A COMPARATIVE STUDY OF LINEAR AND NONLINEAR
MIMO FEEDBACK CONFIGURATIONS

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Abstract

In this paper, we compare several feedback configurations which have appeared in the literature (e.g. unity-feedback, model-reference, etc.). We first consider the linear time-invariant multi-input multi-output case. For each configuration, we specify the stability conditions, the set of all achievable I/O maps and the set of all achievable disturbance-to-output maps, and study the effect of various subsystem perturbations on the system performance. In terms of these considerations, we demonstrate that one of the configurations considered is better than all the others. The results are then extended to the nonlinear multi-input multi-output case.
Table of Contents

I. Introduction 1

II. Single-degree of freedom design: $^1S(P,C)$ 4
   II.1. The system I/O map 4
   II.2. Stability conditions 4
   II.3. Properties of $^1S(P,C)$ 5

III. Two-degree of freedom design group 1: the configurations $\Sigma_a$, $\Sigma_b$, $\Sigma_c$, and $\Sigma_d$ 8
   III.1. The system I/O map 9
   III.2. Stability conditions 9
   III.3. Properties of $\Sigma_a$, $\Sigma_b$, $\Sigma_c$, and $\Sigma_d$ 11

IV. Two-degree of freedom design group 2: the configurations $\Sigma_e$ and $\Sigma_f$ 16
   IV.1. The system I/O map 16
   IV.2. Stability conditions 18
   IV.3. Properties of $\Sigma_e$ and $\Sigma_f$ 18

V. The configurations $\Sigma_a$, $\Sigma_b$, $\Sigma_c$, and $\Sigma_d$ -- the nonlinear case 21
   V.1. The partial system I/O map 23
   V.2. Stability conditions 24
   V.3. Properties 25

VI. Conclusions 29

References 31

Appendix 35

List of Footnotes

List of Figure Captions
I. Introduction

The control system designer must meet various design specifications and to achieve them he has many design configurations to choose from. The standard unity-feedback linear system is the subject of most control textbooks [D'Az. 1, Dor. 1, etc.]. Horowitz discusses briefly a number of different configurations and, in particular the "two-degree of freedom" designs [Hor. 1]. We note also the two-input one-output controller proposed by Astrom [Ast. 1] and developed by Pernebo [Per. 1] and by Desoer and Gustafson [Des. 1] as well as the controller structures used in the model reference adaptive control systems [Lan. 1, Sas. 1].

In this paper, we compare, in a systematic way, several design configurations which have been proposed in the literature. We study first the linear multi-input multi-output case; some of the results are then extended to the nonlinear case.

We adopt the following notations throughout this paper. Let \( \mathbb{R}(\mathbb{C}) \) denote the field of real (complex, resp.) numbers. Let \( \mathbb{R}(s) (\mathbb{P}(s), \mathbb{P}_0(s)) \) denote the set of all rational functions (proper rational functions, strictly proper rational functions, resp.) in \( s \) with real coefficients. Let \( \mathbb{P}(s)^{m \times n}(\mathbb{P}_0(s)^{m \times n}, \mathbb{C}^{m \times n}) \) denote the set of \( m \times n \) matrices with elements in \( \mathbb{R}(s)(\mathbb{P}_0(s), \mathbb{C}, \text{resp.}) \). For \( P \in \mathbb{R}(s)^{m \times n} \), let \( \mathcal{P}[P] \) \( \mathcal{Z}[P] \) denote the list of all poles (all zeros, resp.) of \( P \). For \( A \in \mathbb{C}^{m \times n} \), let \( \|A\| \) denote the maximal singular value of \( A \).

For the given linear time-invariant multi-input multi-output plant \( P(s) \), any linear output feedback design can be represented as the system \( \Sigma(P, K) \) shown in Fig. 1, where the compensator \( K \) has two inputs \( u_1 \) and \( y_2 \), and one output \( y_1 \). Since \( K \) is linear, it is uniquely specified by the transfer function from \( u_1 \) to \( y_1 \) and the transfer function from \( y_2 \) to \( y_1 \) denoted respectively by \( k_{y_1 u_1} \) and \( k_{y_1 y_2} \). More precisely, with
The matrix $H_{yu}$ in (1.1) shows that only two submatrices of $H_{yu}$ can be independently specified by a suitable choice of $\pi$ and $F$. Therefore, however complicated the structure of the linear compensator $K$ may be, there are only two closed-loop maps that can be independently specified.

In most design problems, the two most important maps are $H_{y_2u_1}$ and $H_{y_2d_0}$: $H_{y_2u_1}$ is the map from input $u_1$ to output $y_2$ and $H_{y_2d_0}$ is the map from output-disturbance $d_0$ to output $y_2$. They specify respectively the servo-performance and regulator-performance of the feedback system $\Sigma(P,K)$.

In general, the compensator $K$ is implemented as interconnections of several subsystems. Different interconnections of such subsystems result in different feedback configurations. Following Horowitz, [Hor. 1] we say that a feedback configuration is a two-degree of freedom design iff an appropriate choice of the compensation subsystems (i.e. any subsystems that are not the given plant) will change the input-output map $H_{y_2u_1}$ without affecting the disturbance-to-output map $H_{y_2d_0}$, or vice versa. A feedback configuration is said to be a single-degree of freedom design iff it is not a two-degree of freedom design. A transfer function $H(s) \in \mathbb{R}(s)^{m \times n}$ is said to be exp. stable iff a) $H(s)$ is proper and b) all its poles have negative real part. A linear time-invariant feedback
configuration is said to be \textit{exp. stable} iff the system I/O map from any exogenous input to any subsystem input and to any subsystem output is exp. stable.

Throughout Section I-Section IV, we assume that

(A.1) All subsystems which make up the feedback configuration under study are represented by transfer functions \( P(s), C(s), \ldots \) etc. with elements in \( \mathbb{R}_p(s) \); furthermore none of these subsystems have unstable hidden modes;

(A.2) \( P_0(s), P(s) \in \mathbb{R}_p(s)^{n_0 \times n_1} \); \( C(s), C_0(s), C_1(s), C_2(s), Q(s), Q_0(s) \) and \( Q_1(s) \in \mathbb{R}_p(s)^{n_0 \times n_1} \).

We say that the map \( H \) is an \textit{achievable I/O map} (disturbance-to-output map, resp.) of the linear feedback configuration \( ^1\Sigma(P,K) \) iff by some appropriate choice of the compensation subsystems satisfying (A.1),

(i) \( H_{y_2u_1} = H \), \( H_{y_2d_0} = H \), resp.;

(ii) \( ^1\Sigma(P,K) \) is exp. stable.

For each feedback configuration studied in this paper, we obtain stability conditions, the set of all achievable I/O maps and the set of all achievable disturbance-to-output maps; we compute the effects of various subsystem perturbations on the I/O map \( H_{y_2u_1} \). Based on these considerations, we demonstrate that the configuration \( \Sigma_b \) (in Section III) is the best among the configurations considered.

The paper is organized as follows: Section II reviews the properties of the unity-feedback configuration \( ^1\Sigma(P,C) \); the various two-degree of freedom design configurations are studied in Section III and Section IV. Section V extends the results of Section III to the nonlinear case. Section VI is a brief summary of the paper.
II. Single-degree of freedom design: the unity feedback system $S(P,C)$

The unity feedback system $S(P,C)$ shown in Fig. 3 has been studied extensively in the control literature [For. 1, Kai. 1, Oga. 1, Cal. 2, Des. 2, Doy. 1, Vid. 1, Chen. 1]. In this section, we review some of the properties associated with this configuration for the linear time-invariant lumped multi-input multi-output case. Equation (2.1) below shows that $S(P,C)$ is a single-degree of freedom design.

II.1. The system I/O map

Let $P$ and $C$ satisfy (A.2). For $S(P,C)$, the system I/O map $H_{yu} : (u_1, u_2, d_0) \rightarrow (y_1, y_2)$ is given by

$$H_{yu} = \begin{bmatrix}
C(I+PC)^{-1} & -CP(I+CP)^{-1} & -C(I+PC)^{-1} \\
PC(I+PC)^{-1} & P(I+CP)^{-1} & (I+PC)^{-1}
\end{bmatrix}$$

(2.1)

Assumption (A.2) guarantees that all the inverses are well-defined matrices with elements in $\mathbb{R}_p(s)$.

II.2. Stability conditions of $S(P,C)$

It is easy to check (using the summing node equations) that $S(P,C)$ is exp. stable iff $H_{yu}$ is exp. stable. Hence, by inspection of (2.1) and the identity $I-M(I+M)^{-1} = (I+M)^{-1}$, we have that

$S(P,C)$ is exp. stable $\Leftrightarrow C(I+PC)^{-1}, (I+CP)^{-1}, (I+PC)^{-1}$ and $P(I+CP)^{-1}$ are exp. stable

(2.3)

Note that any one of the four maps in (2.3) can be unstable, while the other three are exp. stable [Des. 2]. It is well-known [Des. 2] that if $P$ is exp. stable, then
\( S(P,C) \) is exp. stable \( \iff \) \( C(I+PC)^{-1} \) is exp. stable \hspace{1cm} (2.4)

For the discussions to follow, it is convenient to note that [Des. 3]

a) \( Q := C(I+PC)^{-1} \leftrightarrow C = Q(I-PQ)^{-1} \) \hspace{1cm} (2.5)

b) \( Q \) is proper (strictly proper) if and only if \( C \) is proper

(strictly proper, resp.); and

c) with \( Q := C(I+PC)^{-1} \),

\[
H_{yu} = \begin{bmatrix}
Q & -QP & -Q \\
-1 & -1 & -1 \\
PQ & P(I-QP) & I-PQ
\end{bmatrix}
\] \hspace{1cm} (2.1a)

The importance of Eq. (2.1a) is that all the I/O properties of \( S(P,C) \)
are specified by \( P \) and \( Q \), without requiring any inverse.

II.3. Properties of \( S(P,C) \)

- Dependence of the I/O map and the disturbance-to-output map

Equation (2.1a) shows that the choice of \( Q := C(I+PC)^{-1} \) determines
simultaneously the I/O map \( H_{y_2u_1} = PQ \) and the disturbance-to-output map
\( H_{y_2d_0} = I-PQ \); clearly we have

\[
H_{y_2u_1} + H_{y_2d_0} = I
\] \hspace{1cm} (2.6)

- Achievable I/O and disturbance-to-output maps

Recall that the map \( H \) is an achievable I/O map (disturbance-to-output map) of \( S(P,C) \) iff for some choice of \( C \in \mathbb{R}^{n_x \times n_o} \), (i) \( H_{y_2u_1} = H \),
\( H_{y_2d_0} = H \), resp.); (ii) \( S(P,C) \) is exp. stable. Let \( \mathcal{H}_{y_2u_1} \) (\( \mathcal{H}_{y_2d_0} \))
denote the set of all achievable I/O maps \( H_{y_2u_1} \) (the set of all
achievable disturbance-to-output maps \( H_{y_2d_0} \), resp.). Then clearly,
\( \mathcal{H}_{y_2u_1}(P) = \{PQ|Q := C(I+PC)^{-1}, \text{where } C \text{ is such that } \mathcal{I}S(P,C) \text{ is exp. stable}\} \) \hspace{1cm} (2.7a)

\( \mathcal{H}_{y_2d_0}(P) = \{I-PQ|Q := C(I+PC)^{-1}, \text{where } C \text{ is such that } \mathcal{I}S(P,C) \text{ is exp. stable}\} \) \hspace{1cm} (2.7b)

Let \( \mathcal{G}(P) \) be the set of all compensators \( C \) that result in \( \mathcal{I}S(P,C) \) exp. stable. Equations (2.7) show that \( \mathcal{G}(P) \) completely characterizes \( \mathcal{H}_{y_2u_1}(P) \) and \( \mathcal{H}_{y_2d_0}(P) \).

