GENERAL STUDY OF DISCRETE-TIME CONVOLUTION CONTROL SYSTEMS

by

V. H. L. Cheng and C. A. Desoer

Memorandum No. UCB/ERL M80/49

15 August 1980

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720
GENERAL STUDY OF DISCRETE-TIME CONVOLUTION CONTROL SYSTEMS

V.H.L. Cheng and C.A. Desoer

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California
Berkeley, California 94720

Abstract

In this report, we formulate a general class of linear time-invariant discrete-time distributed systems; and we study in depth these systems from the control system design point of view. We consider both single-input single-output (SISO) and multi-input multi-output (MIMO) systems, investigate their analytic properties, and establish design procedures for these systems.

The input-output (I/O) behavior of a linear time-invariant system is specified by its transfer function: For a causal lumped SISO system, its transfer function is a proper rational function. Such rational transfer functions have been extensively studied as functions of a complex variable, which led to many important control theory results (e.g. Nyquist theory, Bode plot, etc.). Vidyasagar pointed out that a proper rational function can be expressed as a ratio of two elements in some algebra other than the algebra of polynomials (e.g. the algebra of proper "stable" rational functions): This observation has led to a broad effort to investigate the relationship between the important properties of linear time-invariant systems and their algebraic structures. Extending this idea to continuous-time (distributed) convolution systems, F. M. Callier and C. A. Desoer (1978) developed an algebra of transfer
functions $\hat{B}(\sigma_0)$ for describing their transfer functions: here every element of $\hat{B}(\sigma_0)$ is expressed as a ratio of two elements in an algebra $\hat{A}(\sigma_0)$, the subalgebra of causal $\sigma_0$-stable transfer functions.

In this research, we study discrete-time (distributed) convolution systems, by making full use of the algebraic tools that have proved to be useful in the study of the other system representations, we develop a commutative algebra of transfer functions, $\hat{B}(\rho_0)$, for a general class of SISO discrete-time convolution systems, which covers sampled distributed-systems and, of course, lumped systems as a special case. Each element of $\hat{B}(\rho_0)$ is formulated as a ratio of two elements in an algebra $\hat{\mathcal{L}}^{-1}(\rho_0)$ of causal $\rho_0$-stable transfer functions. We demonstrate that $\hat{\mathcal{L}}^{-1}(\rho_0)$ is indeed a Euclidean ring; we give necessary and sufficient conditions for coprimeness between elements in $\hat{\mathcal{L}}^{-1}(\rho_0)$; and we study the concepts of poles and zeros for elements in $\hat{B}(\rho_0)$. In contrast to the existing theory on transfer functions corresponding to $\ell_1$-sequences, the algebra $\hat{B}(\rho_0)$ includes both stable and unstable systems; and since $\rho_0 < 1$, this formulation allows us to study the dominant poles inside the unit disc of the complex plane.

With the SISO theory well established, we study MIMO systems whose transfer functions are matrices with elements in $\hat{B}(\rho_0)$, and we establish the matrix fraction representation theory. Consequently, matrix multiplication introduces many additional problems: commutativity is lost, zero divisors are present, and the ring structure is lost in the case of nonsquare matrices.

We then investigate in detail many results of MIMO $\hat{B}(\rho_0)$-systems that have similar counterparts in the other system descriptions: In particular, we obtain the dynamic interpretation of poles and
transmission zeros. We consider interconnections of such MIMO systems, with feedback as a special case. We introduce the notion of characteristic functions to study the overall stability of any such interconnection (an idea similar to but not identical with that of characteristic polynomial); and we obtain necessary and sufficient conditions for $\mathcal{L}_p$-stability, $\forall p \in [1,\infty]$. The matrix fraction representation also allows us to obtain procedures for designing feedback systems with controllers to achieve stabilization (analogous to arbitrary closed-loop eigenvalue assignment), asymptotic tracking and disturbance rejection; finally, for the case of stable square plants (which can be obtained from an unstable one by the stabilization procedure), we are able to achieve complete decoupling with detailed pole assignment and finite settling-time, subject to, of course, the limitations imposed by the plant transmission zeros outside the open unit disc.
# Table of Contents

1. Introduction ........................................... 2
   Introduction and notation

2. \( \tilde{\mathcal{B}}(\rho_0) \), the Class of Transfer Functions .......... 5
   2.1 Convolution Systems ............................... 5
   2.2 The Class of Sequences \( \mathcal{L}_1(\rho_0) \) .......... 6
   2.3 The Class of Sequences \( \mathcal{L}_{1^-}(\rho_0) \) ...... 8
   2.4 The Class of Sequences \( \mathcal{L}_{1^\infty}(\rho_0) \) .... 10
   2.5 The Transfer Functions in \( \tilde{\mathcal{B}}(\rho_0) \) ........ 11
   2.6 Examples ........................................... 19

3. Matrix Fraction Representation Theory .......................... 24
   Coprimeness and Representations

4. Poles, Zeros and Their Dynamic Interpretation ................. 34
   4.1 McMillan Degree of Poles, Smith and McMillan Forms .... 34
   4.2 Dynamic Interpretation of Poles ..................... 40
   4.3 Zeros and Their Dynamic Interpretation ............... 41
   4.4 Example ............................................ 44

5. Interconnected Systems and Characteristic Functions ............ 48
   Example and procedure for analyzing interconnected systems;
   characteristic function \( \tilde{\chi} \) and dynamic interpretation of its
   zeros; \( \mathcal{I}_p \)-stability of interconnected systems, \( p \in [1,\infty] \)

6. Feedback System Stability .................................. 54
   \( \mathcal{I}_p \)-stability for feedback systems, \( p \in [1,\infty] \)

7. Compensator Design for Stabilization, Tracking and
   Disturbance Rejection ................................... 58
   7.1 Preliminary Algebraic Result ......................... 58
   7.2 Problem of Stabilization, Tracking and Disturbance
   Rejection ............................................. 59
GENERAL STUDY OF DISCRETE-TIME CONVOLUTION CONTROL SYSTEMS†

V.H.L. Cheng and C.A. Desoer

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California
Berkeley, California 94720

†Research sponsored by the National Science Foundation Grant ENG78-09032-A01.
1. Introduction

Consider a discrete-time convolution system whose weighting sequence is obtained by sampling the impulse response of a continuous-time linear time-invariant distributed system. Such a sampled system cannot, in general, be represented by a rational z-transfer function. In this paper, we develop a general theory to cover such cases. Our approach includes the rational transfer functions as a special case, and in many instances, the analysis exhibits some resemblance with the existing techniques for the rational case.

In Section 2, we develop a model for a class of such systems, whose transfer functions are elements of an algebra denoted by $\tilde{b}(\rho_0)$. The model encompasses both stable and unstable systems in the input-output (I/O) context. We discuss some properties of the poles and zeros for such systems, and we give some examples to demonstrate this more general model of system description. We consider in Section 3 multi-input multi-output (MIMO) systems whose transfer functions are matrices with elements in $\tilde{b}(\rho_0)$: we examine the notion of coprimeness (left- and right-coprime) and derive the matrix fraction representation theory for these systems. In Section 4, we consider the poles and define (transmission) zeros for MIMO systems and exhibit their dynamic interpretations; an example is given to demonstrate the claimed properties of the transmission zeros. Interconnected systems are considered in Section 5: here we introduce the notion of characteristic function for studying I/O stability. As a special case of interconnected systems, feedback systems and their I/O stability are studied in Section 6. In Section 7, we study the problem of controller design for feedback
systems to satisfy specifications on stabilization, tracking and disturbance rejection; an example is provided to demonstrate the step-by-step procedure to obtain the controller transfer function. In Section 8, we extend the findings of [Des 5] to study feedback decoupling when the given plant is square and stable. We conclude this paper by some discussions in Section 9.

**Notation**

Let $\mathbb{R}$ (resp. $\mathbb{C}$) be the field of all real (complex) numbers; let $\mathbb{N} := \{0,1,2,...\}$ be the set of all natural numbers, and $\mathbb{N}^* := \mathbb{N}\{0\} = \{1,2,3,...\}$ be the set of all positive integers. We denote by $\mathbb{R}_{\mathbb{N}}$ (respectively $\mathbb{C}_{\mathbb{N}}$) the set of all real (resp. complex) sequences on $\mathbb{N}$, i.e. $\mathbb{R}_{\mathbb{N}}$ (resp. $\mathbb{C}_{\mathbb{N}}$) := \{(g(0),g(1),g(2),...) | g(k) \in \mathbb{R}$ (resp. $\mathbb{C}$), $\forall k \in \mathbb{N}\}$; and we denote by $\mathbb{R}_{\mathbb{N}}^*$ (resp. $\mathbb{C}_{\mathbb{N}}^*$) the subset of all z-transformable sequences in $\mathbb{R}_{\mathbb{N}}$ (resp. $\mathbb{C}_{\mathbb{N}}$), i.e. $g \in \mathbb{R}_{\mathbb{N}}^*$ (resp. $\mathbb{C}_{\mathbb{N}}^*$) belongs to $\mathbb{R}_{\mathbb{N}}^*$ (resp. $\mathbb{C}_{\mathbb{N}}^*$) if and only if the series $\sum_{k=0}^{\infty} g(k)z^{-k}$ converges for some $z \in \mathbb{C}$. For any $k \in \mathbb{N}$, we define $\delta_k \in \mathbb{C}_{\mathbb{N}}^*$ as the complex sequence on $\mathbb{N}$ with $\delta_k(k) = 1$, and $\delta_k(i) = 0$ $\forall i \neq k$. Let the superscript $^-$ denote z-transforms: if $g \in \mathbb{C}_{\mathbb{N}}^*$, then $\tilde{g}(z) := \sum_{k=0}^{\infty} g(k)z^{-k}$ is defined for $z \in \mathbb{C}$ wherever the series converges; if $S \subseteq \mathbb{C}_{\mathbb{N}}^*$, then $S := \{\tilde{g}|g \in S\} \subseteq \mathbb{C}_{\mathbb{N}}^*$. For any nonzero $g \in \mathbb{C}_{\mathbb{N}}^*$, we define the order [Kuc 1] of its z-transform $\tilde{g}$ as $\text{ord}(\tilde{g}) := \text{index of the first nonzero component of } g$. Let $\mathbb{C}[z]$ be the ring of all polynomials in the complex variable $z$ with coefficients in $\mathbb{C}$, $\mathbb{C}(z)$ be the field of all rational functions, and $\mathbb{C}_p(z)$ be the subset of all proper elements of $\mathbb{C}(z)$. The spaces of n-tuples and matrices are specified by superscripts in the usual manner, e.g. $\mathbb{C}^n$, $\mathbb{R}(z)^{mxm}$, ... Let $\theta_n$
be the zero element of $\mathbb{C}^n$. For $a \in \mathbb{C}[z]$, let $\deg a$ denote the degree of $a$; if $v \in \mathbb{C}[z]^n$, then $\deg v$ denotes the maximum degree of the components of $v$. For $M \in \mathbb{C}[z]^{n_0 \times n_i}$, we denote by $\deg_{c_i} M$ the $i$th column degree of $M$ (i.e. the maximum degree of the components in the $i$th column of $M$). Similarly, for $a \in \mathbb{C}[z^{-1}]$, a polynomial in $z^{-1}$, we denote by $\deg a$ the degree of $a$ as a polynomial in $z^{-1}$. Let $K \subseteq \mathbb{C}$ be open: for $f: K \rightarrow \mathbb{C}$, $Z[f]$ denotes the set of zeros of $f$; for $F: K \rightarrow \mathbb{C}^{n_0 \times n_i}$, $P[F]$ denotes the set of poles of all components of $F$. Let $\rho_0 > 0$; we denote by $D(\rho_0) := \{z \in \mathbb{C} : |z| < \rho_0\}$ the open disc with radius $\rho_0$ about the origin in the complex plane.
2. \( \mathcal{B}(\rho_0) \), the Class of Transfer Functions

2.1 Convolution systems

The I/O characterization of discrete-time causal convolution systems is most conveniently done through their weighting sequences.

