Geometric Constructions of Mapping Cones in the Fukaya Category

by

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Abstract

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We present two geometric constructions of Lagrangian surgeries between two Lagrangian submanifolds intersecting cleanly along a 1-dimensional submanifold. We show in a concrete example in 4-dimensions that the two constructions are isomorphic in the Fukaya Category and represent a mapping cone.
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Chapter 1

Introduction

Exact triangles and mapping cones in the Fukaya Category of a symplectic manifold have been useful tools to help us understand the Fukaya Category as a whole, and seeking a geometric understanding of such mapping cones has been helpful in furthering our understanding of them. Geometric interpretations of mapping cones have been understood for a few cases. One such example was presented by Seidel. Seidel showed that in an exact symplectic manifold, the Dehn twist of a Lagrangian submanifold $L$ about $S$, a Lagrangian sphere, is an example of a mapping cone of a certain evaluation map [5].

Another important way to understand mapping cones in the Fukaya Category geometrically is through Lagrangian surgery. The Lagrangian surgery between two Lagrangian submanifolds of a symplectic manifold intersecting transversely in a single point was proposed by Polterovich [4]. Fukaya-Oh-Ohta-Ono proposed that the Lagrangian surgery between two Lagrangian submanifolds intersecting transversely in a single point $p$ is quasi-isomorphic to $\text{Cone}(p)$ [3].

In this paper, we study the case when two Lagrangian submanifolds of a symplectic manifold, instead of intersecting in a single point, intersect cleanly along a 1-dimensional submanifold. We construct two different surgeries between such Lagrangian submanifolds: the Morse surgery and the Morse-Bott surgery, and demonstrate in a 4-dimensional example that they are isomorphic and both represent a mapping cone.

The structure of this paper is as follows: In Section 2, we introduce the setup — the symplectic manifolds and Lagrangian submanifolds considered, the moduli spaces that will be counted, and the higher products needed. In Section 3, we define the two different Lagrangian surgeries. In Section 4, we present the main result, which is a detailed computation in 4-dimensions.
Chapter 2

Setup

Consider \((M, \omega, \sigma, J)\), where \(M\) is a manifold, \(\omega = d\sigma\) a symplectic form, \(J\) an \(\omega\)-compatible almost complex structure. Let \(i : L \to M\) be a Lagrangian immersion with only transverse double points. Assume there exists \(f : L \to \mathbb{R}\) such that \(df = i^* \sigma\). Let \(R = \{(p, q) \in L \times L : i(p) = i(q), p \neq q\}\) be the self intersections. Let \(g\) be a Morse function on \(L\).

Definition 1. The cochain complex is

\[ CF(L, L) = CM(g) \oplus \bar{R} \]

where \(CM(g) = \bigoplus \mathbb{Z}/2 \cdot q\) generated by \(q\), the critical points of \(g\) just like in Morse chain complex, \(\bar{R} = \bigoplus (\mathbb{Z}/2 \cdot \gamma_- \oplus \mathbb{Z}/2 \cdot \gamma_+)\) where \(\gamma_-, \gamma_+\) are two generators corresponding to \(\gamma\) at each double point.

The immersed Lagrangian may possibly bound holomorphic disks. We first force a condition on the behavior of such disks.

Given a point \(x \in CF(L, L)\), consider maps \(u : (D, \partial D) \to (M, L)\) satisfying

1. \(u : (D, \partial D) \to (M, L)\) non-constant holomorphic disk. \(D\) the closed unit disk in \(\mathbb{C}\) with one marked point on the boundary \(-1 \in \partial D\)

2. If \(x \in CM(g)\), there exists a \(-\nabla g\) flow line from \(x\) to \(u(-1)\)

3. If \(x \in \bar{R}\), then \(u(-1) \to x\)

The moduli space of all such \(u\)'s representing a given class \(A\) will be denoted by \(M(x, A)\).

The moduli space \(M(x, A)\) need not be regular, but when it is, it is a smooth manifold of dimension \(Ind(A) - 2\) when \(x \in \bar{R}\), where \(Ind\) denotes the Fredholm index of the holomorphic disk, and dimension \(\mu(A) + |x| - 2\) when \(x \in CM(g)\), where \(\mu\) is the Maslov index and \(|x|\) is the Morse index of \(x\).

Condition 2. For any \(x \in \bar{R}\) (resp. \(x \in CM(g)\) with \(|x| = 0\)) and \(A \in H_2\) with positive symplectic area we have either
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1. \( \text{Ind} \geq 3 \) (resp. \( \mu(A) \geq 3 \)), or

2. \( \text{Ind} = 2 \) (resp. \( \mu(A) = 2 \)) but \( M(x, A) \) is regular for all such \( A \). We require that the count of index 2 disks with marked points mapping to \( x \) sum to 0 (counting mod 2). Namely

\[
\sum_A |M(x, A)| = 0
\]

Given \( x, y \in CF(L, L) \), \( A \in H_2(M, L) \), consider \( (u_1, \ldots, u_l) \):

1. \( u_i : (D, \partial D) \to (M, L) \) non-constant holomorphic disk. \( D \) the closed unit disk in \( \mathbb{C} \)
2. If \( x, y \in CM(g) \), there exists \( -\nabla g \) flow lines from \( x \) to \( u_1(-1) \), from \( u_i(1) \) to \( u_{i+1}(-1) \), and from \( u_l(1) \) to \( y \)
3. If \( x \in \bar{R} \), then \( u_1(-1) \to x \). If \( y \in \bar{R} \), then \( u_l(1) \to y \)
4. \( [u_1] + \ldots + [u_l] = A \)

The moduli space of all such sequences will be denoted by \( M(x, y, A) \).

![Figure 2.1: The pearly configurations in \( M(x, y, A) \)](image)

In the special case when \( L \) is an embedded monotone Lagrangian with minimal Maslov number \( \mu \geq 2 \), Biran and Cornea showed in [2] the following:

**Statement 3.** Let \( g : L \to \mathbb{R} \) be a Morse function and \( \rho \) a Riemannian metric on \( L \) such that \( (g, \rho) \) is Morse-Smale. Then there exists an almost complex structure \( J_{\text{reg}} \) such that for every \( x, y \in \text{Crit}(g) \) with \( \mu(A) + |x| - |y| - 1 \leq 1 \):
CHAPTER 2. SETUP

1. All the elements \((u_1, \ldots, u_l)\) are simple and absolutely distinct. The moduli space \(M(x, y, A)\) is either empty or a smooth manifold of dimension \(\mu(A) + |x| - |y| - 1\). In particular, if \(\mu(A) + |x| - |y| - 1 < 0\), then the moduli space is empty.

2. If \(\mu(A) + |x| - |y| - 1 = 0\), then \(M(x, y, A)\) is a compact 0-dimensional manifold and hence consists of finite number of points.

