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ERRATA

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pg. 5, 7th line from bottom: for "qualities" read "quantities".

pg. 5, 8th line from top: for "limit cycle in" read "limit cycle in Σ ".

pg. 11, 2nd line from top: for "(2.2)" read "(2.3)".

Figure Captions: correct order of figures is "1,3,2,4".

GLOBAL BEHAVIOR OF INTERCONNECTED POWER SYSTEMS: PART 1

by

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Global Behavior of Interconnected Power Systems: Part 1

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Abstract This study introduces the global qualitative behavior of the classical swing equation model for several interconnected generators. For small levels of excess power supply, every trajectory converges to an equilibrium. For larger levels a trajectory may connect saddle points, and for still larger levels it may bifurcate to a limit cycle. Limit cycles cannot develop through a Hopf bifurcation. There remain some important gaps in understanding the complete behavior.

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1. Introduction. The study of power system transient stability is almost invariably conducted through the use of Lyapunov functions or simulation. The former provides a role for analysis and can determine properties of groups of trajectories. The latter, which relies on the numerical calculation of individual trajectories, can on the other hand, treat models that are more detailed and realistic. Both approaches, however, are used in the same context which can be described as follows. One supposes that at some time the system is in an equilibrium state say x_e . The system is then subjected to a disturbance for a short time interval (for example, a line trips due to a fault and then recloses after the fault is cleared). At the end of the disturbance the system is therefore in another state x , and one asks whether the system, now starting at x , will eventually return to the pre-fault equilibrium x_e . Thus both approaches are concerned with estimating the "size" of the attractor of x_e . The context of transient stability analysis is therefore the local system behavior, that is, behavior in the neighborhood of a prespecified equilibrium point.

In contrast to this focus on local behavior, this paper presents preliminary results of a study of the qualitative properties of global behavior. Since the power system has many equilibria, Lyapunov function techniques are unsuitable for global analysis. The presence of multiple equilibria also implies that the asymptotic behavior of trajectories does not vary continuously with initial conditions. Consequently it is hazardous to infer from the numerical calculation of an individual trajectory starting in a state x , the asymptotic behavior of trajectories starting at states near x . Such inference can be supported by a classification of the possible asymptotic system behavior, and it is the ultimate aim of this study to arrive at such a classification.

This paper falls short of this aim in two respects. First, and as discussed in detail in the next section, the system model used in the study makes two unrealistic simplifications with respect to generator dynamics and the load. Second, we are as yet unable to prove that our classification is complete. We strongly believe, nevertheless, that the three asymptotic behaviors discussed here, namely, convergence to an equilibrium, trajectories connecting saddles and limit cycles, exhaust the possibilities.

The paper is organized as follows. The model is presented in the next section. The case of a single generator connected to an infinite bus is reviewed in section 3. Section 4 is devoted to the closed orbits which are possible only in the special case of zero damping. Section 5 discusses complete stability. Concluding remarks and suggestions for further research are collected in Section 6.

2. The Model. We consider a multimachine system with $1+g$ nodes indexed $i = 0, 1, \dots, g$. Node 0 is a slack bus, and the other nodes are generator buses. The departure from synchronism of generator i is assumed to be governed by the classical swing equation,

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - f_i(\theta), \quad i = 1, \dots, g. \quad (2.1)$$

$M_i (D_i)$ = constant of inertia (damping), with $M_i > 0$, $D_i \geq 0$,

θ_i = generator voltage angle measured relative to slack bus voltage angle,

$\dot{\theta}_i$ = departure of generator frequency from reference frequency ω_0 ,

P_i = exogenously specified mechanical input power minus loss due to damping, $\omega_0 D_i$, minus electrical power demanded at node i . (P_i is called the excess power supply at i),

$$f_i(\theta) := \sum_{\substack{j=0 \\ j \neq i}}^g B_{ij} \sin(\theta_i - \theta_j) \quad (2.2)$$

is the power injected at node i into the rest of the network. In (2.2), $\theta := (\theta_1, \dots, \theta_g)$, the reference angle $\theta_0 \equiv 0$, and B_{ij} is the magnitude of the admittance of the (lossless) transmission line between nodes i and j .

This model has two deficiencies. The first has to do with the neglect of load nodes. Consideration of load nodes leads to a system model with a differential equation similar to (2.1), but in addition the state must satisfy certain algebraic constraints. In turn, this creates both conceptual and technical difficulties (since the trajectory is not always defined, and it may be discontinuous) which we do not as yet know how to treat adequately [8]. The second deficiency of the model concerns the classical swing equation model itself. The classical model presupposes constancy of the generator main field-winding flux linkage, the absence of voltage regulation, and the constancy of mechanical input power. All of these assumptions lead to inaccuracies, especially for the long time period behavior which is the object of this study [2, p. 46]. More accurate models, resulting in a differential equation for each generator more complex than the second order equation (2.1), do not introduce conceptual difficulties. However, these model do not possess the special Hamiltonian-like properties of (2.1) which are exploited in the subsequent analysis.