When an input sequence \( u \in \mathbb{C}^\mathbb{N} \) is applied to a causal convolution system with weighting sequence \( h \in \mathbb{C}^\mathbb{N} \), the output sequence \( y = (y(k))_{k=0}^\infty := h \ast u \) is given by the convolution formula

\[
y(k) = \sum_{i=0}^{k} h(i)u(k-i) = \sum_{i=0}^{k} h(k-i)u(i) \quad \forall k \in \mathbb{N} \quad (2.1)
\]

where the summing variable \( i \) represents the age variable.

For any \( p \in [1,\infty) \), let \( |\cdot|_p \) be the usual norm defined on the normed space \( \ell^p \subset \mathbb{C}^\mathbb{N} \). It is a well-known fact [Des 1, p.244] that if \( h \in \ell_1 \), then, \( \forall p \in [1,\infty) \),

\[
u \in \ell^p \Rightarrow y := h \ast u \in \ell^p \, ,
\]

and in fact, \( \forall p \in [1,\infty) \),

\[
|h \ast u|_p \leq |h|_1 |u|_p \, . \quad (2.2)
\]

The relationship, however, lacks the useful information of how fast the sequence \( y \) decays, even when \( u \) is zero except for a finite number of components.\(^{\dagger}\)

Suppose now that the sequence \( h \) satisfies the stronger condition that, for some \( \rho_0 \in [0,1[ \), the sequence \( \bar{h} \) defined by

\[
\bar{h} := (h(k)^{\rho_0})_{k=0}^\infty \quad \forall k \in \mathbb{N}
\]

belong to \( \ell_1 \). If the input sequence \( u \) has

\(^{\dagger}\)The sequence \( g \in \mathbb{C}^\mathbb{N} \) is said to decay to 0 exponentially at a rate (at least) \( \mu \) iff \( \exists \mu > 1, \exists M > 0 \) such that \( |g(i)| \leq M(1/\mu)^i \) \( \forall i \in \mathbb{N} \).
finite support (i.e. there exists least \( N \in \mathbb{N} \) such that \( u(k) = 0 \) \( \forall k > N \)), then for any \( k > N \),

\[
|y(k)| = \left| \sum_{i=0}^{k} h(i)u(k-i) \right| \\
\leq \sum_{i=0}^{k} |h(i)||u(k-i)| \\
= \sum_{i=k-N}^{k} |h(i)||u(k-i)| \quad \text{since } u(i) = 0 \text{ for } i > N \\
\leq |u|_{\infty} \sum_{i=k-N}^{k} |h(i)| \quad \text{since } |u|_{\infty} = \max_{0 \leq i \leq N} |u(i)| \\
\leq |u|_{\infty} \rho_0^{-N} \sum_{i=k-N}^{k} |h(i)| \rho_0^{-i} \\
|y(k)| \leq |u|_{\infty} |\bar{h}|_{1} \rho_0^{-N} \quad \forall k > N. \quad (2.3)
\]

Hence, the output \( y \) decays exponentially to 0 at a rate at least \( \rho_0^{-1} \), where the constant\(^{\dagger} \) \( M \) may depend on \( N \) (from (2.3), we may take \( M := \max(|y(0)|, |y(1)| \rho_0^{-1}, \ldots, |y(N-1)| \rho_0^{-(N-1)}; |u|_{\infty} |\bar{h}|_{1} \rho_0^{-N}) \).

2.2 The Class of Sequences \( \ell_1(\rho_0) \)

The preceding discussion leads us to consider the class of weighted sequences

\[
\ell_1(\rho_0) := \{ g \in \ell^\infty | \sum_{k=0}^{\infty} |g(k)| \rho_0^{-k} < \infty \} \subset \ell^\infty \quad (2.4)
\]

where typically \( \rho_0 \in [0,1[ \). The properties of this class of sequences are given below. Detailed proofs of these properties and the properties in the next subsection, namely 2.3, are given in Appendix A.

\(^{\dagger}\)See footnote of previous page.
$\ell_1(\rho_0)$ is a complex vector space. It forms a complete normed space with the norm $\| \cdot \|_{\rho_0} : \ell_1(\rho_0) \rightarrow \mathbb{R}_+$ defined by

$$\|g\|_{\rho_0} := \sum_{k=0}^{\infty} |g(k)| \rho_0^{-k} \quad \forall g \in \ell_1(\rho_0) . \quad (2.5)$$

(2.2.1) If we choose as multiplication in $\ell_1(\rho_0)$ the convolution operator, $\ell_1(\rho_0)$ is a commutative Banach algebra with neutral element (unit, multiplicative unit, multiplicative identity) $\delta_0 := (1,0,0,...)$.

(2.2.2) For $0 < \rho_1 < \rho_0$, $\ell_1(\rho_1) \subset \ell_1(\rho_0)$.

(2.2.3) $\ell_1(\rho_0)$ has no divisors of zero and is thus an integral domain (entire ring [Lan 1]).

(2.2.4) For any $g \in \ell_1(\rho_0)$,

(i) the series $\sum_{k=0}^{\infty} g(k)z^{-k}$ converges absolutely for all $z \in D(\rho_0)^C$ and is bounded there by $\|g\|_{\rho_0}$;

(ii) $\forall \varepsilon > 0$, it converges uniformly in $D(\rho_0+\varepsilon)^C$, hence $\bar{g}(\cdot)$ is analytic in $\overline{D(\rho_0)}^C$;

(iii) as $|z| \rightarrow \infty$, $\bar{g}(z) \rightarrow g(0)$.

(2.2.5) $\tilde{\ell}_1(\rho_0)$ is a commutative algebra of functions analytic in $\overline{D(\rho_0)}^C$ and bounded in $D(\rho_0)^C$, with pointwise addition and multiplication, with neutral element $\tilde{\delta}_0(z) = 1 \quad \forall |z| \geq \rho_0$, and with no divisors of zero.

(2.2.6) Inversion Theorem

$g \in \ell_1(\rho_0)$ has an inverse in $\ell_1(\rho_0)$ \quad (2.6)
\[ \inf_{|z| \geq \rho_0} |\tilde{g}(z)| > 0 \quad (2.7) \]

\[ g(0) \neq 0 \quad (2.8) \]

\[ \tilde{g}(z) \neq 0 \quad \forall |z| \geq \rho_0 \]

Note that if \( h \in l_1(\rho_0) \) is the inverse of \( g \), then
\[ \tilde{h}(z) = 1/\tilde{g}(z). \] The next property will be useful for proving the coprimeness condition of (2.3.6).

(2.2.7) Given \( f, g \) in the Banach algebra \( l_1(\rho_0) \). \( \exists u, v \in l_1(\rho_0) \)
such that
\[ u*f + v*g = \delta_0, \quad (2.10a) \]

or equivalently,
\[ \overline{u}\tilde{f} + \overline{v}\tilde{g}(z) = 1 \quad \forall |z| \geq \rho_0 \quad (2.10b) \]

\[ \inf_{|z| \geq \rho_0} |\tilde{(f(z),\tilde{g}(z))}| > 0 \quad (2.11) \]

\[ (i) \quad |(f(0),g(0))| > 0 \quad (2.12) \]

\[ (ii) \quad |(\tilde{f}(z),\tilde{g}(z))| > 0 \quad \forall |z| \geq \rho_0 \]

where \(|\cdot|\) is any norm on \( \mathbb{C}^2 \).

\[ 2.3 \quad \text{The Class of Sequences } l_{1-}(\rho_0) \]

For \( \rho_0 > 0 \), typically \( \rho_0 \leq 1 \), we define a class of complex sequences on \( \mathbb{N} \) by
\[ l_{1-}(\rho_0) := \bigcup_{0 \leq \rho_1 < \rho_0} l_1(\rho_1) \subset \mathbb{C}^\mathbb{N}. \quad (2.14) \]

Note that \( l_{1-}(\rho_0) \subset l_1(\rho_0) \), in view of definition (2.14) and Property (2.2.2).
(2.3.1) \( l_1(\rho_0) \) is a normed commutative subalgebra of \( l_1(\rho_0) \) with norm \( \| \cdot \|_{\rho_0} \), with neutral element \( \delta_0 \), and with no divisors of zero. Similarly \( \tilde{l}_1(\rho_0) \) is a commutative pointwise-product subalgebra of \( \tilde{l}_1(\rho_0) \), with neutral element \( \tilde{\delta}_0(z) = 1 \ \forall |z| \geq \rho_0 \), and with no divisors of zero. Consequently, \( l_1(\rho_0) \) and \( \tilde{l}_1(\rho_0) \) are both integral domains.

(2.3.2) If \( g \in l_1(\rho_0) \), then

(i) \( g(\cdot) \) is analytic in \( D(\rho_0)^c \) for some \( \rho_g < \rho_0 \); in particular, it is analytic in \( D(\rho_0)^c \);

(ii) \( \tilde{g}(\cdot) \) is bounded on \( D(\rho_g)^c \supset D(\rho_0)^c \);

(iii) \( \tilde{g}(\cdot) \) has a finite number of zeros in \( D(\rho_0)^c \).

(2.3.3) \( g \in l_1(\rho_0) \) has an inverse in \( l_1(\rho_0) \) (2.15)

\[
\iff \inf_{|z| \geq \rho_0} |\tilde{g}(z)| > 0 \quad (2.16)
\iff (i) \ g(0) \neq 0 \quad (2.17)
\iff (ii) \ \tilde{g}(z) \neq 0 \ \forall |z| \geq \rho_0.
\]

(2.3.4) \( \tilde{l}_1(\rho_0) \) is a Euclidean ring (hence a principal ideal ring [Sig 1, p.133]), with a gauge [Sig 1, p.132] [Her 1, p.143] (or stathm [McD 1, p.30]) \( \gamma: \tilde{l}_1(\rho_0) \setminus \{0\} \to \mathbb{N} \) defined for all nonzero \( \tilde{g} \in \tilde{l}_1(\rho_0) \) by

\[
\gamma(\tilde{g}) := \text{ord}(\tilde{g}) + \text{number of zeros of } \tilde{g} \text{ in } D(\rho_0)^c, \text{ counting (2.18) multiplicities.}
\]

The Euclidean algorithm is given in Procedure A.1 of Appendix A.