However, the \(L\) in our setting is an immersed Lagrangian submanifold so at best the theorem of Biran-Cornea considers the cases \(x, y \in CM(g)\) but doesn’t apply to the cases when \(x\) or \(y \in \bar{R}\). As it turns out the moduli space when we include the configurations where \(x\) or \(y \in \bar{R}\) need not be regular in general. When all the elements \((u_1, \ldots, u_l)\) are simple and absolutely distinct there will not be a problem. However, \(M(x, y, A)\) can contain configurations where \(u_1\) and \(u_l\) are the same exact holomorphic disk passing through \(x\) and \(y \in \bar{R}\) the two generators associated to the same double point. In practice, we will address this issue by choosing our Morse function \(g\) so no such configurations can exist. If regularity holds then \(M(x, y, A)\) is a smooth manifold, and its dimension will be \(\mu(A) + |x| + \text{Ind}(u_1) - |y| - n - 1\) for \(x \in CM(g)\) and \(y \in \bar{R}\), and \(\mu(A) + \text{Ind}(u_1) + \text{Ind}(u_l) - n - 1\) for \(x, y \in \bar{R}\) and \(l > 1\) (where \(\mu(A)\) is the total Maslov index of the disk components without striplike ends).

We will state the definition of the differential \(\delta\) assuming transversality (which we will address in our example later). The differential is given by

\[
\delta : CF(L, L) \to CF(L, L) : a \mapsto \sum |M(b, a, A)| \cdot b
\]

summing through all \(M(b, a, A)\) that have dimension 0 and counting with mod 2 coefficients.

**Statement 4.** \(\delta^2 = 0\), assuming \(L\) satisfies condition 2 and transversality is achieved.

A proof of the above statement for our specific example will be given explicitly in statement 12. In our case, \(L\) is exact, so all disks with boundary on \(L\) must involve double points.

The Floer cohomology is

\[
HF(L, L) = H(CF(L, L), \delta)
\]

We will also need to define some higher products \(\mu^2\) for our computations later. Assume \(L_1\) is another embedded monotone Lagrangian submanifold. In order to define \(\mu^2 : CF(L, L_1) \otimes CF(L_1, L) \to CF(L, L)\) we consider the moduli space \(M(L, L_1, L, a, b, c)\) where \(a \in CF(L_1, L), b \in CF(L, L_1),\) and \(c \in CF(L, L)\) consisting of

1. Holomorphic triangles with the vertices mapping to \(a, b, c\) and the edges mapping to \(L_1, L, L\).

2. A holomorphic bigon with the boundaries mapping to \(L_1, L\) and the vertices mapping to \(a, b\). On the edge of the holomorphic bigon that maps to \(L\) is a point \(y'\) where a pearly trajectory in \(M(c, y', A)\) is attached.
The map $\mu^2 : CF(L, L_1) \otimes CF(L_1, L) \to CF(L, L)$ is defined by

$$\mu^2(b, a) = \sum |M(L, L_1, L, a, b, c)| \cdot c$$

summing through all the $M(L, L_1, L, a, b, c)$ of dimension 0.

The map $\mu^3 : CF(L, L_1) \otimes CF(L_1, L_2) \otimes CF(L_2, L) \to CF(L, L)$ is defined in the same manner.
Chapter 3

Surgery Construction

There are two types of Lagrangian surgery which we will perform in this section. The first one we refer to as Morse gluing or just regular Lagrangian surgery. The second one we refer to as Morse-Bott gluing. We will give an overview of the two constructions in this section.

Consider $(M, \omega)$ where $M$ is a manifold and $\omega$ is a symplectic form on $M$. $L_1, L_2$ are two Lagrangians in $M$.

Suppose first that $L_1$ and $L_2$ intersect transversely at some point $p$. We will describe a local model for Lagrangian surgery at $p$, as first introduced by Polterovich [4].

There is always a Darboux chart in a neighborhood $U$ of $p$ and $i: U \to V \subset \mathbb{C}^n$ where

$$i(L_1 \cap U) = \mathbb{R}^n \cap V, i(L_2 \cap U) = \sqrt{-1} \mathbb{R}^n \cap V$$

Let $\epsilon$ be a sufficiently small real number, and let $f_\epsilon: \mathbb{R}^n - \{0\} \to \mathbb{R}$ be

$$f_\epsilon = \epsilon \log |x|$$

The graph of $df_\epsilon(x)$, which we will call $H_\epsilon$ in coordinates $z_j = x_j + \sqrt{-1} y_j$ can be described as

$$H_\epsilon = \left\{ (z_1, ..., z_n) : y_j = \frac{\epsilon x_j}{|x|^2}, j = 1, ..., n \right\}$$

$H_\epsilon$ is a Lagrangian submanifold of $\mathbb{C}^n$ which is asymptotic to $\mathbb{R}^n$ as $|x| \to \infty$, and approaches $\sqrt{-1} \mathbb{R}^n$ as $|x| \to 0$.

We will modify the above description a little bit to make sure that the Lagrangian we construct doesn’t just approach $L_1, L_2$ asymptotically. Let $\tau: \mathbb{C}^n \to \mathbb{C}^n$ be the map that reflects along the diagonal $\Delta = \{(z_1, ..., z_n) : x_i = y_i\}$, sending $x_i + \sqrt{-1} y_i \mapsto y_i + \sqrt{-1} x_i$. Note that $\tau(H_\epsilon) = H_\epsilon$. $\tau$ is an anti-symplectomorphism: $\tau^* \omega_0 = -\omega_0$, which maps Lagrangians to Lagrangians. Instead of log, we consider a slightly different map $\rho: \mathbb{R}^+ \to \mathbb{R}$ by

$$\rho(r) = \begin{cases} \log r - |\epsilon| & \text{if } r \leq \sqrt{|\epsilon|} S_0 \\ \log \sqrt{|\epsilon|} S_0 & \text{if } r \geq 2\sqrt{|\epsilon|} S_0 \end{cases}$$
\[ \rho'(r) \geq 0, \rho''(r) \leq 0 \]

here \( S_0 \) is a fixed sufficiently large number and \( \epsilon \) satisfies \( \sqrt{\epsilon}S_0 \) is sufficiently small. The modified function \( \bar{f}_\epsilon : \mathbb{R}^n - \{0\} \rightarrow \mathbb{R} \) is defined by

\[ \bar{f}_\epsilon = \epsilon \rho(|x|) \]

Construct a new Lagrangian manifold \( H_\epsilon \) of \( \mathbb{C}^n \) that satisfies \( \tau(H_\epsilon) = H_\epsilon \) and

\[ \{(z_1, ..., z_n : x_i \geq y_i \forall i) \} \cap H_\epsilon = \{(z_1, ..., z_n : x_i \geq y_i \forall i) \} \cap \text{graph } d\bar{f}_\epsilon \]

Since the graph of \( d\bar{f}_\epsilon \) coincides with that of \( df_\epsilon \) for \( |x| \in (\sqrt{\epsilon}S_0, \sqrt{\epsilon}S_0) \), \( H_\epsilon \) defined in this manner is smooth and coincides with \( H_\epsilon \) inside the ball of radius \( \sqrt{\epsilon}S_0 \) around 0. Outside of the ball \( B^{2n}(2\sqrt{\epsilon}S_0) \) around 0, \( \bar{H}_\epsilon = \mathbb{R}^n \cup \sqrt{-1}\mathbb{R}^n \).

