A MODIFIED NYQUIST STABILITY TEST FOR USE IN
COMPUTER AIDED DESIGN

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A Modified Nyquist Stability Test for Use in Computer Aided Design*

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ABSTRACT

This note shows that the Nyquist stability criterion is not a convenient tool for use in computer-aided design of feedback systems. A substitute graphical test is proposed which is more suitable for use in CAD.

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1. Introduction.

One of the interesting observations that has emerged in recent years is that procedures that are efficient for "hand" computations are frequently either inefficient or inappropriate for use on a digital computer. This observation obviously applies to such well known "manual" techniques as the inversion of a matrix by Cramer's rule and the determination of complete controllability of a single-input system by constructing the controllability matrix and attempting to determine if it is nonsingular.

The Nyquists stability criterion has served for years as a principal "manual" tool for determining linear system stability. However, it turns out to be quite incompatible with modern design techniques which make use of semi-infinite optimization, because it cannot be transcribed into a semi-infinite inequality. Nevertheless, it does lead directly to an alternative graphical stability test which is totally compatible with the requirements of semi-infinite optimization.


Consider the design of the simple, single-input single-output (SISO) closed loop system shown in Fig. 1, where \( P(s) = \frac{n_p(s)}{d_p(s)} \), \( C(x,s) = \frac{n_c(s)}{d_c(s)} \) and \( F(x,s) = \frac{n_p(s)}{d_p(s)} \) are real, proper rational functions in \( s \), with the vector \( x \in \mathbb{R}^n \) denoting the compensator coefficients to be determined by the designer. Let \( S \) be an open unbounded set in the \( s \)-plane, symmetrical with respect to the real axis, e.g., as in Fig. 2, to which the closed loop poles are required to be confined. We shall assume that \( S \) has a right boundary \( \partial S \) which is given by an expres-
sion of the form

$$\mathfrak{S} = \{ s \in \mathbb{C} | s = \sigma + j\omega, \ \sigma = f(\omega), \ -\infty < \omega < \infty \} \tag{1}$$

where $f: \mathbb{R} \to \mathbb{R}$ is a negative, piecewise continuously differentiable function which monotonically decreases as $|\omega|$ increases, with $f(\omega) \to \infty$ as $|\omega| \to \infty$. Consequently, the set $\mathfrak{S}$ has the characterization

$$\mathfrak{S} = \{ s \in \mathbb{C} | s = \sigma + j\omega, \ \sigma - f(\omega) < 0, \ -\infty < \omega < \infty \} \tag{2}$$

For example, suppose that the set $\mathfrak{S}$ is defined by

$$\mathfrak{S} = \{ s \in \mathbb{C} | s = \sigma + j\omega, \ \sigma < -k_1 |\omega| - k_2, \ -\infty < \omega < \infty \} \tag{3}$$

where $k_1, k_2 > 0$.

**Definition 1:** Let $n(x,s) = n(s)n_c(x,s)n_t(x,s)$ and $d(x,s) = d(s)d_c(x,s)d_t(x,s)$. We shall say that the closed loop system in Fig. 1 is $\mathfrak{S}$-stable if all the zeros of its characteristic polynomial

$$c(x,s) \triangleq n(x,s) + d(x,s) \tag{4}$$

are in $\mathfrak{S}$. 

We begin by generalizing the Nyquist stability criterion so that it can be used as a test of $\mathfrak{S}$-stability for the SISO system given in Fig. 1. We need to define an equation for an indented boundary of $\mathfrak{S}$.

Let $x^*$ be a given set of compensator coefficients. Suppose that the polynomial $d(x^*,s)$ has $k$ zeros, $p_1, \ldots, p_k$, on the boundary of $\mathfrak{S}$. Let $\varepsilon > 0$. For $i = 1, 2, \ldots, k$, let $I_i$ be an open interval defined by

$$I_i = (\text{im}(p_i) - \varepsilon, \text{im}(p_i) + \varepsilon), \ i = 1, 2, \ldots, k \tag{5}$$
and let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function such that \( f(w) = f(w) \) for all \( w \notin \mathbb{U} \), and \( f'(w) < f(w) \) for all \( w \in \mathbb{U} \). Furthermore, suppose that 
\[
d(x^*, f(w) + jw) \neq 0 \quad \text{for all} \quad w > 0.
\]
Then 
\[
\exists \tilde{S} = \{ s \in \mathbb{C} | s = \sigma + jw, \sigma = f(w), -\infty < w < \infty \} 
\]
is an indented boundary of \( S \) (indented so as to include, in the resulting enlarged complement of \( S \), all the zeros \( \omega' + j\omega' \) of \( d(x^*, s) \) which satisfy \( \omega' = f(w') \)).

The following result is obvious in view of the ordinary Nyquist stability criterion [1], see also [2].

**Theorem 1** (Extended Nyquist stability criterion): Let \( x^* \) be a given set of compensator coefficients. Suppose that the polynomial \( d(x^*, s) \) has \( m \) zeros, \( p_1, \ldots, p_m \), in \( S^c \), the complement of \( S \). Let the indented boundary of \( S \) be defined as in (6). Then all the zeros of \( c(x^*, s) \) are in \( S \) if and only if

(i) the zeros of \( F(x^*, s) \) and \( C(x^*, s) \) which are in \( S \) do not cancel any poles of \( d(x^*, s) \) which are in \( S \), and

(ii) the locus of 
\[
t(x^*, w) = \frac{n(x^*, w) + d(x^*, w)}{d(x^*, w) + jw},
\]
traced out for \( w \) taking the values from \(-\infty\) to \(+\infty\), encircles the origin counterclockwise \( m \) times. #

Let us consider how we might attempt to verify by computer whether the locus of \( t(x^*, w) \) encircles the origin exactly \( m \) times.