Consequently, \( l_1(\rho_0) \) is a Euclidean ring (and thus a principal ideal ring) with the same gauge defined for \( \tilde{l}_1(\rho_0) \).
(2.3.5) **Definition.** Given \( f, g \) in the commutative Euclidean ring \( \ell_1(\rho_0) \). Then \( f, g \) are said to be \( \rho_0 \)-coprime iff any greatest common divisor of \( f \) and \( g \), denoted by \( \gcd(f,g) \), is an invertible element of \( \ell_1(\rho_0) \) [Sig 1, p.142] [McLl, p.154].

\( \tilde{f}, \tilde{g} \in \tilde{\ell}_1(\rho_0) \) are also said to be \( \rho_0 \)-coprime if and only if \( f, g \in \ell_1(\rho_0) \) are \( \rho_0 \)-coprime.

(2.3.6) Given \( f, g \in \ell_1(\rho_0) \). \( f, g \) are \( \rho_0 \)-coprime, 

\[ \Rightarrow \exists u, v \in \ell_1(\rho_0) \text{ such that} \]

\[ u\tilde{f} + v\tilde{g} = \delta_0, \] (2.20a)

or equivalently, \( (\tilde{u}\tilde{f} + \tilde{v}\tilde{g})(z) = 1 \quad \forall |z| \geq \rho_0, \) (2.20b)

\[ \Rightarrow \inf_{|z| \geq \rho_0} |(\tilde{f}(z),\tilde{g}(z))| > 0 \] (2.21)

\[ \Rightarrow (i) \quad |(f(0),g(0))| > 0 \]

\[ (ii) \quad |(\tilde{f}(z),\tilde{g}(z))| > 0 \quad \forall |z| \geq \rho_0, \] (2.22)

where \( |\cdot| \) is any norm on \( \mathbb{C}^2 \).

2.4 **The Class of Sequences** \( \ell_\infty(\rho_0) \)

With \( \ell_1(\rho_0) \) defined above as in (2.14), we define a subset of it by

\[ \ell_\infty(\rho_0) := \{ g \in \ell_1(\rho_0) \mid \lim_{|z| \to \infty} g(z) = g(0) \neq 0 \} . \] (2.23)

Note that \( \ell_\infty(\rho_0) \) and \( \tilde{\ell}_\infty(\rho_0) \) are multiplicative subsets [Lan 1, p.66] [Zar 1, p.46] of \( \ell_1(\rho_0) \) and \( \tilde{\ell}_1(\rho_0) \), respectively.

**Remark 2.1.** Consider Property (2.3.6). A necessary condition for \( f, g \in \ell_1(\rho_0) \) to be \( \rho_0 \)-coprime is that at least one of them must

...
belong to $\mathcal{L}_1^\infty(\rho_0)$. Under this condition, $f$ and $g$ are $\rho_0$-coprime iff $|(\tilde{f}(z), \tilde{g}(z))| > 0 \forall |z| \geq \rho_0$, i.e. $\tilde{f}$ and $\tilde{g}$ have no common zeros in $D(\rho_0)^c$.

2.5 The Transfer Functions in $b(\rho_0)$

We now define a class of complex sequences on $\mathbb{N}$ whose z-transforms form the class of (stable or unstable) transfer functions we are concerned with.

**Definition 2.1.** Given the convolution algebra $\mathcal{L}_1^\infty(\rho_0)$ and the multiplicative subset $\mathcal{L}_1^\infty(\rho_0)$, $0 < \rho_0 \leq 1$, the algebra of fractions $b(\rho_0)$ [Zar 1, p.46] [Lan 1, p.66] is defined by

$$b(\rho_0) := \mathcal{L}_1^\infty(\rho_0)[\mathcal{L}_1^\infty(\rho_0)]^{-1} = \{\tilde{g} = \tilde{n}/\tilde{d} | \tilde{n} \in \mathcal{L}_1^\infty(\rho_0), \tilde{d} \in \mathcal{L}_1^\infty(\rho_0)\}.$$  

Let $b(\rho_0)$ be the set of complex sequences on $\mathbb{N}$ defined as

$$b(\rho_0) := \{g \in \mathbb{C}^\mathbb{N} | \tilde{g} \in b(\rho_0)\} \subset \mathbb{C}^\mathbb{N}.$$  

**Remark 2.2.** (i) The z-transform is a linear bijective map from $b(\rho_0)$ onto $b(\rho_0)$. The definition (2.25) shows immediately that it is a linear map from $b(\rho_0)$ into $b(\rho_0)$. This map is bijective because every $\tilde{g} \in b(\rho_0)$ can be expressed as a Taylor series (necessarily unique) about infinity, thus specifying a unique sequence in $b(\rho_0) \subset \mathbb{C}^\mathbb{N}$. More precisely, $\tilde{g} = \tilde{n}/\tilde{d}$ where $\tilde{n} \in \mathcal{L}_1^\infty(\rho_0)$ and $\tilde{d} \in \mathcal{L}_1^\infty(\rho_0)$ are both analytic and bounded in $D(\rho_0)^c$. Since $\tilde{d}$ has a finite number of zeros in $D(\rho_0)^c$ and $\lim_{|z| \to \infty} |\tilde{d}(z)| = d(0) \neq 0$, then $\exists \rho_d \geq \rho_0$ such that $\inf_{|z| \geq \rho_d} |\tilde{d}(z)| > 0$. Thus $\tilde{g}$ is analytic and bounded in the "annulus" $|z| \geq \rho_d$.
given by \(|z| \in [\rho_d, \infty[\). Hence \(\forall |z| \geq \rho_d\), \(\tilde{g}(z)\) can be expanded as a unique Laurent series

\[
\tilde{g}(z) = \sum_{k=0}^{\infty} g(k)z^{-k} + \sum_{k=1}^{\infty} g'(k)z^k
\]  

(2.26)

where, \(\forall \rho \geq \rho_d\),

\[
g(k) = \frac{1}{2\pi i} \oint_{|z|=\rho} \tilde{g}(z)z^{-k-1}dz \quad \forall k \in \mathbb{N} \quad (2.27)
\]

and

\[
g'(k) = \frac{1}{2\pi i} \oint_{|z|=\rho} \tilde{g}(z)z^{-k-1}dz \quad \forall k \in \mathbb{N}^* \quad . (2.28)
\]

Since the value of the contour integral in (2.28) is independent of \(\rho \geq \rho_d\), and since \(\tilde{g}(z)\) is bounded by some \(\tilde{g}_{\text{max}} < \infty\) in \(D(\rho_d)^c\), hence for \(k \geq 1\), as \(\rho \to \infty\), the integral in (2.28) goes to zero. Hence \(g'(k) = 0 \ \forall k \in \mathbb{N}^*\), and (2.26) represents \(\tilde{g}\) as a power series in \(z^{-1}\); thus \(\tilde{g}\) specifies a unique sequence \((g(k))_{k=0}^{\infty}\) in \(b(\rho_0)\).

(ii) From the proof of the preceding remark, it follows that if \(\tilde{g}: D(\rho)^c \to \mathbb{C}\) is analytic and bounded on \(D(\rho_0)^c\) for some \(\rho_0 > \rho\), then \(\tilde{g} \in \mathbb{\tilde{x}}_{\text{1-}}(\rho_0)\).

(2.5.1) It is well known [Zar 1, p.46] [Lan 1, p.66] that \(\tilde{b}(\rho_0)\) is a commutative algebra of fractions with pointwise sum and product, and neutral element given by \(\tilde{\delta}_0(z) = 1, \ \forall |z| \geq \rho_0\). Consequently, \(b(\rho_0)\) is a commutative convolution algebra of complex sequences on \(\mathbb{N}\) with neutral element \(\delta_0 := (1,0,0,\ldots)\).

(2.5.2) For any \(\tilde{g} = \bar{n}/\bar{d} \in \tilde{b}(\rho_0)\) with \(\bar{n} \in \tilde{x}_{\text{1-}}(\rho_0)\) and \(\bar{d} \in \tilde{x}_{\text{1-}}^{\infty}(\rho_0)\), since \(\bar{n}, \bar{d}\) are analytic in \(D(\rho_0)^c\) and both have only a finite number of zeros in \(D(\rho_0)^c\), \(\tilde{g}\) has a finite number of zeros in \(D(\rho_0)^c\)
and is analytic except for a finite number of poles in $D(p_0)^C$ (i.e. $\tilde{g}$ is meromorphic in $D(p_0)^C$). Moreover, $\tilde{g}$ is bounded at $\infty$ because
\[ \lim_{|z| \to \infty} \tilde{g}(z) = n(0)/d(0) \text{ and } |n(0)| < \infty, \quad d(0) \neq 0. \]

\[ \square \]

**Definition 2.2.** The pair $(\tilde{n}, \tilde{d})$ is called a $\rho_0$-representation ($\rho_0$-r.) of $\tilde{g} \in \tilde{b}(p_0)$ iff

1. $\tilde{n} \in \tilde{x}_1(p_0)$, $\tilde{d} \in \tilde{x}_1^\infty(p_0)$,
2. $\tilde{g} = \tilde{n}/\tilde{d}$,
3. $\tilde{n}, \tilde{d}$ are $\rho_0$-coprime.

\[ \square \]

**Lemma 2.1.** If $\tilde{g} \in \tilde{b}(p_0)$, then $\tilde{g}$ admits a $\rho_0$-representation. \[ \square \]

**Note.** The proofs of lemmas and theorems are relegated to Appendix B.

**Lemma 2.2.** Given $\tilde{g} \in \tilde{b}(p_0)$, let $(\tilde{n}, \tilde{d})$ be one of its $\rho_0$-representations (whose existence is guaranteed by Lemma 2.1). Then for any $p \in D(p_0)^C$,

1. $\tilde{g}$ has an $m$th order zero at $p$ iff $\tilde{n}$ has an $m$th order zero at $p$;
2. $\tilde{g}$ has an $m$th order pole at $p$ iff $\tilde{d}$ has an $m$th order zero at $p$.

\[ \square \]

Recall that $\mathbb{C}_p(z)$ denotes the set of all proper rational functions in the complex variable $z$ with coefficients in $\mathbb{C}$, and let
\[ r(\rho_0) := \mathbb{C}_p(z) \cap \tilde{x}_1(p_0) \] \[ r^\infty(\rho_0) := r(\rho_0) \cap \tilde{x}_1^\infty(p_0) = \mathbb{C}_p(z) \cap \tilde{x}_1^\infty(p_0). \]

It has been shown in [Mor 1] that $r(\rho_0)$ is a principal ideal ring. In fact, $r(\rho_0)$ is a Euclidean ring, with a gauge
\( \gamma: \mathbb{r}(\rho_0) \setminus \{0\} \to \mathbb{N} \) defined as in (2.18) when \( \mathbb{r}(\rho_0) \) is viewed as a subset of \( \tilde{\mathbb{r}}_1(\rho_0) \); equivalently, for any nonzero \( a \in \mathbb{r}(\rho_0) \),

\[
\gamma(a) = \text{number of poles of } a \text{ in } D(\rho_0) - \text{number of zeros of } a \text{ in } D(\rho_0).
\]

The Euclidean algorithm for \( \mathbb{r}(\rho_0) \) is similar to the one for \( \mathbb{R}(\sigma_0) \) given in [Cal 2] [Cal 3], by noting that the role of \( D(\rho_0) \) with respect to \( \mathbb{r}(\rho_0) \) is the same as the role of \( \mathbb{R}(\sigma_0) \) with respect to \( \mathbb{R}(\sigma_0) \).