Thus for the given \( L_1 \) and \( L_2 \) intersecting transversely at some point \( p \) and the Darboux chart \( U \) in a neighborhood of \( p \), we construct Lagrangian submanifold \( L_\epsilon \subset M \) by replacing \( U \) with the local model \( H_\epsilon \) we constructed, i.e.

\[ L_\epsilon - U = L_1 \cup L_2 - U, \quad i(L_\epsilon \cap U) = \bar{H}_\epsilon \cap V \]

**Definition 5.** Given \( L_1, L_2 \), the Lagrangian surgery of \( L_1 \) and \( L_2 \) at \( p \in L_1 \cap L_2 \) is the Lagrangian manifold \( L_\epsilon \), denoted as \( L_\epsilon = L_1 \#_\epsilon L_2 \).

Now suppose the Lagrangian submanifolds \( L_1, L_2 \), instead of intersecting transversely at a point, now intersect cleanly along a 1 dimensional isotropic submanifold \( K \subset L_1 \cap L_2 \). We will also require \( L_1, L_2 \) to be orientable submanifolds.

Because \( K = L_1 \cap L_2 \) is a one dimensional submanifold, and since \( L_1 \) is orientable, the normal bundle \( K \) of inside \( L_1 \) is trivial. Thus there is a neighborhood of \( K \) in \( L_1 \) that is diffeomorphic to an open subset of \( K \times \mathbb{R}^{n-1} \). Applying the Weinstein neighborhood theorem on \( L_1 \), we see that a neighborhood of \( L_1 \) inside \( M \) is symplectomorphic to \( T^*L_1 \). Thus locally we can view a neighborhood of \( K \) as \( K \times \mathbb{R} \times \mathbb{C}^{n-1} \) with the standard symplectic form, where \( L_1 \) corresponds to \( K \times 0 \times \mathbb{R}^{n-1} \). Denote the coordinates of \( K \times \mathbb{R} \) by \( (s, t) \), and the coordinates of \( \mathbb{C}^{n-1} \) by \( (x_i, y_i) \) for \( i = 1,...,n-1 \). \( K \) is the \( s \)-axis, \( L_1 \) is the \( (s, x_1, ..., x_{n-1}) \)-plane, and the symplectic form is \( ds \wedge dt + \sum dx_i \wedge dy_i \)

Since \( L_2 \) intersects \( L_1 \) cleanly along \( K \), locally \( L_2 \) is a graph in the sense that \( (t, x_1, ..., x_{n-1}) \) are functions of \( (s, y_1, ..., y_{n-1}) \) where the functions vanish on \( K \). Since \( L_2 \) is a Lagrangian submanifold and a graph, it is a graph of a closed 1-form over the \( s, y \) axes. Since \( K \subset L_2 \), the integral of this 1-form over \( K \times 0 \) is 0, the 1-form is exact. Thus there is some function \( h(s, y) \) such that \( L_2 \) is a graph of \( dh \), given by \( x_i = -\frac{dh}{dy_i} \) and \( t = \frac{dh}{ds} \). Moreover, \( h \) is constant along \( K \times 0 \) and its derivative vanishes on \( K \times 0 \). So subtracting a constant we can assume that \( h = O(|y|^2) \).

Consider the Hamiltonian \( H = h(y, s) \). Its time 1 flow maps

\[ (s, t, x, y) \mapsto \left( s, t - \frac{dh}{ds}, x + \frac{dh}{dy_i}, y \right) \]
which is identity on $L_1$ and maps the graph of $dh$ to the $(s, y)$ coordinate plane. Using this modification, we have arranged a neighborhood of $K$ to be $K \times \mathbb{R} \times \mathbb{C}^{n-1}$ where $L_1$ is $K \times 0 \times \mathbb{R}^{n-1}$ (the $s, x$ axes) and $L_2$ is $K \times 0 \times \sqrt{-1}\mathbb{R}^{n-1}$ (the $s, y$ axes).

Under this arrangement of $K, L_1, L_2$, we perform regular Lagrangian surgery on the $x, y$ coordinates. Given a small fixed $\epsilon$, for every value of the $s$-coordinate we perform Lagrangian surgery on $s \times 0 \times \mathbb{R}^{n-1}$ and $s \times 0 \times \sqrt{-1}\mathbb{R}^{n-1}$. Since $\epsilon$ is a fixed small constant, the resulting submanifold is Lagrangian. This Lagrangian submanifold is called the Morse-Bott surgery or the Morse-Bott gluing in the following sections.

**Remark 6.** Even though in our definition we only construct the Morse-Bott surgery for two Lagrangian submanifolds $L_1, L_2$ intersecting cleanly along a one dimensional submanifold $K$, we only used the fact that $K$ is one dimensional for the argument that the normal bundle of $K$ inside $L_1$ is topologically trivial. Thus the construction can be applied whenever the normal bundle to $K$ inside $L_1$ is trivial, and dropping the assumption that $\text{dim}(K) = 1$. In the case where the normal bundle of $K$ in $L_1$ is not trivial, one can still define Morse-Bott surgery, but it will not be described here as it is not needed in the following sections.
Chapter 4

$M_1 \times M_2$

Let $M_1$ be a 2-dimensional cylinder and $M_2$ a punctured 2-torus, both with the standard symplectic form. Here we construct and compute examples of surgeries in $M = M_1 \times M_2$ with product symplectic form. Denote the $S^1$ in $M_1$ by $N$, and the longitude and the meridian of $M_2$ by $C_1, C_2$. Let $*$ be the intersection of $C_1$ and $C_2$. Let $T_1$ be the 2-torus $N \times C_1$, and $T_2$ the 2-torus $N \times C_2$. These are two Lagrangian tori intersecting along $N \times \{*\}$.

The Morse gluing of $T_1$ and $T_2$ is constructed as follows: Think of $M_1$ like $T^*S^1$ where $N$ is the zero section. First choose a Morse function $f$ on $N$ with a max $y$ and a min $x$. Let $N_1$ and $N_2$ be $\text{graph}(df)$ and $\text{graph}(-df)$ respectively. $\tilde{T}_1 = N_1 \times C_1$ and $\tilde{T}_2 = N_2 \times C_2$ now intersect at the two points $(y, *), (x, *)$. Doing a regular Lagrangian surgery on $(y, *)$ gives the Morse gluing.

The Morse-Bott gluing, on the other hand, does a regular Lagrangian surgery on the intersection of $C_1$ and $C_2$ in $M_2$, and products $C' = C_1 \# C_2$ with $N$.

