Essays on Reference-Dependent Preferences

by

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Abstract

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This dissertation consists of two chapters exploring the economic implications of reference-dependent preferences over incentive design and belief formation.

The first chapter studies the intertemporal allocation of incentives in a repeated moral hazard model. Beside consumption utility, reference-dependent agents experience utility from changes in their expectations about present and future income caused by the performance measure realization. In contrast to the prediction with classical preferences but consistent with real-world contracts, this paper shows that if consumption utility is not too concave and if changes in expectations about present income carries sufficiently larger weight in utility than changes in expectations about future income, the optimal contract defers all present incentives into future payments by setting a present fixed wage. Despite this prediction, I further prove that several standard features of the contract with classical preferences—no rents to the agent, conditions to achieve first-best cost and non-optimality of random contracts—still hold.

The second chapter studies the temporal path of subjective beliefs when a reference-dependent agent who experiences standard anticipatory utility and utility from changes in these anticipatory feelings waits T periods for a binary outcome realization. Following the optimal beliefs literature, in each period the agent chooses a belief about her likelihood of success to maximize her intertemporal utility. Consistent with the empirical evidence, the model predicts that optimism decreases as the pay-off date approaches if the outcome is important enough or if the agent is sufficiently loss averse. Intuitively, when the pay-off date is distant disappointment is less salient than the joy of hoping favorable outcomes; as the realization date gets closer, however, the threat of disappointment becomes important. Applying the model to the optimal timing of productivity bonuses, I find these should be granted as frequently as possible because optimism acts as a non-pecuniary motivator that allows the principal to induce the desired effort path at a cheaper expected cost.

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Chapter 1

Intertemporal Incentives Under Loss Aversion

1.1 Introduction

Classical models of dynamic moral hazard predict that both future and present payments must be made contingent on current performance (Lambert (1983), Rogerson (1985), Murphy (1986), Malcomson and Spinnewyn (1988), Chiappori, Macho, Rey, and Salanié (1994).) This prediction, however, is at odds with the observation that real-world contracts seldom use current payments to motivate the agent's present action. For instance, MacLeod and Parent (1999) and Parent (2001) find that only 4-10% U.S workers are paid through commissions or piece-rates.[1] In contrast, contracts completely deferring incentives into future payments are common. End of the year wage revisions, future promotions (Baker, Jensen, and Murphy (1988), Baker, Gibbs, and Holmstrom (1994), Treble, Van Gameren, Bridges, and Barmby (2001)), yearly productivity bonuses (Joseph and Kalwani (1998), Steenburgh (2008)), etc. in combination with present-fixed wages are widely used in real-word contracts. For instance, based on the Executive Compensation Surveys for 1974-1986, Jensen and Murphy (1990) find that for each $1000 change in the shareholder's wealth, the CEO's present and next year salary and bonus increase by 2 cents. In contrast, their wealth from salary revisions, outstanding stock options and performance-related dismissals increase in about 75 cents suggesting that present incentives are negligible relative to incentives deferred into future payments.

One explanation to this puzzle relies on the difficulty to find objective outcome measures. When a formal output-based contract is not verifiable incentives must be set through–possibly informal–contracts rewarding long-term performance rather than short-term output (Eaton and Rosen (1983), Huck, Seltzer, and Wallace (2004).)[2] Equivalently, hidden information about the agent's ability can push the principal to use future incentives rather than present ones. Despite these explanations are certainly at play in the real world, evidence suggests that principals tend to defers incentives even in highly repetitive jobs with well defined and easily

[1] In Parent (2001) the percentage depends on the survey used. Adding a productivity bonus component and excluding profit sharing plans, the percentage increases to 20% using the National Longitudinal Survey of Youth. Barkume (2004) finds similar results. Using data for 2001 on the Bureau of Labor Statistics Employment Cost Index (ECI), he estimates that only 7% of the average total employment in skilled production use commission, piece-rates or productivity bonuses to motivate employees. These numbers, however, are lower bounds since the incentive-pay definition of the ECI only includes individual payments (commission, piece-rates and production bonuses), omitting profit-sharing distributions, all-employee payments and other non-production bonuses.

[2] The observation that finding verifiable performance measures is difficult because of monitoring costs (Barkume (2004), Lazear (1999)), incentives towards skill acquisition (for a survey see Gibbons (1998)) and action complexity (Holmström and Milgrom (1991), MacLeod (2000)), shifted away the contract theory literature from the risk-incentive trade-off towards reviewing the assumptions over the economic environment. As a result, the literature developed a whole array of models that greatly improved the match between theoretical contracts and those observed in reality such as those on relational contracts (MacLeod and Malcomson (1989), Levin (2003)), multitasking (Holmström and Milgrom (1991)) and skill acquisition and career concerns (Holmström (1999), Gibbons and Murphy (1992)).

measurable outputs.[3]

This paper presents an alternative explanation to the optimality of fully deferred incentives. To this end I adapt and integrate the general model of dynamic reference-dependent preferences by Kőszegi and Rabin (2009) into the standard repeated moral hazard model of Rogerson (1985). The paper shows that under mild restrictions over the utility function a contract using only future payments to motivate the agent's present action is optimal. The intuition is as follows. Besides consumption utility, reference-dependent agents experience utility from the changes in their expectations about present and future consumption caused by the outcome realization. When the agent has rational expectations and he is loss averse, however, the expected utility of these changes in expectations is negative as the prospect of being disappointed by the outcome realization equals that of being pleasantly surprised by it. As a consequence, if the agent is more sensitive to receive news about his present payments relative to receive news about his future payments, a contract totally deferring incentives into future payments is optimal: it saves the agent from his most unpleasant source of changes in expectations while replaces lost incentives by increasing the sensitivity of future payments to the present outcome realization.

After a review of the related literature, Section 1.3 presents the basic framework. The set up presents a two-period model with a finite set of outcomes and effort levels. The agent is paid at the end of each period once the outcome has been realized. Following Rogerson (1985) the model assumes the agent has no access to credit, he cannot save between periods and both the agent and the principal can credibly commit to stay in the relationship until the end of period two.

The structure of the utility function and the timing of the principal-agent interaction are as follows. In an initial period zero, given the contract offered by the risk and loss-neutral principal, the agent forms effort plans to be executed in the upcoming two periods. Following Kőszegi and Rabin (2009), the model assumes the agent has rational expectations and thus his effort plans must be credible: he only forms effort plans he is willing to follow given the income expectations they generate and the future preferences they induce. With these effort plans and income expectations, the agent decides whether to accept or reject the contract.

At the beginning of period one, the agent executes an action, observes the outcome realization and receives his period-one payment. Immediately, he experiences two types of utility: standard consumption utility and reference-dependent utility. The reference-dependent utility represents the utility the agent gets from changes in his income expectations, triggered by the income news brought by the outcome realization.[4] Because of the dynamic nature of the model,

[3]Hutchens (1987) using data from the National Longitudinal Survey tests the hypothesis that delayed payments to motivate the agent are less likely to occur in jobs involving repetitive tasks as a proxy for low monitoring costs. He finds that workers in these type of jobs have a 9% smaller probability of a pension, 8% smaller probability of mandatory retirement and 18% shorter length of tenure. Even though suggestive that monitoring costs do play a role in the intertemporal allocation of incentives, these magnitudes indicate that other effects must also be in place. Lazear and Moore (1984) contrasts the steepness of the earnings temporal profiles of self-employed workers against those of wage and salary workers to investigate whether the evolution of earnings can be completely explained by human capital accumulation and skill acquisition. They find that salaries grow faster than productivity, suggesting that skill acquisition alone cannot account for the temporal behavior of wages. Similar results are found in Kotlikoff and Gokhale (1992) and Ilmakunnas, Maliranta, and Vainiomäki (2004). From the theoretical perspective, Akerlof and Katz (1989) explicitly show that in a classical framework future incentives are not a perfect substitute for incentives coming from present payments. As a consequence, models such as those in career concern and termination threats (Stiglitz and Weiss (1983), Shapiro and Stiglitz (1985), MacLeod and Malcomson (1988), Sen (1996), Banks and Sundaram (1998)) simply assumed that incentives were totally deferred into future payments. Lazear (1981) is the only exception. See Section 1.2.

[4]Because this paper's results depend crucially on the reference-dependent utility experienced from the income domain, I omit from this introduction the description of the reference-dependent utility in the effort domain. Such

this reference-dependent utility or "gain-loss utility" has two components. First, "contemporaneous gain-loss utility" capturing the utility the agent gets by comparing the expectation he formed in period zero about his period-one payments with the actual payment received in period one.[5] Second, whenever second-period contracts depend on the first-period outcome, the agent also experiences "prospective gain-loss utility" capturing the utility the agent gets by comparing the expectation formed in period zero about his period-two payments with the expectation he holds after the first-period outcome realization about these future payments . In all these comparisons the agent is loss averse: receiving a payment lower than expected hurts more than receiving an income higher than expected is pleasant. Finally, at the end of the period, the agent updates his effort plans for the second period and forms the corresponding expectations about his future income.[6,7]

In a second and final period, the agent executes effort, receives the corresponding payments, and the relationship ends. Just as in period one, after the outcome realization the agent experiences consumption utility along with contemporaneous gain-loss utility from comparing his actual payment with the income expectations he formed at the end of period one.

Two parameters index the total instantaneous utility composed of the sum of consumption utility and gain-loss utility. First, a reference-dependent parameter $\eta \geqslant 0$ shows the importance of gain-loss utility–contemporaneous plus prospective–relative to consumption utility. When $\eta = 0$, this preference structure reduces to classical reference-independent preferences. Second, a parameter $\gamma \geqslant 0$ shows the relative importance of contemporaneous gain-loss utility relative to prospective gain-loss utility. Intuitively it shows how sensitive the agent is to receive information—and thus to changes in his expectations—about present payments relative to receive information about his future payments. This parameter γ is the most important parameter of the model as predictions crucially depend on its magnitude. Thus, to isolate its role, the model assumes the agent maximizes the undiscounted sum of his instantaneous utilities.[8]

Section 1.4 characterizes some basic properties of the optimal long-term contract with dynamic reference-dependent preferences. I start exploring the shape of the optimal contract when the state perfectly reveals the action. I prove that in these circumstances, the optimal contract does not depend on η, the reference-dependent parameter. Intuitively, when the outcome perfectly reveals the agent's action there is no uncertainty and thus the agent cannot be surprised by the payment he actually gets and thus the reference-dependent component of the utility function is zero. As a consequence, the first-best contract with reference-dependent preferences equals that with classical preferences. Next, I provide a basic characterization of

exclusion is only made for the sake of clarity as the general model in Section 1.3 includes reference-dependent utility in both domains.

[5]Notice that this description implicitly defines the reference point as previous-period beliefs. Since the agent forms his effort plans and income beliefs based on the contract offered by the principal, this reference point specification is in the same spirit as Hart and Moore (2008). It differs from their model in that the reference point is endogenously determined and considers all payments rather than one particular scenario to which–for exogenous reasons–the agent feels entitled to.

[6]In general, gain-loss utility captures the idea that news changing previous beliefs about present and future effort and income generate a present utility flow that is part of the agent's decision utility. This basic idea that agents get utility from changes in consumption levels rather than level themselves was introduced in the seminal prospect theory paper by Kahneman and Tversky (1979). They further stated that decreases are more heavily felt than same-sized increases, phenomenon they called loss aversion.

[7]Prospective gain-loss utility is a form of anticipatory utility in the spirit of Caplin and Leahy (2001) and many others. The key difference is that this is a referent-dependent feeling of anticipation where the reference point corresponds to the previous beliefs for the corresponding period. Thus, the use of this type of utility should not be regarded as a radical departure from the behavioral literature.

[8]Besides assuming that γ is non-negative, the model does not impose restrictions over the value of this parameter. Kőszegi and Rabin (2009) assume $\gamma < 1$ from the start.

the second-best contracts. Broadly, I show that several features of the optimal contract with classical preferences still hold when agents are reference-dependent. I start proving that the optimal contract exists when agents are reference-dependent, to then show that such a contract is unique if the consumption utility function is strictly concave, just as with classical preferences. Furthermore, and because of a very similar argument to that with classical preferences, the optimal contract with reference-dependent preferences does not leave rents to the agent. Then I prove that the second-best contract achieves the first-best cost when the principal wants to implement the minimum effort in every period (least cost path) and when there is at least one outcome with probability zero under the chosen action but positive under the lower cost actions (shifting support). However, and most importantly, in general first-best is not achieved when the consumption utility function is linear because the agent is loss averse and thus the risk-incentive trade-off is still present.

Section 1.5 presents the main prediction: when the agent is more sensitive to changes in his expectations about present income than about his future income (γ small) and consumption utility is not too concave, the optimal contract defers all incentives for present effort into future payments. The intuition behind this result relies on reference-dependent agents disliking fluctuations on their expectations about present and future income: in equilibrium the prospect of being disappointed by the outcome realization equals that of being pleasantly surprised by it. As a consequence, the expected utility of the reference-dependent component of the utility function is always negative for loss averse agents. This has strong implications over how the principal wants to allocate incentives across periods: setting a first-period fixed wage increases the total utility the agent gets out of the contract as expectations about fix payments cannot fluctuate with rational expectations. Setting a fixed-first-period wage, however, is done at the expense of shutting down incentives coming from first-period consumption and contemporaneous gain-loss utilities. To replace the incentives lost the principal increases as much as necessary incentives coming from second-period payments by making them more sensitive to the first-period outcome so to trigger incentives from second-period consumption utility. This, however, has the indirect effect of increasing the prospective gain-loss disutility as now future payments fluctuate more. If γ is small, however, the gains from shutting down contemporaneous gain-loss disutility outweighs the losses from increasing prospective gain-loss disutility. As a consequence, the principal can decrease some payments and still implement the desired effort path.

As the latter logic illustrates, the reference-dependent component of the utility function pushes towards totally deferring incentives into the future. Consumption utility, to the contrary, pushes towards spreading incentives across periods since splitting incentives across periods keeps the agent's marginal utility high, decreasing the risk premium the principal has to pay to the risk averse agent for bearing risk.[9] As a consequence, when consumption utility is concave a sufficient condition for the first-period fixed-wage contract to be optimal is the consumption utility function being not too-concave so that the reference-dependent effect dominates the consumption-utility effect. Evidence of this assumption can be found in Rabin (2000) who shows that for reasonable risk taking behavior the agent's consumption utility function is compatible with risk neutrality for wealth levels of the order of an average monthly salary.[10]

Section 7 explores the convenience of using performance measures that are uncorrelated with the agent's unobserved action when incentives are fully deferred into future payments. I prove that, just as in the case of classical preferences, using ex-post-random contracts is never optimal since a contract paying in each period the certainty equivalent of the period-random

[9]Such an intertemporal risk-sharing argument drives the prediction of classical-repeated-moral-hazard models where is optimal to make present and future payments contingent on current performance.

[10]A reasonable length period for this model is a month because such period is long enough so that the agent can form plans, but not so long that the agent gets used to his income beliefs.

contract allows the principal to decrease the expected cost of implementing the desired action path. Contrary to the classical case, this result is valid even if consumption utility is linear.

Finally, to capture the intuition that the information the agent gets in period one about period-two payments may be relevant to review the agent's effort plans, Section 1.7 assumes prospective gain-loss utility is realized after second-period effort plans are updated. In a two-action set up, I show that it is harder to motivate the agent after a first-period success than it is to motivate him after a first-period failure. This is because, even though the realization of the prospective gain-loss utility is always increasing the second-period effort plans, loss aversion implies that the effect is stronger after a first-period failure. As a consequence the second-period incentive compatibility restriction is easier to satisfy after a loss as the agent will try to compensate the forgone payment increasing the likelihood of higher future payments. A similar phenomena occurs with classical preferences due to decreasing marginal utility, however, with reference-dependent preferences the prediction depends crucially on the agent having planned to exert high effort and also in the timing between the review of plans and the actual execution of the action. If the principal allows enough time after X_1 has been observed and the action execution, the agent may get used to his effort expectations, rendering the realization of prospective gain-loss utility irrelevant for planning.

Section 1.8 summarizes and discusses some extensions.

1.2 Related Literature

This paper builds on two different literatures. First, the classic contract-theory literature on dynamic moral hazard. Lambert (1983) and Rogerson (1985) were the first to prove that when the principal and the agent engage in a repeated interaction in which the agent is paid at the end of every period the optimal contract displays "memory in wages": second-period payments must be contingent on the first-period outcome realization. This result was latter generalized to T periods by Murphy (1986). Since then a vast amount of work has explored several features of the optimal contract, such as the role of commitment, the conditions for the long-term contract to be implementable as a series of one-period contracts (Malcomson and Spinnewyn (1988), Fudenberg, Holmstrom, and Milgrom (1990), Rey and Salanie (1990), Chiappori, Macho, Rey, and Salanié (1994)) or the role of the discount factor in achieving efficiency when the principal-agent relationship is infinitely repeated (Radner (1986), Spear and Srivastava (1987)). For the purposes of this paper, however, all this literature predicts that optimal contract spreads incentives across periods or concentrates them into present payments when the long-term contract is spot implementable.

Second, this paper builds on the literature on reference-dependent preferences. This literature started with the seminal work of Kahneman and Tversky (1979) who proposed that agents experience utility from changes in consumption levels where losses hurt more than same-sized gains are pleasant, phenomenon they called loss aversion. Subsequently, a whole array of reference-dependent models were developed. Bell (1985), Loomes and Sugden (1986), Gul (1991), Munro and Sugden (2003), Sugden (2003), Matthey (2005), Kőszegi and Rabin (2006) and De Giorgi and Post (2008) among others have presented static models of reference-dependent preferences. The model presented in Kőszegi and Rabin (2009) extends prospect theory in two ways. First, they provide a generalization of the gain-loss utility function to a dynamic setting by introducing the notion of prospective gain-loss utility. Second, they endogenize the reference point by assuming the agent has rational expectations, and thus the reference point corresponds to the agent's rational beliefs derived from the economic environment in which the problem is embodied.[11]

[11] Kőszegi and Rabin (2009) is to the best my knowledge the only one that explicitly considers reference-dependent preferences in a dynamic setting besides the models in Barberis and Huang (2001) and Barberis,

There is in fact a small but growing body of literature on behavioral contract theory exploring the role of reference-dependent preferences on hidden-action models. deMeza and Webb (2007) explore the implications of several static models of reference-dependent preferences on a static moral hazard model. They find that optimal contracts have intervals where payments are insensitive to performance and that rewards are more frequently used than punishments. The model of Herweg, Müller, and Weinschenk (2008) is the closest to this work. They build a model of static moral hazard with static reference-dependent preferences as in Kőszegi and Rabin (2006). They conclude that (1) the optimal contract has two levels of payments when the agent has linear consumption utility and (2) that the contract under concavity is significantly simpler than the one predicted by classical theory. My model differs from theirs mainly in its dynamic nature which allows me to focus on the intertemporal allocation of incentives rather than the contract's complexity. Moreover, my modeling approach to the economic environment and the formal approach to solve the principal's problems are also different. Herweg, Müller, and Weinschenk (2008) use a continuous action set up with a linear specification of the conditional probability distribution and thus they use the first-order approach to solve the principal's problem. To the contrary, I use the discrete approach of Grossman and Hart (1983) and impose no restrictions over the probability distribution besides the usual monotone-likelihood property.[12]

Finally the optimality of deferred incentives has received scant attention in the literature. Lazear (1981) is to the best of my knowledge the only paper that has explicitly explored the optimality of deferred compensation. His motivation was to explain why wage's age-tenure profiles are steeper than productivity tenures and the existence of compulsory retirement policies. he proposes a model that builds on the intuition that the firm can increase the workers' effort by overpaying him in the late stage of his career and underpaying him during his early years, because the agent will work hard to avoid the possibility of being dismissed. This compensation structure, however, increases the incentives of the firm to terminate the contract before the stipulated date. The optimal wage-profile thus trade-offs the cost for the firm of not honoring the contract while maximizes the worker's lifetime earnings subject to a zero-profit condition. Lazear's model thus, greatly differs from the model presented in this paper. First, his model gives up on the risk incentive trade-off to explain the optimality of delayed incentives. But most importantly, nothing in the model's basic logic prevents the optimality of present-contingent payments, a very convincing argument with classical preferences from a consumption smoothing perspective (see Akerlof and Katz (1989).)

1.3 The Model

The principal-agent relationship lasts three periods, $t \in \{0, 1, 2\}$. In period zero the agent (he) receives a take-it-or-leave-it offer (TIOLI) from the principal (she) and decides whether to accept or reject it. If he accepts, both parties credibly commit to stay in the relationship until the end of period two. In each of the two subsequent periods the agent privately chooses an effort level e_t from a finite set $\mathcal{E} = \{e_1, \ldots, e_J\} \subset \mathbb{R}$. Next, a verifiable *iid* performance measure X_t is drawn from the finite set $\mathcal{X} = \{x_1, \ldots, x_N\} \subset \mathbb{R}$, where $x_1 < x_2 < \cdots < x_N$. Let $\Pi^X(x_n|e_j)$ be the associated distribution function conditional on effort and let $\pi_n^j \equiv \Pi^X(x_n|e_j) - \Pi^X(x_{n-1}|e_j) > 0$ $\forall n, j$ represent its density. I assume that the agent is paid at the end of each period and he has

Huang, and Santos (2001) which are specific to the financial market. The model of Matthey (2005) is the closest as she also introduces the notion that agents derive utility from changes in beliefs about future consumption.

[12] Iantchev (2005) and Daido and Itoh (2005) also build models of static moral hazard with reference-dependent preferences to explain the Pygmalion Effect and contract complexity respectively.

no access to credit nor can he save between periods.[13,14]

The principal is assumed to be risk and loss neutral. Her gross benefit equals the realization of the performance measure X_t, and her only cost corresponds to the payment made to the agent.[15] A long-term contract denoted by $S = (S_1(X_1), S_2(X_1, X_2))$, $S_t \subset \mathbb{R}$ for $t = 1, 2$, is a set of N+N^2 contingent payments governing the principal-agent relationship during periods one and two. I refer to S_1 as the first period contract, where s_n represents its realization if $X_1 = x_n$. I refer to S_2 as the second period contract or "continuation contract" possibly depending on both X_1 and X_2 where s_{nm} represents its realization if $X_1 = x_n$ and $X_2 = x_m$. If $s_{nm} \neq s_{n'm}$ for at least one $n \neq n'$ and m, it is said that the contract displays *memory in wages* (Chiappori, Macho, Rey, and Salanié (1994)).

I depart from this standard framework solely by assuming the agent has reference-dependent preferences as in Kőszegi and Rabin (2009). Accordingly, agents experience two types of utilities, standard consumption utility and utility from changes in their beliefs about their present and future consumption. In a moral hazard model, the two standard consumption domains are payments and effort and thus period t utility flow corresponds to:

$$v_t = V(s_t, e_t) + \eta \sum_{\tau=t}^{2} \gamma_{t,\tau} G(\widetilde{s}_{t,\tau}, \widetilde{e}_{t,\tau} | \widetilde{s}_{t-1,\tau}, \widetilde{e}_{t-1,\tau}) \tag{1.1}$$

The first component, $V(s_t, e_t)$, represents classical reference-independent consumption utility from current payment s_t and current executed effort e_t. Assumption 1 presents the structure imposed over this function.

Assumption 1 *(Consumption Utility)*
The consumption utility function is of the form $V(s_t, e_t) = u(s_t) - c(e_t)$ where $u(s_t)$ is twice continuously differentiable with $u'(s) > 0$, $u''(s) \leqslant 0$ for all $s \geqslant \underline{s}$ where $\lim_{s \downarrow \underline{s}} u(s) = -\infty$ and $c(e_j) > c(e_k)$ for all e_j, e_k satisfying $e_j > e_k$.

Assumption 1 is standard in the contract theory literature. It states that the agent's preferences are separable across effort and income and that consumption utility in payments is strictly increasing and weakly concave, whereas in the effort domain is strictly increasing and always non-negative. Assumption 1 also ensures that the agent can be motivated by assuming the existence of a payment $\underline{s}$ that is a thereat to the agent.

The second term in equation (1.1), $G(\widetilde{s}_{t,\tau}, \widetilde{e}_{t,\tau} | \widetilde{s}_{t-1,\tau}, \widetilde{e}_{t-1,\tau})$, corresponds to the reference-dependent component of the utility function or "gain-loss" utility. It represents the utility the

[13]Relaxing this assumption does not affect the characteristics of the optimal contract with standard preferences that concerns us, i.e., second-period wages depending on first-period outcome and contingent payments in every period (Chiappori, Macho, Rey, and Salanié 1994, Malcomson and Spinnewyn 1988).

[14]Since my model focuses on formal or written incentive schemes, verifiability is sufficient. However, along the paper, I will refer interchangeably between verifiability and observability. The assumption of X_t being *iid* is standard in the literature and avoids complications due to learning. Extending the analysis to X_t in the Euclidean space $\mathbb{R}^N$, N>1 and $e_j \in \mathbb{R}^J$, J>1 may not be innocuous since issues related to effort substitution and task conflict arise, as the literature in multitasking has acknowledged (see for example Dewatripont, Jewitt, and Tirole (2000) and Holmström and Milgrom (1991).) Even though generalizations of my model to learning or multitasking may be interesting, they are not the focus of this paper.

[15]The assumption of the principal being risk neutral can be justified in this paper's interest in employment contracts. In such environment it is reasonable to assume that employers can diversify risk as they have several employees. Extending the analysis though to a risk-averse principal, where her gross benefits are represented by the existence of a well behaved function $b : \{x_1, \ldots, x_N\} \longrightarrow \mathbb{R}$, $b''(\cdot) < 0$ is straightforward when establishing the existence of conditions needed for the main result, but it may not be innocuous to establish the strength of those conditions. See a discussion in Section 1.5.

agent gets today from departures of present and future consumption from the effort plans and income expectations the agent made in the previous period for the upcoming periods. Formally, at the end of periods $t = 0, 1$, the agent forms an effort plan $\widetilde{e}_{t,\tau} \in \mathcal{E}$ for *all* future periods $\tau > t$. Notice that since the continuation contract S_2 depends on X_1, second period effort plans must be contingent on the first period realization, i.e., $\widetilde{e}_{t,2} \equiv \widetilde{e}_{t,2}(X_1)$, $t = 0, 1$. Given the contract, and given knowledge of the probability distribution over outcomes these effort plans induce an expected probability distribution for future payments, $\widetilde{\Pi}^S_{t,\tau} \equiv \prod_{k=t+1}^{\tau} \Pi^S_k(\widetilde{e}_{t,k})$ for $t = 0$ and $\tau = 1, 2$ and for $t = 1$ and $\tau = 2$. For the ease of notation I refer to such distribution by its realization $\widetilde{s}_{t,\tau}$. Thus, $G(\widetilde{s}_{t,\tau}, \widetilde{e}_{t,\tau}|\widetilde{s}_{t-1,\tau}, \widetilde{e}_{t-1,\tau})$represents the gain-loss utility the agent gets in period t from departures of present and future consumption $(\widetilde{s}_{t,\tau}, \widetilde{e}_{t,\tau})$ from the consumption beliefs formed in$t-1$ about present and future consumption, i.e.$(\widetilde{s}_{t-1,\tau}, \widetilde{e}_{t-1,\tau})$ for all$\tau \geqslant t$. [16]

For this paper purposes it is very important to distinguish two types of gain-loss utility arising from the previous definition. First, "contemporaneous gain-loss utility" from departures of period t consumption from the plans made in the previous period $t-1$ for the future period $\tau = t$ denoted by $G(s_t, e_t|\widetilde{s}_{t-1,t}, \widetilde{e}_{t-1,t}) \equiv G(\widetilde{s}_{t,t}, \widetilde{e}_{t,t}|\widetilde{s}_{t-1,t}, \widetilde{e}_{t-1,t})$. Second ""prospective gain-loss utility" from departures of the plans made in period t for the upcoming periods $\tau > t$ from the plans made in the previous period $t-1$ for these same future periods, denoted $G(\widetilde{s}_{t,\tau}, \widetilde{e}_{t,\tau}|\widetilde{e}_{t-1,\tau}, \widetilde{s}_{t-1,\tau})$ $\tau > t$.

Two parameters index this utility function. First, a parameter $\gamma_{t,\tau}$ represents the relative importance of the two types of gain-loss utility in the current utility flow, i.e., the relative weight the agent assigns to changes in his expectations about present consumption relative to changes in expectations about future consumption. I normalize $\gamma_{t,t} = 1$ $\forall t$ and denote $\gamma_{1,2} \equiv \gamma > 0$ and make no further assumptions over its value. Second, the parameter $\eta \geqslant 0$ is a "reference-dependence parameter" indicating the strength of gain-loss utility—the reference-dependent component of the utility function—relative to consumption utility. Notice that if $\eta = 0$, these preferences reduce to classical reference-independent preferences.

To illustrate gain-loss utility in the payment domain consider the following example (the effort domain is analogous.) A worker with linear consumption utility has historically been paid every Friday a salary of 100 and expects to be paid the same in the future. Unexpectedly—and for reasons that are not of interest—this Friday he finds out that his weekly salary has been permanently reduced to 80. I want then to compare how does that same Friday utility flow changes when the agent has reference-dependent preferences relative to classical preferences. If the worker has classical reference-independent preferences that same Friday utility flow decreases 20. If the worker has reference-dependent preferences, his Friday utility flow also decreases by 20 because of consumption utility. Because of gain-loss utility, however, his utility flow also diminishes from two extra sources. First, he experiences a painful sensation of loss from comparing the 80 he will actually earn today instead of the 100 he expected to earn. Second, the agent also experiences a utility decrease from knowing that in all *upcoming* Fridays he will be earning 20 less. Thus, in total the reference-dependent utility flow decreases by $20+\eta[\mathrm{G}(80|100)+\gamma 99\mathrm{G}(80|100)]>20$.[17]

Assumption 2 presents the structure imposed over the gain-loss utility function. Con-

[16] The assumption that the agent experiences utility directly over money and not over the related consumption is standard in moral hazard models but is not neutral in the Kőszegi and Rabin (2009) model. In fact, the original model is capable of explaining why agents derive utility form money: because money brings news about future consumption. To further conform to the contract theory literature, I omit such interesting observation.

[17] Evidence of this preference structure can be found in Loewenstein (1988), who finds that people are willing to pay more to avoid delaying the delivery of a good when they expect to receive it immediately than to speed up the delivery when they expected to receive it later. For a more detailed description of the evidence for the relevance of prospective gain-loss utility as decision utility, see Kőszegi and Rabin (2009) and Matthey (2005). For recent evidence of expectations as the reference point, see Abeler, Falk, Götte, and Huffman (2009).

sider the payment domain. For $p \in (0,1)$, let $s_{\widetilde{\Pi}^S_{t,\tau}}(p;\widetilde{e}_{t,\tau})$ be the payment level at percentile p of the lottery of payments $\widetilde{\Pi}^S_{t,\tau}(\widetilde{e}_{t,\tau})$ defined as usual by $\widetilde{\Pi}^S_{t,\tau}(s_{\widetilde{\Pi}^S_{t,\tau}}) \geqslant p$ and $\widetilde{\Pi}^S_{t,\tau}(s) < p \ \forall s < s_{\widetilde{\Pi}^S_{t,\tau}}$ (see Section 3.1 for the analogous specification of gain-loss utility in effort.)

Assumption 2 *(Gain-loss Utility)*

(i) $G(\widetilde{s}_{t,\tau}, \widetilde{e}_{t,\tau} | \widetilde{e}_{t-1,\tau}, \widetilde{s}_{t-1,\tau}) = G(\widetilde{s}_{t,\tau}(\widetilde{e}_{t,\tau}) | \widetilde{s}_{t-1,\tau}(\widetilde{e}_{t-1,\tau})) + G(\widetilde{e}_{t,\tau} | \widetilde{e}_{t-1,\tau})$

(ii) for $t = 1$ *and* $\tau = 1, 2$ *and for* $t = 2$ *and* $\tau = 2$*, the* $G(\cdot|\cdot)$ *function corresponds to:*

$$G(\widetilde{s}_{t,\tau}(\widetilde{e}_{t,\tau}) | \widetilde{s}_{t-1,\tau}(\widetilde{e}_{t-1,\tau})) = \int_0^1 \mu\Big(u\big(\widetilde{s}_{\widetilde{\Pi}^S_{t,\tau}}(p;\widetilde{e}_{t,\tau})\big) - u\big(\widetilde{s}_{\widetilde{\Pi}^S_{t-1,\tau}}(p;\widetilde{e}_{t-1,\tau})\big)\Big) dp \qquad (1.2)$$

(iii) where $\mu(x) = x$ *if* $x \geqslant 0$ *and* $\mu(x) = \lambda x$ *if* $x < 0$ *where* $\lambda \geqslant 1$*.*

Assumption 2 part (i) states that the gain-loss utility function is separable between consumption domains.[18] Part (ii) defines how the agent compares his old expectations with the new ones. Following Kőszegi and Rabin (2009), I assume the agent makes a rank order comparison between the expected probability distribution and that held at the beginning of the period once the payments news have arrived. Intuitively, the agent compares the worst percentile of his old beliefs with the worst percentile of his new beliefs, the second worst percentile of his old beliefs with the second worst percentile of his new beliefs, and so on for all percentiles.[19] Finally, part (iii) states that the value function $\mu(\cdot)$ in equation (1.2) is a piece-wise linear function with a slope of one in the gain domain and a slope of $\lambda > 1$ in the loss domain. The parameter λ is the "loss-aversion parameter", representing how intense losses are felt relative to gains.[20,21]

To illustrate how the agent makes this rank order comparison consider the following example in the payments domain (the effort domain is analogous.) A worker lives and works during two periods. His payments are governed by the following long-term contract: the first-period contract pays 100 after a first-period success and 80 after a failure. If a success occurs in the first-period, in the second period the agent gets the same contract. If a failure occurs in the first period then the second period contract pays 60 or 40 according to the second-period outcome. Assume further that success and failures are equally likely in every period and that η =0.1 and γ =1.

[18] Notice that the notation for gain-loss utility in payments makes explicit the effort plan generating the corresponding income belief.

[19] Notice that for the (t,τ) pairs (1,1) and (2,2), equation (1.2) corresponds to contemporaneous gain-loss utility. In such a case, given x_n, the agent compares a particular payment against a non-degenerate payment distribution (the distribution of payments before x_n is realized corresponding to S_1 or S_2 depending on whether t is 1 or 2). For the pair (t,τ)=(1,2) equation (1.2) corresponds to prospective gain-loss utility. In such case, given x_n, the agent compares two non-degenerate distributions: the distribution of second-period payments before x_n is observed with that after x_n is observed.

[20] To capture the intuition that the strength of the reference-dependent utility is proportional to the importance of consumption utility, present and past beliefs are stated in their consumption utility equivalents. To see the importance of this assumption, consider the following example in Kőszegi and Rabin (2006). When choosing between whether to accept or reject a 50% lottery of winning or loosing 100 dollars, versus a 50% lottery of winning or loosing a paper clip, it is reasonable to assume that the agent will be more loss-averse over the first lottery rather than the second.

[21] Whenever a piece-wise linear value function is assumed, loss aversion becomes a necessity to experience reference-dependent preferences. See formulas in Section 3.1. In a more general specification of the value function, in which the gain domain is assumed to be concave and the loss domain is assumed to be convex–phenomenon called diminishing sensitivity–the agent can experience reference-dependent preferences without being loss averse.

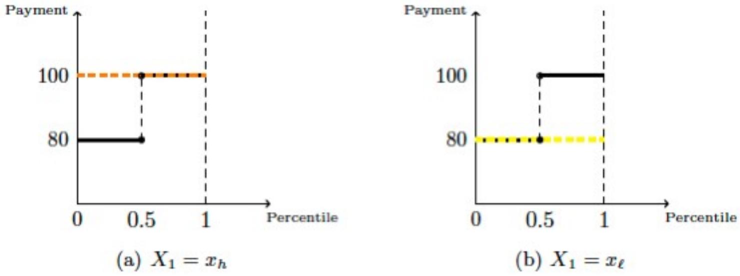


Figure 1.1: First-Period Contemporaneous Gain-Loss Utility Realization

Consider first contemporaneous gain-loss utility. Panel (a) in Figure 1.1 illustrates first-period contemporaneous gain-loss utility if a success is observed in the first period. The black solid line represents the agent's period-zero expectations about period-one payments: with equal probability he expects to get 100 or 80. The dotted line represents the agent's updated expectations. Since a first-period success is observed then his—stochastic—expectations degenerate into a degenerate probability distribution putting probability one in 100. Thus for all the percentiles between 0 and 0.5 he expected to get 80 but he actually gets 100 whereas for all percentiles 0.5 to 1 he expected to get 100 and he actually gets 100. Thus, in the gain scenario he gets 0.5μ(100-80)+0.5μ(100-100)=0.5×20η=10η. Panel (b) in (a) in Figure 1.1 illustrates first-period contemporaneous gain-loss utility if a failure is observed in the first period. Notice that the expected payment distribution is the same as before. This time, however, the updated distribution corresponds to a degenerate probability distribution putting probability one in 80. Thus for all the percentiles between 0 and 0.5 he expected to get 80 and he actually gets 80 whereas for all percentiles 0.5 to 1 he expected to get 100 and he actually gets 80. Thus, in the loss scenario he gets 0.5μ(80-80)+0.5μ(80-100)=-0.5×$\lambda\eta$20=-$\lambda\eta$10. A similar analysis is valid for second-period gain-loss utility under any continuation contract.[22]

Consider now prospective gain-loss utility. Panel (a) in Figure 1.2 illustrates prospective gain-loss utility if a success is observed in the first period. The black solid line represents the agent's period-zero expectations about period-one payments: the agent expects to be paid 40, 60, 80 or a 100 dollars, each with probability 0.25. The dotted line represents the agent's updated expectations. Since a first-period success is observed then he now expects to be paid 100 or 80 with equal probability. Thus the prospective gain-loss utility gain equals 0.25μ(80-40)+0.25μ(80-60)+0.25μ(100-80)+0.25μ(100-100)=0.25×η(40+20+20)=η20 because under the old distribution the first 25 percentiles of payments equal 40 dollars, meanwhile under the new distribution the first 25 percentiles pay 80 dollars; the second percentiles of the old distribution pay 60 dollars meanwhile the second 25 percentiles pay 80 dollars and so on. Panel (b) in (a) in Figure 1.1 prospective gain-loss utility if a failure is observed in the first period. Notice that the expected payment distribution is the same as before. This time, however, the updated distribution corresponds to a probability distribution having equal probability on 40 and 60. Doing the same rank comparison, we have that the prospective gain-loss utility loss corresponds to 0.25μ(40-40)+0.25μ(40-60)+0.25μ(60-80)+0.25μ(60-100)=0.25×$\lambda\eta$(-20-20-40)=-$\lambda\eta$40.

[22]This specification of the gain-loss utility is equivalent to that used in Kőszegi and Rabin (2006, 2007). It is also worth noticing that in the effort domain things are simpler. Because of the assumption made on $\widetilde{e}_t \in \mathcal{E}$, that is, conditional on the past, the agent only forms effort plans in pure strategies, contemporaneous gain-loss utility in effort reduces to $\mu(c(\widetilde{e}_{t-1,\tau}) - c(\widetilde{e}_{t,\tau}))$. As a consequence, in period one, given a chosen effort or in period two given the realization of x_n, all quartiles of the (degenerated) effort lottery make the same comparison.

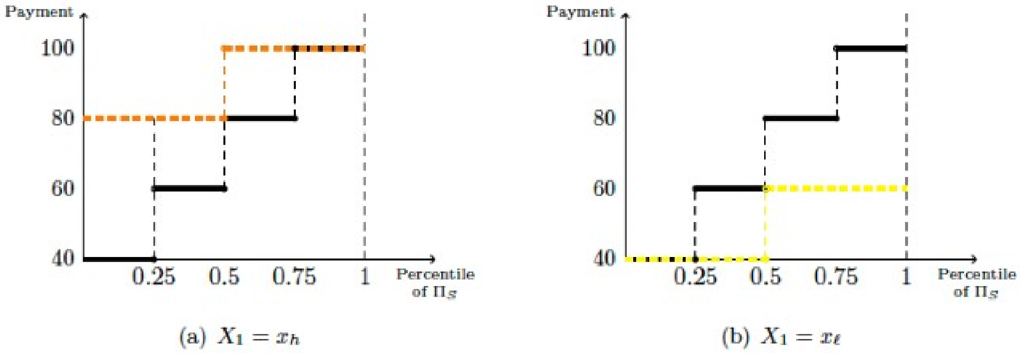


Figure 1.2: Prospective Gain-Loss Utility Realizations

The final piece of the agent's utility function corresponds to the assumption that the agent in period t makes his decisions to maximize the sum of his instantaneous utilities:

$$U_t = \sum_{\tau=t}^{2} v_\tau$$

where there there is no discounting to highlight the role of γ.[23]

Before turning to the agent's behavior, Figure 1.3 summarizes the timing of the principal-agent interaction together and the utility realizations. In an initial period zero the principal makes the contract offer, the agents forms effort plans $(\widetilde{e}_{0,1}, \widetilde{e}_{0,2}(X_1))$ and income expectations for the two upcoming periods.[24] Then he accepts or rejects the contract. In the beginning of period one the agent chooses and exerts first-period effort, e_1. Immediately after X_1 and first-period payments are realized and the agent experiences consumption and gain-loss utilities. At the end the same period the agent reviews his effort plans for the upcoming period and thus forms $\widetilde{e}_{1,2}(x_1)$. At the beginning of period two the agent chooses and exerts second-period effort, e_2. Immediately after X_2 and second-period payments are realized and the agent experiences consumption and contemporaneous gain-loss utility. [25]

I now turn to describe the agent's behavior. Since the agent's utility depends on his effort plans and consequent income expectations, an assumption about how the agent forms these expectations is needed. The model of reference-dependent preferences assumes the agent has rational expectations: the agent correctly anticipates the consequences of his effort decision

[23] It is worth considering the difference between a traditional discount factor and γ. Following Kőszegi and Rabin (2009) one may justify the first in terms of a decision maker putting lower weight on future utility flows because something may cause those future dates to be irrelevant. Time discounting is thus a discounting measure *between* utility flows. Contrarily, the gain-loss utility weight γ is a *within* utility flow "discounting". It may justified in terms of the decision maker putting a different weight into events happening in the future, not necessarily because they may not take place but simply because awareness is decreasing in the date of experience. See Section 1.8 for a discussion on the implications of including an exponential discount factor.