**Definition 2.3.** A \( \rho_0 \)-representation \( (\tilde{n}, \tilde{d}) \) of \( \tilde{g} \in \tilde{\mathbb{b}}(\rho_0) \) is said to be **normalized** iff

(i) \( \tilde{d} \in \mathbb{r}^\infty(\rho_0) \)

(ii) \( \lim_{|z| \to \infty} \tilde{d}(z) = 1 \)

(iii) \( \mathbb{Z}[\tilde{d}] \subset D(\rho_0)^c \)

**Remark 2.3.** (i) Observe that \( \tilde{d} \in \mathbb{r}^\infty(\rho_0) \) is a rational function whose numerator and denominator polynomials have the same degree, and all the poles of \( \tilde{d} \) are inside the open disc \( D(\rho_0) \). Hence a \( \rho_0 \)-representation \( (\tilde{n}, \tilde{d}) \) of \( \tilde{g} \in \tilde{\mathbb{b}}(\rho_0) \) is normalized if and only if \( \tilde{d}(z) \) can be expressed as a finite product of rational factors of the form

\[
(z-p)/(z-a)
\]

where \( p \in D(\rho_0)^c \) and \( a \in D(\rho_0) \).

(ii) If \( (\tilde{n}, \tilde{d}) \) is a \( \rho_0 \)-representation of \( \tilde{g} \in \tilde{\mathbb{b}}(\rho_0) \) with \( \tilde{d} \in \mathbb{r}^\infty(\rho_0) \), we can easily obtain a normalized \( \rho_0 \)-representation \( (\tilde{n}, \tilde{d}) \) of \( \tilde{g} \in \tilde{\mathbb{b}}(\rho_0) \) by adjusting the factors in \( \tilde{d} \): more precisely, put \( \tilde{d}(z) \) in the form
\[ \tilde{d}(z) = d(0) \prod_{i=1}^{m_1} \frac{(z-p_i)}{(z-a_i)} \prod_{i=m_1+1}^{m_2} \frac{(z-a_i)}{(z-a_i)} \]  

(2.32)

where \( d(0) \in \mathcal{C}, \ d(0) \neq 0; \ a_i \in D(\rho_0), \ i = 1, 2, \ldots, m_2; \ p_i \in D(\rho_0), \ i = 1, 2, \ldots, m_1; \ p_i \in D(\rho_0)^c, \ i = m_1+1, \ldots, m_2. \) Note that

\[ \tilde{c} := \frac{m_1(z-p_i)}{d(0) \prod_{i=1}^{m_1} (z-a_i)} \in \mathcal{R}(\rho_0) \]  

(2.33)

is an invertible element of \( \mathcal{R}(\rho_0) \subset \tilde{\mathcal{L}}(\rho_0), \) and \((\tilde{n}, \tilde{d})\) given by

\[ \tilde{n} := \tilde{n}c^{-1}, \ \tilde{d} := \tilde{d}c^{-1} \]  

(2.34)

is a normalized \( \rho_0 \)-representation of \( \tilde{g}. \)

\[ \square \]

**Theorem 2.1.** If \( \tilde{g} \in \tilde{b}(\rho_0), \) then \( \tilde{g} \) admits a normalized \( \rho_0 \)-representation. One such representation can be obtained by the following procedure.

**Procedure 2.1. Normalized \( \rho_0 \)-representation.**

**Given** \( \tilde{g} \in \tilde{b}(\rho_0) \)

**Step 1.** Obtain a \( \rho_0 \)-representation \((\tilde{n}, \tilde{d})\) of \( \tilde{g}. \)

**Step 2.** Determine all \( v \) not-necessarily-different zeros of \( \tilde{d} \) in \( D(\rho_0)^c, \) call them \( p_{\alpha}, \ \alpha = 1, 2, \ldots, v. \)

**Step 3.** Let \( \tilde{d} = \tilde{d}c \) where \( \tilde{d}(z) := \prod_{\alpha=1}^{v} \frac{(z-p_{\alpha})}{(z-a_{\alpha})} \) and we adopt the convention that \( \prod_{\alpha=1}^{v} \frac{z}{(z-p_{\alpha})} = 1. \) Note that \( \tilde{c} := \tilde{d}/\tilde{d} \) is invertible in \( \tilde{\mathcal{L}}(\rho_0). \) (Observe that \( \tilde{d} \) can also be chosen to be \( \tilde{d}(z) := \prod_{\alpha=1}^{v} \frac{(z-p_{\alpha})}{(z-a_{\alpha})} \) for any choice of \( a_{\alpha} \in D(\rho_0), \ \alpha = 1, 2, \ldots, v. \))

**Step 4.** Define \( \tilde{n} := \tilde{n}c^{-1} \in \tilde{\mathcal{L}}(\rho_0), \) then \((\tilde{n}, \tilde{d})\) is a normalized \( \rho_0 \)-representation of \( \tilde{g}. \)

**Stop.**

\[ \square \]
Remark 2.4. \( \mathcal{X}(\rho_0) \) is a multiplicative subset [Zar 1, p.46] [Lan 1, p.66] of both \( \mathcal{X}_p(z) \) and \( \mathcal{X}_1(z) \). In view of Theorem 2.1, we conclude that 
\[ \tilde{b}(\rho_0) = [\tilde{z}_1(\rho_0)][\mathcal{X}_1(\rho_0)]^{-1} = [\tilde{z}_1(\rho_0)][\mathcal{X}_\infty(\rho_0)]^{-1}. \]

Theorem 2.2. Let \( g \in \mathbb{C}_z^N \). Then
\[ \tilde{g} \in \tilde{b}(\rho_0) \quad (2.35) \]
if and only if \( \exists \rho_1 < \rho_0 \) such that
\[ \tilde{g}(z) = \tilde{r}(z) + \tilde{q}(z) \quad \forall z \in D(\rho_1)^C \quad (2.36) \]
where
\[
\begin{align*}
(i) & \quad \tilde{q} \in \mathcal{X}_1(\rho_0); \\
(ii) & \quad \tilde{r} \in \mathcal{X}_p(z) \text{ is strictly proper, and is zero if and only if } \tilde{g} \in \mathcal{X}_1(\rho_0); \\
(iii) & \quad \text{if } \tilde{g} \notin \mathcal{X}_1(\rho_0), \text{ then } \tilde{r} \text{ is the sum of the principal parts of the Laurent expansions of } \tilde{g} \text{ at its poles in } D(\rho_0)^C, \text{ in particular:} \\
& \quad (a) \text{ all poles of } \tilde{r} \text{ are in } D(\rho_0)^C, \text{ and} \\
& \quad (b) \tilde{g} \text{ has an } m \text{th order pole at } p \in D(\rho_0)^C \text{ if and only if } \tilde{r} \text{ has an } m \text{th order pole at } p \in D(\rho_0)^C. \quad (2.39)
\end{align*}
\]

The proof of Theorem 2.2 is given in Appendix B. Sufficiency is proved by construction, and a procedure for obtaining a normalized \( \rho_0 \)-representation \((\tilde{n},\tilde{d})\) for \( \tilde{g} \) described by (2.36) through (2.39b) is given next.

Procedure 2.2. Normalized \( \rho_0 \)-representation from \( \tilde{g} = \tilde{r} + \tilde{q} \).

Given \( \tilde{g} \) in the form (2.36)-(2.39b).
Step 1. If \( g \in \tilde{X}_1(\rho_0) \), i.e. \( r = 0 \), set

\[
\tilde{n} := g, \quad \tilde{d} = 1
\]

and stop.

Step 2. Let \( \bar{r} := \eta /d_r \) define a coprime factorization of \( \bar{r} \) in the ring of polynomials in \( z \), with \( d_r \) monic.

Determine \( v := \deg(d_r) \).

Step 3. Define \((\tilde{n}, \tilde{d})\) by

\[
\tilde{d}(z) := d_r(z)/z^v
\]

\[
\tilde{n}(z) := [n_r(z) + \tilde{q}(z)d_r(z)]/z^v \quad |z| > \rho_1
\]

and stop.

Observation. In both (2.40) and (2.43)-(2.44), \((\tilde{n}, \tilde{d})\) is a normalized \( \rho_0 \)-representation of \( g \).

Remark 2.5. In step 3 of Procedure 2.2, instead of using \( z^v \) as denominator of both \( \tilde{n}(z) \) and \( \tilde{d}(z) \), it can be generalized to be any \( v \)th order polynomial in the form

\[
\prod_{\alpha=1}^v (z-a_{\alpha})
\]

where \( a_{\alpha} \in D(\rho_0), \, \alpha = 1, 2, \ldots, v \).

Theorem 2.3. Let \( \tilde{g} \in \tilde{\tilde{X}}(\rho_0) \), and let \((\tilde{n}, \tilde{d})\) and \((\tilde{n}, \tilde{d})\) be two \( \rho_0 \)-representations of \( \tilde{g} \).

U.t.c.

(i) \( \exists \tilde{h} \in \tilde{X}_1(\rho_0) \), invertible in \( \tilde{X}_1(\rho_0) \), such that

\[
\tilde{n} = \tilde{n}\tilde{h},
\]

\[
\tilde{d} = \tilde{d}\tilde{h}
\]
(ii) if, in addition, the representations are both normalized, then \( \tilde{h} \) is rational with all poles and zeros in \( D(\rho_0) \), in particular,

(a) if \( \tilde{d} \equiv 1 \), then \( \tilde{h} \equiv 1 \)

(b) if \( \tilde{d} \not\equiv 1 \), let

\[
\tilde{r} = \frac{n_r}{d_r} \quad (2.46)
\]

be a coprime polynomial factorization of \( \tilde{r} \) given in (2.36)-(2.39b) with \( d_r \) being a monic polynomial, then

\[
\tilde{d} = \frac{d_r}{n_h}, \quad \tilde{\alpha} = \frac{d_r}{d_h}, \quad \tilde{h} = \frac{n_h}{d_h} \quad (2.47)
\]

with \( d_r, n_h, d_h \) monic polynomials in \( \mathbb{C}[z] \) of the same degree, and such that \( n_h, d_h \) have zeros only in \( D(\rho_0) \).