If the plant \( P \) is exp. stable, then examination of (2.4), (2.5) and (2.5a) shows that Eqs. (2.7) can be more explicitly written as

\[ \mathcal{H}_{y_2u_1}(P) = \{PQ|Q \text{ is exp. stable}\} \] \hspace{1cm} (2.8a)

\[ \mathcal{H}_{y_2d_0}(P) = \{I-PQ|Q \text{ is exp. stable}\} \] \hspace{1cm} (2.8b)

If the plant is not exp. stable, then (2.1a) shows that further constraints on \( Q \), in addition to exp. stability, are needed to ensure exp. stability of \( \mathcal{I}S(P,C) \). There are two-approaches in the literature in characterizing the class of all compensators \( C \) which result in an exp. stable \( \mathcal{I}S(P,C) \) for a given unstable plant \( P \). The first approach is the two-step stabilization scheme proposed by Zames [Zam. 1] and extended by Desoer and Lin [Des. 4]. The second approach uses fractional representations for the plant and the compensator. Youla, Bongiorno and Jabr [You. 1] used polynomial factorization to characterize the class of all stabilizing compensators for a given linear lumped (not necessarily stable) plant. Using more general factorizations Callier and Desoer extended the results to the linear distributed case [Cal. 2]. Further extension into a general algebraic setting was obtained by Desoer, Liu, Murray and Saeks [Des. 5], and by Vidyasagar, Schneider and Francis.
For the special case where the unstable plant $P$ contains only one or a few unstable poles, Desoer and Gustafson [Des. 6] obtained $\gamma y_2 u_1 (P)$ by explicitly specifying the additional constraints on $Q$ required for stability.

- **Plant perturbation**

In practice, the given plant $P$ is usually not known exactly, therefore the design must be based on a certain nominal value of the plant, say $P_0$. Plant variation also contributes to make $P$ different from $P_0$. By plant perturbation, we mean the difference between the actual plant $P$ and the nominal $P_0$. For $\mathcal{L}_S(P_0, C)$ with the given nominal plant $P_0$, the plant perturbation $P_0 \rightarrow P_0 + \Delta P := P$ entails

$$
\Delta H y_2 u_1 := H y_2 u_1 - H^0 y_2 u_1 = (I + PC)^{-1} \Delta P C (I + P_0 C)^{-1}
$$

(2.9)

where $H^0 y_2 u_1$ is the nominal input-output map. Standard derivation of (2.9) can be found in [Cru. 1, Cal. 1].

- **Remark**

Equation (2.6) of the $\mathcal{L}_S(P, C)$ configuration constrains the design, hence a compromise between servo performance and regulation (desensitization) is necessary. For example, suppose the design objectives are

(i) $\overline{\sigma[H y_2 d_0 (j\omega)]} = \overline{\sigma[I - PQ(j\omega)]} \ll 1$ for all $\omega \in [0, \omega_d]$; and

(ii) $\overline{\sigma[H y_2 u_1 (j\omega)]} = \overline{\sigma[PQ(j\omega)]} \ll 1$ for all $\omega \in (\omega_0, \infty)$, with $\omega_0 < \omega_d$.

It is clear that there are conflicting requirements over the frequency interval $(\omega_0, \omega_d)$: objective (i) requires that the product $PQ$ be close to the identity matrix over $(\omega_0, \omega_d)$, while objective (ii) requires that $PQ$ be close to the zero matrix over $(\omega_0, \omega_d)$. 

-7-
In this section, we study the four feedback configurations 
\( \Sigma_a, \Sigma_b, \Sigma_c, \) and \( \Sigma_d \) shown in Fig. 4. It is assumed that \( C_0 \) and \( Q_0 \) are related by \( Q_0 = C_0(I+P_0C_0)^{-1} \) or equivalently \( C_0 = Q_0(I-P_0Q_0)^{-1} \). As shown in Fig. 4, each of these four configurations falls into the scheme of Fig. 1: \( K \) is the two-input, namely \( u_1 \) and \( y_2 \), one-output, namely \( y_1 \), compensator. In these four cases, \( K_{y_1u_1} = (I+C_1P_0)Q_0 \) and \( K_{y_1y_2} = -C_1 \). Therefore \( \Sigma_a, \Sigma_b, \Sigma_c, \) and \( \Sigma_d \) have the same system I/O map \( H_{yu}(u_1, u_2, d_0) \mapsto (y_1, y_2) \). Equation (3.1) below shows that each of the four configurations is a two-degree of freedom design: indeed, for \( P = P_0 \), \( H_{y_2u_1} = P_0Q_0 \), \( H_{y_2d_0} = (I+P_0C_1)^{-1} \).

\( \Sigma_a \) has a model reference structure: \( P \) is the given plant, \( P_0 \) is the nominal plant model, \( Q_0 \) is the precompensator, and \( C_1 \) is the "comparator." Note that if the plant is nominal (i.e. \( P = P_0 \)) and if there is no disturbance (i.e. \( n_1 = d_0 = u_2 = 0 \)), then there is no feedback in this configuration. \( \Sigma_a \) has been called conditional feedback in [Hor. 1, p. 246] for the single-input single-output case.

\( \Sigma_b \) has also a model reference structure. The important difference between \( \Sigma_a \) and \( \Sigma_b \) is the following: in \( \Sigma_b \), the map \( H_{\xi_1u_1} : u_1 \mapsto \xi_1 \) is the result of a closed-loop configuration, whereas in \( \Sigma_a \), \( H_{\xi_1u_1} \) is the result of an open-loop configuration. The structure of \( \Sigma_b \) has been used by Meyer et al. in the design of flight control systems [Mey. 1]. For the configuration \( \Sigma_b \), it is easy to see that \( H_{\xi_1u_1} = Q_0 \).

\( \Sigma_c \) consists of the given plant \( P \), the precompensator \( (I+C_1P_0)Q_0 \), and the feedback compensator \( C_1 \). We assume that the compensator \( (I+C_1P_0)Q_0 \) is built as one transfer function. Zames used the structure of \( \Sigma_c \) in the study of effects of plant uncertainty [Zam. 1, p. 316].
\( \Sigma_d \) is obtained from \( \Sigma_c \) by introducing the transfer function pair \( \pi \) and \( \pi^{-1} \) as shown in Fig. 4d. We assume that the precompensator \( \pi^{-1}(I+C_1P_0)Q_0 \) and the feedback compensator \( \pi^{-1}C_1 \) are each built as a single transfer function, and b) have all their elements in \( \mathbb{R}_p(s) \).

When the given plant is nominal i.e., \( P = P_0 \), we call the resulting nominal feedback configuration and denote it by \( \Sigma_a^0, \Sigma_b^0, \Sigma_c^0, \) and \( \Sigma_d^0 \) respectively. We use \( H_{yu}^0, H_{y2u1}^0, \) and \( H_{y2d0}^0 \) to denote respectively the system I/O map, the input-output map and the disturbance-to-output map of \( \Sigma_a^0, \Sigma_b^0, \Sigma_c^0, \) and \( \Sigma_d^0 \).

III.1. The system I/O map

The system I/O map \( H_{yu} : (u_1, u_2, u_0) \mapsto (y_1, y_2) \) of \( \Sigma_a, \Sigma_b, \Sigma_c, \) and \( \Sigma_d \) is given by (3.1), for the nominal system, and by (3.2) for the case where \( P \neq P_0 \).

\[
H_{yu}^0 = \begin{bmatrix}
Q_0 & -C_1P_0(I+C_1P_0)^{-1} & -C_1(I+P_0C_1)^{-1} \\
-P_0Q_0 & P_0(I+C_1P_0)^{-1} & (I+P_0C_1)^{-1}
\end{bmatrix}
\]

(3.1)

When \( P \neq P_0 \), (see derivation in Appendix)

\[
H_{yu}^0 = \begin{bmatrix}
(I+C_1P)^{-1}(I+C_1P_0)Q_0 & -C_1P(I+C_1P)^{-1} & -C_1(I+PC_1)^{-1} \\
P(I+C_1P)^{-1}(I+C_1P_0)Q_0 & P(I+C_1P)^{-1} & (I+PC_1)^{-1}
\end{bmatrix}
\]

(3.2)

III.2. Stability conditions of \( \Sigma_a^0, \Sigma_b^0, \Sigma_c^0, \) and \( \Sigma_d^0 \)

Recall the definition of the exp. stability of a linear feedback configuration. It can be easily checked (using the summing node equations) that \( \Sigma_a^0 \) is exp. stable iff the map \( H^0 : (u_1, u_2, d_0, n_1) \mapsto (y_1, y_2, y_0, z_1) \) of
$\Sigma_a^0$ is exp. stable. Hence, by (i) Eq. (3.1), (ii) inspection of the configuration $\Sigma_a$ in Fig. 4a, and (iii) that the composition of exp. stable maps is exp. stable, we conclude that

$$\Sigma_a^0 \text{ is exp. stable } \Leftrightarrow P_0, Q_0, \text{ and } l^1S(P_0, C_1) \text{ are exp. stable} \quad (3.3)$$

Similarly, we have

$$\Sigma_b^0 \text{ is exp. stable } \Leftrightarrow l^1S(P_0, C_0) \text{ and } l^1S(P_0, C_1) \text{ are exp. stable} \quad (3.4)$$

$$\Sigma_c^0 \text{ is exp. stable } \Leftrightarrow (I+C_1P_0)Q_0 \text{ and } l^1S(P_0, C_1) \text{ are exp. stable} \quad (3.5)$$

With the system $l^1S(P, \pi, \pi^{-1}C_1)$ defined in Fig. 5,

$$\Sigma_d^0 \text{ is exp. stable } \Leftrightarrow \pi^{-1}(I+C_1P_0)Q_0 \text{ and } l^1S(P_0, \pi, \pi^{-1}C_1) \text{ are stable} \quad (3.6)$$

The following fact relates the exp. stability of $\Sigma^0_c$ and the exp. stability of $\Sigma^0_d$.

