since all of the roots of $\hat{p}$ are contained in $[-R,R]$. \qed

**Corollary 3.4.9.** Fix real-rooted $p,q,r \in \mathbb{R}[x]$. If all polynomials involved are of degree at least 1, then:
$$
\lambda_1(p \boxplus^n q \boxplus^n r) + \lambda_1(r) \leq \lambda_1(p \boxplus^n r) + \lambda_1(q \boxplus^n r)
$$

Note that the following condition is equivalent to the degree restriction:
$$
2n < \deg(p) + \deg(q) + \deg(r) \iff (n - \deg(p)) + (n - \deg(q)) + (n - \deg(r)) < n
$$

**Proof.** Consider polynomials of degree $n$ whose roots limit to the roots of $p,q,r$ and extra roots limit to $-\infty$. The previous theorem and continuity (and use of Lemma 3.4.1 to bound the largest roots away from $+\infty$) then imply the result. \qed

**Further Directions**

Despite its connections to important problems like the paving conjecture and the entropy conjecture, it is still not fully understood how the additive convolution affects the roots
of real-rooted polynomials. In [26], Marcus, Spielman, and Srivastava began the study of
root movement by investigating the effect of differential operators of the form \( I - \alpha D \) on
the largest root. In this chapter we extended their result to all differential operators which
preserve real-rootedness. This extension alone doesn’t have any immediate applications we
are aware of.

The resolution of Horn’s conjecture by Knutson and Tao (see [62]) gave a full charac-
terization of the eigenvalues of the sum of two Hermitian matrices. We were able to obtain
Horn’s inequalities for the additive convolution as well via hyperbolicity, but understanding
the full effect of the additive convolution on roots remains a mystery. The entropy conjecture,
which quantifies the effect of the additive convolution on the discriminant of a polynomial,
is one approach to understanding the effect of the roots holistically. Our submodular ma-
ajorization (and generalized Horn’s inequalities) conjectures provide another insight into the
workings of the inner roots. Because submodularity is unique to the additive convolution,
we believe it will require a new framework (beyond traditional hyperbolicity tools) to tackle
these conjectures.

Another possible future direction is extending submodularity results to the \( b \)-additive
convolution, in which derivatives are replaced by certain finite differences. Such convolu-
tions have an intimate connection to the mesh of a real-rooted polynomial, which is the
minimal distance between any two roots (e.g., see [17] and [63]). In our testing we found
several submodularity relations among such \( b \)-additive convolutions. The additive convolu-
tion can be obtained by limiting \( b \to 0 \), and so any results for the \( b \)-additive convolution
are strictly stronger than the conjectures in this chapter. The advantage of trying to prove
these statements in the finite difference case comes in the limited structures available: fewer
operations interact nicely with the mesh of a polynomial compared to those operations which
preserve real-rootedness, and this may better direct the study of the roots.

Finally in the multivariate realm, little is known. And, many of the natural extensions
of these results seem to fail in the multivariate case. The state of the art in this direction is
currently the ad hoc barrier function arguments used by MSS in their resolution of Kadison-
Singer. That said, an important next step for their work is to encapsulate their techniques
in a more coherent theory. We believe that our results and conjectures are a step in the right
direction.
Chapter 4

Connecting the q-Multiplicative and the Finite Difference Convolution

In [17], the authors show that the additive convolution can only increase root mesh, which is defined as the minimum absolute difference between any pair of roots of a given polynomial. That is, the mesh of the output polynomial is at least as large as the mesh of either of the input polynomials. They use similar arguments to show that, for polynomials with non-negative roots, the multiplicative convolution can only increase logarithmic root mesh. This is similarly defined as the minimum ratio (greater than 1) between any pair of positive roots of a given polynomial.

Regarding mesh and logarithmic mesh, there are natural generalized convolution operators which also preserve such properties. The first will be called the $q$-multiplicative convolution, and it was shown to preserve logarithmic root mesh of at least $q$ in [16]. The second will be called the $b$-additive convolution (or finite difference convolution), and the main concern of this chapter is to demonstrate that it preserves root mesh of at least $b$.

An Analytic Connection

Our first method of proof of Theorem 1.3.5 will demonstrate a way to pass root properties of the $q$-multiplicative convolution to the $b$-additive convolution. As this is interesting in its own right, we state the most general version of this result here.

**Theorem 4.0.1.** Fix $b \geq 0$ and let $p, r$ be polynomials of degree $n$. We have the following, where convergence is uniform on compact sets.

$$
\lim_{q \to 1} (1 - q)^n \left[ E_{q,b}(p) \boxplus^q_{q} E_{q,b}(r) \right] (q^x) = p \boxplus^b_{b} r
$$

Where $E_{q,b}$ is defined in 4.1.11. Note that for $b = 0$, this result pertains to the classical convolutions.

Here, the $E_{q,b}$ are certain linear isomorphisms of $\mathbb{C}[x]$, to be explicitly defined below. Notice that uniform convergence allows us to use Hurwitz’ theorem to obtain root properties
in the limit of the left-hand side. That is, any information about how the $q$-multiplicative convolution acts on roots will transfer to some statement about how the $b$-additive convolution acts on roots. As it turns out, a special case of Lamprecht’s result (Theorem 3 from [16]) will become our result (Theorem 1.3.5) in the limit. We discuss this transfer process in more detail in §4.1.

Extending Lamprecht’s Method

Our second method of proof of Theorem 1.3.5 is an extension of the method used by Lamprecht to prove the log mesh result for the $q$-multiplicative convolution. Specifically, he demonstrates that the $q$-multiplicative convolution preserves a root-interlacing property for $q$-log mesh polynomials. More formally he proves the following result which gives Theorem 1.3.5 as a corollary.

**Theorem 4.0.2** (Lamprecht Interlacing-Preserving). Let $f, g \in \mathbb{R}_n[x]$ be $q$-log mesh polynomials of degree $n$ with only negative roots. Let $T_g : \mathbb{R}_n[x] \rightarrow \mathbb{R}_n[x]$ be the real linear operator defined by $T_g : r \mapsto r \star^n_q g$. Then, $T_g$ preserves the set of polynomials whose roots interlace the roots of $f$.

We achieve an analogous result for the $b$-additive convolution using techniques similar to those found in Lamprecht’s paper. We state it formally here.

**Theorem 4.0.3.** Let $f, g \in \mathbb{R}_n[x]$ be $b$-mesh polynomials of degree $n$. Let $T_g : \mathbb{R}_n[x] \rightarrow \mathbb{R}_n[x]$ be the real linear operator defined by $T_g : r \mapsto r \boxplus^b g$. Then, $T_g$ preserves the set of polynomials whose roots interlace the roots of $f$.

In both cases, mesh and log mesh properties can be shown to be equivalent to root interlacing properties ($f(x)$ interlaces $f(x-b)$ for $b$-mesh, and $f(x)$ interlaces $f(q^{-1}x)$ for $q$-log mesh). The above theorems then immediately imply the desired mesh preservation properties for the respective convolutions. We discuss this further in §4.2.

4.1 First Proof Method: The Finite Difference Convolution as a Limit of $q$-Multiplicative Convolutions

In what follows we establish a general analytic connection between the multiplicative (Grace-Szego) and additive (Walsh) convolutions on polynomials of degree at most $n$. We then extend this connection to the $q$-multiplicative convolution and the $b$-additive convolution (Theorem 4.0.1). Using this connection, we transfer root information results of the multiplicative convolution ($q$ or classical) to the additive convolution ($b$ or classical). Specifically, we use this connection to prove Theorem 1.3.5, which is the conjecture of Brändén, Krasikov, and Shapiro mentioned above.
To begin we state an observation of Vadim Gorin demonstrating an analytic connection in the classical case using matrix formulations of the classical convolutions given in [26]:

\[ \chi(A) \otimes^n \chi(B) = \mathbb{E}_P \left[ \chi(APBP^T) \right] \]
\[ \chi(A) \oplus^n \chi(B) = \mathbb{E}_P \left[ \chi(A + PBP^T) \right] \]

Here, \( A \) and \( B \) are real symmetric matrices, \( \chi \) denotes the characteristic polynomial, and the expectations are taken over all permutation matrices. We then write:

\[
\lim_{t \to 0} t^{-n} \left[ \chi(e^{tA}) \otimes^n \chi(e^{tB}) \right] (tx + 1) = \lim_{t \to 0} t^{-n} \mathbb{E}_P \left[ \det \left( txI + I - e^{tA}Pe^{tB}P^T \right) \right]
\]
\[
= \lim_{t \to 0} t^{-n} \mathbb{E}_P \left[ \det \left( txI - t(APP^T + PBP^T) + O(t^2) \right) \right]
\]
\[
= \lim_{t \to 0} \mathbb{E}_P \left[ \chi(A + PBP^T + O(t)) \right]
\]
\[
= \chi(A) \oplus^n \chi(B)
\]

This connection is suggestive and straightforward, but seemingly confined to the classical case. Therefore, we instead state below a slightly modified (but equivalent) version of this observation for the classical convolutions (Theorem 4.1.2) which we are able to then generalize to the \( q \)-multiplicative and \( b \)-additive convolutions.

The Classical Convolutions

We begin by sketching the proof of the connection between the classical additive and multiplicative convolutions. We then state rigorously the more general result for the \( q \)-multiplicative and \( b \)-additive convolutions. In this section, many quantities will be defined with \( b = 0 \) in mind (this corresponds to the classical additive convolution), with the more general quantities given in subsequent sections. Further, we will leave the proofs of the lemmas to the generic \( b \) case, omitting them here.

To go from the multiplicative world to the additive world, we use a linear map which acts as an exponentiation on roots, and a limiting process which acts as a logarithm. In particular, we will refer to the following algebra endomorphism on \( \mathbb{C}[x] \) as our “exponential map”:

\[ E_{q,0} : x \mapsto \frac{1 - x}{1 - q} \]

In what follows, any limiting process will mean uniform convergence on compact sets in \( \mathbb{C} \), unless otherwise specified. This will allow us to extract analytic information about roots using the classical Hurwitz’ theorem. In particular, the following result hints at the analytic information provided by the exponential map \( E_{q,0} \).

**Proposition 4.1.1.** We have the following for any \( p \in \mathbb{C}[x] \).

\[
\lim_{q \to 1} [E_{q,0}(p)](q^t) = p
\]
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DIFFERENCE CONVOLUTION

Proof. We first consider \( [E_{q,0}(x)](q^x) = \frac{1 - q^x}{1 - q} \), for which we obtain the following by the generalized binomial theorem:

\[
\lim_{q \to 1} [E_{q,0}(x)](q^x) = \lim_{q \to 1} \frac{1 - q^x}{1 - q} = \lim_{q \to 1} \sum_{m=1}^{\infty} \binom{x}{m} (q - 1)^{m-1} = \binom{x}{1} = x
\]

To show that convergence here is uniform on compact sets, consider the tail for \(|x| \leq M\):

\[
\left| \sum_{m=2}^{\infty} \binom{x}{m} (q - 1)^{m-1} \right| \leq \sum_{m=0}^{\infty} |q - 1|^{m+1} \cdot \frac{x(x - 1) \cdots (x - m)}{(m + 2)!} \\
\leq |q - 1| \sum_{m=0}^{\infty} |q - 1|^m \prod_{k=1}^{m+2} \left( 1 + \frac{|x|}{k} \right) \\
\leq |q - 1|(1 + M)^2 \sum_{m=0}^{\infty} (|q - 1|(1 + M))^m \\
= \frac{|q - 1|(1 + M)^2}{1 - |q - 1|(1 + M)}
\]

This limits to zero as \( q \to 1 \), which proves the desired convergence.

Since \( E_{q,0} \) is an algebra morphism, we can use the fundamental theorem of algebra to complete the proof. Specifically, letting \( p(x) = c_0 \prod_k (x - \alpha_k) \) we have:

\[
\lim_{q \to 1} \left[ E_{q,0} \left( c_0 \prod_k (x - \alpha_k) \right) \right] (q^x) = c_0 \prod_k \left( \lim_{q \to 1} [E_{q,0}(x)](q^x) - \alpha_k \right) = c_0 \prod_k (x - \alpha_k)
\]

\[ \square \]

We now state our result in the classical case, which gives an analytic connection between the additive and multiplicative convolutions. As a note, many of the analytic arguments used in the proof of this result will have a flavor similar to that of the proof of Proposition 4.1.1.

**Theorem 4.1.2.** Let \( p, r \in \mathbb{C}[x] \) be of degree at most \( n \). We have the following.

\[
\lim_{q \to 1} (1 - q)^n [E_{q,0}(p) \boxtimes^n E_{q,0}(r)](q^x) = p \boxplus^n r
\]

**Proof Sketch**

We will establish the above identity by calculating it on basis elements. Specifically, we will expand everything into powers of \((1 - q)\). To prove the theorem, it then suffices to show that: (1) the negative degree coefficients are all zero, (2) the series has the desired constant term, and (3) the tail of the series converges to zero uniformly on compact sets. Our first step towards establishing this is expanding \( q^{kx} \) in terms of powers of \((1 - q)\).
CHAPTER 4. CONNECTING THE Q-MULTIPLICATIVE AND THE FINITE DIFFERENCE CONVOLUTION

Remark 4.1.3. Since we will only be considered with behavior for $q \approx 1$, we will use the notation $q \approx 1$ to indicate there exists some $\epsilon > 0$ such that the statement holds for $q \in (1 - \epsilon, 1 + \epsilon)$.

Definition 4.1.4. We define a constant, which will help us to simplify the following computations.

$$\alpha_{q,0} := \frac{\ln q}{1-q}$$

Note that $\lim_{q \to 1} \alpha_{q,0} = -1$.

Lemma 4.1.5. Fix $k \in \mathbb{N}_0$. For $q \approx 1$, we have the following.

$$q^{kx} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \alpha_{q,0}^m k^m (1-q)^m$$

For fixed $q \approx 1$, this series has a finite radius of convergence, and this radius approaches infinity as $q \to 1$.

Notice that this is not a true power series in $(1-q)$, as $\alpha_{q,0}$ depends on $q$. Using this, we can calculate the series obtained after plugging in specific basis elements.

Lemma 4.1.6. Fix $q \approx 1$ in $\mathbb{R}_{>0}$ and $j, k, n \in \mathbb{N}_0$ such that $0 \leq j \leq k \leq n$. We have the following.

$$(1-q)^n [(1-x)^j \boxtimes (1-x)^k] (q^x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} \alpha_{q,0}^m (1-q)^{n+m-j-k} \sum_{i=0}^{j} \frac{(j)}{(i)} \left( \frac{k}{i} \right) (-1)^i i^m$$

We use interpolation arguments to handle the terms of this series, which are combinatorial in nature. In particular we show that this series has no nonzero negative degree terms, as seen in the following.

Proposition 4.1.7. Fix $j, k, m, n \in \mathbb{N}_0$ such that $j \leq k$ and $n + m - j - k \leq 0$. We have the following identity.

$$\sum_{i=0}^{j} \frac{(j)}{(i)} \left( \frac{k}{i} \right) (-1)^i i^m = \begin{cases} (-1)^{n-j-k} \frac{j! k!}{m!} & m = j + k - n \\ 0 & m < j + k - n \end{cases}$$

To deal with the tail of the series, we then use crude bounds to get uniform convergence on compact sets.

Lemma 4.1.8. Fix $M > 0$, and $j, k, n \in \mathbb{N}_0$ such that $j \leq k \leq n$. For $|x| \leq M$, there exists $\gamma > 0$ such that the following bound holds for $q \in (1 - \gamma, 1 + \gamma)$.

$$\left| \sum_{m>j+k-n} \frac{x^m}{m!} \alpha_{q,0}^m (1-q)^{n+m-j-k} \sum_{i=0}^{j} \frac{(j)}{(i)} \left( \frac{k}{i} \right) (-1)^i i^m \right| \leq c_0 c_1 \sum_{m=1}^{\infty} c_2^m |1-q|^m$$

Here, $c_0, c_1, c_2$ are independent of $q$. 
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With this, we can now complete the proof of the theorem by comparing the desired quantity to the constant term in our series in \((1 - q)\).

Proof of Theorem 4.1.2. For \(j, k, n \in \mathbb{N}_0\) such that \(0 \leq j \leq k \leq n\), we can combine the above results to obtain the following. Recall that \(\lim_{q \to 1} a_{q,0} = -1\).

\[
\lim_{q \to 1} (1 - q)^n \left[ \frac{(1 - x)^j}{(1 - q)^j} \right] \left[ \frac{(1 - x)^k}{(1 - q)^k} \right] (q^x) = \frac{j!k!}{n!(j + k - n)!} x^{j+k-n} = x^j \boxplus^n x^k
\]

By symmetry, this demonstrates the desired result on a basis. Therefore, the proof is complete. \(\square\)

General Connection Preliminaries

We now prove the previous results in more generality, which allows for extension to these generalized convolutions. First though, we give some preliminary notation.

Definition 4.1.9. Fix \(q \in \mathbb{R}_{>0}\) and \(x \in \mathbb{C}\). We define \((x)_q := \frac{1 - q^x}{1 - q}\). Note that \(\lim_{q \to 1} (x)_q = x\), using the generalized binomial theorem on \(q^x = (1 + (q - 1)x)\).

Specifically, for any \(n \in \mathbb{Z}\), we have:

\[
(n)_q := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}
\]

We then extend this notation to \((n)!: = (n)_q(n-1)_q \cdots (2)_q(1)_q\) and \(\binom{n}{k}_q := \frac{(n)_q!}{(k)_q!(n-k)_q!}\). We also define a system of bases of \(\mathbb{C}[x]\) which will help us to understand the mesh convolutions.