[24] Notice that this specification abuses a bit of the word "plan", because if the agent does not accept the contract he will never get to execute these actions. Thus, one can think of the period-zero plans are "conditional plans". We do so to avoid introducing further notation to denote the agent's intended actions.

[25] The length of the period is neutral in models with classical preferences. However, this is not the case with reference-dependent preferences. In fact, an implicit assumption of the model is that the periods are long enough so that the agent can form plans but not that long so that the agent does not consider deviations from the references as in the "choice acclimating equilibrium" in Kőszegi and Rabin (2007). A month is a credible long-term period for this model.

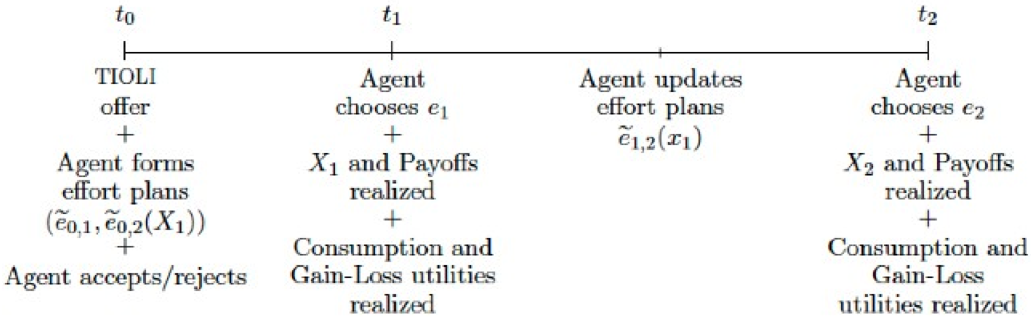


Figure 1.3: Timing of the Principal-Agent Interaction

and thus cannot form effort plans he knows he will not carry through given the implications these plans have over payments and future preferences. In this moral hazard model this assumption implies that the agent's equilibrium behavior is a complete contingent effort plan for period one and two, from which the agent will never want to deviate when actually executing it—given that the effort plan constitutes the reference—nor when updating it as information arrives. Moreover, when updating his effort plans, the agent will only consider deviating to plans that he knows he is willing to follow, that is, to rational or consistent plans.[26]

Definition 1 *(Agent's Equilibrium Behavior)*
Given a contract S, the effort path $(e_1^{PE}, e_2^{PE}(X_1))$ *is a "preferred personal equilibrium" (PPE) for the agent iff*

$$\begin{array}{llll} (i) & EU_0(e_1^{PE}, e_2^{PE}(X_1)|e_1^{PE}, e_2^{PE}(X_1)) & \geqslant EU_0(e_1, e_2(X_1)|e_1, e_2(X_1)) & \forall (e_1, e_2(X_1)) \in \mathcal{E}^{PE} \\ (ii) & EU_1(e_1^{PE}, e_2^{PE}(X_1)|e_1^{PE}, e_2^{PE}(X_1)) & \geqslant EU_1(e_1, e_2^{PE}(X_1)|e_1^{PE}, e_2^{PE}(X_1)) & \forall e_1 \in \mathcal{E} \\ (iii) & EU_2(e_2^{PE}(x_n)|e_2^{PE}(x_n); x_n) & \geqslant EU_2(e_2|e_2^{PE}(x_n); x_n) & \forall e_2 \in \mathcal{E}, \forall x_n \end{array}$$

where $\mathcal{E}^{PE} \equiv \{e, e_2(X_1)|(ii)$ *and* (iii) *hold for* $e = e_1^{PE}$ *and* $e_2 = e_2(X_1)^{PE}\}$.

Definition 1 says that an effort path is a personal equilibrium for the agent if (1) given x_n, in the second period the agent does not want to deviate from his second-period planned effort to any other second-period rational plan (2) in the first period the agent does not want to deviate from his planned effort nor change his plans for the second period to another second-period effort he knows he is willing to follow and (3) in period zero he knows that he will be willing to plan and implement the effort plan in the upcoming periods since it is the rational plan that gives him the highest utility among all rational plans.[27,28]

[26]Notice that assuming the agent has rational expectations is by no means a contradiction with the assumption that agents have reference-dependent preferences. In fact, reference-dependent preferences is by no means a form of irrationality but simply a different preference structure, which the completely rational agent correctly maximizes given the economic environment and restrictions he faces.

[27]This definition of personal equilibrium slightly differs to that in Kőszegi and Rabin (2009). In part (i) I assume the agent forms his period zero reference point at the same time he forms his plans, as in the static model in Kőszegi and Rabin (2006), meanwhile Kőszegi and Rabin (2009) assume that in period zero the agent has an arbitrary set of plans for future periods. They show that the predictions of the model are robust to this specification. This should not make any difference for the model predictions.

[28]Notice that because of the assumption that plans are updated after gain-loss utility is realized, the condition that the agent does not want to deviate to other second-period credible plans at the end of period one is ensured by (i). I relax this assumption in Section 1.7.

To actually compute the equilibrium, the agent must proceed backwards: in period zero he considers which action will be convenient to implement in period two given the continuation contract and each realization of the first period performance measure x_n. This will constitute the effort plan he will hold at the end of period one, $\widetilde{e}_{1,2}(x_n)$. Because of rational expectations, however, he must also check that—given this plan— once in period two he will be willing to carry this plan through, that is, $e_2 = \widetilde{e}_{1,2}(x_n)$. With e_2 and $\widetilde{e}_{1,2}(x_n)$ the agent can consider which effort will be optimal to implement in period one. This will constitute his effort plan for the period $\widetilde{e}_{0,1}$. Then, just as before, he must make sure that—given this belief and those for period two—he will actually implement the action, that is $e_1 = \widetilde{e}_{0,1}$. Thus, since in period zero he knows that $\widetilde{e}_{1,2}$ is a rational plan, then he sets $\widetilde{e}_{0,2} = \widetilde{e}_{1,2}$. An effort path $(e_1^{PE}, e_2^{PE}(X))$ thus will be an equilibrium for the agent if $\widetilde{e}_{0,1} = e_1^{PE}$ and $\widetilde{e}_{0,2}(x_n) = \widetilde{e}_{1,2}(x_n) = e_2^{PE}(x_n)\ \forall x_n$.

I finally turn to the definition of an implementable path and the principal's problem.

Definition 2 *(Implementable Effort Path)*
An effort path $(e_1, e_2(X_1))$ is said to be implementable by a contract S iff
(i) $EU_0(e_1, e_2(x_n)) \geqslant U_R$.
(ii) $(e_1, e_2(X_1))$ is a preferred personal equilibrium for the agent.
where $U_R = U_R^1 + U_R^2$ corresponds to the sum of the agent's per-period reservation utilities.

Part (i) corresponds to the standard individual rationality constraint: the agent accepts the offer if and only if the total utility he gets from the contract is not lower than his reservation utility. Part (ii) corresponds to the incentive compatibility constraints implied by the agent's personal equilibrium, ensuring that the agent executes the desired effort path. Notice, however, that the notion of incentive compatibility implied by PPE is more complex than that with classical reference-independent preferences. This is because when the agent has reference-dependent preferences the contract not only has to ensure that he actually executes the contingent desired actions but also that he plans those desired actions and does not deviate from his plan as information about the performance measure arrives. As a consequence, the requirement of the effort path being a personal equilibrium implies two sets of incentive compatibility restrictions: (1) those that, given the reference, make sure the agent actually implement the desired action and (2) those that make sure the agent does not deviate to other rational plans when reviewing his future plans as information arrives. I call the first set of incentive compatibility restrictions "planning incentive compatibility restrictions" (part (i) in Definition 1), and the second set of restrictions "executing incentive compatibility restrictions" (part (ii) and (iii) in Definition 1.)

Finally, the principal's problem corresponds to find the contract that implements the desired effort path at the minimum expected cost. Formally, for a given effort path $(e_1, e_2(X_1))$ the first step corresponds to the program (P)[29]

$$\min_{\{s_n s_{nm}\}_{m=1}^{N}} \quad \sum_{n=1}^{N} \pi_n(e_1)[s_n + \sum_{m=1}^{N} \pi_m(e_2) s_{nm}]$$

$$\text{subject to} \qquad EU_0(e_1, e_2(X_1)|e_1, e_2(X_1)) \geqslant U_R$$
$$(e_1, e_2(X_1)) \quad \text{is a preferred personal equilibrium}$$

[29]Since Grossman and Hart (1983), is standard to consider the principal's problem as a two-step decision problem. First, find the contract that implements a given effort path $(e_1, e_2(X_1))$ at the minimum cost and second, find the effort path that maximizes profits. I skip the formal statement of the second step as focusing on the first step is sufficient for this paper purposes.

1.4 Basic Properties of the Optimal Contract

Broadly this section aim is to present a basic characterization of the optimal contract when agents are reference-dependent and show that this optimal contract shares many features of the standard features of the contract with classical preferences.

1.4.1 First-best Contracts

I start by considering the shape of the optimal contract when the state perfectly reveals the agent's action, i.e. the first-best contract. Since the difference between classical and reference-dependent preferences corresponds to gain-loss utility, Lemma 1 presents gain-loss utility under certainty.

Lemma 1 *(Gain-loss Utility under Certainty)*
Assume A1-A-2 and that for every effort $e_j \in \mathcal{E}$ there exists an $n \in \{1, \ldots, N\}$ such that $\pi_n(e_j) = 1$ and $\pi_n(e_k) = 0$ for all $e_k \neq e_j$. Then $\int G(\widetilde{s}_{t,\tau}, \widetilde{e}_{t,\tau} | \widetilde{s}_{t-1,\tau}, \widetilde{e}_{t-1,\tau}) d\Pi_\tau^S = 0$ for $t = 1$ and $\tau = 1, 2$ and for $t = 2$ and $\tau = 2$, $\forall \lambda \geqslant 1$.

Lemma 1 states that gain-loss utility under certainty is zero. Intuitively, if the state perfectly reveals the action the agent can never be surprised by the payment he receives since with rational expectations the only credible income expectation is the payment he will receive with certainty given the effort he exerted.[30] This lemma extends Proposition 3 in the static model of reference-dependent preferences by Kőszegi and Rabin (2006). With this lemma at hand, the following proposition describes the first-best contract (S^{FB}.)

Proposition 1 *(First-Best Contracts)*
Assume A1-A2 and that for every effort $e_j \in \mathcal{E}$ there exists an $n \in \{1, \ldots, N\}$ such that $\pi_n(e_j) = 1$ and $\pi_n(e_k) = 0$ for all $e_k \neq e_j$. Then the optimal contract S implementing $(e_1, e_2(X_1))$ does not depend on η. Moreover, S can be such that $s_{nm} = s_{n'm}$ for $n \neq n'$ $\forall m$.

Proposition 1 presents two basic characteristics of the first best contract. First, the contract with reference-dependent preferences equals to that with classical reference-independent preferences as it does not depend on η. The intuition comes straight from Lemma 1: under certainty gain-loss utility is zero and thus reference-dependent agents are behaviorally equivalent to consumption utility maximizers. Second, the first-best contract using a forcing contract in every period. Intuitively, when the contract can be written upon effort the principal does not face the standard risk-incentive trade-off and thus memory plays no role in alleviating it.[31]

[30] A similar analysis is valid in the effort domain. Notice, however, that since effort is a choice variable gain-loss utility is always zero in the equilibrium path.

[31] Notice that without discounting, the first cost can also be implemented with a forcing long-term contract, i.e., a contract paying $\underline{s}$ in the second period if the desired first-period outcome is not observed and paying a fixed first-period wage of $u^{-1}(c(e_1) + U_{R_1})$. This contract, however, does not help highlighting the role of memory when the agent's action is not observable. See Proposition 4. For a proof of the optimality of first-best contracts without memory with classical preferences see Lambert (1983).

1.4.2 Second-Best Contracts

I now study the shape of the optimal contract in the more realistic case when the outcome measure does not reveal the agent's unobserved action, i.e. the second-best contract. I prove that several standard features of the second-best contract under classical reference-independent preferences still hold when agents have reference-dependent preferences. I start proving existence and conditions for uniqueness.

Assumption 3 *(Outcome does not Reveal the Action)*
For every $n \in \{1, \ldots, N\}$ there are at least two actions $e_j \neq e_k$ such that $\pi_n(e_j) \neq 0$ and $\pi_n(e_k) \neq 0$.

Proposition 2 *(Existence and Uniqueness of Second Best Contracts)*
Assume A1-A3 and fix $\eta > 0$ and an implementable effort path $(e_1, e_2(X_1))$. Then, there exists a contract S implementing $(e_1, e_2(X_1))$ at minimum expected cost for the principal. Moreover, if $u''(\cdot) < 0$, S is unique.

Just as in the reference-independent case, the assumption of consumption utility being strictly increasing allows us to uniquely define the principal's problem in a utility equivalent problem, which under Assumption 2 (piece-wise linear value function) corresponds to the minimization of well-behaved convex function with linear constraints. Similarly, the uniqueness argument is equivalent to that with classical preferences[32]

Proposition 3 states that some of the basic characteristic of the second-best contract with classical reference-independent preferences also hold with reference-dependent preferences.

Proposition 3 *(Basic Properties of the Second-Best Contract)*
Assume A1-A3 and fix $\eta > 0$ and an implementable effort path $(e_1, e_2(X_1))$. Then, in the optimal contract S implementing the effort path
(i) the IR is binding.
(ii) the expected cost equals that of the first best contract if $e_1 = e_2(x_n) = e_{\min}$ $\forall n$ where e_{min} is such that $c(e_{min}) = \min\{c(\mathcal{E})\}$ or if there exists a subset $\mathcal{X}_j \subseteq \mathcal{X}$ such that $\sum_{x_n \in \mathcal{X}_j} \pi_n^j = 0 < \sum_{x_n \in \mathcal{X}_j} \pi_n^k$ $\forall e_k \in \mathcal{E} \backslash e_j$ where $c(e_k) < c(e_j)$ for $e_j = e_1$ and equivalently for $e_j = e_2(x_n)$ $\forall x_n$.

Part (i) states that the contract that minimizes the principal's expected cost of implementing the desired effort path leaves no rents to the agent. Much like with reference-independent preferences, if the IR does not bind the principal can decrease the utility equivalent of all payments in a given period contract without affecting incentives neither those coming from consumption utility nor those from gain-loss utility.[33] Part (ii) states conditions under which

[32] The assumption of $\mathcal{E}$ being a discrete set is also important since it avoids the first order approach. See Herweg, Müller, and Weinschenk (2008) for a model of static moral hazard with Kőszegi and Rabin (2006) preferences where the action is assumed to be continuous and the first order approach is used. As the authors point out, assumptions over the cost function must be made for the first order approach to be valid due to problems arising from equilibrium gain-loss utility being negative (see Lemma 2 in the next section). Finally, the assumption of $\mathcal{E}$ being finite is not crucial. In fact, proving the existence of the contract when $\mathcal{E}$ is infinite only requires proving that the expected cost function is lower semicontinuous instead of simply continuous as I have done in the proof. I have opted to assume a finite set because there seem to be no additional interesting insights when $\mathcal{E}$ is assumed to be infinite.

[33] Consumption utility incentives remain unaltered by the same argument used in the classical model. Those coming from gain-loss utility are unaffected because gain-loss utility depends on the distance between the consumption utility of the actual payment and the reference one. Notice moreover that this result is not surprising: despite being reference-dependent, the agent's preferences over income lotteries are still independent of his effort choice due to the separability assumption (see Assumption 1).

the second-best contract achieves the same cost as a first-best contract. There are two cases here. First, if the principal wants to implement the least-cost effort path. The intuition is the same as with classical reference-independent preferences: if the interest of the agent and those of the principal are aligned there is no trade-off between risk-allocation and incentives. Second, if there is a—possibly different—shifting support for each of the desired actions in the effort path; that is, if for every desired action there is a certain range of the performance measure which cannot be achieved under the desired action but only under lower cost actions. In this case the principal can write a sequence of forcing contracts with a sufficiently high penalization so that the agent will never take the risk of not executing the desired action path.[34]

It is interesting to note that with reference-dependent preferences, however, linear consumption utility does not imply that the second-best contract achieves first-best cost as it is the case with reference-independent preferences. In fact, recall that with risk neutral classical preferences the principal can "sell the firm" at a fixed price to the agent—equal to her expected utility—and achieve first best. With reference-dependent preferences this is no longer true: even if consumption utility is linear agents are still loss averse and thus they will not be willing to absorb the risk that such agreement implies.

1.5 Fixed-Wage Contracts

This section presents the paper's main prediction related to the intertemporal allocation of incentives. To motivate, notice that the question of how to allocate incentives across periods is an interesting one in a repeated moral hazard model since in a long-term contract the principal not only has present payments to motivate the agent's present action—as it is the case in a static model of moral hazard—but she can also make future payments contingent in the present outcome realization. Proposition 4 presents the benchmark: the shape of optimal long-term contracts when agents have standard reference-independent preferences.

Proposition 4 *(Long-Term Contract with Reference-Independent Classical Preferences) Assume A1 and A3 and fix* $\eta = 0$ *and an effort path* $(e_1, e_2(X_1))$. *Then, the optimal contract* S *implementing* $(e_1, e_2(X_1))$ *is such that*
(i) if $u^{''} < 0$, $s_n \neq s_{n'}$ *for at least one* $n \neq n'$, *and* $s_{nm} \neq s_{n'm}$ *for* $n \neq n'$ *and at least one* m.
(ii) if $u^{''} = 0$, $s_n \neq s_{n'}$ *for at least one* $n \neq n'$ *and* $s_{nm} = s_{n'm}$ *for all* m *and for at least one* $n \neq n'$.

Proposition 4 part (i) says that when the reference-independent agent is risk averse optimal payments are contingent on the whole outcome history and the contract is said to display "memory in wages". This result replicates that in Rogerson (1985) (see also Lambert (1983), Murphy (1986), Malcomson and Spinnewyn (1988), Chiappori, Macho, Rey, and Salanié (1994)). Most important for this paper's purpose, this result implies that the optimal contract *spreads* the incentives to implement the fist-period action between present and future payments. Equivalently, the principal will never want to cluster all the incentives into future payments. The intuition for this result is well known: by spreading incentives across periods the principal increases the insurance provided to the risk averse agent ameliorating the risk-incentives trade-off and thus the risk premium she must provide to the risk averse agent for bearing risk. Notice further that adding a standard discount factor would not affect the optimality of spreading incentives. Part (ii) corroborates this intuition by stating that if the agent is risk neutral the optimal contract

[34] Notice though that this relaxes the initial assumption that $\pi_n^j > 0$ $\forall n, j$. However, just as in the case of classical reference-independent preferences, this may create some existence problem since proving that the set of IC and IR contracts is not bounded becomes harder. For details see Grossman and Hart (1983).

displays no memory: the agent is not bothered by risk and thus there are no gains from intertemporal risk-sharing. In this case the principal can write a sequence of forcing-short term contract and implement the effort path at firs-best cost.[35]

Before turning to the main proposition I present a useful lemma and add one final assumption upon the behavior of the outcome probability distribution. Lemma 2 generalizes the result in Kőszegi and Rabin (2009) by describing the value of the gain-loss utility in the equilibrium.

Lemma 2 *(Equilibrium Gain-loss Utility is Non-Positive)*
Fix $\eta > 0$ and an effort path $(e_1, e_2(X_1))$. Then, for the optimal contract S implementing $(e_1, e_2(X_1))$

$$\int G(\widetilde{s}_{t,\tau}(\widetilde{e}_{t,\tau})|\widetilde{s}_{t-1,\tau}(\widetilde{e}_{t-1,\tau}))d\Pi^S_\tau \leqslant 0$$

for $t = 1$ and $\tau = 1, 2$ and for $t = 2$ and $\tau = 2$ with equality if $\lambda = 1$.

Lemma 2 says that if the action path is a personal equilibrium, expected gain-loss utility in payments is negative. I illustrate Lemma 2 with an example in contemporaneous gain-loss utility. An agent works every Friday and is paid the same day 100 with probability $\pi(e)$, and 80 otherwise. The day before working he forms an effort plan $\widetilde{e}$ to be executed the next day. The income expectations is then to earn 100 with probability $\pi(\widetilde{e})$ and 80 otherwise. By comparing the percentiles in the old and new distributions, the agent will get a gain $(1-\pi(\widetilde{e_h}))\eta 20$ if he gets paid 100 and a loss $\pi(\widetilde{e}) - \lambda\eta 20$ if he gets paid 80 (see example in Section 1.3.) The expected utility therefore corresponds to

$$\pi(e)(1 - \pi(\widetilde{e_h}))\eta 20 + (1 - \pi(e))\pi(\widetilde{e})(-\lambda\eta 20)$$

where $\pi(e)$ is the probability of success given the executed action. Then, the probability of a gain is $\pi(e_h)(1 - \pi(\widetilde{e_h}))$ while that of the loss is $(1 - \pi(e_h))\pi(\widetilde{e})$. In equilibrium, however, the agent implements his plan and thus $\pi(e)(1 - \pi(\widetilde{e})) = (1 - \pi(e))\pi(\widetilde{e})$. Thus the expected utility corresponds to $\pi(e)(1 - \pi(e))(1 - \lambda)\eta 20$, which is negative since $\lambda > 1$.

Intuitively, the agent is hurt by the prospect of fluctuations in his income beliefs: he expects the payment realization to disappoint him with the same probability that he expects it to pleasantly surprise him, so due to loss aversion the expected utility from changes in income beliefs is unpleasant. By noticing that beliefs can only change in risky environments, this feature of the preference structure implies that the agent dislikes risk from the reference-dependent component of his utility function.[36]

Finally, Assumption 4 imposes some structure on the outcome probability distribution.

Assumption 4 *(MLRP)*
For any e_k, $e_j \in \mathcal{E}$ such that $e_k \leqslant e_j$, the ratio $\pi_n(e_k)/\pi_n(e_j)$ is non-increasing in $n \in \{1, \ldots, N\}$.

Intuitively, Assumption 4 states that exerting more costly actions makes it more likely to exert higher outcome levels. This is a standard assumption and is a very reasonable one for a model

[35] From Proposition 2 its is know that the optimal contract is not unique when the agent has a linear consumption utility function. As discussed in footnote 31, the first-best cost can also be achieved with a contract using memory. However, such contract does not highlight the use of memory as alleviating the risk-incentive trade-off. Thus, I opt to emphasize the first-best contract without memory.

[36] Campbel and Kamlani (1997) present related evidence that agents dislike not having certainty about how their payments will change. They survey 184 firms in Business Week 1000 corporations and explicitly asked ask to rate between 1 to 4 (from no important to very important) how important is the statement that "workers dislike *unpredictable changes* in income" (italics are mine.) The average response, depending on the type of worker hired by the firm, ranged from 2.59 to 2.79, which they interpret as reasonably strong support to the statement.

where the actions correspond to effort and the outcome is the performance measure: the higher the effort the agent exerts the more likely better performance measures are observed.

I now present the main result. Proposition 5 describes the intertemporal allocation of incentives in the optimal contract when the agent has reference-dependent preferences.

Proposition 5 *(Long-Term Contracts with Reference-Dependent Preferences)*
Assume A1-A4 and fix $\eta > 0$ and an effort path $(e_1, e_2(X_1))$. Then there is a $\bar{\gamma} > 0$ and a $M_1 > 0$ such that if $\gamma < \bar{\gamma}$ and $|u^{''}(s)| \leqslant M_1$ for all $s \geqslant \underline{s}$, the optimal contract implementing $(e_1, e_2(X_1))$ sets $s_n = s_{n'}$ for all $n, n^{'}$ and a first-period fixed wage, $s_{nm} \neq s_{n'm}$ for all $n \neq n^{'}$ and at least one $n, n^{'}$.

Proposition 5 says that when the agent is more sensitive to changes in his expectations about present income than about his future income (γ small) and consumption utility is not too concave, the optimal contract with reference-dependent preferences defers all the incentives for period-one effort into future payments.

To illustrate the mechanism behind the optimality of deferred incentives, consider a two-action, two-effort model with $\gamma = 0$. For simplicity and without loss of generality—as the interest is on how to set first-period incentives—I assume the second-period action is verifiable. Assume further that consumption utility is linear and that the principal wants to implement high effort in both periods at a cost c per period. Normalize the cost of low effort to zero. In this simple set up I show that for any contract using contingent first-period payments to implement high effort in the first period there exists a profitable deviation contract using only future payments to motivate the agent.

Define the contingent contract S as follows. The first-period contract pays s_h after a first-period success and $s_\ell < s_h$ after a failure. The second-period contract—for simplicity unrelated to first-period performance—is a forcing contract paying $\bar{s}$ after a second-period success and $\underline{s}$ after a second-period failure. Consider now a deviation contract $\hat{S}$ paying a fixed first-period payment equal to the expected value of the first-period payments in the contingent contract, $s = \pi(e_h)s_h + (1 - \pi(e_h))s_\ell$, where $\pi(e_h)$ is the probability of success given high effort. The second period contract corresponds to a first-period contingent forcing contract: if a low outcome is observed in the second period the contract pays $\underline{s}$ for any first-period outcome, meanwhile if a high outcome is observed in the second period the contract pays $\bar{s} + \rho_h$ after a first-period success and $\bar{s} - \rho_\ell$ after a first-period failure where $\rho_h, \rho_\ell \in \mathbb{R}_+$. Notice that since the fixed-first-period payment under contract $\hat{S}$ equals the first-period expected consumption utility of contract S, the total expected cost of the two contracts is equal if $\pi_h\rho_h - (1 - \pi_h)\rho_\ell = 0$. As a consequence, the principal will be indifferent between the two contracts and thus $\hat{S}$ is a profitable deviation from S it suffices to show that under the alternative contract $\hat{S}$ (1) high effort is a PPE and (2) the agent is better-off relative to contract S.

I start by exploring the conditions over ρ_h and ρ_ℓ such that $\hat{S}$ implements high effort in both periods. Notice that because second-period contracts are forcing contracts the only credible action for period two is high effort (otherwise the agent gets infinite disutility from $\underline{s}$.) Thus, the second-period implementing IC is trivially satisfied. Consider now the first-period implementing IC. Start considering how does the contingent contract sets incentives: under contract S, having planned to exert high effort, the increase in period-one expected utility from exerting high effort instead of low effort corresponds to:[37]

$$EU_1(e_h|e_h;S) - EU_1(e_\ell|e_h;S) = [\pi(e_h) - \pi(e_\ell)]\big\{(s_h - s_\ell) + \phi(s_h - s_\ell) + 0\big\} \geqslant c + \mu(c)$$

where $\phi \equiv \eta[(1-\pi_h) + \lambda\pi_h]$. The first term in the curly bracket $(s_h - s_\ell)$ corresponds to incentives coming from first-period consumption utility: implementing high effort increases the probability of getting the high payment and decreases the probability of a low payment. The second term $\phi(s_h - s_\ell)$ shows incentives coming from first-period contemporaneous gain-loss utility: implementing high effort increases the probability of a gain and decreases the probability of a loss. The last term equal to zero corresponds to incentives coming from future consumption utility: since forcing contracts in S are not contingent on the first-period outcome—and γ is assumed to be zero—there are no incentives coming from future payments. The right hand side shows the increase in cost from exerting high effort rather than low effort. The c corresponds to the consumption disutility increase and $\mu(c)$ represents the reference-dependent cost capturing the forgone gain of a pleasant surprise of working less than expected. Under contract $\hat{S}$, the implementing first-period IC corresponds to:[38]

$$EU_1(e_h|e_h;\hat{S}) - EU_1(e_\ell|e_h;\hat{S}) = [\pi(e_h) - \pi(e_\ell)]\big\{0 + 0 + (\rho_h + \rho_\ell)\big\} - c - \mu(c) \geqslant 0$$

The first and second zeros in the curly bracket represent the incentives lost from fixing the first-period payment: consumption utility does not change with the first-period outcome and the agent's expectations about his first-period payment do not change as the gent has rational expectations. The third term $(\rho_h + \rho_\ell)$ represent the new incentives coming from second-period consumption utility: if the agent executes high effort in the first period he increases the probability of getting the high second-period payment $\bar{s} + \rho_h$ relative to the low one $\bar{s} - \rho_\ell$.

Because S implements the desired effort path, from the two equations above is straightforward to see that if

$$(\rho_h + \rho_\ell) = (1+\phi)(s_h - s_\ell) \tag{1.3}$$

the period-one implementing IC under contract $\hat{S}$ holds: having planned to execute high effort, the agent will actually work hard. Notice that this equation together with $\pi(e_h)\rho_h + (1 - \pi(e_h))\rho_\ell = 0$ uniquely define ρ_h^* and ρ_ℓ^*.[39] The final step for high effort to be a PPE under $\hat{S}$ is

[37]To derive this equation start building contemporaneous gain-loss utility. For a given effort plan made in period zero to be executed in period one, $\widetilde{e}_{0,1}$, if a success is observed in the first period the agent will experience a gain equal to $\eta(1-\pi(\widetilde{e}_{0,1}))(s_h - s_\ell)$. If a failure is observed the agent gets a loss equal to $-\lambda\eta\pi(\widetilde{e}_{0,1})(s_h - s_\ell)$ (see example in Section 1.3). When the executed action is high effort the gain scenario happens with probability $\pi(e_h)$ and the loss scenario happens with probability $(1-\pi(e_h))$. Then, the agent's total period-one expected utility of exerting e_h corresponds to:

$$EU(e_h|\widetilde{e}_{0,1}, S) = \pi(e_h)[s_h + \eta(1-\pi(\widetilde{e}_{0,1}))(s_h - s_\ell) + \bar{s}] + (1-\pi(e_h))[s_\ell - \lambda\eta\pi(\widetilde{e}_{0,1})(s_h - s_\ell) + \bar{s}] - c + \mu(-c - c(\widetilde{e}_{0,1}))$$

The equation follows simply by setting $\widetilde{e}_{0,1} = e_h$, constructing the same equation for e_ℓ given $\widetilde{e}_{0,1} = e_h$ and then taking differences between the two. Notice then that ϕ in fact corresponds to $\eta[(1-\pi(\widetilde{e}_{0,1})) + \lambda\pi(\widetilde{e}_{0,1})]$. Finally notice that because the second-period action is observable, the notation of all the utility functions in this example suppress the second-period action.

[38]To derive this equation notice that for a given effort plan made in period zero to be executed in period one, $\widetilde{e}_{0,1}$, the agent's total period-one expected utility of exerting e_h corresponds to:

$$EU_1(e_h|\widetilde{e}_{0,1}, \hat{S}) = \pi(e_h)[s + 0 + \bar{s} + \rho_h] + (1-\pi(e_h))[s + 0 + \bar{s} - \rho_\ell] - c + \mu(-c - c(\widetilde{e}_{0,1}))$$

The equation follows simply by first setting $\widetilde{e}_{0,1} = e_h$ in the equation above, constructing the same equation for e_ℓ and then taking differences between the two.

[39]Straight algebra shows that the optimal difference between the second-period continuation contracts corresponds to $\rho_h^* = (1-\pi_h)(1+\phi)(s_h - s_\ell)$ and $\rho_\ell^* = \pi_h(1+\phi)(s_h - s_\ell)$.

to show that the period-zero planning IC holds. Start noticing that if the disutility the agent gets in period zero from planning low effort but then deviating to high effort is big enough, an effort path with low effort in the first period may also be credible. Thus, it must be the case that under the deviation contract $\hat{S}$ the total period-zero expected utility of planning and exerting high effort is greater than that of planning and exerting low effort in the first-period. In fact, under the alternative contract $\hat{S}$ this condition is implied by the first-period implementing IC since this contract only relies on consumption utility–and thus on the implemented action–to put incentives.[40]

I now compare the total period-zero expected utility the agent gets under the contingent contract and the deviation contract for ρ_h^* and ρ_ℓ^*. Under the original contract S the total period-zero equilibrium expected utility corresponds to:

$$EU_0(e_h,|e_h;S) = \underbrace{\pi(e_h)s_h + (1-\pi(e_h))s_\ell}_{} - \underbrace{\eta(\lambda-1)\pi(e_h)(1-\pi(e_h))(s_h - s_\ell)}_{} + \underbrace{\bar{s}}_{} - 2c$$

The first and third brackets corresponds to first and second-period expected consumption utility. The second bracket corresponds to period-one expected contemporaneous gain-loss utility, which because of loss aversion is negative (see Lemma 2). Notice further that there is no prospective gain-loss utility nor second-period contemporaneous gain-loss utility because $\gamma = 0$ and second-period contracts do not depend upon first-period performance. Equivalently, under contract $\hat{S}$, the total period-zero expected utility corresponds to

$$EU_0(e_h,|e_h;\hat{S}) = \underbrace{s+0}_{} + \underbrace{\bar{s} + \pi(e_h)\rho_h - (1-\pi(e_h))\rho_\ell}_{} - 2c = \pi(e_h)s_h + (1-\pi(e_h))s_\ell + \bar{s} - 2c$$

The first bracket corresponds to period-one utility, which consists of first-period expected consumption utility and no contemporaneous gain-loss utility because the first-period payment is fixed. The second bracket corresponds to second-period expected utility which consists, again, uniquely of second-period expected consumption utility because second-period contracts are forcing contracts.

Taking the difference between the total period-zero expected utility under $\hat{S}$ and S corresponds to:

$$EU_0(e_h|e_h;\hat{S},\rho_h^*,\rho_\ell^*) - EU_0(e_h|e_h;S) = -\eta(1-\lambda)\pi(e_h)(1-\pi(e_h))(s_h - s_\ell) > 0$$

which is positive because the agent is loss averse ($\lambda > 1$).

Intuitively, when $\gamma = 0$, the agent is not hurt by fluctuations in his beliefs about future income. As a consequence, the principal can defer all the risk—and thus all the incentives—into the future to maximize the utility the agent gets from the contract. To replace the incentives lost the new contract increases the sensitivity of future payments to first-period performance. This modification of future payments is optimal as it substitutes incentives using second-period consumption utility rather that the more expensive incentives coming from second-period gain-loss utility.[41]

[40]To see this notice that, by adding and subtracting conveniently in the period-zero planning IC can be written as

$$EU_0(e_h|e_h;\hat{S}) - EU_0(e_\ell|e_\ell;\hat{S}) = [EU_1(e_h|e_h;\hat{S}) - EU_1(e_\ell|e_h;\hat{S})] - [EU_1(e_\ell|e_\ell;\hat{S}) - EU_1(e_\ell|e_h;\hat{S})]$$

Furthermore, with very little work one can see that under this contract $EU_1(e_\ell|e_\ell;\hat{S}) - EU_1(e_\ell|e_h;\hat{S}) = -\mu(c)$ and thus $EU_0(e_h|e_h;\hat{S}) - EU_0(e_\ell|e_\ell;\hat{S}) = [EU_1(e_h|e_h;\hat{S}) - EU_1(e_\ell|e_h;\hat{S})] + \mu(c)$.

[41]Notice that in the case that the second-period action is not observable the principal can achieve this by increasing/decreasing *all* payments within a continuation contract by the same amount. Such modification increases incentives from consumption utility without affecting second-period incentives.

The latter example shows that when the agent is risk neutral the reference-dependent component of the utility function pushes towards totally deferring incentives into future payments. When the agent is risk-averse, however, an intertemporal risk-sharing argument pushes towards spreading incentives across periods as discussed in Proposition 4. As a consequence, an extra assumption is needed to ensure that the effect coming from the reference-dependent component outweighs that coming from consumption utility. Such a restriction corresponds to a not-too-concave restriction over the consumption utility function ensuring that the cost of deferring payments to the second period do not outweigh the benefits of the fixed first-period wage.

When $\gamma > 0$ increasing the sensitivity of future payments to first-period performance creates an indirect negative effect over the agent's period-zero expected utility: since the distance between future payments increases so does the reference-dependent disutility the agent gets from changes in his beliefs about future payments, i.e. from prospective gain-loss utility. However, if γ is small enough the increase in the total expected utility from shutting down first-period contemporaneous gain-loss utility outweighs the increase in the disutility from prospective gain-loss utility.[42]

Two final remarks are due. First, it is important to notice that even though γ captures the relative weight of a present utility experience with respect to a future one, γ is not a discount factor. Furthermore, γ generates exactly the opposite effect over the intertemporal allocation of incentives than that of a standard exponential discount: adding a classical discount factor over consumption utility would implied that, ceteris paribus, the principal will want to allocate more incentives into *present* payments rather than future ones. Second, and to the contrary of the classical model, the length of the period is important in this model. The equilibrium notion used, however, suggests that the length of the period cannot be arbitrarily set as it has to be long enough so that the agent has time to form his plans and it has to be short enough so that the agent does not get used to his expectations. This all suggests that probably a reasonable length for a time period in this model is something between a week or a month.

Whether γ is big or small is an empirical matter, despite it seems reasonable that agents have a greater concern about present payments relative to future payments. I now further explore the importance of γ in the shape of the optimal contract by considering the case where changes in beliefs about future consumption resonate more strongly than changes in beliefs about present consumption. In such a case the increase in the continuation contract distance needed to preserve incentives would be small since prospective gain-loss utility is strong and helps to set incentives. However, this also implies a strong decrease in total period-zero total expected utility.

[42] This result is more likely to hold if the agent experiences diminishing sensitivity. Since the deviation contract does not rely on second-period contemporaneous gain-loss utility to replace incentives and since a lower prospective gain-loss disutility can be replaced with (cheaper) incentives coming from second-period consumption utility, a non-piece-wise-linear value function should increase the range of γ for which the result holds. To the contrary, the assumption of the principal being risk neutral is relevant since a risk averse principal is not willing to absorb the risk of the fixed-present wage for free. As a consequence, an assumption would have to be made about the principal being non too risk averse for the result to hold. This is no different from the classical model, however, where the optimal contract also depends on the principal's risk aversion assumption. This result also depends on the separability assumption made over the gain-loss utility function. In particular, if the gain-loss disutility in the second period depends negatively upon that in the first period, then fixing the first-period wage may trigger a higher prospective gain-loss disutility that would make the fixed-wage contract too expensive. Finally, the assumption of the agent consuming all his period income is also important. If the agent receives income news but he is forbidden to spend it, then the Kőszegi and Rabin (2009) model predicts the agent should not experience contemporaneous gain-loss disutility, as such news brings no information about present consumption. As a consequence, the principal would have no incentives to set a present-fixed wage.

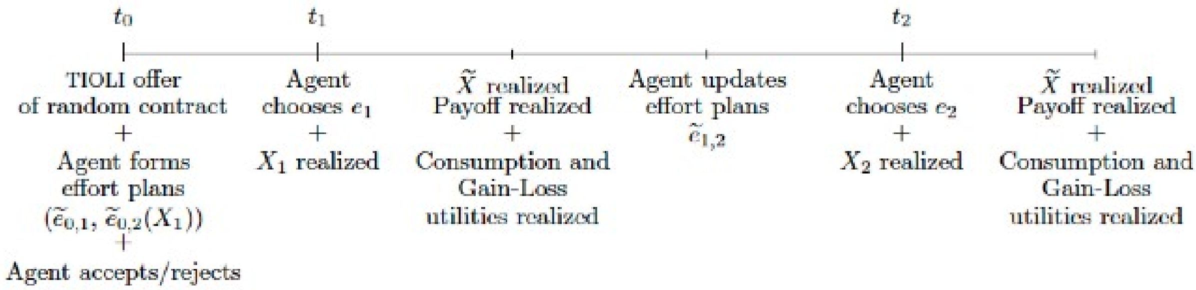


Figure 1.4: Timing of the Principal-Agent Interaction with a Random Contract

Proposition 6 *(No Memory)*
Fix $\eta > 0$ and an effort path $(e_1, e_2(X_1))$. Then there is a $\underline{\gamma} > 0$ such that if $\gamma > \underline{\gamma}$, the optimal contract implementing $(e_1, e_2(X_1))$ sets $s_n \neq s_{n'}$ for at least one $n \neq n^{'}$ and $s_{nm} = s_{n^{'}m}$ for all $n \neq n^{'}$, $\forall m$.