Remark 2.6. From Theorem 2.3, if \( (\tilde{n}, \tilde{d}) \) is any normalized \( \rho_0 \)-representation of \( \tilde{g} \in \tilde{B}(\rho_0) \), then \( \tilde{d} = d_r/p \), for some monic polynomial \( p \) of the same degree as \( d_r \) and has zeros only in \( D(\rho_0) \). By (2.43)-(2.45) in Procedure 2.2, we can write

\[
\tilde{g} = \frac{n}{d} := \left( \frac{n_r + \tilde{q}d_r}{p} \right) / \left( d_r/p \right) \quad (2.48)
\]

and \( p \) appears as a common divisor (polynomial in \( \mathbb{C}[z] \)) in defining \( \tilde{n} \) and \( \tilde{d} \). Hence the choice of \( p \) does not affect \( \tilde{g} \), and \( p \) could thus be called a scaling polynomial in defining the normalized \( \rho_0 \)-representation: the restrictions of \( p \) being that it is a monic polynomial with \( \mathcal{O}(p) = \mathcal{O}(d_r) \) and \( \mathbb{Z}[p] \subset D(\rho_0) \). Consequently, we conclude that a normalized \( \rho_0 \)-representation is unique up to a scaling polynomial. \( \square \)
Theorem 2.4. Let $\tilde{g} \in \tilde{B}(\rho_0)$. Then

$\tilde{g}$ is an invertible element of $\tilde{B}(\rho_0)$ \hspace{1cm} (2.49)

if and only if

$$g(0) = \lim_{|z| \to \infty} \tilde{g}(z) \neq 0 .$$ \hspace{1cm} (2.50)

\[ \square \]

2.6 Examples. $g = (g(k))_{k=0}^\infty \in \mathbb{C}^\mathbb{N}$.

(2.6.1) $g_1(0) := 1$

$$g_1(1) := -\frac{1}{2}$$

$$g_1(k) := -\frac{1}{2} \prod_{m=2}^{k} \frac{2m-3}{2m}, \quad k = 2,3,4,\ldots .$$ \hspace{1cm} (2.51)

By [Dwi 1, formula 5.3],

$$\sqrt{1-x} = \sum_{k=0}^{\infty} g_1(k)x^k \quad \forall |x| \leq 1 .$$ \hspace{1cm} (2.52)

Hence evaluating (2.52) at $x = 1$, we obtain

$$\sum_{k=0}^{\infty} g_1(k) = 0 .$$ \hspace{1cm} (2.53)

Note that $g_1(0) = 1$ and $g_1(k) < 0$ for $k = 1,2,\ldots$, hence by

(2.53)

$$\sum_{k=1}^{\infty} g_1(k) = -1 \quad \text{and} \quad \sum_{k=1}^{\infty} |g_1(k)| = 1 ,$$

i.e.

$$\sum_{k=0}^{\infty} |g_1(k)| = 2 .$$ \hspace{1cm} (2.54)

Therefore,

$$g_1 \in \ell_1(1) .$$ \hspace{1cm} (2.55)

By Property (2.2.4), the series defining $g_1$ converges absolutely in $D(\rho_1)^\mathbb{C}$; hence using (2.52), we obtain
\[ \tilde{g}_1(z) = \sum_{k=0}^{\infty} g_1(k) z^{-k} = \frac{\sqrt{z-1}}{z} \quad \forall z \in D(1)^C. \quad (2.56) \]

However, \( \tilde{g}_1 \) in (2.56) is not analytic at \( z = 1 \), hence

\[ g_1 \notin \xi_1(1). \quad (2.57) \]

Since \( g_1(0) \neq 0 \), thus

\[ g_1 \in \xi_1^\infty(\rho_0) \subset \xi_1(\rho_0) \quad \forall \rho_0 > 1. \quad (2.58) \]

(2.6.2) Consider the slight variation of example (2.6.1):

\[ \tilde{g}_2(z) := \frac{\sqrt{z-0.5}}{z}, \quad z \in D(0.5)^C; \quad (2.59) \]

then \( \tilde{g}_2 \in \bar{\xi}_1(0.5) \), and \( \tilde{g}_2 \notin \bar{\xi}_1(0.5) \); but \( \tilde{g}_2 \in \bar{\xi}_1^\infty(\rho_0) \subset \bar{\xi}_1(\rho_0) \)

for all \( \rho_0 > 0.5 \).

(2.6.3) For any fixed \( a \in \mathbb{C} \), consider

\[ g_3(k) = \frac{a^k}{k!}, \quad k = 0,1,2,\ldots. \quad (2.60) \]

Hence, for all \( z \neq 0 \),

\[ \tilde{g}_3(k) := \sum_{k=0}^{\infty} \frac{a^k}{k!} z^{-k} = e^{az^{-1}}, \quad (2.61) \]

i.e.

\[ \sum_{k=0}^{\infty} |g_3(k)| \rho_0^{-k} = e^{|a|\rho_0^{-1}} < \infty \quad \forall \rho_0 > 0; \]

(2.62)

furthermore, by noting that \( g_3(0) = 1 \), hence nonzero, we conclude that

\[ g_3 \in \xi_1^\infty(\rho_0) \subset \xi_1(\rho_0) \subset \xi_1(\rho_0) \quad \forall \rho_0 > 0. \quad (2.63) \]

(2.6.4) Let

\[ g_4(0) := 0 \]

\[ g_4(k) = \frac{1}{k}, \quad k = 1,2,\ldots. \]

(2.64)
By [Han 1, (5.13.4)],
\[
\sum_{k=1}^{\infty} \frac{1}{k} x^k = -\ln(1-x) \quad \forall |x| < 1, \quad x \neq 1, \quad (2.65)
\]
i.e.
\[
\sum_{k=0}^{\infty} |g_4(k)|\rho_0^{-k} = -\ln(1-\rho_0^{-1}) < \infty, \quad \forall \rho_0 > 1. \quad (2.66)
\]
Hence,
\[
g_4 \in \ell_1(\rho_0) \subset \ell_1(\rho_0), \quad \forall \rho_0 > 1; \quad (2.67)
\]
but since the series in (2.65) does not converge for $x = 1$,
\[
g_4 \notin \ell_1(1). \quad (2.68)
\]
By (2.65) and using the absolute-convergence property as in example (2.6.1), we conclude that
\[
\tilde{g}(z) = -\ln\left(\frac{z-1}{z}\right), \quad \forall z \in \overline{B(1)^c}. \quad (2.69)
\]
(2.6.5) Let
\[
g_5(0) := 0 \quad (2.70)
\]
\[
g_5(k) := \sum_{i=1}^{k} \frac{1}{i}, \quad k = 1, 2, \ldots . \quad (2.71)
\]
Note that the sequence of positive numbers $(g_5(k))_{k=0}^{\infty}$ is unbounded, hence
\[
g_5 \notin \ell_1(1). \quad (2.72)
\]
By [Han 1, (5.13.21)],
\[
\sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} \frac{1}{i} \right)x^k = \frac{1}{x-1}\ln(1-x), \quad \forall |x| < 1, \quad (2.72)
\]
i.e.
\[
\sum_{k=0}^{\infty} |g_5(k)|\rho_0^{-k} = \frac{1}{\rho_0^{-1}-1}\ln(1-\rho_0^{-1}) < \infty, \quad \forall \rho_0 > 1. \quad (2.73)
\]
Hence,
\[
g_5 \in \ell_1(\rho_0) \subset \ell_1(\rho_0), \quad \forall \rho_0 > 1. \quad (2.74)
\]
Using similar arguments as before, we have

\[ \tilde{g}_5(z) = \frac{z}{1-z} \ln \left( \frac{z-1}{z} \right), \quad \forall z \in \mathbb{D}(1)^c. \]  

(2.6.6) \hspace{1cm} g_6(0) := 0 \hspace{1cm} (2.76)

\[ g_6(k) := \frac{1}{k^2}, \quad k = 1,2,... \]

By [Han 1, (5.12.43)],

\[ \sum_{k=1}^{\infty} \frac{1}{k^2} x^k = -\int_{0}^{x} t^{-1} \ln(1-t) dt, \quad \forall |x| < 1. \]  

(2.77)

In particular,

\[ \sum_{k=0}^{\infty} |g_6(k)| = \sum_{k=0}^{\infty} \frac{1}{k^2} = -\int_{0}^{1} t^{-1} \ln(1-t) dt = \frac{\pi^2}{6}; \]  

hence \[ g_6 \in \mathcal{L}_1(1). \]  

(2.79)

However, the series in (2.77) does not converge for \(|x| > 1\), hence

\[ g_6 \notin \mathcal{L}_1(\rho_0), \quad \forall \rho_0 < 1, \]  

(2.80)

and thus \[ g_6 \notin \mathcal{L}_1^-(1). \]  

(2.81)

(2.6.7) \hspace{1cm} g_7(0) := 0 \hspace{1cm} (2.82)

\[ g_7(k) := \frac{1}{k(k-1)}, \quad k = 1,2,... \]

By [Han 1, (5.9.16)],

\[ \sum_{k=1}^{\infty} \frac{1}{k(k+1)} x^k = 1 + \left( \frac{1}{x} - 1 \right) \ln(1-x), \quad |x| \leq 1, \]  

(2.83)

i.e. \[ \sum_{k=0}^{\infty} |g_7(k)| \rho_0^{-k} = 1 + \left( \frac{1}{\rho_0^{-1}} - 1 \right) \ln(1-\rho_0^{-1}) < \infty, \quad \forall \rho_0 \geq 1. \]  

(2.84)

The case when \( \rho_0 = 1 \) is best calculated by using [Han 1, (5.9.17)]
\[
\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}; \quad (2.85)
\]

hence
\[
\sum_{k=0}^{\infty} |g_7(k)| = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 < \infty, \quad (2.86)
\]

and so \( g_7 \in \mathcal{L}_1(1) \).