Definition 4.1.10. For \(b \geq 0\) and \(q \in \mathbb{R}_{>0}\), we define the following bases of \(\mathbb{C}[x]\).

\[
v_{q,b} := (1 - x)(1 - q^b x) \cdots (1 - q^{(k-1)b} x) \quad\frac{1}{(1 - q)^k}
\]

\[
v_{b} := x(x + b)(x + 2b) \cdots (x + (k - 1)b)
\]

We demonstrate the relevance of these bases to the generalized convolutions by giving alternate definitions. Consider a linear map \(A_b\) on \(\mathbb{C}[x]\) defined via \(A_b : v_{q,b}^k \mapsto v_b^k\). That is, \(A_b : x^k \mapsto x(x + b) \cdots (x + (k - 1)b)\). We can then define the \(b\)-additive convolution as follows:

\[
p \boxplus^n_b r := A_b(A_b^{-1}(p) \boxplus^n A_b^{-1}(r))
\]

That is, the \(b\)-additive convolution is essentially a change of basis of the classical additive convolution. Note that equivalently, one can conjugate \(\partial_x\) by \(A_b\) to obtain \(\Delta_b : p \mapsto \frac{p(x) - p(x-b)}{b}\) which demonstrates the definition of \(\boxplus^n_b\) in terms of finite difference operators.
Similarly, the $q$-multiplicative convolution can be seen as a change of basis of the classical multiplicative convolution. Consider a linear map $M_q^{(n)}$ on $\mathbb{C}[x]$ defined via $M_q^{(n)} : \binom{n}{k} x^k \mapsto \binom{n}{k} q^{\binom{k}{2} \cdot x^k}$, which has the property that $M_q^{(n)} : (1 - x)^n \mapsto (1 - q)^n v_{q,b}^n$. We can then define the $q$-multiplicative convolution as follows:

$$p \boxtimes q^n r := M_q^{(n)} \left( (M_q^{(n)})^{-1}(p) \boxtimes (M_q^{(n)})^{-1}(r) \right)$$

These bases will be used to simplify the proof of the general analytic connection for the mesh (non-classical) convolutions. In what follows they will play the role that the basis elements $x^k$ and $(1 - x)^k$ did in the classical proof sketch above.

**Main Result for Mesh Convolutions**

We now generalize the results from the classical ($b = 0$) setting.

**Definition 4.1.11.** Consider the following generalized “exponential map” using the basis elements defined above:

$$E_{q,b} : \nu_b^k \mapsto v_{q,b}^k$$

Note for $b > 0$ these are no longer algebra morphisms. Specialization to $b = 0$ recovers the original “exponential map”. Also notice that for any $p$, the roots of $E_{q,b}(p)$ approach 1 as $q \to 1$ (multiply the output polynomial by $(1 - q)^{\deg(p)}$). In all that follows, previously stated results can be immediately recovered by setting $b = 0$.

**Proposition 4.1.12.** We obtain the same key relation for the generalized exponential maps:

$$\lim_{q \to 1} [E_{q,b}(p)](q^x) = p$$

**Proof.** We compute on basis elements, using Proposition 4.1.1 in the process:

$$\lim_{q \to 1} [E_{q,b}(\nu_b^k)](q^x) = \lim_{q \to 1} [v_{q,b}^k](q^x)$$

$$= \lim_{q \to 1} \frac{(1 - q^x)(1 - q^{x+b}) \cdots (1 - q^{x+(k-1)})}{(1 - q)^k}$$

$$= \prod_{j=0}^{k-1} \lim_{q \to 1} \frac{1 - q^{x+jb}}{1 - q} = \prod_{j=0}^{k-1} (x + jb) = \nu_b^k$$

As in Proposition 4.1.1, one can interpret the $E_{q,b}$ maps as a way to exponentiate the roots of a polynomial. The inverse to these maps is given in the previous proposition by plugging in an exponential and limiting, which corresponds to taking the logarithm of the roots. This discussion will be made more precise in §4.1.

We now state and prove the main result, which gives an analytic link between the $b$-additive and $q$-multiplicative convolutions. We follow the proof sketch of the classical result given above, breaking the following full proof up into more manageable sections.
Theorem 4.1.13. Fix $b \geq 0$ and let $p, r$ be polynomials of degree $n$. We have the following.

$$\lim_{q \to 1} (1 - q)^n \left[ E_{q,b}(p) \boxtimes_q^n E_{q,b}(r) \right] (q^x) = p \boxtimes_b^n r$$

Series Expansion

In order to prove this theorem, we first expand the left-hand side of the expression in a series in $(1 - q)^m$. As above, this is not quite a power series in $(1 - q)^m$ as $\alpha_{q,b}$ (which we now define) depends on $q$.

Definition 4.1.14. We define the $b$-version of the $\alpha_{q,0}$ constants as follows.

$$\alpha_{q,b} := \begin{cases} 
-\frac{b}{q(b-1)} & b > 0 \\
\frac{\ln q}{1-q} & b = 0 
\end{cases}$$

Note that $\lim_{b \to 0} \alpha_{q,b} = \alpha_{q,0}$ for $q \in \mathbb{R}_{>0}$, and $\lim_{q \to 1} \alpha_{q,b} = -1$ for fixed $b \geq 0$.

We now need to understand how exponential polynomials in $q$ relate to our basis elements.

Lemma 4.1.15. Fix $b \geq 0$, and $k \in \mathbb{N}_0$. For $q \approx 1$ in $\mathbb{R}_{>0}$, we have the following.

$$q^{kx} = \sum_{m=0}^{\infty} \nu_b^m \binom{m}{k}_{q^{-b}} (1-q)^m$$

For fixed $q \approx 1$, this series has a finite radius of convergence, and this radius approaches infinity as $q \to 1$.

Proof. For $b > 0$, we use the generalized binomial theorem to compute:

$$q^{kx} = (q^{-bk} - 1 + 1)^{-x/b} = \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} x(x+b) \cdots (x+b(m-1))(q^{-bk} - 1)^m$$

$$= \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \nu_b^m (k)_{q^{-b}} (1-q)^m$$

$$= \sum_{m=0}^{\infty} \frac{\nu_b^m}{m!} \left( \frac{-(b)_{q^{-1}}}{qb} \right)^m (k)_{q^{-b}} (1-q)^m$$

For $b = 0$, manipulating the Taylor series of $q^{kx} = e^{kx \ln q}$ gives the result.

For fixed $q \approx 1$, let $\delta > 0$ be small enough such that $|\alpha_{q,b}| < \frac{1+\delta}{q}$ and $(k)_{q^{-b}} < k + \delta$. Consider:

$$|\nu_b^m| = |x(x+b) \cdots (x+(m-1)b)| \leq m!(|x|+b)^m$$

From this, we obtain:

$$|q^{kx}| \leq \sum_{m=0}^{\infty} \left( \frac{|x|+b(1+\delta)(k+\delta)}{q} \right)^m |1-q|^m$$
It is then easy to see that the radius of convergence of this series limits to infinity as \( q \to 1 \).

We will now proceed by proving the main result on a basis. To that end, we will prove a number of results related to basis element computations. Most of these are rather tedious and not very illuminating. Perhaps this can be simplified through some more detailed and generalized theory of \( q- \) and \( b- \) polynomial operators.

**Lemma 4.1.16.** Fix \( q \approx 1 \) in \( \mathbb{R}_{>0} \), \( b \geq 0 \), and \( j, k, n \in \mathbb{N}_0 \) such that \( 0 \leq j \leq k \leq n \). We have the following.

\[
(1 - q)^n \left[ \tau_{q,b}^j \mathcal{B}_q^m v_{q,b}^k \right] (q^x) = \sum_{m=0}^{\infty} \frac{\nu^m}{m!} \alpha_{q,b}(1 - q)^{n + m - j - k} \sum_{i=0}^{j} \frac{(j)}{(i)} q^b(i-1)/2(-1)^i(i)_q^m
\]

**Proof.** We compute:

\[
(1 - q)^n \left[ \tau_{q,b}^j \mathcal{B}_q^m v_{q,b}^k \right] (q^x) = (1 - q)^{n - j - k} \sum_{i=0}^{j} \frac{(j)}{(i)} q^b(i-1)/2(-1)^i q^{ix} = \sum_{i=0}^{j} \frac{(j)}{(i)} q^b(i-1)/2(-1)^i q^{ix} \sum_{m=0}^{\infty} \frac{\nu^m}{m!} \alpha_{q,b}(1 - q)^{n + m - j - k} \sum_{i=0}^{j} \frac{(j)}{(i)} q^b(i-1)/2(-1)^i (i)_q^m
\]

Q-Lagrange Interpolation

To prove convergence in Theorem 4.0.1, we break up the infinite sum of Lemma 4.1.16 into two pieces. For \( n + m - j - k \leq 0 \), we use an interpolation argument to obtain the following identity. Note that this generalizes a similar identity (for \( q = 1 \)) found in [64].

**Proposition 4.1.17.** Fix \( q \approx 1 \), \( b \geq 0 \), and \( j, k, m, n \in \mathbb{N}_0 \) such that \( j \leq k \) and \( n + m - k \leq j \). We have the following identity.

\[
\sum_{i=0}^{j} \frac{(j)}{(i)} q^b(i-1)/2(-1)^i (i)_q^m = \begin{cases} (1 - q)^{n - j - k} q^{b(n/2) - b(k/2) - b(k/2)} (j) q^b(k) q^n/(n)_q^m & m = j + k - n \\ 0 & m < j + k - n \end{cases}
\]

We first give a lemma, and then the proof of the proposition will follow. Let \([t^j]p(t)\) denote the coefficient of \( p \) corresponding to the monomial \( t^j \).

**Lemma 4.1.18.** Fix \( p \in \mathbb{C}_j[x] \). We have the following identity:

\[
(-1)^j q^{-j/2} \cdot [t^j]p(t) = \sum_{i=0}^{j} p((i)_q^{-1}) \frac{(-1)^i}{(i)_q^{j-i}} q^{i/2}
\]
Proof. Using Lagrange interpolation, the following holds for any polynomial of degree at most \( j \):

\[
[t^j]p(t) = \sum_{i=0}^{j} p((i)_q) \frac{(-1)^{j-i}}{(i)_q!(j-i)_q!} q^{-i} q^{i(j-i)}
\]

Using the identity \((i)_q^{-1} = q^{-j}(i)_q! \) (via \((i)_q^{-1} = q^{-i+1}(i)_q \)) and replacing \( q \) by \( q^{-1} \) gives:

\[
[t^j]p(t) = \sum_{i=0}^{j} p((i)_q^{-1}) \frac{(-1)^{j-i}}{(i)_q! (j-i)_q!} q^{2j} q^{j(j-i)} q^{(j+i)}
\]

\[
= \sum_{i=0}^{j} p((i)_q^{-1}) \frac{(-1)^{j-i}}{(i)_q! (j-i)_q!} q^{(i)} q^{(j)}
\]

The result follows. \( \square \)

**Proof of Proposition 4.1.17.** Consider the polynomial \( p(t) = t^m((n)_q - t)((n-1)_q - t) \cdots ((k+1)_q - t) \), which is of degree \( m + n - k \leq j \). So, \([t^j]p(t) = (-1)^{n-k} q^{-m}\delta_{m=j+k-\gamma} \). Also, recall the identity \((i)_q^{-b} = q^{-b(i)} (i)_q^b \). Using the previous lemma and replacing \( q \) by \( q^b \), we obtain:

\[
(-1)^j q^{-b(i)} (-1)^{n-k} q^{-m} \delta_{m=j+k-\gamma} = \sum_{i=0}^{j} p((i)_q^{-b}) \frac{(-1)^j}{(i)_q^b! (j-i)_q^b!} q^{(i)} q^{(j)}
\]

\[
= \sum_{i=0}^{j} q^{-b(n-k)} (n-i)_q^{-b}! \frac{(-1)^j}{(k-i)_q^{-b}! (i)_q^b! (j-i)_q^b!} q^{b(i)} q^{(i)}
\]

\[
= \sum_{i=0}^{j} q^{b(i)} q^{-b(i)} (n-i)_q^b! \frac{(-1)^j}{(k-i)_q^b! (i)_q^b! (j-i)_q^b!} q^{b(i)} q^{(i)}
\]

\[
= \sum_{i=0}^{j} q^{b(i)} q^{-b(i)} (n)_q^b \frac{q^b (i)_q^b}{(j)_q^b (k)_q^b} \frac{(j)_q^b (k)_q^b}{(i)_q^b} (n)_q^b (-1)^j (i)_q^{-b} m
\]

The result follows. \( \square \)

**Tail of the Series**

For \( n + m - j - k > 0 \), we show that the tail of the infinite series in Lemma 4.1.16 is bounded by a geometric series in \( \epsilon \to 0 \) as \( q \to 1 \). The proof, is somewhat similar to the discussion of convergence in the proof of Lemma 4.1.15.

**Lemma 4.1.19.** Fix \( b \geq 0 \), \( M > 0 \), and \( j, k, n \in \mathbb{N}_0 \) such that \( j \leq k \leq n \). For \( |x| \leq M \), there exists \( \gamma > 0 \) such that the following bound holds for \( q \in (1 - \gamma, 1 + \gamma) \):

\[
\left| \sum_{m=j+k-n}^{\infty} \frac{\nu_m}{m!} a^m_q (1 - q)^{n+j-k} \sum_{i=0}^{j} \frac{(j)_q^b (k)_q^b}{(i)_q^b} q^{b(i-1)/2} (-1)^i (i)_q^{-b} \right| \leq c_0 c_1 \sum_{m=1}^{\infty} c_2^m |1 - q|^m
\]
Here, \( c_0, c_1, c_2 \) are independent of \( q \).

**Proof.** Fix \( n+m-j-k>0 \) with \( j \leq k \leq n \) and \( q \approx 1 \). We have the following bound, where \( c_0 \) is some positive constant independent of \( q \):

\[
\left| \sum_{i=0}^{j} \binom{j}{i} q^{i} \binom{n}{i} q^{b(i-1)/2} (-1)^{i} (i)_{q-b}^{m} \right| \leq \sum_{i=0}^{j} c_{0}(i+\delta)^{m} \leq c_{0}(n+\delta)^{m+1}
\]

For \( |x| \leq M \), we have:

\[
|\nu_{b}^{m}| = |x(x+b) \cdots (x+(m-1)b)| \leq |M(M+b) \cdots (M+(m-1)b)| \leq m!(M+b)^{m}
\]

This then implies the following bound on the tail. Let \( c_{1} := (n+\delta)(1+\delta)(M+b)(n+\delta)^{j+k-n} \)

and \( c_{2} := (1+\delta)(M+b)(n+\delta) \), where small \( \delta > 0 \) is needed to deal with limiting details.

\[
\left| \sum_{m=j+k+1-n}^{\infty} \frac{\nu_{b}^{m}}{m!} a_{q,b}^{m}(1-q)^{n+m-j-k} \sum_{i=0}^{j} \binom{j}{i} q^{i} \binom{n}{i} q^{b(i-1)/2} (-1)^{i} (i)_{q-b}^{m} \right|
\]

\[
\leq \sum_{m=j+k+1-n}^{\infty} \frac{\nu_{b}^{m}}{m!} a_{q,b}^{m}(1-q)^{n+m-j-k} c_{0}(n+\delta)^{m+1}
\]

\[
\leq c_{0} \sum_{m=j+k+1-n}^{\infty} (1+\delta)^{m}(M+b)^{m}|1-q|^{n+m-j-k}(n+\delta)^{m+1}
\]

\[
\leq c_{0} c_{1} \sum_{m=j+k+1-n}^{\infty} [(1+\delta)(M+b)(n+\delta)|1-q|]^{n+m-j-k}
\]

\[
= c_{0} c_{1} \sum_{m=1}^{\infty} c_{2}^{m} |1-q|^{m}
\]

So, for any \( \epsilon > 0 \) we can select \( q \) close enough to 1 such that \( |1-q| < \frac{\epsilon}{c_{2}} \). This implies the above series is geometric with terms bounded by \( \epsilon^{m} \).

The above lemma in particular demonstrates that the tail of the series in Lemma 4.1.16 converges to 0 uniformly on compact sets. With this, we can now complete the proof of the theorem.

**Proof of Theorem 4.0.1.** For \( j,k,n \in \mathbb{N}_{0} \) such that \( 0 \leq j \leq k \leq n \), we can combine the above results. When we expand our limit as a sum of powers of \( (1-q) \), we have shown that everything limits to zero except for the constant term. Recall that \( \lim_{q \to 1} \alpha_{q,b} = -1 \).

\[
\lim_{q \to 1} (1-q)^{n} [\nu_{q,b}^{j} \boxtimes_{q}^{n} \nu_{q,b}^{k}] (q^{x}) = \lim_{q \to 1} \frac{\nu_{b}^{j+k-n} \alpha_{q,b}^{j+k-n}}{(j+k-n)!} (-1)^{j+k-n} q^{b(n_{2})-b(j)+b(k)} \frac{(j)! \phi!(k)!}{(n)! \phi!}
\]

\[
= \frac{j! k!}{n!(j+k-n)!} \nu_{b}^{j+k-n}
\]

\[
= \nu_{b}^{j} \boxtimes_{b}^{n} \nu_{b}^{k}
\]
By symmetry, this demonstrates the desired result on a basis. Therefore, the proof is complete.

Applications To Previous Results

The main motivation for the multiplicative to additive convolution connection was to be able to relate seemingly analogous root information results. The following table outlines the results we proceed to connect.

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All of these connections have a similar flavor, and rely on the following elementary facts about exponential polynomials. We say $f(x) = \sum_{k=0}^{n} c_k q^{kx}$ is an exponential polynomial of degree $n$ with base $q$. A real number $x$ is a root of $f$ if and only if $q^x$ is a root of $\sum_{k=0}^{n} c_k x_k$. Because of this we can bootstrap the fundamental theorem of algebra.

**Definition 4.1.20.** We call $\{ x \in \mathbb{C} : -\pi/\ln(q) < \text{Im}(x) < \pi/\ln(q) \}$ the principal strip (with respect to $q$). Let $p(q^x)$ be an exponential polynomial of degree $n$ with base $q$. The number of roots of $p(q^x)$ in the principal strip is the same as the number of roots of $p$ in $\mathbb{C} \setminus (-\infty, 0]$. We call the roots in the principal strip the principal roots.

**Lemma 4.1.21.** The principal roots of $E_{q,b}(p)[q^x]$ converge to the roots of $p$ as $q \to 1$. In particular, $E_{q,b}(p)[q^x]$ has $\deg(p)$ principal roots for $q \approx 1$.

**Proof.** This follows from the fact that, as $q \to 1$, $E_{q,b}(p)[q^x]$ converges uniformly on compact sets to $p$ and the principal strip grows towards the whole plane.

We can analyze the behavior of this convergence when $p$ is real rooted with distinct roots.