Fact 3.1: If $\pi$ and $\pi^{-1}$ are exp. stable, then

$$\Sigma^0_c \text{ is exp. stable } \Leftrightarrow \Sigma^0_d \text{ is exp. stable.} \quad (3.7)$$

Proof: (see Appendix)

Remarks

a) $\Sigma_a$ is the only configuration that requires the nominal plant $P_0$ be stable, because there is no feedback around the model $P_0$.

b) The stability conditions (3.3)-(3.6) are robust in the following sense: suppose that in the configuration $\Sigma_a^0, \Sigma_b^0, \Sigma_c^0$ and $\Sigma_d^0$, we impose arbitrary but small (in the graph topology) perturbations on all subsystems, then [Vid. 1, Chen 2] each of the resulting perturbed systems $\Sigma_a, \Sigma_b, \Sigma_c$ and $\Sigma_d$ is also exp. stable.
III.3. Properties of $\Sigma_a^0$, $\Sigma_b^0$, $\Sigma_c^0$, and $\Sigma_d^0$

- Nominal design

The four nominal configurations $\Sigma_a^0$, $\Sigma_b^0$, $\Sigma_c^0$, and $\Sigma_d^0$ have the same system I/O map $H_{yu}^0$; furthermore $Q_0$ specifies the nominal I/O map

$$H_{yu}^0 = P_0 Q_0; \quad (3.8)$$

$c_1$ specifies the nominal disturbance-to-output map

$$H_{yd}^0 = (I + P_0 c_1)^{-1} \quad (3.9)$$

Any achievable I/O map (disturbance-to-output map) must have the form specified by (3.8) for some $Q_0$ ((3.9) for some $c_1$, resp.) where $Q_0$, ($c_1$, resp.) satisfies (A.2) and each configuration satisfies the stability requirements.

- Achievable I/O maps

We denote the set of all achievable I/O maps for $\Sigma_a^0$, $\Sigma_b^0$, $\Sigma_c^0$, and $\Sigma_d$ by $\mathcal{H}_{y_2u_1}^a$, $\mathcal{H}_{y_2u_1}^b$, $\mathcal{H}_{y_2u_1}^c$, and $\mathcal{H}_{y_2u_1}^d$, respectively.

(a) For $\Sigma_a^0$:

(i) If $P_0$ is not exp. stable, then (3.3) shows that the configuration $\Sigma_a^0$ is unstable for all $Q_0$ and $c_1$ satisfying (A.2).

(ii) If $P_0$ is exp. stable, then

$$\mathcal{H}_{y_2u_1}^a (P_0) = \{P_0 Q_0 | Q_0 \text{ is exp. stable}\} \quad (3.11a)$$

Proof of (ii):

Let $\mathcal{H} := \{P_0 Q_0 | Q_0 \text{ is exp. stable}\}$. It is clear from (3.8) and (3.3) that every achievable I/O map of $\Sigma_a^0$ is of the form $P_0 Q_0$ for some exp. stable $Q_0$. Hence, $\mathcal{H}_{y_2u_1}^a (P_0) \subset \mathcal{H}$. To show $\mathcal{H} \subset \mathcal{H}_{y_2u_1}^a (P_0)$, we note
that 1) for any \( H \in \mathcal{H} \), there exists an exp. stable \( Q_0 \) such that 
\[ H^0_{y_2u_1} = P_0Q_0 = H, \]
2) from (3.3), given that \( P_0 \) and \( Q_0 \) are exp. stable, \( \Sigma_a \) is exp. stable iff \( ^1S(P_0,C_1) \) is exp. stable, 3) there are many \( C_1 \)'s such that \( ^1S(P_0,C_1) \) is exp. stable; for example \( C_1 = 0.1, 2), \) and 3) together show that \( H \in \mathcal{H} \) implies \( H \in \mathcal{H}^a_{y_2u_1}(P_0) \), hence \( \mathcal{H} \subset \mathcal{H}^a_{y_2u_1}(P_0) \).

This proves the assertion.

(b) For \( \Sigma_b^0 \):

Since \( P_0 \in \mathbb{R}_{p_0\cdot 0}(s)^{n_0\times n_1} \), there exists \( C_1 \in \mathbb{R}_{p(s)}^{n_1\times n_0} \) such that \( ^1S(P_0,C_1) \) is exp. stable [Bra.1, You.1]. By using similar arguments as those in the proof of (3.11a), it is easily shown that

\[
\mathcal{H}^b_{y_2u_1}(P_0) = \left\{ P_0Q_0 \mid Q_0 = C_0(I+P_0C_0)^{-1} \text{ where } C_0 \text{ is such that } ^1S(P_0,C_0) \text{ is exp. stable} \right\} (3.11b)
\]

(c) For \( \Sigma_c^0 \): By Eq. (3.8) and the stability condition (3.5), we see that

\[
\mathcal{H}^c_{y_2u_1}(P_0) = \left\{ P_0Q_0 \mid Q_0 \text{ is such that } (I+C_1P_0)Q_0 \text{ is exp. stable for some } C_1 \text{ which yields } ^1S(P_0,C_1) \text{ exp. stable} \right\} (3.11c)
\]

Note that the \( Q_0 \)'s in (3.11c) are necessarily exp. stable, because

\[ \mathcal{Z}[I+C_1P_0] = \rho[(I+C_1P_0)^{-1}] \subset \mathcal{Z} \]

(d) For \( \Sigma_d^0 \) with \( \pi \) and \( \pi^{-1} \) exp. stable: Since in this case \( \Sigma_d^0 \) is exp. stable iff \( \Sigma_c^0 \) is exp stable (Fact 3.1), and \( \Sigma_d^0 \) and \( \Sigma_c^0 \) have the same input-output map \( H_{y_2u_1} \), we have

\[
\mathcal{H}^d_{y_2u_1}(P_0) = \mathcal{H}^c_{y_2u_1}(P_0) \quad (3.11d)
\]
Achievable disturbance-to-output maps

We denote the set of all achievable disturbance-to-output maps for \( \Sigma_a, \Sigma_b, \Sigma_c, \) and \( \Sigma_d \) by \( \mathcal{A}_y, \mathcal{A}_y^b, \mathcal{A}_y^c, \) and \( \mathcal{A}_y^d, \) respectively.

(a) For \( \Sigma_a \):
   (i) If \( P_0 \) is not exp. stable, then no stable design is possible.
   (ii) If \( P_0 \) is exp. stable, then (3.3) and (3.9) imply that
   \[
   \mathcal{A}_y^a(P_0) = \{ (I+P_0C_1)^{-1} \mid C_1 \text{ is such that } \mathcal{S}(P_0,C_1) \text{ is exp. stable} \} \tag{3.13a}
   
   Alternatively, if we set \( Q_1 := C_1(I+P_0C_1)^{-1} \) -- hence \( \mathcal{S}(P_0,C_1) \) is exp. stable iff \( Q_1 \) is exp. stable -- then,
   \[
   \mathcal{A}_y^a(P_0) = \{ I-P_0Q_1 \mid Q_1 \text{ is exp. stable} \}.
   
(b) For \( \Sigma_b \): The stability condition (3.4) and Eq. (3.9) show that
   \[
   \mathcal{A}_y^b(P_0) = \left\{ (I+P_0C_1)^{-1} \mid C_1 \text{ is such that } \mathcal{S}(P_0,C_1) \text{ is exp. stable; and } \mathcal{S}(P_0,C_0) \text{ is exp. stable for some } C_0 \right\}
   
   Since \( P_0 \in \mathbb{R}_p^{n_0 \times n_1} \), there always exists \( C_0 \in \mathbb{R}_p^{n_1 \times n_0} \) such that \( \mathcal{S}(P_0,C_0) \) is exp. stable. Therefore, the above expression simplifies to
   \[
   \mathcal{A}_y^b(P_0) = \{ (I+P_0C_1)^{-1} \mid C_1 \text{ is such that } \mathcal{S}(P_0,C_1) \text{ is exp. stable} \} \tag{3.13b}
   
(c) For \( \Sigma_c \): The stability condition (3.5) and Eq. (3.9) show that
   \[
   \mathcal{A}_y^c(P_0) = \left\{ (I+P_0C_1)^{-1} \mid C_1 \text{ is such that } \mathcal{S}(P_0,C_1) \text{ is exp. stable and } \text{ such that } (I+C_1P_0)Q_0 \text{ is exp. stable for some } Q_0 \right\} \tag{3.13c}
   
-13-
(d) For $\Sigma_d$ with both $\pi$ and $\pi^{-1}$ exp. stable: Since in this case, $\Sigma_d$ is exp. stable iff $\Sigma_c$ is exp. stable (Fact 3.1), we have that

$$\mathcal{H}_{y_2d_0}(P_0) = \left\{ \begin{array}{l}
(C_1 \text{ such that } s(P_0, C_1) \text{ is exp. stable and such that } (I+C_1P_0)Q_0 \text{ is exp. stable for some } Q_0 \end{array} \right\}
$$

and

$$= \mathcal{H}_{y_2d_0}(P_0)$$

Remarks

(i) For the configurations $\Sigma_a$ and $\Sigma_b$, we can simultaneously achieve any $H_{y_2u_1}^a \in \mathcal{H}_{y_2u_1}(P_0)$, $(\mathcal{H}_{y_2u_1}(P_0))$, resp.) and any $H_{y_2d_0}^a \in \mathcal{H}_{y_2d_0}(P_0)$, $(\mathcal{H}_{y_2d_0}(P_0))$, resp.) i.e., the choices of $H_{y_2u_1}^a$ and $H_{y_2d_0}^a$ (hence the choices of $Q_0$ and $C_1$) are independent. For the configurations $\Sigma_c$ and $\Sigma_d$, the choices of $H_{y_2u_1}^c$ and $H_{y_2d_0}^c$ are constrained: indeed, $Q_0$ and $C_1$ must be chosen so that the transfer function $[(I+C_1P_0)Q_0]$ is exp. stable.

(ii) Although $\Sigma_c$ and $\Sigma_d$ have the same achievable I/O maps and achievable disturbance-to-output maps, $\Sigma_d$ offers more flexibility in implementation: for example, $\pi$ may be used to adjust the signal level at the summing node.

- Plant perturbation

For $\Sigma_a$, $\Sigma_b$, $\Sigma_c$, and $\Sigma_d$, the plant perturbation $P_0 + P_0 + \Delta P := P$ entails

$$\Delta H_{y_2u_1} := H_{y_2u_1} - H_{y_2u_1}^0 = (I+PC_1)^{-1}\Delta PQ_0$$

(3.17) follows by the same calculation for (2.9).

- Model perturbation (for $\Sigma_a$ and $\Sigma_b$

By model perturbation we mean any inaccuracy and variation in the model $P_0$ (of $\Sigma_a$ and $\Sigma_b$).
(a) For $\Sigma_a^0$, the I/O map $H_{y_2u_1}^0$ is sensitive to perturbations in the model $P_0$. Indeed, the perturbation $P_0 \rightarrow P_0 + \Delta P_m$ in the model implies that (see Appendix)

$$\Delta h_{y_2u_1}^a := h_{y_2u_1}^a - H_{y_2u_1}^0 = P_0C_1(I+P_0C_1)^{-1}\Delta P_m Q_0 \quad (3.18)$$

where $\Omega_d$ is the frequency band of interest for disturbance rejection. Note that an arbitrary small but unstable $\Delta P_m$ will in general cause system instability.