However, since the series in (2.83) does not converge for \(|x| > 1\),
\[
g_7 \notin \mathcal{L}_1(\rho_0) \quad \forall \rho_0 < 1. \quad (2.87)
\]

By (2.83) and the absolute convergence property as before,
\[
\bar{g}_7(z) = 1 + (z-1)\ln\left(\frac{z-1}{z}\right), \quad \forall z \in D(1)^c. \quad (2.88)
\]

(2.6.8) Recall that \( r(\rho_0) \subset \mathcal{H}_1(\rho_0) \); thus
\[
z \mapsto \frac{z-1}{z} \in \mathcal{H}_1^\infty(\rho_0) \subset \mathcal{H}_1(\rho_0), \quad \forall \rho_0 > 0, \quad (2.89)
\]

furthermore, it is also an invertible element of \( \mathcal{H}_1(\rho_0), \forall \rho_0 > 1 \). (2.90)

Using (2.89) and example (2.6.2), if
\[
\bar{g}(z) := \frac{z}{z-1} \cdot \sqrt{\frac{z-0.5}{z}}, \quad \forall z \in D(0.5)^c, \quad (2.91)
\]

then
\[
g \in b(\rho_0), \quad \forall \rho_0 > 0.5; \quad (2.92)
\]

note that \( \bar{g} \) has a pole at \( z = 1 \). Finally, from (2.90),
\[
g \in \mathcal{H}_1(\rho_0) \subset \mathcal{L}_1(\rho_0), \quad \forall \rho_0 > 1. \quad (2.93)
\]
3. Matrix Fraction Representation Theory

From this point on, we are concerned with multi-input multi-output (MIMO) convolution systems whose transfer functions are matrices with elements in \( \mathbb{C}_z \), \( \mathcal{L}(\rho_0) \), \( \mathcal{H}_1(\rho_0) \) or \( \mathcal{B}(\rho_0) \).

Observe that \( (\mathbb{C}_z)^{n \times n} \), \( \mathcal{L}(\rho_0)^{n \times n} \), \( \mathcal{H}_1(\rho_0)^{n \times n} \) and \( \mathcal{B}(\rho_0)^{n \times n} \) are all algebras with a pointwise sum and a non-commutative (pointwise) product, with unit \( I_n \).

**Lemma 3.1.** \( \mathcal{G} \in \mathcal{L}(\rho_0)^{n \times n} \) (respectively \( \mathcal{H}_1(\rho_0)^{n \times n} \)) is invertible in \( \mathcal{L}(\rho_0)^{n \times n} \) (resp. \( \mathcal{H}_1(\rho_0)^{n \times n} \)) if and only if

\[
\inf_{|z| \geq \rho_0} |\det \mathcal{G}(z)| > 0 \tag{3.1}
\]

i.e. \( \det \mathcal{G} \) is invertible in \( \mathcal{L}(\rho_0) \) (resp. \( \mathcal{H}_1(\rho_0) \)).

**Comment.** Such \( \mathcal{G} \) is called a unimodular matrix in \( \mathcal{L}(\rho_0)^{n \times n} \) (resp. \( \mathcal{H}_1(\rho_0)^{n \times n} \)).

**Lemma 3.2.** \( \mathcal{G} \in \mathcal{B}(\rho_0)^{n \times n} \) is invertible in \( \mathcal{B}(\rho_0)^{n \times n} \) if and only if

\[
\lim_{|z| \to \infty} \det \mathcal{G}(z) \neq 0, \tag{3.2}
\]

i.e. \( \det \mathcal{G} \) is invertible in \( \mathcal{B}(\rho_0) \).

The next lemma is a multi-input multi-output generalization of Theorem 2.2.

**Lemma 3.3.** Let \( \mathcal{G} \in (\mathbb{C}_z)^{n_0 \times n_1} \). Then \( \mathcal{G} \in \mathcal{B}(\rho_0)^{n_0 \times n_1} \) if and only if for some \( \rho_1 \in [0, \rho_0] \),

\[
\mathcal{G} = \mathcal{R} + \mathcal{Q} \quad \text{in } D(\rho_1)^{n_0 \times n_1} \tag{3.3}
\]
where (i) $\tilde{Q} \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_i}$;

(ii) $\tilde{R} \in \mathcal{G}_p(z)^{n_0 \times n_i}$ is strictly proper, and $\tilde{R} \equiv 0$ if and only if $\tilde{G} \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_i}$;

(iii) if $\tilde{G} \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_i}$ then $\tilde{R} = (\tilde{r}_{ij})$ is the sum of the principal parts of the Laurent expansions of $\tilde{G} = (\tilde{g}_{ij})$ at its poles in $D(p_0)^c$; in particular, $\tilde{g}_{ij}$ has an $m$th order pole at $p \in D(p_0)^c$ if and only if $\tilde{r}_{ij}$ has an $m$th order pole at $p \in D(p_0)^c$. \hfill □

**Definition 3.1.$r$.** Let $N_r \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_i}$ and $\mathcal{D}_r \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_i \times n_i}$. The pair $(N_r, \mathcal{D}_r)$ is said to be $p_0$-right coprime $(p_0$-r.c.) iff any greatest common right divisor (g.c.r.d.) of $N_r$ and $\mathcal{D}_r$ in $\tilde{\mathcal{G}}_{-1}(p_0)^{n_i \times n_i}$ [McD 1, p.35] is an invertible element of $\tilde{\mathcal{G}}_{-1}(p_0)^{n_i \times n_i}$.

**Definition 3.1.$\ell$.** Let $\mathcal{D}_\ell \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_0}$ and $N_\ell \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_i}$. The pair $(\mathcal{D}_\ell, N_\ell)$ is said to be $p_0$-left coprime $(p_0$-l.c.) iff any greatest common left divisor (g.c.l.d.) of $\mathcal{D}_\ell$ and $N_\ell$ in $\tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_0}$ is an invertible element of $\tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_0}$. \hfill □

**Lemma 3.4.$r$.** $N_r \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_i}$ and $\mathcal{D}_r \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_i \times n_i}$ are $p_0$-r.c. if and only if $\exists \mathcal{U}_r \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_i \times n_0}, \mathcal{V}_r \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_i \times n_i}$ such that

$$u_r N_r + v_r \mathcal{D}_r = I_{n_i}. \tag{3.4}$$

**Lemma 3.4.$\ell$.** $\mathcal{D}_\ell \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_0}$ and $N_\ell \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_i}$ are $p_0$-l.c. if and only if $\exists \mathcal{U}_\ell \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_0 \times n_0}, \mathcal{V}_\ell \in \tilde{\mathcal{G}}_{-1}(p_0)^{n_i \times n_0}$ such that

$$N_\ell \mathcal{U}_\ell + \mathcal{D}_\ell \mathcal{V}_\ell = I_{n_0}. \tag{3.5}$$
Definition 3.2(ν). Let \( G \in (\mathbb{C}_z)^{n_0 \times n_1} \). The pair \((N_\nu, D_\nu)\) is said to be a \( \rho_0 \)-right representation \((\rho_0\text{-r.r.) of } \tilde{G}\) iff \( N_\nu \in \tilde{\mathfrak{k}}_1(\rho_0)^{n_0 \times n_1} \) and \( D_\nu \in \tilde{\mathfrak{k}}_1(\rho_0)^{n_1 \times n_1} \) such that

(i) \( \tilde{G} = N_\nu D_\nu^{-1} \)

(ii) \((N_\nu, D_\nu)\) is \( \rho_0 \)-r.c.

(iii) \( \det D_\nu \in \tilde{\mathfrak{k}}_1^\infty(\rho_0) \).

Definition 3.2(ℓ). Let \( G \in (\mathbb{C}_z)^{n_0 \times n_1} \). The pair \((D_\ell, N_\ell)\) is said to be a \( \rho_0 \)-left representation \((\rho_0\text{-l.r.) of } \tilde{G}\) iff \( D_\ell \in \tilde{\mathfrak{k}}_1(\rho_0)^{n_0 \times n_0} \) and \( N_\ell \in \tilde{\mathfrak{k}}_1(\rho_0)^{n_1 \times n_1} \) such that

(i) \( \tilde{G} = D_\ell^{-1} N_\ell \)

(ii) \((D_\ell, N_\ell)\) is \( \rho_0 \)-l.c.

(iii) \( \det D_\ell \in \tilde{\mathfrak{k}}_1^\infty(\rho_0) \).

Remark 3.1. By Cramer's rule and Definition 3.2(ν) (respectively Definition 3.2(ℓ)), if \( G \in (\mathbb{C}_z)^{n_0 \times n_1} \) admits a \( \rho_0 \)-r.r. (respectively \( \rho_0 \)-l.r.), then \( \tilde{G} \in \tilde{\mathfrak{b}}(\rho_0)^{n_0 \times n_1} \). The next Theorem states that the converse is also true.

Theorem 3.1. If \( \tilde{G} \in \tilde{\mathfrak{b}}(\rho_0)^{n_0 \times n_1} \), then \( \tilde{G} \) admits a \( \rho_0 \)-r.r. and a \( \rho_0 \)-l.r. More precisely, there exist matrices with elements in \( \tilde{\mathfrak{k}}_1(\rho_0) \), namely

\[ N_\nu, D_\nu, U_\nu, V_\nu \]

\[ N_\ell, D_\ell, U_\ell, V_\ell \]

such that

(i) \((N_\nu, D_\nu)\) is a \( \rho_0 \)-r.r. of \( \tilde{G} \)

(ii) \((D_\ell, N_\ell)\) is a \( \rho_0 \)-l.r. of \( \tilde{G} \)
(iii) \[ \begin{bmatrix} n_1 & 0 & n_1 & 0 \\ n_0 & -N_\epsilon & n_0 & -N_\epsilon \\ N_\epsilon & D_\epsilon & N_\epsilon & D_\epsilon \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ -I_{n_1} & 0 \\ 0 & I_{n_0} \end{bmatrix} \] (3.6)

**Remark 3.2.** If we call the matrices on the left hand side of (3.6) \( \mathbf{W} \) and \( \mathbf{W}^{-1} \) respectively, then obviously \( \mathbf{W} \) is an invertible element of \( \tilde{\mathcal{L}}_{1-}(\rho_0)^{n_0 \times n_0} \). In particular, we can scale \( \mathbf{W} \) and \( \mathbf{W}^{-1} \) so that

\[
\text{det } \mathbf{W} = \text{det } \mathbf{W}^{-1} = 1.
\] (3.7)

Theorem 3.1 can be proved easily by construction using the Euclidean algorithm for \( \tilde{\mathcal{L}}_{1-}(\rho_0) \). However, this is an unnecessarily difficult way to obtain a \( \rho_0 \)-l.r. and \( \rho_0 \)-r.r. We give instead a proof based on the following procedure for the general case when \( \tilde{G} \not\in \tilde{\mathcal{L}}_{1-}(\rho_0)^{n_0 \times n_1} \):

**Procedure 3.1.** \( \rho_0 \)-r.r. and \( \rho_0 \)-l.r.

*Given* \( \tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_1} \), \( \tilde{G} \not\in \tilde{\mathcal{L}}_{1-}(\rho_0)^{n_0 \times n_1} \)

**Step 1.** Find \( \tilde{R}, \tilde{Q} \) according to Lemma 3.3 so that

\[
\tilde{G} = \tilde{R} + \tilde{Q}
\] (3.8)

with \( \tilde{R} \in \mathcal{L}(\rho_0)^{n_0 \times n_1} \), \( \tilde{Q} \in \tilde{\mathcal{L}}_{1-}(\rho_0)^{n_0 \times n_1} \).