**Lemma 4.1.22.** Suppose $p$ is real with real distinct roots. For $q \approx 1$, we have that $E_{q,b}(p)[q^x]$ has principal roots which are real and distinct (and converging to the roots of $p$).

**Proof.** Since $p$ has real coefficients, the roots of $E_{q,b}(p)[q^x]$ are either real or come in conjugate pairs. (Consider the fact that $q^x = \overline{q}^{-x}$.) If $p$ has real distinct roots, the previous lemma implies the principal roots of $E_{q,b}(p)[q^x]$ have distinct real part for $q$ close enough to 1. Therefore, the principal roots of $E_{q,b}(p)[q^x]$ must all be real.

If we exponentiate (with base $q$) the principal roots of $E_{q,b}(p)[q^x]$, we get the roots of $E_{q,b}(p)$. So if the principal roots of $E_{q,b}(p)[q^x]$ are real, then the roots of $E_{q,b}(p)$ are positive. Considering the above results, this means that $E_{q,b}$ maps polynomials with distinct real roots to polynomials with distinct positive roots for $q \approx 1$. (In fact, the roots will be near 1.)
Root Preservation

The most classical results about the roots are the following:

**Theorem** (Root Preservation).

- If \( p, r \in \mathbb{R}_n[x] \) have positive roots, then \( p \boxast^n r \) has positive roots.
- If \( p, r \in \mathbb{R}_n[x] \) have real roots, then \( p \boxplus^n r \) has real roots.

Neither of these results are particularly hard to prove, but showing how the additive result follows from the multiplicative serves as a prime example of how our theorem connects results on the roots.

**Proof of Additive from Multiplicative.** We can reduce to showing that the additive convolution preserves real rooted polynomials with distinct roots since the closure of polynomials with distinct real roots is all real rooted polynomials.

By Lemma 4.1.22, the roots of \( E_{q,0}(p) \) are real, distinct, and exponentials of the principal roots of \( E_{q,0}[p](q^x) \) for \( q \approx 1 \). This implies that \( E_{q,0}(p) \) has positive real roots. By the multiplicative result, \( E_{q,0}(p) \boxast^n E_{q,0}(r) \) has positive real roots, and therefore \([E_{q,0}(p) \boxast^n E_{q,0}(r)](q^x)\) has real principal roots. By our main result, \((1 - q)^n[E_{q,0}(p) \boxast^n E_{q,0}(r)](q^x)\) converges to \( p \boxplus^n r \). The real-rootedness of \([E_{q,0}(p) \boxast^n E_{q,0}(r)](q^x)\) for \( q \approx 1 \) then implies \( p \boxplus^n r \) is real-rooted.

Triangle Inequality

The next classical theorem relates to the max root of a given polynomial. Given a real-rooted polynomial \( p \), let \( \lambda(p) \) denote its max root. Given an exponential polynomial \( f \) with principal roots all real, let \( \lambda(f) \) denote the largest principal root of \( f \). Also, denote \( \exp_q(\alpha) := q^\alpha \).

**Theorem** (Triangle Inequalities).

- Given positive-rooted polynomials \( p, r \) we have \( \lambda(p \boxast^n r) \leq \lambda(p) \cdot \lambda(r) \)
- Given real-rooted polynomials \( p, r \) we have \( \lambda(p \boxplus^n r) \leq \lambda(p) + \lambda(r) \)

As before, neither of these have particularly complicated proofs, but we can use the multiplicative result to deduce the additive result in the following.

**Proof of Additive from Multiplicative.** As in the previous proof, we can reduce to showing that the result holds for \( p, r \) with distinct roots. For this proof, we only consider \( q > 1 \).

By Lemma 4.1.22, we have that the roots of \( E_{q,0}(p) \) are real, distinct, and exponentials of the principal roots of \( E_{q,0}[p](q^x) \) for \( q \approx 1 \). This implies the roots of \( E_{q,0}(p) \) are positive for \( q \approx 1 \). Additionally, notice that \( \exp_q(\lambda(f(q^x))) = \lambda(f(p)) \) whenever \( f \) is positive-rooted.
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From the multiplicative result and the fact that $\boxtimes$ preserves positive-rootedness, we have the following for $q \approx 1$:

\[
\exp_q(\lambda([E_{q,0}(p) \boxtimes^n E_{q,0}(r)](q^x))) = \lambda(E_{q,0}(p) \boxtimes^n E_{q,0}(r))
\leq \lambda(E_{q,0}(p)) \cdot \lambda(E_{q,0}(r))
= \exp_q(\lambda(E_{q,0}[p](q^x)) + \lambda(E_{q,0}[r](q^x)))
\]

Therefore, $\lambda([E_{q,0}(p) \boxtimes^n E_{q,0}(r)](q^x)) \leq \lambda(E_{q,0}[p](q^x)) + \lambda(E_{q,0}[r](q^x))$. By our main result, $(1 - q)^n[E_{q,0}(p) \boxtimes^n E_{q,0}(r)](q^x)$ converges to $p \boxplus^n r$, and therefore $\lambda([E_{q,0}(p) \boxtimes^n E_{q,0}(r)](q^x))$ converges to $\lambda(p \boxplus^n r)$. Similarly $\lambda(E_{q,0}[p](q^x))$ converges to $\lambda(p)$, and the result follows.

Application to Mesh Preservation Conjecture

Recall the log mesh result of Lamprecht in [16] regarding the $q$-multiplicative convolution.

**Theorem 1.3.4.** Fix $q > 1$. Given positive-rooted polynomials $p, r \in \mathbb{R}_n[x]$ with $\text{lmesh}(p), \text{lmesh}(r) \geq q$, we have:

\[
\text{lmesh}(p \boxtimes_q^n r) \geq q
\]

In [17], Brändén, Krasikov, and Shapiro conjectured the analogous result for the $b$-additive convolution (for $b = 1$). Using our connection we will confirm this conjecture:

**Theorem 1.3.5.** Given real-rooted polynomials $p, r \in \mathbb{R}_n[x]$ with $\text{mesh}(p), \text{mesh}(r) \geq b$, we have:

\[
\text{mesh}(p \boxplus_b^n r) \geq b
\]

**Proof.** We will prove this claim for polynomials $p, r$ with $\text{mesh}(p), \text{mesh}(r) > b$. Since we can approximate any polynomial with $\text{mesh}(p) = b$ by polynomials with larger mesh, the result then follows.

By Lemma 4.1.22, $E_{q,b}[p](q^x)$ has real roots which converge to the roots of $p$ for $q \approx 1$. Since the roots of $p$ satisfy $\text{mesh}(p) > b$, the principal roots of $E_{q,b}[p](q^x)$ will have mesh greater than $b$ for $q \approx 1$. Further, $\text{lmesh}(E_{q,b}(p)) = \exp_q(\text{mesh}(E_{q,b}[p](q^x))) > q^b$. (All of this discussion holds for $r$ as well.) By our main result, we have:

\[
\lim_{q \to 1}(1 - q)^n \left[E_{q,b}(p) \boxtimes_q^n E_{q,b}(r)\right](q^x) = p \boxplus_b^n r
\]

By the previous theorem, the $q^b$-multiplicative convolution of $E_{q,b}(p)$ and $E_{q,b}(r)$ has logarithmic mesh at least $q^b$. Precomposition by $q^x$ then yields an exponential polynomial with mesh (of the principal roots) at least $b$. The principal roots of this exponential polynomial then converge to $p \boxplus_b^n r$, and hence $p \boxplus_b^n r$ has mesh at least $b$. □
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4.2 Second Proof Method: A Direct Proof using Interlacing

While the previous framework generically transferred Lamprecht’s multiplicative result to prove the conjectured result in the additive realm, one might desire a direct proof to gain insight on the underlying structure of the convolution. In what follows, we first outline the preliminary knowledge required to understand a special case of Lamprecht’s argument. Then we outline his approach in the multiplicative case and extend this approach to the additive realm to prove the desired conjecture.

Lamprecht’s Approach

In what follows, we follow Lamprecht’s approach to proving that the space of $q$-log mesh polynomials is preserved by the $q$-multiplicative convolution. Here, we are only interested in proving this result for $q$-log mesh polynomials with non-negative roots, which simplifies the proof. (Lamprecht demonstrates this result for a more general class of polynomials.) The main structure of the proof is: (1) establish properties of two distinguished polar derivatives, (2) show how these derivatives relate to the $q$-multiplicative convolution, and (3) use this to prove that the $q$-multiplicative convolution preserves certain interlacing properties. In the next section, we will emulate this method for $b$-mesh polynomials and the $b$-additive convolution.

$q$-Polar Derivatives

In [16], Lamprecht defines the following $q$-derivative operators, which generalize the operators $\partial_x$ and $-\partial_y$ on homogeneous polynomials. Here, $q > 1$ is always assumed, as above. (As a note, Lamprecht uses the $\Delta$ symbol for these derivatives, and actually gives different definitions as his convention is $q \in (0, 1)$.)

$$(d_{q,n}f)(x) := \frac{f(qx) - f(x)}{q^{1-n}(q^n - 1)x} \quad (d^*_{q,n}f)(x) := \frac{f(qx) - q^n f(x)}{q^n - 1}$$

He then goes on to show that these “derivative” operators have similar preservation properties to that of the usual derivatives. In particular, he obtains the following.

**Proposition 4.2.1.** The operators $d_{q,n} : \mathbb{R}_n[x] \to \mathbb{R}_{n-1}[x]$ and $d^*_{q,n} : \mathbb{R}_n[x] \to \mathbb{R}_{n-1}[x]$ preserve the space of $q$-log mesh polynomials and the space of strictly $q$-log mesh polynomials. Further, we have that $d_{q,n}f \ll f$ and $d^*_{q,n}f \ll d_{q,n}f$.

The above result is actually spread across a number of results in Lamprecht’s paper. We omit the proof for now, referring the reader to Section 6 in [16], mainly Theorems 25 and 28 (in the arXiv version). Note also that our definitions of $d_{q,n}$ and $d^*_{q,n}$ are slightly different than that of Lamprecht.
We prove the theorem by induction. For \( n = 1 \) the result is straightforward, as \( \mathbb{R}^1 \equiv \mathbb{R}^1 \). For \( m > 1 \), we inductively assume that the result holds for \( n = m-1 \). By Corollary 1.4.7 and the fact that \( f \) has \( n \) simple roots, we only need to show that \( T_g[f_{\alpha_k}] \ll T_g[f] \) for all roots \( \alpha_k \) of \( f \). That is, we want to show \( f_{\alpha_k} \ll m g \ll f \ll q,m g \) for all \( k \).

By Proposition 4.2.1, we have that \( d_{q,m} g \) and \( d_{q,m}^* g \) are q-log mesh and \( d_{q,m}^* g \ll d_{q,m} g \). Further, \( d_{q,m} g \) and \( d_{q,m}^* g \) are of degree \( m-1 \) and have no roots at 0. The inductive hypothesis and symmetry of \( \boxtimes_q \) then imply:

\[
 f_{\alpha_k} \ll m d_{q,m}^* g \ll f_{\alpha_k} \ll m d_{q,m} g
\]

The fact that these polynomial have leading coefficients with the same sign means that the max root of \( f_{\alpha_k} \ll m d_{q,m} g \) is larger than that of \( f_{\alpha_k} \ll m d_{q,m}^* g \). Further, since all roots are positive we obtain:

\[
 f_{\alpha_k} \ll m d_{q,m}^* g \ll x(f_{\alpha_k} \ll m d_{q,m} g)
\]

By properties of \( \ll \), this gives:

\[
 f_{\alpha_k} \ll m d_{q,m} g \ll x(f_{\alpha_k} \ll m d_{q,m} g) - \alpha_k(f_{\alpha_k} \ll m d_{q,m} g)
\]

By the above identities and the fact that \( f(x) = (x - \alpha_k)f_{\alpha_k}(x) \), this is equivalent to \( f_{\alpha_k} \ll m g \ll f \ll m g \). \( \square \)

**Corollary 1.3.4.** Let \( f, g \in \mathbb{R}_n[x] \) be q-log mesh polynomials (with non-negative roots), not necessarily of degree \( n \). Then, \( f \ll q,m g \) is q-log mesh.

**Proof.** First suppose \( f, g \) are of degree \( n \) with only positive roots. Since \( f \ll f(q^{-1}x) \), the previous theorem implies:

\[
 f \ll n g \ll f(q^{-1}x) \ll n g = (f \ll n g)(q^{-1}x)
\]
That is, \( f \boxtimes_q^n g \) is \( q \)-log mesh.

Otherwise, suppose \( f \) is of degree \( m_f \leq n \) with \( z_f \) roots at 0 and \( g \) is of degree \( m_g \leq n \) with \( z_g \) roots at zero. Intuitively, we now add roots “near 0 and \( \infty \)” and limit. Let new polynomials \( F \) and \( G \) be given as follows:

\[
F(x) := f(x) \cdot x^{-z_f} \prod_{j=1}^{z_f} \left( x - \frac{1}{\alpha_j} \right) \cdot \prod_{j=m_f+1}^{n} \left( \frac{x}{\alpha_j} - 1 \right)
\]

\[
G(x) := g(x) \cdot x^{-z_g} \prod_{j=1}^{z_g} \left( x - \frac{1}{\beta_j} \right) \cdot \prod_{j=m_g+1}^{n} \left( \frac{x}{\beta_j} - 1 \right)
\]

Here, \( \alpha_j \) and \( \beta_j \) are any large positive numbers such that \( F \) and \( G \) are \( q \)-log mesh polynomials of degree \( n \). By the previous argument, \( F \boxtimes_q^n G \) is \( q \)-log mesh. Letting \( \alpha_j \) and \( \beta_j \) limit to \( \infty \) (while preserving \( q \)-log mesh) implies \( F \boxtimes_q^n G \rightarrow f \boxtimes_q^n g \) root-wise, which implies \( f \boxtimes_q^n g \) is \( q \)-log mesh.

Lamprecht is actually able to remove the degree \( n \) with positive roots restriction earlier in the line of argument, albeit at the cost of a more complicated proof. We have elected here to take the simpler route. He also proves similar results for a class of \( q \)-log mesh polynomials with possibly negative roots, which we omit here.

**b-Additive Convolution**

The main structure of Lamprecht’s argument revolves around the two “polar” \( q \)-derivatives, \( d_{q,n} \) and \( d_{q,n}^* \). The key properties of these derivatives are: (1) they preserve the space of \( q \)-log mesh polynomials, and (2) they recursively work well with the definition of the \( q \)-multiplicative convolution. So, when extending this argument to the \( b \)-additive convolution we face an immediate problem: there is only one natural derivative which preserves the space of \( b \)-mesh polynomials. This stems from the fact that 0 and \( \infty \) have special roles in the \( q \)-multiplicative world, whereas only \( \infty \) is special in the \( b \)-additive world. The key idea we introduce then is that given a fixed \( b \)-mesh polynomial \( f \), we can pick a polar derivative with pole “close enough to \( \infty \)” so that it maps \( f \) to a \( b \)-mesh polynomial. The fact that we use a different polar derivative for each fixed input \( f \) does not affect the proof method.

We now give a few facts about the finite difference operator \( \Delta_b \), which plays a crucial role in the definition of the \( b \)-additive convolution. Recall its definition:

\[
(\Delta_{b,n} f)(x) \equiv (\Delta_b f)(x) := \frac{f(x) - f(x - b)}{b}
\]

(We use the notation \( \Delta_{b,n} \) when we want to restrict the domain to \( \mathbb{R}_n[x] \), as in Proposition 4.2.3 below.) This operator acts on rising factorial polynomials as the usual derivative acts on monomials. That is, for all \( k \):

\[
\Delta_b x(x + b) \cdots (x + (k - 1)b) = k x(x + b) \cdots (x + (k - 2)b)
\]
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This operator has preservation properties similar to that of the usual derivative and the $q$-derivatives. The following result, along with many others regarding mesh and log-mesh polynomials, can be found in Chapter 8 of [65].

**Proposition 4.2.3.** [Fisk: Lemma 8.8] The operator $\Delta_{b,n} : \mathbb{R}_n[x] \rightarrow \mathbb{R}_{n-1}[x]$ preserves the space of $b$-mesh polynomials, and the space of strictly $b$-mesh polynomials. Further, we have $\Delta_b f \ll f$ and $\Delta_b f \ll f(x - b)$. If $f$ is strictly $b$-mesh, then these interlacings are strict.

**Finding Another Polar Derivative**

We now define another “derivative-like” operator that is meant to generalize $\partial_y$ and $d_{q,n}^*$. Notice that unlike $\Delta_b$, this operation depends on $n$.

$$(\Delta_{b,n}^* f)(x) := nf(x - b) - (x - b)\Delta_b f(x)$$

Unfortunately, this operator does not preserve $b$-mesh. However, it does generalize other important properties of $\partial_y$. In particular, it maps $\mathbb{R}_n[x]$ to $\mathbb{R}_{n-1}[x]$, and as $b \to 0$ it limits to $\partial_y f$, the polar derivative of $f$ with respect to $0$. Further, we have the following results.

**Lemma 4.2.4.** Fix $f \in \mathbb{R}_n[x]$ and write $f = \sum_{k=0}^{n} a_k x(x + b) \cdots (x + (k - 1)b)$. Then:

$$(\Delta_{b,n}^* f)(x + b) = \sum_{k=0}^{n-1} (n - k)a_k x(x + b) \cdots (x + (k - 1)b)$$

This next lemma is a generalization of the corollary following it.

**Lemma 4.2.5.** Fix monic polynomials $f, g \in \mathbb{R}[x]$ of degree $m$ and $m - 1$, respectively, such that $g$ is strictly $b$-mesh and $g \ll f$ strictly. Denote $h_{a,t}(x) := af(x) - (x - t)g(x)$ for $a \geq 1$ and $t > 0$. For all $t$ large enough, we have $g \ll h_{a,t}$ strictly, $h_{a,t} \ll f$ strictly, and $h_{a,t}$ is strictly $b$-mesh.