(b) For $\Sigma_b^0$, the I/O map $H_{y_2u_1}^0$ is relatively insensitive to perturbations in the model $P_0$, compared to $\Sigma_a^0$. Indeed, the perturbation $P_0 \rightarrow P_0 + \Delta P_m := P$, in the model implies that (see Appendix)

$$\Delta h_{y_2u_1}^b := h_{y_2u_1}^b - H_{y_2u_1}^0 = [(I+P_0C_0)^{-1}-(I+P_0C_1)^{-1}]\Delta P_m C_0(I+P_0C_0)^{-1} \quad (3.19)$$

$$= [(I+P_0C_0)^{-1}-(I+P_0C_1)^{-1}](I+\Delta P_mQ_0)^{-1}\Delta P_mQ_0 \quad (3.19a)$$

Note that if the perturbation is small, more precisely, if

$\tilde{1}[\Delta P_m(j\omega)] \tilde{1}[Q_0(j\omega)] \ll 1$ for all $\omega \in [0, \infty)$, then (3.19a) shows that

$$\Delta h_{y_2u_1}^b \approx [(I+P_0C_0)^{-1}-(I+P_0C_1)^{-1}]\Delta P_mQ_0 \quad (3.19b)$$

- Perturbation in precompensator

For $\Sigma_a$, $\Sigma_c$ and $\Sigma_d$, the precompensator is not under feedback hence the I/O map $H_{y_2u_1}^0$ is sensitive to perturbations in the precompensator: namely, $Q_0$ in $\Sigma_a$, $(I+C_1P_0)Q_0$ in $\Sigma_c$, and $P^{-1}(I+C_1P_0)Q_0$ in $\Sigma_d$. 

-15-
Conclusions

a) $\Sigma_b$ is better than $\Sigma_a$:
   1) $\Sigma_a$ is more sensitive to changes in the model $P_0$ (see (3.18) and (3.19)).
   2) $\Sigma_b$ can accommodate unstable $P_0$'s.

b) $\Sigma_b$ is better than $\Sigma_a$, $\Sigma_c$, and $\Sigma_d$:
   $\Sigma_a$, $\Sigma_c$, and $\Sigma_d$ are sensitive to changes in the precompensator. (In $\Sigma_b$, the precompensator is realized as a feedback configuration, hence $\Sigma_b$ is less sensitive, if well designed).

c) $\Sigma_b$ is better than $\Sigma_c$ and $\Sigma_d$:
   In $\Sigma_b$, the choices of the I/O map $H^0_{y_2u_1}$ and the disturbance-to-output map $H^0_{y_2d_0}$ are independent, whereas in $\Sigma_c$ and $\Sigma_d$, the choices are constrained.

IV. Two-degree of freedom design-group 2: the configurations $\Sigma_e$ and $\Sigma_f$

The configuration $\Sigma_e$ has the same model reference structure as that of $\Sigma_a$ except that the output of the comparator $C_2$ in $\Sigma_e$ is feedback to the input of $Q_0$, rather than as in $\Sigma_a$, to the plant input. For the single-input single-output case, $\Sigma_e$ has been called model feedback by Horowitz [Hor. 1, p. 246].

The structure of $\Sigma_f$ has been considered by Cruz, et al. [Cru. 1] among others. Note that for the special case when $C_2 = I$, the configuration $\Sigma_f$ reduces to the unity-feedback configuration $^1S(P,C_0)$, with $C_0 := Q_0(I-P_0Q_0)^{-1}$.

We use $\Sigma^0_e$ and $\Sigma^0_f$ to denote the nominal feedback configurations, and $H^0_{yu}$ to denote the nominal I/O map.

IV.1. The system I/O map

For the nominal system $\Sigma^0_e$ (i.e., when $P = P_0$), the system I/O map $H^0_e : (u_1, u_2, d_0, n_1) \mapsto (y_1, y_2, y_0, e_1)$ is given by
For the nominal system $\Sigma^0_f$ (i.e., when $P = P_0$), the system I/O map $H^0_f : (u_1, u_2, d_0) \mapsto (y_1, y_2, e^f)$ is given by

\[
H^0_f = \begin{bmatrix}
Q_0 & -Q_0C_2P_0 & -Q_0C_2 \\
-P_0Q_0C_2P_0 & P_0(I-Q_0C_2P_0) & I-P_0Q_0C_2 \\
-C_2P_0 & -P_0Q_0C_2P_0 & P_0 \\
-I & -C_2P_0 & -C_2 & C_2P_0
\end{bmatrix}
\] (4.1)

When $P \neq P_0$, $\Sigma_e$ and $\Sigma_f$ have the same system I/O map $H_{yu} : (u_1, u_2, d_0) \mapsto (y_1, y_2)$: indeed, with $\Delta P := P - P_0$, (see Appendix),

\[
H_{yu} = \begin{bmatrix}
Q_0(I+C_2\Delta PQ_0)^{-1} & -(I+Q_0C_2\Delta P)^{-1}Q_0C_2P & -Q_0C_2(I+\Delta P Q_0C_2)^{-1} \\
-P_0Q_0(I+C_2\Delta PQ_0)^{-1} & (I-P_0Q_0C_2)(I+\Delta P Q_0C_2)^{-1}P & (I-P_0Q_0C_2)(I+\Delta P Q_0C_2)^{-1}
\end{bmatrix}
\] (4.3)
IV.2. Stability conditions of $\Sigma_e^0$ and $\Sigma_f^0$

It can be checked (using the summing node equations) that

the nominal configuration $\Sigma_e^0$ ($\Sigma_f^0$) is exp. stable
iff the system I/O map $H_e^0$, ($H_f^0$, resp.), is exp. stable.

Hence, by inspection of (4.1), we have that

$$\Sigma_e^0 \text{ is exp. stable } \iff P_0, Q_0 \text{ and } C_2 \text{ are exp. stable} \quad (4.5)$$

To test the exp. stability of $\Sigma_f$, we have to check all the submatrices in (4.2). However, in the special case where $P_0$ is exp. stable, $Q_0$ and $C_2$ are exp. stable implies that $\Sigma_f^0$ is exp stable.

IV.3. Properties of $\Sigma_e^0$ and $\Sigma_f^0$

- Nominal design

For $\Sigma_e^0$ and $\Sigma_f^0$, $Q_0$ specifies the nominal I/O map

$$H_e^{0} y_2 u_1 = P_0 Q_0 ; \quad (4.6)$$

$C_2$ and $Q_0$ together specifies the nominal disturbance-to-output map

$$H_d^{0} y_2 d_0 = I - P_0 Q_0 C_2 = I - H_e^{0} y_2 u_1 C_2 \quad (4.7)$$

In the following, we specify the set of all achievable I/O maps and the set of all achievable disturbance-to-output map for $\Sigma_e^0$ and $\Sigma_f^0$.

- Achievable I/O maps

(a) For $\Sigma_e^0$:

(i) If $P_0$ is not exp. stable, then $\Sigma_e^0$ is not exp. stable for any choice of $Q_0$ and $C_2$ satisfying (A.2).

(ii) If $P_0$ is exp. stable, then (4.5) and (4.6) together show that
(b) For $\Sigma_f^0$: By the stability condition (4.4) and Eq. (4.6), we have that

$$H_{y_2u_1}^f(P_0) = \{P_0Q_0|Q_0 \text{ is such that } C_2 \text{ which yields } H_f^0 \text{ exp. stable}\}$$

(4.10a)

For the special case where $P_0$ is exp. stable, it can be easily checked that

$$H_{y_2u_1}^f(P_0) = \{P_0Q_0|Q_0 \text{ is exp. stable}\}$$

(4.10b)

* Achievable disturbance-to-output maps*

(a) For $\Sigma_e^0$: If $P_0$ is exp. stable, then (4.5) and (4.7) together show

$$H_{y_2u_1}^e(P_0) = \{I-P_0Q_0C_2|Q_0 \text{ and } C_2 \text{ are exp. stable}\}
= \{I-P_0Q|Q \text{ is exp. stable}\}$$

(4.11)

(b) For $\Sigma_f^0$: By the stability condition (4.4) and Eq. (4.7), we have that

$$H_{y_2d_0}^f(P_0) = \{I-P_0Q_0C_2|Q_0 \text{ and } C_2 \text{ are such that } H_f^0 \text{ is exp. stable}\}$$

(4.12a)

For the special case where $P_0$ is exp. stable, we have that $Q_0$ and $C_2$ are exp. stable implies that $\Sigma_f^0$ is exp. stable, hence,

$$H_{y_2d_0}^f(P_0) \supset \{I-P_0Q_0C_2|Q_0 \text{ and } C_2 \text{ are exp. stable}\}.$$ Therefore,

$$H_{y_2d_0}^f(P_0) \supset \{I-P_0Q|Q \text{ is exp. stable}\}.$$ (4.12b)

However, the stability condition (4.4) and Eq. (4.2) show that $\Sigma_f^0$ is exp. stable implies that the product $Q_0C_2$ is exp. stable, hence
We conclude from (4.12b) and (4.12c) that if $P_Q$ is exp. stable, then

$$\mathcal{H}_{y_2d_0}^f(P_Q) = \{I-P_Q|Q \text{ is exp. stable}\}$$ (4.12d)

**Plant Perturbation**

For $\Sigma_0^0$ and $\Sigma_f^0$, the plant perturbation $P_0 + P_0 + \Delta P$ entails

$$\Delta H_{y_2u_1} = H_{y_2u_1} - H_{y_2u_1}^0 = (I-P_0Q_2)(I+\Delta PQ_0C_2)^{-1}\Delta PQ_0$$ (4.13)

(see Appendix for the derivation of (4.13))

**Model Perturbation (for $\Sigma_e^0$)**

For $\Sigma_e^0$, the I/O map $H_{y_2u_1}^0$ is sensitive to perturbations in the model $P_0$. Indeed, the model perturbation $P_0 + P_0 + P_m$ implies that $\Delta H_{y_2u_1}^e$, the corresponding change in $H_{y_2u_1}^0$, is given by (see Appendix)

$$\Delta H_{y_2u_1}^e = P_0Q_0C_2P_mQ_0(I-C_2(\Delta P_mQ_0)^{-1})$$ (4.14)

$$= P_0Q_0C_2(I-\Delta P_mQ_0C_2)^{-1}\Delta P_mQ_0$$ (4.15)

If $\Delta H_{y_2d_0}(j\omega) \leq 1$ for all $\omega \in \Omega_d$ and if $\|H_{y_2d_0}(j\omega)\| \leq 1$ for all $\omega \in \Omega_d$, then, from (4.15),

$$\Delta H_{y_2u_1}^e \sim \Delta P_mQ_0 \text{ over } \Omega_d$$ (4.16)

where $\Omega_d$ is the frequency band of interest for disturbance rejection.
• Perturbation in the compensators $Q_0$ and $C_2$ in $\Sigma^0$

By inspection of $\Sigma_e$ in Fig. 6(a), it is clear that when the plant is nominal ($P=\Sigma_0$), there is no feedback in $\Sigma_e$ i.e., $\Sigma_e$ is an open-loop system. If the compensators $Q_0$ and $C_2$ undergo the perturbations $Q_0 + \Delta Q_0$ and $C_2 + \Delta C_2$, then the resulting I/O map $H^C_{y_2u_1}$ and the resulting disturbance-to-output map $H^C_{y_2d_0}$ are given by

\[
H^C_{y_2u_1} = P_0(Q_0+\Delta Q_0), \quad \text{and}
\]

\[
H^C_{y_2d_0} = I - P_0(Q_0+\Delta Q_0)(C_2+\Delta C_2)
\]

• Conclusion

$\Sigma_b$ and $\Sigma_f$ are better than $\Sigma_e$: indeed,

1) $\Sigma_e$ requires that $P_0$ be exp. stable;

2) $\Sigma_e$ is sensitive to changes in the model $P_0$; and

3) $\Sigma_e$ is sensitive to changes in the compensation subsystems $Q_0$ and $C_2$.