**Step 2.** Find \( \hat{N}_\epsilon \in \mathcal{L}(\rho_0)^{n_0 \times n_1} \) and \( \hat{N}_\epsilon \in \mathcal{L}(\rho_0)^{n_1 \times n_1} \) with \( \text{det } \hat{N}_\epsilon \in \mathcal{L}(\rho_0) \) such that

\[
\tilde{R} = \hat{N}_\epsilon \hat{N}_\epsilon^{-1}
\] (3.9)

e.g. set \( \hat{N}_\epsilon = \text{diag}[d_j]_{j=1}^{n_1} \) where for \( j = 1,2,\ldots,n_1 \), \( d_j \) is a \( j \)th
column least common denominator of $\tilde{R}$ with respect to $\pi(\rho_0)$.

**Step 3.** Consider the $(n_i+n_0) \times n_1$ full rank matrix

$$
\hat{M} := \begin{bmatrix}
\hat{P} \\
\hat{Q}
\end{bmatrix} \in \pi(\rho_0)^{(n_i+n_0) \times n_i}. 
$$

(3.10)

By performing elementary row operations based on the Euclidean algorithm in the Euclidean ring $\pi(\rho_0)$, bring $\hat{M}$ to "upper triangular" forms, i.e. find an $(n_i+n_0) \times (n_i+n_0)$ matrix $\tilde{W}$ invertible in $\pi(\rho_0)^{(n_i+n_0) \times (n_i+n_0)}$ and a full rank upper triangular matrix $\tilde{R} \in \pi(\rho_0)^{n_i \times n_i}$ such that

$$
\begin{bmatrix}
\tilde{W} \\
\tilde{R}
\end{bmatrix} \begin{bmatrix}
\hat{M}
\end{bmatrix} = \begin{bmatrix}
n_i \\
n_0
\end{bmatrix} \begin{bmatrix}
\tilde{R}
\end{bmatrix}.$$

(3.11)

Observe that, as in the previous remark, $\tilde{W}$ can be scaled so that $\det \tilde{W} = 1$.

**Step 4.** Partition $\tilde{W}$ and $\tilde{W}^{-1}$ into

$$
\tilde{W} = \begin{bmatrix}
\hat{P} & \hat{Q} \\
\hat{Q} & \hat{R}
\end{bmatrix}, \quad \tilde{W}^{-1} = \begin{bmatrix}
\hat{P} & \hat{Q} \\
\hat{Q} & \hat{R}
\end{bmatrix}.
$$

(3.12)

**Step 5.** Define

$$
\hat{p}_r := \hat{p}_r \\
\hat{p}_l := \hat{p}_l \\
\hat{n}_r := \hat{n}_r + \hat{q}_r \hat{p}_r \\
\hat{n}_l := \hat{n}_l + \hat{q}_l \hat{p}_l \\
\hat{u}_r := \hat{u}_r - \hat{q}_r \hat{p}_r \\
\hat{u}_l := \hat{u}_l - \hat{q}_l \hat{p}_l
$$

(3.13)

and stop.
Comments. (i) The eight matrices in (3.12) with elements in $\mathcal{P}(\rho_0) \subset \mathcal{X}_1(\rho_0)$, namely

$$\begin{align*}
\tilde{N}_\epsilon, \tilde{D}_\epsilon, \tilde{U}_\epsilon, \tilde{V}_\epsilon \\
\tilde{N}_\ell, \tilde{D}_\ell, \tilde{U}_\ell, \tilde{V}_\ell
\end{align*}$$

satisfy Theorem 3.1 with $\tilde{G} \leftarrow \tilde{R}$.

(ii) The eight matrices in (3.13) satisfy the conclusions of Theorem 3.1.

Remark 3.3. Observe that in Procedure 3.1, which is used in the Proof of Theorem 3.1, we actually obtain

$$\begin{align*}
\mathcal{P}_\epsilon &\in \mathcal{P}(\rho_0)^{n_\epsilon \times n_\epsilon} \\
\det \mathcal{P}_\epsilon &\in \mathcal{P}(\rho_0)^{\infty}
\end{align*}$$

and

$$\begin{align*}
\mathcal{P}_\ell &\in \mathcal{P}(\rho_0)^{n_\ell \times n_\ell} \\
\det \mathcal{P}_\ell &\in \mathcal{P}(\rho_0)^{\infty}
\end{align*}$$

i.e. the denominator matrices of the $\rho_0$-r.r. $(N_\epsilon, \mathcal{P}_\epsilon)$ and the $\rho_0$-l.r. $(\mathcal{P}_\ell, N_\ell)$ are rational. \( \square \)

The next corollary follows from Theorem 3.1 and Remark 3.1.

Corollary 3.1a. Let $G \in (\mathbb{C}\mathbb{Z})^{n_0 \times n_i}$; then

$$\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i} \quad (3.14)$$

$$\Rightarrow \tilde{G} \text{ admits a } \rho_0\text{-r.r. } (N_\epsilon, \mathcal{P}_\epsilon) \quad (3.15)$$

$$\Rightarrow \tilde{G} \text{ admits a } \rho_0\text{-l.r. } (\mathcal{P}_\ell, N_\ell) \quad (3.16)$$

\( \square \)

Remark 3.4. In view of Corollary 3.1a, we have

$$\tilde{b}(\rho_0)^{n_0 \times n_i} = \{G \in (\mathbb{C}\mathbb{Z})^{n_0 \times n_i} | \tilde{G} \text{ admits a } \rho_0\text{-r.r. or } \rho_0\text{-l.r.} \} \quad (3.17)$$
The following corollaries are the MIMO generalization of Remark 2.1.

Corollary 3.1b(τ). Let \( N_\tau \in \mathcal{I}_{1-}(\rho_0)^{n_0 \times n_1} \) and \( D_\tau \in \mathcal{I}_{1-}(\rho_0)^{n_1 \times n_1} \) with \( \det D_\tau \in \mathcal{I}_{1-}(\rho_0) \). Then \((N_\tau, D_\tau)\) is \( \rho_0-r.c. \) if and only if

\[
\text{rank} \begin{bmatrix} D_\tau(z) \\ -N_\tau(z) \end{bmatrix} = n_1 \quad \forall z \in D(\rho_0)^c. \tag{3.18}
\]

Corollary 3.1b(ℓ). Let \( N_\ell \in \mathcal{I}_{1-}(\rho_0)^{n_0 \times n_1} \) and \( D_\ell \in \mathcal{I}_{1-}(\rho_0)^{n_0 \times n_0} \) with \( \det D_\ell \in \mathcal{I}_{1-}(\rho_0) \). Then \((D_\ell, N_\ell)\) is \( \rho_0-l.c. \) if and only if

\[
\text{rank} \begin{bmatrix} D_\ell(z) \\ N_\ell(z) \end{bmatrix} = n_0 \quad \forall z \in D(\rho_0)^c. \tag{3.19}
\]

In view of Corollary 3.1b(τ) (Corollary 3.1b(ℓ)), we present next an algorithm to obtain a \( \rho_0-r.r. \) (respectively, \( \rho_0-l.r. \)) for \( \mathcal{G} \) given by (3.3) and (3.8) that does not use the Euclidean algorithm in \( \mathcal{I}(\rho_0) \) (which is used in Steps 2 and 3 of Procedure 3.1).

Procedure 3.2(τ). \( \rho_0-r.r. \) for \( \mathcal{G} \in \mathcal{B}(\rho_0)^{n_0 \times n_1} \)

Given

\[
\mathcal{G} = \mathcal{R} + \mathcal{Q} \tag{3.20}
\]

as in Lemma 3.3.

Step 1. Find \( \bar{N}_\tau \in \mathcal{C}[z]^{n_0 \times n_1}, \bar{D}_\tau \in \mathcal{C}[z]^{n_1 \times n_1} \) such that

(i) \( \bar{R} = \bar{N}_\tau \bar{D}_\tau^{-1} \)

(ii) \((\bar{N}_\tau, \bar{D}_\tau)\) is right coprime in the ring of polynomials \( \mathcal{C}[z] \)

(iii) \( \det \bar{D}_\tau \neq 0. \)

Step 2. Find \( M \in \mathcal{C}[z]^{n_1 \times n_1} \) unimodular such that

\[
D_\tau := \bar{D}_\tau M \tag{3.21}
\]
is column-reduced [Wol 1, Thm. 2.5.7]. Let

\[ N_\mathcal{H} := \tilde{N}_\mathcal{H} M . \] (3.22)

Now \( N_\mathcal{H} D_\mathcal{H}^{-1} \) is also a right coprime factorization of \( \tilde{R} \).

**Step 3.** For \( i = 1, 2, \ldots, n_i \), let

\[ \gamma_i := \alpha_{c_i} [D_\mathcal{H}] \] (3.23)

and let \( \pi_i \in \mathcal{C}[z] \) be defined by

\[ \pi_i(z) := z_i \] (3.24)

Define

\[ S := \text{diag}(\pi_i) \in \mathcal{C}[z]^{n_i \times n_i} . \] (3.25)

**Step 4.** Define

\[ \tilde{N}_\mathcal{H} := N_\mathcal{H} S^{-1} \in \mathcal{K}(\rho_0)^{n_0 \times n_i} \] (3.26)

\[ \tilde{D}_\mathcal{H} := D_\mathcal{H} S^{-1} \in \mathcal{K}(\rho_0)^{n_i \times n_i} . \] (3.27)

**Comment.** \((\tilde{N}_\mathcal{H}, \tilde{D}_\mathcal{H})\) is a \( \rho_0 \)-r.r. of \( \tilde{R} \) with elements in \( \mathcal{K}(\rho_0) \).

**Step 5.** Define

\[ N_\mathcal{H} := \tilde{N}_\mathcal{H} + \tilde{D}_\mathcal{H} \in \mathcal{K}_1(\rho_0)^{n_0 \times n_i} \] (3.28)

\[ D_\mathcal{H} := \tilde{D}_\mathcal{H} \in \mathcal{K}(\rho_0)^{n_i \times n_i} \subset \mathcal{K}_1(\rho_0)^{n_i \times n_i} . \] (3.29)

**Comment.** \((N_\mathcal{H}, D_\mathcal{H})\) is a \( \rho_0 \)-r.r. of \( \tilde{G} \).

**Procedure 3.2(\( \ell \)).** \( \rho_0 \)-r.r. of \( \tilde{G} \in \tilde{\tilde{G}}(\rho_0)^{n_0 \times n_i} \)

A \( \rho_0 \)-r.r. \((D_\mathcal{G}, N_\mathcal{G})\) of \( \tilde{G} \) in (3.20) can be obtained through obvious modification of Procedure 3.2(\( \alpha \)).
Theorem 3.2. Let \( \tilde{G} \in \tilde{B}(\rho_0)^{n_0 \times n_1} \). Then for any \( \rho_0 \)-l.r. \((D_L, N_L)\) of \( \tilde{G} \) and any \( \rho_0 \)-r.r. \((N_r, D_r)\) of \( \tilde{G} \), there exist matrices with elements in \( \tilde{X}_1(\rho_0) \), namely
\[
U_r, V_r, U_L, V_L
\]
such that
\[
\begin{bmatrix}
n_1 & n_0 \\
n_0 & n_1 \\

\end{bmatrix}
\begin{bmatrix}
V_r & U_r \\
-N_r & D_r \\

\end{bmatrix}
= 
\begin{bmatrix}
I_{n_1} & 0 \\
0 & I_{n_0} \\

\end{bmatrix}.
\]
(3.30)

The next corollaries follow immediately from Theorem 3.1 and Theorem 3.2.