**Proof.** Denote $h_{a,t}(x) := af(x) - (x - t)g(x)$. Since $f, g$ are monic, we have that $h_{a,t}$ is of degree at most $m$ with positive leading coefficient (for large $t$ if $a = 1$). Further, if $\alpha_1 < \cdots < \alpha_{m-1}$ are the roots of $g$ and $\beta_1 < \cdots < \beta_m$ are the roots of $f$, then $g \ll f$ strictly and $t$ large implies:

$$h_{a,t}(\alpha_{m-1}) = af(\alpha_{m-1}) < 0 \quad \quad h_{a,t}(\beta_m) = -(\beta_m - t)g(\beta_m) > 0$$
$$h_{a,t}(\alpha_{m-2}) = af(\alpha_{m-2}) > 0 \quad \quad h_{a,t}(\beta_{m-1}) = -(\beta_{m-1} - t)g(\beta_{m-1}) < 0$$
$$h_{a,t}(\alpha_{m-3}) = af(\alpha_{m-3}) < 0 \quad \quad h_{a,t}(\beta_{m-2}) = -(\beta_{m-2} - t)g(\beta_{m-2}) > 0$$
$$\vdots \quad \quad \vdots$$

The alternating signs imply $h_{a,t}$ has an odd number of roots in the interval $(\alpha_k, \beta_{k+1})$ and an even number of roots in the interval $(\beta_k, \alpha_k)$ for all $1 \leq k \leq m - 1$. Since the degree of $h_{a,t}$ is at most $m$, each of these intervals must contain exactly one root and zero roots,
respectively. If \( h_{a,t} \) is of degree \( m \), then it has one more root which must be real since \( h_{a,t} \in \mathbb{R}[x] \). Additionally, since \( h_{a,t} \) has positive leading coefficient, this last root must lie in the interval \((-\infty, \beta_1)\) (and not in \((\beta_m, \infty)\)). Therefore, \( g \ll h_{a,t} \) strictly and \( h_{a,t} \ll f \) strictly.

Finally, \( h_{a,t} \to g \) as \( t \to \infty \) coefficient-wise, and so therefore also in terms of the zeros. This means that the root in the interval \((\alpha_k, \beta_{k+1})\) will limit to \( \alpha_k \) from above (for all \( k \)). Further, the possible root in the interval \((-\infty, \beta_1)\) will then limit to \(-\infty\), as \( g \) is of degree \( m - 1 \). Since \( g \) is strictly \( b \)-mesh, this implies \( h_{a,t} \) is also strictly \( b \)-mesh for large enough \( t \).

**Corollary 4.2.6.** Let \( f \in \mathbb{R}_n[x] \) be strictly \( b \)-mesh. Then for all \( t > 0 \) large enough, we have that \((t\Delta_{b,n} + \Delta^*_{b,n})f\) is strictly \( b \)-mesh and \( \Delta_{b,n}f \ll (t\Delta_{b,n} + \Delta^*_{b,n})f \) strictly.

**Proof.** Consider \((t\Delta_{b,n} + \Delta^*_{b,n})f = nf(x-b) - (x-b-t)\Delta_{b,n}f\). Note that \( \Delta_{b,n}f \in \mathbb{R}_{n-1}[x] \) is strictly \( b \)-mesh and of degree one less than \( f \), and \( \Delta_{b,n}f \ll f(x-b) \) strictly by Proposition 4.2.3. Now assume WLOG that \( f \) is monic and of degree at least 1. Letting \( c \) denote the leading coefficient of \( \Delta_{b,n}f \), we have \( 1 \leq c \leq n \). We can then write:

\[
\frac{1}{c}(t\Delta_{b,n} + \Delta^*_{b,n})f = \frac{n}{c}f(x-b) - (x-b-t)\frac{\Delta_{b,n}f}{c}
\]

Applying the previous lemma to \( f(x-b) \) and \( \frac{\Delta_{b,n}f}{c} \) with \( a = \frac{n}{c} \) gives the result.

This corollary says that \( t\Delta_{b,n} + \Delta^*_{b,n} \) preserves \( b \)-mesh, even though \( \Delta^*_{b,n} \) does not. The operator \( t\Delta_{b,n} + \Delta^*_{b,n} \) can be thought of as the polar derivative with respect to \( t \), since by limiting \( b \to 0 \) we obtain the classical polar derivative.

**Recursive Identities**

The \( \Delta^*_{b,n} \) operator is also required to obtain \( b \)-additive convolution identities similar to Lamprecht’s given above.

**Lemma 4.2.7.** Fix \( f \in \mathbb{R}_{n-1}[x] \) and \( g \in \mathbb{R}_n[x] \). We have:

\[
f \boxplus^ng = f \boxplus^{n-1}_b \Delta_{b,n}g \quad \quad (xf) \boxplus^ng = x(f \boxplus^{n-1}_b \Delta_{b,n}g) + f \boxplus^{n-1}_b \Delta^*_{b,n}g
\]

**Proof.** The first identity is straightforward from the definition of \( \boxplus^ng \). As for the second, we compute:

\[
\Delta^k_b(xf) = \Delta^{k-1}_b(x\Delta_b f + f(x-b)) = \cdots = x\Delta^k_b f + k\Delta^{k-1}_b f(x-b)
\]
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Notice that $\Delta_b$ commutes with shifting, so this is unambiguous. This implies:

$$(xf) \boxplus_b^n g = \sum_{k=0}^{n} (x\Delta_b^k f + k\Delta_b^{k-1} f(x-b)) \cdot (\Delta_b^{n-k} g)(0)$$

$$= x(f \boxplus_b^{n-1} \Delta_{b,n} g) + \sum_{k=1}^{n} k\Delta_b^{k-1} f(x-b) \cdot (\Delta_b^{n-k} g)(0)$$

$$= x(f \boxplus_b^{n-1} \Delta_{b,n} g) + \sum_{k=0}^{n-1} \Delta_b^{n-1-k} f(x-b) \cdot ((n-k)\Delta_b^k g)(0)$$

$$= x(f \boxplus_b^{n-1} \Delta_{b,n} g) + f(x-b) \boxplus_b^{n-1} (\Delta_{b,n}^* g)(x+b)$$

The last step of the above computation uses Lemma 4.2.4 and the fact that $(\Delta_b^k g)(0)$ picks out the coefficient corresponding to the $k^{th}$ rising factorial term. Finally:

$$f(x-b) \boxplus_b^{n-1} (\Delta_{b,n}^* g)(x+b) = (f \boxplus_b^{n-1} (\Delta_{b,n}^* g)(x+b))(x-b) = f \boxplus_b^{n-1} \Delta_{b,n}^* g$$

This implies the second identity. \qed

With this we can now emulate Lamprecht’s proof to prove interlacing preserving properties of the $b$-additive convolution.

Lamprecht-Style Proof

Theorem 4.2.8. Let $f, g \in \mathbb{R}_n[x]$ be strictly $b$-mesh polynomials of degree $n$. Let $T_g : \mathbb{R}_n[x] \rightarrow \mathbb{R}_n[x]$ be the real linear operator defined by $T_g : r \mapsto r \boxplus_b^n g$. Then, $T_g$ preserves interlacing with respect to $f$.

Proof. We prove the theorem by induction. For $n = 1$ the result is straightforward, as $\boxplus_b^1 \equiv \boxplus^1$. For $m > 1$, we inductively assume that the result holds for $n = m - 1$. By Corollary 1.4.7, we only need to show that $T_g[f_{\alpha_k}] \ll T_g[f]$ for all roots $\alpha_k$ of $f$. That is, we want to show $f_{\alpha_k} \boxplus_b^m g \ll f \boxplus_b^m g$ for all $k$.

By Proposition 4.2.3 and Corollary 4.2.6, we have that $\Delta_{b,m} g$ and $(t\Delta_{b,m} + \Delta_{b,m}^*) g$ are strictly $b$-mesh and $\Delta_{b,m} g \ll (t\Delta_{b,m} + \Delta_{b,m}^*) g$ strictly for large enough $t$. Further, $\Delta_{b,m} g$ and $(t\Delta_{b,m} + \Delta_{b,m}^*) g$ are of degree $m - 1$. The inductive hypothesis and symmetry of $\boxplus_b^n$ then imply:

$$f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g \ll f_{\alpha_k} \boxplus_b^{m-1} (t\Delta_{b,m} + \Delta_{b,m}^*) g$$

It is easy to see, (e.g. from 1.4.5)

$$f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g \ll (x - \alpha_k - t)(f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g)$$

By properties of $\ll$, this gives:

$$f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g \ll (x - \alpha_k - t)(f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g) + f_{\alpha_k} \boxplus_b^{m-1} (t\Delta_{b,m} + \Delta_{b,m}^*) g$$

$$= (x - \alpha_k)(f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g) + f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m}^* g$$
By the above identities and the fact that \( f(x) = (x - \alpha_k)f_{\alpha_k}(x) \), this is equivalent to 
\[ f_{\alpha_k} \boxplus_b^m g \ll f \boxplus_b^m g. \]

**Corollary 1.3.5.** Let \( f, g \in \mathbb{R}_n[x] \) be strictly \( b \)-mesh polynomials. Then, \( f \boxplus_b^n g \) is \( b \)-mesh.

**Proof.** First suppose \( f, g \) are of degree \( n \). Since \( f \ll f(x - b) \) strictly, the previous theorem implies the following:
\[ f \boxplus_b^n g \ll f(x - b) \boxplus_b^n g = (f \boxplus_b^n g)(x - b) \]
That is, \( f \boxplus_b^n g \) is \( b \)-mesh.

If \( f, g \) are not of degree \( n \), then the result follows by adding new roots and limiting them to \( \infty \), in a fashion similar to the proof of Corollary 1.3.4 given above. \( \square \)

### 4.3 Conclusion

**Extensions of other classical convolution results**

In this chapter we investigate connections between the additive and multiplicative convolutions and their mesh generalizations. Looking forward, it is natural to look at other results in the classical case and ask for mesh generalizations. To the authors knowledge, there are two classical results which have been extended to mesh analogues: in [17], the authors explore extensions of the Hermite-Poulain theorem to the 1-mesh world, and in [16], Lamprecht extends classical results for the multiplicative convolution to the \( q \)-log mesh world.

An important related result in the classical case is the triangle inequality, which we discuss in §4.1. To our knowledge, there is not a known generalization of the triangle inequality to the mesh and log mesh cases. If one could establish such a result for the \( q \)-multiplicative convolution, it would automatically extend to the \( b \)-additive convolution using our analytic connection. Establishing this is the first step towards potentially getting a grasp on \( b \)-additive submodularity.

**Extensions of other \( q \)-multiplicative convolution results**

In addition to log mesh preservation, Lamprecht proves other results about the \( q \)-multiplicative convolution. Here we comment on these and their relation to the mesh world.

Beyond the finite degree case, Lamprecht discusses the extension of Laguerre-Polya functions to the \( q \)-multiplicative world, and then establishes a \( q \)-version of Polya-Schur multiplier sequences via a power series convolution. Since we are not aware of analogous power series results for the classical additive convolution, we have not explored the connections to the \( b \)-additive case.

Additionally, Lamprecht classifies log-concave sequences in terms of \( q \)-log mesh polynomials using the Hadamard product and a limiting argument. There might be an analogue
result in the mesh world for concave sequences, but it is unclear what would take the place of the Hadamard product.

Lamprecht details the classes of polynomials that the $q$-multiplicative convolution preserves. Most of these results come from the presence of two poles in the $q$-multiplicative case, yielding derivative operators which preserve negative- and positive-rootedness respectively. The $b$-additive case does not have such complications. In our simplification of Lamprecht’s argument, we assume the input polynomials to be generic (strictly $b$-mesh), and then limit to obtain the result for all $b$-mesh polynomials. By keeping track of boundary case information, Lamprecht is able to get more precise results about boundary elements of the space of $b$-mesh polynomials. We believe it is likely possible to emulate this in the above proof with more bookkeeping.

The analytic connection applied to other known classical results

There are other results known about the classical multiplicative convolution which we believe could be transferred to the additive convolution using our generic framework. Specifically in [26], Marcus, Spielman, and Srivastava establish a refinement of the triangle inequality for both the additive and multiplicative convolutions. These refinements parallel the well studied transforms from free probability theory. We have not yet worked out the details of this connection.

Further directions for the generic analytic connection

Finally, it is worth noting that our analytic connection can only transfer results about the multiplicative convolution to the additive convolution. The main obstruction is finding the appropriate analogue to the exponential map. The following limiting connection between exponential polynomials and polynomials motivated our investigation:

$$\lim_{q \to 1} \frac{1 - q^x}{1 - q} = x$$

Finding the appropriate “logarithmic analogue” could yield a way to pass results from the additive convolution to the multiplicative convolution. That said, some heuristic evidence suggests that such an analogue might not exist.

Above all, our analytic connection still remains rather mysterious. We suspect that there exists a more general theory which provides better intuition for this multiplicative-to-additive connection. While developing this connection, we found multiple candidate exponential maps which experimentally worked. We settled on the ones introduced in this chapter due to their relatively nice combinatorial properties. Ideally, an alternative approach would avoid proving the result on a basis and better explain the role of these “exponential maps”.
Chapter 5

Real Stability Testing

Two reoccurring problems in the modern study of real-rooted polynomials are the following:

**Problem 1.** Given a bivariate polynomial\(^1\) \( p \in \mathbb{R}_n[x, y] \), is \( p \) real stable?

**Problem 2.** Given a linear operator \( T : \mathbb{R}_n[x] \to \mathbb{R}_m[x] \), does \( T \) preserve real-rootedness?

As we have seen in 1.5.2, Problem 1 and Problem 2 are equivalent. The main result of this chapter is a strongly polynomial time algorithm that solves Problem 1.

**Theorem 5.0.1 (Main).** Given the coefficients of a bivariate polynomial \( p \in \mathbb{R}_n[x, y] \), there is a deterministic algorithm which decides whether or not \( p \) is real stable in at most \( O(n^5) \) arithmetic operations, assuming exact arithmetic.

To give the reader a feel for the objects at hand, we remark that the set of real stable polynomials in any number of variables is a nonconvex set with nonempty interior [66]. In the univariate case, the interior of the set of real-rooted polynomials simply corresponds to polynomials with distinct roots, and its boundary contains polynomials which have roots with multiplicity greater than one. With regards to Problem 2, the prototypical example of an operator which preserves real rootedness is differentiation. Recent applications such as [67] rely on finding more elaborate differential operators with this property.

We now describe the main ideas in our algorithm. It turns out that testing bivariate real stability is equivalent to testing whether a certain **two parameter** family of polynomials is real rooted. It is not clear how to check this continuum of real-rootedness statements in strongly polynomial, or even in exponential time. To circumvent this, we use a deep convexity result from the theory of hyperbolic polynomials to reduce the two parameter family to a one parameter family of degree \( n \) polynomials, whose coefficients are themselves polynomials of degree \( n \) in the parameter. We then use a characterization of real-rootedness

\[^1\text{We use } \mathbb{R}_n[x_1, \ldots, x_k] \text{ to denote the vector space of real polynomials in } x_1, \ldots, x_k \text{ of degree at most } n \text{ in each variable.}\]
as positive semidefiniteness of certain moment matrices to further reduce this to checking that a finite number of univariate polynomials are nonnegative an interval. Finally, we solve each instance of the nonnegativity problem using Sturm sequences and a bit of algebra.

The set of polynomials nonnegative on an interval forms a closed convex cone, so the last step of our algorithm may be viewed as a strongly polynomial time membership oracle for this cone. We would not be surprised if such a result is already known (at least as folklore) but we were unable to find a concrete reference in the literature, so this component of our method may be of independent interest.

We see this result as being both mathematically fundamental, as well as useful for researchers who work with stable polynomials, particularly since many of their known applications so far (e.g. [68]) put special emphasis on properties of bivariate restrictions. More speculatively, it is possible that being able to test membership in the set of real stable polynomials is a step towards being able to optimize over them.

Related Work

Problem 1 was solved in the univariate case by C. Sturm in 1835 [49], who described a now well-known method that can be turned into a strongly polynomial quadratic time algorithm given the coefficients of \( p \) [69]. We are unaware of any published work regarding algorithms for the bivariate case or for Problem 2. We remark that Thorsten Theobald has observed (informal communication) that the quantifier elimination techniques of [70] can be used to obtain weakly polynomial time algorithms for Problems 1 and 2.

The paper [71] studied the problem of testing whether a bivariate polynomial is real zero (a special case of real stability). It reduced that problem to testing PSDness of a one-parameter family of matrices which it then suggested could be solved using semidefinite programming, but without quite proving a theorem to that effect. This work is partly inspired by ideas in [71].

The paper [72] gives semidefinite programming based algorithms that can test whether certain restricted classes of multiaffine polynomials are real stable (in more than 2 variables).

The problem of certifying that a univariate polynomial is nonnegative is typically stated (for instance, in lecture notes) as being the solution to a semidefinite program. If one were able to work out the appropriate error to which the SDP has to be solved, this could give a weakly polynomial time algorithm for nonnegativity, which we suspect must be known as folklore. The paper [73] analyzes a semidefinite programming based algorithm in the special case when the polynomial is nondegenerate in an appropriate sense.

5.1 Parameter Reduction via Hyperbolicity

In this section we use the properties of hyperbolic polynomials to reduce real stability of a bivariate polynomial to testing real rootedness of a one parameter family of polynomials.
Theorem 5.1.1 (Reduction to One-Parameter Family). A nonzero bivariate polynomial \( p \in \mathbb{R}_n[x, y] \) is real stable if and only if following two conditions hold:

1. The one-parameter family of univariate polynomials \( q_{\gamma} \in \mathbb{R}[t] \) given by,
   \[ q_{\gamma}(t) = p(\gamma + t, t) \in \mathbb{R}[t] \]
   are real rooted for all \( \gamma \in \mathbb{R} \).

2. The univariate polynomial
   \[ t \mapsto p_H(t, 1 - t, 0) \]
   is strictly positive on the interval \((0, 1)\).

Proof. (real-stability of \( p \) \( \implies \) (1) & (2))

By 1.7.2, \( p_H \) is hyperbolic with respect to the positive orthant \( \mathbb{R}^2_{>0} \times \{0\} \). Since \( (1, 1, 0) \in \mathbb{R}^2_{>0} \times \{0\} \), this implies that for all \( (x, y, z) \in \mathbb{R}^3 \),

\[ q(t) = p_H(x + t, y + t, z) \]

is real-rooted. Setting \( x = \gamma, y = 0 \) and \( z = 1 \) we get that \( q_\gamma(t) = p_H(\gamma + t, t, 1) = p(\gamma + t, t) \) is real-rooted for all \( \gamma \in \mathbb{R} \) which is condition (1). Finally, since

\[ \{(t, 1 - t, 0) | t \in (0, 1)\} \subset \mathbb{R}^2_{>0} \times \{0\} \]

and \( p_H \) is hyperbolic with respect to \( \mathbb{R}^2_{>0} \times \{0\} \), it follows that \( p_H(t, 1 - t, 0) > 0 \) for all \( t \in (0, 1) \).