• Generalization

So far, in studying feedback configurations, we restrict ourselves to the continuous linear time-invariant lumped systems. However, it should be noted that in deriving stability conditions and various properties of each configuration, the only necessary restrictions are linearity and time-invariance. Hence, all the results developed in the present section and Section II and III can be easily generalized to the discrete linear time-invariant case and to the continuous linear time-invariant distributed case.

V. Configurations $\Sigma_a$, $\Sigma_b$, $\Sigma_c$, and $\Sigma_d$: the nonlinear case

In Section III, we compare the four configurations $\Sigma_a$, $\Sigma_b$, $\Sigma_c$, and $\Sigma_d$ for the linear case. We specify the set of all achievable I/O maps
and the set of all achievable disturbance-to-output maps, and study the
effects of various subsystem perturbations on the I/O map $H_{y_2u_1}$. In
this section, we do the same comparison for this four configurations in
the nonlinear context. We shall see that, under suitable assumptions,
most of the results in Section III still hold for the nonlinear case.

We use an input-output description of the nonlinear system. Let
$$(\mathcal{L}, \| \cdot \|)$$
be a normed space of "time functions": $\mathcal{J} \to \mathcal{U}$ where $\mathcal{J} \subseteq \mathbb{R}_+$
($\mathcal{J} = \mathbb{R}_+ \ (\mathbb{N}, \text{ resp.})$ for the continuous-time case (discrete-time case, resp.)),
$\mathcal{U}$ is a normed space and $\| \cdot \|$ is the chosen norm in $\mathcal{L}$. Let $\mathcal{L}_e$ be the
corresponding extended space [Wil. 1], [Des. 7], [Vid. 2]. A function
$\phi: \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $K$ (denoted by $\phi \in K$) iff $\phi$ is
continuous and increasing. $\phi$ is said to belong to class $K_0$ iff $\phi \in K$
and $\phi(0) = 0$. A nonlinear causal map $H: \mathcal{L}_e^n \to \mathcal{L}_e^0$ is said to be $\mathcal{J}$-stable
iff $\exists \phi \in K$ s.t. $\forall x \in \mathcal{L}_e^n, \forall t \in \mathcal{J}$,

$$\| Hx \|_T \leq \phi(\| x \|_T)$$

$H$ is said to be incrementally $\mathcal{J}$-stable (incr. $\mathcal{J}$-stable) iff
(i) $H$ is $\mathcal{J}$-stable, (ii) $\exists \bar{\phi} \in K_0$ s.t. $\forall x, x' \in \mathcal{L}_e^n, \forall t \in \mathcal{J}$,

$$\| Hx - Hx' \|_T \leq \bar{\phi}(\| x - x' \|_T)$$

Note that if $\phi: x \to \gamma x$, $\gamma$ constant ($\bar{\phi}: x \to \bar{\gamma} x$, $\bar{\gamma}$ constant), then we have
finite-gain stability, (finite incremental gain stability, resp.). It
can be easily checked that the sum and the composition of $\mathcal{J}$-stable maps,
(incr. $\mathcal{J}$-stable maps) are $\mathcal{J}$-stable, (incr. $\mathcal{J}$-stable, resp.).

We make the following assumptions throughout this section:
(N.1) $P_0, P: \mathcal{L}_e^n \to \mathcal{L}_e^0$ and $Q_0: \mathcal{L}_e^0 \to \mathcal{L}_e^n$ are nonlinear causal maps;
(N.2) $C: \mathcal{L}_e^0 \to \mathcal{L}_e^n$ is linear and causal;
(N.3) \( \pi^{-1}, \pi: \mathcal{L}_e^n \rightarrow \mathcal{L}_e^n \) are \textbf{linear and causal};

(N.4) for each configuration, both the nominal and perturbed system are \textbf{well-posed} i.e., the relation from the exogenous inputs into each subsystem variable (i.e., input or output) is a \textbf{well-defined} nonlinear causal map between the corresponding extended spaces;

(N.5) The nonlinear maps \( C_0 \) and \( Q_0 \) are related by

\[ C_0 = Q_0(I-P_0Q_0)^{-1} \text{ or equivalently } Q_0 = C_0(I+P_0C_0)^{-1}. \]

We say that a \textbf{well-posed} feedback configuration is \( \mathcal{S} \)-\textbf{stable} iff the map from the exogenous inputs to any subsystem variable (i.e., input or output) is \( \mathcal{S} \)-\textbf{stable}. The map \( H: \mathcal{L}_e^n \rightarrow \mathcal{L}_e^n \) is said to be an \textbf{achievable} \textbf{I/O map} (achievable disturbance-to-output map, resp.) of the nonlinear feedback configuration \( \Sigma_a \) (\( \Sigma_b \), \( \Sigma_c \), \( \Sigma_d \), resp.) iff by some appropriate choice of the compensation subsystems satisfying (N.1)-(N.5),

(i) \( H_yu_1 = H, (H_yd_0 = H, \text{ resp.}); \)
(ii) \( \Sigma_a, (\Sigma_b, \Sigma_c, \Sigma_d, \text{ resp.}) \) is \( \mathcal{S} \)-\textbf{stable}.

It is crucial to note that although the formulas belows have the same form as those in the linear case, they have here a completely different meaning: for example in the previous sections \( PC \) meant the \textbf{product} of the transfer function \( P \) with the transfer function \( C \), in the nonlinear case \( PC \) means the composition of the function \( P \) with the function \( C \): e.g., when we write \( PCe \), we mean \( P(C(e)) \) or equivalently \( P \circ C(e) \).

V.1. The system I/O map

With the assumption that \( C_1 \) and \( \pi \) are \textbf{linear}, it can be easily verified that the \textbf{partial} system I/O maps of the four configurations are given by the Eqs. (5.1) and (5.2) below; each entry of (5.1) and (5.2) is
composition of nonlinear causal maps. By assumption (N.4), all the inverses in (5.1) and (5.2) are well-defined causal maps. Let for k = 1,2, \( F_k : (u_1, u_2, d_0) \rightarrow y_k \); so \( F_1 \) and \( F_2 \) specify the closed-loop map.

We denote the partial maps by the same notation as in Section III: for example in terms of partial maps, we have \( H_{y_1 u_1} := F_1(u_1, 0, 0) \) and \( H_{y_2 d_0} := F_2(0, 0, d_2) \). When \( P = P_0 \), the partial maps relating \( (u_1, u_2, d_0) \) to \( (y_1, y_2) \) are given by

\[
\begin{bmatrix}
H_{y_1 u_1}^0 & H_{y_1 u_2}^0 & H_{y_1 d_0}^0 \\
H_{y_2 u_1}^0 & H_{y_2 u_2}^0 & H_{y_2 d_0}^0 \\
\end{bmatrix} = \begin{bmatrix}
Q_0 & -C_1 P_0 (I + C_1 P_0)^{-1} & -C_1 [I - P_0 (-C_1)]^{-1} \\
P_0 Q_0 & P_0 (I + C_1 P_0)^{-1} & [I - P_0 (-C_1)]^{-1} \\
\end{bmatrix}
\]

(5.1)

When \( P \neq P_0 \),

\[
\begin{bmatrix}
H_{y_1 u_1} & H_{y_1 u_2} & H_{y_1 d_0} \\
H_{y_2 u_1} & H_{y_2 u_2} & H_{y_2 d_0} \\
\end{bmatrix} = \begin{bmatrix}
(I + C_1 P)^{-1} (I + C_1 P_0) Q_0 & -C_1 P (I + C_1 P)^{-1} & -C_1 [I - P (-C_1)]^{-1} \\
P (I + C_1 P)^{-1} (I + C_1 P_0) Q_0 & P (I + C_1 P)^{-1} & [I - P (-C_1)]^{-1} \\
\end{bmatrix}
\]

(5.2)

In the following all the symbols \( \Sigma_a^0, \Sigma_b^0, \Sigma_e^0, \Sigma_d^0 \) have the same meaning as in Section III except that they are associated with the nonlinear configurations \( \Sigma_a, \Sigma_b, \ldots \) etc.

V.2. Stability conditions of the nominal nonlinear feedback configurations \( \Sigma_a^0, \Sigma_b^0, \Sigma_e^0 \) and \( \Sigma_d^0 \)

Unlike the linear case, each partial map of (5.1) being \( \mathcal{A} \)-stable does not imply that the nominal nonlinear feedback configurations are
$\mathcal{S}$-stable. The following stability conditions can be obtained by

(i) that the composition of $\mathcal{S}$-stable maps are $\mathcal{S}$-stable, and

(ii) inspection of the block diagrams of the nonlinear configurations in Fig. 4.

(a) $\Sigma_a^0$ is $\mathcal{S}$-stable $\Leftrightarrow$ $Q_0$, $P_0$ and $\tilde{1}S(C_1,P_0)$ are $\mathcal{S}$-stable \hspace{1cm} (5.3a)

(b) $\Sigma_b^0$ is $\mathcal{S}$-stable $\Leftrightarrow$ $\tilde{1}S(P_0,C_0)$ and $\tilde{1}S(C_1,P_0)$ are $\mathcal{S}$-stable \hspace{1cm} (5.3b)

(c) $\Sigma_c^0$ is $\mathcal{S}$-stable $\Leftrightarrow$ $(I+C_1P_0)Q_0$ and $\tilde{1}S(C_1,P_0)$ are $\mathcal{S}$-stable \hspace{1cm} (5.3c)

(d) $\Sigma_d^0$ is $\mathcal{S}$-stable $\Leftrightarrow$ $\pi^{-1}(I+C_1P_0)Q_0$ and $\tilde{1}S(P_0,\pi,\pi^{-1}C_1)$ are $\mathcal{S}$-stable \hspace{1cm} (5.3d)

Fact 5.1. If $\pi$ and $\pi^{-1}$ are linear and incr. $\mathcal{S}$-stable, then

$$\Sigma_d^0 \text{ is } \mathcal{S}\text{-stable } \Leftrightarrow \Sigma_c^0 \text{ is } \mathcal{S}\text{-stable} \hspace{1cm} (5.3e)$$

Proof: (See Appendix).