We denote $L_1(\epsilon)$ to be the Morse gluing ($\epsilon$ is the gluing parameter) and $L_2$ the Morse-Bott gluing. $L_2$ is an embedded Lagrangian torus which is $N$ in the first factor and $C_1 \# C_2$ in the second factor, a product of Lagrangian $S^1$’s. $L_1(\epsilon)$ is an immersed Lagrangian which topologically is a genus two surface with one self intersection.

Statement 7. $L_2$ bounds no disc in $M_1 \times M_2$.

Proof. $L_2$ is a product Lagrangian. Each factor bounds no disk. □
Upon Lagrangian surgery, the two strips bounded by $N_1$ and $N_2$ in $M_1$ give rise to two teardrop-shaped regions with boundary on $L_1(\epsilon)$. The boundary loops of the two teardrops, shown in red and blue on Figure 4.1, run once through the double point and once through the neck of the surgery.

**Statement 8.** Consider $L_1(\epsilon) \subset M$, the immersed genus two surface with one self intersection (as opposed to the abstract genus 2 surface $S$ that is the domain of this immersion). $H_2(M, L_1(\epsilon))$ is generated by the two teardrops, denoted by $A_+, A_-$, and a third generator $a$, the class of the small disc bounded by the neck of the surgery at $(y, \ast)$.

**Proof.** Consider the following section in the long exact sequence

$$
\cdots \to H_2(L_1(\epsilon)) \to H_2(M) \to H_2(M, L_1(\epsilon)) \to H_1(L_1(\epsilon)) \to H_1(M) \to \cdots
$$

In the first map $H_2(L_1(\epsilon)) \to H_2(M)$, we have $H_2(L_1(\epsilon)) = \mathbb{Z}$, $H_2(M) = H_2(M_1 \times M_2) = H_1(M_1) \otimes H_1(M_2) = \mathbb{Z}^2$. The map $H_2(L_1(\epsilon)) \to H_2(M)$ is injective with cokernel $\mathbb{Z}$. $H_1(L_1(\epsilon)) = \mathbb{Z}^5$, generated by the 4 generators of $H_1(S)$, where $S$ is the abstract genus two surface, and a loop $\gamma_1$ that starts and end on the immersed double point (say the red loop in Figure 4.1). We specify one of the four generators of $H_1(S)$ to be the sum of loop red and blue, denoted by $\gamma_2$. $H_1(M) = H_1(M_1 \times M_2) = (H_1(M_1) \otimes H_0(M_2)) \oplus (H_0(M_1) \otimes H_1(M_2)) = \mathbb{Z}^3$. The last map $H_1(L_1(\epsilon)) \to H_1(M)$ is onto, and its kernel is generated by $\gamma_1$ and $\gamma_2$. We conclude that $H_2(M, L_1(\epsilon)) = \mathbb{Z}^3$. Two of the generators of $H_2(M, L_1(\epsilon))$, denoted by $A_+, A_-$, are disks with boundaries on $\gamma_1$ and $\gamma_2 - \gamma_1$ (the red loop and the blue loop respectively). The last generator of $H_2(M, L_1(\epsilon))$ is a disk bounded by the neck of the surgery at $(y, \ast)$, depicted as the grey disk in Figure 4.1. \qed
We will show that only the classes $A_+, A_-$ (and their multiples) can be represented by holomorphic disks.

**Statement 9.** $L_1(\epsilon)$ bounds two holomorphic teardrops, and no other somewhere injective holomorphic disk.

**Proof.** Let $L_1 = \overline{T}_1 \cup \overline{T}_2$ be $L_1(\epsilon)$ before gluing. $L_1$ is the union of $N_1 \times C_1$ and $N_2 \times C_2$. In the $M_1$ factor, the Riemann mapping theorem tells us that $N_1 \cup N_2$ bounds two homotopy classes of simple holomorphic maps, each with a unique holomorphic disk (mod reparametrization) and multiplicity 1. In the $M_2$ factor, $C_1 \cup C_2$ bounds no nonconstant holomorphic disk. If $u : (D, \partial D) \to (M, L_1)$ is a holomorphic disk that $L_1$ bounds, then $\pi_1 \circ u$ and $\pi_2 \circ u$ will be holomorphic maps in the first and second factor, and so any holomorphic disk that $L_1$ bounds must be constant in the second factor, and one of the two holomorphic disks in the first factor.

Now we move from disks in $L_1$ to disks with boundary in $L_1(\epsilon)$. Away from an arbitrarily small $\epsilon$ neighborhood of the self intersection $y$ where we perform the gluing, the Lagrangian stays the same. Consider the local model for the Lagrangian gluing. In coordinates $z_i = x_i + \sqrt{-1}y_i$, the local gluing model can be written as

$$\left\{ (z_1, z_2) : y_i = \frac{\epsilon x_i}{x_1^2 + x_2^2}, i = 1, 2 \right\}$$

When projected to either of the two factors, it looks like $\left( x_i, \frac{\epsilon x_i}{x_1^2 + x_2^2} \right)$, covering all areas between $\left( x_i, \frac{\epsilon}{x_1} \right)$ and the axes.

![Figure 4.2: On the left: the local model for Morse surgery. On the right: What it looks like in the $M_1$ factor. The holomorphic "teardrops" are the regions marked by $u_1, u_2$. $x$ is the self intersection, and $y$ is where Morse surgery is performed.](image-url)
In the $M_1$ factor of $L_1(\epsilon)$, there are two holomorphic $w_1, w_2 : (D, \partial D) \to (M_1, L_1(\epsilon))$. They pass through the “outermost” part of the projected gluing neck (the $\left( x_1, \frac{\epsilon}{x_1} \right)$ part in the local model). Let $u_i : (D, \partial D) \to (M, L_1(\epsilon))$ be the holomorphic disks that coincide with $w_i$ in $M_1$ and constant in $M_2$. They are of homotopy class $A_+, A_-$ respectively. Let $u : (D, \partial D) \to (M, L_1(\epsilon))$ be a holomorphic disk with boundary on $L_1(\epsilon)$ with homotopy class $A_+$. The image of $w_1$ is a subset of the image of $\pi_1 \circ u$. Since holomorphic disks are area-minimizing in their homotopy class, $u_1$ and $u$ have to be the same holomorphic disk. The same argument can be applied to show that a holomorphic $(D, \partial D) \to (M, L_1(\epsilon))$ with homotopy class $A_-$ will have to be $w_2$.