((1) & (2) \( \implies \) real-stability of \( p \))

First, we claim that the polynomial \( p_H \) is hyperbolic with respect to \((1, 1, 0)\). By (2) we have \( p_H(1/2, 1/2, 0) > 0 \) so homogeneity implies that \( p_H(1, 1, 0) > 0 \). It remains to show that \( q_{x,y,z}(t) = p_H(x + t, y + t, z) \) is real-rooted for all \( (x, y, z) \in \mathbb{R}^3 \). First, consider the case of \( (x, y, z) \in \mathbb{R}^3 \) with \( z \neq 0 \).

\[ \forall (x, y, z) \in \mathbb{R}^3 \text{ with } z \neq 0, p_H(x + t, y + t, z) \text{ is real-rooted} \]

\[ \iff \forall (x, y, z) \in \mathbb{R}^3 \text{ with } z \neq 0, p_H(\frac{x}{z} + \frac{t}{z}, \frac{y}{z} + \frac{t}{z}, 1) \text{ is real-rooted} \]

\[ \iff \forall (x, y, z) \in \mathbb{R}^3 \text{ with } z \neq 0, p_H(\frac{x}{z} + t, \frac{y}{z} + t, 1) \text{ is real-rooted (replacing } t/z \text{ with } t) \]

\[ \iff \forall (x, y) \in \mathbb{R}^2, p_H(x + t, y + t, 1) \text{ is real-rooted} \]

\[ \iff \forall (x, y) \in \mathbb{R}^2, p_H(x + t, t, 1) \text{ is real-rooted (replacing } t \text{ with } t - y) \]

\[ \iff \forall \gamma \in \mathbb{R}, p(\gamma + t, t) \text{ is real-rooted} \]

By Hurwitz’s theorem, the limit of any sequence of real-rooted polynomials is real-rooted. Therefore, if \( q_{x,y,z}(t) \) is real-rooted for all \( (x, y, z) \in \mathbb{R}^3 \) with \( z \neq 0 \) then \( q_{x,y,z}(t) \) is real-rooted for all \( (x, y, z) \in \mathbb{R}^3 \).
Given that $p_H$ is hyperbolic with respect to $e = (\frac{1}{2}, \frac{1}{2}, 0)$, its hyperbolicity cone $K(p_H, e)$ is a convex cone containing $(1, 1, 0)$. Condition (2) implies that the connected component of $\{x | p(x) \neq 0\}$ containing $(1, 1, 0)$ contains the open line segment from $(1, 0, 0)$ to $(0, 1, 0)$. Together, this implies that the positive quadrant $\mathbb{R}^2 \times \{0\} \subseteq K(p_H, e)$. By 1.7.2, this implies that $p$ is real-stable.

Thus, our algorithmic goal is reduced to testing whether a one-parameter family is real-rooted, and whether a given univariate polynomial is positive on an interval. We solve these problems in the sequel.

### 5.2 Real-rootedness of one-parameter families

In this section we present two algorithms for testing real-rootedness of a one-parameter family of polynomials. Both algorithms reduce this problem to verifying nonnegativity of a finite number of polynomials on the real line. The first algorithm produces $n$ polynomials of degree roughly $O(n^3)$, and has the advantage of being very simple, relying only on elementary techniques and standard algorithms such as fast matrix multiplication and the discrete Fourier transform. The second algorithm produces $n$ polynomials of degree roughly $O(n^2)$ and runs significantly faster, but uses somewhat more specialized (but nonetheless classical) machinery from the theory of resultants.

#### A Simple $O(n^{3+\omega})$ Algorithm

The first algorithm is based on the observation that real-rootedness of a single polynomial is equivalent to testing positive semidefiniteness of its moment matrix, which in turn is equivalent to testing nonnegativity of the elementary symmetric polynomials of that matrix. In the more general case of a one-parameter family, the latter polynomials turn out to be polynomials of bounded degree in the parameter, and it therefore suffices to verify that these are nonnegative everywhere.

We begin by recalling the Newton Identities, which express the moments of a polynomial in terms of its coefficients.

**Lemma 5.2.1 (Newton Identities).** If

$$p(x) = \sum_{k=0}^{n} (-1)^k x^{n-k} c_k = c_0 \prod_{i=1}^{n} (x - x_i) \in \mathbb{R}[x]$$

with $c_0 \neq 0$ is a univariate polynomial with roots $x_1, \ldots, x_n$, then the moments

$$m_k := \sum_{i=1}^{n} x_i^k$$
satisfy the recurrence:

\[ m_k = (-1)^{k-1} \frac{c_k}{c_0} + \sum_{i=1}^{k-1} (-1)^{k-1+i} \frac{c_{k-i}}{c_0} m_i \quad 0 \leq k \leq n, \]

\[ m_k = \sum_{k-n}^{k-1} (-1)^{k-1+i} \frac{c_{k-i}}{c_0} m_i \quad k > n, \]

\[ m_0 = n. \]

The following consequences of 5.2.1 will be relevant to analyzing our algorithm.

**Corollary 5.2.2.** 1. The moments \( m_0, \ldots, m_{2n-2} \) of a degree \( n \) polynomial can be computed from its coefficients in \( O(n^2) \) arithmetic operations.

2. Suppose \( p(x) = \sum_{k=0}^{n} (-1)^{k} x^{n-k} c_k(\gamma) \) is a polynomial whose coefficients are polynomials \( c_0(\gamma), \ldots, c_n(\gamma) \in \mathbb{R}_d[\gamma] \) in a parameter \( \gamma \). Then the moments of \( p \) are given by

\[ m_k(\gamma) = r_k(\gamma)/c_0(\gamma)^k, \]

for some polynomials \( r_k \in \mathbb{R}_{dk}[\gamma] \).

**Proof.** The first claim follows because each application of the recurrence requires at most \( n \) arithmetic operations. For the second claim, observe that each ratio \( \frac{c_{k-i}(\gamma)}{c_0(\gamma)} \) is a rational function with a numerator of degree at most \( d \) and denominator \( c_0(\gamma) \). Thus, each application of the recurrence increases the degree of the numerator by at most \( d \) and introduces an additional \( c_0 \) in the denominator.

As a subroutine, we will also need the following standard result in linear algebra.

**Theorem 5.2.3** (Keller-Gehrig [74]). Given an \( n \times n \) complex matrix \( A \), there is an algorithm which computes the characteristic polynomial of \( A \) in time \( O(n^\omega \log n) \).

We now specify the algorithm and prove its correctness.

**Theorem 5.2.4.** A polynomial \( p_\gamma(x) = \sum_{k=0}^{n} (-1)^k x^{n-k} c_k(\gamma) \) is real-rooted for all \( \gamma \in \mathbb{R} \) if and only if the polynomials \( q_0, \ldots, q_n \) output by \SimpleRR are nonnegative on \( \mathbb{R} \). Moreover, \SimpleRR runs in time \( \tilde{O}(dn^{2+\omega} + d^2n^3) \).

**Proof.** We first show correctness. Let \( m_k(p) \) denote the \( k^{th} \) moment of the roots of a polynomial. By Sylvester’s theorem [69, Theorem 4.58], a real polynomial

\[ p_\gamma(x) = \sum_{k=0}^{n} (-1)^k x^{n-k} c_k(\gamma) \]
is real-rooted if and only if the corresponding moment matrix
\[ M(\gamma)_{k,l} := m_{k+l-2}(p_\gamma) \]
is positive semidefinite. Since \( \nu \) is even and \( c_0 \) has real coefficients, we have for every \( \gamma \in \mathbb{R} \) that is not a root of \( c_0 \):
\[ M(\gamma) \succeq 0 \iff c_0(\gamma)H(\gamma) \succeq 0. \]
Since \( c_0 \) has only finitely many roots and a limit of PSD matrices is PSD, we conclude that
\[ M(\gamma) \succeq 0 \forall \gamma \in \mathbb{R} \iff H(\gamma) \succeq 0 \forall \gamma \in \mathbb{R}. \]
Note that by 5.2.2 the entries of \( H(\gamma) \) are polynomials of degree at most \( d(\nu + 2n - 2) \) in \( \gamma \).

We now recall a well-known\(^2\) (e.g., [75]) characterization of positive semidefiniteness as a semialgebraic condition: an \( n \times n \) real symmetric matrix \( A \) is PSD if and only if \( e_k(A) \geq 0 \) for all \( k = 1, \ldots, n \), where
\[ e_k(A) = \sum_{|S|=k} \det(A_{S,S}) \]
is the sum of all \( k \times k \) principal minors of \( A \). Thus, \( p_\gamma \) is real-rooted for all \( \gamma \in \mathbb{R} \) if and only if the polynomials
\[ q_k(\gamma) := e_k(H(\gamma)) \]
for \( k = 1, \ldots, n \) are nonnegative on \( \mathbb{R} \).

Since each \( q_k \) is a sum of determinants of order at most \( n \) in \( H(\gamma) \) it has degree at most \( n \) in the entries of \( H(\gamma) \), and we conclude that \( q_1, \ldots, q_n \in \mathbb{R}[\gamma] \). Thus, the \( q_k \) can be recovered by interpolating them at the \( N^\text{th} \) roots of unity. Since the \( k^\text{th} \) elementary symmetric function of a matrix is the coefficient of \( z^{n-k} \) in its characteristic polynomial, this is precisely what is achieved in Step 2.

For the complexity analysis, it is clear that Step 1 takes \( O(dn^2) \) time. Constructing each Hankel matrix \( H(s_i) \) takes time \( O(dn + n^2) \) by 5.2.2, and computing its elementary symmetric functions via the characteristic polynomial takes time \( O(n^\omega \log n) \), according to 5.2.3. Thus, the total time for each iteration is \( O(n^\omega \log n + dn) \), so the time for all iterations is \( O(dn^{2+\omega} \log n + d^2n^3) \). The final step requires \( O(N \log N) \) time for each \( e_k \) using fast polynomial interpolation via the discrete Fourier transform, for a total of \( O(dn^3 \log n) \). Thus, the total running time is \( O(dn^{2+\omega} + d^2n^3) \), suppressing logarithmic factors.

\(^2\)Here is a short proof: \( A \) is PSD iff \( \det(zI - A) \) has only nonnegative roots. Since \( A \) is symmetric we know the roots are real. We now observe that a real-rooted polynomial has nonnegative roots if and only if its coefficients alternate in sign.
Algorithm SimpleRR

Input: \((n + 1)\) univariate polynomials \(c_0, \ldots, c_n \in \mathbb{R}_d[\gamma]\) with \(c_0 \neq 0\).

Output: \(n\) univariate polynomials \(q_1, \ldots, q_n \in \mathbb{R}_{3n^2d}[\gamma]\)

1. Let \(\nu\) be the first even integer greater than or equal to \(n\) and let \(N = nd(2n - 2 + \nu) = O(n^2d)\). Let \(s_1, \ldots, s_N \in \mathbb{C}\) be the \(N\)th roots of unity.

2. For each \(i = 1, \ldots, N\):
   - Compute the \(n \times n\) Hankel matrix \(H(s_i)\) with entries
     \[H(s_i)_{k,l} := c_0(s_i)^\nu m_{k+l-2}(p_{s_i}),\]
     by applying the Newton identities (5.2.1).
   - Compute the characteristic polynomial
     \[\det(zI - H(s_i)) = \sum_{k=0}^{n} (-1)^k z^{n-k} e_k(H(s_i))\]
     using the Keller-Gehrig algorithm (5.2.3).

3. For each \(k = 1, \ldots, n\): Use the points \(e_k(H(s_1)), \ldots, e_k(H(s_N))\) to interpolate the coefficients of the polynomial
   \[q_k(\gamma) := e_k(H(\gamma)).\]

Output \(q_1, \ldots, q_n\).

A Faster \(O(n^4)\) Algorithm Using Subresultants

The algorithm of the previous section is based on the generic fact that a matrix is PSD if and only if its elementary symmetric polynomials are nonnegative. In this section we exploit the fact that our matrices have a special structure – namely, they are moment matrices – to find a different finite set of polynomials whose nonnegativity suffices to certify their PSDness. These polynomials are called subdiscriminants, and turn out to be related to another class of polynomials called subresultants, for which there are known fast symbolic algorithms.

Let \(M_p\) denote the \(n \times n\) moment matrix corresponding to a polynomial \(p\) of degree \(n\). Recall that \(M_p = VV^T\) where \(V\) is the Vandermonde matrix formed by the roots of \(p\). Let \((M_p)_{i,i}\) denote the leading principal \(i \times i\) minor of \(M_p\). We define subdiscriminants of a polynomial, and then show their relation to the leading principal minors of the moment.
matrix. For the remainder of this section it will be more convenient to use the notation
\[ p(x) = \sum_{k=0}^{n} a_k x^k \]
for the coefficients of a polynomial, with roots \( x_1, \ldots, x_n \) and \( a_n \neq 0 \).

**Definition 5.2.5.** The \( k \)th subdiscriminant of a polynomial \( p \) is defined as
\[
\text{sDisc}_k(p) = a_n^{2k-2} \sum_{S \subseteq \{1, \ldots, n\}, |S| = k} \prod_{i<j} (x_i - x_j)^2
\]

**Lemma 5.2.6.** The leading principal minors of the moment matrix are multiples of the subdiscriminants,
\[
(M_p)_i = a_n^{2-2k} \text{sDisc}_k(p) = \sum_{S \subseteq \{1, \ldots, n\}, |S| = k} \prod_{i<j} (x_i - x_j)^2
\]

*Proof.* Let
\[
V_i = \begin{bmatrix}
1 & \ldots & 1 \\
x_1 & \ldots & x_n \\
\vdots & \ddots & \vdots \\
x_1^{i-1} & \ldots & x_n^{i-1}
\end{bmatrix}
\]
Then \( (M_p)_i = \det(V_i V_i^T) \). By Cauchy-Binet, this determinant is the sum over the determinants of all submatrices of size \( i \times i \). These submatrices are exactly the Vandermonde matrices formed by subsets of the roots of size \( i \). Then the identity follows from the formula for the determinant of a Vandermonde matrix.

Equipped with this we can provide an alternative characterization of real rootedness. Define the sign of a number, denoted \( \text{sgn} \) to be +1 if it is positive, −1 if it is negative, and 0 otherwise.

**Lemma 5.2.7.** \( p \) is real-rooted if and only if the sequence \( \text{sgn}(\text{sDisc}_1(p)), \ldots, \text{sgn}(\text{sDisc}_n(p)) \) is first 1’s and then 0’s.

*Proof.* Note that since \( a_n \neq 0 \) we have \( \text{sgn}(\text{Disc}_k) = \text{sgn}(a_n^{2(1-k)} \text{Disc}_k) = \text{sgn}((M_p)_i) \). It is clear from the definition of the subdiscriminants that if \( p \) is real-rooted with \( k \) distinct roots then \( \text{sDisc}_i \) is positive if \( i \leq k \) and \( \text{sDisc}_i = 0 \) if \( i > k \).

Conversely, given a polynomial \( p \) with \( k \) distinct roots, then if \( i > k \) we have all the minors of size \( i \) in \( V_i V_i^T \) contain two identical rows, and hence \( V_i V_i^T \) does not have full rank, so \( V_i V_i^T \) is singular. Let \( x_1, x_2, \ldots, x_j \) be the real distinct roots of \( p \) and \( y_1, \bar{y}_1, \ldots, y_l, \bar{y}_l \) be the
distinct complex conjugate pairs of $p$ where $j + l = k$. Suppose the multiplicities of $x_i$ are $n_i$ and $y_i$ are $m_i$. Then the top left $k \times k$ submatrix of $M_p$ is

\[
= \sum_{i} n_i \begin{bmatrix} 1 \\ x_i \\ \vdots \\ x_i^{k-1} \end{bmatrix} [1 \ x_i \ \cdots \ x_i^{k-1}] + \sum_{i} m_i \begin{bmatrix} 1 \\ y_i \\ \vdots \\ y_i^{k-1} \end{bmatrix} [1 \ y_i \ \cdots \ y_i^{k-1}] + \begin{bmatrix} 1 \\ \bar{y}_i \\ \vdots \\ \bar{y}_i^{k-1} \end{bmatrix} \]

This shows that this submatrix is positive definite if and only if the distinct roots are all real. Note that by Sylvester’s criterion this submatrix is positive definite if and only if all the leading principal minors of size $\leq k$ are positive.

We now obtain a formula for the subdiscriminants of a polynomial in terms of its coefficients. The connection is provided by another family of polynomials called the subresultants.

**Definition 5.2.8.** Let $p = \sum_{k=0}^{n} a_k x^k$ where $a_n \neq 0$. The $k$th subresultant of $p$, denoted $\text{sRes}_k(p, p')$ is the determinant of the submatrix obtained from the first $2n - 1 - 2k$ columns of the following $(2n - 1 - 2k) \times (2n - 1 - k)$ matrix:

\[
\begin{bmatrix}
a_n & \cdots & \cdots & \cdots & a_0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 \\
\vdots & \ddots & a_n & \cdots & \cdots & \cdots & a_0 \\
\vdots & 0 & n a_n & \cdots & \cdots & \cdots & a_1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
na_n & \cdots & \cdots & a_1 & 0 & \cdots & 0
\end{bmatrix}
\]

We will use two properties of subresultants. The first is a good bound on their degree as a consequence of the determinant formula above. The second is quick algorithm to compute them. We refer the reader to [69] for a more detailed discussion of subresultants.

In this chapter we will only be interested in subresultants of a polynomial with its derivative. We are interested in this because of its relation to our leading principal minors:

**Lemma 5.2.9 ([69] Proposition 4.27).** Let $p(x) = \sum_{k=0}^{n} a_k x^k$ where $a_n \neq 0$

\[
\text{sRes}_k(p, p') = a_n \text{sDisc}_{n-k}(p)
\]
Corollary 5.2.10. Since the first column of the determinant used to define the subresultant is divisible by $a_n$, we get $\text{Disc}_k(p)$ is a polynomial in our coefficients $a_n, \ldots, a_0$ of degree at most $2n$.