Proposition 6 states that if prospective gain-loss utility is strong enough, then the optimal contract will display two basic characteristics. First, it will use contingent payments in the first period. The intuition here is the reverse of that in Proposition 5: if news about future consumption resonates more than news about current consumption, then it would be too expensive to set a first-period fixed wage since the savings from shutting down first-period contemporaneous gain-loss utility do not compensate the costs in increasing prospective gain-loss utility, as mentioned before. Second, wages will not display memory. The intuition is that using memory necessarily implies the agent will experience prospective gain-loss utility, which is negative (see Lemma 2). If γ is big enough, this creates a big disutility from the period-zero perspective, so big in fact that is it cheaper to use only first-period payments to motivate the agent.

1.6 Random Contracts

I finally explore the convenience of paying the agent using a random contract, i.e., a contract that makes payments contingent on a measure uncorrelated with the agent's action. Specifically, a random contract offers the agent a menu of contracts, one per each possible realization of the outcome measure. In each period, once the outcome is realized, a pure randomization device uncorrelated with the agent's action is used to determine the payment from the contract previously chosen by the outcome realization.[43]

The timing of the principal-agent interaction is presented in Figure 1.4. In period zero the principal makes a TIOLI offer consisting of the random contract $S(X_1, X_2, \widetilde{X})$ where $\widetilde{X}$ drawn from the finite set $\widetilde{\mathcal{X}}$ correspond to the random device uncorrelated with the agent's action. For simplicity I assume that $\widetilde{X}$ is used in both period one and two. Given the contract, the agent forms an effort plan $\widetilde{e}_{0,1}$ and $\widetilde{e}_{0,2}$ for the upcoming two periods. With these effort plans and the associated income beliefs he accepts or rejects the long-term contract. In the first period the agent chooses and exerts effort e_1 and the performance measure X_1 is realized

[43]This type of random contract is usually denoted as "ex-post random" contract since the wheel is spun *after* the agent has executed the action. A second type of randomness can be included if the wheel is spun *before* the agent executes the action. Such a contracts are called "ex-ante" random contracts, and under specific assumptions over the third derivative of the consumption utility function, they can be optimal when the agent has classical preferences. See Fellingham, Kwon, and Newman (1984) and Arnott and Stiglitz (1988).

according to which one of the contracts in the first-period menu is chosen. Next $\widetilde{X}$ is realized and thus one of the payments in the previously chosen contract is selected. Immediately, the agent experiences consumption and gain-loss utility in payments. At the end of period one the agent updates his effort plans for the second period given the realization of X_1. At the beginning of the second-period the agent chooses and exerts effort e_2 and the performance measure X_2 is realized according to which one of the contracts in the second-period menu is chosen. Next, $\widetilde{X}$ is realized for the second time and the agent receives his second-period payments and corresponding utilities.

Proposition 7 shows that it is not optimal for the principal to use a contract using randomness to implement the desired effort path when the optimal contract uses a fixed-first-period wage.

Proposition 7 *(Non-Optimality of Random Contracts)*
Assume A1-A4 and fix $\eta > 0$ and an effort path $(e_1, e_2(X_1))$. Then, there is a $M_1 > 0$ such that if $\gamma = 0$ and $|u^{''}(s)| \leqslant M_1$ for all $s \geqslant \underline{s}$, the expected cost of the contract $S(X_1, X_2, \widetilde{X})$ is greater than that of the non-random contract $S(X_1, X_2)$.

Intuitively, and just as with classical preferences, the principal can increase the agent's utility by offering him the certainty equivalent of the random payment. This is because the random device is not correlated with the agent's action–and thus it does not help to provide incentives–and the agent dislike risk. Then, because the principal's objective function is convex, the result follows by Jensen's inequality.[44] Hence, just as with classical reference-independent preferences, the principal will never use performance measures that are uncorrelated with the agent's unobserved effort.

Finally, it is interesting to notice that the certainty equivalent associated with the random device of an agent with reference-dependent preferences is smaller than that of an agent with classical preferences because he not only dislikes risk from the concavity of the consumption utility function, but also from the reference-dependent component of the utility function (see Lemma 2). As a consequence he is willing to accept even less money than a classical agent with the same consumption utility function to give up on the ex-post randomness of the contract.

1.7 Timing of the Prospective Gain-Loss Utility Realization

This section modifies the timing of the prospective gain-loss utility realization. Up to now, I have assumed that prospective gain-loss utility is realized after the agent reviews his effort plans for the second period. To capture the intuition that experiencing a loss or gain from news about second-period payments may affect second-period effort plans, I now assume prospective gain-loss utility is experienced after the agent has reviewed his second-period effort plans.[45] To keep the structure of incentives tractable, I further reduce the analysis to a two-effort $e_t \in \{e_\ell, e_h\}$, two-outcomes $x_t \in \{x_\ell, x_h\}$ model in which the principal wants to implement high effort in both periods. Let $S(X_1) = (s_h, s_\ell)$ and $S(X_1, X_2) = (s_{nh}, s_{n\ell})$ for $n = h, \ell$ correspond to the first and second-period contracts. The rest of the interaction equals that in the original set up. Figure 1.5 shows the new timeline and Proposition 8 presents how second period incentives are affected by the timing of the prospective gain-loss utility realization.

[44] Notice that Proposition 7 assumes $\gamma = 0$. This assumption is made only for simplicity and should be straightforwardly generalizable to small values of γ as in the main proposition. Moreover, I conjecture that the non-optimality of random contracts should also hold if γ is big so that a fixed-first-period wage is not optimal.

[45] Notice that the equilibrium concept must be extended when changing the timing of prospective gain-loss utility realization. This is because now period-zero planning IC does not necessarily imply that the agent will form the correct plans. As consequence, a period-two planning restriction must be added. See the proof in Section 3.2.

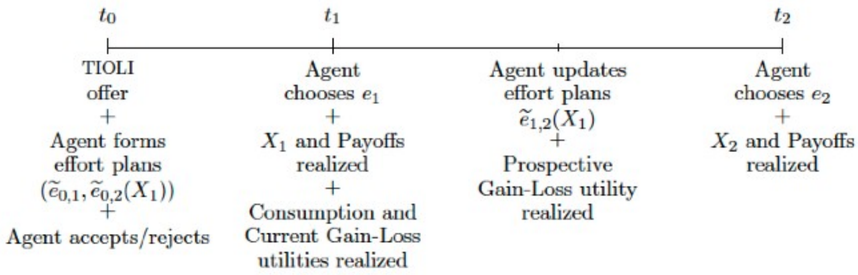
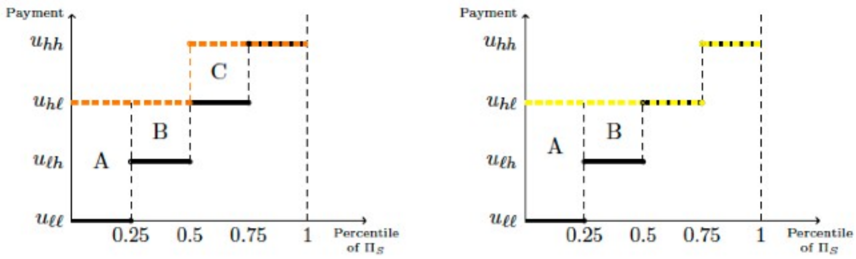


Figure 1.5: Timing of the Principal-Agent Interaction when Prospective Gain-Loss Utility Affects Second-Period Plans



(a) $\widetilde{e}_{1,2} = e_h$ (b) $\widetilde{e}_{1,2} = e_\ell$

Figure 1.6: Prospective Gain-Loss Utility Realization when $X_1 = x_h$ as a Function of $\widetilde{e}_{1,2}$

Proposition 8 *(Slacking After Success)*
Fix $\eta > 0$ and assume the principal wants to implement $e_1 = e_2(x_\ell) = e_2(x_h) = e_h$. Then, the optimal contract implementing (e_h, e_h) is such that $s_{hh} - s_{h\ell} \geqslant s_{\ell h} - s_{\ell\ell}$.

Proposition 8 says that it is harder to provide incentives for high effort after a period success than it is after a failure. The explanation comes from the fact that the prospective gain-loss utility realization depends on the agent's effort plans for the second period when it is realized after the agent reviews his second-period plans. To understand this, panel (a) in Figure 1.6 computes the the prospective gain-loss utility gain when the agent plans to exert high effort and the probability of success is 0.5. The solid line represents the agent's previous expectations about second-period payments meanwhile the dashed line represents the updated expectation about second-period payments after $X_1 = x_h$ was observed. Since the agent compares the percentiles in the reference distribution with those in the expected one (see Assumption 2), the prospective gain-loss utility gain corresponds to the area A+B+C. Equivalent,y, panel (b) shows the size of the realized prospective gain if the agent plans to exert low effort in period two when the probability of success under low effort corresponds to 0.25. As before, the solid line represents expected second-period payments meanwhile the dashed line corresponds to the updated expectation. In this case the prospective gain-loss utility gain corresponds to the area A+B because planning to exert low effort in period two decreases the probability of being pleasantly surprised by u_{hh} given that $u_{h\ell}$ was planned. As this example shows planning high effort maximizes the distance between the reference and the actual distribution of second-period payments. Equivalently, if a failure is observed in the first period, the agent experiences a loss from knowing that in the second period he will be paid with the low continuation contract. The size of such loss is decreasing in the agent's second-period effort plans since planning high effort minimizes the distance between the actual second-period payments distribution and the reference distribution. In particular, planning high effort decreases the probability of being unpleasantly surprised by $u_{\ell\ell}$ given that $u_{h\ell}$ was planned. As consequence, the prospective gain-loss utility realization makes it easier to plan high effort for any realization of the first-period outcome. However, because of loss aversion, these two scenarios do not weight the same in the agent's planning decision. As a consequence the period-two planning IC is harder to satisfy after a gain than it is after a loss as a first-period failure sets incentives to compensate the loses experienced in the first-period with higher second-period payments.

There are three important things to notice. First, this result mimics that with classical preferences where is easier to motivate the agent after a failure because after a failure the marginal utility is higher. However, the mechanism is different: with reference-dependent preferences it is loss aversion what generates the prediction. As a consequence, and to the contrary with reference-independent preferences, this result holds even if consumption utility is linear. Second, with reference-dependent preferences the temporal distance between the planning and the action execution is relevant. In fact, if the principal allows enough time after X_1 as been observed and the action execution, the agent may get used to his effort beliefs, rendering the realization of prospective gain-loss utility irrelevant for planning. Third, this result holds given that the agent planned in period zero to exert high effort in period two.[46]

1.8 Extensions

This paper explores the intertemporal allocation of incentives in a repeated moral hazard model when agents have dynamic reference-dependent preferences. I show that in contrast to the prediction with classical preferences under reasonable assumptions over the utility function, the optimal contract defers all incentives into future payments. This prediction reconciles the apparent dissociation between the models based on the risk-incentive trade-off and the shape of real contracts using present fixed wages. I further prove that, despite difference in the timing of incentive allocation, the optimal contract with reference-dependent preferences shares many features of the optimal contract with classical preferences: it does not leave rents to the agent, achieves first best when actions are least cost and there is a shifting support and ex-post random contracts are never optimal.

I emphasize that this model is about the principal's decision of whether to fully defer incentives into the future. Future work could shed light on (1) how far incentives should be deferred or (2) how those deferred incentives should be spread among future periods. To answer those questions the model must be generalized to T periods. As the model stands, however, if the sensitivity to income news is decreasing in the income period to which the news refers, it will predict that the principal will always want to defer risk as far into the future as possible. In these circumstances thus the inclusion of an standard discount factor δ would become crucial. If $\delta < 1/r$—where r is the discount rate at which the principal can borrow or save—using future consumption utility to replace present incentives would get more and more expensive as the agent is less sensitive to incentives coming from future consumption utility the further in the future these incentives are deferred. As a consequence, for a given small γ, there will be an optimal delay of incentives. As a result, optimal payments in a T period model will be fixed for given intervals of time and thus it will look as an step function where the length of the steps

[46] Changing the time of the prospective gain-loss utility would not affect the main result. To see this in pour Section 1.5 example, notice that at the end of period one, when the agent is reviewing his the second-period effort plan given the first-period outcome realization, there will be no other credible plans to deviate to and thus the first-period planning IC is also trivially satisfied under both S and $\hat{S}$.

will depend on the interaction between γ and δ. Crucially, however, the jumps from one fixed wage to the other will be a function of past rather than present performance.

This prediction of the T period model may explain why there is less contract renegotiation than that predicted by the classical theory, where it is optimal for the parties to renegotitate after the action is executed and before the outcome is observed. With reference-dependent preferences and an optimal payment function that resembles a step function there will be no incentives to renegotitate the contract as the agent is perfectly insured in the present and cannot renegotiate future dates since actions have not been executed yet.

A third interesting question that can be explored in a T-period model relates to the optimal timing of feedback through the outcome realization. Until now, the model assumes that the principal makes the performance measure public as soon as it has been realized. An extension of the model may deal with the principal's decision of whether to disclose the performance measure at the end of period one. Notice that, contrarily to the case with classical preferences, this is possible even if the agent is paid in every period. Notice further that in the classical model the preference structure is robust to the disclosure policy: only the informational structure of the problem changes. This is no longer true for dynamic reference-dependent preferences. In fact, the disclosure policy is a relevant choice variable for the principal because it determines whether the agent experiences gain-loss utility: if the principal decides not to disclosure x_n, then the agent only experiences second-period consumption utility and second-period current gain-loss utility. To the contrary, if the principal disclosures x_n in period one, then the agent will experience prospective gain-loss utility at the end of period one.

Chapter 2

Dynamic Beliefs

2.1 Introduction

A recent body of literature proposes agents distort beliefs to maximize utility. Works by Akerlof and Dickens (1982), Brunnermeier and Parker (2005), Gollier and Muerman (2006), Brunnermeier, Gollier, and Parker (2007) argue that agents who are willing and capable of manipulating their expectations trade-off the benefits of savoring an optimistic future with the cost of suboptimal decision making when choosing incorrect but otherwise "optimal" beliefs. This literature has been successful in providing an structural explanation for overconfidence and optimism, widely documented facts in the empirical literature. However, some related issues still remain unexplored; in particular, underconfidence, pessimism and subjective-beliefs time dynamics.

This paper studies the temporal path of subjective beliefs. To this end, it presents a model that enriches the preference structure in an optimal beliefs model by considering reference-dependent agents who experience anticipatory utility from beliefs about future expected utility and from changes in those anticipatory feelings. The agent waits T periods for a binary random outcome realization. In each period she freely chooses a subjective belief about her probability of success in the realization period so to maximize her intertemporal utility. Dynamics in optimal beliefs arise because the two anticipatory utilities have different implications over the agent's choice. On the one hand, beliefs about future expected utility push towards optimism because agents enjoy thinking of a promising future. On the other hand, changes in beliefs about future expected utility bound optimism because agents prefer to limit today their optimistic beliefs in order not get to the realization period holding great expectations. The model predicts that optimism decreases as the realization date approaches if the lottery is important enough or if the agent is sufficiently loss averse: when the pay-off date is distant there are many periods to enjoy optimistic thoughts making the cost associated to possible disappointment less salient. As the realization date approaches, however, less periods are left to enjoy optimism and thus the threat of experiencing a painful last-period disappointment becomes noticeable.

After a summary of the optimal beliefs literature, Section 2.3 presents the model. The agent waits during T periods for a binary outcome realization, which takes a high or low value in period T+1. In each period she chooses an optimal belief about the probability of observing the high outcome. The model sets the agent's problem of choosing the optimal path of subjective beliefs as a sequential game with perfect information between the agent's T selves. Each self optimal strategy maximizes her total intertemporal utility corresponding to the sum of her own utility flow and those of future selves, taking as given previous selves optimal beliefs and all future selves best responses to her choice.

The utility flows are specified as follows. In period T+1 the agent experiences two types of utility: standard consumption utility and reference-dependent utility from comparing the actual random outcome to his previous-period subjective belief about the same lottery. Periods $t \in \{1, \ldots, T\}$ are waiting periods in which no value of the binary random outcome is drawn and no relevant information about the realization period arrives. The model assumes that during these periods the agent experiences two types of anticipatory utility. First, she experiences standard anticipatory utility, i.e., utility from her beliefs about the realization-period expected utility. This component captures the idea that agents experience today feelings about outcomes that will only be realized in the future by bringing them into present in an standard expected utility form (see Loewenstein (1987), Caplin and Leahy (2001), Bénabou and Tirole (2004), Bernheim and Thomadsen (2005), Bénabou and Tirole (2006), Kőszegi (2009) among others.) Second, because of the reference-dependent nature of her preferences, the agent also experiences utility from changes in these anticipatory feelings. This component of the utility flow captures the idea that agents experience utility from changes in their beliefs (Matthey (2005), Kőszegi and Rabin (2009).) To emphasize, I refer to the former type of anticipatory utility as *reference-independent* anticipatory utility and to the latter as *reference-dependent* anticipatory utility.

To illustrate these two types of anticipatory utilities consider the following example. A worker has been offered a productivity bonus at the end of the month. Suppose today she thinks she will receive the bonus with probability 0.8 whereas yesterday she thought she would receive the bonus with probability 0.5. According to this model the agent experiences two sources of anticipatory utility: she is pleased with the fact the bonus is likely (reference-independent anticipatory utility proportional to 0.8) but also she experiences a gain because she thinks the bonus is more likely than she previously thought (reference-dependent anticipatory utility is proportional to 0.3). If instead today she thinks the probability of getting the bonus is the same as yesterday, she still experiences utility from the fact the bonus is likely (reference-independent anticipatory utility proportional to 0.5), but since her expectations of success have not changed, reference-dependent anticipatory utility equals zero.

The model's framework builds on standard assumptions of the optimal beliefs literature. First, anticipatory utility is a prominent element of the agent's decision utility. Second, agents choose and hold subjective beliefs that contradict the rational processing of the objective data. Whereas there seem to be a growing consensus among economists on the relevance of the first assumption, the second one is still controversial. I view the process through which agents choose subjective beliefs as the result of the operation of a "fast, automatic, effortless, associative, implicit and often emotionally charged system of thought" (Kahneman (2003).) Under such view, optimal beliefs may be seen as the result of a secondary process responding to our most basic instinct of pleasure-seeking. Even though there is still a long way to fully understand the "technology" agents use to distort beliefs, this interpretation seems reasonable under the view of the human mind as based in this dual system of thinking and is compatible with the basic notion of humans as utility maximizers.[1]

Section 2.4 presents the model predictions. I start analyzing the self t's optimal strategy by showing that in this model the agent's selves are *perfect strategic complements*: self t optimal reaction to an increase in self $t-1$ belief is to increase her belief of success in the same magnitude. As a result, increasing current optimal belief increases *all* future selves optimal beliefs. This feature of the optimal strategy is important as it shapes the costs and benefits of optimism

[1]This is the view Brunnermeier and Parker (2005) also embrace. A second possible interpretation arises in the context of ambiguity (see Camerer and Weber (1992) for a review). Under ambiguity, the agent does not know the true probability distribution but a family of such distributions and a probability distribution over the family. Because of the linearity of the expected utility function on the true probability, the agent uses as a proxi for the objective belief her best guess: the expected mean of the compounded probability distribution. As a consequence, the agent's optimal beliefs are a credible choice because, by construction, optimal beliefs are drawn from the family of feasible distributions. However, in a model without information arrival like ours, this is a rather trivial extension and would not make any new predictions.

during waiting periods. Consider first the costs. Because the agent experiences reference-dependent anticipatory utility, choosing a high belief today puts a burden into the next self as reference-dependent anticipatory utility is decreasing in the reference. However, because of the selves perfect complementarity, self $t+1$—and all those after her—will optimally put-off this cost avoiding thus changes in beliefs. As a consequence, the real cost of optimism only comes in period T+1 when a high period T belief unavoidably decreases the last period reference-dependent expected utility. To the contrary, benefits of optimism come from reference-independent anticipatory utility. The more optimistic the agent is today the higher the utility she gets from anticipating the future outcome as reference-independent anticipatory utility is increasing in present optimism. Moreover, the selves perfect complementarity reinforces this benefit as a high belief today implies all future selves will also set a high belief.

Next, I show that optimal beliefs decrease as the realization date approaches when the lottery outcome is important enough or the agent is sufficiently loss averse. The intuition is as follows. Because agents are perfect strategic complements having a high belief today implies a high period T belief. This decreases the probability of a gain and increases the probability of a loss in the realization period. When the high outcome is important enough or the agent is sufficiently loss averse this cost of optimism is high and thus setting a high belief today is expensive. When far away from the pay-off date, however, high optimism has great benefits since it increases not only today's expectations of a positive future but also those of upcoming periods because of the agents strategic complementarity. As the realization date approaches, however, this benefit decreases and thus the cost that current optimism imposes on last-period reference-dependent expected utility becomes relatively important. For a sufficiently high cost of optimism, this implies the agent will optimally choose a decreasing path of beliefs.

Evidence in the psychology literature supports this prediction. Shepperd, Ouellette, and Fernandez (1996) conducted a field experiment where 144 students estimated their score on a exam on four occasions: once 1 month before the exam and then 5 days, 50 minutes and 3 seconds before the grades were due. To control for information arrival, the last two measures were obtained in the same lecture. In between them the teacher lectured about material that was not related to the exam. They find students displayed overoptimism in the initial measures but that optimism decreased steadily until it becomes pessimism right before they find out their grades. The authors report that all the differences are statistically significant.[2] Taylor and Shepperd (1998) conducted an experiment were the agents believed they would be tested for a medical condition which could be severe or not. At the beginning of the experiment, subjects who anticipated being tested believed they would receive their results in 3 to 4 weeks. At the end of the test a treatment group learned they would receive their results in few moments. Subjects where tested the most pessimistic when anticipated immediate feedback for a deficiency with severe consequences. Similar evidence of subjective beliefs that evolve from optimistic to pessimistic without information arrival–or that depend on the proximity to the realization date–can be found in Nisan (1972), Pyszczynski (1982), Manger and Teigen (1988), Gilovich et al. (1993), Gilovich and Medvec (1995), Radhakrishnan, Arrow, and Sniezek (1996), Shepperd, Ouellette, and Fernandez (1996), Savitsky, Medvec, Charlton and Gilovich (1998), Beyer (1999), Sanna (1999), Shepperd (2006), Sweeny, Carroll, and Shepperd (2006), and Jö rgensen, Faugli, and Gruschke (2007).[3]

[2] Gilovich, Kerr and Medvec (1993) present evidence of heterogeneity in these results. Whit the same experimental design as in the study by Shepperd, Ouellette, and Fernandez (1996) they find that 69% of the students displayed this decrease in optimism, meanwhile 15% made steady predictions and the rest had increasing predictions.

[3] This result has also been applied on the axiomatic literature of self control and commitment. Epstein and Kopylov (2007) build a model where in period 0 the agent chooses a menu of acts from where she will pick an

The empirical evidence also supports the prediction that the decreasing path of optimism depends on the importance of the outcome. van Dijk, Zeelenberg, and van der Pligt (2003) had psychology students take a test that was described as predictive either for a career as a psychologist or for a career as a lawyer. Students were asked to give performance estimates directly after completing the test and again just before they were to receive feedback about their performance. Consistent with Shepperd, Ouellette, and Fernandez (1996) and others, participants lowered their expectations when they were closer to learning the result of the test. However, they did this only when the test they had taken was predictive for their career as a psychologist. Similar evidence can be found in Taylor and Shepperd (1998).

I then explore other possible paths of optimism. In particular, when there is no disappointment in the realization period–because the realization period expected utility is not referent dependent–the optimal path of beliefs is perfectly optimistic or increasing across time depending on the relative importance of the two types of anticipatory utility: the path will be increasing if it is convenient for the agent to delay perfect optimism to take advantage of reference-dependent anticipatory utility, i.e., if the utility the agent gets from positive surprises compensate the forgone gain in reference-independent anticipatory utility.

I further explore the relationship between the model structural parameters and the path of optimal beliefs. First, I show that optimism is increasing in the high outcome true probability: the more likely the high outcome is, the more optimistic the agent will be. This feature of the model is important as it shows some reasonability in subjective beliefs. Second, I prove that optimism is greater for lotteries with shorter waiting periods. I interpret such result as shorter horizons keeping the agent engaged or excited about the lottery she is holding. Finally, I study the relationship between optimism and the reward size. I show that when the probability of getting the high outcome is high, then optimism is increasing in the reward size. To the contrary, optimism is decreasing in the size of the reward when the true probability is low.

The model has several interesting economic applications. Consider for example the behavior of small investors who are deciding when to sell a stock. When the price of the stock depends on the realization of infrequent information—e.g, macroeconomic variables or the launch of a technological product—the optimal selling time will depend on the interaction between information arrival and her distorted beliefs, leading to the agent to sell at—ex ante probably suboptimal—times. Other possible examples concern the decision of taking medical treatment. If the decision of taking a primary treatment depends on the result of upcoming medical information, biased beliefs may cause the agent to take a suboptimal decision regarding the initial treatment. Even though this and other applications may be worth explored, Section 2.5 applies the problem of a risk neutral principal who wants to create a productivity bonus scheme to induce the agent to exert high effort during T periods at the cheapest cost. I prove that when the agent's effort choice greatly affects the probability of getting the bonus, and the positive effect over optimism of a shorter horizon outweighs the negative effect of a smaller payment, it is optimal for the principal to pay small but frequent bonuses. This is because frequent bonuses keep the agent optimistic about her chances of success, which acts as a non-pecuniary motivator allowing the principal to induce high effort at a cheaper cost. Section 2.6 ends with a discussion.

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action in the second period. Therefore, ex-ante choices can limit ex-post actions. In both periods agent pick up beliefs, choosing the "correct" ones in period zero and (by assumption) pessimistic ones in period one. Temptation, therefore, comes from changes in beliefs which creates an incentive for self zero to limit the choices pessimistic self 1 will face.

2.2 Related Literature

The optimal-beliefs literature builds on the assumption that people can choose optimal subjective beliefs that depart from the true probabilities. Two branches of the literature can be distinguished according on how the costs and benefits of optimal belief distortion are modeled.

The first branch relates to the cognitive dissonance literature. This literature proposes agents trade-off a taste for consistency and the desires of enhancing the desirability of their past actions. To model this taste for consistency an extra term is added to the utility function which depends on some measure of distance between objective and subjective beliefs. Akerlof and Dickens (1982), who were to the best of my knowledge the first to propose that agents optimally distort beliefs, build a decision making model for workers in a hazardous profession. Workers decide whether to work in a high risk job or in a safe job. If they choose to work in the risky job, in a second period they choose how much safety equipment to buy. If they choose to buy the equipment the (known) probability of an accident is greatly reduced. To introduce anticipatory feelings the authors add an extra component to the utility function: fear of accident risk. The key assumption is that this fear is inversely related to a subjective probability of an accident and directly related to the true probability of the same event. The worker then chooses her optimal beliefs by trading the benefits of diminishing emotions of fear and the cost that these distorted beliefs have over choosing the wrong industry or buying less than objective optimal amount of security equipment. Yariv (2005) develop a generic model that explores the implications of agents trading taste for consistency and the desire of making their actions more agreeable. His model shows that agents can be under and over-confident, they prefer less accurate signals and are even willing to pay to forgo information. In the same line, Bracha (2004) develops a model where agents selects an optimal risk perception to balance two contradicting forces: the desire to hold favorable personal risk perception (optimism) and a taste for consistency. She applies the model to insurance decisions. Finally, Rabin (1994) develops a model of cognitive dissonance where the agent chooses how much to consume of an activity and her belief about the level over which such consumption becomes immoral. He shows that when the agents beliefs affect other's beliefs, increasing the cost from consuming more than the threshold cause members of society to convince each other that immoral activities are not such increasing thus the level of the immoral activity.[4]

The second branch of the optimal beliefs literature characterizes the benefits of distorted beliefs by assuming that agents have anticipatory feelings.[5] Brunnermeier, Gollier, and Parker (2007) have written a sequence of papers where they apply the concept of optimal beliefs with anticipatory utility to several risk taking decisions. Brunnermeier and Parker (2005) write a model of subjective beliefs and build two applications; a simple portfolio choice where agents choose between a risky and a safe asset and a consumption-saving problem with stochastic income. The basic logic is as usual: agents with anticipatory utility care about expected future utility flows, and hence have higher current felicity if they believe that better outcomes are more likely. On the other hand, biased expectations lead to poorer decisions and worse realized outcomes on average. Optimal expectations balance these forces by maximizing average felicity. Their model concludes that a small bias in beliefs typically leads to first-order gains due to increased anticipatory utility and only to second-order costs due to distorted behavior. In the portfolio choice problem, agents overestimate the return on their investment inducing them to purchase too much of the risky asset compared to what would maximize the objective expected

[4] Yariv (2005), Bracha (2004) and Rabin (1994) are actually models of cognitive dissonance: agents have a direct disutility of not being cognitive consistent. Akerlof and Dickens (1982) use the concept of cognitive dissonance not to picture this direct disutility, but just to point out that optimal beliefs differ from objective beliefs.

[5] Note that this is in fact very similar to what is proposed by Akerlof and Dickens (1982). Their extra term of fear of accident risk is nothing but an anticipatory emotion related to future states!

utility. They also show that agents exhibit a preference for skewness. In the consumption-saving application they conclude that agents are both overconfident and overoptimistic. Gollier (2007) extends the analysis by examining the same portfolio choice problem in the context of complete markets. A further extension is made in Brunnermeier, Gollier, and Parker (2007) who build a general equilibrium model with complete markets. They show that when investors hold optimal beliefs that balance anticipatory benefits and costs of distorted actions, portfolio holdings and asset prices match some observed empirical patterns such as investor being imperfectly diversified and over-investing in the most skewed securities, which causes that more skewed asset can have lower average returns.

Finally, the closest work to this is that of Gollier and Muerman (2006). They develop a two-period model of risk taking in which people extract utility from anticipation and also experience ex-post disappointment as in Gul (1991). In the first period the agent makes two choices: an objective lottery whose outcome is realized in the second period and a subjective lottery which is an optimal distortion of the probability distribution of the objective lottery. The agent problem is to make her choices so to maximize her inter-temporal welfare defined as the weighted sum of the first period and second period preferences. Preferences in the first period are solely composed of anticipatory utility defined as the expected utility using the subjective probability lottery: the more optimistic the subjective lottery is, the happier the agent will be in the first period. Preferences in the second period have a expected utility form using the objective lottery for the relevant probabilities (as is common to every paper in distorted beliefs.) To introduce disappointment this second period's utility function is negatively related to an "anticipated payoff" or reference point, which is defined as the certain equivalent of the subjective lottery. Optimal beliefs then balance the ex-ante pleasure of being optimistic and the desire to scape from disappointment. They conclude that these preferences are compatible with the Allai's paradox and show that the agent will take less risk compared to a standard utility maximizer. This paper's model is an extension of Gollier and Muerman (2006) as it includes not only last period disappointment but also reference-dependent anticipatory utility in a T periods setting, which allows me to study optimal beliefs dynamics.

2.3 The Model

There are T+1 periods and a single agent. A random outcome X is realized in period T+1. The outcome takes a high or a low value leading to consumption utilities $u_L = 0$ and $u_H \equiv u > 0$. Let $\theta \in (0, 1)$ represent the true probability of observing the high outcome in the realization period. Periods $1 \leqslant t \leqslant T$ are "waiting" periods: no value of X is drawn and no information arrives. The agent's problem of choosing an optimal path of beliefs corresponds to a sequential game with perfect information between the agent's T selves. In each period t, current self chooses a belief $\widetilde{\theta}_t$ about the probability of observing the high outcome in the realization period. Each self makes her choice having observed all past selves choices $\{\widetilde{\theta}_s\}_{s=1}^{t-1}$ and future selves optimal reaction to her choice, i.e., the game structure and relevant functions are common knowledge.

2.3.1 The Utility Function

The agent's intertemporal utility consists of two types of utility flows: the realization period utility flow and those during waiting periods. Consider first period T+1 expected utility flow. I assume that in this period the agent experiences two types of utility: standard consumption utility and reference-dependent utility from comparing the true outcome lottery with the lottery he expected in period T to get in period T+1. Moreover, I assume that to compare these two lotteries, the agent compares each possible outcome with each possible realization of the reference lottery. This specification follows the structure of the utility function in Kőszegi and

Rabin (2006).[6] Thus the total expected utility in the realization period T corresponds to:

$$\begin{aligned} v_{T+1} &= \theta[u + \widetilde{\theta}_T\mu(0) + (1-\widetilde{\theta}_T)\mu(u))] + (1-\theta)[0 + \widetilde{\theta}_T\mu(-u) + (1-\widetilde{\theta}_T)\mu(0)] \\ &= \underbrace{\theta u} + [\underbrace{\theta(1-\widetilde{\theta}_T)\mu_+ + (1-\theta)\widetilde{\theta}_T\mu_-}] \end{aligned} \tag{2.1}$$

where $\mu_+ \equiv \mu(u_H - u_L) = \mu(u)$ and $\mu_- \equiv \mu(u_L - u_H) = \mu(-u)$.

The first bracket corresponds to standard expected consumption utility. The second bracket corresponds to the expected value of the reference-dependent component of the utility function where $\mu(\cdot)$ is the value function comparing the actual outcome with the expected one. It has two components. First, a potential gain in case the high outcome is realized when the low outcome was expected (μ_+ with probability $\theta(1-\widetilde{\theta}_T)$), and second, a potential loss in case the low outcome is realized when the high outcome was expected (μ_- with probability $(1-\theta)\widetilde{\theta}_T$.) Finally, and for generality, I write equation (2.1) as $\theta u + \gamma^{RD}[\theta(1-\widetilde{\theta}_T)\mu_+ + (1-\theta)\widetilde{\theta}_T\mu_-]$ were weight γ^{RD} has been added to the reference dependent component. Such weight shows the importance of expected gain-loss utility relative to standard expected consumption utility in the realization period. Hence, if $\gamma^{RD} = 0$ these preferences reduces to standard reference-independent preferences.

To model utility flows during waiting periods $1 \leqslant t \leqslant T$, I join two literatures on belief-based preferences. First, the literature on standard anticipatory utility, which proposes that forward looking agents care today about utility flows that will only be realized in the future. This literature further assumes that agents bring these prospects to present in an standard expected consumption utility form. Therefore period *t reference-independent anticipatory utility* corresponds to:

$$\widetilde{\theta}_t u_H + (1-\widetilde{\theta}_t)u_L \equiv \widetilde{\theta}_t u$$

This utility source has two important features. First, its size depends on the importance of the lottery the agent is playing. If the lottery is rather irrelevant, then u will be small and thus $\widetilde{\theta}_t u$ will be small. Second, and most important, its size depends *positively* on the agent's current belief about future success: the more likely the agent thinks that she will succeed in the future, the happier she is today.

Second, I follow Matthey (2005) and Kőszegi and Rabin (2009) who propose agents also experience anticipatory utility from *changes* in their beliefs. Building on this second notion of anticipatory utility, I assume the agent also experience utility from changes in their expected utility from intrinsic consumption during waiting periods.[7] Therefore this period *t reference-*

[6] The Kőszegi and Rabin (2006) model differs from this in that they assume the agent has rational expectations when forming their beliefs, which leads them to a completely different equilibrium concept.

[7] This model greatly differs from that in Kőszegi and Rabin (2009). They develop a dynamic model of reference-dependent preferences where agents experience utility from three sources. First, directly from present consumption, second, from "contemporaneous gain loss utility" relating previous period's beliefs about present consumption with actual current consumption, and third, from "prospective gain-loss utility" which captures utility from changes in their past beliefs about future consumption. Just as their static model however, they assume the agent has rational expectations in forming these beliefs and changes in beliefs are computed over changes in consumption, not over changes in future expected consumption utility.

dependent anticipatory utility[8,9] corresponds to:

$$\mu\big((\widetilde{\theta}_t u_H + (1-\widetilde{\theta}_t)u_L) - (\widetilde{\theta}_{t-1} u_H + (1-\widetilde{\theta}_{t-1})u_L)\big) = \mu\big((\widetilde{\theta}_t - \widetilde{\theta}_{t-1})u\big)$$

This utility source has two important features. First, just as consumption utility, its size depends on the importance of the lottery the agent is playing. If the lottery is rather irrelevant, then u will be small and thus $\mu(\cdot)$ will be small. Second, and most important, its size depends *negatively* on the previous period belief: if the agent holds a great expectation about her success in the realization period during period t, then self $t+1$ will have to set an even higher expectation if she wants to create a gain from this utility source.

Finally, Assumption 1 presents the structure imposed over the gain-loss utility function $\mu(\cdot)$.

Assumption 5 *(Concavity of the Gain-Loss Utility Function)*
Let $\mu : [-u, u] \longrightarrow \mathbb{R}$ *be:*

(i) *continuous, twice differentiable and* $\mu(0) = 0$
(ii) *strictly increasing* $\forall x$ *and* $\mu'(0) = 1$
(iii) $\mu'(-x) = \lambda(x)\mu'(x)$ *where* $\lambda(x) > 1$, $\lambda'(x) \geqslant 0$ $\forall x \in (0, u]$ *and* $\lim_{u \to 0} \lambda(u) = 1$
(iv) $\mu''(x) \leqslant 0 \quad \forall x$

Part (i) and (ii) are regularity assumptions ensuring that the value function is well-behaved. Part (iii) says the agent is loss averse, that is, she is more sensitive to losses than to same sized gains. Finally, part (iv) assumes the gain-loss utility function is concave in all its domain.[10]

Given these two types of anticipatory utility, each self's total utility flow v_t during waiting periods corresponds to the weighted sum of reference-independent and reference-dependent anticipatory utilities:

$$v_t = \gamma^{RI}_{T-t+1}\widetilde{\theta}_t u + \gamma^{RD}_{T-t+1}\mu((\widetilde{\theta}_t - \widetilde{\theta}_{t-1})u) \tag{2.2}$$

where γ^{RI}_{T-t+1} and γ^{RD}_{T-t+1} are the corresponding weights representing how important is for current self reference-independent and reference-dependent anticipatory utility respectively. Note

[8]Alternatively I could have follow Kőszegi and Rabin (2009) and name this term as *gain-loss anticipatory utility*. I choose the name of reference-dependent anticipatory utility only for sake of emphasis with respect to the first type of anticipatory utility.

[9]Notice that this functional form for reference-dependent anticipatory utility differs from the natural extension of (2.1), $\widetilde{\theta}_{t-1}(1-\widetilde{\theta}_t)\mu_+ + (1-\widetilde{\theta}_{t-1})\widetilde{\theta}_t\mu_-$ $\forall t$. This functional form, even though directly related to (2.1), is actually not capable of reflecting that increases (decreases) in the probability of success imply a positive (negative) utility. To see this, consider the following example. If the probability of success increases from 0.2 to 0.5, then the agent would experience a reference-dependent expected utility of $(0.2 \times 0.5)\mu_+ + (0.8 \times 0.5)\mu_- < 0$, which corresponds to a loss rather than again.

[10]This assumption differs to those in Kőszegi and Rabin (2006, 2007, 2009). I argue, however, that this is not binding in this framework. Two main reasons support the claim. First, the function's domain is not necessarily a large interval making the distinction between small and big stakes a non-central issue. Second, I can still assume μ is sufficiently steep near zero from the left so this concave gain-loss utility function still displays loss aversion for small changes in beliefs (see Assumption 5.*4*). The key difference with Kőszegi and Rabin gain loss utility function is therefore, that Assumption 5 rules out diminishing sensitivity in losses. I justify this in that for an argument of the type $\propto (\widetilde{\theta}_t - \widetilde{\theta}_{t-1})$ is credible to assume disutility increases as losses increase. In fact, it seems reasonable to think a decay in the probability of success hurts more when falling from 0.2 to 0 than from 0.6 to 0.4. This is supported by the shape of Kahneman & Tversky (1979) weighting function that is steeper in the extremes and flatter in the middle. The concavity assumption has great technical advantages. It implies $\mu'^{-1}(\cdot)$ exists and is well defined.[11] In particular $\mu'^{-1}(\cdot) : [\mu'(u), \mu'(-u)] \longrightarrow [-u, u]$ and is of the form $\mu'^{-1}(x) \geqslant 0$ if $x \leqslant \mu'(0)$ and $\mu'^{-1}(x) < 0$ if $x > \mu'(0)$ where $\mu'(0) > 0$ is a location parameter. For simplicity Assumption 5.*5* also includes the normalization $\mu'(0) = 1$. This simplification is useful to clarify the intuition behind the results although is not critical.

these weights are indexed with respect to the distance from the realization period (in period t there are $T-t+1$ periods left to period T+1). Assumption 6 impose reasonable restrictions over temporal behavior of these weight sequences.

Assumption 6 *(Non-negative and Non-decreasing Feelings of Anticipation)*
$\gamma^{RI}_{T-t+1} \geqslant \gamma^{RI}_{T-t+2} \geqslant 0$ *and* $\gamma^{RD}_{T-t+1} \geqslant \gamma^{RD}_{T-t+2} \geqslant 0 \quad \forall\, t \in \{1,\ldots,T\}$.

Assumption 6 establishes feelings of anticipation are non-negative and do not decrease as the outcome realization date approaches. In period t, when there are $T-t+1$ periods left to the realization period, feelings of anticipation are stronger than in previous period $t-1$, when there were $T-t+2$ waiting periods left. Therefore today's weight γ_{T-t+1} is greater than yesterday's weight $\gamma_{T-(t-1)+1} = \gamma_{T-t+2}$ for both types of anticipatory utility. It is important to notice that these weights are not related to time discounting as they do not reflect how the future is less important than the present, but exactly the opposite.