V.3. Properties of nonlinear configurations $\Sigma_a^0$, $\Sigma_b^0$, $\Sigma_c^0$, and $\Sigma_d^0$

* Nominal design

As in the linear case, for the nonlinear feedback configurations $\Sigma_a^0$, $\Sigma_b^0$, $\Sigma_c^0$, and $\Sigma_d^0$, $Q_0$ specifies the nominal I/O map

$$H_{y_2u_1}^0 = P_0Q_0 ; \hspace{1cm} (5.4)$$

$C_1$ specifies the nominal disturbance-to-output map

$$H_{y_2d_0}^0 = [I - P_0(-C_1)]^{-1} \hspace{1cm} (5.5)$$

Remark: With $C_1$ linear, $\forall x \in \mathcal{L}^n_e$,

$$[I-P_0(-C_1)](x) = x-P_0C_1(-x) = -(x) - P_0C_1(-x) = -(I+P_0C_1)(-x).$$

In the following, we specify the set of all achievable I/O maps and the set of all achievable disturbance-to-output maps for each
configuration. We assume that there exists a linear \( C_1 \) such that \( ^1S(C_1,P_0) \) is \( \mathcal{S} \)-stable, and that there exists a \( C_0 \) such that \( ^1S(P_0,C_0) \) is \( \mathcal{S} \)-stable.

- **Achievable I/O maps**

(a) For \( \Sigma_a^0 \):

(i) If \( P_0 \) is not \( \mathcal{S} \)-stable, (5.3a) shows that the configuration \( \Sigma_a \) is not \( \mathcal{S} \)-stable.

(ii) If \( P_0 \) is incr. \( \mathcal{S} \)-stable, then [Des. 8]

\[
\mathcal{H}_{y_2u_1}^a(P_0) = \{ P_0 Q_0 \mid Q_0 \text{ is } \mathcal{S} \text{-stable} \} \tag{5.6}
\]

(b) For \( \Sigma_b^0 \):

\[
\mathcal{H}_{y_2u_1}^b(P_0) = \left\{ P_0 Q_0 \mid Q_0 = C_0(I+P_0C_0)^{-1} \text{ where } C_0 \text{ is such that } ^1S(P_0,C_0) \text{ is } \mathcal{S} \text{-stable} \right\} \tag{5.7}
\]

(c) For \( \Sigma_c^0 \):

\[
\mathcal{H}_{y_2u_1}^c(P_0) = \left\{ P_0 Q_0 \mid Q_0 \text{ is such that } (I+C_1P_0)Q_0 \text{ is } \mathcal{S} \text{-stable for some } C_1 \text{ which yields } ^1S(C_1,P_0) \text{ } \mathcal{S} \text{-stable} \right\} \tag{5.8}
\]

(d) For \( \Sigma_d^0 \) with \( \pi \) and \( \pi^{-1} \) linear and incr. \( \mathcal{S} \)-stable:

\[
\mathcal{H}_{y_2u_1}^d(P_0) = \left\{ P_0 Q_0 \mid Q_0 \text{ is such that } (I+C_1P_0)Q_0 \text{ is } \mathcal{S} \text{-stable for some } C_1 \text{ which yields } ^1S(C_1,P_0) \text{ } \mathcal{S} \text{-stable} \right\} \tag{5.9}
\]

\[= \mathcal{H}_{y_2u_1}^c(P_0)\]
Remark: For $\Sigma_c^0$ and $\Sigma_d^0$, we did not need that $C_1$ be linear.

- Achievable disturbance-to-output maps

(a) For $\Sigma_a^0$:

If $P_0$ is incr. $\mathcal{J}$-stable, then

$$\mathcal{A}_{y_2^d 0}^a (P_0) = \left\{ \left[ I - P_0 (-C_1) \right]^{-1} \left| \begin{array}{l} \text{If } \text{PQ is incr. } A\text{-stable, then} \\ C_1 \text{ is linear and is such that} \\ 1S(C_1, P_0) \text{ is } \mathcal{J}\text{-stable} \end{array} \right. \right\}$$  (5.10)

(b) For $\Sigma_b^0$:

$$\mathcal{A}_{y_2^d 0}^b (P_0) = \left\{ \left[ I - P_0 (-C_1) \right]^{-1} \left| \begin{array}{l} \text{If } \text{PQ is incr. } A\text{-stable, then} \\ C_1 \text{ is linear and is such that} \\ 1S(C_1, P_0) \text{ is } \mathcal{J}\text{-stable} \end{array} \right. \right\}$$  (5.11)

(c) For $\Sigma_c^0$:

$$\mathcal{A}_{y_2^d 0}^c (P_0) = \left\{ \left[ I - P_0 (-C_1) \right]^{-1} \left| \begin{array}{l} \text{If } \text{PQ is incr. } A\text{-stable, then} \\ C_1 \text{ is such that } 1S(C_1, P_0) \text{ is } \mathcal{J}\text{-stable} \\ \text{and that } (I+C_1 P_0) Q_0 \text{ is } \mathcal{J}\text{-stable for some } Q_0. \end{array} \right. \right\}$$  (5.12)

(d) For $\Sigma_d^0$ with $\pi$ and $\pi^{-1}$ linear and incr. $\mathcal{J}$-stable:

$$\mathcal{A}_{y_2^d 0}^d (P_0) = \left\{ \left[ I - P_0 (-C_1) \right]^{-1} \left| \begin{array}{l} \text{If } \text{PQ is incr. } A\text{-stable, then} \\ C_1 \text{ is such that } 1S(C_1, P_0) \text{ is } \mathcal{J}\text{-stable.} \\ \text{and that } (I+C_1 P_0) Q_0 \text{ is } \mathcal{J}\text{-stable for some } Q_0. \end{array} \right. \right\}$$  (5.13)

Remarks

(i) From (5.6), (5.7), (5.10), and (5.11), it is clear that for the configurations $\Sigma_a^0$ and $\Sigma_b^0$, we can simultaneously achieve any $H_{y_2^d 0} \in \mathcal{A}_{y_2^d 0}^a (P_0)$, $(\mathcal{A}_{y_2^d 0}^b (P_0)$, resp.) and any $H_{y_2^d 0} \in \mathcal{A}_{y_2^d 0}^a (P_0)$,
(\mathcal{H}^0 y_{2d_0}(P_0), \text{resp.}) i.e., the choices of \(H^0_{y_2u_1}\) and \(H^0_{y_2d_0}\), (hence the choices of \(Q_0\) and \(C_1\)) are independent. For the configurations \(\Sigma^0_c\) and \(\Sigma^0_d\), the choices of \(H^0_{y_2u_1}\) and \(H^0_{y_2d_0}\) are constrained: indeed, \(Q_0\) and \(C_1\) must be chosen so that the map \([\pi^{-1}(I+C_1P_0)Q_0]\), \([\pi^{-1}(I+C_1P_0)Q_0]\), resp. is \(\mathcal{S}\)-stable.

(ii) As in the linear case, \(\Sigma^0_d\) offers more flexibility in implementation than \(\Sigma^0_c\) does.

**Plant perturbation**

For each of the four configurations \(\Sigma^0_a, \Sigma^0_b, \Sigma^0_c, \text{and } \Sigma^0_d\), where \(C_1\) is assumed linear, the plant perturbation \(P_0 \leftarrow P_0 + \Delta P := P\) has the same effect on the I/O map \(H^0_{y_2u_1}\). More precisely, let \(\Delta H_{y_2u_1} := H_{y_2u_1} - H^0_{y_2u_1}\).

Then for any input \(u_1 \in E^0\),

\[
\Delta H_{y_2u_1}(u_1) = \int_0^1 [I+D(P)C_1]^{-1} d\alpha \cdot \Delta P Q_0(u_1) \tag{5.14}
\]

where \(D(P)\) is the Frechet derivative of \(P\), (see [Die. 1], [Des. 9]), and is evaluated at \((I+C_1P)^{-1}[(I+C_1P_0)Q_0(u_1) + \alpha C_1 \Delta P Q_0(u_1)]\) with \(\alpha \in [0,1]\).

See Appendix for derivation of (5.14).

**Remark:** Equation (5.14) tells us that if the linear compensator \(C_1\) is chosen so that along the trajectory, defined in (5.14), where \(D(P)\) is evaluated, all the linear maps \(D(P)C_1\) has "large gain," then, for \(\Sigma^0_a, \Sigma^0_b, \cdots \Sigma^0_d\), the output \(y_2\) (corresponding to the fixed input \(u_1\)) is very insensitive to perturbations in the nominal plant \(P_0\) (in comparison with the equivalent open-loop system).

**Model perturbation**

(i) For \(\Sigma^0_a\), let \(\Delta H^a_{y_2u_1}\) be the change in the I/O map \(H^0_{y_2u_1}\) caused by the model perturbation \(P_0 \leftarrow P_0 + \Delta P_m\), then \(\forall u_1 \in E^0\),

-28-
\[ \Delta_{y_2u_1}(u_1) = \int_{0}^{1} [I-(I+D(P_0)C_1)^{-1}]d\alpha \cdot \Delta_{m}P_0(u_1) \]  

(5.15)

where \( D(P_0) \) is evaluated at \( (I+C_1P_0)^{-1}[((I+C_1P_0)Q_0(u_1)+\alpha C_1\Delta mP_0(u_1)] \)

with \( \alpha \in [0,1] \). See Appendix.

(ii) For \( \Sigma_b \), if we assume that both \( C_0 \) and \( C_1 \) are linear, then it can be checked that \( Vu_1 \in \Sigma_b \), \( \Delta_{y_2u_1}(u_1) = P_0(I+C_1P_0)^{-1}[(C_1-C_0) \cdot \int_{0}^{1}(I+D(P)C_0)^{-1}d\alpha \Delta_mC_0(I+P_0C_0)^{-1}(u_1)] \) where \( P := P_0+\Delta m \), and \( D(P) \) is evaluated at

\[ C_0(I+PC_0)^{-1}(u_1+\alpha\Delta mC_0(I+P_0C_0)^{-1}(u_1)) \text{ for } \alpha \in [0,1]. \]

• Perturbation in the precompensators

For \( \Sigma_a, \Sigma_c \) and \( \Sigma_d \), the I/O map \( H^0_{y_2u_1} \) is sensitive to changes in the precompensators, namely \( Q_0 \) in \( \Sigma_a \), \( (I+C_1P_0)Q_0 \) in \( \Sigma_c \) and \( \pi^{-1}(I+C_1P_0)Q_0 \) in \( \Sigma_d \), since they are outside the feedback loop.