The benefit of studying the principal minors instead of the coefficients of the characteristic polynomial for our moment matrix is that we can use an algorithm from subresultant theory to quickly calculate all the minors at once.

Theorem 5.2.11 ([69] Algorithm 8.21). There exists an algorithm which, given a polynomial $p$ of degree $n$ returns a list of all of its subresultants $\text{Res}_k(p, p')$ for $k = 1, \ldots, n$ in $O(n^2)$ time.

Remark 5.2.12. Many computer algebra systems (e.g., Mathematica, Macaulay2) have built-in efficient algorithms to compute subresultants.

We now combine the above facts to obtain a crisp condition for real-rootedness of a one-parameter family. Recall that by 5.1.1, we are interested in testing when a family of polynomials $p_\gamma(x)$ are real-rooted for all $\gamma \in \mathbb{R}$, where

$$p_\gamma(x) = \sum_{k=0}^{n} a_k(\gamma)x^k$$

with $c_k \in \mathbb{R}_n[\gamma]$. Let $c_m(\gamma)$ be the highest coefficient that is not identically zero. We are only interested in the case when $m \geq 2$.

Proposition 5.2.13. If $p_\gamma(x) = \sum_{k=0}^{n} x^k c_k(\gamma)$ with $c_k \in \mathbb{R}_d[\gamma]$, then $\text{Disc}_k(p_\gamma)$ is a polynomial in $\gamma$ of degree at most $2dn$.

Proof. From our previous lemma, we know that $\text{Disc}_k$ is a polynomial in the coefficients of $p$ of degree at most $2n$. Since each of these coefficients $c_k(\gamma)$ is a polynomial in $\gamma$ of degree at most $d$, our result follows. ☐

We now extend our characterization of real-rootedness in terms of the signs of the principal minors of a fixed polynomial to a characterization for coefficients which are polynomials in $\gamma$.

Theorem 5.2.14. $p_\gamma(x)$ is real-rooted for all $\gamma \in \mathbb{R}$ if and only if there exists a $k$ such that $\text{Disc}_i(p_\gamma)$ is a nonnegative polynomial which is not identically zero for all $i \leq k$ and $\text{Disc}_i(p_\gamma)$ is identically zero for $i > k$.

Proof. First suppose that $p_\gamma(x)$ is real rooted for all $\gamma \in \mathbb{R}$. Observe that $c_m(\gamma)$ vanishes for at most finitely many points $Z_1$. Moreover, the degree $m$ discriminant of $p_\gamma$ is a polynomial in $\gamma$, and is zero for at most finitely many points — call them $Z_2$. Thus, for $\gamma \notin Z_1 \cup Z_2$, we know that $p_\gamma$ has exactly $m$ distinct real roots, so by 5.2.7 $\text{Disc}_i(p_\gamma)$ is strictly positive for $i \leq m$ and zero for $i > m$ on this set. By continuity this implies that $\text{Disc}_i(p_\gamma)$ is
nonnegative and not identically zero on \( \mathbb{R} \) for \( i \leq m \), and \( s\text{Disc}_i(p_\gamma) \) is identically zero for \( i > m \), as desired.

To prove the converse, note that for \( i \leq k \), \( s\text{Disc}_i(p_\gamma(t)) \) is not identically zero, and hence there are finitely many \( \gamma \) away from which \( s\text{Disc}_i(p_\gamma) \) is positive for all \( i \leq k \), and then all zero. By 5.2.7 we get that \( p_\gamma(x) \) is real rooted for all these \( \gamma \). Since real-rootedness is preserved by taking limits (by Hurwitz’s theorem), we conclude that \( p_\gamma(x) \) is real rooted for all \( \gamma \in \mathbb{R} \).

Combining these observations, and using the \( O(n^2) \) time algorithm to compute the subdiscriminants, we arrive at the following \( O(n^4) \) time algorithm for computing all the subdiscriminants.

### Algorithm FastRR

**Input:** \((n + 1)\) univariate polynomials \( c_0, \ldots, c_n \in \mathbb{R}_d[\gamma] \) with \( c_0 \neq 0 \).

**Output:** \( n \) univariate polynomials \( q_1, \ldots, q_n \in \mathbb{R}_2dn[\gamma] \)

1. Find distinct points \( \gamma_1, \ldots, \gamma_{2dn} \in \) such that \( c_m(\gamma_i) \neq 0 \).
2. For each \( \gamma_i \) use the subresultant algorithm (5.2.11) to compute all of the \( s\text{Res}_k(p_\gamma_i) \), with \( k = 1, \ldots, n \).
3. Use the above values to compute \( 2dn \) different values \( q_k(\gamma_1), \ldots, q_k(\gamma_{2dn}) \) for each of the polynomials

\[
q_k(\gamma) := s\text{Disc}_k(p_\gamma) = c_m(\gamma)^{-1}s\text{Res}_{m-k}(p_\gamma),
\]

\( k = 1, \ldots, n \).
4. Use fast interpolation to compute the coefficients of \( q_1, \ldots, q_n \).

**Output** \( q_1, \ldots, q_n \).

### Theorem 5.2.15. FastRR runs in \( O(n^4) \) time.

**Proof.** Since \( c_n(\gamma) \) is of degree at most \( d \) we can test \( 2dn + d \) points to find \( 2dn \) points on which \( c_n(\gamma) \) doesn’t vanish. Each evaluation takes \( O(d) \) times, so total this takes \( O(d^2n) \) time. To compute \( s\text{Res}_k(p_\gamma) \) for each \( 0 \leq k \leq n - 1 \) and \( 1 \leq i \leq 2dn \) takes \( O(dn^3) \) time by 5.2.11. Then to scale all the subresultants, since we have \( O(dn^2) \) data points and have already computed \( c_n(\gamma_i) \) takes \( O(dn^2) \) time. Finally, since the degrees of the \( q_k \) are at most \( 2dn \), the total time to interpolate all of them is \( O(dn^2 \log n) \).
5.3 Univariate Nonnegativity Testing

In this section, we describe an algorithm to test non-negativity of a univariate polynomial over the real line.

Let $p \in \mathbb{R}[x]$ denote a univariate polynomial of degree $d$. The goal of the algorithm is to test if $p(x) \geq 0$ for all $x \in \mathbb{R}$. A canonical approach for the problem would be to use a Sum-of-Squares semidefinite program to express $p$ as a sum of squares of low-degree polynomials. Unfortunately, the resulting algorithm is not a symbolic algorithm, i.e., its runtime is not strongly polynomial in the degree $d$, since semidefinite programming is not known to be strongly polynomial.

We will now describe a strongly polynomial time algorithm to test non-negativity of the polynomial $p$. Our starting point is an algorithm to count the number of real roots of a polynomial using Sturm sequences. We refer the reader to Basu et al. [69] for a detailed presentation of Sturm sequences and algorithms to compute them. For our purposes, we will need the following lemma.

**Lemma 5.3.1.** Given a univariate polynomial $p \in \mathbb{R}[x]$, the algorithm based on computing Sturm sequences uses $O(\text{deg}(p)^2)$ arithmetic operations to determine the number of real roots of $p$.

The polynomial $p$ is positive, i.e., $p(x) > 0$ for all $x \in \mathbb{R}$, if and only if it has no real roots. Therefore, 5.3.1 yields an algorithm to test positivity using in $O(d^2)$ arithmetic operations. To test non-negativity, the only additional complication stems from the roots of the polynomial $p$. We begin with a simple observation.

**Fact 5.3.2.** If $p \in \mathbb{R}[x]$ is monic then $p(x) \geq 0$ for all $x \in \mathbb{R}$ if and only if $p$ has no real roots of odd multiplicity.

**Definition 5.3.3.** A square-free decomposition of a polynomial $p \in \mathbb{R}[x]$ of degree $d$, is a set of polynomials $\{a_1, \ldots, a_d\} \in \mathbb{R}[x]$ such that

$$p(x) = \prod_{i=1}^{d} a_i(x)^i,$$

and each $a_i$ has no roots with multiplicity greater than one. Alternately, for each $i \in [d]$, $a_i(x)^i$ consists of all roots of $p$ with multiplicity exactly $i$.

Square-free decompositions can be computed efficiently using gcd computations. Yun [50] carries out a detailed analysis of square-free decomposition algorithms. In particular, he shows that an algorithm due to Musser can be used to compute square-free decompositions at the cost of constantly many gcd computations.

Now, we are ready to describe an algorithm to test non-negativity.
Algorithm Nonnegative

*Input.* A monic polynomial \( p \in \mathbb{R}[x] \), \( \deg(p) = d \)

*Goal.* Test if \( p(x) \geq 0 \) for all \( x \in \mathbb{R} \).

1. Using Musser’s algorithm, compute the square-free decomposition of \( p \) given by,

\[
p = \prod_{i \in [d]} a_i^i
\]

where \( a_i \in \mathbb{R}[x] \) has no roots with multiplicity greater than 1.

2. For each \( i \in \left[ \left\lceil \frac{d}{2} \right\rceil \right] \)
   - Using Sturm sequences, test if \( a_{2i-1} \) has real roots. If \( a_{2i-1} \) has real roots \( p \) is NOT non-negative.

**Runtime** Let \( T_{gcd}(d) \) denote the time-complexity of computing the \( \text{gcd} \) of two univariate polynomials of degree \( d \). The runtime of Musser’s square-free decomposition algorithm is within constant factors of \( T_{gcd}(d) \). Let \( S_{\text{real}}(\ell) \) denote the time-complexity of determining if a degree \( \ell \) polynomial has no real roots. Observe that

\[
\sum_{i} \deg(a_i) \leq \deg(p) = d
\]

Since \( S_{\text{real}}(\ell) \) is super-linear in \( \ell \), we have \( \sum_{i \in [d]} S_{\text{real}}(a_i) \leq S_{\text{real}}(d) \). The run-time of the algorithm is given by \( O(T_{gcd}(d) + S_{\text{real}}(d)) \). Using Sturm sequences, \( S_{\text{real}}(d) = O(d^2) \) elementary operations on real numbers (see [69]). Using Euclid’s algorithm, \( T_{gcd}(d) = O(d^2) \) elementary operations on real numbers. This yields an algorithm for non-negativity that incurs at most \( O(d^2) \) elementary operations.

### 5.4 Further Remarks

Finally, we combine the ingredients from sections 3, 4, and 5 to obtain the proof of our main theorem.

**Proof of 5.0.1.** Given the coefficients of \( p \), we can compute the coefficients of the one-parameter family in (1) of 5.1.1 in time at most \( O(n^3) \). By 5.2.15, \textbf{FastRR} produces the polynomials \( q_1, \ldots, q_n \) in time \( O(n^4) \). We check that some final segment of these polynomials are identically zero by evaluating each one at \( O(n^2) \) points. These polynomials have degree \( O(n^2) \), so \textbf{Nonnegative} requires time \( O(n^4) \) to check nonnegativity of each remaining one, for a total running time of \( O(n^5) \).

For part (2) of 5.1.1, we simply use a Sturm sequence to ensure that there are no roots in \((0, 1)\), and then evaluate the polynomial at a single point to check that the sign is positive. \( \square \)
CHAPTER 5. REAL STABILITY TESTING

The algorithm in this chapter offers a starting point in the area of polynomial time algorithms for real stability. In addition to the obvious possibility of improving the running time to say $O(n^4)$ or below, several natural open questions remain:

- Can the algorithm be generalized to 3 or more variables? The bottleneck to doing this is that we do not know how to check real rootedness of 2-parameter families, or equivalently, nonnegativity of bivariate polynomials.

- Is there an algorithm for testing whether a given polynomial is hyperbolic with respect to some direction, without giving the direction as part of the input?

- Is there an algorithm for testing stability of bivariate polynomials with complex coefficients?

Perhaps leaving the realm of strongly polynomial time algorithms, the major open question in this area is the following: a famous theorem of Helton and Vinnikov [76] asserts that every bivariate real stable polynomial can be written as

$$p(x, y) = \det(xA + yB + C)$$

for some positive semidefinite matrices $A, B$ and real symmetric $C$. Unfortunately, their proof does not give an efficient algorithm for finding these matrices. Can the ideas in this chapter, perhaps via using SDPs to find sum-of-squares representations of certain nonnegative polynomials derived from $p$, be used to obtain such an algorithm?
Chapter 6

Exponential Lower Bounds on Spectrahedral Representations of Hyperbolicity Cones

Linear and semidefinite programming are central to modern algorithm design. While every semidefinite program can be approximated by a linear program with potentially many more constraints, there is a blowup in complexity in doing this; moreover, SDPs give a different algorithmic perspective on relaxations of combinatorial problems which is valuable in its own right. In this chapter we study a third kind of cone programming, called hyperbolic programming (HP), which generalizes SDP — in particular, we examine the issue of whether there is in general a complexity blowup in passing from HP to SDP representations.

Recall that the Hausdorff distance between two cones \( K \) and \( K' \) is defined as

\[
\text{hdist}(K, K') = \max_{x \in B \cap K, y \in B \cap K'} (d(x, K') + d(y, K)),
\]

where \( B \) is the unit ball in \( \mathbb{R}^n \). We say that a spectrahedral cone \( K' \) is an \( \eta \)-approximate spectrahedral representation of another cone \( K \) if \( \text{hdist}(K, K') \leq \eta \). Our main theorem is the following.

Theorem 6.0.1 (Main Theorem). There exists an absolute constant \( \kappa > 0 \) such that for all sufficiently large \( n,d \), there exists an \( n \)-variate degree \( d \) hyperbolic polynomial \( p \) whose hyperbolicity cone \( K_p \) does not admit an \( \eta \)-approximate spectrahedral representation of dimension \( B \leq (n/d)^{\kappa d} \), for \( \eta = 1/n^{4d} \).

This result is analogous to the works showing that there exists \( \{0,1\} \)-polytopes that need exponential sized LP extended formulations [77] and SDP extended formulations [78]. A crucial difference is that our main theorem lower bounds the size of spectrahedral representations as opposed to SDP extended formulations. Our proof is analytic and does not rely on algebraic obstacles to representability; in fact the polynomials we construct have very simple coefficients (they are essentially binary perturbations of \( e_d \)). However, since
CHAPTER 6. EXPONENTIAL LOWER BOUNDS ON SPECTRAHEDRAL REPRESENTATIONS OF HYPERBOLICITY CONES

there are \( \binom{n+d}{d} \) coefficients, the examples require \( \Omega(n^d) \) bits to write down. It is still unknown whether \( e_d \) itself admits a low dimensional spectrahedral representation, though Parrilo and Saunderson [79] have shown that it admits an SDP extended formulation of size \( \text{poly}(n, d) \).

Algebraically, the notion of (exact) spectrahedral representation of a hyperbolicity cone \( K_p \) for a polynomial \( p \) corresponds to an algebraic identity of the form,

\[
p(x) \cdot q(x) = \det\left( \sum_{i \in [n]} C_i x_i \right),
\]

where \( K_q \) contains \( K_p \). Thus, our main theorem implies the existence of a degree \( d \) polynomial \( p \) such that the degree of any identity of the above form is at least \( (n/d)^{\kappa d} \).

Proof Overview

The starting point for our proof is the theorem of Nuij [42], which says that the space of hyperbolic polynomials of degree \( d \) in \( n \) variables has a nonempty interior, immediately implying that it has dimension \( n^d \). The Generalized Lax Conjecture concerns the cones of these polynomials, which are geometric rather than algebraic objects. If we could show that this space of cones also has large “dimension” in some appropriate quantitative sense, and that the maps between the hyperbolicity cones and their spectrahedral representations are suitably well-behaved, then it would rule out the existence of small spectrahedral representations for all of them uniformly, since such representations are parameterized by tuples of \( n B \times B \) matrices, which have dimension \( nB^2 \).

The difficulties in turning this idea into a proof are: (1) There are no quantitative bounds on Nuij’s theorem. (2) the space of hyperbolicity cones is hard to parameterize and it is not clear how to define dimension. (3) the mapping from hyperbolicity cones to their representations can be arbitrary, and needn’t preserve the usual notions of dimension anyway. We surmount these difficulties by a packing argument, which consists of the following steps.

1. Exhibit a large family of hyperbolicity cones of size \( 2^{(n/d)^{\Omega(d)}} \), every pair of which are at least \( \epsilon \) apart from each other in the Hausdorff metric between the cones.

2. Show that the spectrahedral representations of two distant cones \( K \) and \( K' \) are distant from each other, once the representations are appropriately normalized. Formally, the matrices \( \{ C_i \}_{i \in [n]} \) and \( \{ C'_i \}_{i \in [n]} \) representing the two cones are at least \( \epsilon' \) away in operator norm if \( \text{hdist}(K, K') \geq \epsilon \) (see Lemma 6.4.3).

We work only with cones which contain the positive orthant in order to ensure that they have normalized representations.

3. By considering the volume, there is a \((B/\epsilon')^{nB^2}\) upper bound on the number of pairwise \( \epsilon' \)-distant spectrahedral representations in \( B \times B \) matrices, thus giving the lower bound on \( B \).
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By far, the most technical part of the proof is the first step exhibiting a large family of pairwise distant hyperbolicity cones. The set of hyperbolic polynomials is known to have a non-empty interior in the set of all degree $d$ homogenous polynomials. Although this implies the existence of a full-dimensional family of hyperbolic polynomials, it is not clear how far apart they are quantitatively. Moreover, without any further understanding of the structure of the polynomials, it is difficult to argue that their cones are far in the Hausdorff metric. To this end, we work with an explicit family of hyperbolic polynomials which are all perturbations of the elementary symmetric polynomials, whose cones we are able to understand. Specifically, we will show the following:

1. There exists an explicit family $\mathcal{P}$ of $2^{\Omega\left(\binom{n}{d}\right)}$ perturbations of the degree $d$ elementary symmetric polynomial $e_d(x) = \sum_{S, |S| \leq d} \prod_{i \in S} x_i$ that are all hyperbolic, and pairwise distant from each other. These perturbations are indexed by a hypercube of dimension $\Omega\left(\binom{n}{d}\right)$, as depicted in Figure 1. The subspace of perturbations is carefully chosen to preserve real-rootedness of all the restrictions, thereby preserving the hyperbolicity of the polynomial $e_d(x)$ (see Section 6.1), as well as to yield perturbations of an especially simple structure.