Before presenting the total utility function, Assumption 7 presents a technical requirement establishing a relationship between the level of the $\mu(\cdot)$ and the relevant parameters of the model.

Assumption 7 *(Arguments of the Gain-Loss Utility function Belong to the Domain)*
Define $B(\theta) \equiv \gamma^{RD}[\theta\mu_+ + (1-\theta)|\mu_-|]$. *For any set of parameters* $(\{\gamma^{RI}\},\{\gamma^{RD}\},u,\theta,T)$ *it must be the case that* $\min_t \left\{ \frac{B(\theta)-\sum_{s=t}^{T}\gamma^{RI}_{T-s+1}u}{\gamma^{RD}_{T-t+1}u} \right\} \geqslant \mu'(u)$ *and* $\max_t \left\{ \frac{B(\theta)-\sum_{s=t}^{T}\gamma^{RI}_{T-s+1}u}{\gamma^{RD}_{T-t+1}u} \right\} \leqslant \mu'(-u)$.

Assumption 7 has two purposes. First, it ensures equilibrium marginal utilities are positive, and second, it guarantees the relationship between the parameters $(\{\gamma^{RI}\},\{\gamma^{RD}\},u,\theta,T)$ have the correct size so the gain-loss utility function arguments belong to its domain $[-u,u]$ as defined in Assumption 5.[12]

The final piece of the utility description relates the utility flows among selves. We assume self t total utility corresponds to the sum of her own utility flow and those of future selves, the natural assumption in a game where players are incarnations of a unique agent. Therefore:

$$U_t \equiv \sum_{s=t}^{T+1} v_t$$

Note this specification assumes the agent does not discount utilities from future selves. We make such assumption to isolate the basic mechanism behind the relationship between reference-independent and reference-dependent anticipatory utility. The whole analysis remains if an exponential discount is included.

2.3.2 Optimal Strategies and Equilibrium

An optimal strategy for self t is a function providing the best belief choice given future selves optimal responses and the game history. Formally, let $h_{t-1} \equiv \{\widetilde{\theta}^*_s\}_{s=1}^{t-1} \in [0,1]^{t-1}$ represent a game history up to period $t-1$. An strategy $s^*_t : h_{t-1} \longrightarrow [0,1]$ is optimal if it is self t best response to future selves optimal strategies $\{s^*_{t+1}, s^*_{t+2}, \ldots, s^*_T\}$ for any given game history, that is:

[12]The quantity $B(\theta) = \gamma^{RD}[\theta\mu_+ + (1-\theta)|\mu_-|]$ will appear often in the analysis. It corresponds to the change in period T+1 utility flow (equation (2.1)) due to changes in last period beliefs $(\partial\theta(1-\widetilde{\theta}_T)\mu_+ + (1-\theta)\widetilde{\theta}_T\mu_-/\partial\widetilde{\theta}_T = -B(\theta))$. Because of reasons that will be clear thereafter, $B(\theta)$ will actually correspond to the marginal cost of optimism.

$$\widetilde{\theta}_t^* \equiv s_t^*(h_{t-1}/s_{t+1}^*,\ldots,s_T^*) \in \text{Argmax}_{\widetilde{\theta}_t\in[0,1]} \sum_{s=t}^{T+1} v_t(s_{t+1}^*(\widetilde{\theta}_t), s_{t+2}^*(s_{t+1}^*(\widetilde{\theta}_t)),\ldots,s_T^*(s_{T-1}^*(\ldots s_{t+1}^*(\widetilde{\theta}_t)\ldots)))\quad(2.3)$$

where

$$\begin{aligned}\sum_{s=t}^{T+1} v_t(s_{t+1}^*(\widetilde{\theta}_t),\ldots,s_T^*(s_{T-1}^*(\ldots s_{t+1}^*(\widetilde{\theta}_t)\ldots))) &= \sum_{s=t}^{T}\gamma_{T-t+1}^{RI}\widetilde{\theta}_s^* u + \theta u + \sum_{s=t}^{T}\gamma_{T-s+1}^{RD}\mu((\widetilde{\theta}_s^*-\widetilde{\theta}_{s-1}^*)u)\\ &+ \gamma^{RD}[\theta(1-\widetilde{\theta}_T^*)\mu_+ + (1-\theta)\widetilde{\theta}_T^*\mu_-]\end{aligned}$$

where for the fist period θ_0 is an exogenous initial belief. Lemma 3 establishes the existence and uniqueness of optimal strategies and therefore of the game's equilibrium.[13] Lemma 4 characterizes each self optimal strategy for any given history of the game.

Lemma 3 *(Existence and Uniqueness of Equilibrium)*
Under Assumption 5.1-1.3 an optimal strategy $\widetilde{\theta}_t^* \equiv s_t^*(h_{t-1}/s_{t+1}^*,\ldots,s_T^*)$ solving self t maximization problem in (2.3) exists and is unique. Therefore a subgame perfect equilibrium $\{\widetilde{\theta}_t^*\}_{t=1}^T$ in pure strategies exists and is unique.

Lemma 4 *(Optimal Strategies)*
Under Assumption 5.1-1.4 and Assumption 6.1, for any given $h_{t-1} \equiv \{\widetilde{\theta}_s^*\}_{s=1}^{t-1} \in [0,1]^{t-1}$, self t optimal strategy corresponds to:

$$s_t^*(h_{t-1}) = \begin{cases} 0 & \text{if} \quad \frac{B(\theta)}{u} \geqslant \gamma_{T-t+1}^{RD}\mu'(-\widetilde{\theta}_{t-1}^*u) + \sum_{s=t}^T \gamma_{T-s+1}^{RI} \\ \widetilde{\theta}_{t-1}^* + \frac{1}{u}\mu'^{-1}(A_t) & \text{if} \quad \gamma_{T-t+1}^{RD}\mu'((1-\widetilde{\theta}_{t-1}^*)u) + \sum_{s=t}^T \gamma_{T-s+1}^{RI} < \frac{B(\theta)}{u} < \gamma_{T-t+1}^{RD}\mu'(-\widetilde{\theta}_{t-1}^*u) + \sum_{s=t}^T \gamma_{T-s+1}^{RI} \\ 1 & \text{if} \quad \frac{B(\theta)}{u} \leqslant \gamma_{T-t+1}^{RD}\mu'((1-\widetilde{\theta}_{t-1}^*)u) + \sum_{s=t}^T \gamma_{T-s+1}^{RI} \end{cases}$$

where

$$A_t \equiv \frac{B(\theta) - \sum_{s=t}^T \gamma_{T-s+1}^{RI} u}{\gamma_{T-t+1}^{RD} u} \qquad \textit{and} \qquad B(\theta) \equiv \gamma^{RD}[\theta\mu_+ + (1-\theta)|\mu_-|]$$

The optimal strategy in Lemma 4 has several interesting features. First, $\widetilde{\theta}_{t-1}^*$ contains all the information current self t needs to know from the game history, so that $s_t^*(h_{t-1}) = s_t^*(\widetilde{\theta}_{t-1}^*) = \widetilde{\theta}_t^*$. Second and most important, in this model selves are *strategic complements* (Bulow et al. (1985).) Intuitively, selves are said to be strategic complements when an increase (decrease) in self $t-1$ belief makes a higher belief more (less) desirable for self t.[14] This can be seen directly seen from s_t^* in Lemma 4. For an interior solution, the relationship between $\widetilde{\theta}_{t-1}^*$ and $\widetilde{\theta}_t^*$ is positive and 1:1. In the corner solution cases, the upper bound defining a corner in zero is increasing in $\widetilde{\theta}_{t-1}^*$, that is, the higher the previous belief, the less likely is that the current optimal belief

[13]Notice that because the self's game corresponds to a a sequential game with perfect information its solution can be found by backward induction.

[14]Formally, selves are said to be strategic complements whenever increasing self $t-1$'s belief increases self t's marginal utility, i.e., whenever $\frac{\partial^2 U_t}{\partial\widetilde{\theta}_{t-1}\partial\widetilde{\theta}_t} \geqslant 0$. In this model $\frac{\partial U_t}{\partial\widetilde{\theta}_t} = \gamma_{T-t+1}^{RI}u + \gamma_{T-t+1}^{RD}\mu'((\widetilde{\theta}_t - \widetilde{\theta}_{t-1})u)u + \gamma_{T-t}^{RD}\mu'((\widetilde{\theta}_{t+1}-\widetilde{\theta}_t)u)u \Rightarrow \frac{\partial^2 U_t}{\partial\widetilde{\theta}_{t-1}\partial\widetilde{\theta}_t} = -\gamma_{T-t+1}^{RD}\mu''((\widetilde{\theta}_t-\widetilde{\theta}_{t-1})u)u^2 \geqslant 0$. If players are not strategic complements, then they are said to be strategic substitutes whenever $\frac{\partial^2 U_t}{\partial\widetilde{\theta}_{t-1}\partial\widetilde{\theta}_t} \leqslant 0$

is zero. Equivalently, the lower bound defining a corner in one is increasing in $\widetilde{\theta}^*_{t-1}$, that is, the higher the previous belief, the higher the lower bound and thus the more likely is that the current choice is optimally equal to one. Why are agents strategic complements? Because of reference-dependent anticipatory utility. Since agents dislike falling behind the reference, ceteris paribus, the optimal reaction to an increase in the reference is to increase current belief in the same magnitude. In fact, notice that that loss aversion is not responsible for this result; agents dislike falling behind the reference independently from how strong this disutility is compared to a same sized gain.

The fact selves are strategic complements has two important implications that will be at the core of the model predictions. First, since reference-dependent anticipatory utility is strictly decreasing in the reference $\frac{\partial \mu((\widetilde{\theta}_t - \widetilde{\theta}_{t-1})u)}{\partial \widetilde{\theta}_{t-1}} = -\mu'((\widetilde{\theta}_t - \widetilde{\theta}_{t-1})u)u < 0$, choosing a high belief today puts a burden on the next-period self. However, because selves are strategic complements the optimal reaction of the next-period self will be to increase her belief in the same magnitude. Intuitively, each self then puts-off the cost of an increase in previous belief to avoid a loss in reference-dependent anticipatory utility. As a consequence, the real cost of optimism during waiting periods comes in period T when there is no subsequent belief to choose and hence a high period T belief unavoidably decreases last period reference-dependent expected utility. Using equation (2.1) is straight to see that the *marginal cost of optimism* in any period t corresponds to

$$\frac{\partial v_{T+1}}{\partial \widetilde{\theta}_T} = \gamma^{RD}[\theta\mu_+ + (1-\theta)|\mu_-|] \equiv B(\theta)$$

Crucially, notice that this marginal cost of optimism is increasing in the agent's loss aversion and in the size of the reward and in how sensitive the agent is in the realization period to the reference-dependent component of her utility function. Second, and despite the latter negative effect over future selves utility flow, increasing current belief has a twofold positive effect over self t total utility. The more optimistic the agent is today the higher the utility she gets from anticipating the future outcome since reference-independent anticipatory utility is increasing in the self's belief. Moreover, because selves are strategic complements, setting a high belief today implies all future selves will set a high belief, increasing thus self t total intertemporal utility.

One last interesting feature of the selves' optimal strategy corresponds to the nature of the bounds separating perfect optimism/pessimism from an interior solution. These bounds establish and upper and a lower threshold for the marginal cost of optimism ($B(\theta)$) such that below (above) the lower (upper) threshold the agent will be perfectly optimistic (pessimistic). These upper and lower bounds for $B(\theta)$ correspond to the marginal benefit associated with perfect pessimism and perfect optimism respectively. For any given previous period belief $\widetilde{\theta}^*_{t-1}$, the agent will choose a corner solution in zero whenever the marginal cost of optimism is too high relative to the highest marginal utility the agent may get corresponding to perfect pessimism. Conversely, the agent will choose a corner solution at one when the marginal cost of optimism is smaller than the smallest marginal benefit she may get corresponding to perfect optimism. Furthermore, notice the difference between these bounds, $\gamma^{RD}_{T-t+1}[\mu(-\widetilde{\theta}^*_{t-1}u) - \mu((1-\widetilde{\theta}^*_{t-1})u)]$, only depends on the gain-loss utility function and the value of $\{\gamma^{RD}\}$ and u. Therefore, agents who are more loss averse or who have a greater consumption valuation for the outcome have a wider range of marginal costs for which an interior solution holds.

2.4 Model Predictions

2.4.1 Intertemporal Path of Beliefs

I start considering the agent's problem of choosing an optimal belief in a given period of time t. The FOC for self t maximization problem in (2.3) corresponds to:

$$\underbrace{\sum_{s=t}^{T}\gamma^{RI}_{T-s+1}u}_{MB^{RI}_t}+\underbrace{\gamma^{RD}_{T-t+1}\mu'((\widetilde{\theta}^*_t-\widetilde{\theta}^*_{t-1})u)u}_{MB^{RD}_t} \quad = \quad \underbrace{B(\theta)}_{MC} \qquad \forall t \tag{2.4}$$

where MB^{RI}_t and MB^{RD}_t stand for reference-independent and reference-dependent marginal benefits of optimism meanwhile MC stands for optimism marginal cost.

The two implications of the selves strategic complementarity can be seen in equation (2.4). The functional form of MB^{RI}_t is a reflection of the first consequence of the selves strategic complementarity: current reference-independent marginal benefit is not just period t reference-independent marginal utility but the sum of all present to last period reference-independent marginal utilities. MC corresponds to the marginal cost of optimism as described before. The fact that this cost is common to all selves corresponds to the second consequence of the strategic complementarity: the real impact of choosing a high belief today is the burden imposed in the period T belief, the reference for the realization period reference-dependent utility. Finally, the second bracket in the LHS is the reference-dependent marginal benefit. This is the only endogenous component of the total marginal benefit and is therefore in charge of ensuring the equality in an interior solution.

To understand the agent's problem from a dynamic perspective notice from Lemma 4 that the agent's problem in an interior solution corresponds to the decision of increase, maintain or decrease the previous belief. As expected, self t will choose the same belief as self $t-1$ if the marginal cost from her choice is equal to the total marginal benefit associated with $\widetilde{\theta}^*_{t-1}=\widetilde{\theta}^*_t$. Formally, from equation (2.4), this will happen iff:

$$\sum_{s=t}^{T}\gamma^{RI}_{T-s+1}u+\gamma^{RD}_{T-t+1}u \quad = \quad B(\theta) \tag{2.5}$$

Analogously, self t will choose a higher (lower) belief than the previous self if the marginal cost is lower (higher) than the associated marginal utility. As a consequence, to solve this dynamic problem it is necessary to understand how the marginal cost and benefits of choosing same beliefs as in the previous period change from period to period.

Consider first the reference-independent component in equation (2.5), $\sum_{s=t}^{T}\gamma^{RI}_{T-s+1}u$. As the outcome realization period approaches this term decreases as the summation losses one term as it approaches one more period towards T+1. Intuitively, the closer the agent is to the realization period, the less periods are left to enjoy the positive effect current optimism has in the future selves choice due to their strategic complementarity. How much does this term decreases from period $t-1$ to t? Exactly $\sum_{s=t-1}^{T}\gamma^{RI}_{T-s+1}u-\sum_{s=t}^{T}\gamma^{RI}_{T-s+1}u=\gamma^{RI}_{T-t+2}u$. Therefore, the magnitude of this component decrease depends on the *size* of the reference-independent weights $\{\gamma^{RI}\}$. Second, reference-dependent marginal benefit, $\gamma^{RD}_{T-t+1}\mu'(0)u=\gamma^{RD}_{T-t+1}u$ corresponds to the change in reference-dependent utility the agent gets by choosing $\widetilde{\theta}^*_{t-1}=\widetilde{\theta}^*_t$. This benefit increases in time because feelings of anticipation are increasing in time (Assumption 6). How much this term increases from period $t-1$ to t? Exactly $(\gamma^{RD}_{T-t+1}-\gamma^{RD}_{T-t+2})\mu'(0)u$. Therefore, the magnitude of this component increase depends on the *pace* at which $\{\gamma^{RD}\}$ increases;

the faster $\{\gamma^{RD}\}$ increases, the greater the increase of reference-dependent marginal utility of choosing previous period belief. How does the total marginal utility evolve across time hence depends on the relative speeds at which the reference-independent marginal benefit decreases and the reference-dependent marginal benefit increases. If the level of the reference-independent weights outweigh (is outweighed by) the change in reference-dependent weights, then the total marginal utility will be decreasing (increasing) in time. Finally, recalling that the marginal cost of optimism—the decrease in the realization period reference-dependent expected utility due to an increase in period T beliefs—is constant in time, it is clear that the temporal behavior of beliefs will depend on how the level marginal cost of optimism relates to the temporal sequence of marginal benefits.

Proposition 9 establishes conditions over the marginal cost of optimism such that the optimal path of beliefs is decreasing in time for any relative behavior of the sequences $\{\gamma^{RI}\}$ and $\{\gamma^{RD}\}$.

Proposition 9 *(Decreasing Optimism)*
Assume an interior solution. Under Assumptions 1-3, if
(1) $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}\ \forall t > 1$ *and* $\frac{B(\theta)}{u} \geqslant \gamma^{RD}_1 + \gamma^{RI}_1$ *or*
(2) $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < \gamma^{RI}_{T-t+2}\ \forall t > 1$ *and* $\frac{B(\theta)}{u} \geqslant \gamma^{RD}_T + \sum_{s=1}^{T} \gamma^{RI}_{T-s+1}$
then the optimal path of beliefs is such that $\widetilde{\theta}^*_t \geqslant \widetilde{\theta}^*_{t+1}\ \forall t \leqslant T-1$.

Proposition 9 states that for any relative behavior of the reference-dependent and reference-independent weights a big enough marginal cost of optimism will cause optimal beliefs to decrease in time. In further detail, if the total marginal benefit of choosing the same previous-period belief is decreasing in time—because the level of the reference-independent weights outweigh the change in the reference-dependent weights—then a high enough marginal cost inducing the agent to decrease her beliefs in the first period will also induce her to decrease her choice in all subsequent periods. Equivalently, if the total marginal benefit of choosing the same previous-period belief is increasing in time—because the level of the reference-independent is outweighed by the change in reference-dependent weights—then a marginal cost inducing the agent to decrease her belief in the last period will also be sufficient for her to decrease her belief choice in all previous periods. Formally, recall $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2}$ is proportional to the reference-dependent marginal utility increase, whereas γ^{RI}_{T-t+2} is proportional to the reference-independent marginal utility decrease.[15] Therefore, the condition $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}$ implies that the total marginal benefit of choosing last period belief is non-decreasing in time. The associated condition $\frac{B(\theta)}{u} \geqslant \gamma^{RD}_1 + \gamma^{RI}_1$ ensures the marginal cost of optimism is greater than the total marginal benefit of choosing previous period belief in every period and, therefore, the agent will optimally decrease her belief choice with respect to the previous period belief in every period. Similarly, the condition $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < \gamma^{RI}$ represents the alternative case in which total marginal utility is decreasing in time. The associated condition $\frac{B(\theta)}{u} \geqslant \gamma^{RD}_T + \sum_{s=t}^{T} \gamma^{RI}_{T-s+1}$ ensures that in every period the agent will optimally choose a belief smaller than the previously chosen optimal belief since total marginal benefit of keeping previous belief is always smaller than its marginal cost.[16]

[15] Note the conditions hold for $t > 1$ since for the initial period there is no previous period. Therefore, increases in the total marginal utility count from period 2 onwards.

[16] An alternative (but more cumbersome) explanation to Proposition 9 comes from examining the evolution of the FOC across time. Recall that in an interior solution recall self t FOC: $\sum_{s=t}^{T} \gamma^{RI}_{T-s+1} u + \gamma^{RD}_t \mu'((\widetilde{\theta}^*_{T-t+1} - \widetilde{\theta}^*_{t-1})u)u = B(\theta)$. If the RHS is big enough, the LHS must also be high. When far away form the realization period MB^{RI} is big and therefore the agent does not have to set a loss (or big loss): MB^{RD} can be small. As time goes

Intuitively, when the high outcome is important enough or the agent is sufficiently loss averse—so that the marginal cost of optimism is high—a small increase in period T belief implies a big decrease in last period reference-dependent expected utility, making current optimism expensive. When far away from the pay-off date high optimism has great benefits since it increases not only today's expectations of a positive future but also those of upcoming periods. As the realization date approaches, however, this benefit decreases and the cost current optimism imposes on last period reference-dependent expected utility becomes relatively important. For a sufficiently high cost of optimism, this implies the agent will optimally choose a decreasing path of beliefs from the very first period.

To strength this intuition, consider the following questions. First, how is the case the agent is willing to afford loses in equilibrium? and second, if the agent wants to decrease her beliefs, why is it the case she does not decrease them immediately to zero? The answer to both questions relies on reference-independent anticipatory utility. First, even though the agent is experiencing painful losses from reference-dependent anticipatory utility these losses are compensated by reference-independent anticipatory utility, which makes optimal to avoid perfect pessimism. Equivalently, the agent does not decrease her beliefs to success to zero from the start because if reference-independent anticipatory utility is important enough, it will be convenient to afford the losses from a decreasing path to experience the joy of hoping a promising realization period.[17,18]

The importance of the degree of loss aversion and that of the reward size in Proposition 9 can also be seen from their impact on the feasible range of marginal cost of optimism. Because of Assumption 7, when $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}$, it is known $B(\theta) \in [\gamma^{RD}_{T}\mu'(u)u + \sum_{s=t}^{T}\gamma^{RI}_{T-s+1}u, \gamma^{RD}_{1}\mu'(-u)u + \gamma^{RI}_{1}u]$. Therefore, all costs in the interval $[\gamma^{RD}_{1} + \gamma^{RI}_{1}, \gamma^{RD}_{1}\mu'(-u) + \gamma^{RI}_{1}]$ lead to a decreasing pattern of beliefs. How big this interval is depends on the degree of loss aversion and the consumption utility size the agent gets from the reward; higher loss aversion and higher reward imply a larger interval and therefore a higher likelihood of observing a decreasing pattern of beliefs. The intuition is straightforward. The more loss averse the agent is, the more she has to lose in expectation by setting a higher last period reference. As a consequence, she is more likely to choose a decreasing path of beliefs to avoid the possibility of a painful loss in the realization period. Equivalently, if the size of the reward is big–or if the agent has a high valuation for the outcome–she has more to loose for a given reference beliefs and therefore is more likely she will choose a decreasing path of beliefs. The case of a decreasing total marginal utility is analogous. When $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < \gamma^{RI}_{T-t+2}$, then $B(\theta) \in [\gamma^{RD}_{1}\mu'(u)u + \gamma^{RI}_{1}u, \gamma^{RD}_{T}\mu'(-u)u + \sum_{s=t}^{T}\gamma^{RI}_{T-s+1}u]$. Therefore, all costs in the interval $[\gamma^{RD}_{T}\mu'(0) + \sum_{s=t}^{T}\gamma^{RI}_{T-s+1}, \gamma^{RD}_{T}\mu'(-u) + \sum_{s=t}^{T}\gamma^{RI}_{T-s+1}]$ lead to a decreasing pattern of beliefs. As before, such interval increases with the degree of loss aversion and the reward size or the level of the intrinsic consumption the agent derives from the high outcome.

Finally, notice Proposition 9 assumptions $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}$ and $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < \gamma^{RI}_{T-t+2}$ are related, but are not implied by the relative speeds at which $\{\gamma^{RD}\}$ and $\{\gamma^{RI}\}$ grow. This can be seen by assuming that reference-independent weights grow at a constant

by, MB^{RI} decreases thus MB^{RD} must increase to preserve the equality. If γ^{RD} grows fast (equivalently if γ^{RD} is big compared to γ^{RI}) the path of decrease can be softer since there are less pressures for the $\mu'((\widetilde{\theta}^{*}_{T-t+1} - \widetilde{\theta}^{*}_{t-1})u)$ to increase (since the γ^{RD}_{T-t+1} increases stronger). Alternatively, if $\{\gamma^{RD}\}$ grows fast a smaller marginal cost can sustain the same decreasing path compared to the case when $\{\gamma^{RD}\}$ grows slow.

[17]Notice that last period reference-dependent expected utility is not in fact sufficient for the agent to hold losses: for last period expected utility purposes all that matters is period T belief.

[18]The importance of reference-dependent marginal benefit in this intuition is secondary and only determines the speed at which beliefs will decrease. When reference-dependent anticipatory utility increases faster than reference-independent marginal decreases, then beliefs will decrease at an increasing rate. Conversely, when total marginal utility increase, the optimal path of beliefs decrease at a decreasing rate.

pace. Re-expressing the second condition as $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < (t-1)\gamma^{RI}$ it can be seen that this condition implies $\{\gamma^{RD}\}$ growing slower than $\{\gamma^{RD}\}$ $-\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < \gamma^{RI} = \gamma^{RI}_{T-t+1} - \gamma^{RI}_{T-t+2}-$. However, the condition $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}$ is stronger than $\{\gamma^{RD}\}$ growing faster than $\{\gamma^{RI}\}$; it is needed for $\{\gamma^{RD}\}$ to grows much faster than $\{\gamma^{RI}\}$ to ensure total marginal utility decreases in time.

It is important to emphasize that all three components of the utility function–last period reference-dependent expected utility, reference-dependent and reference-independent anticipatory utilities–are *together* necessary and sufficient to create beliefs time dynamics in this model. Removing any of these components of the utility function imply beliefs that cannot change smoothly in time. To see this, first consider the case where the agent has last period reference-dependent preferences and reference-independent anticipatory utility but does not experience utility from changes in beliefs. [19] Proposition 15—in Appendix B—establishes that in this case the agent will be perfectly optimistic in every period, except for period T when she will choose to be perfectly pessimistic. The intuition is straight. Without reference-dependent anticipatory utility optimism is free lunch since current beliefs are not related to future ones. Second, consider the case when there is no reference-independent anticipatory utility. Proposition 16—in Appendix B—states that under Proposition 9 assumptions, the optimal sequence of beliefs when $\{\gamma^{RI}\} = \{0\}$ cannot be decreasing. To see this, notice other paths of beliefs such as $\widetilde{\theta}^*_t = 0$ $\forall t$ give the agent higher utility than a decreasing one. This result highlights the importance of reference-independent anticipatory utility: the agent is only willing to afford losses in the reference-dependent domain if these are compensated through reference-independent anticipatory utility.

Finally, another interesting case occurs when the agent does not experience reference-dependent expected utility in the realization period. I refer to this case as the "no day or reckoning" case. This may happen in the extreme case when the agent never finds out which outcome has been realized. Economic application of this assumption are bequests models, where agents engage in economic activities such as portfolio selection or stock acquisition with the purpose of delegating to future generations. Proposition 10 shows that when no day of reckoning, the optimal path of beliefs is increasing or perfectly optimistic depending on the relationship between the anticipatory utility weights.

[19] In my framework, this replicates the work by Gollier and Muerman (2006) who propose agents have reference-independent anticipatory utility and last period disappointment but the agent does derive utility from changes in her anticipatory emotions.

Proposition 10 *(No date of Reckoning)*
Suppose $\gamma^{RD} = 0$. Under Assumptions 1-2 if
(1) $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}$ $\forall t > 1$ and $\lambda(u) \geqslant \max\left\{\frac{\gamma_1^{RD}-\gamma_2^{RI}}{\gamma_2^{RD}}, \frac{\gamma_{T-1}^{RD}-\gamma_T^{RI}}{\gamma_T^{RD}}\right\}$ then $\widetilde{\theta}^*_t \leqslant \widetilde{\theta}^*_{t+1}$ $\forall t$.
(2) $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < \gamma^{RI}_{T-t+2}$ $\forall t > 1$ then the optimal pattern of beliefs is $\widetilde{\theta}^*_t = 1$ $\forall t$

The intuition behind Proposition 10 is straight. If last period expected utility is not reference-dependent, there is no disappointment and thus no bound on optimism. Whether the equilibrium is perfectly optimistic or increasing, depends on the relative importance of the two types of anticipatory utility. The path will be increasing if it is convenient for the agent to delay perfect optimism to take advantage of reference-dependent anticipatory utility, i.e., if the utility the agent derives from positive surprises in beliefs compensate the forgone gains in reference-independent anticipatory utility. As expected, this will happen whenever $\{\gamma^{RI}\}$ has small components compared to the speed art which $\{\gamma^{RD}\}$ grows. Conversely, the path will be perfectly optimistic whenever is not convenient for the agent to forgone reference-independent anticipatory utility. Not surprisingly, this will happen whenever $\{\gamma^{RI}\}$ are big relative to the speed at which $\{\gamma^{RD}\}$ grows. Finally, notice that whenever the optimal path is increasing, an extra condition must be imposed to ensure the path of beliefs will not display wiggles. To see why a certain degree of loss aversion is needed, notice that in this model decreasing today's belief and increase it in the next period actually does convey a gain from reference-independent anticipatory utility since $\{\gamma^{RI}\}$ is increasing. Therefore, for this strategy not to be optimal, the reference-dependent disutility of such strategy has to be big enough so to outweigh the reference-independent gain. Formally, the condition $\lambda(u) \geqslant \max\left\{\frac{\gamma_1^{RD}-\gamma_2^{RI}}{\gamma_2^{RD}}, \frac{\gamma_{T-1}^{RD}-\gamma_T^{RI}}{\gamma_T^{RD}}\right\}$ sets a lower bound into the agent's loss aversion parameter $\lambda(\cdot)$ (see Assumption *5.4*) such that it makes it a bad deal for the agent to decrease her belief today to increase it tomorrow; if the agent is sufficiently loss averse, then the loss from the decrease will outweigh the gains from the increase.

Nygren, Isen, Tyalor, and Dulin (1996) present evidence of agents being significantly more optimistic whenever there is no reckoning. In a series of laboratory experiments individuals were required to state probability estimates for 12 different three-outcome lotteries. In the control group the agents stated their estimates of the probability distribution of different lotteries being informed that no subsequent betting would take place. In the treatment group they made similar estimates for similar lotteries, knowing that they had to make a subsequent bet. Across all the experiments the estimated probabilities of success in the post-betting treatment where considerably higher compared to the non-betting one.

One final interesting question arises from considering the case when the agent reaches perfect pessimism before T+1. Proposition 17—in Appendix B—shows that when the agent is sufficiently loss averse, then under the assumption of Proposition 9 once perfect pessimism is reached the agent remains perfectly pessimistic in all subsequent periods. The intuition behind this result is as follows. Suppose the agent deviates from perfect pessimism in t_1 having reached it in t_0, setting therefore $\widetilde{\theta}^d_{t_1} > \widetilde{\theta}^*_{t_1} = 0$. Under Proposition 9 assumptions, future selves optimal response is to gradually decrease beliefs once again. The extra utility from the deviation has a positive component because of the increase in reference-independent anticipatory utility $\gamma^{RI} u \sum_{s=t_1} \widetilde{\theta}^*_s(\widetilde{\theta}^d_{t_1}) > 0$ and because of the positive surprise she gets in period $t-1$ from reference-independent anticipatory utility, $\gamma^{RD}_{t-T-1+1}\mu(\widetilde{\theta}^d u) > 0$. However, the deviation utility also has a negative term because of all the losses the agent must bear due to the decreasing path of beliefs, $\sum_{s=t_1} \gamma^{RD}_{T-s+1}\mu((\widetilde{\theta}^*(\widetilde{\theta}^d_{t-1})_s - \widetilde{\theta}^*(\widetilde{\theta}^d_{t-1})_{s-1})u)$. Proposition 17 shows that whenever the agent is sufficiently loss averse, then the losses due to the subsequent decrease outweigh the gains. As in Proposition 10, a minimum degree of loss aversion is needed because when reference-independent utility is stronger than the increase in reference-dependent utility, then those periods of positive optimism are a big source of positive utility that is not necessarily outweighed by the losses in reference-dependent anticipatory utility when loss aversion is not prominent.

2.4.2 Optimal Beliefs Sensitivity to Objective Probabilities

I now explore how subjective beliefs respond to changes in the objective probability. Proposition 11 establishes optimism increases with the true probability θ.

Proposition 11 *(Subjective Beliefs Sensitivity to Objective Probabilities)*
Assume Proposition 9 assumptions hold. Let θ^0, θ^1 be two objective probabilities and define $\phi \equiv \frac{\gamma^{RD}[\mu_+ - |\mu_-|]}{\gamma^{RD}_{T-t+1} u}$.
(1) If $\theta^0 < \theta^1$, then $\widetilde{\theta}^*_t(\theta^0) < \widetilde{\theta}^*_t(\theta^1)$ $\forall t$.
(2) Let $\theta_1 = \theta_0 + \triangle$ where $\triangle > 0$. If $\frac{1}{u}\frac{\partial \mu'^{-1}(A_t(\theta_0))}{\partial A_t}\phi = 1$ then $\widetilde{\theta}^*(\theta 1) - \widetilde{\theta}^*(\theta_0) = \triangle$. The agent overreact if $\frac{1}{u}\frac{\partial \mu'^{-1}(A_t(\theta_0))}{\partial A_t}\phi > 1$ and underreacts if $\frac{1}{u}\frac{\partial \mu'^{-1}(A_t(\theta_0))}{\partial A_t}\phi < 1$ where A_t is defined in Lemma 4.

Proposition 11 part (1) states there is a reasonable correspondence between true probabilities and optimal beliefs. Despite beliefs distortion, the agent considers true probabilities in a reasonable manner: higher probability of success imply higher optimism. The mechanism behind this result lies in the relationship between the objective probability and the marginal cost of optimism $B(\theta)$. The less likely the high outcome is, the higher the marginal cost and hence the lower optimal beliefs are. It is important to notice that loss aversion is at the heart of this claim since $\frac{\partial B(\theta)}{\partial \theta} = \gamma^{RD}[\mu_+ - |\mu_-|]$ is only negative whenever the agent is loss averse. As a consequence, the marginal cost of optimism—and therefore, optimal beliefs—only respond to changes in the true probability whenever losses hurt more than same sized gains.

This result can be related to the empirical work linking task difficulty with optimism. Kruger (1999) and Kruger and Burrus (2004) show that people believe that they are below average in unicycle riding, computer programming, and their chances of living past 100. More recently Moore and Cain (2007) present evidence that people believe they are below average on skill-based tasks that are difficult. Noticing that harder tasks can be interpreted as events with lower probability of success, the model predicts agents are less optimistic the harder the task is.

Part (2) establishes conditions over the model parameters for the agent to over/subestimate the change in the true probability. The more loss averse the agent is or the higher the importance of the reference-dependent component of last period expected utility, the more likely is that beliefs will overreact to the change in true probabilities. The reason is that the more loss averse the agent is or the greater γ^{RD} is, the stronger is the reaction of the marginal cost of optimism for a same sized change in θ.

2.4.3 Optimal Beliefs Sensitivity to Time Horizon

I now explore how optimism depends on the waiting period length. Proposition 12 states the agent will be more optimistic if waits one less period. Furthermore, for shorter horizons the optimal path of beliefs is less steep than for longer waiting horizons.

Proposition 12 *(Time Horizon impact over Optimism)*
Assume an interior solution. Under Assumptions 1-2, if
(1) $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}\ \forall t > 1$ *and* $\frac{B(\theta)}{u} \geqslant \gamma^{RD}_1 + \gamma^{RI}_1$ *or if*
(2) $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < \gamma^{RI}_{T-t+2}\ \forall t > 1$ *and* $\frac{B(\theta)}{u} \geqslant \gamma^{RD}_T \frac{\gamma^{RI}_T}{\gamma^{RD}_{T-1} - \gamma^{RD}_T} + \sum_{s=1}^T \gamma^{RI}_{T-s+1}$
then $\widetilde{\theta}^*_t(T) < \widetilde{\theta}^*_t(T-1)\ \forall t \leqslant T-1$ *and* $\widetilde{\theta}^*_{t-1}(T-1) - \widetilde{\theta}^*_{t-1}(T) < \widetilde{\theta}^*_t(T-1) - \widetilde{\theta}^*_t(T)\ \forall t \leqslant T-1$

Proposition 12 gives sufficient conditions over the marginal cost of optimism for the temporal decrease in optimism to be smaller for shorter horizons of uncertainty resolution. In short, Proposition 12 states that for shorter horizons, the agent will be more optimistic. Moreover, the difference in optimism for lotteries with different horizon in given period is increasing in the common periods.

To understand this proposition, note two things. First, from Lemma 4, in an interior solution the difference between previous and current belief is a weighted function of the difference between the marginal cost of optimism and reference-independent marginal utility, where the weight corresponds to reference-dependent marginal utility. The bigger this quantity is—the bigger $\frac{B(\theta) - \sum_t^T \gamma^{RI}_{T-t+1} u}{\gamma^{RD}_{T-t+1} u}$ is—the stronger the decrease in beliefs when the marginal cost is sufficiently big. Second, the agent who waits during T periods has a $\{\gamma^{RD}\}$ string equal to $\{\gamma^{RD}_T, \gamma^{RD}_{T-1}, \ldots, \gamma^{RD}_1\}$ whereas the agent who waits T-1 periods has a one period shorter string $\{\gamma^{RD}_{T-1}, \gamma^{RD}_{T-2}, \ldots, \gamma^{RD}_1\}$ equal to the last T-1 components of the first sequence and the same for the $\{\gamma^{RI}\}$ sequence. Therefore, an agent who waits one less period has a smaller reference-independent marginal benefit, but has a greater reference-dependent marginal benefit in the same period. Consequently, both the numerator and the denominator in $\frac{B(\theta) - \sum_t^T \gamma^{RI}_{T-t+1} u}{\gamma^{RD}_{T-t+1} u}$ are bigger when the agent faces a one period shorter horizon of uncertainty. Therefore, whether this quantity decreases or increases depends on the relative strength of these two increases. Proposition 12 assumptions give us the conditions for the numerator to increase less than the denominator when T decreases. When $\{\gamma^{RD}\}$ grows fast relative to the size of $\{\gamma^{RI}\}$, the decrease in reference-independent marginal utility is relatively small whereas the increase in reference-dependent marginal utility is greater. The condition over the marginal cost of optimism is therefore only necessary for the optimal path of beliefs to decrease in time. Notice this condition is equal to that in Proposition 9. When $\{\gamma^{RD}\}$ grows slow relative to the size of $\{\gamma^{RI}\}$, then only for high enough marginal cost of optimism $\frac{B(\theta) - \sum_t^T \gamma^{RI}_{T-t+1} u}{\gamma^{RD}_{T-t+1} u}$ will be smaller for shorter horizons. Notice further the needed condition over the cost is more demanding than that in Proposition 9, $\gamma^{RD}_T \frac{\gamma^{RI}_T}{\gamma^{RD}_{T-1} - \gamma^{RD}_T} + \sum_{s=1}^T \gamma^{RI}_{T-s+1} > \gamma^{RD}_T + \sum_{s=1}^T \gamma^{RI}_{T-s+1}$. Finally, the results in Proposition 12 give as a different interpretation for optimal beliefs. In his model, when the marginal cost of optimism is big enough, chosen beliefs can be thought as a proxi of the agent's level of positive arousal or excitement. On other words, shorter horizons of uncertainty resolution keep agents excited or engaged in the lottery they hold.

2.4.4 Optimal Beliefs Sensitivity to Reward Size

The next proposition explores the relationship between optimal beliefs and the reward size u. It shows that when the high outcome probability is very high, the agent holds more optimistic beliefs for higher payments. To the contrary, higher payments relate to lower optimism when the probability of getting the high outcome is low.

Proposition 13 *(Sensitivity to Reward Size)*
Assume Proposition 9 assumptions hold. Let $u_0 < u_1$*. If*

(1) $\theta \to 1$*, then* $\widetilde{\theta}^*_t(u_0) < \widetilde{\theta}^*_t(u_1)$ $\forall t$.
(2) $\theta \to 0$*, then* $\widetilde{\theta}^*_t(u_0) > \widetilde{\theta}^*_t(u_1)$ $\forall t$.

To understand the intuition behind Proposition 13, notice that two opposite forces are created when the reward size is slightly increased. First, the marginal cost of optimism increases—$\frac{\partial B(\theta)}{\partial u} > 0$—pushing optimism down. Second, the total marginal benefit of optimism also increases since it increases both the reference-dependent and reference-independent marginal benefits, pushing towards higher optimism (see equation (2.4).) How strong or weak these two effects are depend on the value of the true probability. If θ is very big the force from the marginal cost of optimism pushing towards pessimism is weak—as it is unlikely that the agent will actually experience disappointment—and the total marginal benefit force pushing towards optimism outweighs it and hence the agent will hold more optimistic beliefs for higher payments. To the contrary, whenever the true probability of a high outcome is very low the marginal cost of optimism is as big as possible outweighing the positive benefit through a higher total marginal utility. As a consequence, optimal beliefs will be lower for higher payments.

Whenever the true probability of success lies away from the boundaries, the response of optimal beliefs to changes in the reward size depend on the degree of loss aversion and on the absolute size of the reward size. For example, for $\theta \in (0, \frac{1}{2})$, even moderate degrees of loss aversion and moderate payments sizes could lead the agent to hold more pessimistic beliefs for lower payments. For $\theta \in (\frac{1}{2}, 1)$ very high degrees of loss aversion or very big rewards sizes would be needed to obtain the same result.