Equation (2.1) can be expressed as a system of $2g$ first-order equations,

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -M^{-1}D\omega + M^{-1}(P-f(\theta)) \end{aligned} \quad (2.3)$$

where $M(D)$ is a $g \times g$ diagonal matrix with entries $M_i(D_i)$, and $P, f(\theta)$ are g -dimensional vectors with components $P_i, f_i(\theta)$. Observe from (2.2) that $f(\theta) = f(\phi)$ whenever $\theta_i - \phi_i = 0 \pmod{2\pi}$ for all i . Therefore we may

regard (ω, θ) as a member of the state space $X := \mathbb{R}^g \times \mathbb{R}^g$ or $\Sigma := \mathbb{R}^g \times \mathbb{T}^g$ where $\mathbb{T} := [0, 2\pi]$ with the endpoints $0, 2\pi$ identified. In the sequel both state spaces are used. To motivate these two state spaces we introduce a definition. A trajectory $\theta(t)$, $t \geq 0$ is said to be a (closed) orbit of the first kind, respectively second kind, of period $T > 0$, if $\theta(t+T) = \theta(t)$, respectively $\theta(t+T) = \theta(t) \pmod{2\pi}$, for $t \geq 0$. Thus a closed orbit of the first kind is a limit cycle in X (and hence in Σ), whereas a closed orbit of the second kind is a limit cycle in (but not necessarily in X).

Our aim is to study the qualitative properties of the trajectories of (2.3). In particular, we are interested in the changes in these properties as the parameters M, D, P change. In the mathematical literature this is called a study of the (dynamic) bifurcations of (2.3).

We begin with the case of a single generator, $g = 1$.

3. One Generator Case. Now (2.1) simplifies to

$$M\ddot{\theta} + D\dot{\theta} = P - B \sin \theta, \quad (3.1)$$

in which all quantities are scalars. This equation is almost completely analyzed in [3, Chapter 8] and we summarize its conclusions.

Rewriting (3.1) as

$$\dot{\theta} = \omega, \quad M\dot{\omega} + D\omega = P - B \sin \theta, \quad (3.2)$$

and eliminating dt gives the behavior in phase coordinates,

$$\begin{aligned} \frac{d\omega}{d\theta} &= \frac{-D\omega + P - B \sin \theta}{M\omega} \\ &= \frac{-\alpha\omega + \beta - \gamma \sin \theta}{\omega}, \quad \omega \neq 0, \end{aligned} \quad (3.3)$$

where $\alpha := D/M$, $\beta := P/M$, $\gamma := B/M$. Since one of these three parameters can be eliminated by a change of time scale, it is enough to study the behavior as any two of these, say α and β , vary. We first vary α or the damping constant, and then β or the excess power supply.

3.1. Cyclic Saddle Connection Bifurcation. Suppose that $0 \leq \beta < \gamma$. The state (ω, θ) is an equilibrium if and only if $\omega = 0$ and $\theta = \phi_k$ or $\theta = \psi_k$, where

$$\phi_k := 2k\pi + \theta_0, \quad \psi_k = (2k-1)\pi + \theta_0,$$

and $\beta - \gamma \sin \theta_0 = 0$, $0 \leq \theta_0 < \frac{\pi}{2}$. See Figure 1a. $(\omega = 0, \theta = \phi_k)$ is stable while $(\omega = 0, \theta = \psi_k)$ is a saddle. In the state space $\Sigma = R^1 \times T^1$, which is now a cylinder, there are only two equilibria corresponding to ϕ_0 and ψ_1 .

The curve Γ_0 in Figure 1a is the unstable invariant manifold of the saddle $(\omega = 0, \theta = \psi_0)$. It intersects the vertical line through ϕ_0 at height H_0 . The curve Γ_1 is the stable invariant manifold of the saddle $(\omega = 0, \theta = \psi_1)$ intersecting the same vertical at height H_1 . Remember that on the cylinder $R^1 \times T^1$ these two saddles coincide. The relative magnitudes of H_0 and H_1 depend on the damping α . There are three possibilities.

Closed Orbit. For α small enough one gets $H_0 > H_1$ as in Figure 1a. Then (3.2) has a unique, stable solution (the curve 0 in Figure 1a), $(\omega(t), \theta(t))$, with period T , such that

$$\omega(t+T) = \omega(t) > 0, \quad \theta(t+T) = \theta(t) + 2\pi, \quad t \geq 0. \quad (3.4)$$

This is a closed orbit of the second kind. Finally, every trajectory of (3.2) on the cylinder $R^1 \times T^1$ converges to one of these critical

elements, namely, the orbit 0, the stable equilibrium ϕ_0 or the saddle ψ_1 .

Define $\alpha_0(\beta)$ such that $H_0 > H_1$ if and only if $\alpha < \alpha_0(\beta)$.

Complete Stability. If the damping is so large that $\alpha > \alpha_0(\beta)$, then every trajectory converges to an equilibrium point. The system is said to be completely stable. In Figure 1b, S_0 and S_1 are the invariant stable manifolds of the saddles $(0, \psi_0)$ and $(0, \psi_1)$ respectively, and the region between S_0 and S_1 is the region of attraction of the stable equilibrium $(0, \phi_0)$. On the cylinder $R^1 \times T^1$ the curves S_0 and S_1 coincide and all points outside this one-dimensional manifold converge to ϕ_0 .

Saddle Connection. When $\alpha = \alpha_0(\beta)$, $H_0 = H_1$, and the curves Γ_0, Γ_1 coincide as in Figure 1c. On the cylinder $R^1 \times T^1$ the curve Γ is a closed curve connecting the saddle $(0, \psi_1)$ to itself. This is called a cyclic saddle connection, see [1, p. 496]. As α decreases to $\alpha_0(\beta)$, the closed orbit 0 of Figure 1a converges to the cyclic saddle connection Γ and the period T of the orbit increases to infinity. In this sense Γ may be regarded as a closed orbit of infinite period. (The trajectory S in Figure 1c gradually approaches Γ .) On the other hand if α increases to $\alpha_0(\beta)$, the closed "orbit" seems to appear "out of the blue" [1, p. 567]. Thus the cyclic saddle connection separates or bifurcates the presence and absence of orbits. It occurs at the exceptional value $\alpha = \alpha_0(\beta)$ and the slightest perturbation in α will destroy it. This "non-genericity" of the cyclic saddle connection is a special case of a result of Kupka and Smale [1, p. 533].