• Conclusions. For nonlinear \( P_0, P \), linear \( C_1 \) and \( \pi \),

(i) \( \Sigma_b \) is better than \( \Sigma_a \) in that \( \Sigma_b \) can accommodate unstable plants.

(ii) \( \Sigma_b \) is better than \( \Sigma_a, \Sigma_c \) and \( \Sigma_d \): the latter are sensitive to changes in the precompensator. (In \( \Sigma_b \), the precompensator is realized as a feedback configuration, hence is less sensitive if well-designed).

(iii) \( \Sigma_b \) is better than \( \Sigma_c \) and \( \Sigma_d \): In \( \Sigma_b \), the choices of the I/O map \( H^0_{y_2u_1} \) and the disturbance-to-output map \( H^0_{y_2d_0} \) are independent, whereas in \( \Sigma_c \) and \( \Sigma_d \) the choices are constrained.

Conclusions

In this paper, we study several feedback configurations which have appeared in the control literature. We start with the definitions of two-degree of freedom design and of achievable I/O and disturbance-to-
output map. In section II, we show the basic limitation of linear unity feedback configuration $I^S(P,C)$, namely the dependence of the I/O and disturbance-to-output map. We study the four two-degree of freedom design configurations $Z_a$, $Z_b$, $Z_c$ and $Z_d$ in section III, in terms of their achievable I/O maps and disturbance-to-output maps and their sensitivity to subsystem perturbations, we demonstrate that $Z_b$ is better than $Z_a$, $Z_c$ and $Z_d$. In section IV, the two-degree of freedom design configurations $Z_e$ and $Z_f$ are studied and compared to $Z_b$. In our discussion, we have restricted ourselves to the linear time-invariant lumped case, however the same results hold for the linear time-invariant distributed and the linear time-invariant discrete-time cases. Finally, we study $Z_a$, $Z_b$, $Z_c$ and $Z_d$ in the nonlinear context, it is seen that some of the linear properties are also held for the nonlinear case.

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References


-32-


Appendix

Derivation of (3.2):

(i) For $\Sigma_a$ and $\Sigma_b$:

(a) Let $u_2 = d_0 = n_1 = 0$, then we obtain successively

\[
y_1 = Q_0 u_1 - C_1 [P y_1 - P_0 Q_0 u_1]
\]

\[(I+C_1 P) y_1 = (I+C_1 P_0) Q_0 u_1
\]

\[y_1 = (I+C_1 P)^{-1} (I+C_1 P_0) Q_0 u_1 \quad (A.1)
\]

\[y_2 = P (I+C_1 P)^{-1} (I+C_1 P_0) Q_0 u_1 \quad (A.2)
\]

From (A.1) and (A.2), we have

\[H y_2 u_1 = P (I+C_1 P)^{-1} (I+C_1 P_0) Q_0
\]

\[H y_1 u_1 = (I+C_1 P)^{-1} (I+C_1 P_0) Q_0.
\]

(b) Let $u_1 = d_0 = n_1 = 0$, then $y_1 = y_0 = 0$. Thus by inspection,

\[H y_2 u_2 = P (I+C_1 P)^{-1}
\]

\[H y_1 u_2 = -C_1 P (I+C_1 P)^{-1}
\]

(c) Let $n_1 = u_1 = u_2 = 0$, then $y_1 = y_0 = 0$. Again by inspection

\[H y_2 d_0 = (I+P C_1)^{-1}
\]

\[H y_1 d_0 = -C_1 (I+P C_1)^{-1}
\]

(ii) For $\Sigma_c$ and $\Sigma_d$, Eq. (3.2) can be easily verified by inspection.

\[\Box\]
Proof of Fact 3.1: It can be seen from Fig. 5 that the system $s(P_0, p, \pi^{-1}C_1)$ is exp. stable iff the map $H_{eu}: (u_1, u_2, d_0) \rightarrow (e_1, e_2, y_2)$ is exp. stable. By simple calculation, we have

$$H_{eu} = \begin{bmatrix}
\pi^{-1}(I+C_1P_0)^{-1} & -\pi^{-1}C_1P_0(I+C_1P_0)^{-1} & -\pi^{-1}C_1(I+P_0C_1)^{-1} \\
(I+C_1P_0)^{-1} & (I+C_1P_0)^{-1} & -C_1(I+P_0C_1)^{-1} \\
P_0(I+C_1P_0)^{-1} & P_0(I+C_1P_0)^{-1} & (I+P_0C_1)^{-1}
\end{bmatrix} \quad (A.3)$$

By assumption, $\pi$ and $\pi^{-1}$ are exp. stable, hence (i) $(I+C_1P_0)Q_0$ is exp. stable $\Leftrightarrow (I+C_1P_0)^{-1}Q_0$ is exp. stable; and from (A.3), (ii) $s(P_0, C_1)$ is exp. stable $\Leftrightarrow s(\pi, \pi^{-1}C_1)$ is exp. stable. Therefore, (3.7) follows from (i), (ii), (3.5) and (3.6).

Derivation of (3.18): By computation, we have that the corresponding perturbed I/O map $m_{H^a} = P_0(I+C_1P_0)^{-1}(I+C_1P)Q_0$. Therefore,

$$\Delta m_{H^a} = m_{H^a} - H^0 = P_0(I+C_1P_0)^{-1}(I+C_1P)Q_0 - P_0Q_0$$
$$= P_0(I+C_1P_0)^{-1}(I+C_1P_0+C_1\Delta P_m)Q_0 - P_0Q_0$$
$$= P_0Q_0 + P_0(I+C_1P_0)^{-1}C_1\Delta P_mQ_0 - P_0Q_0$$
$$= P_0C_1(I+P_0C_1)^{-1}\Delta P_mQ_0$$

Derivation of (3.19): By computation, we have that the corresponding perturbed I/O map $m_{H^b} = [P_0(I+C_1P_0)^{-1} + P_0C_1(I+P_0C_1)^{-1}P]C_0(I+P_0C_1)^{-1}$. Therefore,
\[
\Delta_{H}^{m,b} y_{2}u_{1} := m_{H}^{b} y_{2}u_{1} - H^{0}
\]

\[
= [P_{0}(I+C_{1}P_{0})^{-1}+P_{0}C_{0}(I+P_{0}C_{0})^{-1}P]_{C_{0}}(I+P_{0}C_{0})^{-1} - P_{0}C_{0}(I+P_{0}C_{0})^{-1}
\]

\[
= (I+P_{0}C_{1})^{-1}P_{0}C_{0}(I+P_{0}C_{0})^{-1} + [I-(I+P_{0}C_{1})^{-1}]P_{0}C_{0}(I+P_{0}C_{0})^{-1} - P_{0}C_{0}(I+P_{0}C_{0})^{-1}
\]

\[
= (I+P_{0}C_{1})^{-1}P_{0}C_{0}(I+P_{0}C_{0})^{-1} - (I+P_{0}C_{1})^{-1}P_{0}C_{0}(I+P_{0}C_{0})^{-1} + P_{0}C_{0}(I+P_{0}C_{0})^{-1}
\]

\[
- P_{0}C_{0}(I+P_{0}C_{0})^{-1}
\]

\[
= -(I+P_{0}C_{1})^{-1}P_{0}C_{0}(I+P_{0}C_{0})^{-1} + (I+P_{0}C_{0})^{-1}[I-(I+P_{0}C_{1})^{-1}]P_{0}C_{0}(I+P_{0}C_{0})^{-1}
\]

\[
= -(I+P_{0}C_{1})^{-1}P_{0}C_{0}(I+P_{0}C_{0})^{-1} + (I+P_{0}C_{0})^{-1}(I+P_{0}C_{0})^{-1}[I-(I+P_{0}C_{1})^{-1}]
\]

\[
\Delta_{m}^{p} C_{0}(I+P_{0}C_{0})^{-1}
\]

\[
= -(I+P_{0}C_{1})^{-1}P_{0}C_{0}(I+P_{0}C_{0})^{-1} + (I+P_{0}C_{0})^{-1}(I+P_{0}C_{0})^{-1}(I+P_{0}C_{1})^{-1}
\]

\[
\Delta_{m}^{p} C_{0}(I+P_{0}C_{0})^{-1}(I+P_{0}C_{0})^{-1}
\]

\[
= [(I+P_{0}C_{0})^{-1}-(I+P_{0}C_{1})^{-1}]P_{0}C_{0}(I+P_{0}C_{0})^{-1}
\]

**Derivation of (4.3):**

(i) For \( \Sigma_{e} \):

(a) Let \( n_{1} = u_{2} = d_{0} = 0 \), and \( u_{1} \neq 0 \), then we obtain successively

\[
e_{e} = u_{1} - C_{2}(P-P_{0})Q_{0}e_{1}
\]

\[
[I+C_{2}(P-P_{0})Q_{0}]e_{1} = u_{1}
\]

\[
e_{1} = (I+C_{2}PQ_{0})^{-1}u_{1}
\]

\[
y_{1} = Q_{0}(I+C_{2}PQ_{0})^{-1}u_{1}
\]

(A.6)
\[ y_2 = PQ(I+C_2\Delta P Q_0)^{-1} u_1 \]  

(A.7)

From (A.6) and (A.7), we have

\[ H_y y_1 u_1 = Q_0 (I+C_2\Delta P Q_0)^{-1} \]

\[ H_y y_2 u_1 = PQ_0 (I+C_2\Delta P Q_0)^{-1} \]

(b) Let \( u_1 = n_1 = d_0 = 0 \), and \( u_2 \neq 0 \), then

\[ y_1 = -Q_0 C_2 e_d = -Q_0 C_2 (P(y_1+u_2) - P_0 y_1) = -Q_0 C_2 (\Delta P y_1 + P u_2). \]

Hence,

\[ y_1 = -(I+Q_0 C_2 \Delta P)^{-1} Q_0 C_2 P u_2; \text{ thus} \]

\[ H_y y_1 u_2 = -(I+Q_0 C_2 \Delta P)^{-1} Q_0 C_2 P \]

Since \( y_2 = P(y_1+u_2) \),

\[ H_y y_2 u_2 = P(H_y y_1 u_2 + I) \]

\[ = P[-(I+Q_0 C_2 \Delta P)^{-1} Q_0 C_2 P + I] \]

\[ = [I-PQ_0 C_2 (I+\Delta P Q_0 C_2)^{-1}] P \]

\[ = (I+\Delta P Q_0 C_2 - PQ_0 C_2)(I+\Delta P Q_0 C_2)^{-1} P \]

\[ = (I-PQ_0 C_2)(I+\Delta P Q_0 C_2)^{-1} P. \]

(c) Let \( u_1 = u_2 = n_1 = 0 \), and \( d_0 \neq 0 \), then

\[ y_1 = -Q_0 C_2 e_d = -Q_0 C_2 (P y_1 + d_0 - P_0 y_1) = -Q_0 C_2 (\Delta P y_1 + d_0) \]

Hence,

\[ y_1 = -(I+Q_0 C_2 \Delta P)^{-1} Q_0 C_2 d_0; \text{ thus} \]

\[ H_y y_1 d_0 = -(I+Q_0 C_2 \Delta P)^{-1} Q_0 C_2 = -Q_0 C_2 (I+\Delta P Q_0 C_2)^{-1} \]
Since $y_2 = Py_1 + d_0$

$$H_{y_2d_0} = I + PH_{y_1d_0} = I - PQ_0C_2(I + APQ_0C_2)^{-1}$$

$$= (I + APQ_0C_2 - PQ_0C_2)(I + APQ_0C_2)^{-1}.$$ 

$$= (I - PQ_0C_2)(I + APQ_0C_2)^{-1}$$

(ii) For $\Sigma_f$, (4.3) can be easily verified by inspection of Fig. 6(b) and simple computations. 