Two final comments. First, notice that a high reward can be indistinctly caused by a high size of the reward or high valuation for it. Proposition 13 predicts that, ceteris paribus, agents with a higher consumption valuation for the same reward should be more optimistic about their likelihood of success. This takes us to the second comment. Just as Proposition 12, Proposition 13 gives us an alternative interpretation of beliefs in this model: optimal beliefs are related to the engagement or excitement the agent feels about the lottery she is holding. The intuition is that when the agent holds a lottery which she most likely will win, increasing the size of the reward creates a positive arousal that in this model is translated into greater optimism. When the probability of getting the high outcome is low, increasing the reward fires back and creates a bigger sensation of loss, triggering negative arousal which in this model is translated into lower optimism.

2.5 Productivity Bonus Design

This section applies the model to the problem of a risk neutral principal who wants to design a productivity bonus schema to induce a desired string effort at the cheapest cost. The main concern is the optimal timing in the bonus schema: what is the optimal frequency of productivity bonuses? which are the key considerations to set this periodicity?

2.5.1 The Set Up

There are T periods. In each period the agent takes an effort decision not observable by the principal. Effort can be high or low $e_t \in \{e_H, e_L\}$ with cost zero or $c > 0$ respectively. Once the agent has exerted effort, a productivity signal is realized taking a high or low value $s_t \in \{s_H, s_L\}$ where $\mathbb{P}(s_t = s_H|e_H) > \mathbb{P}(s_t = s_H|e_L) > 0$.[20] I further assume that the productivity signal is only observed by the principal.

[20]For example, the signal can be binary $s_t \in \{0, 1\}$, therefore, $s_t \sim$Bernoulli$(\pi(e_t))$ where $\pi(e_t)$ is the probability of the high signal whenever the agent has exerted effort e_t and $\pi(e_H) > \pi(e_L)$.

A bonus scheme B_n is a pair (T_n, b_n) where T_n corresponds to the number of working periods considered in the bonus and $b_n \in \mathbb{R}_{++}$ is the monetary reward the agent receives at the end of period T_n if the sum of the productivity signals over the time interval T_n, $S(T_n) = \sum_{t=1}^{T_n} 1(s_t(e_t) = s_H) \in \{0, \ldots, T_n\}$—where $1(\cdot)$ is the usual indicator function—reaches a threshold $\overline{S_n}$, which for simplicity is considered exogenous.[21] As a consequence, the principal grants $N \in \{1, \ldots T\}$ productivity bonuses B_n, $n = 1, \ldots, N$ such that $T_1 + \cdots + T_N = T$. For every bonus B_n, periods $t \in \{1, \ldots, T_n\}$ $\forall n$ are "waiting" periods: no other payments of any kind are made to the agent. Finally, the structure of the bonus scheme is fully known by the agent since the first period.[22]

2.5.2 The Principal's Problem

I assume the principal wants to induce high effort in all T periods. The principal's problem corresponds to find the optimal number of productivity bonuses N^*, their time spans $\{T_n\}_{n=1}^{N^*}$ and associated payments $\{b_n^*\}_{n=1}^{N^*}$ inducing the agent to exert high effort at minimum expected cost. To set up the problem, consider first the principal's objective function. Let θ_n represent the probability of giving bonus $B_n$when the agent exerts high effort in every period, $\theta_n(e_H, \ldots, e_H) \equiv \mathbb{P}(S(e_H, \ldots, e_H; T_n) \geqslant \overline{S}_n)$ where $(e_H, \ldots, e_H)$ represents a string of T_n high efforts.[23]

Consider the set up of the incentive compatibility constraints (IC). For each productivity bonus, the T_n IC restrictions make sure the optimal payments are such that deviating from high to low effort is not profitable for the agent in any period. From the agent's point of view, each bonus B_n is equivalent to hold a lottery paying b_n with probability θ_n in an future realization period T_n.[24] If the agent gets the bonus, she obtains consumption utility $u(b_n) \equiv u_n$ and $u(0) = 0$ otherwise. According to this model in Section 2.3.2, in each period the agent not

[21]See discussion on the intuitive determinants of this target at the end of section 5.3

[22]Two caveats apply to this set up. Firs, the assumption of the agent observing effort but not the productivity signal (and vice-versa) is important. To see this, suppose the agent does observe s_t. For a given target $\bar{S}$, if the agent gets to observe $S = \bar{S}$ in period $t_0 <$ T+1, then the bonus will not be effective since the agent will not work in subsequent periods $t > t_0$. Therefore the target that would induce high effort in every period would be $\bar{S} = T + 1$, a non-realistic target if effort does not perfectly translates into production. Is this assumption realistic? There are many productivity bonuses for which this is a reasonable assumption. In many, real world productivity bonuses, the relevant targets are subjective and the bar is set by the principal's overall evaluation of the agent. This is the case for productivity bonuses given to executives and more qualified workers for whom an objective signal is not readily available. There are many other productivity bonuses for which this assumption is incorrect. Those given to sales or field force are a clear example. In this case, measures of productivity are readily available to both the agent and the principal. In this case, though, productivity bonuses are usually set in the context of a "progressive plan",where other extra bonuses are given when the agent reaches the target early. Therefore, I think of this assumption of non-observable productivity signal as the replacement to directly model the productivity bonus progressive plans. A second concern is the principal's risk neutrality assumption. In some real world examples, the principal is just concerned with the target productivity measure, independently of how much effort does it cost to the agent. This sounds more reasonable in the context of a risk averse principal who rather get a certain productivity for sure.

[23]For example, if s_t is a Bernoulli random variable with probability $\pi(e_t)$, then the total productivity signal has a distribution $S \sim$ Binomial(π). Therefore, the probability of observing a realization s of the random variable S, for any given time span T_n, corresponds to: $\mathbb{P}(S = s/e_H, \ldots, e_H) = \left(\begin{array}{c} T_n \\ s \end{array} \right) \pi(e_H)^s (1 - \pi(e_H))^{(1-s)}$ therefore, if the agent works hard in every period, the probability of getting that bonus B_n corresponds to:

$$\begin{aligned} \theta_n = \theta(e_H, \ldots, e_H) &\equiv P(\overline{S} \leqslant S \leqslant T_n/e_H, \ldots, e_H) \\ &= \sum_{j=\overline{S}}^{T} \left(\begin{array}{c} T_n \\ j \end{array} \right) \pi(e_H)^j (1 - \pi(e_H))^{(T_n - j)} \end{aligned}$$

[24]Notice the slight change with respect to previous section where the realization period was T+1.

only chooses optimal effort, but also beliefs of success about the relevant probability bonus. For any given path of effort, the agent computes her beliefs by maximizing the sum of all her present and future utility flows which consider not only reference-dependent expected utility but also reference-independent and reference-dependent feelings of anticipation. The key in this agency framework is the agent can influence the true probability of getting the reward through her effort choice. Using the fact the optimal productivity bonus design induces the agent to work hard in every period, the agent takes her effort decision assuming all future selves will exert high effort.[25] Therefore the utility the agent gets in period t when paid with productivity bonus $B_n = (b_n, T_n)$ and exerts high effort corresponds to:

$$\begin{aligned} U^*_{t,n}(e_H, T_n) &= \sum_{s=t}^{T_n} \gamma^{RI}_{T_n-t+1} \widetilde{\theta}^*_s u + \theta_n(e_H, e_{-H})u + \sum_{s=t}^{T_n} \gamma^{RD}_{T_n-s+1} \mu((\widetilde{\theta}^*_s - \widetilde{\theta}^*_{s-1})u) \\ &+ \gamma^{RD}[\theta(e_H)_n(1-\widetilde{\theta}^*_{T_n})\mu_+ + (1-\theta_n(e_H)\widetilde{\theta}^*_{T_n}\mu_-] \end{aligned}$$

where $\theta(e_H) \equiv \theta(e_H, e_{-H})$ represents the probability of getting the bonus when all selves work hard and $\{\widetilde{\theta}^*_t(e_H, e_{-H})\}_{t=1}^T$ constitutes a subgame perfect equilibrium as shown in Lemma 3 in Section 2.3.2. In case self t deviates and goofs-off, the agent gets a utility of

$$\begin{aligned} U^*_{t,n}(e_L, T_n) &= \sum_{s=t}^{T_n} \gamma^{RI}_{T_n-t+1} \widetilde{\theta}^*_s u + \theta(e_L, e_{-H})u + \sum_{s=t}^{T_n} \gamma^{RD}_{T_n-s+1} \mu((\widetilde{\theta}^*_s - \widetilde{\theta}^*_{s-1})u) \\ &+ \gamma^{RD}[\theta(e_L)(1-\widetilde{\theta}^*_{T_n})\mu_+ + (1-\theta(e_L)\widetilde{\theta}^*_{T_n}\mu_-] \end{aligned}$$

were $\theta(e_L) \equiv \theta(e_L, e_{-H}) < \theta(e_H)$ represents the probability of getting the bonus when the agent goofs-off and all other selves work hard and $\{\widetilde{\theta}^*_t(e_L, e_{-H})\}_{t=1}^T$ constitutes a subgame perfect equilibrium as shown in Lemma 3.[26]

Finally, the principal's problem can be written as follows:

$$\begin{aligned} &\min_{N,\{u_{T_n}\}} \sum_{i=1}^{N} \theta_n u^{-1}(u_{T_n}) \\ s.t. \quad &U^*_{t,n}(e_H) - U^*_{t,n}(e_L) \geqslant c \qquad 1 \leqslant t \leqslant T_n, \quad 1 \leqslant n \leqslant N \end{aligned} \tag{IC}$$

where $u_{T_n} = u(b_n)$ represents the utility equivalent of the bonus payment covering a T_n periods. Notice that since there are N productivity bonuses, each covering a span of T_n periods, the T IC constraints can be grouped into N groups of T_n ICs. Finally, Lemma 5 explores some key features of the principal's problem which will be relevant to determine the optimal frequency of productivity bonuses.

[25] Focusing on the agent's problem given the optimal payment implies that even if the agent has feelings of anticipation with respect to future effort, these will be canceled in the IC constraints.

[26] In case the productivity signal is binary, then when choosing low effort today, the probability of getting the bonus is calculated as:

$$\begin{aligned} \theta(e_L, e_{-H}) &\equiv P(\overline{S} \leqslant S \leqslant T / e_H, \ldots, e_H, e_L, e_H, \ldots, e_H) \\ &= \sum_{j=\overline{S}}^{T} \binom{T}{j} \pi(e_H)^{j-s_t}(1-\pi(e_H))^{(T-1)-(j-s_t))} \pi(e_L)^{s_t}(1-\pi(e_L))^{(1-s_t)} \\ &< \theta(e_H, e_{-H}) \end{aligned}$$

Lemma 5 *(Constant and Binding IC)*
Assume an interior solution for the agent's problem of finding optimal beliefs. If $\theta(e_H) \longrightarrow 1$*, then*
(1) For $1 \leqslant t \leqslant T_n \quad \forall n$

$$U^*_{t,n}(e_H, u^*_{T_n}) - U^*_{t,n}(e_L, u^*_{T_n}) = [\theta(e_H) - \theta(e_L)] \left[u^*_{T_n} + \gamma^{RD}(1 - \widetilde{\theta}^*_{T_n}(u^*_{T_n}))\mu(u^*) + \widetilde{\theta}^*_{T_n}(u^*_{T_n})|\mu(-u^*_{T_n})| \right]$$

(2)For $\forall u, n, t$

$$\frac{\partial [U^*_{t,n}(e_H) - U^*_{t,n}(e_L)]}{\partial u} > 0$$

(3)For $1 \leqslant t \leqslant T_n \quad \forall n$

$$U^*_{t,n}(e_H, u^*_{T_n}) - U^*_{t,n}(e_L, u^*_{T_n}) \approx \frac{\partial U^*_t(h+1)}{\partial h}|_h \quad = \quad \frac{\partial \theta(h)}{\partial h} \left[u + \gamma^{RD}\mu_+ + \gamma^{RD}\widetilde{\theta}^*_T(h)[|\mu_-| - \mu_+] \right] = c$$

Lemma 5.*1* tell us two important things. First, the utility difference between working hard and goofing off corresponds to the change in the last (relevant for bonus n) period reference-dependent expected utility due to changes in the true probability. Second, it says this quantity is constant across periods for a given productivity bonus time span. These two are a straight consequence of the envelope theorem. To see this, note the impact of θ over U^* has two sources. A direct impact through last period reference-dependent expected utility and an indirect effect through optimal beliefs. Since beliefs are optimally computed, it is only the common direct effect what drives changes in utility.[27] Finally, Lemma 5.*1* holds independently of the assumption $\theta(e_h) \longrightarrow 1$. Lemma 5.*2* says the utility difference from working hard and goofing-off is increasing in the optimal payment. To see this, recall last period expected utility has two components, both strictly increasing in the true probability: the reference-independent one–θu– and the reference-dependent one–$\theta(1 - \widetilde{\theta}^*_T)\mu_+ + (1-\theta)\widetilde{\theta}^*_T|\mu_-|$–for which an increase in the true probability increases the probability of the gain scenario and decreases the probability of a loss. Finally, Lemma 5.*3* states the IC is binding in the optimal payment for any given productivity bonus B_n. If the true probability of getting the bonus is high when the agent chooses to work hard, then the utility difference between working hard and goofing off is strictly increasing in payment. Therefore, if the IC is not binding the principal can decrease the payment and still get an IC contract that is cheaper than the first.

Because of Lemma 5, the final principal's problem corresponds to:

$$\min_{N,\{u_n\}} \sum_{n=1}^{N} \theta_n u^{-1}(u_n) \tag{2.6}$$

$$s.t. \qquad U^*_{T_n}(e_H) - U^*_{T_n}(e_L) = c \qquad 1 \leqslant n \leqslant N \tag{IC}$$

where there are only N IC constraints.

2.5.3 Productivity Bonuses Optimal Timing

There are two associated effects governing the relationship between the length of the bonus time span and the optimal payment. Proposition 12 in Section 2.4.3 showed that shorter horizons

[27] If the true probability is small, i.e., if $\theta(e_H) \longrightarrow 0$, an extra condition for the utility difference to be increasing in payment in the whole domain is needed $-\frac{\partial \widetilde{\theta}^*_T}{\partial u} > -\frac{1+\gamma^{RD}\mu'(u)}{\gamma^{RD}[|\mu_-| - \mu_+]}-$. Such condition imposes an upper bound to the sensitivity of beliefs to payments, requiring beliefs not to decrease too much for a given payment increase.

increase optimism for any given payment. I call this the *horizon effect*: shorter waiting times keep the agent engaged in the bonus, implicating a smaller payment is needed to induce high effort in the periods left. Two factors, therefore, decrease the optimal payment in a shorter bonus. First, the bonus covers less periods of effort and second, higher optimism due to the horizon effect. This total decrease in payment, however, triggers a second effect over optimism. Proposition 13 in Section 2.4.4 showed that when the true probability of getting the bonus was high, then a decrease in payment reduced optimism. I call this the *reward effect*: lower payments decrease the agent's optimism, decreasing, therefore, the agent's motivation to work hard. Whether it is convenient for the principal to set more frequent but smaller bonuses depends on the relative strength of the horizon and the reward effects. When the horizon effect is strong relative to the reward effect, decreasing the horizon span of the productivity bonus makes high effort in the common periods cheaper since, despite the decrease in payment, the agent gets more optimistic to the realization period compared to the longer bonus. Contrarily, if the reward effect is high enough, it can outweigh the horizon effect and therefore the principal has to pay the same or more for the common periods of effort due to the agent's lack of motivation due to a lower payment.

To understand these relevant forces defining the optimal frequency of productivity bonuses, consider the case when the principal gives just one productivity bonus at the end of period T. In this case, the principal's problem in (2.6) has just one binding IC constraint, which is strictly increasing in the optimal payment because of Lemma 5. Slightly rearranging the IC, the unique payment u_T^* solves:

$$U_T^*(e_H, u_T^*) - U_T^*(e_L, u_T^*) = [\theta(e_H) - \theta(e_L)]\left[u_T^* + \gamma^{RD}\mu(u_T^*) + \gamma^{RD}\widetilde{\theta}_T^*(u_T^*, T)[|\mu(-u_T^*)| - \mu(u_T^*)]\right] = c \quad (2.7)$$

where the notation u_T^* represents the productivity bonus optimal payment for a bonus of length T and $\widetilde{\theta}_T^*(u_T^*, T)$ corresponds to period T beliefs (last period) when payment is u_T^* and there are T waiting periods before the relevant payment is made.

Several parameter values affect the optimal payment level. A straight application of the implicit function theorem show us u_T^* is increasing in the importance of last period reference-dependent component of expected utility, on loss aversion and on high effort marginal cost[28]. For my purposes, however, the most important thing to notice is that the level of optimism in the realization period effects the optimal payment level. This implies the bonus timing is relevant to determine the optimal payment since in this model optimism is related to the bonus time horizon.

Now consider the principal's decision of giving the productivity bonus in period $T-1$ instead of period T. In such case, for the same payment u_T^* solving for (2.7), it can be seen thaht:

$$[\theta(e_H) - \theta(e_L)]\left[u_T^* + \gamma^{RD}\mu(u_T^*) + \gamma^{RD}\widetilde{\theta}_T^*(u_T^*, T-1)[|\mu(-u_T^*)| - \mu(u_T^*)]\right] > c$$

since

$$\widetilde{\theta}_T^*(u_T^*, T) < \widetilde{\theta}_T^*(u_T^*, T-1)$$

were $\widetilde{\theta}_T^*(\cdot, s)$ represents last period optimism when there are s waiting periods and the inequality

[28]The intuition is straight. If the marginal cost of working hard is high, the principal must compensate the agent with a higher payment. If γ^{RD} is big or if the agent is more loss averse, then for the same payment the agent gets a lower last period expected utility and therefore the principal must give a bigger payment to convince the agent to exert high effort. Contrarily, the more responsive is the true probability θ to one more day of hard work, the lower the optimal payment must be to convince the agent of working hard. The intuition is that a big increase in the true probability is a good substitute of payment as a motivator to exert high effort.

follows under Proposition 12 assumptions. The difference $\widetilde{\theta}_T^*(u_T^*, T-1) - \widetilde{\theta}_T^*(u_T^*, T)$ corresponds to the horizon effect. Since equation (2.7) is increasing in the reward u because of Lemma 5, decreasing one period allows the principal to set a lower payment and still get the agent to work in all T-1 periods. That is, there exists a payment u_{T-1}^* such that:

$$[\theta(e_H) - \theta(e_L)] \left[u_{T-1}^* + \gamma^{RD}\mu(u_{T-1}^*) + \gamma^{RD}\widetilde{\theta}_T^*(u_{T-1}^*, T-1)[|\mu(-u_{T-1}^*)| - \mu(u_{T-1}^*)] \right] = c$$

where

$$u_{T-1}^* < u_T^*$$

because of Lemma 5.*3*. Intuitively, this decrease in payment has two sources: the horizon effect and the fact the bonus covers one less period of high effort. The reward effect is reflected in the difference $\widetilde{\theta}_T^*(u_{T-1}^*, T-1) - \widetilde{\theta}_T^*(u_T^*, T-1)$.

The decision between a bonus plan consisting of two productivity bonuses B_1 and B_2; B_1 paying u_{T-1}^* for the first T-1 periods and B_2 paying u_1^* in the last period and a second schema paying just one bonus of u_T^* in period T, depends on the comparison between u_T^* and $u_{T-1}^* + u_1^*$. If the horizon effect outweighs the reward effect –$\widetilde{\theta}_T^*(u_T^*, T-1) - \widetilde{\theta}_T^*(u_T^*, T) > \widetilde{\theta}_T^*(u_{T-1}^*, T-1) - \widetilde{\theta}_T^*(u_T^*, T-1)$–, then $u_{T-1}^* + u_1^* < u_T^*$ because $u_T^* - u_{T-1}^*$ is big, and the principal will choose the bonus schema with two rather than one productivity bonus. Following the latter logic, a bonus schema granting a bonus in each period will be optimal from the principal's perspective if $T_n u_1^* < u_{T_n}^*$ $\forall n$. The following proposition is the main section result. It states sufficient conditions for the principal to be optimal to grant frequent productivity bonuses.

Proposition 14 *(Optimal Timing of Productivity Bonuses)*
Assume $\theta(e_H) \longrightarrow 1$ and that conditions in Proposition 12 hold. If the gain-loss utility function is $\mu(Nx) = N^k\mu(x)$ such that $k \longrightarrow 1$ and

$$\theta(e_H) - \theta(e_L) > \frac{c}{\alpha}$$

where $\alpha = \gamma^{RD}[|\mu(-u_1^*)| - \mu(u_1^*)]\mu'^{-1}\left(\frac{B(u_1^*) - \gamma_1^{RI}u_1^*}{\gamma_1^{RD}u_1^*}\right)$ then the solution to the principal's problem in (2.6) *is $N^* = T$ and $b_n = u_1^*$ $\forall n$ where u_1^* solves $U_1^*(e_H, u_1^*) - U^*(e_L, u_1^*) = c$.*

Proposition 14 states that whenever the agent can affect the probability of getting the bonus through her effort choice and the gain loss utility function is homogenous of degree one, the principal will award productivity bonuses in every period. Proposition 14 is supported by the marketing literature on sales incentives. Zoltners, Sinha, and Lorimer (2006) recommend that "as a general rule, an incentive plan should pay incentives as frequently as is possible without compromising customer focus, generating excessive administrative overhead or making the award size trivial".[29]

Consider now the reasonability of Proposition 14 assumptions. Assuming Proposition 12 conditions hold ensure the marginal cost of optimism is high enough such that optimal beliefs are decreasing in time and on the horizon length. This is an assumption has been made in most of the paper and only requires the agent being sufficiently loss averse or having a high valuation

[29] In fact, in light of Zoltners, Sinha, and Lorimer (2006) recommendation, a reasonable interpretation of Proposition 14 result corresponds to the principal giving productivity as often as *feasible* (rather than the textual every period); in fact, one can think of unit period as function of the job nature. In sales forces, for example, a productivity measure is readily available –weekly and even daily–, contrarily to more other jobs where productivity signals are fuzzy and therefore it takes longer time to the principal to collect them.

for the reward. The assumption over the gain-loss utility function being homogenous of degree one is sufficient but by no means necessary for Proposition 14 to hold. It ensures the horizon effect dominates the reward effect in every time span so the agent gets more optimistic to the realization period for short horizons. For many other parameter values, however, this conclusion can hold for gain-loss utility functions of much lower degree of homogeneity.[30]

The assumption $\theta(e_H) \longrightarrow 1$ is an important one. It states that whenever the agent exerts high effort in every period, it must be almost certain she will get the productivity bonus. Note this is related to how the principal sets the target defining whether the agent gets the bonus. Given the conditional probability distribution of the signal given effort, this condition implies the principal must set reasonable and easy to understand productivity targets so the agent is aware of her capacity of affecting the bonus likelihood through her effort.

The following Corollary of Proposition 14 states that if the agent works hard in every period but it is still unlikely she gets the bonus, then there is no payment that may convince the agent to work hard. In such case the principal will never want to reward the agent through productivity bonuses.

Corollary 1 *(Effort does not Affect the Probability of Getting the Bonus)*
If $\theta(e_H) \longrightarrow 0$ *then* $\nexists u^*$ *that solves* (2.7) *for any* T_n.

This result can be seen straight from Lemma 5.*1*. If $\theta(e_H) \longrightarrow 0$, then $\theta(e_H) - \theta(e_L) \approx 0$ and therefore $U^*_{t,n}(e_H, u^*) - U^*_{t,n}(e_L, u^*) \approx 0$. Intuitively, if the agent cannot affect the probability of getting the bonus through her effort choice, then the utility she gets from working hard is always less than the utility she gets from goofing-off, since the cost of effort does not pay back by increasing probability of getting the bonus.

Finally consider the condition over $\theta(e_H) - \theta(e_L)$ being sufficiently big. Intuitively, this requires the agent must be able to affect the likelihood of getting the bonus through her effort choice. In other words, if $\theta(e_L) \longrightarrow 1$, the reward to exert high effort vanishes and the agent will always choose to goof-off. Notice this requirement is related, but it is definitely not the same as the requirement $\theta(e_H) \longrightarrow 1$. The relevant lower bound is directly proportional to high effort cost c: the more costly the unit of hard work, the more such unit must pay in back in terms of increasing the likelihood of getting the bonus. The numerator of the lower bound is the component related to the bonus frequency. The term $\mu'^{-1}\left(\frac{B(u_1^*)-\gamma_1^{RI}u_1^*}{\gamma_1^{RD}u_1^*}\right)$ corresponds to a lower bound of the difference between the horizon effect and the reward effect: the stronger the horizon effect is over the reward effect, the smaller the difference in true probabilities and it will still be convenient for the principal to give frequent rewards. This term is weighted by $\gamma^{RD}[|\mu(-u_1^*)|-\mu(u_1^*)]$, which is a consequence of the temporal component in the IC constraint. The magnitude of the numerator in Proposition 14 is what actually drives how often the principal will want to grant productivity bonuses. When the numerator is being because the horizon effect outweigh the reward effect greatly, the lower bound is small and $\theta(e_H) - \theta(e_L)$ could be small. Equivalently, for any given impact of high effort in the true probability of getting the bonus, the stronger the horizon effect over the reward effect and therefore the more likely is the condition will hold. Notice further that one implication of the condition over $\theta(e_H) - \theta(e_L)$ is that the optimal payment in the one period productivity bonus (u_1^*) cannot be negligible. To see this, note that if $u_1^* \longrightarrow 0$, then $|\mu(-u_1^*)| - \mu(u_1^*) \approx 0$ and thus the upper bound becomes zero. Recalling the citation from Zoltners, Sinha, and Lorimer (2006), this is an intuitive requirement. If the optimal payment is negligible, then the incentives toward high effort vanish.

[30] Kahneman and Tversky (1979) parametrized utility function is homogenous of degree 0.6.

A final word relates to the applicability of the model to different job types. The key to this application is that shorter bonuses keep the agent motivated in the bonus. This implies the principal is using optimism as a replacement for other types of motivators, such as intrinsic ones. This makes the model appropriate for the dynamics of the field force or other jobs where there are no other types of incentives driving the agent's engagement with the job.

2.6 Discussion

This paper studies the temporal behavior of subjective optimal beliefs. Building on the interaction between two types of anticipatory feelings—standard anticipation of expected consumption utility and utility from changes in these anticipatory feelings—the paper shows that optimism decreases as the pay-off date of a risky prospect approaches when the agent is sufficiently loss averse or the outcome is important enough. The paper further proves that (1) even though the agent holds incorrect beliefs, these are still reasonable: optimism is increasing in the true probability of success, (2) optimism is decreasing in the time horizon and increasing in the size of the reward. I conclude by applying the model to the optimal timing of productivity bonuses.

This paper aims to be a contribution to the literature on distorted beliefs by exploring the extent to which self-serving biases can rationalize relevant economic phenomena. My analysis rests on the crucial assumption that agents can freely depart from the true probability distribution to maximize utility. This assumption, however, most probably overstates the degree of control agents have over their beliefs. To reduce the degree of beliefs malleability, the model could have included in the utility function an exogenous cost function mapping from the distance between true probabilities and optimal beliefs as is common in the cognitive dissonance literature. However, unless people's distaste from deviating from true probabilities change across time, including such function would only scale the path of beliefs without modifying their temporal path.

Besides exaggerating the degree of control gents have over their subjective beliefs, it is also legitimate to wonder about the assumption of agents knowing the true probability distribution of the relevant random process. Even though this assumption seems unrealistic in the context of risk, it is not in the context of ambiguity. In this case, the agent does not know an exact probability distribution but does know that it belongs to a known set of probability distributions. Extending the model to ambiguity would be straightforward. In such a case the agent would choose her optimal belief in a restricted set equal to the set of family distributions characterizing ambiguity. In other words, the agent would choose incorrect, but still *feasible* optimal beliefs, which depart from the true probability on reasonable magnitudes.

A final comment relates to the fact the model builds on a rational intuition of utility maximizing. Whether this is the correct approach to study belief formation is an open question and much more research on the area, both empirical and theoretical, is needed. One possible argument in favor of this view is the growing interests among psychologists on the the topic of emotion regulation. This relatively new field of study acknowledges that—conscious or unconsciously— people do have a degree of control over their emotions (Gross (2007).) I view, therefore, this rational approach with its consequent implication over beliefs malleability as a useful first-approach to the topic of beliefs formation.

Chapter 3

Appendix

3.1 Formal Description of Preferences in Chapter 1

3.1.1 Gain-loss Utility in Effort

This section presents gain-loss utility in the effort domain. Making a similar definition of p relative to that for the payments domain but for the effort distribution (see Assumption 2), I have that gain-loss utility in the effort domain corresponds to

$$G(\widetilde{e}_{t,\tau}|\widetilde{e}_{t-1,\tau}) = \int_0^1 \mu\big(-c(e_{t,\tau}(p)) + c(\widetilde{e}_{t-1,\tau}(p))\big)dp$$

for $t=0$ and $\tau=1,2$ and for $t=1$ and $\tau=2$ where I have suppressed the tilde notation because $e_{t,\tau}$ represents the actually implemented action. Consider first contemporaneous gain-loss utility for $t=1$. Then, I have that if the agent implements action e_1 having planned $\widetilde{e}_{0,1}$, contemporaneous gain-loss utility corresponds to

$$G(e_1|\widetilde{e}_{0,1}) = \mu(c(\widetilde{e}_{0,1}) - c(e_1))$$

To see this, recall that $\widetilde{e} \in \mathcal{X}$ and thus for the first period the actual and the reference effort distributions are degenerate. Second-period contemporaneous gain-loss utility is defined analogously. For $(t,\tau)=(1,2)$, I have that if the agent implements effort e_1 having planned $(\widetilde{e}_{0,1}, \widetilde{e}_{0,2})$, prospective gain-loss utility corresponds to

$$G(\widetilde{e}_{t,\tau}|\widetilde{e}_{t-1,\tau}) = \int_0^1 \mu\big(-c(e_{t,\tau}(p)) + c(\widetilde{e}_{t-1,\tau}(p))\big)dp$$

3.1.2 Total Utilities Description

From now on I rewrite gain-loss utility in effort and income in its discrete equivalent. Such conversion is done using the quantile comparison described in Section 1.3. Start by considering total period-two expected utility. Having observed $X_1 = x_n$ and thus having planned $\widetilde{e}_{1,2}$ at the end of period one, the total expected utility of implementing effort e_2 corresponds to

$$EU_2(e_2(x_n)|\widetilde{e}_{1,2}(x_n); x_n) = \sum_m \pi_m^2 u_{nm} + \sum_m \sum_\ell \pi_m^2 \pi_\ell^{1,2} \mu(u_{nm} - u_{n\ell}) - c(e_2) + \mu(c(\widetilde{e}_{1,2}) - c(e_2)) \quad (3.1)$$

where $\pi_m^2 \equiv \pi_m(e_2)$, $\pi_\ell^{1,2} \equiv \pi_\ell(\widetilde{e}_{1,2})$, $u_{nm} \equiv u(s_{nm})$ and s_{nm} are second-period payments given $X_1 = x_n$ and $X_2 = x_m$. The first term corresponds to the period-two expected consumption utility given the agent executes effort e_2. The second term corresponds to contemporaneous gain-loss utility as presented in Assumption 2. The last two terms represent the total cost. The term $c(e_2)$ corresponds to the consumption utility cost of exerting effort e_2, meanwhile the last term is the gain-loss utility of deviating from the planned effort made at the end of period one, $\widetilde{e}_{1,2}$ to e_2.

At the end of period one, having exerted e_1 in period one and having observed $X_1 = x_n$, the total expected utility of planning the credible action $\widetilde{e}_{1,2}$ corresponds to

$$EU_2(\widetilde{e}_{1,2}(x_n)|\widetilde{e}_{1,2}(x_n); x_n) = \sum_m \pi_m^{1,2} u_{nm} + \sum_m \sum_\ell \pi_m^{1,2} \pi_\ell^{1,2} \mu(u_{nm} - u_{n\ell}) - c(e_2) \tag{3.2}$$

The first term corresponds to second-period expected consumption utility of planning to exert $\widetilde{e}_{1,2}$. The second term is second-period contemporaneous gain-loss utility, knowing that $\widetilde{e}_{1,2}$ will be the the next period reference and knowing that he will execute the plan. Finally, notice there is no contemporaneous gain-loss utility in effort since the agent has rational expectations.

At the beginning of period one, before observing X_1 and having planned $(\widetilde{e}_{0,1}, \widetilde{e}_{0,2}(X_1))$ in period zero, the total expected utility of executing e_1 corresponds to

$$\begin{aligned} EU_1(e_1 \widetilde{e}_{0,2}(X_1)|\widetilde{e}_{0,1}, \widetilde{e}_{0,2}(X_1)) = &\sum_n \pi_n^1 u_n + \sum_n \sum_\ell \pi_n^1 \pi_\ell^{0,1} \mu(u_n - u_\ell) + \gamma \sum_n \sum_\ell \sum_m \pi_n^1 \pi_\ell^{0,1} \pi_m^{0,2} \mu(u_{n\ell} - u_{\ell m}) \\ &+ \sum_n \pi_n^1 EU_2(\widetilde{e}_{0,2}(x_n)|\widetilde{e}_{0,2}(x_n); x_n) \\ &- c(e_1) + \mu(c(\widetilde{e}_{0,1}) - c(e_1)) + \gamma \sum_n \sum_\ell \pi_n^1 \pi_\ell^{0,1} \mu(c(\widetilde{e}_2(x_\ell)) - c(\widetilde{e}_2(x_n))) \end{aligned} \tag{3.3}$$

where $\pi_n^1 \equiv \pi_n(e_1)$ and $\pi_n^{0,1} \equiv \pi_n(\widetilde{e}_{0,1})$, $u_n \equiv u(s_n)$ and s_n are first-period payments and others defined as before. The first line represents period-one total expected utility from exerting e_1 having planned to exert $\widetilde{e}_{0,1}$. As before, the first term is expected consumption utility and the second term is contemporaneous gain-loss utility. The third term corresponds to prospective gain-loss utility as defined in Assumption 2. The second line represents period-two total expected utility. The third and final line represents total period-one expected utility from effort. The first and second term correspond to consumption and gain-loss utilities from implementing action e_1 having planned $\widetilde{e}_{0,1}$ and the last term corresponds to prospective gain-loss utility in effort from second period effort being a contingent strategy.

Finally, in period zero, the agent's total expected utility corresponds to

$$\begin{aligned} EU_0(\widetilde{e}_{0,1} \widetilde{e}_{0,2}(X_1)|\widetilde{e}_{0,1}, \widetilde{e}_{0,2}(X_1)) = &\sum_n \pi_n^{0,1} u_n + \sum_n \sum_\ell \pi_n^{0,1} \pi_\ell^{0,1} \mu(u_n - u_\ell) + \gamma \sum_n \sum_\ell \sum_m \pi_n^{0,1} \pi_\ell^{0,1} \pi_m^{0,2} \mu(u_{n\ell} - u_{\ell m}) \\ &+ \sum_n \pi_n^{0,1} EU_2(\widetilde{e}_{0,2}(x_n)|\widetilde{e}_{0,2}(x_n); x_n) \\ &- c(\widetilde{e}_{0,1}) + \gamma \sum_n \sum_\ell \pi_n^{0,1} \pi_\ell^{0,1} \mu(c(\widetilde{e}_2(x_\ell)) - c(\widetilde{e}_2(x_n))) \end{aligned} \tag{3.4}$$

The description of period-zero total expected utility follows that of period one. The only difference is that in this period the agent does not execute an action but forms plans. Since he has rational expectations, those plans must coincide with the actual action he plans to implement, thus he does not experience contemporaneous gain-loss utility in effort and the probabilities are defined in term of the plans uniquely. As a consequence $EU_0(\widetilde{e}_{0,1}\widetilde{e}_{0,2}(X_1)|\widetilde{e}_{0,1}, \widetilde{e}_{0,2}(X_1)) = EU_1(\widetilde{e}_{0,1}\widetilde{e}_{0,2}(X_1)|\widetilde{e}_{0,1}, \widetilde{e}_{0,2}(X_1))$.

3.2 Chapter 1 Proofs

Proof of Lemma 1

Consider first-period contemporaneous gain-loss utility. Start noticing that if for $e_j \in \mathcal{E}$ $\pi_n^j = 1$, then $\pi_{n'}^j = 0$ $\forall n' \neq n$. Consider first the equilibrium path. In this case $e_1 = \widetilde{e}_{0,1}$ and thus $\forall e_1 \in \mathcal{E}$ $\exists!$

$n(e_1) \in \{1, \ldots, N\}$ such that $\pi^1_{n(e_1)}\pi^{0,1}_{n(e_1)} \neq 0$ and $\pi^1_{n(e_1)}\pi^{0,1}_{\ell} = 0$ $\forall \ell \neq n(e_1)$. Therefore,

$$\sum_{n=1}^{N}\sum_{\ell=1}^{N}\pi^1_n\pi^{0,1}_\ell\mu(u_n - u_\ell) = \pi^1_{n(e_1)}\pi^{0,1}_{n(e_1)}\mu(u_{n(e_1)} - u_{n(e_1)}) = 0$$

Second, consider the outside equilibrium path. In this case $e_1 \neq \widetilde{e}_{0,1}$, and thus $\pi^1_n\pi^{0,1}_\ell = 0$ $\forall n, \ell$ and thus $\sum_{n=1}^{N}\sum_{\ell=1}^{N}\pi^1_n\pi^{0,1}_\ell\mu(u_n - u_\ell) = 0$. The proof for second-period contemporaneous gain-loss utility is analogous $\forall x_n$. Consider now prospective gain-loss utility from period one perspective. Consider first the equilibrium path. If $e_1 = \widetilde{e}_{0,1} = \widetilde{e}_{0,2}$, then $\forall e_1 \in \mathcal{E}$ $\exists!$ $n(e_1) \in \{1, \ldots, N\}$ such that $\pi^1_{n(e_1)}\pi^{0,1}_{n(e_1)}\pi^{0,2}_{n(e_1)} \neq 0$ and $\pi^1_{n(e_1)}\pi^{0,1}_\ell\pi^{0,2}_m = 0$ $\forall \ell \neq n(e_1)$, $m \neq n(e_1)$ or $\ell \neq m$. Therefore,

$$\sum_{n=1}^{N}\sum_{\ell=1}^{N}\sum_{m=1}^{N}\pi^1_n\pi^{0,1}_\ell\pi^{0,2}_m\mu(u_{n\ell} - u_{\ell m}) = \pi^1_{n(e_1)}\pi^{0,1}_{n(e_1)}\pi^{0,2}_{n(e_1)}\mu(u_{n(e_1)n(e_1)} - u_{n(e_1)n(e_1)}) = 0$$

Second, if $e_1 = \widetilde{e}_{0,1} \neq \widetilde{e}_{0,2}$, then $\pi^1_n\pi^{0,1}_\ell\pi_m(\widetilde{e}_{0,2}) = 0$ $\forall n, \ell, m$. Consider now the outside equilibrium path. In such case $e_1 \neq \widetilde{e}_{0,1}$ and thus $\pi^1_n\pi^{0,1}_\ell = 0$ $\forall n, \ell$ and thus $\sum_{n=1}^{N}\sum_{\ell=1}^{N}\sum_{m=1}^{N}\pi^1_n\pi^{0,1}_\ell\pi^{0,2}_m\mu(u_{n\ell} - u_{\ell m}) = 0$.. From period zero perspective, the proof is analogous.

■

Proof of Proposition 1
From Lemma 1 I have that equation (1.1) becomes $v_t = V(s_t, e_t)$ $\forall \eta$. Thus the optimal contract does not depend on η. To see that the optimal contract implementing $(e_1, e_2(X_1))$ does not necessarily displays memory, notice that when the action is observable for e_1 $\exists!$ $n(e_1) \in \{1, \ldots, N\}$ such that $\pi^1_n = 1$ and thus $\pi^1_{n'} = 0$ $\forall n' \neq n$. Defining $m(e_2) \in \{1, \ldots, N\}$ in analogous manner, I have that the IR restriction to implement $(e_1, e_2(x_n))$ corresponds to

$$\sum_{n=1}^{N}\pi_n(e_1)[u_n + \sum_{m=1}^{N}\pi_m(e_2)u_{nm}] - c(e_1) - c(e_n) = u_{n(e_1)} + u_{m(e_2)} - c(e_1) - c(e_n) \geqslant U_{R_1} + U_{R_2}$$

Thus, the principal can write a long-term-forcing contract as

$$S_1 = \begin{cases} u^{-1}(U^1_R + c(e_1)) & \text{if } X_1 = x_{n(e_1)} \\ \underline{s} & \text{otherwise.} \end{cases} \quad \text{and} \quad S_2 = \begin{cases} u^{-1}(U^2_R + c(e_2)) & \text{if } X_2 = x_{m(e_2)} \\ \underline{s} & \text{otherwise.} \end{cases}$$

where $\underline{s}$ is defined in Assumption 1. It is straight to see that such contract implements the action path and is least cost since the principal's objective function is decreasing in the agent's payments. Notice that in fact, there is a family of forcing contracts implementing the first best, as any payment $u^{-1}(\underline{s}) \leqslant s < u^{-1}(U^1_R + c(e_1))$ implements e_1 at first best cost. Same for the second-period contract.