3.2. The Saddle-Node Bifurcation. The damping parameter $\alpha > 0$ is now fixed and the excess power supply β is varied. For $0 \leq \beta < \gamma$ we saw that on the cylinder the critical elements consist of the stable equilibrium or node $(0, \phi_0)$, the saddle $(0, \psi_1)$ and, possibly, the closed orbit 0. As β increases ϕ_0 and ψ_1 approach each other, and at $\beta = \gamma$ they annihilate each other in the saddle-node bifurcation [1, p. 550]. The (stable) invariant manifold is drawn in Figure 1d. There is also another critical element, the stable orbit 0. If $\beta > \gamma$, then there is no equilibrium and the orbit 0 is the only critical element.

The saddle-node bifurcation is static, since it corresponds to the appearance or disappearance of equilibrium points and hence it is concerned with qualitative changes in the solution of the "power flow" equation, $P - B \sin \theta = 0$. Static bifurcations in the case of several generators are examined in [4]. In contrast, the saddle connection bifurcation is dynamic. The study of dynamic bifurcations is much more complex since it requires an understanding of critical elements that are not equilibrium points, for example, limit cycles.

The qualitative behavior of (3.2) can be summarized in the "bifurcation diagram" of Figure 2, where E indicates presence of equilibria and 0 the presence of an orbit of the second kind. Although the proof is not given here, it is possible to show as in the diagram, that $\alpha_0(\beta)$ increase with β , $\alpha_0(\beta) \rightarrow 0$ as $\beta \rightarrow 0$, and its graph approaches the vertical through $\beta = \gamma$.

4. Orbits of the First Kind. In the previous section it was seen that in the single generator case it is impossible to have an orbit of the first kind when the damping is positive. Such orbits will arise in the absence of damping.

4.1. No Damping. Since modern generators have very little damping it is frequently considered valid to assume $D = 0$ in (3.1). The system is now conservative. As shown in Figure 3, the equilibrium $(0, \phi_0)$ is now a center, while $(0, \psi_1)$ is still a saddle. The stable and unstable manifolds of this saddle coincide forming a closed orbit Γ of infinite period or zero frequency. The center $(0, \phi_0)$ is surrounded by infinitely many closed orbits of the first kind whose frequency decreases from Ω to zero as one approaches Γ . Ω can be calculated by linearizing (3.1) around the equilibrium $(0, \theta_0)$. This linear system has two imaginary eigenvalues $\pm i\Omega$ which are the solutions of the characteristic equation $M\lambda^2 + B \cos \theta_0 = 0$. Observe that Ω is a good approximation to the frequency only for the orbits of very small magnitude. For larger orbits the linearized analysis can give poor answers. In this sense the discussion in [2, pp. 59-63] can be misleading without additional qualifications. The result depicted in Figure 3 extends to the case of several generators as follows. Suppose $D = 0$ in (2.3) and let $\theta_0 \in \mathbb{R}^g$ such that $P - f(\theta_0) = 0$. Then $(\omega=0, \theta_0) \in \mathbb{R}^{2g}$ is an equilibrium. Suppose this equilibrium is stable in the Lyapunov sense. This is equivalent to the condition that the symmetric matrix $F(\theta_0) := \frac{\partial f}{\partial \theta}(\theta_0)$ is positive definite (see e.g. [8]). Now linearize (2.3) around the equilibrium $(\omega=0, \theta_0)$. The characteristic equation of the resulting system is

$$\det[\lambda^2 M + F(\theta_0)] = 0,$$

and, since M is diagonal with positive entries and $F(\theta_0)$ is positive definite, so this equation has $2g$ imaginary roots, say $\pm i\Omega_1, \dots, \pm i\Omega_g$. Assume that $\Omega_1, \dots, \Omega_g$ are all distinct. Then corresponding to each Ω_i , the nonlinear system (2.3) has a two-dimensional invariant manifold

consisting of infinitely many closed orbits of the first kind. Moreover the frequency of these orbits varies continuously as one moves away from the equilibrium. (It is not possible to assert as in the case of $g = 1$ that this frequency decreases.) As one approaches the equilibrium this frequency converges to Ω_1 . Finally, the tangent space to this invariant manifold at the equilibrium is the two-dimensional eigenspace corresponding to the eigenvalues $\pm \Omega_1$. These results, which follow from the Lyapunov sub-center manifold theorem [1, p. 580], justifies the linearized approximation in [2, pp. 59-63]. Note, however, that it is necessary to assume that $\Omega_1, \dots, \Omega_g$ are distinct.

4.2. Damping eliminates Hopf Bifurcations. Consider the system (2.3) and let $(M(\alpha), D(\alpha), P(\alpha))$, $-1 < \alpha < 1$, denote a continuous one-dimensional parametric change in the system parameters. Let $(\omega = 0, \theta_0(\alpha))$ be an equilibrium point and suppose that $\theta_0(\alpha)$ varies continuously with α . Linearize the system (2.3) around this equilibrium and let $\Lambda(\alpha)$ be the resulting set of eigenvalues. Suppose $\Lambda(\alpha)$ is in the open left half plane for $\alpha < 0$. Suppose moreover that at $\alpha = 0$ there occurs a pair of purely imaginary eigenvalues $\pm i\Omega$, $\Omega > 0$, which cross into the right half plane for $\alpha > 0$. Such a situation often gives rise to an orbit of the first kind and with a frequency approximately equal to Ω . This phenomenon is known as the Hopf bifurcation (see [5],[6]).