2. The hyperbolicity cones for every pair of polynomials in $\mathcal{P}$ are far from each other. In order to lower bound the Hausdorff distance between these cones, we identify an explicit set of $\Omega\left(\binom{n}{d}\right)$ points on the boundary of the hyperbolicity cone for $e_d(x)$ as markers. We will lower bound the perturbation of these markers as the polynomial is perturbed, in order to argue that the corresponding hyperbolicity cone is also perturbed (see Section 6.3). Again, the structure of the chosen subspace ensures that there are no “interactions” between the markers, and the analysis is reduced to understanding the perturbation of a single univariate Jacobi polynomial.
6.1 Many Hyperbolic Perturbations of $e_d$

In this section we prove that even though $e_d(x_1, \ldots, x_n)$ is not in the interior of the set of Hyperbolic polynomials, there is a large subspace $\Pi_d$ of homogeneous polynomials of degree $d$ in $n$ variables such that all sufficiently small perturbations of $e_d$ in this subspace remain hyperbolic.

The subspace will be spanned by certain homogeneous polynomials corresponding to matchings. For any matching $M$ containing $d$ edges on $\{1, \ldots, n\}$, define the polynomial

$$q_M(x_1, \ldots, x_n) := \prod_{i<j \in M} (x_i - x_j).$$

We say that a $d$−matching on $[n]$ fully crosses a $d$−subset $S$ of $[n]$ if every edge of $M$ has exactly one endpoint in $S$.

**Lemma 6.1.1** (Many Uniquely Crossing Matchings). There is a set $\mathcal{M}_d$ of $d$−matchings on $[n]$ of size at least

$$|\mathcal{M}_d| =: N \geq \binom{n}{d} \cdot \frac{4}{2^d}$$

and a set of $d$−subsets $S_d$ such that for every $S \in S_d$ there is a unique matching $M \in \mathcal{M}_d$ which fully crosses it, and for every $M \in \mathcal{M}_d$ there is at least one $S \in S_d$ which it fully crosses.

Moreover, for every indicator vector $\mathbf{1}_S$, $S \in S_d$ there is a unique $M \in \mathcal{M}_d$ such that $q_M(\mathbf{1}_S) \neq 0$, so the dimension of the span of

$$\Pi_d := \{q_M : M \in \mathcal{M}_d\}$$

is exactly $N$.

**Proof.** Let $\mathcal{M}$ denote the set of all $d$−matchings on $[n]$ and let $S$ be the set of all $d$−subsets of $[n]$. Let

$$E := \binom{n-d}{d} \cdot d!$$

be the number of matchings fully crossing a fixed set $S \in S$. Let $\mathcal{M}_d$ be a random subset of $\mathcal{M}$ in which each matching is included independently with probability $\alpha/E$ for some $\alpha \in (0, 1)$ to be determined later, and let $X_M$ be the indicator random variable that $M \in \mathcal{M}_d$. For any set $S$, define the random variable

$$\deg(S) := \sum_{M \text{ fully crossing } S} X_M,$$

and observe that

$$\mathbb{E} \deg(S) = \alpha.$$
CHAPTER 6. EXPONENTIAL LOWER BOUNDS ON SPECTRAHEDRAL REPRESENTATIONS OF HYPERBOLICITY CONES

Call a set \(S\) good if \(\deg(S) = 1\) and let \(G\) be the number of good \(S\). Observe that

\[
\mathbb{E} G = \binom{n}{d} \cdot \mathbb{P}[\deg(\{1, \ldots, d\} = 1] = \binom{n}{d} \cdot E \cdot (1 - (\alpha/E))^{E-1} \cdot \frac{\alpha}{E} \geq \left(\binom{n}{d}\right) \alpha (1 - \alpha \cdot (1 - 1/E)).
\]

Setting \(\alpha = 1/2\) we therefore have

\[
\mathbb{E} G \geq \left(\binom{n}{d}\right) / 4,
\]

so with nonzero probability there are at least \(\binom{n}{d} / 4\) good subsets.

Let \(S_d\) be the set of good subsets. Finally, remove from \(M_d\) all the matchings that do not fully cross a good subset. Since there are at least \((1/4)\binom{n}{d}\) good subsets, every good subset fully crosses at least one matching in \(M_d\), and at most \(2^d\) subsets cross any given matching, the number of matchings which remain in \(M_d\) is at least

\[
(1/4)\binom{n}{d} / 2^d,
\]

as desired.

The moreover part is seen by observing that \(q_M(1_S) = 0\) if and only if \(M\) fully crosses \(S\).

**Remark 6.1.2.** In fact, the dimension of the span of all of the polynomials \(q_M\) is exactly \(\binom{n}{d} - \binom{n}{d-1}\), by properties of the Johnson Scheme, but in order to obtain the additional property above we have restricted to \(M_d\).

Henceforth we fix \(\{q_M : M \in M_d\}\) to be a basis of \(\Pi_d\) and define the \(\ell_1\) norm of a polynomial

\[
q = \sum_{M \in M_d} s_M q_M \in \Pi_d
\]

as \(\|q\|_1 := \|s\|_1\). We will use the notation

\[
m_d(x_1, \ldots, x_n) = \max_{|S|=d} \prod_{i \in S} |x_i|,
\]

to denote the product of the \(d\) largest entries of a vector in absolute value, and occasionally we will write \(e_d(p)\) and \(m_d(p)\) for a real-rooted polynomial \(p\), which means applying \(e_d\) or \(m_d\) to its roots. The operator \(D\) refers to differentiation with respect to \(t\).

The main theorem of this section is:

**Theorem 6.1.3** (Effective Relative Nuij Theorem for \(e_d\)). If \(q \in \Pi_d\) satisfies

\[
\|q\|_1 \leq \frac{\binom{n}{d}}{2^n \cdot R^{(d+1)(n-d)}} =: R
\]

then \(e_d + q\) is hyperbolic with respect to \(1\).
Recalling the definition of hyperbolicity, our task is to show that all of the restrictions
\[ t \mapsto e_d(t \mathbf{1} + x), \quad x \in \mathbb{R}^n \]
remain real-rooted after perturbation by \( q \). Many of these restrictions lie on the boundary of the set of (univariate) real-rooted polynomials, or arbitrarily close to it, so it is not possible to simply discretize the set of \( x \) by a net and choose \( q \) to be uniformly small on this net; one must instead carry out a more delicate restriction-specific analysis which shows that for small \( R \), the perturbation \( q(t \mathbf{1} + x) \) is less than the distance of \( e_d(t \mathbf{1} + x) \) to the boundary of the set of real-rooted polynomials, for each \( x \in \mathbb{R}^n \). Since we are comparing vanishingly small quantities, it is not a priori clear that such an approach will yield an effective bound on \( R \) depending only on \( n \) and \( d \); Lemma 6.1.4 shows that this is indeed possible.

**Proof of Theorem 6.1.3.** Fix any nonzero vector \( x \in \mathbb{R}^n \) and perturbation \( q \in \Pi_d \) with \( \|q\|_1 \leq R \) and consider the perturbed restriction:
\[ r(t) := e_d(t \mathbf{1} + x) + q(t \mathbf{1} + x). \]

Let \( p(t) := e_d(t \mathbf{1} + x) \) and observe that since \( q \) is translation invariant, we have \( q(t \mathbf{1} + x) = q(x) \), so in fact
\[ r(t) = p(t) + q(x). \]

Let \( \gamma \geq 0 \) be the largest constant such that \( p(t) + \delta \) is real-rooted for all \( \delta \in [-\gamma, \gamma] \) (note that \( \gamma \) could be zero if \( p \) has a repeated root). It is sufficient to show that \( |q(x)| \leq \gamma \). Observe that
\[ \gamma = \min_{t : p'(t) = 0} |p(t)|, \tag{6.1} \]
since the boundary of the set of real-rooted polynomials consists of polynomials with repeated roots, and at any double root \( t_0 \) of the \( p + \gamma \) we have \( p'(t_0) = (p + \gamma)'(t_0) = (p + \gamma)(t_0) = 0 \). Let \( t_0 \) be the minimizer in (6.1) and replace \( x \) by \( x - t_0 \mathbf{1} \), noting that this translates \( r(t) \) to \( r(t - t_0) \) and does not change \( \gamma \) or \( q(x) \), so that we now have:
\[ p'(0) = d \cdot e_{d-1}(x) = 0 \]
and
\[ \gamma = |p(0)| = |e_d(x)|. \]

On the other hand, observe that:
\[
|q(x)| \leq \sum_{M \in \mathcal{M}_d} |s_M| \prod_{i,j \in M} |x_i - x_j|
\leq \|q\|_1 \cdot \max_{M \in \mathcal{M}_d} \prod_{i,j \in M} (|x_i| + |x_j|)
\leq \|q\|_1 \cdot 2^d \cdot m_d(x).
\]
Thus we have $\gamma \geq |q(x)|$ as long as $m_d(x) = 0$ or
\[ \|q\|_1 \leq \frac{|e_d(x)|}{2^d m_d(x)}, \]
which is implied by
\[ \|q\|_1 \leq \frac{\binom{n}{d}}{2^d (2n^{d+1})(n-d)} \]
by Lemma 6.1.4, as advertised.

The following lemma may be seen as a quantitative version of the fact that if a real-rooted polynomial has two consecutive zero coefficients $e_d = e_{d-1} = 0$ then it must have a root of multiplicity $d+1$ at zero.

**Lemma 6.1.4.** If $x \in \mathbb{R}^n$ satisfies $e_{d-1}(x) = 0$ then
\[ |e_d(x)| \geq \frac{\binom{n}{d}}{(2n^{d+1})(n-d)} |m_d(x)|. \]

**Proof.** Let $p(t) := \prod_{i=1}^{n} (t - x_i)$ and let $q_k(t) := D^{n-k}p$, noting that $q_k$ is real-rooted of degree exactly $k$. Assume for the moment that all of the $x_i$ are distinct and that $q_k(0) \neq 0$ for all $k = n, \ldots, d$ (note that this is equivalent to assuming that the last $d+1$ coefficients of $p$ are nonzero). Note that these conditions imply that all of the polynomials $q_k$ have distinct roots, since differentiation cannot increase the multiplicity of a root.

If $n = d$ then the claim is trivially true since
\[ e_d(x) = e_d(q_d) = m_d(q_d) = m_d(x). \]  
(6.2)

Observe that $e_d$ behaves predictably under differentiation:
\[ e_d(q_d) = e_d(D^{n-d}p) = \frac{(n-d)!}{n \cdots (d+1)} e_d(p) = \binom{n}{d}^{-1} e_d(q_n). \]  
(6.3)

We will show by induction that:
\[ m_d(q_d) \geq \frac{1}{2n^{d+1}} m_d(q_{d+1}) \geq \cdots \geq \frac{1}{(2n^{d+1})^{k-d}} m_d(q_k) \geq \cdots \geq \frac{1}{(2n^{d+1})^{n-d}} m_d(q_n), \]
which combined with (6.2) and (6.3) yields the desired conclusion.

**Case k = d + 1.** Let $z_-$ and $z_+$ be the smallest (in magnitude) negative and positive roots of $q_d = Dq_{d+1}$, respectively. Let $w \neq 0$ be the unique root of $q_{d+1}$ between $z_-$ and $z_+$; assume without loss of generality that $w > 0$ (otherwise consider the polynomial $p(-x)$). Let $x_- < z_-$ and $x_+ < z_+$ be the smallest in magnitude negative and positive roots of $q_{d+1}$
other than \( w \), so that \( x_- < 0 < w < x_+ \). There are two subcases, depending on whether or not \( w \) is close to zero — if it is, then it prevents any root from shrinking too much under differentiation, and if it is not, the hypothesis \( e_{d-1}(p) = 0 \) shows that \( |z_-| \) and \( |z_+| \) are comparable, which also yields the conclusion.

- **Subcase \(|w| \leq |x_-|/2n\).** By Lemma 6.1.6, we have
  \[
  |z_-| \geq |x_-| - (|x_-| + |w|)(1 - 1/n) \geq |x_-|(1 - (1 + 1/2n)(1 - 1/n)) \geq |x_-|/2n.
  \]
  For every root of \( q_{d+1} \) other than \( x_- \) there is another root of \( q_{d+1} \) between it and zero, so Lemma 6.1.6 implies that for every such root the neighboring (towards zero) root of \( Dq_{d+1} \) is smaller by at most \( 1/n \). Thus, we conclude that
  \[
  m_d(q_d) = m_d(Dq_{d+1}) \geq \frac{m_d(q_{d+1})}{2n^d}.
  \]

- **Subcase \(|w| > |x_-|/2n\).** In this case we may assume that \( m_d(q_{d+1}) \) is witnessed by the \( d \) roots of \( q_{d+1} \) excluding \( x_- \), call this set \( W \), losing a factor of at most \( 1/2n \). Observe that every root in \( W \setminus \{w\} \) is separated from zero by another root of \( q_{d+1} \), so such roots shrink by at most \( 1/n \) under differentiation by Lemma 6.1.6. Noting that \( x_- \notin W \), we have by interlacing that:
  \[
  \prod_{q_d(z)=0,z\neq z_-} |z| \geq \frac{1}{n^{d-1}} \prod_{x \in W \setminus \{w\}} |x|,
  \]
  and our task is reduced to showing \( |z_-| \) is not small compared to \( |w| \).
  
  The hypothesis \( e_{d-1}(q_{d+1}) = 0 \) implies that \( q'_d(0) = 0 \); applying Lemma 6.1.6, we find that the magnitudes of the innermost roots of \( q_d \) must be comparable:
  \[
  |z_-| \vee |z_+| \leq d \cdot (|z_-| \wedge |z_+|).
  \]
  We now have
  \[
  |z_-| > |z_+|/n > |w|/n,
  \]
  so we conclude that
  \[
  m_d(q_d) \geq \frac{m_d(q_{d+1})}{2n^d},
  \]
  as desired.

**Case \( k \geq d + 2 \).** We proceed by induction. Assume \( m_d(q_k) \) is witnessed by a set of \( d \) roots \( L \cup_R \), where \( L \) contains negative roots and \( R \) contains positive ones. If there is a negative root not in \( L \) and a positive root not in \( R \) then as before every root in \( L \cup R \) is separated from zero by another root of \( q_k \), and by Lemma 7 we have
  \[
  m_d(Dq_k) \geq \frac{1}{n^d} m_d(q_k),
  \text{(6.4)}
  \]
so we are done. So assume all of the negative roots are contained in $L$; since $|L \cup R| = d$ this implies that there are at least two positive roots not in $R$; let $z_*$ be the largest positive root not contained in $R$. Let $z_-$ and $z_+$ be the negative and positive roots of $q_k$ of least magnitude. There are two cases:

- $|z_+| > |z_-|/2n$. This means that we can delete $z_-$ from $L$ and add $z_*$ to $R$, and reduce to the previous situation, incurring a loss of at most $1/2n$, which means by (6.4):

$$m_d(q_{k-1}) \geq \frac{1}{2n} \frac{1}{nd} m_d(q_k).$$

- $|z_+| \leq |z_-|/2n$. By Lemma 7, the smallest in magnitude negative root of $Dq_k$ has magnitude at least

$$(1 - (1 - 1/n)(1 + 1/2n))|z_-| \geq |z_-|/2n,$$

and all the positive roots decrease by at most $1/n$ upon differentiating by Lemma 7, whence we have

$$m_d(Dq_k) \geq \frac{1}{2n^d} m_d(q_k).$$

To finish the proof, the requirements that $q_k(0) \neq 0$ for all $k$ and that all coordinates of $x$ are distinct may be removed by a density argument, since the set of $x$ for which this is true is dense in the set of $x \in \mathbb{R}^n$ satisfying $e_{d-1}(x) = 0$.

Remark 6.1.5. We suspect that the dependence on $n$ and $d$ in the above lemma can be improved, and it is even plausible that it holds with a polynomial rather than exponential dependence of $R$ on $n$. Since we do not know how to do this at the moment, we have chosen to present the simplest proof we know, without trying to optimize the parameters.

The following lemma is a quantitative version of the fact that the roots of the derivative of a polynomial interlace its roots.

Lemma 6.1.6 (Quantitative Interlacing). If $p$ is real rooted of degree $n$ then every root of $p'$ between two distinct consecutive roots of $p$ divides the line segment between them in at most the ratio $1 : n$.

Proof. Begin by recalling that if $p$ has distinct roots $z_1 < \ldots < z_n$ then the roots $z'_1 < \ldots < z'_{n-1}$ of $p'$ satisfy for $j = 1, \ldots, n-1$:

$$\sum_{i \leq j} \frac{1}{z'_j - z_i} = \sum_{i > j} \frac{1}{z_i - z'_j}.$$
CHAPTER 6. EXPONENTIAL LOWER BOUNDS ON SPECTRAHEDRAL REPRESENTATIONS OF HYPERBOLICITY CONES

Note that the solution $z'_j$ is monotone increasing in the $z_i$ on the LHS and monotone decreasing in the $z_i$ on the RHS. Thus, $z'_j$ is at least the solution to:

$$\frac{1}{z'_j - z_j} = \frac{n}{z_{j+1} - z'_j},$$

which means that it is at least $z_j + \frac{z_{j+1} - z_j}{n}$. A similar argument shows that it it at most $z_j + (1 - 1/n)(z_{j+1} - z_j)$. Adding the common roots of $p$ and $p'$ back in, we conclude that these inequalities are satisfied by all of the $z'_j$.