■

Proof of Proposition 2
Let $\mathcal{U} = \big(\mathcal{U}_1(X_1), \mathcal{U}_2(X_1, X_2)\big)$ be the utility equivalent of the contract S. Because $u'(\cdot) > 0$ (see Assumption 1) such representation is unique. Thus, I can write the principal problem as

$$\min_{\left\{\{u_n\}_{n=1}^N, \{u_{nm}\}_{m=1}^N\right\}} \quad EC_1(u_n) + EC_2(u_{nm}) = \sum_n \pi^1_n u^{-1}(u_n) + \sum_n\sum_m \pi^2_m u^{-1}(u_{nm})$$

$$\text{subject to} \qquad EU_0(e_1, e_2(X_1)|e_1, e_2(X_1)) \geqslant U_R$$

$$(e_1, e_2(X_1)) \quad \text{is a Preferred Personal Equilibrium}$$

where the objective function is convex. Moreover, because the function $\mu(\cdot)$ is piece-wise linear (Assumption 2), then is straightforward to see that all the expected utilities in the restriction set are linear in u_n and u_{nm} $\forall n, m$ (see Definition 1 and Section 3.1). Thus the principal's problem is a convex problem with linear constraints. Consider the set of all contracts $\mathcal{U}$ implementing the action path

$$\mathcal{C}(e_1, e_2(X_1)) \equiv \{\sum_{n=1}^{N} \pi_n(e_1)[u^{-1}(u_n) + \sum_{m=1}^{N} \pi_m(e_2) u^{-1}(u_{nm})] | \mathcal{U} \text{ implements } (e_1, e_2(X_1))\}$$

I are done if show that $\mathcal{C}(e_1, e_2(X_1))$ has an infimum that is achieved. I prove that the set is bounded below. Start by noticing that, because U_R is bounded, there must exist a finite $\widetilde{U}_{R_2}(x_n)$ $\forall n$ so that $EU_2(e_2(x_n)|e_2(x_n); x_n) = \widetilde{U}_{R_2}(x_n)$ (recall the agent is fully committed in period zero to stay in the relationship and thus the principal is not committed to $\mathrm{EU}_2(e_2(x_n)|e_2(x_n); x_n) \geqslant U_{R_2}$ $\forall n$). Thus, using equation (3.1) in equilibrium, I have

$$\sum_{m} \pi_m^2 u_{nm} - c(e_2) \geqslant \sum_{m} \pi_m^2 u_{nm} + \sum_{m} \sum_{\ell} \pi_m^2 \pi_\ell^2 \mu(u_{nm} - u_{n\ell}) + c(e_2) \geqslant \widetilde{U}_{R_2}(x_n) \geqslant \widetilde{U}_{R_2}^{\max}$$

where $\widetilde{U}_{R_2}^{\max} \equiv max\{\widetilde{U}_{R_2}(x_n)\}_{n=1}^{N}$ and the second inequality comes from the fact that contemporaneous gain-loss utility is negative in equilibrium (see Lemma 2). Applying the definition of convexity twice, I have that a lower bound for the second-period contract corresponds to:

$$\sum_{n} \sum_{m} \pi_n^1 \pi_m^2 u^{-1}(u_{nm}) \geqslant \sum_{n} \pi_n(e_1) u^{-1}\Big(\sum_{m} u_{nm}\Big) \geqslant u^{-1}\Big(\widetilde{U}_{R_2}^{\max} + c(e_2)\Big)$$

Doing a similar analysis for period-one payments, I have that

$$\sum_{n} \pi_n^1 u^{-1}(u_n) \geqslant u^{-1}\Big(\sum_{n} \pi_n^1 u_n\Big) \geqslant u^{-1}\Big(U_R + c(e_1)\Big)$$

Thus the set $\mathcal{C}(e_1, e_2(X_1))$ is bounded below. Since the set is closed by construction, it as an infimum. The rest of the proof follows exactly that in Grossman and Hart (1983) by proving that this lower bound is achieved and then applying Weierstrass' theorem.

I now prove that the optimal contract is unique. By contradiction, suppose not. Let $U' = (U_1^{'}, U_2^{'})$ be also an optimal contract implementing $(e_1, e_2(x_n))$ where $u_n^{'} \neq u_n$ and $u_{nm}^{'} \neq u_{nm}$ for at least one n and one m. Define a new contract U^α as $u_n^\alpha = \alpha u_n + (1-\alpha) u_n^{'}$ and $u_{nm}^\alpha = \alpha u_{nm} + (1-\alpha) u_{nm}^{'}$, $\alpha \in (0,1)$ (where without loss of generality I have assumed α is the same for fist and second- period contracts). I start proving that such contract still implements the same effort path. Consider first the IR restriction under U^α. First period consumption utility can be written as $\sum_{n=1}^{N} \pi_n^{0,1} u_n^\alpha = \sum_{n=1}^{N} \pi_n^{0,1}(\alpha u_n + (1-\alpha) u_n') = \alpha \sum_{n=1}^{N} \pi_n^{0,1} u_n + (1-\alpha) \sum_{n=1}^{N} \pi_n^{0,1} u_n^{'}$, and equivalently for second-period consumption utility. From equation (3.5) in the proof of Lemma 2, because the value function is assumed to be piece-wise linear, first period gain-loss utility can be written in linear form as $(1-\lambda) \sum_{n=1}^{N} \sum_{\ell<n} \pi_n^{0,1}(u_n^\alpha - u_\ell^\alpha) = \alpha(1-\lambda) \sum_{n=1}^{N} \sum_{\ell<n} u_n + (1-\alpha)(1-\lambda) \sum_{n=1}^{N} \sum_{\ell<n} u_n^{'}$, and equivalently for second-period contemporaneous gain-loss utility. Also from the proof of Lemma 2, prospective gain-loss utility can be separated into its positive and negative components so find a shape for prospective gain-loss utility under U^α as function of U and $U^{'}$ payments. Consider first from equation (3.7) its positive component,

$$\begin{aligned} &\sum_{n=1}^{N} \sum_{\ell<n}^{N} \sum_{m=1}^{N} \pi_n^1 \pi_\ell^{0,1} \pi_m^{0,2}(u_{n\ell}^\alpha - u_{\ell m}^\alpha) + \sum_{n=1}^{N} \sum_{\ell=n}^{N} \sum_{m<\ell}^{N} \pi_n^1 \pi_\ell^{0,1} \pi_m^{0,2}(u_{n\ell}^\alpha - u_{\ell m}^\alpha) \\ &= \alpha \Big[\sum_{n=1}^{N} \sum_{\ell<n}^{N} \sum_{m=1}^{N} \pi_n^1 \pi_\ell^{0,1} \pi_m^{0,2}(u_{n\ell} - u_{\ell m}) + \sum_{n=1}^{N} \sum_{\ell=n}^{N} \sum_{m<\ell}^{N} \pi_n^1 \pi_\ell^{0,1} \pi_m^{0,2}(u_{n\ell} - u_{\ell m})\Big] \\ &+(1-\alpha)\Big[\sum_{n=1}^{N} \sum_{\ell<n}^{N} \sum_{m=1}^{N} \pi_n^1 \pi_\ell^{0,1} \pi_m^{0,2}(u_{n\ell}^{'} - u_{\ell m}^{'}) + \sum_{n=1}^{N} \sum_{\ell=n}^{N} \sum_{m<\ell}^{N} \pi_n^1 \pi_\ell^{0,1} \pi_m^{0,2}(u_{n\ell}^{'} - u_{\ell m}^{'})\Big] \end{aligned}$$

Working the negative terms from equation (3.8) in the same fashion, and putting together first and second-period consumption utility with first and second-period contemporaneous gain-loss utility, is easy to see that

$$EU_0(e_1, e_2(x_n)|e_1, e_2(x_n); U^{\alpha}) = \alpha EU_0(e_1, e_2(x_n)e_1, e_2(x_n); U) + (1-\alpha)EU_0(e_1, e_2(x_n)|e_1, e_2(x_n); U^{'})$$

Thus, since the IR holds for U and $U^{'}$, it must also hold for U^{α}. Exactly the same logic shows that the IC restrictions are all satisfied for U^{α} as they are satisfied for U and $U^{'}$. As a consequence, U^{α} implements the same effort path as U and U'. Finally, notice that because of Jensen's inequality,

$$EC_t(u_n^{\alpha}) \leqslant \alpha EC_t(u_n) + (1-\alpha)EC_t(u_n') = EC_t(u_n) \qquad t = 1, 2$$

with at least one inequality strict if $u^{''}(\cdot) < 0$ which contradicts the optimality of U.

■

Proof of Proposition 3

I start proving that the IR binds in the principal's problem. By contradiction, assume the contract S is the least cost contract implementing the effort path $(e_1, e_2(X_1))$ where $EU_0(e_1, e_2(X_1)|e_1, e_2(X_1); S) > U_R$. Consider an alternative contract $\hat{S} = (s_n - \varepsilon, s_{nm})$ $\varepsilon > 0$. Notice that because $\hat{S}$ does not modify second-period payments, second-period implementing IC is satisfied. Consider first-period implementing IC. By equation (3.3), first-period contemporaneous utility under contract $\hat{S}$ corresponds to $\sum_{n=1}^{N}\sum_{\ell=1}^{N}\pi_n^1\pi_\ell^{0,1}\mu((u_n - \varepsilon) - (u_\ell - \varepsilon)) = \sum_{n=1}^{N}\sum_{\ell=1}^{N}\pi_n^1\pi_\ell^{0,1}\mu(u_n - u_\ell)$, which is equal to contemporaneous gain-loss utility under S. Thus, first-period implementing IC holds under $\hat{S}$ with the same argument as in the classical model. The same analysis is valid for period-zero planning IC. The result follows from $\mathrm{EC}_1(S)$ being decreasing in first period payments and thus S cannot be a least-cost contract.

Consider now part (*ii*). Consider first the case when the principal wants to implement $e_1 = e_2(x_n) = e_{\min}$ $\forall n$ where e_{min} is such that $c(e_{min}) = \min\{c(\mathcal{E})\}$. The proof follows straight from noticing that a contract S^{min}, $s_n^{min} = u^{-1}(U_R^1 + c(e_{min}))$ $\forall n$ and $s_{nm}^{min} = u^{-1}(U_R^2 + c(e_{min}))$ $\forall n, m$ implements the action path and at first best cost (for a description of the first-best cost, see proof of Proposition 1). To see that the contract is IR, notice from equation (1.2) that gain-loss utility (contemporaneous and prospective) is zero with fixed wages, thus

$$EU_0(e_{\min}, e_{\min}|e_{\min}, e_{\min}; S^{min}) = \sum_n \pi_n^{min}(U_R^1 + c(e_{min})) + \sum_n\sum_m \pi_m^{min}(U_R^2 + c(e_{min})) - 2c(e_{min}) = U_R$$

where $\pi_n^{min} \equiv \pi_n(e_{min})$. An equivalent analysis show that under S^{min} all the IC constraints are met.

Consider now the case where there exists a subset $\mathcal{X}_j \subseteq \mathcal{X}$ such that $\sum_{x_n\in\mathcal{X}_j}\pi_n^j = 0 < \sum_{x_n\in\mathcal{X}_j}\pi_n^k$ $\forall e_k \in \mathcal{E}\backslash e_j$ such that $c(e_k) < c(e_j)$ for $e_j = e_1$ and equivalently for $e_j = e_2(x_n)$ $\forall x_n$ where we use the notation $\mathcal{X}_2(x_n)$ to make clear that the set $\mathcal{X}_2$ may depend on x_n. The proof follows straight from noticing that contract S consisting of the following two spot implementable forcing contracts

$$S_1 = \begin{cases} u^{-1}(U_R^1 + c(e_1)) & \text{if } x_n \notin \mathcal{X}_1 \\ \underline{s} & \text{otherwise.} \end{cases} \quad \text{and} \quad S_2(x_n) = \begin{cases} u^{-1}(U_R^2 + c(e_2)) & \text{if } x_n \notin \mathcal{X}_2(x_n) \\ \underline{s} & \text{otherwise.} \end{cases} \quad \forall n$$

implements the desired action path and achieves first best contract. To do so, is straight to check the IR and IC constraints hold.

■

Proof of Proposition 4

I start proving part (*i*), following the proof in Rogerson (1985). To see that first-period payments are contingent, let $U = (u_n, u_{nm})$ be the unique utility equivalent of the optimal contract implementing the effort path $(e_1, e_2(x_n))$. Consider the following deviation contract $\hat{U} = (\hat{u}_n, \hat{u}_{nm})$ where $\hat{u}_n = u_n$ and $\hat{u}_{nm} = u_{nm}$ and $\hat{u}_{n'} = u_{n'} - \rho$ and $\hat{u}_{n'm} = u_{n'm} + \rho$ for one $n' \neq n$ for all $m \in \{1, \ldots, N\}$. Notice that the agent's expected utility is the same under both contracts since $\pi_{n'}^1[(u_{n'} - \rho) + \sum_m \pi_m^2(u_{n'm} + \rho)] =$

$\pi^1_{n'}[u_{n'} + \sum_m \pi^2_m u_{n'm}]$. Moreover, since ρ does not depend on the action, $\hat{U}$ also implements $(e_1,e_2(x_n))$ and the agent is indifferent between the two contracts. Then, for U to be the optimal contract, the value $\rho = 0$ must minimize $u^{-1}(u_n - \rho) + \sum_m \pi_m(e_2(x_n))u^{-1}(u_{nm} + \rho)$ Taking first derivatives and using the fact $\rho = 0$ in the optimal contract, I have that the following condition must hold:

$$\frac{1}{u'(u^{-1}(u_n))} = \sum_{m=1}^{N} \frac{\pi_m(e_2(x_n))}{u'(u^{-1}(u_{nm}))}$$

Because the right hand side depends on x_n (as the notation emphasizes), then it must be the case that $u_n \neq u_{n'}$ for $n \neq n'$. To see that the optimal contract will display memory, suppose not. Then, it must be the case that $u_{nm} = u_{km}$ $\forall n, k$ and thus $e_2(x_n) = e_2(x_k)$. Therefore, from the equation above,

$$\frac{1}{u'(u^{-1}(u_n))} = \sum_{m=1}^{N} \frac{\pi_m(e_2(x_n))}{u'(u^{-1}(u_{nm}))} = \sum_{m=1}^{N} \frac{\pi_m(e_2(x_k))}{u'(u^{-1}(u_{km}))} = \frac{1}{u'(u^{-1}(u_k))}$$

implying that $u_n = u_k$, which is a contradiction.

To prove part (ii) it suffices to show that when $u^{''}(\cdot) = 0$, the optimal long term contract without memory equals the first-best contract. Let $u_{nm} = u_{n'm} \equiv u_m$ be the second period payments in the contract without memory. The expected cost of implementing $(e_1, e_2(x_n))$ under such contract corresponds to:

$$\begin{aligned}
\sum_n \pi^1_n u^{-1}(u_n) + \sum_m \pi^2_m u^{-1}(u_m) &= u^{-1}\Big(\sum_n \pi^1_n u_n + \sum_m \pi^2_m u_m\Big) \\
&= u^{-1}\big(U_R + c(e_1) + c(e_2)\big) \\
&= u^{-1}\big(U^1_R + c(e_1)\big) + u^{-1}\big(U^2_R + c(e_2)\big)
\end{aligned}$$

where the first line follows from the fact that the IR constraint binds because of the usual arguments. Then, from Proposition 1 we the principal can implement the action path at the same cost of the first-best contract. As a consequence, memory cannot decrease the principal's expected cost .

■

Proof of Lemma 2

Let $u_n \equiv u(s_n)$, $u_{nm} \equiv u(s_{nm})$ be the utility equivalent of contract S. Because of Assumption 1, such representation is unique. Order this utility contract in ascending order with respect to n. Consider first expected contemporaneous gain-loss utility, that is $\int G(\widetilde{s}_{t\tau}|\widetilde{s}_{t-1,\tau})d\Pi^S_\tau$ for $(t,\tau) = (1,1)$ and $(t,\tau) = (2,2)$. Consider $(t,\tau) = (1,1)$,

$$\begin{aligned}
\int G(s_1|\widetilde{s}_{0,1})d\Pi^S_1 &= \int\int_0^1 \mu\big(u(s_1(p)) - u(\widetilde{s}_{0,1}(p))\big)dpd\Pi^S_1 \\
&= \sum_{n=1}^{N}\sum_{\ell=1}^{N} \pi_n(e_1)\pi_\ell(\widetilde{e}_{0,1})\mu(u_n - u_\ell) \\
&= \sum_{n=1}^{N}\sum_{\ell\leqslant n} \pi_n(e_1)\pi_\ell(\widetilde{e}_{0,1})\mu(u_n - u_\ell) + \sum_{n=1}^{N}\sum_{\ell\geqslant n} \pi_n(e_1)\pi_\ell(\widetilde{e}_{0,1})\mu(u_n - u_\ell) \\
&= \sum_{n=1}^{N}\sum_{\ell\leqslant n} \pi_n(e_1)\pi_\ell(\widetilde{e}_{0,1})\mu(u_n - u_\ell) + \sum_{n=1}^{N}\sum_{\ell\leqslant n} \pi_n(\widetilde{e}_{0,1})\pi_\ell(e_1)\mu(u_\ell - u_n) \\
&= \sum_{n=1}^{N}\sum_{\ell\leqslant n} \pi_n(e_1)\pi_\ell(e_1)[\mu(u_n - u_\ell) + \mu(u_\ell - u_n)] \leqslant 0 \qquad (3.5)
\end{aligned}$$

where in the last line I have used the fact that in equilibrium $\widetilde{e}_{0,1} = e_1$. Notice further that if $\lambda = 1$, then $\mu(u_n - u_\ell) = |\mu(u_\ell - u_n)|$ and thus $\int G(s_\tau|\widetilde{s}_{t-1,\tau})d\Pi^S_\tau = 0$. For $(t,\tau) = (2,2)$ the proof is analogous by considering $\int G(\widetilde{s}_{2,2}|\widetilde{s}_{1,2})d\Pi_2^S = \int\int_0^1 \mu\big(u(s_2(p)) - u(\widetilde{s}_{1,2}(p))\big)dpd\Pi_2^S = \sum_{m=1}^N \sum_{\ell=1}^N \pi_m(e_2)\pi_\ell(\widetilde{e}_{1,2})\mu(u_{nm} - u_{n\ell})$ for any given n. Finally, notice that the latter proof does not rely on the assumption of the value function $\mu(\cdot)$ being piece-wise linear.

Consider now expected prospective gain-loss utility (gain-loss utility for $(t,\tau) = (1,2)$). Notice that the difference between this utility source and contemporaneous gain-loss utility for period two is that, for the latter, and given the realization of x_n, second-period payments are still a non-degenerate distribution (a continuation contract $S_2(x_n)$), whereas in second-period contemporaneous gain-loss utility the second-period payment is a degenerate distribution since x_n has already been observed. Thus, using the fact that prospective gain-loss utility is realized before plans are updated and thus the only relevant case is $e_2(x_n) = \widetilde{e}_2(x_n)$, then

$$\begin{aligned}\gamma\int G(s_2|\widetilde{s}_{0,2})d\Pi_2^S \propto \int G(s_2|\widetilde{s}_{0,2})d\Pi_2^S &= \int\int_0^1 \mu\big(u(s_2(p)) - u(\widetilde{s}_{0,2}(p))\big)dpd\Pi_2^S \\ &= \sum_{n=1}^N\sum_{\ell=1}^N\sum_{m=1}^N \pi_n(e_1)\pi_\ell(\widetilde{e}_{0,1})\pi_m(\widetilde{e}_{0,2})\mu(u_{n\ell} - u_{\ell m}) \quad (3.6)\end{aligned}$$

Ordering the payments as $s_{nm} \geqslant s_{nm'}$ for $m \geqslant m'$ $\forall n$ and that $s_{nN} \geqslant s_{(n-1)1}$ for $\forall n$, I can use the same strategy as in contemporaneous gain-loss utility of splitting the positive terms from the negative ones in equation (3.6) as

$$\sum_{n=1}^N\sum_{\ell<n}^N\sum_{m=1}^N \pi_n^1\pi_\ell^{0,1}\pi_m^{0,2}\mu(u_{n\ell} - u_{\ell m}) + \sum_{n=1}^N\sum_{\ell=n}^N\sum_{m<\ell}^N \pi_n^1\pi_\ell^{0,1}\pi_m^{0,2}\mu(u_{n\ell} - u_{\ell m}) \quad (3.7)$$

$$+\sum_{n=1}^N\sum_{\ell>n}^N\sum_{m=1}^N \pi_n^1\pi_\ell^{0,1}\pi_m^{0,2}\mu(u_{n\ell} - u_{\ell m}) + \sum_{n=1}^N\sum_{\ell=n}^N\sum_{m>\ell}^N \pi_n^1\pi_\ell^{0,1}\pi_m^{0,2}\mu(u_{n\ell} - u_{\ell m}) \quad (3.8)$$

where I have shortened the notation as $\pi_n(e_1) \equiv \pi_n^1$, $\pi_\ell(\widetilde{e}_{0,1}) = \pi_\ell^{0,1}$ and $\pi_m(\widetilde{e}_{0,2}) \equiv \pi_m^{0,2}$, and those terms in (3.7) are positive whereas those in equation (3.8) are negative. Contrarily to the case of contemporaneous gain-loss utility, the terms in (3.7) are not the same as those in (3.8). To make them comparable, I add an subtract so to create neighboring payments and use the value function specification $\mu(x) = x$ if $x > 0$ and $\mu(x) = \lambda x$ if $x < 0$. Consider first equation (3.7). For $n > \ell$, $s_{n\ell} - s_{\ell m} = s_{n\ell} \pm s_{n(\ell-1)} \pm \cdots \pm s_{n1} \pm s_{(n-1)N} \pm \ldots s_{(n-1)1} \pm \cdots \pm s_{\ell N} \pm \cdots \pm s_{\ell(m+1)} - s_{\ell m}$; for $n = \ell > m$, $s_{nn} - s_{nm} = s_{nn} \pm s_{n(m+1)} \pm \cdots \pm s_{n(n-1)} - s_{nm}$, and for a given x_n, with a lot of tedious work I can rewrite (3.7) as

$$\begin{aligned}= \quad & \sum_{n=1}^{N-1}\sum_{\ell\neq 1}^{N}\Big[\sum_{i=1}^{n-1}\pi_i^{0,1} + \pi_n^{0,1}\sum_{j=1}^{\ell-1}\pi_j^{0,2}\Big](s_{n\ell} - s_{n(\ell-1)}) + \sum_{n=2}^{N}\sum_{i=1}^{n-1}\pi_i^{0,1}(s_{n1} - s_{(n-1)N}) \\ + \quad & \sum_{\ell=1}^{N-1}\Big[\sum_{i=\ell+1}^{N-1}\pi_i^{0,1} + \pi_N^{0,1}\sum_{j=1}^{\ell}\pi_j^{0,2}\Big](s_{N(\ell+1)} - s_{N\ell})\end{aligned}$$

Adding and subtracting in a similar fashion, I can rewrite (3.8) as:

$$\begin{aligned}= \quad - \quad & \lambda\sum_{n=2}^{N}\sum_{\ell\neq 1}^{N}\Big[\sum_{i=n+1}^{N}\pi_i^{0,1} + \pi_n^{0,1}\sum_{j=\ell}^{N}\pi_j^{0,2}\Big](s_{n\ell} - s_{n(\ell-1)}) - \lambda\sum_{n=2}^{N}\sum_{i=n}^{N}\pi_i^{0,1}(s_{n1} - s_{(n-1)N}) \\ - \quad & \lambda\sum_{\ell=1}^{N-1}\Big[\sum_{i=2}^{\ell}\pi_i^{0,1} + \pi_1^{0,1}\sum_{j=\ell+1}^{N}\pi_j^{0,2}\Big](s_{1(\ell+1)} - s_{1\ell})\end{aligned}$$

Then, changing the notation from n to k so to sum over the realizations of x_n and rearranging, I have that expected prospective gain-loss utility can be written in terms of neighboring payments as:

$$\begin{aligned}
&\Big\{\sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\sum_{\ell>1}^{N}\pi_n^1\Big[\sum_{i=1}^{k-1}\pi_i^{0,1}+\pi_k^{0,1}\sum_{j=1}^{\ell-1}\pi_j^{0,2}\Big](s_{k\ell}-s_{k(\ell-1)})+\sum_{n=1}^{N}\sum_{k=2}^{n}\sum_{i=1}^{k-1}\pi_n^1\pi_i^{0,1}(s_{k1}-s_{(k-1)N}) \\
&+\sum_{k=2}^{N}\sum_{\ell>1}^{k}\pi_k^1\Big[\sum_{i=\ell}^{k-1}\pi_i^{0,1}+\pi_k^{0,1}\sum_{j=1}^{\ell-1}\pi_j^{0,2}\Big](s_{k\ell}-s_{k(\ell-1)})\Big\}- \\
\lambda\quad&\Big\{\sum_{k=2}^{N}\sum_{n=1}^{k-1}\sum_{\ell>1}^{N}\pi_n^1\Big[\sum_{i=k+1}^{N}\pi_i^{0,1}+\pi_k^{0,1}\sum_{j=\ell}^{N}\pi_j^{0,2}\Big](s_{k\ell}-s_{k(\ell-1)})+\sum_{n=1}^{N}\sum_{k=n+1}^{N}\sum_{i=k}^{N}\pi_n^1\pi_i^{0,1}(s_{k1}-s_{(k-1)N}) \\
&+\sum_{k=1}^{N-1}\sum_{\ell=k+1}^{N}\pi_k^1\Big[\sum_{i=k+1}^{\ell-1}\pi_i^{0,1}+\pi_k^{0,1}\sum_{j=\ell}^{N}\pi_j^{0,2}\Big](s_{k\ell}-s_{k(\ell-1)})\Big\}
\end{aligned}\tag{3.9}$$

Now I can prove that the latter expression is non-positive. In order to do so, I will conveniently sum up the terms. Consider first the sum of the second and the fifth terms in equation (3.9):

$$\begin{aligned}
&\sum_{n=1}^{N}\sum_{k=2}^{n}\sum_{i=1}^{k-1}\pi_n^1\pi_i^{0,1}(s_{k1}-s_{(k-1)N})-\lambda\sum_{n=1}^{N}\sum_{k=n+1}^{N}\sum_{i=k}^{N}\pi_n^1\pi_i^{0,1}(s_{k1}-s_{(k-1)N}) \\
=\;&\sum_{n=1}^{N}\Big[\sum_{k=2}^{n}\sum_{i=1}^{k-1}\pi_n^1\pi_i^{0,1}-\lambda\sum_{k=n+1}^{N}\sum_{i=k}^{N}\pi_n^1\pi_i^{0,1}\Big](s_{k1}-s_{(k-1)N}) \\
=\;&(1-\lambda)\sum_{k=2}^{N}\sum_{n=k}^{N}\sum_{i=1}^{k-1}\pi_n\pi_i(s_{k1}-s_{(k-1)N})\leqslant 0
\end{aligned}$$

where in the last line I have used the fact that in equilibrium $\widetilde{e}_{0,1}=e_1$. Consider now the first and last terms in equation (3.9) for $k=1$:

$$\sum_{n=2}^{N}\sum_{\ell>1}^{N}\pi_n^1\Big[\sum_{i=1}^{0}\pi_i^{0,1}+\pi_1^{0,1}\sum_{j=1}^{\ell-1}\pi_j^{0,2}\Big](s_{1\ell}-s_{1(\ell-1)})-\lambda\sum_{\ell=2}^{N}\pi_1^1\Big[\sum_{i=2}^{\ell-1}\pi_i^{0,1}+\pi_1^{0,1}\sum_{j=\ell}^{N}\pi_j^{0,2}\Big](s_{1\ell}-s_{1(\ell-1)})\Big\}\tag{3.10}$$

$$= \sum_{n=2}^{N}\sum_{\ell>1}^{N}\pi_n^1\pi_k^{0,1}\sum_{j=1}^{\ell-1}\pi_j^{0,2}(s_{1\ell}-s_{1(\ell-1)})-\lambda\sum_{\ell=2}^{N}\pi_1^1\Big[\sum_{i=2}^{\ell-1}\pi_i^{0,1}+\pi_1^{0,1}\sum_{j=\ell}^{N}\pi_j^{0,2}\Big](s_{1\ell}-s_{1(\ell-1)})\Big\}$$

$$= \pi_1^1\sum_{\ell>1}^{N}\Big\{\sum_{n=2}^{N}\sum_{j=1}^{\ell-1}\pi_n^1\pi_j^{0,2}-\lambda\Big[\sum_{i=2}^{\ell-1}\pi_i^{0,1}+\pi_1^{0,1}\sum_{j=\ell}^{N}\pi_j^{0,2}\Big]\Big\}(s_{1\ell}-s_{1(\ell-1)})\tag{3.11}$$

where in the last line I have used the fact that in equilibrium $\pi_1^1=\pi_1^{0,1}$. Consider now the curly bracket in equation (3.11). Changing the notation for convenience, adding and subtracting conveniently, and then rearranging, I have

$$\begin{aligned}
&\sum_{n=2}^{N}\sum_{j=1}^{\ell-1}\pi_n^1\pi_j^{0,2}-\lambda\Big[\sum_{i=2}^{\ell-1}\pi_i^{0,1}+\pi_1^{0,1}\sum_{j=\ell}^{N}\pi_j^{0,2}\Big] \\
=\;&\pi_1^{0,2}\sum_{n=2}^{N}\pi_n^1+\sum_{n=2}^{N}\sum_{j=2}^{\ell-1}\pi_n^1\pi_j^{0,2}\pm\pi_1^{0,2}\sum_{n=2}^{\ell-1}\pi_n^1-\lambda\Big[\sum_{j=1}^{N}\sum_{n=2}^{\ell-1}\pi_n^{0,1}\pi_j^{0,2}+\pi_1^{0,1}\sum_{j=\ell}^{N}\pi_j^{0,2}\Big] \\
=\;&(1-\lambda)\Big[\pi_1\sum_{n=\ell}^{N}\pi_n+\sum_{n=1}^{N}\sum_{j=2}^{\ell-1}\pi_n\pi_j\Big]
\end{aligned}$$

where in the last line I have used $e^1 = \widetilde{e}^{0,1}$. Thus equation (3.10) corresponds to

$$(1-\lambda)\pi_1 \sum_{\ell>1}^{N} \Big\{ \sum_{n=\ell}^{N} \pi_n + \sum_{n=1}^{N}\sum_{j=2}^{\ell-1} \pi_n \pi_j \Big\} (s_{1\ell} - s_{1(\ell-1)}) \leqslant 0$$

Second, consider the sum of the third and fourth terms on (3.9) for $k = N$.

$$\begin{aligned}
& \sum_{\ell>1}^{N} \pi_N^1 \Big[\sum_{i=\ell}^{N-1} \pi_i^{0,1} + \pi_N^{0,1} \sum_{j=1}^{\ell-1} \pi_j^{0,2} \Big] (s_{N\ell} - s_{N(\ell-1)}) - \lambda \sum_{n=1}^{N-1} \sum_{\ell>1}^{N} \pi_n^1 \Big[\sum_{i=N+1}^{N} \pi_i^{0,1} + \pi_N^{0,1} \sum_{j=\ell}^{N} \pi_j^{0,2} \Big] (s_{k\ell} - s_{k(\ell-1)}) \quad (3.12) \\
= & \sum_{\ell>1}^{N} \pi_N^1 \Big[\sum_{i=\ell}^{N-1} \pi_i^{0,1} + \pi_N^{0,1} \sum_{j=1}^{\ell-1} \pi_j^{0,2} \Big] (s_{N\ell} - s_{N(\ell-1)}) - \lambda \sum_{n=1}^{N-1} \sum_{\ell>1}^{N} \pi_n^1 \pi_N^{0,1} \sum_{j=\ell}^{N} \pi_j^{0,2} (s_{k\ell} - s_{k(\ell-1)}) \\
= & \pi_N^1 \sum_{\ell_1}^{N} \Big\{ \sum_{i=\ell}^{N-1} \pi_i^{0,1} + \pi_N^{0,1} \sum_{j=1}^{\ell-1} \pi_j^{0,2} - \lambda \sum_{n=1}^{N-1} \pi_n^1 \sum_{j=\ell}^{N} \pi_j^{0,2} \Big\} (s_{N\ell} - s_{N(\ell-1)}) \quad (3.13)
\end{aligned}$$

where in the last line I have used the fact that in equilibrium $\pi_N^1 = \pi_N^{0,1}$. Consider now the curly bracket in equation (3.13). Slightly changing the notation for convenience, adding and subtracting conveniently, and then rearranging, I have

$$\begin{aligned}
\sum_{i=\ell}^{N-1} \pi_i^{0,1} + \pi_N^{0,1} \sum_{j=1}^{\ell-1} \pi_j^{0,2} - \lambda \sum_{n=1}^{N-1} \pi_n^1 \sum_{j=\ell}^{N} \pi_j^{0,2} &= \sum_{j=\ell}^{N-1} \pi_j^{0,1} + \pi_N^{0,1} \sum_{j=1}^{\ell-1} \pi_j^{0,2} \pm \pi_N^{0,1} \sum_{j=\ell}^{N} \pi_m^{0,2} \\
&= (1-\lambda)(1-\pi_N) \sum_{j=\ell}^{N} \pi_j
\end{aligned}$$

where in the last lien I have used $\pi_1^1 = \pi_1^{0,1}$. Thus equation (3.12) corresponds to

$$(1-\lambda)(1-\pi_N)\pi_N^1 \sum_{\ell=1}^{N} \sum_{j=\ell}^{N} \pi_j (s_{N\ell} - s_{N(\ell-1)}) \leqslant 0$$

Finally, consider all terms $2 \leqslant k \leqslant N-1$ in the first, third , fourth and last term in equation (3.9). For a given k and a given ℓ, I have

$$\begin{aligned}
& \Big\{ \sum_{n=k+1}^{N} \pi_n^1 \Big[\sum_{i=1}^{k-1} \pi_i^{0,1} + \pi_k^{0,1} \sum_{j=1}^{\ell-1} \pi_j^{0,2} \Big] + \pi_k^1 \Big[\sum_{i=\ell}^{k-1} \pi_i^{0,1} + \pi_k^{0,1} \sum_{j=1}^{\ell-1} \pi_j^{0,2} \Big] \Big\} (s_{k\ell} - s_{k(\ell-1)}) \\
- & \lambda \Big\{ \sum_{n=1}^{k-1} \pi_n^1 \Big[\sum_{i=k+1}^{N} \pi_i^{0,1} + \pi_k^{0,1} \sum_{j=\ell}^{N} \pi_j^{0,2} \Big] + \pi_k^1 \Big[\sum_{i=k+1}^{\ell-1} \pi_i^{0,1} + \pi_k^{0,1} \sum_{j=\ell}^{N} \pi_j^{0,2} \Big] \Big\} (s_{k\ell} - s_{k(\ell-1)})
\end{aligned}$$

Five possible cases arise here depending on the relative magnitudes of ℓ, $k-1$ and $k+2$. With some further work, one can prove that all the terms above are negative in each of the five cases, which concludes the proof.

■

Before proving the main result, we present the following lemma.

Lemma 6 *(Outside Equilibrium Contemporaneous Gain-Loss Utility)*
For any actions $e_j \neq e_k \in \mathcal{E}$, *if* $\Pi_\tau^S(e_j) \succsim_{LRD} \Pi_\tau^S(e_k)$, *then*
(i) $\int G(\widetilde{s}_{t,\tau}(e_k) | \widetilde{s}_{t-1,\tau}(e_j)) d\Pi_\tau^S \leqslant 0$

(ii) $\int G(\widetilde{s}_{t,\tau}(e_j)|\widetilde{s}_{t-1,\tau}(e_j))d\Pi_\tau^S - \int G(s_\tau(e_k)|\widetilde{s}_{t-1,\tau}(e_j))d\Pi_\tau^S \leqslant 0$
for $t = 1$ *and* $\tau = 1$ *and* $t = 2$ *and* $\tau = 2$.

Proof of Lemma 6
To keep notation simple, I will use first period contemporaneous gain-loss utility. To prove part (i), notice

$$\begin{aligned}\sum_n \sum_\ell \pi_n^k \pi_\ell^j \mu(u_n - u_\ell) &= \eta \sum_n \sum_{\ell<n} \pi_n^k \pi_\ell^j (u_n - u_\ell) - \lambda\eta \sum_n \sum_{\ell>n} \pi_n^k \pi_\ell^j (u_\ell - u_n) \\ &= \eta \sum_n \sum_{\ell<n} \pi_n^k \pi_\ell^j (u_n - u_\ell) - \lambda\eta \sum_n \sum_{\ell<n} \pi_n^j \pi_\ell^k (u_n - u_\ell) \\ &= \eta \sum_n \sum_{\ell<n} [\pi_n^k \pi_\ell^j - \lambda \pi_n^j \pi_\ell^k](u_n - u_\ell)\end{aligned}$$

Because $\Pi_\tau^S(e_j) \succsim_{LRD} \Pi_\tau^S(e_k)$,then for all n, $n > \ell$

$$\pi_n^k \pi_\ell^j \leqslant \pi_n^j \pi_\ell^k \leqslant \lambda \pi_n^j \pi_\ell^k \leqslant$$

where the last inequality holds because $\lambda \geqslant 1$. Then it must be the case that $[\pi_n^k \pi_\ell^j - \lambda \pi_n^j \pi_\ell^k] \leqslant 0$, which concludes the proof. Part (ii) is obvious from the fact that gains are increasing in n whereas losses are decreasing in n.

■

Proof of Proposition 5
Let S, $u_n \neq u_{n'}$ for at least one $n \neq n^{'}$ and $u_{nm} = u_{n'm}$, $n \neq n'$ for at least one m be the optimal contract implementing the effort path $(e_1, e_2(x_n))$. I prove that the alternative contract $\hat{S}$, $\hat{u}_n = \hat{u}_{n'} = \sum_n \pi^1 u_n \equiv \bar{u}$ $\forall n \neq n'$ and $\hat{u}_{nm} = u_{nm} + \rho_n$, $\forall n, m$ $\rho_n \in \mathbb{R}$ implements $(e_1, e_2(x_n))$ at a lower expected cost than S. Consider first period-two implementing IC under contract S. Ordering the payments as in Lemma 2 and using equation (3.1) from Section 3.1.2 and Definition 1, these restrictions correspond to

$$\begin{aligned}EU_2(e_2(x_n)|e_2(x_n);x_n,S) - EU_2(e_j|e_2(x_n);x_n,S) &= \sum_m [\pi_m^2 - \pi_m^j]\big(u_{nm} + \sum_\ell \pi_\ell^2 \mu(u_{nm} - u_{n\ell})\big) \\ &\geqslant c(e_2(x_n)) - c(e_j) \qquad \forall e_j \in \mathcal{E}\backslash e_2(x_n) \qquad \forall n\end{aligned}$$

where recall $\pi_m^2 \equiv \pi_m(e_2(x_n))$ and $\pi_m^j \equiv \pi_m(e_j)$. The agent's expected utility under the alternative contract $\hat{S}$ corresponds to

$$\begin{aligned}EU_2(e_2(x_n)|e_2(x_n);x_n,\hat{S}) &= \sum_m \pi_m^2(u_{nm} + \rho_n) + \sum_m \sum_\ell \pi_m^2 \pi_\ell^{1,2} \mu(u_{nm} - u_{n\ell}) - c(e_2(x_n)) \\ &= EU_2(e_2(x_n)|e_2(x_n);x_n,S) + \rho_n\end{aligned}$$

Thus, for all $e_j \in \mathcal{E}\backslash e_2(x_n)$ and for all n,

$$EU_2(e_2(x_n)|e_2(x_n);x_n,\hat{S}) - EU_2(e_j|e_2(x_n);x_n,\hat{S}) = EU_2(e_2(x_n)|e_2(x_n)|e_2(x_n);x_n,S) - EU_2(e_j|e_2(x_n);x_n,S)$$

so $\hat{S}$ does not distort second period incentives and thus period-two implementing IC is satisfied under $\hat{S}$. Consider now first-period implementing IC. Using equation (3.3) from Section 3.1.2 and Definition 1, under S these restrictions correspond to

$$\begin{aligned}&EU_1(e_1, e_2(X_1)|e_1,e_2(X_1);S) - EU_1(e_j, e_2(X_1)|e_1, e_2(X_1);S) = \\ &\sum_n [\pi_n^1 - \pi_n^j]\Big\{u_n + \sum_\ell \pi_\ell^1 \mu(u_n - u_\ell) + \gamma \sum_\ell \sum_m \pi_\ell^1 \pi_m^2 \mu(u_{n\ell} - u_{\ell m}) \\ &\qquad + EU_2(e_2(x_n)|e_2(x_n);x_n,S) + \gamma \sum_\ell \pi_\ell^1 \mu\big(c(e_2(x_\ell) - c(e_2(x_n)))\big)\Big\} \geqslant c(e_1) - c(e_j) \quad \forall e_j \in \mathcal{E}\backslash e_1\end{aligned}$$

where notice that the difference between the utilities is taken for the implemented action given the reference. Under the alternative contract, the agent's first-period utility corresponds to:

$$\begin{aligned}EU_1(e_1,e_2(X_1)|e_1,e_2(X_1);\hat{S}) = \sum_n \pi_n^1\Big\{\bar{u} + \gamma\sum_\ell\sum_m \pi_\ell^{0,1}\pi_m^{0,2}\mu((u_{n\ell}+\rho_n)-(u_{\ell m}+\rho_\ell))\\ + EU_2(e_2(X_1)|e_2(x_n);S)+\rho_n\\ +\gamma\sum_\ell \pi_\ell^1\mu\big(c(e_2(x_\ell)-c(e_2(x_n))\big)\Big\} - c(e_1) - c(\widetilde{e}_{0,2})\end{aligned}$$

Working for the period-one implementing IC under this alternative contract and operating I have

$$\begin{aligned}&EU_1(e_1,e_2(X_1)|e_1,e_2(X_1);\hat{S}) - EU_1(e_j,e_2(X_1)|e_1,e_2(X_1);\hat{S}) =\\ &EU_1(e_1,e_2(X_1)|e_1,e_2(X_1);S) - EU_1(e_j,e_2(X_1)|e_1,e_2(X_1);S) - \sum_n[\pi_n^1-\pi_n^j]\Big\{u_n + \sum_\ell \pi_\ell^1\mu(u_n-u_\ell)\Big\}\\ &+\gamma\sum_n[\pi_n^1-\pi_n^j]\Big\{\sum_\ell\sum_m \pi_\ell^1\pi_m^2[\mu(u_{n\ell}-u_{\ell m}+\rho_n-\rho_\ell)-\mu(u_{n\ell}-u_{\ell m})]\Big\} + \sum_n[\pi_n^1-\pi_n^j]\rho_n \geqslant 0 \quad \forall e_j\in\mathcal{E}\backslash e_1\end{aligned}$$

where the first curly bracket corresponds to the incentive lost from fixing the first-period wage, the second one–weighted by γ–corresponds to the incentives gained from prospective gain-loss utility whereas the last corresponds to the incentives gained from second-period consumption utility. To simplify the notation, define $\phi(\rho_n,\rho_\ell)\equiv\mu(u_{n\ell}-u_{\ell m}+\rho_n-\rho_\ell)-\mu(u_{n\ell}-u_{\ell m})$. Then, incentives are preserved if the sequence $\{\rho_n\}_{n=1}^N$ satisfies

$$\sum_n[\pi_n^1-\pi_n^j]\Big\{u_n+\sum_\ell\pi_\ell^1\mu(u_n-u_\ell)\Big\} = \gamma\sum_n[\pi_n^1-\pi_n^j]\Big\{\sum_\ell\sum_m\pi_\ell^1\pi_m^2\phi(\rho_n,\rho_\ell)\Big\} + \sum_n[\pi_n^1-\pi_n^j]\rho_n \quad \forall e_j\in\mathcal{E}\backslash e_1 \tag{3.14}$$

Existence of such sequence can be straightly proved by showing setting the system of $J-1$ equations as $A\rho = B$–where A is a $(J-1)\times N$ matrix and B is a $(J-1)$ vector–and showing that B can be written as a linear combination of the cols of A.