In a numerical analysis of the power system of the Powerton station [10], the investigators determined that the oscillations observed in practice arise from a Hopf bifurcation. The model that they used differs from (2.3) in one crucial respect: several of the generators are equipped with voltage regulators. The proposition below shown that Hopf bifurcations cannot occur for the model (2.3). This leads to the surprising conclusion that voltage regulators can induce oscillations in

a power system.

Lemma 4.1. Suppose $D > 0$. Let $(\omega=0, \theta_0)$ be an equilibrium of (2.2). Let λ be an eigenvalue of the system linearized around this equilibrium. If $\text{Re } \lambda \geq 0$, then λ is real.

Proof. Let $F(\theta_0) := \frac{\partial f}{\partial \theta}(\theta_0)$. Let $(x, y) \in \mathbb{C}^{2g}$ be an eigenvector corresponding to λ . Then

$$\begin{bmatrix} 0 & I \\ -M^{-1} & -M^{-1}D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix},$$

which implies

$$\lambda^2 Mx + \lambda Dx + F(\theta_0)x = 0.$$

Multiplying on the left by x^* , the conjugate transpose of x , gives

$$\lambda^2 x^* Mx + \lambda x^* Dx + x^* F(\theta)x = 0. \quad (4.1)$$

This equation has two roots λ_1, λ_2 with

$$\lambda_1 + \lambda_2 = -\frac{x^* Dx}{x^* Mx} < 0.$$

It follows that if $\text{Re } \lambda_1 \geq 0$, then $\text{Re } \lambda_2 \leq 0$ and so $\text{Im } \lambda_1 = 0$. \square

Corollary. If $D > 0$, then (2.3) cannot have oscillations induced through a Hopf bifurcation.

A more careful study of (4.1) gives some more information about the eigenvalues of the linearized system. Recall that $F(\theta_0)$ is symmetric so all its eigenvalues are real.

Lemma 4.2. Suppose $D > 0$. Let $F(\theta_0)$ have m non-negative and $g-m$ negative eigenvalues. Then the linearized system has $g+m$ eigenvalues in

the closed left-half plane and g -m positive and real eigenvalues.

This proposition is a special case of a more general result in [9] proved by a different method.

While Lemma 4.1 precluded the possibility of a Hopf bifurcation one can show more directly and more generally that damping eliminates closed orbits of the first kind.

Lemma 4.3. Suppose $D > 0$. Then (2.3) can have no closed orbits of the first kind.

Proof. By way of contradiction suppose that $(\omega(t), \theta(t))$, $t \geq 0$ is a solution of (2.3) such that for some $T > 0$, $\omega(T) = \omega(0)$, $\theta(T) = \theta(0)$. Integrate (2.1) along this solution to obtain

$$\int_0^T \langle M\ddot{\theta}, d\theta \rangle + \int_0^T \langle D\dot{\theta}, d\theta \rangle = \int_0^T \langle P, d\theta \rangle - \int_0^T \langle f(\theta), d\theta \rangle$$

which evaluates to

$$\int_0^T \langle D\dot{\theta}, d\theta \rangle = - \int_0^T \langle f(\theta), d\theta \rangle .$$

But from (2.2) $f(\theta) = \nabla V(\theta)$, where $V(\theta) := -\sum B_{ij} \cos(\theta_i - \theta_j)$, hence the equality above gives

$$\int_0^T \langle D\dot{\theta}, d\theta \rangle = V(\theta(0)) - V(\theta(T)) = 0 ,$$

and since $D > 0$ this implies $\dot{\theta}(t) \equiv 0$. □

5. Complete Stability. Throughout this section it is assumed that $D > 0$. Also the state space of the system is taken to be $(\omega, \theta) \in \mathbb{R}^g \times \mathbb{R}^g$. The following notation is used.

If ξ, η are in \mathbb{R}^g , then $\langle \xi, \eta \rangle$ is their inner product and $|\xi| := \langle \xi, \xi \rangle^{1/2}$. If $\xi(t), t \geq 0$ is a function with values in \mathbb{R}^g , then for $T < \infty$,

$$|\xi|_{2,T} := \left(\int_0^T |\xi(t)|^2 dt \right)^{1/2}, \quad |\xi|_2 := \left(\int_0^\infty |\xi(t)|^2 dt \right)^{1/2},$$

$$|\xi|_1 := \int_0^\infty |\xi(t)| dt, \quad |\xi|_\infty := \text{ess sup}_t |\xi(t)|.$$

One says $\xi \in L_{2,T}$, respectively L_2, L_1, L_∞ if $|\xi|_{2,T}$, respectively $|\xi|_2, |\xi|_1, |\xi|_\infty$, is finite. One says $\xi \in C_0$ if $\xi(\cdot)$ is continuous and $\lim_{t \rightarrow \infty} \xi(t) = 0$.

Recall that $f(\theta) = \nabla V(\theta)$ where $V(\theta) = -\sum B_{ij} \cos(\theta_i - \theta_j)$.

Lemma 5.1. Let $\theta(t), t \geq 0$ be a solution of

$$M\ddot{\theta} + D\dot{\theta} = P - \nabla V(\theta) + \phi(t)$$

where $\phi \in L_\infty$. Then $\dot{\theta} \in L_\infty$.

Proof. Let $\psi(t) := P - \nabla V(\theta(t)) + \phi(t)$. Then $\psi \in L_\infty$. Since

$$\dot{\theta}_i(t) = \left[\exp - \frac{D_i}{M_i} t \right] \dot{\theta}_i(0) + \int_0^t \left[\exp - \frac{D_i}{M_i} (t-s) \right] \psi_i(s) ds \quad (5.1)$$

it follows that $\dot{\theta} \in L_\infty$. □

The next result was inspired by the arguments in [7].