There are no other simple holomorphic disks with boundary on $L_1(\epsilon)$ with other homotopy classes. To show this, let $u$ be a holomorphic disk in $M$ with boundary on $L_1(\epsilon)$. Its image in the $M_1$ factor, $\pi_1 \circ u$, has boundary on $\pi_1(L_1(\epsilon))$. Open mapping principle tells us that $\pi_1 \circ u$ either covers $w_1$ or $w_2$ a certain number of times but not both, or stays in a small region. Thus the homology class of $u$ in $H_2(M, L_1(\epsilon))$ must be either $k \cdot A_+ + l \cdot a$ or $k \cdot A_- + l \cdot a$ for $k \geq 0$ and $l$ integers. Suppose the homology class is $k \cdot A_+ + l \cdot a$ (the case for $k \cdot A_- + l \cdot a$ works the same way). The symplectic area of the disk $u$ is equal to $k$ times the area of $u_1$. Since the image of $\pi_1 \circ u$ covers $k$ times the entire image of $w_1$, we see that the disc is a $k$-fold cover of $u_1$ in the first factor and constant in the second factor. Thus $u$ represents the class $k \cdot A_+$, and $l = 0$.

**Statement 10.** The two teardrops are regular with index 2:

*Proof.* In the previous statement we showed that the holomorphic disks with boundaries on $L_1(\epsilon)$ are “teardrops” in the first $M_1$ factor and constant in the $M_2$ factor. The tangent planes to the Lagrangian $L_1(\epsilon)$ along the boundary of the teardrop are products of lines in $M_1$ and $M_2$. Away from the point where gluing happens, $L_1(\epsilon)$ coincides with $L_1 = \tilde{T}_1 \cup \tilde{T}_2$, which is a union of two product Lagrangians $\tilde{T}_1 = N_1 \times C_1$ and $\tilde{T}_2 = N_2 \times C_2$, and thus locally the tangent planes split as a product of lines. On the part where we glue, we will take a look at the local model again. On the local model, when passing though the gluing, the boundary of the teardrop lives in

$$\left\{ (z_1, z_2) : y_i = \frac{\epsilon x_i}{x_1^2 + x_2^2}, i = 1, 2, z_2 = 0 \right\}$$

On $z_2 = 0$ the tangent space is spanned by $(1, \frac{\epsilon}{x_1}, 0, 0), (0, 0, 1, \frac{\epsilon}{x_1})$. Thus the tangent planes to the $L_1(\epsilon)$ along the teardrops are indeed product of lines in each $M_1, M_2$ factor.

The linearized $\bar{\partial}$ operator splits into the direct sum of two $\bar{\partial}$ operators on $\mathbb{C}$-valued functions with boundary conditions given by a family of real lines: the real lines of the tangent direction to $L$.

The Fredholm index of $u_1$ will be computed as follows: We will consider $u_1$ as a holomorphic disk with one mark output point on the self intersection $x$. On the $M_1$ factor, after closing the family of boundary conditions to a closed loop in the Lagrangian Grassmannian
by adding a short counterclockwise path at the double point, the Lagrangian tangent line
rotates a full $2\pi$. On the $M_2$ factor, again closing the family of boundary conditions to a
closed loop in the Lagrangian Grassmannian in the same way, the Lagrangian tangent line
ends up not rotating. The Fredholm index of $u_2$ can be computed the same way.

As the computation shows, the Fredholm index of a teardrop for each factor is non-
negative. In the $M_1$ factor, the nonconstant teardrop has Fredholm index 2, while in the $M_2$
factor, the constant map has index 0. Since the linearized $\bar{\partial}$ operator splits into the direct
sum of two $\bar{\partial}$ operators on $\mathbb{C}$-valued functions with boundary conditions given by a family of
real lines, we can apply Lemma 11.5 in Seidel’s book [6] for $\bar{\partial}$ operators on line bundles. The
lemma states that in dimension 1, if the index is less than 0, then $\bar{\partial}$ is injective. Considering
the adjoint operator which is also a $\bar{\partial}$ operator (with a different boundary condition), the
lemma then states that if the index is $\geq 0$, then the operator is surjective.

**Statement 11.** $L_1(\epsilon)$ satisfies the teardrop cancelling condition (Condition 2):

**Proof.** Given that there is only one self intersection $x$, and the previous computation shows
that there are only two index 2 holomorphic teardrops, one in $M(x_+,A_+)$ and the other in
$M(x_-,A_-)$. Here $x_+$ is one of the two generators of $CF(L_1(\epsilon),L_1(\epsilon))$ that will be assigned
to the self intersection $x$. Thus counting in $\mathbb{Z}/2$, $\sum_A |M(x,A)| = 0$

Define the chain complex. Choose a Morse function $g$ on the genus 2 surface $S$ which
is the domain of the Lagrangian immersion $i : S \to L_1(\epsilon)$. $CF(L_1(\epsilon),L_1(\epsilon)) = CM(g) \oplus \bar{R}$
where $CM(g)$ has the usual 6 generators, and $\bar{R}$ has 2 generators: $x_+$ and $x_-$. The degree
of the Morse critical points will be the Morse index, while the degrees of $x_+$ and $x_-$ are 2
and 0 respectively. Here we choose our Morse function $g$ so that the boundary of the two
teardrops all live in a level set of the Morse function $g$ (note that the boundaries of the two
teardrops taken together form a homologically nontrivial embedded simple closed curve on
$S$). We want to ensure that there are no Morse trajectories connecting the boundary of a
holomorphic teardrop to itself so the moduli space considered in the proof that $\delta \circ \delta = 0$ is
regular. Consider the 0-dimensional moduli space $M(p,q,A)$, where $\text{deg}(p) - \text{deg}(q) = 1$,
consisting of configurations of the form:

1. A teardrop passing through self intersection $p$ with homotopy class $A$, followed by a
   gradient flow line of $-\nabla g$ that goes from the boundary of the teardrop to a Morse
critical point $q$.

2. A gradient flow line of $-\nabla g$ that goes from the Morse critical point $p$ to the boundary
   of a teardrop that passes through a self intersection $q$.

3. A gradient flow line of $-\nabla g$ that goes from $p$ to $q$. 
Transversality:

We showed that the two teardrops themselves are both regular. Our choice of $g$, which makes the boundary of teardrops live in a level set, ensures that the gradient flow lines will intersect the boundaries of teardrops transversely.

The differential is given by

$$\delta : CF(L_1(\epsilon), L_1(\epsilon)) \to CF(L_1(\epsilon), L_1(\epsilon)) : q \mapsto \sum_{\dim(M(p,q,A))=0} |M(p,q,A)| \cdot p$$

**Statement 12.** $\delta \circ \delta = 0$

**Proof.** Consider a 1-dimensional moduli space $M(p,q,A)$. The boundary of $M(p,q,A)$ consists of the following configurations:

1. A teardrop passing through self intersection $p$ with homotopy class $A$, followed by a broken gradient flow line of $-\nabla g$ that goes from the boundary of the teardrop to another Morse critical point $q'$, and finally to the Morse critical point $q$.

2. Same as above, except starting with a broken gradient flow line from $p$ to $p'$, and then to the boundary of the teardrop passing through self intersection $q$ with homotopy class $A$.