Consider first the set $\widetilde{\mathcal{E}} = \{e\in\mathcal{E}|c(e)\leqslant c(e_1)\}\subseteq\mathcal{E}$ and consider period-zero planning IC. Adding and subtracting, and noticing that the equilibrium total period one and period zero coincide, is straightforward to see that the period-zero planning IC can be written as

$$\begin{aligned}&EU_0(e_1,e_2(X_1)|e_1,e_2(X_1);\hat{S}) - EU_0(e_j,e_k(X_1)|e_j,e_k(X_1);\hat{S})\\ =&\Big(EU_1(e_1,e_2(X_1)|e_1,e_2(X_1);\hat{S}) - EU_1(e_j,e_2(X_1)|e_1,e_2(X_1);\hat{S})\Big) \quad -\Big(EU_1(e_j,e_k(X_1)|e_j,e_k(X_1);\hat{S})\\ &\qquad\qquad -EU_1(e_j,e_2(X_1)|e_1,e_2(X_1);\hat{S})\Big)\end{aligned}$$

Because under the optimal sequence $\{\rho^*\}$ the implementing IC holds, it is sufficient to find conditions for the second line to be negative. For this, notice that this planning IC is necessary iff $(e_j,e_k(x_n))$ is a credible plan, i.e. if $EU_0(e_j,e_k(x_n)|e_j,e_k(x_n))\geqslant EU_0(e_j,e_2|e_j,e_k(x_n))$. Thus,

$$\begin{aligned}&EU_1(e_j,e_k(x_n)|e_j,e_k(x_n);\hat{S}) - EU_1(e_j,e_2|e_1,e_2(x_n);\hat{S})\\ \leqslant&EU_1(e_j,e_2(x_n)|e_j,e_k(x_n);\hat{S}) - EU_1(e_j,e_2(x_n)|e_j,e_k(x_n);\hat{S})\\ \leqslant&EU_0(e_j,e_2(x_n)|e_j,e_2(x_n)) - EU_1(e_j,e_2|e_1,e_2(x_n))\end{aligned}$$

where the last line follows from the fact that second-period IC holds. Thus, if

$$\gamma\leqslant\frac{\min_j\{\mu(c(e_1)-c(e_j))\}}{\max_j\{\sum_n\sum_\ell\sum_m[\pi_n^j\pi_\ell^j\pi_m^2-\pi_n^j\pi_\ell^1\pi_m^2]\phi(\rho_n^*,\rho_\ell^*)}=\bar{\gamma}_1 \tag{3.15}$$

the period-zero planning IC is implied by the period-one implementing IC. Furthermore, notice that $\bar{\gamma}_1>0$ because the numerator is positive for all j since prospective gain-loss utility is decreasing in the reference distribution.

Consider now the difference between total period-zero utility between contracts for the optimal sequence $\{\rho_n^*\}$. Modifying equation (3.4) in Section 3.1.2 and using Definition 1, I have that under contract $\hat{S}$ the total period-zero expected utility corresponds to,

$$\begin{aligned} EU_0(e_1, e_2(X_1)|e_1, e_2(X_1); \hat{S}) = & \sum_n \pi_n^1 \bar{u} + \gamma \sum_n \sum_\ell \sum_m \pi_n^1 \pi_\ell^1 \pi_m^2 \mu(u_{n\ell} - u_{\ell m} + \rho_h - \rho_\ell) \\ & + \sum_n \pi_n^1 [EU_2(e_2(x_n)|e_2(x_n); x_n; S) + \rho_n] \\ & - c(\widetilde{e}_{0,1}) + \gamma \sum_n \sum_\ell \pi_n^{0,1} \pi_\ell^{0,1} \mu(c(e_2(x_\ell)) - c(e_2(x_n))) \end{aligned} \tag{3.16}$$

Taking differences between equation (3.4) in Section 3.1.2 and equation (3.16), I have that the deviation contract gives a higher expected utility if

$$\gamma \sum_n \sum_\ell \sum_m \pi_n^1 \pi_\ell^1 \pi_m^2 \phi(\rho_n^*, \rho_\ell^*) + \sum_n \pi_n^1 \rho_n \leqslant - \sum_n \sum_\ell \pi_n^1 \pi_n^1 \sum_\ell^1 \mu(u_n - u_\ell) \tag{3.17}$$

where the RHS and LHS are positive because of Lemma 2 and because $\{\rho_n^*\}$ is increasing, respectively. Using equations (3.14) for the optimal sequence, I can rewrite equation (3.17) as

$$\sum_n [\pi_n^1 - \pi_n^j](u_n - \rho_n^*) - \sum_n \sum_\ell^1 \mu(u_n - u_\ell) \geqslant \gamma [- \sum_n \sum_\ell \sum_m \pi_n^j \pi_\ell^1 \pi_m^2 \phi(\rho_n^*, \rho_\ell^*)] \tag{3.18}$$

Thus, if $\gamma \leqslant \min\{\bar{\gamma}_1, \bar{\gamma}_2\}$ where

$$\bar{\gamma}_2 = \frac{\min_j \{\sum_n [\pi_n^1 - \pi_n^j](u_n - \rho_n^*) - \sum_n \sum_\ell^1 \mu(u_n - u_\ell)\}}{\max_j \{- \sum_n \sum_\ell \sum_m \pi_n^j \pi_\ell^1 \pi_m^2 \phi(\rho_n^*, \rho_\ell^*)\}} \tag{3.19}$$

where $\bar{\gamma}_2 > 0$ because of equation (3.14) and Lemma 6. Then $\hat{S}$ implements the action path at the same expected cost for the principal and increases the agent's total period-zero utility in the restricted set $\widetilde{\mathcal{E}}$.

I am done in the linear consumption utility case if can show that adding actions does not decrease the principal's expected cost under the contract $\hat{S}$. To see this, first notice that working for the period-zero planning IC for both contracts and operating, I have that period-zero planning IC holds under contract $\hat{S}$ can be written as

$$\begin{aligned} & EU_0(e_1, e_2(X_1)|e_1, e_2(X_1); \hat{S}) - EU_0(e_j, e_k(X_1)|e_j, e_k(X_1); \hat{S}) = \\ & EU_0(e_1, e_2(X_1)|e_1, e_2(X_1); S) - EU_0(e_j, e_k(X_1)|e_j, e_k(X_1); S) - \sum_n [\pi_n^1 - \pi_n^j]\{u_n - \rho_n^*\} \\ & - \sum_n \sum_\ell [\pi_n^1 \pi_\ell^1 - \pi_n^j \pi_\ell^j] \mu(u_n - u_\ell) + \gamma \sum_n \sum_\ell \sum_m [\pi_n^1 \pi_\ell^1 \pi_m^2 - \pi_n^j \pi_\ell^j \pi_m^k] \phi(\rho_n^*, \rho_\ell^*) \geqslant 0 \quad \forall (e_j, e_k(X_1)) \in \mathcal{E}^{PE} \end{aligned} \tag{3.20}$$

where $\mathcal{E}^{PE}$ is as in Definition 1. By contradiction, assume that there exists a $e_j \in \mathcal{E} \backslash \widetilde{\mathcal{E}}$ such that

$$\begin{aligned} & EU_1(e_j, e_2(X_1)|e_1, e_2(X_1); \hat{S}) - EU_1(e_1, e_2(X_1)|e_1, e_2(X_1); \hat{S}) = \\ & EU_1(e_j, e_2(X_1)|e_1, e_2(X_1); S) - EU_1(e_1, e_2(X_1)|e_1, e_2(X_1); S) - \sum_n [\pi_n^j - \pi_n^1] \Big\{ u_n + \sum_\ell \pi_\ell^1 \mu(u_n - u_\ell) \Big\} \\ & + \gamma \sum_n [\pi_n^j - \pi_n^1] \Big\{ \sum_\ell \sum_m \pi_\ell^1 \pi_m^2 \phi(\rho_n^*, \rho_\ell^*) + \sum_n [\pi_n^j - \pi_n^1] \rho_n^* \geqslant 0 \end{aligned}$$

however, since S implements the action path I know that

$$EU_1(e_j, e_2(X_1)|e_1, e_2(X_1); S) - EU_1(e_1, e_2(X_1)|e_1, e_2(X_1); S) < 0$$

which contradicts equation (3.14). An analogous proof holds for period-zero planning IC. This completes the proof for the linear case.

Consider now the case where $u^{''} < 0$. Let $EC(S)$ and $EC(\hat{S})$ correspond to the expected cost of each contract. First, notice that if $\hat{S}$ is the optimal contract, given $\{\rho_n^*\}$, the fixed-wage u^* that maximizes the principal's profits corresponds to

$$u^* = U_R + c(e_1) - \gamma \sum_n \sum_\ell \sum_m \pi_n^1 \pi_\ell^1 \pi_m^2 \mu(u_{n\ell} - u_{\ell m} + \rho_h^* - \rho_\ell^*) - \sum_n \pi_n^1 EU_2(e_2(x_n)|e_2(x_n); S) < \sum_n \pi_n^1 u_n \tag{3.21}$$

where the last inequality follows from the IR under contract S. Thus the difference between the first-period expected cost between contract S and $\hat{S}$

$$EC_1(S) - EC_1(\hat{S}) = \sum_n \pi_n^1 u^{-1}(u_n) - u^{-1}(u^*) > \sum_n \pi_n^1 u^{-1}(u_n) - u^{-1}\Big(\sum_n \pi_n^1 u_n\Big) > 0$$

To consider the difference between the second-period expected cost, recall that the optimal deviation contract sets $\sum_n \pi_n \rho_n^* = 0$. Defining the sets $N_+ = \{n|\rho_n^* \geqslant 0\}$ and $N_- = \{n|\rho_n^* < 0\}$,

$$\begin{aligned} EC_2(S) - EC_2(\hat{S}) &= \sum_n \sum_m \pi_n^1 \pi_m^2 \Big(u^{-1}(u_{nm}) - u^{-1}(u_{nm} + \rho_n^*)\Big] \\ &= \sum_m \pi_m^2 \Bigg(\sum_{n \in N_-} \pi_n^1 \Big[u^{-1}(u_{nm}) - u^{-1}(u_{nm} + \rho_n^*)\Big] - \sum_{n \in N_+} \pi_n^1 \Big[u^{-1}(u_{nm} + \rho_n^*) - u^{-1}(u_{nm})\Big]\Bigg) \end{aligned}$$

Noticing that the first term is positive, for the contract $\hat{S}$ to have a smaller expected cost than S it suffices to find a condition for the following to hold

$$\begin{aligned} \sum_{n \in N_+} \sum_m \pi_n^1 \pi_m^2 \Big[u^{-1}(u_{nm} + \rho_n^*) - u^{-1}(u_{nm})\Big] \leqslant & \sum_n \pi_n^1 u^{-1}(u_n) - u^{-1}(u^*) \\ & + \sum_{n \in N_-} \sum_m \pi_n^1 \pi_m^2 \Big[u^{-1}(u_{nm}) - u^{-1}(u_{nm} + \rho_n^*)\Big] \end{aligned}$$

where I have already showed that the right hand side is positive. For that notice that for all m, $n \in N_+$ there exists an $u_{nm}^+ \in (u_{nm}, u_{nm} + \rho_n^*)$ such that $u^{-1^{'}}(u_{nm}^+)\rho_n^* = u^{-1}(u_{nm} + \rho_n^*) - u^{-1}(u_{nm})$ where $u^{-1^{'}}(\cdot)$ is the inverse's derivative. Defining $u_{\max}^+ = \max\{\{u_{nm}^+\}_{n \in N_+, m}\}$, I can find an upper bound for the increase in the second-period expected cost,

$$\sum_{n \in N_+} \sum_m \pi_n^1 \pi_m^2 \Big[u^{-1}(u_{nm} + \rho_n^*) - u^{-1}(u_{nm})\Big] = \sum_{n \in N_+} \sum_m \pi_n^1 \pi_m^2 u^{-1^{'}}(u_{nm}^+)\rho_n^* \leqslant u^{-1^{'}}(u_{\max}^+) \sum_{n \in N_+} \pi_n^1 \rho_n^*$$

Analogously, I can define a lower bound for the second-period expected cost gain by noticing that for all m, $n \in N_-$ there exists an $u_{nm}^- \in (u_{nm} + \rho_n^*, u_{nm})$ such that $u^{-1^{'}}(u_{nm}^-)(-\rho_n^*) = u^{-1}(u_{nm}) - u^{-1}(u_{nm} + \rho_n^*)$ so that

$$u^{-1^{'}}(u_{\min}^-) \sum_{n \in N_-} \pi_n^1(-\rho_n^*) \leqslant \sum_{n \in N_-} \sum_m \pi_n^1 \pi_m^2 \Big[u^{-1}(u_{nm}) - u^{-1}(u_{nm} + \rho_n^*)\Big]$$

where $u_{\min}^- = \min\{\{u_{nm}^-\}_{n \in N_-, m}\} < u_{\max}^+$ because $\{\rho^*\}$ is nondecreasing. Now, by assumption, there exists a M_1 such that $u^{-1^{''}}(s) \leqslant M_1 \; \forall s$. Thus, using the mean value theorem again, I now have that $u^{-1^{'}}(u_{\max}^+) - u^{-1^{'}}(u_{\min}^-) \leqslant M_1(u_{\max}^+ - u_{\min}^-)$. Thus, the expected cost of $\hat{S}$ is no greater than that of S

if M_1 is such that

$$\begin{aligned} M_1 &\leqslant \frac{\sum_n \pi_n^1 u^{-1}(u_n) - u^{-1}(u^*) + u^{-1'}(u_{\min}^-)\big(\sum_{n\in N_-}\pi_n^1(-\rho_n^*) - \sum_{n\in N_+}\pi_n^1\rho_n^*\big)}{(u_{\max}^+ - u_{\min}^-)} \\ &= \frac{\sum_n \pi_n^1 u^{-1}(u_n) - u^{-1}(u^*)}{(u_{\max}^+ - u_{\min}^-)} \end{aligned}$$

where this upper bound is positive by Jensen's inequality and equation (3.21). This concludes the proof.

■

Proof of Proposition 6

The fact the contract uses contingent payments in the first period follows straight from the proof of Proposition 5 by defining $\gamma > \min\{\bar{\gamma}_1, \bar{\gamma}_2\}$. To see that the contract does not use memory if γ is big enough, suppose the contract S is such that $u_n \neq u_n^{'}$ for at least one $n \neq n^{'}$ and $u_{nm} = u_{n'm}$ for $u_n \neq u_n^{'}$ $\forall m$. Consider now a deviation contract using memory in wages, $\hat{S}$, $\hat{u}_n = u_n + \rho_n$ $\rho_n \in \mathbb{R}$, $\rho_n \neq 0$ for at least one n and $\hat{u}_{nm} \neq \hat{u}_{nm}^{'}$, $n \neq n^{'}$ for at least one m.

Consider the case of linear consumption utility. Building the first-period implementing IC and the period-zero planning IC, using a proof equivalent to that in the proof of Proposition 5, one can show that there exists a $\{\rho_n^*\}$ such that $\hat{S}$ implements the effort path. Consider now the difference between total period-zero expected utility under contract $\hat{S}$ relative to S

$$\begin{aligned} EU_0(\hat{S}) - EU_0(S) = &\sum_n \pi_n^1\rho_n^* + \sum_n\sum_\ell \pi_n^1\pi_\ell^1\left[\mu(u_n - u_\ell + (\rho_n^* - \rho_\ell^*)) - \mu(u_n - u_\ell)\right] \\ &+ \gamma\Big(\sum_n\sum_\ell\sum_m \pi_n^1\pi_\ell^1\pi_m^2\mu(\hat{u}_{n\ell} - \hat{u}_{\ell m}) + \sum_n\sum_\ell \pi_n^1\pi_\ell^1\mu(c(e_2(x_\ell)) - c_2(x_n))\Big) \\ &+ \sum_n \pi_n^1\big(EU_2(e_s(x_n)|e_2(x_n); x_n, \hat{S}) - EU_2(e_s(x_n)|e_2(x_n); x_n, S)\big) \end{aligned}$$

where $EU_0(\hat{S})$ stands for $EU_0(e_1, e_(x_n)|e_1, e_(x_n); \hat{S})$ and the same for s. Noticing that from Lemma 2 the second line is negative, then is straight to see that for any $\{\rho_n^*\}$ and any $EC(\hat{S}) - EC(S)$ there is always a γ such that $EU_0(e_1|e_1; \hat{S}) < EU_0(e_1|e_1; S)$ so that $\hat{S}$ is not a profitable deviation.

■

Proof of Proposition 7

Let $\widetilde{\pi}_m = \mathbb{P}\{\widetilde{X} = \widetilde{x}_m\} > 0$ for all $\widetilde{x}_m \in \mathcal{X}$ represent the density of the random device. Consider first the utilities the agent gets under the random contract S. Let $EU_2^2(e_2(x_n)|e_2(x_n); S)$ represent the total expected utility at the end of period-two once the agent has exerted e_2, which corresponds to

$$EU_2^2(e_2(x_n)|e_2(x_n); S) = \sum_m \widetilde{\pi}_m u_{nm} + \sum_m\sum_\ell \widetilde{\pi}_m\widetilde{\pi}_\ell\mu(u_{nm} - u_{n\ell})$$

Thus, the total expected utility at the beginning of period two having planned to exert $\widetilde{e}_{1,2}(x_n)$ at the end of period one

$$EU_2^1(e_2(x_n)|\widetilde{e}_{1,2}(x_n); S) = \sum_n \pi_n^2\Big[\sum_m \widetilde{\pi}_m u_{nm} + \sum_m\sum_\ell \widetilde{\pi}_m\widetilde{\pi}_\ell\mu(u_{nm} - u_{n\ell})\Big] - c(e_2(x_n)) + \mu\big(c(e) - c(\widetilde{e}_{1,2}(x_n))\big)$$

Thus, the second-period implementing IC for any n corresponds to

$$\sum_n[\pi_n^2 - \pi_n^j]\Big[\sum_m \widetilde{\pi}_m u_{nm} + \sum_m\sum_\ell \widetilde{\pi}_m\widetilde{\pi}_\ell\mu(u_{nm} - u_{n\ell})\Big] - \big(c(e_2(x_n)) - c(e_j)\big) - \mu(c(e_2(x_n)) - c(e_j))$$

(notice that a period-one planning IC equal the latter except for the fact that $\mu\big(c(e_2(x_n)) - c(e_j)\big) = 0\big)$. Because Proposition 5 assumptions hold, let $\bar{u}_m$ be the first-period payment depending on the first period

$\widetilde{X}$ but not on X_1. As before, the expected utility flow of the agent once the agent has executed the first-period action corresponds to $\sum_m \widetilde{\pi}_m \bar{u}_m + \sum_m \sum_\ell \widetilde{\pi}_m \widetilde{\pi}_\ell \mu(\bar{u}_m - \bar{u}_\ell)$. Thus, the total period-one expected utility of implementing action e_1 having planned corresponds to

$$\begin{aligned} EU_1^1(e_1, \widetilde{e}_{0,2}(x_n)|\widetilde{e}_{0,1}, \widetilde{e}_{0,2}(x_n); S) &= \sum_m \widetilde{\pi}_m \bar{u}_m + \sum_m \sum_\ell \widetilde{\pi}_m \widetilde{\pi}_\ell \mu(\bar{u}_m - \bar{u}_\ell) \\ &\quad + \sum_n \pi_n^1 EU_2^1(\widetilde{e}_{0,2}(x_n)|\widetilde{e}_{0,2}(x_n); x_n, S) - c(e_1) - \mu\big(c(\widetilde{e}_{0,1}) - c(e_1)\big) \end{aligned}$$

Thus for all $e_j \in \mathcal{E} \backslash e_1$ the period-one implementing IC corresponds to

$$\begin{aligned} &EU_1^1(e_1, \widetilde{e}_{0,2}(x_n)|\widetilde{e}_{0,1}, \widetilde{e}_{0,2}(x_n); S) - EU_1^1(e_j, \widetilde{e}_{0,2}(x_n)|\widetilde{e}_{0,1}, \widetilde{e}_{0,2}(x_n); S) \\ &= \sum_m \widetilde{\pi}_m \bar{u}_m + \sum_m \sum_\ell \widetilde{\pi}_m \widetilde{\pi}_\ell \mu(\bar{u}_m - \bar{u}_\ell) + \sum_n [\pi_n^1 - \pi_n^j] EU_2^1(\widetilde{e}_{0,2}(x_n)|\widetilde{e}_{0,2}(x_n); x_n, S) - c(e_1) - \mu\big(c(\widetilde{e}_{0,1}) - c(e_1)\big) \end{aligned}$$

where the period-zero planning IC equals this for $\mu\big(c(\widetilde{e}_{0,1}) - c(e_1)\big) = 0$. Then, by noticing that paying $u_{nm} = \sum_m \widetilde{\pi}_m u_{nm} + \sum_m \sum_\ell \widetilde{\pi}_m \widetilde{\pi}_\ell \mu(u_{nm} - u_{n\ell})$ and $\bar{u}_m = \sum_m \widetilde{\pi}_m \bar{u}_m + \sum_m \sum_\ell \widetilde{\pi}_m \widetilde{\pi}_\ell \mu(\bar{u}_m - \bar{u}_\ell)$ all the IC are satisfied and the agent is indifferent between the two contracts, the result follows straight from Jensen's inequality.

■

Proof of Proposition 8

I first restate the equilibrium concept. Now, period-zero planning IC does not ensure that the agent chooses the right plans at the end of period one and thus a period-one planning IC must be explicitly stated.

Definition 3 *(Modified PPE)*
Given a contract S, the effort path $(e_1^{PPE}, e_2^{PE}(X_1))$ *is a "preferred personal equilibrium" (PE) for the agent if*

$$\begin{array}{llcll} (i) & EU_0(e_1^{PE}, e_2^{PE}(X_1)|e_1^{PE}, e_2^{PE}(X_1)) & \geqslant & EU_0(e_1, e_2(X_1)|e, e_2(X_1)) & \forall (e_1, e_2(X_1)) \in \mathcal{E}^{PE} \\ (ii) & EU_1(e_1^{PE}, e_2^{PE}(X_1)|e_1^{PE}, e_2^{PE}(X_1)) & \geqslant & EU_1(e_1, e_2^{PE}(X_1)|e_1^{PE}, e_2^{PE}(X_1)) & \forall e_1 \in \mathcal{E} \\ (iii) & EU_1(e_2^{PE}(x_n)|e_2^{PE}(x_n); x_n) & \geqslant & EU_1(e_2|e_2; x_n) & \forall e_2 \in \mathcal{E}_n^{PE}, \forall x_n \\ (iv) & EU_2(e_2^{PE}(x_n)|e_2^{PE}(x_n); x_n) & \geqslant & EU_2(e_2|e_2^{PE}(x_n); x_n) & \forall e_2 \in \mathcal{E}, \forall x_n \end{array}$$

where $\mathcal{E}^{PE} \equiv \{e \in \mathcal{E}, e_2(X_1) \in \mathcal{E} \times \cdots \times \mathcal{E} | (ii), (iii)$ *and* (iv) *hold for* $e = e_1^{PE}$ *and* $e_2 = e_2(X_1)^{PE}\}$ *and* $\mathcal{E}_n^{PE} \equiv \{e \in \mathcal{E} | (iv)$ *holds for* $e = e_2^{PE}$ *given* $X_1 = x_n\}$.

First consider period-two implementing IC for any given realization of the first-period outcome $x_n \in \{x_h, x_\ell\}$,

$$\begin{aligned} EU_2(e_h|e_h; x_n) - EU_2(e_\ell|e_h; x_n) &= [\pi_h - \pi_\ell](u_{nh} - u_{n\ell}) \\ &\quad + [\pi_h(1-\pi_h) - \lambda \pi_h (1-\pi_h)](u_{nh} - u_{n\ell}) \\ &\quad - [\pi_\ell(1-\pi_h) - \lambda \pi_h (1-\pi_\ell)](u_{nh} - u_{n\ell}) - c - \mu(c) \geqslant 0 \end{aligned}$$

I now build period one planning IC. From the analysis in Section 1.7, I have that after a first period failure the change in the prospective gain-loss utility loss from planning high effort for the second period instead of low, corresponds to $\lambda \eta (\pi_h - \pi_\ell)(u_{\ell h} - u_{\ell\ell}) \geqslant 0$. Thus, the period-one planning IC after a first-period failure corresponds to

$$\begin{aligned} EU_2(e_h|e_h; x_\ell) - EU_2(e_\ell|e_\ell; x_\ell) &= [\pi_h - \pi_\ell](u_{\ell h} - u_{\ell\ell}) \\ &\quad + [\pi_h(1-\pi_h) - \lambda \pi_h (1-\pi_h)](u_{\ell h} - u_{\ell\ell}) \\ &\quad - [\pi_\ell(1-\pi_h) - \lambda \pi_h (1-\pi_\ell)](u_{\ell h} - u_{\ell\ell}) \\ &\quad + \lambda \gamma (\pi_h - \pi_\ell)(u_{\ell h} - u_{\ell\ell}) - c \geqslant 0 \end{aligned}$$

Comparing the planning and implementing ICs, one can see that if

$$c(1-\gamma) > (\pi_h - \pi_\ell)[\lambda(1-\pi_\ell-\gamma)-\pi_\ell] \tag{3.22}$$

then the planning IC implies the implementing IC and thus it must bind (in the opposite case the implementing IC binds). Equivalently, after a first first-period success, the change in prospective gain-loss utility from planning high effort for the second period instead of low, corresponds to $\eta(\pi_h-\pi_\ell)(u_{hh}-u_{h\ell}) \geqslant 0$. Thus, period-one planning IC after a first-period success corresponds to

$$\begin{aligned} EU_2(e_h|e_h;x_h) - EU_2(e_\ell|e_\ell;x_h) = {} & [\pi_h-\pi_\ell](u_{hh}-u_{h\ell}) \\ & + [\pi_h(1-\pi_h) - \lambda\pi_h(1-\pi_h)](u_{hh}-u_{h\ell}) \\ & - [\pi_\ell(1-\pi_h) - \lambda\pi_h(1-\pi_\ell)](u_{hh}-u_{h\ell}) \\ & + \gamma\eta(\pi_h-\pi_\ell)(u_{hh}-u_{h\ell}) - c \geqslant 0 \end{aligned}$$

where have used that the principal wants to implement high effort in both periods. Comparing the planning and implementing ICs, one can see that if

$$c(1-\gamma) > (\pi_h - \pi_\ell)[(1-\pi_\ell-\gamma)-\pi_\ell] \tag{3.23}$$

then the planning IC implies the implementing IC and thus it must bind (in the opposite case the implementing IC binds). Thus, four cases arise here. If both implementing IC's bind, because their RHS are equal, then by making them equal is straight to see that $(u_{\ell h}-u_{\ell\ell}) = (u_{hh}-u_{h\ell})$. If both planning IC's bind, doing the same, one can see that $(u_{\ell h}-u_{\ell\ell}) < (u_{hh}-u_{h\ell})$. The other two cases follow using the same logic and using the conditions in equations (3.22) and (3.23).

■

3.3 Chapter 2 Extra Propositions

Proposition 15 *(No Reference-dependent Anticipatory Utility)*
Suppose $\gamma^{RD}_{T-t+1} = 0$ for $1 \leqslant t \leqslant T$. Then the optimal path of beliefs is $\widetilde{\theta}^*_t = 1$ $\forall 1 \leqslant t < T$ and $\widetilde{\theta}^*_T = 0$.

Proof Straight By backward induction, FOC for each self are:

$$\begin{aligned} \frac{\partial U_1}{\partial \widetilde{\theta}_1} &= \gamma^{RI} u > 0 \\ \frac{\partial U_1}{\partial \widetilde{\theta}_t} &= \gamma^{RI} u > 0 && \forall \quad 1 < t < T \\ \frac{\partial U_1}{\partial \widetilde{\theta}_T} &= \gamma^{RI} u - B(\theta) < 0 \end{aligned}$$

where the last inequality follows straight from Assumption 3.

■

Proposition 16 *(No Reference-independent Anticipatory Utility)*
Assume Proposition 9 assumptions hold. Let $\gamma^{RI} = 0$. Then, for any given parameter sequences the optimal sequence of beliefs cannot be decreasing.

Proof Let $\{\widetilde{\theta}^*_t\}^T_{t=1}$ denote the optimal path of beliefs such that $\widetilde{\theta}^*_t \geqslant \widetilde{\theta}^*_{t+1}$. Let $\widetilde{\theta}^d_t \equiv \widetilde{\theta}^*_t - \triangle$ where $\triangle > 0$ define self's t deviation. I prove that $\widetilde{\theta}^d_t$ is an optimal deviation and therefore $\{\widetilde{\theta}^*_t\}^T_{t=1}$ can not be an optimal path of beliefs. The utility self t gets setting $\widetilde{\theta}^*_t$ corresponds to $U_t(\widetilde{\theta}^*_t) = \sum_{s=t}^T \gamma^{RD}_{T-s+1}\mu((\widetilde{\theta}^*_s -$

$\widetilde{\theta}^*_{s-1})u) + \gamma^{RD}[\theta(1-\widetilde{\theta}^*_T)\mu_+ + (1-\theta)\widetilde{\theta}^*_T\mu_-]$, where $\sum_{s=t}^T \gamma^{RD}_{T-t+1}\mu((\widetilde{\theta}^*_s - \widetilde{\theta}^*_{s-1})u) < 0$ because the optimal path is decreasing. Consider now the deviation. Since selves are still strategic complements, decreasing the optimal beliefs decreases all the other beliefs. This implies that self t total belief corresponds to $U_t(\widetilde{\theta}^d_t) = \sum_{s=t}^T \gamma^{RD}_{T-s+1}\mu((\widetilde{\theta}^*_s(\widetilde{\theta}^d_t) - \widetilde{\theta}^*_{s-1}(\widetilde{\theta}^d_t))u) + \gamma^{RD}[\theta(1-\widetilde{\theta}^*_T(\widetilde{\theta}^d_t))\mu_+ + (1-\theta)\widetilde{\theta}^*_T(\widetilde{\theta}^d_t)\mu_-]$ where $\sum_{s=t}^T \gamma^{RD}_{T-s+1}\mu((\widetilde{\theta}^*_s(\widetilde{\theta}^*_t) - \widetilde{\theta}^*_{s-1}(\widetilde{\theta}^*_t))u) = \sum_{s=t}^T \gamma^{RD}_{T-s+1}\mu((\widetilde{\theta}^*_s(\widetilde{\theta}^d_t) - \widetilde{\theta}^*_{s-1}(\widetilde{\theta}^d_t))u)$ because agents are 1:1 strategic complements. However, $\gamma^{RD}[\theta(1-\widetilde{\theta}^*_T(\widetilde{\theta}^*_t))\mu_+ + (1-\theta)\widetilde{\theta}^*_T(\widetilde{\theta}^*_t)\mu_-] < \gamma^{RD}[\theta(1-\widetilde{\theta}^*_T(\widetilde{\theta}^d_t))\mu_+ + (1-\theta)\widetilde{\theta}^*_T(\widetilde{\theta}^d_t)\mu_-]$ because $\widetilde{\theta}^*_T(\widetilde{\theta}^*_t) > \widetilde{\theta}^*_T(\widetilde{\theta}^d_t)$. Therefore $U_t(\widetilde{\theta}^*_t) < U_t(\widetilde{\theta}^d_t)$.

■

Proposition 17 *(No Wiggles Once Perfect Pessimism is Reached)*
Assume Proposition 9 assumptions hold. If
(1) $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}$ *and* $\lambda(u) \geqslant \max\left\{\frac{\gamma^{RD}_1 - \gamma^{RI}_2}{\gamma^{RD}_2}, \frac{\gamma^{RD}_{T-1} - \gamma^{RI}_T}{\gamma^{RD}_T}\right\}$ *or*
(2) $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < \gamma^{RI}_{T-t+2}$
then if $\widetilde{\theta}^*_{t_0} = 0$ *for any* $t_0 \in \{1, \ldots, T-1\}$, $\widetilde{\theta}^*_t = 0$ $\forall t > t_0$.

Proof If $\widetilde{\theta}^*_{t_0} = 0$ then by Lemma 4 it must be the case that for any $\widetilde{\theta}^*_{t_0-1}$, $\frac{B(\theta)}{u} \geqslant \gamma^{RD}_{T-t_0+1}\mu'(-\widetilde{\theta}^*_{t_0-1}u) + \sum_{s=t_0}^T \gamma^{RI}_{T-s+1}$ which implies $\frac{B(\theta)}{u} \geqslant \gamma^{RD}_{T-t_0+1}\mu'(0) + \sum_{s=t_0}^T \gamma^{RI}_{T-s+1}$ since $\mu'(0) < \mu'(-x)$ $\forall x \in (0, u]$. Now, $\widetilde{\theta}^*_{t_0+1} = 0$ iff $\frac{B(\theta)}{u} \geqslant \gamma^{RD}_{T-t_0}\mu'(0) + \sum_{s=t_0+1}^T \gamma^{RI}_{T-s+1}$. Therefore, it suffices to show that

$$\begin{aligned} \gamma^{RD}_{T-t_0+1}\mu'(0) + \sum_{s=t_0}^T \gamma^{RI}_{T-s+1} &> \gamma^{RD}_{T-t_0}\mu'(0) + \sum_{s=t_0+1}^T \gamma^{RI}_{T-s+1} \\ \Leftrightarrow \quad \gamma^{RD}_{T-t_0+1}\mu'(0) - \gamma^{RD}_{T-t_0}\mu'(0) &> \sum_{s=t_0+1}^T \gamma^{RI}_{T-s+1} - \sum_{s=t_0}^T \gamma^{RI}_{T-s+1} \\ \Leftrightarrow \quad \gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} &< \gamma^{RI}_{T-t+2} \end{aligned}$$

where in the last line I have moved back one period and have used the normalization $\mu'(0)$. Therefore, $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} < \gamma^{RI}_{T-t+2}$ is a sufficient condition.

Now consider the case where $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}$. I need to find conditions such that the sufficient condition $\gamma^{RD}_{T-t_0+1}\mu'(-\widetilde{\theta}^*_{t_0-1}u) + \sum_{s=t_0}^T \gamma^{RI}_{T-s+1} > \gamma^{RD}_{T-t_0}\mu'(0) + \sum_{s=t_0+1}^T \gamma^{RI}_{T-s+1}$ holds. For that notice that because of Assumption *5.4*, for any given $\widetilde{\theta}^*_{t_0-1}$ it must be the case that

$$\mu'(-\widetilde{\theta}^*_{t_0-1}u) = \lambda(\widetilde{\theta}^*_{t_0-1})\mu'(\widetilde{\theta}^*_{t_0-1}) < \lambda(\widetilde{\theta}^*_{t_0-1})\mu'(0) < \lambda(u)\mu'(0)$$

therefore I can write the sufficient condition as

$$\gamma^{RD}_{T-t_0+1}\lambda(u)\mu'(0) + \sum_{s=t_0}^T \gamma^{RI}_{T-s+1} > \gamma^{RD}_{T-t_0}\mu'(0) + \sum_{s=t_0+1}^T \gamma^{RI}_{T-s+1}$$

Using the normalization $\mu'(0) = 1$ and moving forward one period to make the notation familiar, the latter condition implies a lower bound for the loss aversion parameter of the shape:

$$\lambda(u) > \frac{\gamma^{RD}_{T-t_0+1} - \gamma^{RI}_{T-t_0+2}}{\gamma^{RD}_{T-t_0+2}} > 1$$

where the last inequality comes from the fact I are assuming $\gamma^{RD}_{T-t+1} - \gamma^{RD}_{T-t+2} \geqslant \gamma^{RI}_{T-t+2}$ $\forall t$. The proof for the lower bound over loss aversion mimics that in Proposition 10.

■

3.4 Chapter 2 Proofs

Proof of Lemma 3

For any two given sequences of beliefs $\{\widetilde{\theta}^0\}$ and $\{\widetilde{\theta}^1\}$ and $\alpha \in [0,1]$ it is straightforward to prove that

$$\alpha \left[\sum_{s=t}^{T+1} v_t(\widetilde{\theta}^0)\right] + (1-\alpha)\left[\sum_{s=t}^{T+1} v_t(\widetilde{\theta}^1)\right] \leqslant \sum_{s=t}^{T+1} v_t(\alpha\widetilde{\theta}^0 + (1-\alpha)\widetilde{\theta}^1)$$

which is a straight consequence of the concavity assumption of $\mu(\cdot)$ in Assumption 5. Consequently, the FOC are necessary and sufficient for the agent's problem. Since $[0,1]$ is compact, then the associated solution to (2.3) exists and is unique. By Kuhn's Theorem its is known that any sequential game with perfect information has a subgame perfect equilibrium. Moreover, such equilibrium in pure strategies is unique because the agent optimal strategy is unique (selves are not indifferent between beliefs, there is a unique maximum).

■

Proof of Lemma 4

By backward induction. Having observed $h_{T-1} \equiv \{\widetilde{\theta}_t^*\}_{t=1}^{T-1}$, self T solves

$$\begin{aligned} \max\nolimits_{\widetilde{\theta}} U_T = \quad & \gamma_1^{RI}\widetilde{\theta}_T u + \theta u + \gamma_1^{RD}((\widetilde{\theta}_T - \widetilde{\theta}_{T-1}^*)u) + \gamma^{RD}[\theta(1-\widetilde{\theta}_T)\mu_+ + (1-\theta)\widetilde{\theta}_T\mu_-] \\ \text{s/t} \quad & 0 \leqslant \widetilde{\theta}_T \leqslant 1 \end{aligned}$$

The FOC corresponds to

$$\frac{\partial U_T}{\partial \widetilde{\theta}_T} = \gamma_1^{RI} u + \gamma_1^{RD}\mu'((\widetilde{\theta}_T - \widetilde{\theta}_{T-1}^*)u)u - B(\theta)$$

and the SOC ensures a maximum because of Assumption 5 ($\gamma_1^{RD}\mu''((\widetilde{\theta}_T^* - \widetilde{\theta}_{T-1}^*)u)u^2 \leqslant 0 \ \forall\ \widetilde{\theta}_{T-1}^*$). In case of an interior condition $\frac{\partial U_T}{\partial \widetilde{\theta}_T} = 0$ and the optimal belief corresponds to:

$$\widetilde{\theta}_T^* = \widetilde{\theta}_{T-1}^* + \frac{1}{u}\mu'^{-1}(A_T) \qquad \text{where} \qquad A_T \equiv \frac{B(\theta) - \gamma_1^{RI}u}{\gamma_1^{RD}u}$$

A corner solution in 0 will occur if the marginal cost is high enough in equilibrium $B(\theta) > \gamma_1^{RI}u + \gamma_1^{RD}\mu'((\widetilde{\theta}_T^* - \widetilde{\theta}_{T-1}^*)u)u = \gamma_1^{RI}u + \gamma_1^{RD}\mu'(-\widetilde{\theta}_{T-1}^* u)u$. Equivalently, a corner solution in 1 will occur if the marginal cost of optimism is low enough $B(\theta) < \gamma_1^{RI}u + \gamma_1^{RD}\mu'((\widetilde{\theta}_T^* - \widetilde{\theta}_{T-1}^*)u)u = \gamma_1^{RI}u + \gamma_1^{RD}\mu'((1 - \widetilde{\theta}_{T-1}^*)u)u$.