Theorem 5.1. Let $\theta(t), t \geq 0$ be any solution of

$$M\ddot{\theta} + D\dot{\theta} = P - \nabla V(\theta) + \phi, \quad (5.2)$$

where $\phi \in L_2 \cap C_0$. Suppose $\theta \in L_\infty$. Then there exists θ^* such that

$$\lim_{t \rightarrow \infty} (\dot{\theta}(t), \theta(t)) = (0, \theta^*) .$$

Moreover $(0, \theta^*)$ is an equilibrium of (2.1) i.e.

$$P - f(\theta^*) = 0 .$$

Proof. Multiply (5.2) by $\dot{\theta}(t)$ and integrate over $[0, T]$ to obtain

$$\int_0^T \langle M\ddot{\theta}, \dot{\theta} \rangle dt + \int_0^T \langle D\dot{\theta}, \dot{\theta} \rangle dt = \int_0^T \langle P, \dot{\theta} \rangle dt - \int_0^T \langle \nabla V(\theta), \dot{\theta} \rangle dt + \int_0^T \langle \phi, \dot{\theta} \rangle dt ,$$

which, rearranging terms, leads to

$$\begin{aligned} \int_0^T \langle D\dot{\theta}, \dot{\theta} \rangle dt &= -\frac{1}{2} \{ \langle M\dot{\theta}(T), \dot{\theta}(T) \rangle - \langle M\dot{\theta}(0), \dot{\theta}(0) \rangle \} + \langle P, \theta(T) - \theta(0) \rangle \\ &\quad - V(\theta(T)) + V(\theta(0)) + \int_0^T \langle \phi, \dot{\theta} \rangle dt \end{aligned} \quad (5.3)$$

Now $\phi \in L_2 \cap C_0$ implies $\phi \in L_\infty$ and so, by Lemma 5.1, $\dot{\theta} \in L_\infty$. Hence the sum of the first three terms on the right in (5.3) is bounded by some constant K independent of T . Let d be the minimum of the D_i . Then, using the Schwarz inequality for the last term in (5.3) yields the estimate

$$d|\dot{\theta}|_{2,T}^2 \leq K + |\phi|_{2,T}|\dot{\theta}|_{2,T} \leq K + |\phi|_2|\dot{\theta}|_{2,T} ,$$

from which, upon completing squares, one obtains

$$(d^{1/2}|\dot{\theta}|_{2,T} - \frac{1}{2}d^{-1/2}|\phi|_2)^2 \leq K + \frac{1}{4}d^{-1/4}|\phi|_2^2 =: k^2, \text{ say.}$$

Hence

$$d^{1/2}|\dot{\theta}|_{2,T} \leq k + \frac{1}{2}d^{-1/2}|\phi|_2 ,$$

and, since the right hand side is independent of T , this implies $\dot{\theta} \in L_2$. By inspection of (5.1) it is seen that $\ddot{\theta} \in L_\infty$ and so $\dot{\theta}(t)$ is uniformly continuous on $(0, \infty)$. Since $\dot{\theta} \in L_2$ this implies

$$\lim_{t \rightarrow \infty} \dot{\theta}(t) = 0 . \quad (5.4)$$

To show that $\theta(t)$ converges, let $\hat{\omega}(s)$ denote the Laplace Transform of $\omega(t) := \dot{\theta}(t)$. Let $\hat{\xi}(s)$ denote the Laplace Transform of $\xi(t) := P - \nabla V(\theta(t)) + \phi(t)$. Then (5.1) can also be expressed as

$$M(s\hat{\omega}(s) - \omega(0)) + D\hat{\omega}(s) = \hat{\xi}(s) .$$

So,

$$\hat{\omega}(s) = (sM+D)^{-1} \omega(0) + (sM+D)^{-1} \hat{\xi}(s) ,$$

From (5.4) and the Final Value theorem,

$$0 = \lim_{t \rightarrow \infty} \omega(t) = \lim_{s \rightarrow 0} s\hat{\omega}(s) = D^{-1} \lim_{s \rightarrow 0} s\hat{\xi}(s) ,$$

whence,

$$\lim_{t \rightarrow \infty} \xi(t) = 0 .$$

Since $\lim_{t \rightarrow \infty} \phi(t) = 0$ by hypothesis, this implies

$$\lim_{t \rightarrow \infty} \nabla V(\theta(t)) = P .$$

Therefore every limit point θ^* of $\theta(t)$, $t \geq 0$ satisfies $\nabla V(\theta^*) - P = 0$. Since the limit points of $\theta(t)$, $t \geq 0$ form a connected set whereas the solutions of $\nabla V(\theta^*) - P = 0$ are isolated it follows that $\theta(t)$, $t \geq 0$ has a unique limit point θ^* satisfying (5.2). \square

Thus a bounded solution of the swing equations must converge to an equilibrium. An excursion of $\theta_i(t)$ through 2π radians means that

generator i "skips" a cycle relative to the synchronous frame. The previous result can therefore be stated in an intuitive form: if each generator skips only a finite number of cycles, then the system must settle into synchronism.

Theorem 5.1 prompts two important questions. For which values of M, D, P is the system completely stable? And, if $\theta(t)$ is an unbounded solution, what are its asymptotic properties? Both of these questions seem very difficult to answer. Regarding the second question the discussion of Section 3 suggests the conjecture that an unbounded solution must converge to a cyclic saddle connection or to a closed orbit of the second kind. A rather weak result in this direction is given in the following lemma which will also be used later on.