3. Broken Morse trajectory.

4. A teardrop passing through self intersection $p$ with homotopy class $A$, with a Morse trajectory from one of the two preimages in $S$ of the double point $p$ to a Morse critical point $q$. 
Of these four types of configurations, only the first three contribute to \( \delta \circ \delta \). Moreover, \( \delta \circ \delta \) also counts two other types of broken configurations:

5) A Morse trajectory from \( p \) to the boundary of a teardrop passing through double point \( x \), followed by another teardrop passing through \( x \), followed by a Morse trajectory that goes from the boundary of the teardrop to a Morse critical point \( q \).

6) A teardrop passing through self intersection \( p \), followed by a Morse trajectory that goes from the boundary of the teardrop to a Morse critical point \( r \), followed by another Morse trajectory from \( r \) to the boundary of a teardrop that passes through self intersection \( q \).

The configurations in (4) contribute 0 to the count because the condition that forces teardrops count to sum to 0 will also force this count to 0. This is because for each preimage of the double point, the number of such configurations in (4) has the same count as teardrops that pass through the double point \( p \), and hence Condition 2 forces that count to be zero.

Configurations (5), (6) do not occur in our case. Configuration (5) does not occur because the output point of a teardrop is \( x_+ \), but the input point of a teardrop is \( x_- \). On the other hand, configuration (6) also does not occur because we chose our Morse function \( g \) so that the boundary of the teardrops live in a level set of \( g \), so there cannot be any configurations in (6) as no such critical point \( r \) can exist.

Since configurations (5) and (6) do not occur, the remaining configurations (those in (1), (2), (3)) count the coefficient of \( p \) that comes from \( \delta \circ \delta(q) \). Because the signed count of the boundary of a compact smooth 1-dimensional manifold with boundary is 0 and the configurations in (4) cancel out, we see that \( \delta \circ \delta = 0 \).

Let \( HF(L_1(\epsilon), L_1(\epsilon)) = H(CF(L_1(\epsilon), L_1(\epsilon)), \delta) \) be the cohomology of this chain complex.
Statement 13. $HF(L_1(\epsilon), L_1(\epsilon))$ has the cohomology of a torus.

Proof. Denote the index 2 and index 0 Morse critical points as $p, q$ respectively, the index 1 critical points as $\gamma_1, \ldots, \gamma_4$, and the two generators from the self intersection as $x_+, x_-$. The boundary of the two teardrops together form a simple closed curve $\eta$ on the genus two surface $S$ (recall $S$ is the domain of the immersion. $i(S) = L_1(\epsilon)$). Moreover, that simple closed curve lives in a level of our Morse function $g$. For the purpose of this proof, let’s arrange the Morse function $g$ as depicted in Figure 4.5. The boundary of the two teardrops intersects the ascending manifolds of $\gamma_1, \gamma_2, \gamma_3$ each at one point, while it intersects the descending manifolds of another $\gamma$, say $\gamma_4$, at another point. So $\delta(\gamma_1) = x_+$ (and also $\delta(\gamma_2) = x_+, \delta(\gamma_3) = x_+$), $\delta(x_-) = \gamma_4$, while all other $\delta$ are 0.

The differential $\delta : CF^0(L_1(\epsilon), L_1(\epsilon)) \to CF^1(L_1(\epsilon), L_1(\epsilon))$ thus has rank 1, and similarly $\delta : CF^1(L_1(\epsilon), L_1(\epsilon)) \to CF^2(L_1(\epsilon), L_1(\epsilon))$ also has rank 1. $CF^0$ has rank 2, $CF^1$ has rank 4, and $CF^2$ has rank 2. Thus in cohomology, $HF(L_1(\epsilon), L_1(\epsilon))$ now has $HF^0$ rank 1, $HF^1$ rank 2, and $HF^2$ rank 1. So cohomologically, $HF(L_1(\epsilon), L_1(\epsilon))$ looks like that of a 2-torus.

There is another way to compute the differential $\delta$ without arranging the Morse function $g$ in a specific way like we did above. The simple closed curve $\eta$, formed by the boundaries of the two teardrops, is not 0 in homology $H_1(S)$. Since the ascending manifolds of the $\gamma_i$’s form a basis for $H_1(S)$, at least one will have nonzero intersection number with $\eta$, say $\gamma_1$. Then $\delta(\gamma_1) = x_+$. On the other hand, the descending manifolds of the $\gamma_i$’s also form a basis for $H_1(S)$, thus at least one of them will have nonzero intersection number with $\eta$. Therefore, $\delta(x_-)$ will be a nonzero linear combination of the $\gamma_i$’s. An analogous argument to the previous paragraph shows that $HF$ is that of a 2-torus.

The following computations build towards establishing the relation between $L_1(\epsilon)$ and $L_2$, the Morse and Morse-Bott gluing. To accomplish that, we will first work with $L_1$, the union of $N_1 \times C_1$ and $N_2 \times C_2$, which is $L_1(\epsilon)$ before gluing.
CHAPTER 4. $M_1 \times M_2$

Let $p \in (N_2 \times C_2) \cap L_2$, $q \in L_2 \cap (N_1 \times C_1)$, $y \in (N_1 \times C_1) \cap (N_2 \times C_2)$ be the self intersection in $L_1$ that we eventually perform the Lagrangian surgery (Morse surgery), and $x \in (N_1 \times C_1) \cap (N_2 \times C_2)$ the immersed double point. $q, p, x$, and $y$ are depicted in Figure 4.6 below. We start by counting holomorphic triangles with boundaries on $(N_2 \times C_2), L_2, (N_1 \times C_1)$ with the vertices going to $(q, p, y)$ or $(q, p, x)$.

In the $M_2$ factor, there are two simple triangles (depicted on the right of Figure 4.6). On the other hand, in the $M_1$ factor, a few cases can happen:

1. Concave triangles with vertices going to $(q, p, y)$, each in a one-parameter family. There are three possible such triangles, depicted as the shaded grey area on the left of Figure 4.6. The one-parameter family comes from forming slits along the “red” and “blue” direction, with the parameter governing how big the slit is.

2. Convex simple triangles with vertices going to $(q, p, x)$. There is one isolated such triangle in the $M_1$ factor.

Denote $\Delta(q, p, y)$ and $\Delta(q, p, x)$ to be the set of all such triangles with vertices going to $(q, p, y)$ and $(q, p, x)$ respectively.

![Figure 4.6: The holomorphic triangles in $\Delta(q, p, y)$](image)

We will use the above counts of triangles to show:

**Statement 14.** Given any point $\beta$ on $L_2$, it lies on the boundary of some triangle in $\Delta(q, p, y)$. Moreover, for generic $\beta$ the count of such triangles is equal to 1.
**Proof.** Given any point $\beta$ on $L_2$, since $L_2$ is a product Lagrangian, we write $\beta = (\beta_1, \beta_2)$. We want to find a holomorphic triangle $T$ in $\Delta(q,p,y)$ with an extra marked point $\xi$ on the $L_2$ boundary edge such that the holomorphic triangle $T$ maps $\xi$ to $\beta$. Asking that the extra marked point $\xi$ on $T$ maps, in the $M_2$ factor, to $\beta_2$ fixes the position of $\xi$ on the boundary of $T$ as the holomorphic triangles in the $M_2$ factor are rigid. In the $M_1$ factor, it will hit a point $\xi_1$, which we claim that we can make it equal to $\beta_1$.