Derivation of (4.13): By definition and the system I/O map (4.3),

$$\Delta H_{y_2u_1} \triangleq H_{y_2u_1} - H^0_{y_2u_1} = PQ_0(I + C_2\Delta PQ_0)^{-1} - P_0Q_0$$

$$= [PQ_0 - P_0Q_0(I + C_2\Delta PQ_0)](I + C_2\Delta PQ_0)^{-1}$$

$$= (I - P_0Q_0C_2)\Delta PQ_0(I + C_2\Delta PQ_0)^{-1}$$

$$= (I - P_0Q_0C_2)(I + \Delta PQ_0C_2)^{-1}\Delta PQ_0.$$ 

Derivation of (4.14): By simple computations, we have the corresponding perturbed I/O map $m^e_{H_{y_2u_1}} = P_0Q_0(I - C_2\Delta m Q_0)^{-1}$. Hence,

$$\Delta m^e_{y_2u_1} \triangleq m^e_{y_2u_1} - H^0_{y_2u_1}$$

$$= P_0Q_0(I - C_2\Delta m Q_0)^{-1} - P_0Q_0$$

$$= P_0Q_0[(I - C_2\Delta m Q_0)^{-1} - I]$$

$$= P_0Q_0[I - (I - C_2\Delta m Q_0)](I - C_2\Delta m Q_0)^{-1}$$

$$= P_0Q_0C_2\Delta PQ_0(I - C_2\Delta m Q_0)^{-1}.$$ 

-39-
Proof of Fact 5.1:

Since $\pi$ and $\pi^{-1}$ are incr. $S$-stable, $(I+C_1P_0)Q_0$ is $S$-stable if $\pi^{-1}(I+C_1P_0)Q_0$ is $S$-stable. Thus from (5.3c) and (5.3d), to show (5.3e) we only have to show that $^1S(C_0,P_0)$ is $S$-stable iff $^1S(P_0,\pi,\pi^{-1}C_1)$ is $S$-stable.

Since $\pi$ is linear, it can be easily seen, from Fig. 5, that $^1S(P_0,\pi,\pi^{-1}C_1)$ is $S$-stable implies that $^1S(C_0,P_0)$ is $S$-stable. The proof is complete if we show that $^1S(C_0,P_0)$ is $S$-stable implies that $^1S(P_0,\pi,\pi^{-1}C_1)$ is $S$-stable; we prove this next.

Consider the system $^1S(P_0,\pi,\pi^{-1}C_1)$ shown in Fig. 5 with input $(u_1,u_2,d_0)$; write the equations determining $e_2$ and $y_2$:

\begin{align*}
e_2 &= \pi(u_1-\pi^{-1}C_1y_2) + u_2 \\
y_2 &= d_0 + P_0e_2
\end{align*}

(A.8)

(A.9)

Let $\tilde{u}_2 := \pi(u_1-\pi^{-1}C_1y_2) - \pi(-\pi^{-1}C_1y_2)$, and

\begin{align*}
\tilde{d}_0 &:= d_0;
\end{align*}

(A.10)

(A.11)

and then rewrite (A.8) and (A.9) as

\begin{align*}
e_2 &= \tilde{u}_2 + u_2 + \pi(-\pi^{-1}C_1y_2) = \tilde{u}_2 + u_2 - C_1y_2 \\
y_2 &= \tilde{d}_0 + P_0e_2
\end{align*}

(A.12)

(A.13)

where in (A.12) we have used the linearity of $\pi$.

Note that (A.12) and (A.13) describe the system $^1S(C_1,P_0)$ with input $(\tilde{u}_2+u_2,\tilde{d}_0)$. Since by assumption $^1S(C_1,P_0)$ is $S$-stable, for the system $^1S(P_0,\pi,\pi^{-1}C_1)$, the map $H: (d_0,\tilde{u}_2+u_2) \mapsto (e_2,y_2)$ is $S$-stable. Since $\pi$ is incr. $S$-stable, it can be easily shown that the map $\psi: (u_1,u_2,d_0) \mapsto (d_0,\tilde{u}_2+u_2)$ is $S$-stable. Therefore, for $^1S(P_0,\pi,\pi^{-1}C_1)$, the composite
map $\psi: (u_1, u_2, d_0) \mapsto (e_2, y_2)$ is $\mathcal{J}$-stable. Since $y_1 = e_2 - u_2$ and $e_1 = \pi^{-1} y_1$, the map $(u_1, u_2, d_0) \mapsto (e_1, y_1)$ is also $\mathcal{J}$-stable, consequently, the system $^1S(P_0, \pi, \pi^{-1} C_1)$ is $\mathcal{J}$-stable.

Derivation of (5.14): By definition of $\Delta H_{y_2 u_1}$ and Eq. (5.2),

$$\Delta H_{y_2 u_1} = P(I + C_1 P)^{-1} (I + C_1 P_0) Q_0 - P_0 Q_0$$

$$= P(I + C_1 P)^{-1} (I + C_1 P_0) Q_0 - P_0 Q_0 + P_0 Q_0 - P_0 Q_0$$

$$= P(I + C_1 P)^{-1} (I + C_1 P_0) Q_0 - P(I + C_1 P)^{-1} (I + C_1 P) Q_0 + APQ_0$$

For $u_1 \in \mathcal{L}^\eta_\epsilon$, let $\eta_1 := (I + C_1 P_0) Q_0 (u_1)$, $\Delta \eta_1 := C_1 APQ_0(u_1)$, then

$$\Delta H_{y_2 u_1}(u_1) = P(I + C_1 P)^{-1} (\eta_1 + \Delta \eta_1) - P(I + C_1 P)^{-1} (\eta_1 + \Delta \eta_1) + APQ_0(u_1)$$

Using Taylor's formula, [Die. 1, Theorem 8.14.3],

$$\Delta H_{y_2 u_1}(u_1) = [-\int_0^1 D(P)(I + C_1 P)^{-1} C_1 APQ_0(u_1) d\alpha + APQ_0(u_1)]$$

$$= [-\int_0^1 D(P)(I + C_1 D(P))^{-1} \Delta \eta_1, d\alpha + APQ_0(u_1)]$$

where in both instances $D(P)$ is evaluated at $(I + C_1 P)^{-1} (\eta_1 + \alpha \Delta \eta_1)$. Note that in the last step we only used the chain rule, the inverse function rule and the linearity of $C_1$ [Die. 1, Theorems 8.2.1, 8.2.3]. Now, since $C_1$ is linear

$$\Delta H_{y_2 u_1}(u_1) = [-\int_0^1 D(P)(I + C_1 D(P))^{-1} C_1 APQ_0(u_1) d\alpha + APQ_0(u_1)]$$

$$= \int_0^1 [I - D(P) C_1 (I + D(P) C_1)^{-1}] APQ_0(u_1) d\alpha$$

$$= \int_0^1 (I + D(P) C_1)^{-1} d\alpha \cdot APQ_0(u_1)$$
Derivation of (5.15): By definition,

$$\Delta_{y_2u_1} h^m = h^m_{y_2u_1} - h^0_{y_2u_1}$$

$$= P_0(I+C_1P_0)^{-1}(I+C_1P)Q_0 - P_0Q_0$$

$$= P_0(I+C_1P_0)^{-1}(I+C_1P)Q_0 - P_0(I+C_1P_0)^{-1}(I+C_1P_0)Q_0$$

For \( u_1 \in L^0 \), let \( \eta_1 := (I+C_1P_0)Q_0(u_1) \)

$$\Delta \eta_1 := C_1 \Delta P_0 Q_0(u_1)$$, then

$$\Delta_{y_2u_1} h^m(u_1) = P_0(I+C_1P_0)^{-1}(\eta_1 + \Delta \eta_1) - P_0(I+C_1P_0)^{-1}(\eta_1)$$

By using Taylor's expansion,

$$\Delta_{y_2u_1} h^m(u_1) = \int_0^1 [P_0(I+C_1P_0)^{-1}(\eta_1 + \alpha \Delta \eta_1) - \Delta \eta_1] d\alpha$$

$$= \int_0^1 [D(P_0) + D(P_0)C_1] \Delta P_0 Q_0(u_1) d\alpha$$

where \( D(P_0) \) is evaluated at \((I+C_1P_0)^{-1}(\eta_1 + \alpha \Delta \eta_1) \). Now since \( C_1 \) is linear,

$$\Delta_{y_2u_1} h^m(u_1) = \int_0^1 D(P_0)C_1(D(P_0)C_1)^{-1} \Delta P_0 Q_0(u_1) d\alpha$$

$$= \int_0^1 [I-(I+D(P_0)C_1)^{-1}] \Delta P_0 Q_0(u_1)$$
List of Figure Captions

Fig. 1. The system $\Sigma(P,K)$.

Fig. 2. $\Sigma(P,K)$ with the controller $K$ replaced by the two subsystems $\pi$ and $F$.

Fig. 3. Single degree of freedom design: $\Sigma(P,C)$ which takes $(u_1, u_2, d_0)$ into $(y_1, y_2)$.

Fig. 4. Two-degree of freedom designs-group 1: feedback configurations $\Sigma_a$, $\Sigma_b$, $\Sigma_c$, and $\Sigma_d$. It is assumed that $Q_0 = C_0(I+P_0C_0)^{-1}$.
   (a) $\Sigma_a$
   (b) $\Sigma_b$
   (c) $\Sigma_c$
   (d) $\Sigma_d$.

Fig. 5. The system $\Sigma(P,\pi,\pi^{-1}C_1)$.

Fig. 6. Two-degree of freedom designs-group 2: feedback configurations $\Sigma_e$ and $\Sigma_f$.
   (a) $\Sigma_e$
   (b) $\Sigma_f$.
Fig. 1

Fig. 2

Fig. 3
\[ \Sigma(P, \pi, \pi^{-1}C_1) \]

**Fig. 5**

(a) Configuration \( \Sigma_e \)

(b) Configuration \( \Sigma_f \)

**Fig. 6**