Lemma 5.2. If $\theta(t), t \geq 0$ is an unbounded solution of

$$M\ddot{\theta} + D\dot{\theta} = P - \nabla V(\theta)$$

then $\lim_{t \rightarrow \infty} \langle P, \theta(t) \rangle = \infty$.

Proof. Suppose in contradiction that there is an unbounded sequence $\{T_i\}$ with $\langle P, \theta(T_i) \rangle < K$ for all i . From (5.3) one gets

$$\int_0^{T_i} \langle D\dot{\theta}, \dot{\theta} \rangle dt = -\frac{1}{2} \{ \langle M\dot{\theta}(T_i), \dot{\theta}(T_i) \rangle - \langle M\dot{\theta}(0), \dot{\theta}(0) \rangle \} + \langle P, \theta(T_i) - \theta(0) \rangle - \{ V(\theta(T_i)) + V(\theta(0)) \}.$$

From (5.1) it can be seen that the first term on the right is bounded, the second term is bounded by hypothesis, and since V is bounded so is the third term. This proves that $\dot{\theta} \in L_2$. The argument in the proof of Theorem 5.1 now shows that $\theta(t)$ converges. \square

We now discuss the first question. The next result shown that if $P = 0$ the system is completely stable.

Theorem 5.2. Suppose $D > 0$ and $P = 0$. Let $\theta(t)$, $t \geq 0$ be any solution of

$$M\ddot{\theta} + D\dot{\theta} = -\nabla V(\theta) + \phi \quad (5.5)$$

where $\phi \in L_2 \cap C_0$. Then $\theta(t)$ converges to θ^* and $\nabla V(\theta^*) = 0$.

Proof. From (5.5) one obtains

$$\begin{aligned} \int_0^T \langle D\dot{\theta}, \dot{\theta} \rangle dt &= -\frac{1}{2} \{ \langle M\dot{\theta}(T), \dot{\theta}(T) \rangle - \langle M\dot{\theta}(0), \dot{\theta}(0) \rangle \} - V(\theta(T)) + V(\theta(0)) \\ &\quad + \int_0^T \langle \phi, \dot{\theta} \rangle dt \end{aligned}$$

which also leads to an estimate of the form

$$d|\theta|_{2,T}^2 \leq K + |\phi|_{2,T} |\dot{\theta}|_{2,T}.$$

The argument now proceeds exactly as in the preceding proof. \square

The proof of the following stronger version of Theorem 5.2 is more complicated.

Theorem 5.3. There exists $\pi > 0$ such that if $|P| < \pi$ and $\theta(t)$, $t \geq 0$ is any solution of

$$M\ddot{\theta} + D\dot{\theta} = P - \nabla V(\theta), \quad (5.6)$$

then $\theta(t)$ converges to θ^* and $P - \nabla V(\theta^*) = 0$.

Proof. The proof proceeds in several steps.

Step 1. A solution $\theta(t)$ satisfies

$$\dot{\theta}(t) = [\exp^{-M^{-1}Dt}] \dot{\theta}(0) + \int_0^t \exp[-M^{-1}D(t-s)] [P-f(\theta(s))] ds .$$

The first term on the right approaches 0. Hence, if $|P| \leq 1$, and this will be guaranteed by choosing $\pi < 1$, then there is a constant k independent of initial conditions $(\dot{\theta}(0), \theta(0))$ such that $|\dot{\theta}(t)| \leq k$ for all large t . Therefore, for purposes of the proof, one may limit attention to solutions $\theta(t)$ such that $|\dot{\theta}(t)| \leq k$ for all t . Under this limitation there exist constants a_j , $j = 1, 2, 3$ so that

$$\left| \frac{d^j}{dt^j} \theta_i(t) \right| < a_j, \quad 1 \leq i \leq g. \quad (5.7)$$

For positive t, h the mean value theorem gives

$$\dot{\theta}(t+h) - \dot{\theta}(t) - h\ddot{\theta}(t) = \frac{1}{2} h^2 \xi$$

where $\xi_i := \ddot{\theta}_i(t+s_i)$ for some $0 \leq s_i \leq h$. Substitution from (5.6)

$$M\dot{\theta}(t+h) - M\dot{\theta}(t) + h[D\dot{\theta}(t) - P + f(\theta(t))] = \frac{1}{2} h^2 M\xi,$$

whence, using (5.7) and $m := \max_i M_i$,

$$|M\dot{\theta}(t+h) - M\dot{\theta}(t) + h[D\dot{\theta}(t) - \frac{1}{2} P]| > h|f(\theta(t))| - \frac{1}{2} h|P| - \frac{1}{2} h^2 a_3 m .$$

This can be rewritten as

$$\begin{aligned} |M[\dot{\theta}(t+h) - \frac{1}{2} D^{-1}P] - (M-hD)[\dot{\theta}(t) - \frac{1}{2} D^{-1}P]| &> h|f(\theta(t))| \\ &- \frac{1}{2} h|P| - \frac{1}{2} h^2 a_3 m . \end{aligned}$$

Consequently, either

$$m|\dot{\theta}(t+h) - \frac{1}{2} D^{-1}P| > \frac{1}{2} h|f(\theta(t))| - \frac{1}{4} h|P| - \frac{1}{4} h^2 a_3 m,$$

or

$$m_h|\dot{\theta}(t) - \frac{1}{2} D^{-1}P| > \frac{1}{2} h|f(\theta(t))| - \frac{1}{4} h|P| - \frac{1}{4} h^2 a_3 m$$

where

$$m_h := \max_i |M_i - hD_i| .$$

Suppose that $\varepsilon > 0$, $h > 0$ are such that

$$|f(\theta(t))| > 3\varepsilon \quad , \quad T_1 < t < T_2 \quad , \quad (5.8)$$

$$|P| < \varepsilon \quad , \quad h a_3 m < \varepsilon \quad , \quad m_h < m \quad . \quad (5.9)$$