In the $M_1$ factor, as we have listed above, the triangles $\Delta(q,p,y)$ come in three one-parameter families. As we let the parameter vary, the extra marked point $\xi_1$ will trace out a 1 dimensional path in $N$. Consider the top family of concave triangles. On the one end, as the red slit gets closer towards $(N_2 \times C_2)$, the concave holomorphic triangle breaks into a convex holomorphic triangle with vertices on $(q, p, x)$ (a holomorphic triangle as described in (2) previously) and a holomorphic bigon with vertices to $(x, y)$. Denote the position of $\xi_1$ at this configuration to be $p_0 \in N$. On the other end, as the blue slit gets closer towards $(N_1 \times C_1)$, the concave holomorphic triangle breaks into a convex holomorphic triangle with vertices on $(q, q', y)$ and a holomorphic bigon with vertices to $(q', p)$. Denote the position of $\xi_1$ at this configuration to be $p_1 \in N$. The point $\xi$ sweeps all the points in $N$ from point $p_0$ to $p_1$.

In the middle family of concave triangles, on the one end as the red slit gets closer towards $N$, the concave holomorphic triangle breaks into a convex holomorphic triangle with vertices on $(q, q', y)$, and a holomorphic bigon with vertices on $(q', p)$. The convex holomorphic triangle with vertices on $(q, q', y)$ coincides exactly with the convex holomorphic triangle described in the previous paragraph, and hence $\xi_1$ is again at $p_1 \in N$. On the other end of this family, as the blue slit approaches $N$, the concave triangle splits into a convex triangle with vertices on $(p', p, y)$, and a holomorphic bigon with vertices on $(p', q)$. Denote the position of $\xi_1$ on $N$ to be $p_2$. In the second family of concave holomorphic triangles, $\xi_1$ traces all points in $N$ from $p_1$ to $p_2$.

Applying a similar argument as before, following the third family of concave holomorphic disks $\xi_1$ now goes from $p_2$ back to $p_0$, concluding the proof of the statement that any point on $L_2$ lies on the boundary of some triangle in $\Delta(q, p, y)$, and that the count of such triangles is $1 \pmod{2}$. 

Repeating the proof of the previous statement with an extra fourth marked point $\xi$ on the $N_1 \times C_1$ boundary edge (or $N_2 \times C_2$) shows the following statements:

**Statement 15.** Given any point $\beta$ on $N_1 \times C_1$ (or $N_2 \times C_2$), it lies on the boundary of some triangle in $\Delta(q, p, y)$. For generic $\beta$ the count of such triangles is equal to 1.

The $\mu^2$ that we will be using below are between three distinct embedded Lagrangians, which will count the usual rigid (index 0) holomorphic triangles with edges going to the three distinct Lagrangians.

**Statement 16.** $\mu^2(y, q) = \mu^2(p, y) = \mu^2(q, p) = 0$
Proof. Let’s compute $\mu^2(y, q)$. There is a single convex triangle with vertices on $(q', y, q)$ in the $M_1$ factor. In the $M_2$ factor there are two simple triangles. Counting mod 2 the corresponding triangles in $M_1 \times M_2$ contribute 0 to the coefficient in front of $q' \in CF(L_2, N_2 \times C_2)$. A similar argument applies to $\mu^2(p, y)$ and $\mu^2(q, p)$ as well. \qed

The $\mu^3$ considered below will be a specific case of $\mu^3$ described in section 2 where the first and the last Lagrangian are identical. Since $L_2, N_1 \times C_1$, and $N_2 \times C_2$ are embedded, the only contributions are from holomorphic triangles with an extra marked point attached to a Morse flow line (left side of Figure 4.7).

Figure 4.7: Configurations considered in $\mu^3$

**Statement 17.** $\mu^3(q, p, y) = \min(N_2 \times C_2), \mu^3(p, y, q) = \min(L_2), \mu^3(y, q, p) = \min(N_1 \times C_1)$ where $\min$ is the generator corresponding to the minimum of the Morse function chosen to define the Morse complex.

Proof. Let’s compute $\mu^3(p, y, q) = \min(L_2)$. Statement 14 shows that any point $\beta$ on $L_2$ lies on the boundary of some triangle in $\Delta(p, y, q)$. In particular, $\min(L_2)$ lies on the boundary of some triangle in $\Delta(p, y, q)$ and the count of such triangles is equal to one. This contributes a count of one to the coefficient of $\min(L_2)$ when computing $\mu^3(p, y, q)$, and for dimension reasons there are no other contributions to $\mu^3(p, y, q)$. A similar argument applies to $\mu^3(q, p, y)$ and $\mu^3(y, q, p)$ as well. \qed

Recalling that the minima correspond to the identity endomorphisms, using the characterization of exact triangles ([6] Section 3) the last two statements imply:

**Statement 18.**

$$N_2 \times C_2 \xrightarrow{q} L_2 \xrightarrow{p} N_1 \times C_1 \xrightarrow{q} N_2 \times C_2$$

forms an exact triangle.
To move from \(L_1\) to \(L_1(\epsilon)\), we appeal to a result from Fukaya, Oh, Ohta, Ono ([3] Theorem 55.7):

**Theorem 19 (FOOO).** Let \(K\) be a compact subset of \(M(L_0, L_1, L_2, u_{01}, u_{12}, u_{20})\) and \(U\) be a relatively compact open neighborhood of \(K\) inside \(M(L_0, L_1, L_2, u_{01}, u_{12}, u_{20})\). Let \(M(L_\epsilon, L_0, u_{01}, u_{20}, K, \epsilon_2)\) be the set of elements in \(M(L_\epsilon, L_0, u_{01}, u_{20})\) represented by a \(J\)-holomorphic map \(w\) satisfying

\[
\max_{z \in \mathbb{D}^2} \text{dist}(w(z), w_{\text{tri}}(x)) \leq \epsilon_2
\]

for some \(w_{\text{tri}} \in K\).