6.2 Separation in Restriction Distance

For a parameter $\epsilon > 0$ to be chosen later and $s \in \{0,1\}^{\mathcal{M}_d}$ let

$$p_s(x_1, \ldots, x_n) := e_d(x_1, \ldots, x_n) - \epsilon \sum_{M \in \mathcal{M}_d} s_M q_M(x_1, \ldots, x_n).$$

For any polynomial $p$ hyperbolic with respect to $\mathbf{1}$, define the restriction embedding

$$\Lambda(p) := (\lambda_{\text{max}}(p(t \mathbf{1} + 1_S)))_{S \in \mathcal{S}_d}.$$

Lemma 6.2.1. If $0 < \epsilon < R/N$ and $s, s' \in \{0,1\}^{\mathcal{M}_d}$ are distinct then both $p_s$ and $p'_s$ are hyperbolic with respect to $\mathbf{1}$ and

$$\|\Lambda(p_s) - \Lambda(p'_s)\|_\infty \geq \Delta := \epsilon \left( \binom{n}{d} de \right)^{-1}.$$

Proof. Since $\epsilon < R/N$, we have $\|e_d - p_s\|_1, \|e_d - p'_s\|_1 \leq R$ so by Theorem 6.1.3 both of them must be hyperbolic with respect to $\mathbf{1}$.

Since $s \neq s'$, suppose $s_M = 0$ and $s'_M = 1$ for some matching $M \in \mathcal{M}_d$, and let $S \in \mathcal{S}_d$ be a set which fully crosses $M$. By Lemma 6.1.1, $M$ is the only matching in $\mathcal{M}_d$ which crosses $S$, so we have $q_M(1_S) = 1$ and $q_{M'}(1_S) = 0$ for all other $M' \neq M \in \mathcal{M}_d$. Thus, along the restriction $t \mathbf{1} + 1_S$, one has

$$q_s(t \mathbf{1} + 1_S) = e_d(t \mathbf{1} + 1_S) =: J(t) \quad (6.5)$$

and

$$q_{s'}(t \mathbf{1} + 1_S) = e_d(t \mathbf{1} + 1_S) - \epsilon q_M(1_S) = e_d(t \mathbf{1} + 1_S) - \epsilon = J(t) - \epsilon,$$

where we have again used that the $q_M$ are translation invariant. Note that $J(t)$ has positive leading coefficient, so subtracting a constant from it increases its largest root. Thus, our task is reduced to showing that

$$\lambda_{\text{max}}(J(t) - \epsilon) \geq \lambda_{\text{max}}(J(t)) + \Delta.$$
CHAPTER 6. EXPONENTIAL LOWER BOUNDS ON SPECTRAHEDRAL REPRESENTATIONS OF HYPERBOLICITY CONES

To analyze the behavior of this perturbation, first we note that

\[ J(t) = e_d(t_1, \ldots, t_1, t_1+1, \ldots, t_1+1) = \frac{1}{(n-d)!} D^{n-d} e_N(t_1+1_s) = \frac{1}{(n-d)!} D^{n-d} t^{n-d}(t+1)^d. \]

Since \( t^{n-d}(t+1)^d \) has roots in \([-1, 0]\) and the roots of the derivative of a polynomial interlace its roots, we immediately conclude that the zeros of \( J \) satisfy \(-1 \leq z_1 \leq \ldots \leq z_d \leq 0\). Again by interlacing, we see that \( J'(z) \geq 0 \) and \( J''(z) \geq 0 \) for \( z \geq z_d \), whence \( J \) is monotone and convex above \( z_d \). Thus, \( \lambda := \lambda_{\max}(J(t) - \epsilon) \) is the least \( \lambda > z_d \) such that \( J(\lambda) \geq \epsilon \). Let \( \theta > 0 \) be a parameter to be set later. Then either \( \lambda \geq z_d + \theta \) or we have by convexity that

\[ J(z_d + \theta) \leq J(z_d) + \theta J'(z_d + \theta). \]

The first term is zero and we can upper bound the second term as:

\[ |J'(z_d + \theta)| \leq \left( \frac{n}{d} \right) \sum_{i \leq d} \left| \prod_{j \neq i} (z_d + \theta - z_i) \right| \leq \left( \frac{n}{d} \right) d \cdot (1 + \theta)^{n-1}, \]

since \( |z_d - z_i| \leq 1 \) for every \( i \leq d \). Thus, we have

\[ J(z_d + \theta) \leq \left( \frac{n}{d} \right) d \theta (1 + \theta)^{n-1} \leq \left( \frac{n}{d} \right) d \theta e \]

whenever \( \theta < 1/n \), which is less than \( \epsilon \) for \( \theta = \epsilon \left( \frac{n}{d} \right) e^{-1} \). This means that in either case we must have

\[ \lambda \geq z_d + \epsilon \left( \frac{n}{d} \right) e^{-1}, \]

as desired. \( \square \)

6.3 Separation in Hausdorff Distance

Lemma 6.3.1. For \( s, s' \in \{0, 1\}^{Md} \) and

\[ \epsilon < \frac{1}{4n^d(n-d)Nd\sqrt{n}} =: R_2 \]

we have

\[ \text{hdist}(K_{p_s}, K_{p_{s'}}) > \frac{\|\Lambda(p_s) - \Lambda(p_{s'})\|_\infty}{18n^d(n-d)Nd} Nn \]

Proof. Following the same argument as the beginning of Lemma 6.2.1, we see that given any \( s \neq s' \), we have some restriction such that

\[ p_s(t1 + 1_s) = e_d(t1 + 1s) \]
and
\[ p_s'(t1 + 1_S) = e_d(t1 + 1_S) - \epsilon \]

Let \( z, z' \) denote the corresponding points of the intersection of the line \( t1 + 1_S \) with the boundaries of \( K_{p_s} \) and \( K_{p_s'} \), respectively. If \( \lambda \in \mathbb{R} \) is the largest root of the polynomial \( J(t) \) from (6.5), then we note \( z = \lambda 1 + 1_S \) and \( z' = t1 + 1_S \) where \( t > \lambda + \Delta \) by the proof of Lemma 6.2.1. Let \( H \) be the hyperplane tangent to \( K_{p_s'} \) at \( z' \). Let \( v \) be a unit vector normal to the hyperplane at \( H \). Since the hyperbolicity cones are convex, the distance from \( z \) to \( K_{p_s'} \) is at least the distance to \( H \), which is bounded below by
\[ \| \Delta 1 \| \cdot \frac{|\langle 1, v \rangle|}{\| v \| \| z \|} = \Delta \cdot \frac{|\langle 1, v \rangle|}{\| v \|} \cdot \| z \|. \]

Normalizing so that \( z \) is a unit vector, we obtain:
\[ \text{hdist}(K_{p_s}, K_{p_s'}) \geq \Delta \cdot \frac{|\langle 1, v \rangle|}{\| v \| \| z \|}, \tag{6.6} \]

so if we can prove a uniform lower bound on this quantity over all \( v \) and \( z \) corresponding to \( s, s' \), then we are done.

Computing the normal we find that
\[ v = \nabla p_{s'}(z') = \nabla e_d(z') - \epsilon \sum_{M \in M_d} s'_M \nabla q_M(z') = : \nabla e_d(z') + \epsilon w, \]

so that
\[ |\langle 1, v \rangle| \geq |\langle 1, \nabla e_d(z') \rangle| - \epsilon \| 1 \| \| w \|. \]

The first inner product is just the directional derivative of \( e_d \) in direction \( 1 \) at \( z' \). Since \( e_d(t1 + 1_S) = J(t) \), and \( z' \) lies above the maximal root of this restriction, by convexity above maximal root we get this is bounded below by the directional derivative at \( z \), so
\[ |\langle 1, \nabla e_d(z') \rangle| \geq \frac{d}{dt} e_d(t1 + 1_S)|t=\lambda = J'(\lambda) \geq \frac{1}{n^d(n-d)}, \tag{6.7} \]

by Lemma 6.3.2, proven below.

We now prove crude upper bounds on \( \| z \|, \| w \|, \| v \| \), which will be negligible when \( \epsilon \) is small. First we have
\[ \| z \| \leq \lambda \| 1 \| + \| 1_S \| \leq 3\sqrt{n}, \tag{6.8} \]

since \( |\lambda| \leq 1 \) because \( J \) has roots in \([-1, 0]\).

To control \( \| w \| \), we compute the \( i \)th coordinate of \( \nabla q_M(x) \):
\[ \partial_{x_i} q_M(x) = \sigma_i \prod_{k \neq l \in M \setminus \{i\}} (x_k - x_l), \]
where $\sigma_i$ zero if $i \notin M$ and $\pm 1$ if $i \in M$. Since $J(t)$ has constant 1 and $\epsilon < 1$ we conclude $z'$ has coordinates in $[-1,0]$. This implies $|\partial_x q_M(z')| \leq 1$ for all $i \in M$ and
\[
\|\nabla q_M(z')\| \leq \sqrt{2d}.
\]
Applying the triangle inequality and noting that $s_M \in \{0,1\}$ gives
\[
\|w\| \leq |M| \cdot \sqrt{2d} \leq 2Nd. \tag{6.9}
\]
Finally, we have
\[
\|\nabla e_d(z')\| = \sqrt{\sum_{i=1}^{n} e_{d-1}(z'_{i-1})^2},
\]
where $z'_{i-1}$ is the vector obtained by deleting the $i^{th}$ coordinate of $z'$. Since these coordinates are bounded in magnitude by 1, the above norm is bounded by
\[
\sqrt{n} \cdot \binom{n}{d-1} \leq \sqrt{n} \cdot N,
\]
and applying the triangle inequality once more we get
\[
\|v\| \leq \sqrt{n} \cdot N + \epsilon \|w\| \leq 3N\sqrt{n}, \tag{6.10}
\]
since $\epsilon < 1$.

Combining (6.6), (6.7), (6.8),(6.9), and (6.10), we have:
\[
\text{hdist}(K_{p_s}, K_{p_s'}) \geq \frac{n-d(n-d)}{3\sqrt{n} \cdot 3N\sqrt{n}} \geq \frac{\Delta}{18n^{d(n-d)}N^n},
\]
provided
\[
2Nd\sqrt{n} \epsilon \leq (1/2)n^{-d(n-d)},
\]
as desired.

**Lemma 6.3.2** (Sensitivity of Jacobi Root). Let $\lambda$ be the largest root of $J(t)$. Then
\[
J'(\lambda) \geq \frac{1}{n^{d(n-d)}}.
\]

**Proof.** Let $z_1 \leq \ldots \leq z_d = \lambda$ be the roots of $J(t)$. Observe that
\[
J'(\lambda) = \prod_{i<d} (z_d - z_i) \geq |z_d - z_{d-1}|^{d-1},
\]
so if we can prove a lowerbound on the spacing between $z_d$ and $z_{d-1}$ we are done.
We do this by recalling from Lemma 6.5 that:

\[ J(t) = \frac{1}{d!} D^{n-d} t^{n-d} (1 + t)^d. \]

Let \( q_k(t) := D^k t^{n-d} (1 + t)^d \) and note that \( q_{k+1} \) interlaces \( q_k \), and all of these polynomials have real roots in \([-1, 0]\). Let \( y_k \) be the largest root of \( q_k \) that is strictly less than 0, noting that every \( q_k \) for \( k < n - d \) has a root at 0 because \( q_0 \) has a root of multiplicity \( n - d \) at 0. By Lemma 6.1.6, we have for \( k = 1, \ldots, n - d \):

\[ y_{k+1} \leq 0 - \frac{1}{n} |0 - y_k| \leq -|y_k|/n. \]

Noting that \( y_0 = -1 \) and iterating this bound \( n - d - 1 \) times we obtain

\[ y_{n-d-1} \leq -1/n^{n-d-1}, \]

so that \( D^{n-d-1} q_0 \) has a root at zero and a root \( y_{n-d-1} \leq -1/n^{n-d} \). Since \( J(t) = D q_{n-d-1}(t) \), we must have \( z_{d-1} \leq -y_{n-d-1} \) by interlacing. However, applying Lemma 6.1.6 once more, we see that

\[ z_d \geq -y_{n-d-1} + |0 + y_{n-d-1}|/n = -(1 - 1/n)y_{n-d-1}, \]

so the gap must be at least

\[ z_d - z_{d-1} \geq y_{n-d-1}/n = \frac{1}{n^{n-d}}. \]

Thus, we conclude that

\[ J'(\lambda) \geq \frac{1}{n^{(d-1)(n-d)}} \geq \frac{1}{n^{d(n-d)}}, \]

as desired.

6.4 Separation of Matrix Parameterizations

Given \( C = \{ C_i \}_{i \in [n]} \subset \mathbb{R}^{k \times k} \), define the cone

\[ K_C = \{ x \in \mathbb{R}^n \mid \sum_{i \in [n]} C_i x_i \succeq 0 \} \]

Here \( C \) is said to be a spectrahedral representation of the cone \( K_C \).

**Definition 6.4.1.** A spectrahedral representation of a cone \( K \subseteq \mathbb{R}^n \) as \( K = \{ x \in \mathbb{R}^n \mid \sum_i C_i x_i \succeq 0 \} \) is said to be a *normalized* if,

\[ \sum_{i=1}^n C_i = I d_k \]

and \( C_i \succeq 0 \).
Lemma 6.4.2. If a spectrahedral cone $K$ contains the positive orthant $\mathbb{R}_+^n$ then $K$ admits a normalized representation.

Proof. Let $C = \{C_i\}_{i \in [n]}$ be a spectrahedral representation of $K$. Let $U = \bigcap_{i=1}^n \ker(C_i)$ be the subspace in the kernel of all the $C_i$, and let $\Pi_{U^\perp}$ denote the projection on to $U^\perp$. It is easy to check that for all $x \in \mathbb{R}^n$,

$$\sum_i C_i x_i \succeq 0 \iff \sum_i \Pi_{U^\perp} C_i \Pi_{U^\perp} x_i \succeq 0.$$  

By a basis change of $\mathbb{R}^n$, that contains a basis for $U^\perp$, we obtain matrices $C'' = \{C''_i\}_{i \in [n]}$ in $\mathbb{R}^{k' \times k'}$ where $k' = \dim(U^\perp)$ such that $K = K_{C''}$.

Furthermore, since the $i^{th}$ basis vector $e_i$ is in the non-negative orthant, $e_i \in K$. This implies that $C''_i \succeq 0$ for each $i$. For each $u \in U^\perp$, there exists $C_i$ such that $u^T C_i u > 0$, which implies that $\sum_i u^T C''_i u > 0$. In other words, we have $M = \sum_i C''_i \succ 0$ is positive definite.

The normalized representation of $K$ is given by $C'' = \{M^{-1/2} C''_i M^{-1/2}\}_{i \in [n]}$. By definition, the representation is normalized in that $\sum_{i \in [n]} C''_i = \Id_{k'}$.

For two spectrahedral representations given by matrices $C = \{C_i\}_{i \in [n]}$ and $C' = \{C'_i\}_{i \in [n]}$, define the distance between the representations as,

$$\text{mdist}(C, C') = \max_i \|C_i - C'_i\|$$

Lemma 6.4.3. Suppose $C, C'$ are normalized spectrahedral representations of $K_C$ and $K_{C'}$ respectively. Then, $h\text{dist}(K_C, K_{C'}) \leq n^{3/2} \cdot \text{mdist}(C, C')$

Proof. Let $B(0, 1)$ denote the $\ell_\infty$ unit ball in $\mathbb{R}^n$. For every $x \in K_C \cap B(0, 1)$,

$$\sum_{i \in [n]} C'_i x_i = \sum_{i \in [n]} C_i x_i + \sum_{i \in [n]} (C_i - C'_i) x_i$$

$$\geq -n \text{mdist}(C, C') \cdot \Id_k$$
where we are using the fact that $\sum_i C_i x_i \preceq 0$. This implies that the point $x' = x + n \cdot \text{dist}(C, C') \cdot 1 \in K'$. To see this, recall that the representation is normalized in that $\sum_i C'_i = \text{Id}_k$. Therefore, we get
\[
\sum_i C'_i x'_i = \sum_i C'_i x_i + n \cdot \text{dist}(C, C') \sum_i C'_i \succeq 0
\]

The lemma follows by observing that $\|x - x'\|_2 \leq n^{3/2} \cdot \text{dist}(C, C')$.

6.5 Proof of The Main Theorem

Proof of 6.0.1. Set $\epsilon$ smaller than $R$ and $R_2$. Theorem 6.1.3, Lemma 6.2.1 and Lemma 6.3.1 together imply a family of hyperbolic polynomials $P$ with the following properties:

1. $|P| = 2^{|M_d|}$ where $|M_d| \geq (n/d)\kappa d$ for some absolute constant $\kappa > 0$.
2. For all $p \neq p' \in P$, $\text{hdist}(K_p, K'_p) \geq \gamma$ for $\gamma > \frac{1}{n^3}$.
3. For every $p \in P$, the positive orthant $\mathbb{R}^n_+$ is contained in $K_p$.

The last observation follows from the fact that positive orthant is contained in $K_{\epsilon_d}$, and the perturbations of $\epsilon_d$ in $P$ are small enough to keep all the coefficients non-negative.

Suppose each of the hyperbolicity cones $K_p$ admitted a $\gamma/3$–approximate spectrahedral representation in dimension $B$. By Lemma 6.4.2, for each cone in $K_p$, there exists a normalized $\gamma/3$-approximate spectrahedral representation $C_p$ in dimension $B_p \leq B$. Further, by Lemma 6.4.3, for every pair of polynomials $p, p' \in P$, their corresponding $\gamma/3$-approximate spectrahedral representations satisfy $\text{dist}(C_p, C_{p'}) \geq n^{-3/2} \gamma / 3 = \eta$.

Notice that in every normalized spectrahedral representation $C_p$ in dimension $B$, every matrix $C \in C_p$ satisfies $C \succeq 0$ and $C \preceq \text{Id}_B$. This implies that $\|C\|_{\infty} \leq \sqrt{B}$. By a simple volume argument, for every $\eta > 0$, the number of normalized spectrahedral representations in $\mathbb{R}^{B \times B}$ whose pairwise distances are $\geq \eta$ is at most $\left(\sqrt{B/\eta}\right)^{nB^2}$. Since every cone $K_p$ for $p \in P$ admits a normalized spectrahedral representation of dimension at most $B$, we get that
\[
|P| \leq t \left(\sqrt{B/\eta}\right)^{nB^2}
\]
which implies that
\[
B^2 \log B \geq |M_d| / \log(n^{3/2}/\gamma) \geq \frac{1}{n \log n} \cdot (n/d)^{\kappa d}.
\]
which implies the lower bound of $B \geq (n/d)^{\kappa' d}$ for some constant $\kappa' > 0$. 

Bibliography