Then, for $T_1 < t < T_2$, either

$$m|\dot{\theta}(t+h) - \frac{1}{2} D^{-1}P| > \frac{3}{2} h\varepsilon - \frac{1}{4} h\varepsilon - \frac{1}{4} h\varepsilon = h\varepsilon \quad ,$$

or

$$m|\dot{\theta}(t) - \frac{1}{2} D^{-1}P| > h\varepsilon \quad .$$

In either case one obtains the lower bound

$$\int_{T_1}^{T_2+h} |\dot{\theta}(t) - \frac{1}{2} D^{-1}P|^2 dt > \left(\frac{h\varepsilon}{m}\right)^2 (T_2 - T_1) \quad . \quad (5.10)$$

Step 2. Let $\mathbb{H} := \{\theta | f(\theta) = 0\}$. Let $r > 0$ be such that if θ, ψ are two distinct elements in \mathbb{H} , then

$$B(\theta, 2r) \cap B(\psi, r) = \emptyset \quad .$$

Here $B(\theta, r)$ is the open ball of radius r and center θ . Let

$$B(\mathbb{H}, r) := \bigcup_{\theta \in \mathbb{H}} B(\theta, r) \quad .$$

Define $\varepsilon > 0$ by

$$3\epsilon := \inf\{|f(\theta)| \mid \theta \notin B(\mathbb{H}, r)\}, \quad (5.11)$$

and select positive numbers h, δ, π so that

$$ha_3m < \epsilon, \quad m_h < m, \quad 2a_1h < r, \quad (5.12)$$

$$3m^2\delta < d\epsilon^2h^3 \quad \text{where } d = \min_i D_i, \quad (5.13)$$

$$\pi < 1, \quad \pi < \epsilon, \quad 8\pi m^2\pi < \epsilon^2h^3d, \quad 2\pi m < \epsilon hd. \quad (5.14)$$

Note that this choice guarantees (5.8).

Now fix $P, |P| < \pi$, and a solution $\theta(t), t \geq 0$ of (5.6). If θ is bounded then it converges by Theorem 5.1 and there is nothing more to prove. So suppose $\theta \notin L_\infty$. Multiply (5.6) by $\dot{\theta}(t)$ and integrate

$$\begin{aligned} \int_{T_1}^{T_2} \langle D\dot{\theta} - P, \dot{\theta} \rangle dt &= \int_{T_1}^{T_2} |D^{1/2}\dot{\theta} - \frac{1}{2}D^{-1/2}P|^2 dt - \int_{T_1}^{T_2} |\frac{1}{2}D^{-1/2}P|^2 dt \\ &= W(\theta(T_1), \dot{\theta}(T_1)) - W(\theta(T_2), \dot{\theta}(T_2)), \end{aligned} \quad (5.15)$$

where

$$W(\theta, \dot{\theta}) := \frac{1}{2} \langle M\dot{\theta}, \dot{\theta} \rangle + V(\theta). \quad (5.16)$$

Note that

$$\begin{aligned} \int_{T_1}^{T_2} |D^{1/2}\dot{\theta} - \frac{1}{2}D^{-1/2}P|^2 dt &- \int_{T_1}^{T_2} |\frac{1}{2}D^{-1/2}P|^2 dt \\ &> d \int_{T_1}^{T_2} |\dot{\theta} - \frac{1}{2}D^{-1}P|^2 dt - \frac{1}{4d} \pi^2 (T_2 - T_1). \end{aligned} \quad (5.17)$$

Next observe that the function $t \mapsto W(\theta(t), \dot{\theta}(t))$ is bounded. Hence T_1 can be chosen so that

$$W(\theta(T_1), \dot{\theta}(T_1)) - W(\theta(T_2), \dot{\theta}(T_2)) < \delta, T_2 > T_1. \quad (5.18)$$

Using (5.17), (5.18) in (5.15) gives the upper bound

$$\int_{T_1}^{T_2} \left| \dot{\theta} - \frac{1}{2} D^{-1} p \right|^2 dt < \frac{1}{4d^2} \pi^2 (T_2 - T_1) + \frac{\delta}{d}. \quad (5.19)$$

Step 3. There are several cases to be considered. Suppose first that at time T_1 $\theta(T_1) \notin B(\mathbb{H}, r)$. Define T , possibly infinite, by

$$T := \inf\{t > T_1 \mid \theta(t) \in \overline{B(\mathbb{H}, r)}\} \quad (5.20)$$

Then $\theta(t) \notin B(\mathbb{H}, r)$ for $T_1 < t < T$. Hence, because of (5.11), condition (5.8) is fulfilled.