Assume moreover that every \(w_{\text{tri}} \in U\) has multiplicity one at \(u_{12}\) and is Fredholm regular. For each sufficiently small \(\epsilon_2\) and \(\epsilon_1\), there exists an open neighborhood \(M(L_{\epsilon_1}, L_0, u_{01}, u_{20}, K, \epsilon_2)^+\) of \(M(L_\epsilon, L_0, u_{01}, u_{20}, K, \epsilon_2)\) inside \(M(L_\epsilon, L_0, u_{01}, u_{20})\) and a map

\[
\pi : M(L_{\epsilon_1}, L_0, u_{01}, u_{20}, K, \epsilon_2)^+ \to U
\]

such that:

1. Every element of \(M(L_{\epsilon_1}, L_0, u_{01}, u_{20}, K, \epsilon_2)^+\) is Fredholm regular.
2. If \([w] \in M(L_{\epsilon_1}, L_0, u_{01}, u_{20}, K, \epsilon_2)^+\) and \(\pi([w]) = [w_{\text{tri}}]\) then we have

\[
\text{dist}(w(z), w_{\text{tri}}(z)) \leq C\epsilon_2
\]

by re-choosing the representative \(w\) in the class \([w]\) if necessary.
3. If \(\epsilon_1 < 0\) then the restriction \(\pi^{-1}(K) \to K\) of \(\pi\) is a diffeomorphism.

When we apply this theorem, \(L_0\) will be \(L_2\), \(L_1\) and \(L_2\) will be \(N_1 \times C_1\) and \(N_2 \times C_2\) respectively.

The previous list of holomorphic triangles involving the generators \(q\) and \(p\), after Lagrangian surgery, becomes:

1. One-parameter families of concave holomorphic bigons, roughly looking like the concave triangles before, but with one corner rounded.
2. The smaller triangle in \(M_1\) times one of the triangles in \(M_2\). (Unchanged as it does not pass through the self intersection \(y\) to begin with.)

To define

\[
\mu^2 : CF(L_2, L_1(\epsilon)) \otimes CF(L_1(\epsilon), L_2) \to CF(L_1(\epsilon), L_1(\epsilon))
\]

we look at \(M((L_1(\epsilon), L_2, L_1(\epsilon)), a, b, c), \) where \(a \in CF(L_1(\epsilon), L_2), b \in CF(L_2, L_1(\epsilon)), c \in CF(L_1(\epsilon), L_1(\epsilon))\), consisting of:
1. If \( c \in CM(g) \), holomorphic bigons with boundary on \( L_1(\epsilon), L_2 \) and vertices mapping to \( a, b \), followed by a Morse flow trajectory of \( g \) on \( L_1(\epsilon) \) from the \( L_1(\epsilon) \) boundary of the bigon to \( c \). (The left picture in Figure 4.8).

2. If \( c \in \bar{R} \), there are two cases: a holomorphic triangle with boundary in \( L_1(\epsilon), L_2, L_1(\epsilon) \) and vertices mapping to \( a, b, c \); or a holomorphic bigon with boundary on \( L_1(\epsilon), L_2 \) and vertices to \( a, b \), followed by a Morse flow trajectory from the \( L_1(\epsilon) \) boundary of the bigon to the boundary of a teardrop that passes through the self-intersection \( c \). (The other two pictures in Figure 4.8).

\[
\mu^2(b,a) = \sum |M((L_1(\epsilon), L_2, L_1(\epsilon)), a, b, c)| \cdot c
\]

Similarly, we can define \( \mu^2 : CF(L_1(\epsilon), L_2) \otimes CF(L_2, L_1(\epsilon)) \to CF(L_2, L_2) \) by counting points in the 0-dimensional moduli space \( M((L_2, L_1(\epsilon), L_2), a, b, c) \) consisting of holomorphic bigons with boundary on \( L_2, L_1(\epsilon) \) and vertices at \( a, b \), followed by a Morse flow trajectory of \( h \) on \( L_2 \) from the \( L_2 \) boundary of the bigon to \( c \).

---

**Statement 20.** \( L_1(\epsilon) \) and \( L_2 \) are isomorphic

*Proof.* We will proceed by showing that \( \mu^2(q,p) \in CF(L_2, L_2) \) is the minimum of \( h \) in \( CM(h) \) and \( \mu^2(p,q) \in CF(L_1(\epsilon), L_1(\epsilon)) \) is the minimum of \( g \) in \( CM(g) \).

Since \( L_2 \) has no self intersections, \( \mu^2(q,p) \) will only involve holomorphic bigon with boundary on \( L_2, L_1(\epsilon) \) and vertices to \( p, q \) and Morse gradient flow lines in \( L_2 \). Consider \( K \subset M(L_2, N_1 \times C_1, N_2 \times C_2, p, y, q) \) consisting of configurations considered in statement 14 where the triangle is not too close to a degenerate one and the extra marked point \( \xi \) is not too close to the vertices of the holomorphic triangles.

Let \( U \) be a relatively compact open neighborhood of \( K \) in \( M(L_2, N_1 \times C_1, N_2 \times C_2, p, y, q) \). The theorem of FOOO tells us that there exists \( M(L_1(\epsilon), L_2, p, q, K, \epsilon_2)^+ \) and a map \( \pi : \)
$M(L_1(\epsilon), L_2, p, q, K, \epsilon_2)^+ \to U$ that is a diffeomorphism on $\pi^{-1}(K) \to K$. This means that we can find a one dimensional family of holomorphic bigons in $M(L_1(\epsilon), L_2, p, q, K, \epsilon_2)^+$ that is diffeomorphic to $K$. Furthermore, each holomorphic bigon is $\epsilon_2$ close to the corresponding holomorphic triangle in $K$. Note also that the limit as $\epsilon \to 0$ of any bigon with boundary on $L_1(\epsilon)$ and $L_2$ is a triangle with boundary on $L_1$ and $L_2$, so for $\epsilon$ small enough, all bigons of interest are covered.

We’ve shown in statements 14 and 17 that any point (in particular the minimum of $h$ on $L_2$) lies on the boundary of a unique (mod 2) triangle in $M(L_2, N_1 \times C_1, N_2 \times C_2, p, y, q)$. Restricting to the subset $K$ only restricts us to be outside of an arbitrarily small neighborhood of the vertices $p, y, q$ and of the boundaries of the configuration when the domain degenerates. Thus the minimum of $h$ can always be arranged to lie on the boundary of some holomorphic triangle in $K$ and of no triangle outside of $K$. Applying the FOOO theorem now tells us that the minimum of $h$ also lies on the boundary of some holomorphic bigon in $M(L_1(\epsilon), L_2, p, q, K, \epsilon_2)^+$ and no other bigon. Thus in the count of $\mu_2^2(q, p)$, the coefficient of $\min(h)$ will be 1. Thus $\mu_2^2(q, p) = \min(h)$. An identical argument shows that the coefficient of $\min(g)$ in $\mu_2^2(p, q)$ is 1. On the other hand, the two convex triangles with vertices mapping to $(q, p, x)$ contribute 0 mod 2 to the coefficient of $x$ in $\mu_2^2(p, q)$, as shown in Statement 16. Hence $\mu_2^2(p, q) = \min(g)$. Again since the minima correspond to the identity endomorphisms, we see that $\mu_2^2(q, p) = Id$ and $\mu_2^2(p, q) = Id$, proving that $q, p$ are the desired isomorphisms between $L_1(\epsilon)$ and $L_2$. 

$\square$
Bibliography