**Method 1:** Define the integer valued function \( N(x) \) by
\[ N(x) = \lim_{w \to \infty} \frac{\text{arg}[t(x, w)] - \text{arg}[t(x, 0)]}{2\pi} \]  

where \( \text{arg}[z] \) denotes the argument of the complex number \( z \). Then the number of encirclements of the origin by the locus of \( t(x^*, w) \) is given by \( N(x^*) \) and hence all zeros of \( c(x^*, s) \) are in \( S \) if \( N(x^*) = m \# \).

The evaluation of \( N(x^*) \) requires the evaluation of \( \text{arg}[t(x^*, w)] \) for a large number of values of \( w \) in \([0, \infty)\), which is needed so as not to lose any \( 2\pi \) increment. There may be some numerical difficulty in the vicinity of zeros of \( d(x^*, s) \) which are on the boundary of \( S \). However, the major objection to the use of the function \( N(x) \) for counting encirclement on a digital computer stems from the fact that \( N(x) \) is a discontinuous function of \( x \) and therefore incompatible with semi-infinite optimization techniques which may be required for adjusting the compensator coefficient vector \( x \).

**Method 2**: Let \( q: \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function such that \( q(0) < 0 \) and \( q(u) \to \infty \) as \( |u| \to \infty \), e.g., \( q(u) = k_1 u - k_2 \), with \( k_1, k_2 > 0 \), see Fig. 3. Consider the set \( Q \) in the complex plane \( \mathbb{C} \), defined by

\[ Q = \{ z \in \mathbb{C} \mid z = u + jv, v - q(u) > 0 \} \]  

Clearly, \( Q \) contains the origin in its interior and is unbounded in the \( v \)-direction. Consequently, the locus of \( t(x^*, w) \) cannot encircle the origin if it does not penetrate \( Q \). The locus of \( t(x^*, w) \) does not penetrate \( Q \) if and only if

\[ \text{im}(t(x^*, w)) - q(\text{re}(t(x^*, w))) \leq 0 \text{ for all } w \in [0, \infty). \]  

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The geometric interpretation of (10) is given in Fig. 3.

This leads us to a special case. Suppose that \( d(x^*,s) \) has no zeros in \( S^c \) and let \( \varepsilon = 0 \), i.e., \( \tilde{f}(w) = f(w) \) for all \( w \). Then the closed loop system is \( S \)-stable if (10) holds.

Clearly, the narrower the region in the complex plane defined by the inequality \( -q(u) + v > 0 \), the less conservative the test (10) becomes. The advantage of Method 2, assuming that \( d(x^*,s) \) has no zeros in \( S \) for all \( x \) values to be considered, is that the function \( \phi: \mathbb{R}^n \rightarrow \mathbb{R} \) defined by

\[
\phi(x,w) = \text{im}(t(x,w)) - q(\text{re}(t(x,w)))
\]

is differentiable in \( x \) and hence compatible with the use of semi-infinite optimization algorithms for compensator parameter adjustment.

The disadvantages of Method 2 are (i) that it can only be used when \( d(x,s) \) has no zeros in \( S^c \) for all \( x \) of interest and (ii) that it results in a sufficient, rather than both necessary and sufficient condition of \( S \) stability. The first disadvantage can be totally removed and the second one considerably mitigated by replacing Theorem 1 with the following obvious equivalent:

**Theorem 2**: (Modified extended Nyquist criterion). Let \( D(s) \) be any monic polynomial of the same degree as \( d(x^*,s) \), such that all the zeros of \( D(s) \) are in \( S \). Let

\[
T(x^*,w) = \frac{n(x^*,f(w)+jw) + d(x^*,f(w)+jw)}{D(f(w)+jw)}
\]

Then all the zeros of \( c(x^*,s) \) are in \( S \) if and only if the locus of
T(x*,w) traced out for w taking values from $-\infty$ to $\infty$ does not encircle the origin. #

Corollary: Let q(u) be defined as in Method 2. Then all the zeros of c(x*,s) are in S if

$$\text{im}(T(x^*,w)) - q(\text{re}(T(x^*,w))) \leq 0 \quad \text{for all } w \in [0, \infty). \quad (13)$$

The polynomial D(s) can be chosen in such a way as to make the test (13) not only sufficient but also necessary. For example, when P(s) is strictly proper, a reasonable choice for D(s) is a monic polynomial such that D(0) is close to c(x*,0)/2 for the range of x* being considered, so that the plot of T(x*,w) starts at a value of around 2 for w = 0 and goes to 1 as w goes to $\infty$, minimizing the chances that a stable system would violate the test (13).

3. Conclusion.

We have shown that it is possible to extend the classical Nyquist stability test to allow for the verification of S-stability in a numerically well conditioned manner. In the process, we have made it impossible to determine the usual gain and phase margins. However, this is not a great loss, since, by means of semi-infinite optimization, stability robustness can be ensured in a much more sophisticated manner, see [3].

4. References


Fig. 1. Control System Configuration

Fig. 2. Stability Constraint in s-plane.

Fig. 3. Stability Constraint in T(x*, w) plane.