Suppose $T - T_1 \geq 2h$. Then one may set $T_2 = T_1 + h$ in (5.10) and $T_2 = T_1 + 2h$ in (5.19) to obtain respectively

$$\int_{T_1}^{T_1+2h} \left| \dot{\theta} - \frac{1}{2} D^{-1} p \right|^2 dt > \frac{h^3 \epsilon^2}{m^2},$$

$$\begin{aligned} \int_{T_1}^{T_1+2h} \left| \dot{\theta} - \frac{1}{2} D^{-1} p \right|^2 dt &< \frac{\pi^2 h}{2d^2} + \frac{\delta}{d} \\ &< \frac{h^3 \epsilon^2}{8m^2} + \frac{h^3 \epsilon^2}{3m^2}, \text{ by (5.14), (5.13)} \\ &< \frac{h^3 \epsilon^2}{m^2} \end{aligned}$$

which is a contradiction. Hence it must be the case that

$$T - T_1 < 2h \quad (5.21)$$

From (5.20) it follows that there is ψ in \mathbb{H} such that $\theta(T) \in \overline{B(\psi, r)}$. Since $\phi \notin L_\infty$, Lemma 5.2 implies that $\theta(t)$ must eventually leave the ball $B(\psi, r)$. Define (see Figure 4)

$$T' = \max\{t \mid \theta(t) \in \overline{B(\psi, r)}\} \quad (5.22)$$

and let $T_2 := T' + 2h$. Now write

$$\begin{aligned} w &:= W(\theta(T_1), \dot{\theta}(T_1)) - W(\theta(T_2), \dot{\theta}(T_2)) \\ &= W(\theta(T_1), \dot{\theta}(T_1)) - W(\theta(T), \dot{\theta}(T)) \\ &\quad + W(\theta(T), \dot{\theta}(T)) - W(\theta(T'), \dot{\theta}(T')) \\ &\quad + W(\theta(T'), \dot{\theta}(T')) - W(\theta(T_2), \dot{\theta}(T_2)) \\ &=: w_1 + w_2 + w_3, \text{ say.} \end{aligned}$$

Using (5.15) twice, one obtains

$$w_1 > - \int_{T_1}^T \frac{1}{2} |D^{-1}P|^2 dt > - \frac{1}{4d} \pi^2 (T - T_1) > - \frac{\pi^2 h}{2d}, \text{ using (5.21);}$$

$$w_2 = \int_T^{T'} \langle D\dot{\theta} - P, dt \rangle > - \int_T^{T'} \langle P, \dot{\theta} \rangle dt > - |P| |\theta(T') - \theta(T)| > - 2\pi r.$$

To estimate w_3 observe that since $2h|\dot{\theta}| < 2ha_1 < r$ by (5.12), therefore $\theta(t) \notin B(\mathbb{H}, r)$ for $T' < t < T' + 2h$. Consequently (5.8) is fulfilled and one can use (5.10). Then, from (5.15) and (5.17),

$$\begin{aligned} w_3 &> d \int_{T'}^{T_2} \left| \dot{\theta} - \frac{1}{2} D^{-1}P \right|^2 dt - \frac{1}{4d} \pi^2 (T_2 - T') \\ &> \frac{dh^3 \varepsilon^2}{m^2} - \frac{\pi^2 h}{2d}, \text{ by (5.10)} \end{aligned}$$

Adding these estimates gives

$$\begin{aligned}
 w &> -\frac{\pi^2 h}{d} - 2\pi r + \frac{dh^3 \epsilon^2}{m^2} \\
 &> -\frac{dh^3 \epsilon^2}{4m^2} - \frac{dh^3 \epsilon^2}{4m^2} + \frac{dh^3 \epsilon^2}{m^2} = \frac{dh^3 \epsilon^2}{2m^2} > \frac{3}{2} \delta
 \end{aligned}$$

Using (5.14), (5.13). But from (5.18) $w < \delta$ which is a contradiction.

Hence the only remaining possibility is that there is ψ in \mathbb{H} such that $\theta(T_1) \in B(\mathbb{H}, r)$. Define T' as in (5.22) and $T_2 := T' + 2h$.

Then

$$\begin{aligned}
 w &:= W(\theta(T_1), \dot{\theta}(T_1)) - W(\theta(T_2), \dot{\theta}(T_2)) \\
 &= W(\theta(T_1), \dot{\theta}(T_1)) - W(\theta(T'), \dot{\theta}(T')) \\
 &\quad + W(\theta(T'), \dot{\theta}(T')) - W(\theta(T_2), \dot{\theta}(T_2)) \\
 &:= w_2 + w_3 \text{ say.}
 \end{aligned}$$

The same estimates apply giving $w_2 + w_3 > \frac{3}{2} \delta$ and a contradiction is reached. The theorem is proved. \square

6. Conclusions. In our knowledge this paper is the first attempt following the pioneering work published in [3] that deals with the global behavior of power systems. The practical implications of the results reported here have to be drawn with care because of the limitations of the model discussed in Section 2. Nevertheless, two conclusions seem worth emphasizing. First, as pointed out in the Introduction, transient stability studies focus on the determination of the "size" of the region of attraction of a particular, namely the pre-fault, equilibrium. Suppose, however, that it can be shown that the

system is completely stable. In such cases the study of a single equilibrium would seem not to be very important since the system is guaranteed to converge to some equilibrium. Second, the results suggest that in the absence of voltage regulation a Hopf bifurcation of the kind reported in [10] may not arise. Does this have any bearing on the design of such regulators?

Even within the framework of the simple model used here there remain important gaps in our understanding. Suppose $D > 0$. Then the set of excess power supply vectors P for which the system is completely stable includes a neighborhood of the origin. We conjecture that this set is open. Second, if P belongs to the boundary of this set, we conjecture the appearance of cyclic saddle connections. Finally, if P lies outside the boundary, we conjecture the formation of closed orbit of the second kind.

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Figures Captions

- 1a. $\alpha < \alpha_0(\beta)$, stable orbit 0.
- 1b. $\alpha > \alpha_0(\beta)$, complete stability.
- 1c. $\alpha = \alpha_0(\beta)$, saddle connection Γ .
- 1d. $\beta = \gamma$, saddle-node bifurcation
2. Bifurcation diagram for (3.2).
3. $\alpha = 0$, orbits of first kind.
4. Definition of T , T' , T_2 .