Self T-1 uses this strategy to compute her own. Her problem, having observed $h_{T-2} \equiv \{\widetilde{\theta}_t^*\}_{t=1}^{T-2}$ and knowing self T optimal reaction $\widetilde{\theta}_T^* \equiv s_T^*(\widetilde{\theta}_{T-1})$, corresponds to:

$$\begin{aligned} \max_{\widetilde{\theta}} U_{T-1} \quad = \quad & \gamma_2^{RI}\widetilde{\theta}_{T-1} + \gamma_1^{RI}s_T^*(\widetilde{\theta}_{T-1})u + \theta u + \gamma_2^{RD}((\widetilde{\theta}_{T-1} - \widetilde{\theta}_{T-2}^*)u) \\ + \quad & \gamma_1^{RD}((s_T^*(\widetilde{\theta}_{T-1}) - \widetilde{\theta}_{T-1}^*)u) + \gamma^{RD}[\theta(1 - s_T^*(\widetilde{\theta}_{T-1}))\mu_+ + (1-\theta)s_T^*(\widetilde{\theta}_{T-1})\mu_-] \\ \text{s/t} \quad & 0 \leqslant \widetilde{\theta}_{T-1} \leqslant 1 \end{aligned}$$

First assume $\widetilde{\theta}_{T-1}^*$ is such that $\gamma_1^{RI}u + \gamma_1^{RD}\mu'((1-\widetilde{\theta}_{T-1}^*)u)u \leqslant B(\theta) \leqslant \gamma_1^{RI}u + \gamma_1^{RD}\mu'(-\widetilde{\theta}_{T-1}^* u)u$, therefore an interior solution for self T holds. Then, self T-1 problem reduces to

$$\begin{aligned} \max_{\widetilde{\theta}} U_{T-1} \quad = \quad & (\gamma_1^{RI} + \gamma_2^{RI})\widetilde{\theta}_{T-1} + \gamma_2^{RD}((\widetilde{\theta}_{T-1} - \widetilde{\theta}_{T-2}^*)u) - \widetilde{\theta}_{T-1}B(\theta) + \Omega \\ \text{s/t} \quad & 0 \leqslant \widetilde{\theta}_{T-1} \leqslant 1 \end{aligned}$$

where $\Omega = \gamma_1^{RD}\frac{1}{u}\mu'^{-1}(A_T) + \theta u + \gamma_1^{RD}\mu(\mu'^{-1}(A_T)) + \gamma^{RD}\theta\mu_+$ corresponds to all constant terms. The FOC for this maximization problem corresponds to

$$\frac{\partial U_{T-1}}{\partial\widetilde{\theta}_{T-1}} = (\gamma_1^{RI} + \gamma_2^{RI})u + \gamma_2^{RD}\mu'((\widetilde{\theta}_{T-1} - \widetilde{\theta}^*_{T-2})u)u - B(\theta)$$

and the SOC ensures a maximum because of Assumption 5 ($\gamma_2^{RD}\mu''((\widetilde{\theta}^*_{T-1} - \widetilde{\theta}^*_{T-2})u)u^2 \leqslant 0\ \forall\ \widetilde{\theta}^*_{T-2}$). In case of an interior condition $\frac{\partial U_{T-1}}{\partial\widetilde{\theta}_{T-1}} = 0$ and the optimal belief corresponds to:

$$\widetilde{\theta}^*_{T-1} = \widetilde{\theta}^*_{T-2} + \frac{1}{u}\mu'^{-1}(A_{T-1}) \qquad \text{where} \qquad A_{T-1} \equiv \frac{B(\theta) - (\gamma_1^{RI} + \gamma_2^{RI})u}{\gamma_2^{RD}u}$$

A corner solution in 0 will occur if the marginal cost is high enough in equilibrium $B(\theta) > (\gamma_1^{RI} + \gamma_2^{RI})u + \gamma_2^{RD}\mu'((\widetilde{\theta}^*_{T-1} - \widetilde{\theta}^*_{T-2})u)u = (\gamma_1^{RI} + \gamma_2^{RI})u + \gamma_1^{RD}\mu'(-\widetilde{\theta}^*_{T-2}u)u$. Equivalently, a corner solution in 1 will occur if the marginal cost of optimism is low enough $B(\theta) < (\gamma_1^{RI} + \gamma_2^{RI})u + \gamma_2^{RD}\mu'((\widetilde{\theta}^*_{T-1} - \widetilde{\theta}^*_{T-2})u)u = (\gamma_1^{RI} + \gamma_2^{RI})u + \gamma_2^{RD}\mu'((1 - \widetilde{\theta}^*_{T-2})u)u$.

Finally, I prove that the previous conditions over $B(\theta)$ still hold in case there is a corner solution in period T. To see this, first assume $\widetilde{\theta}^*_{T-1}$ is such that $\widetilde{\theta}^*_T = 0$, which is true if and only if $\frac{B(\theta)}{u} \geqslant \gamma_1^{RD}\mu'(-\widetilde{\theta}^*_{T-1}u) + \gamma_1^{RI}$. Self $T-1$ problem is then

$$\begin{aligned} \max_{\widetilde{\theta}} U_{T-1} = \quad & \gamma_2^{RI}\widetilde{\theta}_{T-1}u + \theta u + \gamma_2^{RD}\mu((\widetilde{\theta}_{T-1} - \widetilde{\theta}^*_{T-2})u) + \gamma_1^{RD}(-\widetilde{\theta}^*_{T-1}u) + \gamma^{RD}\theta\mu_+ \\ \text{s/t} \quad & 0 \leqslant \widetilde{\theta}_{T-1} \leqslant 1 \end{aligned}$$

The FOC corresponds to

$$\frac{\partial U_{T-1}}{\partial\widetilde{\theta}_T} = \gamma_2^{RI}u + \gamma_2^{RD}\mu'((\widetilde{\theta}_{T-1} - \widetilde{\theta}^*_{T-2})u)u - \gamma_1^{RD}\mu'(-\widetilde{\theta}^*_{T-1}u)u$$

Then I have three cases. First, $\widetilde{\theta}^*_{T-1} = 0$ which is true if and only if $\gamma_2^{RI} + \gamma_2^{RD}\mu'(-\widetilde{\theta}^*_{T-2}u) < \gamma_1^{RD}\mu'(0)$. Directly from the condition $\widetilde{\theta}^*_T = 0$ I have that $\frac{B(\theta)}{u} - \gamma_1^{RI} \geqslant \gamma_1^{RD}\mu'(0)$, therefore by transitivity I have that $\gamma_2^{RI} + \gamma_2^{RD}\mu'(-\widetilde{\theta}^*_{T-2}u) < \frac{B(\theta)}{u} - \gamma_1^{RI} \Leftrightarrow (\gamma_1^{RI} + \gamma_2^{RI}) + \gamma_2^{RD}\mu'(-\widetilde{\theta}^*_{T-2}u) < \frac{B(\theta)}{u}$ which is the same condition I obtained for an interior solution for the last period. Second, $\widetilde{\theta}^*_{T-1} = 1$, which will happen if and only if $\gamma_2^{RI} + \gamma_2^{RD}\mu'(-\widetilde{\theta}^*_{T-2}u) > \gamma_1^{RD}\mu'(0)$. Directly from the condition $\widetilde{\theta}^*_T = 0$ I have that $\frac{B(\theta)}{u} - \gamma_1^{RI} \geqslant \gamma_1^{RD}\mu'(0)$ therefore is sufficient for $\widetilde{\theta}^*_{T-1} = 1$ that $\gamma_2^{RI} + \gamma_2^{RD}\mu'(-\widetilde{\theta}^*_{T-2}u) > \frac{B(\theta)}{u} - \gamma_1^{RI}$ $\Leftrightarrow (\gamma_1^{RI} + \gamma_2^{RD}) + \gamma_2^{RD}\mu'(-\widetilde{\theta}^*_{T-2}u) > \frac{B(\theta)}{u}$ which is the same condition I obtained for an interior solution for the last period. Third consider the case $\widetilde{\theta}^*_{T-1} = 0 \in (0,1)$,which happens if and only if $\gamma^{RI} + \gamma_2^{RD}\mu'((\widetilde{\theta}^*_{T-1} - \widetilde{\theta}^*_{T-2})u) = \gamma_1^{RD}\mu'(-\widetilde{\theta}^*_{T-1})$. Once more straight from the fact that $\widetilde{\theta}^*_T = 0$ I have that it must be the case that $\gamma_2^{RI} + \gamma_2^{RD}\mu'((\widetilde{\theta}^*_{T-1} - \widetilde{\theta}^*_{T-2})u) = \gamma_1^{RD}\mu'(-\widetilde{\theta}^*_{T-1}) \leqslant \frac{B(\theta)}{u} - \gamma_1^{RI}$ which contains the equality case.

The proof that the conditions obtained assuming an interior solution are also valid in the case $\widetilde{\theta}^*_T = 1$ is analogous. The lemma then follows by induction.

■

Proof of Proposition 9
Because of Lemma 4, in an interior solution the optimal path of beliefs is described by

$$\widetilde{\theta}^*_t = \widetilde{\theta}^*_{t-1} + \frac{1}{u}\mu'^{-1}(A_t) \qquad \forall t$$

where

$$A_t \equiv \frac{B(\theta) - \sum_{s=t}^{T} \gamma_{T-s+1}^{RI} u}{\gamma_{T-t+1}^{RD} u} \qquad \text{and} \qquad B(\theta) \equiv \gamma^{RD}[\theta\mu_+ + (1-\theta)|\mu_-|]$$

The optimal path of beliefs will be decreasing iff $\mu'(0) \leqslant A_t(\gamma, \theta)\ \forall t$ because this implies $\mu'^{-1}(A_t(\gamma, \theta)) \leqslant 0\ \forall t$ since $\mu'^{-1}(\cdot)$ is strictly decreasing and negative for arguments below $\mu'(0)$. Using the definition of A_t, this condition can be rewritten as

$$\frac{B(\theta)}{u} \geqslant \gamma_{T-t+1}^{RD}\mu'(0) + \sum_{s=t}^{T} \gamma_{T-s+1}^{RI} \quad \forall t \tag{3.24}$$

Rename the RHS of (3.24) as $z_t \equiv \gamma_{T-t+1}^{RD}\mu'(0) + \sum_{s=t}^{T} \gamma_{T-s+1}^{RI}$. First consider the case $\gamma_{T-t+1}^{RD} - \gamma_{T-t+2}^{RD} \geqslant \frac{\gamma_{T-t+2}^{RI}}{\mu'(0)}$. In this case z_t is increasing in time because

$$\begin{aligned}
\gamma_{T-t+1}^{RD}\mu'(0) - \gamma_{T-t+2}^{RD}\mu'(0) &\geqslant \gamma_{T-t+2}^{RI} = \sum_{s=t-1}^{T} \gamma_{T-s+1}^{RI} - \sum_{s=t}^{T} \gamma_{T-s+1}^{RI} \\
\Leftrightarrow \gamma_{T-t+1}^{RD}\mu'(0) + \sum_{s=t}^{T} \gamma_{T-s+1}^{RI} &\geqslant \gamma_{T-t+2}^{RD}\mu'(0) + \sum_{s=t-1}^{T} \gamma_{T-t+1}^{RI} \\
\Leftrightarrow z_t &\geqslant z_{t-1} \qquad \forall t
\end{aligned}$$

Therefore $\max\{z_1, \ldots, z_T\} = z_T = \gamma_1^{RD}\mu'(0) + \gamma_1^{RI}$. Consequently, a necessary and sufficient condition for the optimal path of beliefs to be decreasing is

$$\frac{B(\theta)}{u} \geqslant \gamma_1^{RD}\mu'(0) + \gamma_1^{RI}$$

Now assume the sequence $\{\gamma^{RD}\}$ grows slow in time, i.e., $\gamma_{T-t+1}^{RD} - \gamma_{T-t+2}^{RD} < \frac{\gamma_{T-t+2}^{RI}}{\mu'(0)}$. In this case z_t is decreasing in time by analogous reasons to the first case. Therefore $\max\{z_1, \ldots, z_T\} = z_1 = \gamma_T^{RD}\mu'(0) + \sum_{s=1}^{T} \gamma_{T-s+1}^{RI}$. Consequently, a necessary and sufficient condition for the optimal path of beliefs to be decreasing is

$$\frac{B(\theta)}{u} \geqslant \gamma_T^{RD}\mu'(0) + \sum_{s=1}^{T} \gamma_{T-s+1}^{RI}$$

■

Proof of Proposition 10

First note that if $\gamma^{RD} = 0$ then it must be the case that $\widetilde{\theta}_T^* = 1$ since for any given reference belief $\widetilde{\theta}_{T-1}^*$ I have $\frac{\partial U_T}{\partial \widetilde{\theta}_T} = \gamma_T^{RI} u + \gamma_T^{RD}\mu'(\widetilde{\theta}_T - \widetilde{\theta}_{T-1}^*)u > 0$.

I will proceed by contradiction. Assume the path of beliefs is not increasing, in particular, without loss of generality assume $0 < \widetilde{\theta}_{T-1}^* < \widetilde{\theta}_{T-2}^*$, that is, the path of beliefs has a decreasing part. I will find conditions such that it is optimal for self $t-1$ to deviate and increase her belief. Consider how does the indirect utility changes when self t increases her belief:

$$\frac{\partial U_{T-1}^*}{\partial \widetilde{\theta}_{t-1}^*} = \gamma_2^{RI} u + \gamma_2^{RD}\mu'((\widetilde{\theta}_{T-1}^* - \widetilde{\theta}_{T-2}^*)u)u - \gamma_1^{RD}\mu'((1 - \widetilde{\theta}_{T-1}^*)u)u$$

I need to show that

$$\gamma_2^{RI} + \gamma_2^{RD}\mu'((\widetilde{\theta}_{T-1}^* - \widetilde{\theta}_{T-2}^*)u) \geqslant \gamma_1^{RD}\mu'((1 - \widetilde{\theta}_{T-1}^*)u) \tag{3.25}$$

Notice that a lower bound for the LHS of (3.25) corresponds to $\gamma_2^{RI}+\gamma_2^{RD}\mu'(0)$ (since $\widetilde{\theta}^*_{T-1}-\widetilde{\theta}^*_{T-2}<0$) and an upper bound for the RHS corresponds to $\gamma_1^{RD}\mu'(0)$ (since $1-\widetilde{\theta}^*_{T-1}>0$), therefore it is sufficient to show that $\gamma_2^{RI}+\gamma_2^{RD}\mu'(0)>\gamma_1^{RD}\mu'(0)$. Imposing the normalization $\mu'(0)=1$, then this equivalent to $\gamma_2^{RI}+\gamma_2^{RD}>\gamma_1^{RD}\Leftrightarrow\gamma_1^{RD}-\gamma_2^{RD}<\gamma_2^{RI}$. Therefore, $\{\gamma^{RD}\}$ growing slower than the size of $\{\gamma^{RI}\}$ is a sufficient for the agent to be profitable to increase her belief. This assumption is in fact sufficient for the agent to set $\{\widetilde{\theta}^*_t\}_{t=1}^T=\{1\}_{t=1}^T$. To see this, go back to period T-1 FOC. The agent will choose $\widetilde{\theta}^*_{T-1}$ if and only if:

$$\gamma_2^{RI}+\gamma_2^{RD}\mu'((\widetilde{\theta}^*_{T-1}-\widetilde{\theta}^*_{T-2})u)>\gamma_1^{RD}\mu'((1-\widetilde{\theta}^*_{T-1})u)$$

I just proved it must be the case $\widetilde{\theta}^*_{T-1}-\widetilde{\theta}^*_{T-2}\geqslant 0$, therefore looking for a lower bound for the LHS and an upper bound for the RHS, I get the following sufficient condition:

$$\gamma_2^{RI}+\gamma_2^{RD}\mu'(u)>\gamma_1^{RD}\mu'(0)$$

which necessarily holds since it is implied by $\gamma_1^{RD}-\gamma_2^{RD}<\gamma_2^{RI}$. To see this last implication, notice that $\gamma_1^{RD}-\gamma_2^{RD}<\gamma_2^{RI}\Leftrightarrow\gamma_1^{RD}\mu'(0)<\gamma_2^{RI}+\gamma_2^{RD}\mu'(0)$ and that therefore $\gamma_2^{RI}+\gamma_2^{RD}\mu'(u)$ is lower upper bound for $\gamma_1^{RD}\mu'(0)$. By induction it must be the case optimal beliefs are all perfectly optimistic.

Now consider the case $\gamma_1^{RD}-\gamma_2^{RD}\geqslant\gamma_2^{RI}$. Consider once more the case where, without loss of generality $0<\widetilde{\theta}^*_{T-1}<\widetilde{\theta}^*_{T-2}$. In this case I need to find a lower bound for the loss aversion parameter such that the agent has profitable deviation in increasing her belief. Remember that increasing $\widetilde{\theta}^*_{T-1}$ will be profitable if (3.25) holds. Using Assumption *5.4*, I rewrite this condition as:

$$\gamma_2^{RI}+\gamma_2^{RD}\lambda((\widetilde{\theta}^*_{T-2}-\widetilde{\theta}^*_{T-1})u)\mu'((\widetilde{\theta}^*_{T-2}-\widetilde{\theta}^*_{T-1})u)\geqslant\gamma_1^{RD}\mu'((1-\widetilde{\theta}^*_{T-1})u)$$

Noticing that an upper bound for $\lambda((\widetilde{\theta}^*_{T-2}-\widetilde{\theta}^*_{T-1})u)$ corresponds to $\lambda(u)$, one for $\mu'((\widetilde{\theta}^*_{T-2}-\widetilde{\theta}^*_{T-1})u)$ corresponds to $\mu'(0)$ and one for $\mu'((1-\widetilde{\theta}^*_{T-1})u)$ is $\mu'(0)$, a sufficient condition for a profitable deviation show us a sufficient lower bound for loss aversion:

$$\begin{aligned}\gamma_2^{RI}+\gamma^{RD}\lambda(u)\mu'(0) &\geqslant 0\\ \Leftrightarrow\qquad \lambda(u) &\geqslant \frac{\gamma_1^{RD}-\gamma_2^{RI}}{\gamma_2^{RD}}\end{aligned}$$

Generalizing the latter to any period t, I have that the condition for a profitable deviation is $\lambda(u)\geqslant\frac{\gamma^{RD}_{T-t+1}-\gamma^{RI}_{T-t+2}}{\gamma^{RD}_{T-t+2}}$. The behavior of the the lower bound in time depends on whether $\{\gamma^{RD}\}$ increases an an increasing or decreasing rate. To see this, notice that with some algebraic manipulation

$$\frac{\gamma^{RD}_{T-t+1}-\gamma^{RI}_{T-t+2}}{\gamma^{RD}_{T-t+2}}-\frac{\gamma^{RD}_{T-t+2}-\gamma^{RI}_{T-t+3}}{\gamma^{RD}_{T-t+3}}=\frac{\frac{\gamma^{RD}_{T-t+1}}{\gamma^{RD}_{T-t+2}}}{\frac{\gamma^{RD}_{T-t+2}}{\gamma^{RD}_{t-t+3}}}$$

which is greater than one if $\{\gamma^{RD}\}$ increases at an increasing rate and is smaller than one if increases at a decreasing rate. Therefore a lower bound for loss aversion corresponds to:

$$\lambda(u)\geqslant\max\left\{\frac{\gamma_1^{RD}-\gamma_2^{RI}}{\gamma_2^{RD}},\frac{\gamma_{T-1}^{RD}-\gamma_T^{RI}}{\gamma_T^{RD}}\right\}.$$

■

Proof of Proposition 11

Proving (1) is straight. First note $\frac{\partial B(\theta)}{\partial \theta} = \gamma^{RD}[\mu_+ - |\mu_-|] < 0$. Therefore:

$$\begin{aligned}
\theta_0 < \theta_1 \quad &\Leftrightarrow \quad B(\theta_0) > B(\theta_1) \\
&\Leftrightarrow \quad A_t(\theta_0) > A_t(\theta_1) \\
&\Leftrightarrow \quad \widetilde{\theta}_t^*(\theta_0) < \widetilde{\theta}_t^*(\theta_1)
\end{aligned}$$

where A_t and the optimal strategy are defined in Lemma 4. Notice that the previous to last is true under Proposition 9 assumptions. To see (2) note that $B(\theta_1)$ and $A_t(\theta_1)$ can be written as $B(\theta_1) = \gamma^{RD}[\mu_+ - |\mu_-|] + B(\theta_0)$ and $A_t(\theta_1) = \frac{\gamma^{RD}[\mu_+ - |\mu_-|]}{\gamma^{RD}_{T-t+1} u} \triangle + A_t(\theta_0) \equiv \phi \triangle + A_t(\theta_0)$, where $\triangle > 0$ and $\phi < 0$ because of loss aversion. Therefore:

$$\begin{aligned}
\widetilde{\theta}_1^*(\theta_1) - \widetilde{\theta}_1^*(\theta_0) \quad &= \quad \frac{1}{u}\left[\mu'^{-1}(\phi \triangle + A_t(\theta_0)) - \mu'^{-1}(A_t(\theta_0))\right] \\
&= \quad \frac{1}{u}\left[\mu'^{-1}(A_t(\theta_0)) + \frac{\partial \mu'^{-1}(A_t(\theta_0))}{\partial A_t}\phi \triangle - \mu'^{-1}(A_t(\theta_0))\right] \\
&= \quad \frac{1}{u}\frac{\partial \mu'^{-1}(A_t(\theta_0))}{\partial A_t}\phi\triangle > 0
\end{aligned}$$

where the second line uses a first order Taylor on $\mu'^{-1}(\phi\triangle + A_t(\theta_0))$ around $A_t(\theta_0)$. The sign comes from the fact $[\mu_+ - |\mu_-|] < 0$ and $\frac{\partial \mu'^{-1}(A_t(\theta_0))}{\partial A_t} < 0$ since $\mu'^{-1}(\cdot)$ is a decreasing function.

To check whether the agent over or under reacts to the change in θ, notice that the condition of equal reaction $\frac{1}{u}\frac{\partial \mu'^{-1}(A_t(\theta_0))}{\partial A_t}\phi = 1$ can be written as $\frac{\partial \mu'^{-1}(A_t(\theta_0))}{\partial A_t}\gamma^{RD}[\mu_+ - |\mu_-|] = u^2$. This holds with equality then the change in beliefs is equal to the change in true probabilities, if it is $>$ then the agent overreacts and if $<$ she underreacts.

■

Proof of Proposition 12

I start proving (1). First note that the agent who waits during T periods has a $\{\gamma^{RD}\}$ string equal to $\{\gamma_T^{RD}, \gamma_{T-1}^{RD}, \ldots, \gamma_1^{RD}\}$ whereas the agent who waits T-1 periods has a one period shorter string $\{\gamma_{T-1}^{RD}, \gamma_{T-2}^{RD}, \ldots, \gamma_1^{RD}\}$ equal to the last T-1 components of the first sequence. For notational simplicity assume $\mu'(0) = 1$. I start comparing the first period belief. Working the case when $\{\gamma^{RD}\}$ grows fast, $\gamma_{T-1}^{RD} - \gamma_T^{RD} \geqslant \gamma_T^{RI}$, I have

$$\begin{aligned}
\gamma_{T-1}^{RD} - \gamma_T^{RD} \quad &> \quad \gamma_T^{RI} \pm \frac{B(\theta)}{u} \pm \sum_{s=2}^{T}\gamma_{T-s+1}^{RI} \\
&= \quad \frac{(\frac{B(\theta)}{u} - \sum_{s=2}^{T}\gamma_{T-s+1}^{RI})\gamma_{T-1}^{RD}}{\gamma_{T-1}^{RD}} - \frac{(\frac{B(\theta)}{u} - \sum_{s=2}^{T}\gamma_{T-s+1}^{RI} - \gamma_T^{RI})\gamma_T^{RD}}{\gamma_T^{RD}} \\
&\equiv \quad A_1(T-1)\gamma_{T-1}^{RD} - A_1(T)\gamma_T^{RD} \\
\Leftrightarrow (A_1(T) - 1)\gamma_T^{RD} \quad &> \quad (A_1(T-1) - 1)\gamma_{T-1}^{RD} > (A_1(T-1) - 1)\gamma_T^{RD}
\end{aligned}$$

therefore

$$\begin{aligned}
A_1(T) \quad &> \quad A_1(T-1) > \mu'(0) = 1 \\
\Rightarrow \widetilde{\theta}_1^*(T) = \theta_0 + \frac{1}{u}\mu'^{-1}(A_1(T)) \quad &< \quad \theta_0 + \frac{1}{u}\mu'^{-1}(A_1(T-1)) = \widetilde{\theta}_1^*(T-1)
\end{aligned}$$

where $A_1 > \mu'(0)$ for any time horizon comes from the fact $\frac{B(\theta)}{u} \geqslant \frac{\gamma_T^{RD}\gamma_T^{RI}}{\gamma^{RD}} + \sum_{s=1}^{T}\gamma_{T-s+1}^{RI} > \gamma_1^{RD} + \gamma_1^{RI}$ (see implications in Proposition 9). The proof can be easily extended to any period $t \leqslant T-1$.

Consider now the case when $\{\gamma^{RD}\}$ grows slow, i.e., $\gamma^{RD}_{T-1} - \gamma^{RD}_{T} < \gamma^{RI}_{T}$. I want to find conditions such that $A(T) \geqslant A(T-1)$. With some algebraic manipulation it can be show that such condition is equivalent to

$$\begin{aligned} \frac{B(\theta)}{u}\left[\frac{1}{\gamma^{RD}_{T}} - \frac{1}{\gamma^{RD}_{T-1}}\right] &\geqslant \left[\frac{\sum_{s=1}^{T}\gamma^{RI}_{T-s+1}}{\gamma^{RD}_{T}} - \frac{\sum_{s=2}^{T}\gamma^{RI}_{T-s+1}}{\gamma^{RD}_{T-1}}\right] \\ \Leftrightarrow B(\theta) &\geqslant \left[\frac{\gamma^{RD}_{T-1}\sum_{s=1}^{T}\gamma^{RI}_{T-s+1} - \gamma^{RD}_{T}\sum_{s=2}^{T}\gamma^{RI}_{T-s+1}}{\gamma^{RD}_{T-1} - \gamma^{RD}_{T}}\right] \end{aligned} \tag{3.26}$$

With a little algebraic manipulation it can be shown that this condition is equivalent to:

$$B(\theta) \geqslant \gamma^{RD}_{T}\frac{\gamma^{RI}_{T}}{\gamma^{RD}_{T-1} - \gamma^{RD}_{T}} + \sum_{s=1}^{T}\gamma^{RI}_{T-s+1}$$

Notincing that in this case $\gamma^{RD}_{T-1} - \gamma^{RD}_{T} < \gamma^{RI}_{T}$, I have that:

$$B(\theta) \geqslant \gamma^{RD}_{T}\frac{\gamma^{RI}_{T}}{\gamma^{RD}_{T-1} - \gamma^{RD}_{T}} + \sum_{s=1}^{T}\gamma^{RI}_{T-s+1} \geqslant \gamma^{RD}_{T}\mu'(0) + \sum_{s=1}^{T}\gamma^{RI}_{T-s+1}$$

Therefore, the condition in (3.26) is stronger than that in Proposition 9.

I now prove (2). Form the first part recall

$$\widetilde{\theta}^{*}_{1}(T-1) - \widetilde{\theta}^{*}_{1}(T) = \frac{1}{u}\left[\mu'^{-1}(A_1(T-1)) - \mu'^{-1}(A_1(T))\right] > 0$$

Now consider next period belief:

$$\begin{aligned} \widetilde{\theta}^{*}_{2}(T-1) - \widetilde{\theta}^{*}_{2}(T) &= \widetilde{\theta}^{*}_{1}(T-1) - \widetilde{\theta}^{*}_{1}(T) + \frac{1}{u}\left[\mu'^{-1}(A_2(T-1)) - \mu'^{-1}(A_2(T))\right] \\ &= \frac{1}{u}\left[\mu'^{-1}(A_1(T-1)) - \mu'^{-1}(A_1(T))\right] + \frac{1}{u}\left[\mu'^{-1}(A_2(T-1)) - \mu'^{-1}(A_2(T))\right] \\ &> \frac{1}{u}\left[\mu'^{-1}(A_2(T-1)) - \mu'^{-1}(A_2(T))\right] \\ &= \widetilde{\theta}^{*}_{1}(T-1) - \widetilde{\theta}^{*}_{1}(T) \end{aligned}$$

where the last inequality comes from $A_t(T-1) < A_t(T)$. By induction the result follows.

■

Proof of Proposition 13

Recall that in an interior solution $\widetilde{\theta}^{*}_{t}$ corresponds to $\widetilde{\theta}^{*}_{t} = \theta_0 + \sum_{s=1}^{t}\mu'^{-1}(A_t)$ where $A_t \equiv \frac{B(\theta)-(T-t+1)\gamma^{RI}u}{\gamma^{RD}_{T-t+1}u}$. Therefore, $\frac{\partial\widetilde{\theta}^{*}_{1}}{\partial u} = \frac{\partial\mu'^{-1}(A_1)}{\partial A_1}\frac{\partial A_t}{\partial u}$ where $\frac{\partial\mu'^{-1}(A_1)}{\partial A_1} < 0$ and $\frac{\partial\widetilde{\theta}^{*}_{t}}{\partial u} = \sum_t \frac{\partial\mu'^{-1}(A_t)}{\partial A_t}\frac{\partial A_t}{\partial u}$. Consequently, the impact of the size of the reward u depends on the sign of the following derivative:

$$\begin{aligned} \frac{\partial A_t}{\partial u} &= \left[\frac{\partial B(\theta)}{\partial u}u - B(\theta)\right]\frac{1}{\gamma^{RD}_{T-t+1}u^2} \\ &= \left[\theta\left(\mu'(u) - \frac{\mu(u)}{u}\right) + (1-\theta)\left(\mu'(-u) - \frac{|\mu(-u)|}{u}\right)\right]\frac{1}{\gamma^{RD}_{T-t+1}u^2} \end{aligned}$$

For any given reward $u > 0$, take a closed interval $[0, u]$. By the Mean Value Theorem I know $\exists\, c_+ \in (0, u)$ such that $\mu'(c_+)u = \mu(u)$, where I have used the fact that $\mu(0) = 0$. Since $c_+ < u$, then this implies

$\mu'(u) < \frac{\mu(u)}{u}$. Equivalently, define a closed interval $[-u, 0]$. By the Mean Value Theorem I know $\exists$ $c_- \in (-u, 0)$ such that $\mu'(c_-)u = -\mu(-u) = |\mu(u)|$. Since $c_- > -u$, then this implies $\mu'(-u) > \frac{|\mu(-u)|}{u}$. Therefore $\frac{\partial A_t}{\partial u} > 0$ iff

$$\frac{\theta}{(1-\theta)} < \frac{\mu'(-u)u - |\mu(-u)|}{\mu(u) - \mu'(u)u}$$

where the RHS is different from zero from the mean value theorem. First, noticing that $\lim_{\theta \to 1} \frac{\theta}{(1-\theta)} = \infty$ implies the condition above cannot be true, therefore it must be the case that $\frac{\theta}{(1-\theta)} > \frac{\mu'(-u)u - |\mu(-u)|}{\mu(u)-\mu'(u)u}$ implying $\frac{\partial A_t}{\partial u} < 0$ and therefore $\frac{\partial \tilde{\theta}_t^*}{\partial u} > 0$.

Second, noticing that $\lim_{\theta \to 0} \frac{\theta}{(1-\theta)} = 0$ implies the condition above is necessarily true because of the mean value theorem, which implies the RHS of the condition is positive. Therefore $\frac{\partial A_t}{\partial u} > 0$ and thus $\frac{\partial \tilde{\theta}_t^*}{\partial u} < 0$.

■

Proof of Lemma 5

Let h represent the number of periods of exerted high effort. Using a Taylor approximation of the utility of working hard today (and thus accumulated h+1 periods of hard work) around h I have:

$$U_t^*(e_H) - U_t^*(e_L) \equiv U_t(h+1) - U_t(h) \approx \frac{\partial U_t(h+1)}{\partial h}|_h (h+1-h)$$

Now, for any given effort choice, the change in the utility evaluated in the optimal path of beliefs U_t^* from working hard today versus goofing off corresponds to (the envelope theorem):

$$\frac{\partial U_t^*(h+1)}{\partial h} = \frac{\partial U_t(h+1)}{\partial h} + \sum_{s=t}^{T} \frac{\partial U_t(h+1)}{\partial \tilde{\theta}_t^*} \frac{\partial \tilde{\theta}_s^*}{\partial h} = \frac{\partial U_t(h+1)}{\partial h}$$

Therefore

$$U_t^*(e_H) - U_t^*(e_L) \approx \frac{\partial U_t^*(h+1)}{\partial h}|_h = \frac{\partial \theta(h)}{\partial h}\left[u + \gamma^{RD}\mu_+ + \gamma^{RD}\tilde{\theta}_T^*(h)[|\mu_-| - \mu_+]\right] \quad \forall t$$

Since this derivative is equal for all periods, then $U_t^*(e_H) - U_t^*(e_L) = U_{t'}^*(e_H) - U_{t'}^*(e_L)$ $\forall t \neq t'$. To check that the difference between utilities must be binding notice that the difference in utilities from working hard and goofing off is increasing in payment $\forall u$, that is:

$$\frac{U_t^*(e_H) - U_t^*(e_L)}{\partial u} \approx \frac{\partial \theta(h)}{\partial h}\left[1 + \gamma^{RD}\mu'(u) + \gamma^{RD}\tilde{\theta}_T^*(h)[\mu'(-u) - \mu'(u)] + \gamma^{RD}\frac{\tilde{\theta}_T^*(h)}{\partial u}[|\mu_-| - \mu_+]\right]$$

which is certainly positive $\forall u$ if $\theta(h+1) \longrightarrow 1$ because of Lemma 13. If $\theta(h+1) \longrightarrow 0$ and thus by Lemma 13 optimal beliefs decrease as u increase, then this derivative will not necessarily be positive. A sufficient condition for it to be positive $\forall u$ corresponds to $\frac{\partial \tilde{\theta}_T^*}{\partial u} > -\frac{1+\gamma^{RD}\mu'(u)}{\gamma^{RD}[|\mu_-|-\mu_+]}$ for any u. Assume by contradiction that the condition does not bind, ie, $U_t^*(e_H) - U_t^*(e_L) > c$. Because the difference is increasing in payment and the principal's objective function is decreasing in u, then the principal can decrease the payment b_n (decreasing therefore u) and still get an IC contract. Therefore, the IC constraints must bind.

■

Proof of Proposition 14
Let $\triangle U_n^*(u_n^*, T_n) \equiv U_T^*(e_H, u_n^*, T_n) - U_T^*(e_L, u_n^*, T_n)$ where $U_n^*(u_n^*, T_n)$ represents the value of the utility function when evaluated in optimal beliefs $\{\widetilde{\theta}^*(u_n^*, T_n)\}_{t=1}^{T_n}$ for a given productivity bonus $B_n = (u_n^*, T_n)$ where u_n^* solves $\triangle U_n^*(u_n^*, T_n) = c$. Let u_1^* solve $\triangle U_n^*(u_1^*, 1) = c$. That is, u_1^* solves the optimal payment for a productivity bonus of time span of one period.

Because of Lemma 5 I know $\triangle U_n^*(u_n^*, T_n)$ is increasing in u_n^* for any time span T_n. Therefore, if

$$\triangle U_n^*(T_n u_1^*, T_n) < c \qquad \Rightarrow T_n u_1^* < u_n^* \tag{3.27}$$

That is, if the utility difference that determines u_n^*, when evaluated at $T_n u_1^*$ is smaller than u_n^*, then it must be the case that paying T_n one period bonuses is cheaper from the principal's perspective than paying one bonus of u_n^* in period T_n. Equivalently, if $\triangle U_n^*(T_n u_1^*, T_n) > c$ then one bigger bonus is cheaper. The following picture describes this situation:
Let $\mu(\cdot)$ be homogenous of degree $k \in \mathbb{R}$, that is $\mu(Tx) = T^k \mu(x)\ \forall x$. Assume $k \longrightarrow 1$. I look for the conditions for (3.27) to hold. For that, consider:

$$\begin{aligned}
\triangle U_n^*(T_n u_1^*, T_n) &= [\theta(e_H) - \theta(e_L)](T_n u_1^* + \gamma^{RD}\mu(T_n u_1^*) + \gamma^{RD}\widetilde{\theta}_T(T_n u_1^*, T_n)[|\mu(-T_n u_1^*)| - \mu(T_n u_1^*)]) \\
&= (\theta(e_H) - \theta(e_L))[T_n u_1^* + \gamma^{RD} T_n \mu(u_1^*) + \gamma^{RD} T_n(\widetilde{\theta}_T(T_n u_1^*, T_n) \pm \widetilde{\theta}_T(u_1^*, 1))[|\mu(-u_1^*)| - \mu(u_1^*)]) \\
&= T_n c + \gamma^{RD} T_n[\theta(e_H) - \theta(e_L)][\widetilde{\theta}(T_n u_1^*, T_n) - \widetilde{\theta}_T(u_1^*, 1)][|\mu(-u_1^*)| - \mu(u_1^*)]
\end{aligned}$$

where in the last line I have used the fact u_1^* solves $\triangle U_n^*(u_1^*, 1) = c$. Therefore, the principal will establish a bonus productivity in every period if:

$$T_n c + \gamma^{RD} T_n[\theta(e_H) - \theta(e_L)][\widetilde{\theta}(T_n u_1^*, T_n) - \widetilde{\theta}_T(u_1^*, 1)][|\mu(-u_1^*)| - \mu(u_1^*)] < c$$

To find out the sign of $\widetilde{\theta}(T_n u_1^*, T_n) - \widetilde{\theta}_T(u_1^*, 1)$, notice that

$$\begin{aligned}
\frac{B(T_n u_1^*)}{T_n u_1^*} &= \theta \frac{\mu(T_n u_1^*)}{T_n u_1^*} + (1-\theta)\frac{\mu(-T_n u_1^*)}{T_n u_1^*} \\
&\longrightarrow \frac{T_n \mu() u_1^*}{T_n u_1^*} = \frac{B(u_1^*)}{u_1^*}
\end{aligned}$$

where I have used the assumption $\theta \longrightarrow 1$. Therefore:

$$A_t(T_n u_1^*) \longrightarrow A_t(u_1^*)$$

where

$$A_t \equiv \frac{B(\theta) - \sum_{s=t}^{T} \gamma_{T-s+1}^{RI} u}{\gamma_{T-t+1}^{RD} u} \qquad \text{and} \qquad B(\theta) \equiv \gamma^{RD}[\theta \mu_+ + (1-\theta)|\mu_-|]$$

Consequently

$$\begin{aligned}
\widetilde{\theta}_T(T_n u_1^*, T_n) - \widetilde{\theta}_T(u_1^*, 1) &= \theta_0 + \sum_{t=1}^{T_n} \mu'^{-1}(A_t(T_n u_1^*)) - \theta_0 - \mu'^{-1}(A_1(u_1^*)) \\
&= \sum_{t=2}^{T_n} \mu'^{-1}(A_t(T_n u_1^*)) < 0
\end{aligned}$$

where optimal beliefs and the sign comes from Lemma 4. Finally, the condition for $T_n u_n^* > u_1^*\ \forall n$ reduces to:

$$0 < \gamma^{RD}[\theta(e_H) - \theta(e_L)][\widetilde{\theta}_T(u_1^*, 1) - \widetilde{\theta}(T_n u_1^*, T_n)][|\mu(-u_1^*)| - \mu(u_1^*)] > c[1 - \frac{1}{T_n}] \tag{3.28}$$

Noticing that the RHS of (3.28) is greatest when T_n is the greatest and that the LHS is the smallest whenever T_n is the smallest, then a sufficient condition for (3.28) which does not depend on the time span of the bonus corresponds to:

$$c < \gamma^{RD}[\theta(e_H) - \theta(e_L)][|\mu(-u_1^*)| - \mu(u_1^*)][\widetilde{\theta}_T(u_1^*, 1) - \widetilde{\theta}(2u_1^*, 2)]$$

$\Leftrightarrow$

$$c < \gamma^{RD}[\theta(e_H) - \theta(e_L)][|\mu(-u_1^*)| - \mu(u_1^*)]\mu'^{-1}\left(\frac{B(u_1^*) - \gamma_1^{RI}u_1^*}{\gamma_1^{RD}u_1^*}\right)$$

■

Proof of Corollary 1

If $\theta(e_H) \longrightarrow 0$, then since $\theta(e-L) < \theta(e_H)$ it must be the case $\theta(e_H) \longrightarrow 0$. Therefore $\theta(e_H) - \theta(e_L) \approx 0$. Consequently

$$U_{t,n}^*(e_H) - U_{t,n}^*(e_L) \approx 0$$

Therefore for all payments. the IC constraints are approximately zero and there is no payment that convinces the agent to exert high effort.

■

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