Corollary 3.2(1). Let \( \tilde{G} \in \tilde{B}(\rho_0)^{n_0 \times n_1} \). Then for any \( \rho_0 \)-l.r. \((D_L, N_L)\) of \( \tilde{G} \), there exist matrices with elements in \( \tilde{X}_1(\rho_0) \), namely
\[
U_L, V_L; N_r, D_r, U_r, V_r
\]
such that
\[
(i) \ (N_r, D_r) \text{ is a } \rho_0 \text{-r.r. of } \tilde{G}.
\]
\[
(ii) \ \text{Equation (3.30) holds.}
\]

Corollary 3.2(2). A statement similar to Corollary 3.2(1) holds by interchanging the terms "\( \rho_0 \)-l.r." and "\( \rho_0 \)-r.r.", and by interchanging the subscripts "\( L \)" and "\( r \)."

The following theorem is an MIMO generalization of Theorem 2.3.

Theorem 3.3. Let \( \tilde{G} \in \tilde{B}(\rho_0)^{n_0 \times n_1} \) and let \((N_r, D_r)\) and \((N'_r, D'_r)\) be two \( \rho_0 \)-r.r.'s of \( \tilde{G} \) (respectively, let \((D_L, N_L)\) and \((D'_L, N'_L)\) be two \( \rho_0 \)-l.r.'s of \( \tilde{G} \)). Under these conditions, there exists a unimodular matrix
\[ R \in \tilde{X}_1(\rho_0)^{n_1 \times n_1} \]  

(respectively \( L \in \tilde{X}_1(\rho_0)^{n_0 \times n_0} \)) such that

\[ D_r = D_r^1 R, \quad N_r = N_r^1 R \]  

(respectively \( D_\ell = L D_\ell^1, \quad N_\ell = L N_\ell^1 \)).
4. Poles, Zeros and Their Dynamic Interpretation

4.1 McMillan Degree of Poles, Smith and McMillan Forms

Consider a proper rational function matrix \( \tilde{R} \in \mathbb{C}_p(\alpha) \)\(^{n_0 \times n_1} \). It is well known that the McMillan degree of \( \tilde{R} \) is the degree of the characteristic polynomial \( \chi(\alpha) := \det(zI-A) \) of a minimal realization \((A,B,C,D)\) of \( \tilde{R} \). Since the zeros of the characteristic polynomial \( \chi \) are the poles of \( \tilde{R} \), we henceforth define the **McMillan degree of** \( p \in \mathbb{C} \) **as a pole of** \( \tilde{R} \) **to be the order of** \( p \) **as a zero of** \( \chi \). Noting that the McMillan degrees defined for \( \tilde{R} \in \mathbb{C}_p(\alpha)^{n_0 \times n_1} \) are identical to those defined for the strictly proper rational function matrix \( \tilde{R}_0 \) defined by \( \tilde{R}_0(\alpha) := \tilde{R}(\alpha) - \tilde{R}(\infty) \), we consider the following:

**Lemma 4.1.** Let \( \tilde{R} \in \mathbb{C}_p(\alpha)^{n_0 \times n_1} \) be strictly proper, with partial fraction expansion given by

\[
\tilde{R}(\alpha) = \sum_{\alpha=1}^{\nu} \sum_{i=1}^{m_\alpha} \frac{Z_{\alpha i}}{(\alpha - p_\alpha)^i}.
\]

For \( \alpha = 1, 2, \ldots, \nu \), the McMillan degree of \( p_\alpha \) as a pole of \( \tilde{R} \) is equal to the rank \( r_\alpha \) of the matrix

\[
H_\alpha := \begin{bmatrix}
Z_{\alpha 1} & Z_{\alpha 2} & \cdots & Z_{\alpha m_\alpha} \\
Z_{\alpha 2} & Z_{\alpha 3} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
Z_{\alpha m_\alpha} & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{C}^{(m_\alpha n_0) \times (m_\alpha n_1)}.
\]

In view of this lemma, we give the definition of McMillan degrees of poles for matrix transfer functions in \( \tilde{R}(\alpha)^{n_0 \times n_1} \) as follows:
Definition 4.1. Let \( p \in D(\rho_0)^C \) be a pole of \( \tilde{G} \in \mathbb{B}(\rho_0)^{n_0 \times n_1}, \) and let the principal part of the Laurent expansion of \( \tilde{G} \) at \( p \) be given by
\[
\tilde{G}_p(z) = \sum_{i=1}^{m} \frac{Z_i}{(z-p)^i}.
\] (4.3)

The McMillan degree of \( p \) as a pole of \( \tilde{G} \) is defined as the rank of the matrix
\[
H := \begin{bmatrix}
Z_1 & Z_2 & \cdots & Z_m \\
Z_2 & Z_3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
Z_m & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{C}^{(mn_0) \times (mn_1)}. \quad (4.4)
\]

Remark 4.1. (i) If \( p \in D(\rho_0)^C \) is a pole of \( \tilde{R} \in \mathbb{C}_p(z)^{n_0 \times n_1} \subset \mathbb{B}(\rho_0)^{n_0 \times n_1}, \) then, by Lemma 4.1, its McMillan degree as defined in Definition 4.1 agrees with the definition discussed at the beginning of this section.

(ii) For \( \tilde{G} \in \mathbb{B}(\rho_0)^{n_0 \times n_1}, \) let \( \tilde{R} \in \mathbb{C}_p(s)^{n_0 \times n_1} \) be given as in Lemma 3.3, i.e. \( \tilde{R} \) is the sum of the principal parts of the Laurent expansions of \( \tilde{G} \) at its poles in \( D(\rho_0)^C. \) Then the McMillan degree of \( p \in D(\rho_0)^C \) as a pole of \( \tilde{G} \) is equal to the McMillan degree of \( p \) as a pole of \( \tilde{R}. \)

Recall that if \((N_\mathcal{L}, D_\mathcal{L})\) (respectively \((D_\mathcal{L}, N_\mathcal{L})\)) is a right coprime (respectively left coprime) polynomial matrix factorization of \( \tilde{R} \in \mathbb{C}_p(z)^{n_0 \times n_1}, \) then \( \det D_\mathcal{L} \) (respectively \( \det D_\mathcal{L} \)) is equal to the characteristic polynomial of any minimal representation of \( \tilde{R} \) modulo a nonzero constant factor: hence the McMillan degree of the pole \( p \) of \( \tilde{R} \) is the order of \( p \) as a zero of \( \det D_\mathcal{L} \) (respectively \( \det D_\mathcal{L} \)). The next theorem contains a generalization of this result.
Theorem 4.1. Let \( \tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i} \), with a \( \rho_0 \)-r.r. \( (N_\tau, D_\tau) \) and a \( \rho_0 \)-l.r. \( (D, N) \). Under these conditions,

(a) \( p \in D(\rho_0)^C \) is a pole of \( \tilde{G} \) \( \iff \) \( \det D_\tau(p) = 0 \) \( \iff \) \( \det D(p) = 0 \)

(b) If \( p \in D(\rho_0)^C \) is a pole of \( \tilde{G} \), then the order of \( p \) as a zero of \( \det D_\tau \) (respectively of \( \det D \)) is its McMillan degree.

(c) There exists \( \tilde{r} \in \tilde{\lambda}_1(\rho_0) \) invertible in \( \tilde{\lambda}_1(\rho_0) \) such that

\[
\det D_\tau = \tilde{r} \cdot \det D.
\]  

\( \Box \)

We study next the Smith and McMillan forms, as these concepts are closely related to the notion of McMillan degree (see Theorem 4.3 below), and the notion of transmission zeros (to be discussed in subsection 4.3).

Smith Form \([\text{McD 1, p.40}] [\text{McL 1, p.361}] [\text{Sig 1, p.370}]\):

Definition 4.2. Given \( N_1, N_2 \in \tilde{\lambda}_1(\rho_0)^{n_0 \times n_i} \). \( N_1 \) and \( N_2 \) are said to be equivalent iff there exist unimodular matrices \( L \in \tilde{\lambda}_1(\rho_0)^{n_0 \times n_0} \), \( R \in \tilde{\lambda}_1(\rho_0)^{n_i \times n_i} \) such that

\[ N_1 = LN_2R. \]

\( \Box \)

Remark 4.2. Throughout this paper, we say that \( N \in \tilde{\lambda}_1(\rho_0)^{n_0 \times n_i} \) (or \( \tilde{b}(\rho_0)^{n_0 \times n_i} \) has normal rank \( r \) iff \( \text{rank}[N(z)] = r \) for almost all \( z \in D(\rho_0)^C \).

\( \Box \)

Theorem 4.2 \([\text{McD 1, p.40}] [\text{McL 1, p.361}] [\text{Sig 1, p.370}]\). Given \( N \in \tilde{\lambda}_1(\rho_0)^{n_0 \times n_i} \), with normal rank \( r \). Then \( N \) is equivalent to a matrix \( S[N] \in \tilde{\lambda}_1(\rho_0)^{n_0 \times n_i} \) which satisfies
where \( n_i \mid n_{i+1} \), \( i = 1, 2, \ldots, r-1 \).

\[ S[N] = \begin{bmatrix}
    n_1 & & & & \cdots & n_{i-r} \\
    & n_2 & & & \cdots & 0 \\
    & & \ddots & & \cdots & 0 \\
    & & & \ddots & & \vdots \\
    n_0 & & & & & n_r \\
\end{bmatrix} \tag{4.6} \]

Remark 4.3. (i) In Theorem 4.2, \( n_i \mid n_{i+1} \) means that \( n_{i+1} \) is a multiple of \( n_i \) as elements of the ring \( \mathbb{Z}_l(p_0) \).

(ii) In general, the Smith form \( S[N] \) is not unique. To avoid any confusion, however, we can assume that a fixed normalization procedure has been chosen so that \( S[N] \) is unique: for instance, if we have obtained a Smith form \( S[N] \) as in (4.6), we can use Fact A.1 of Appendix A to decompose

\[ n_i = n_{iu} \cdot n_{is} , \quad i = 1, 2, \ldots, r , \]

such that the leading coefficient of \( n_{iu} \) (lowest \( z^{-1} \)-degree term) is 1. Then

\[ S[N] = \begin{bmatrix}
    n_{1u} & & & & \cdots & n_{i-r} \\
    & n_{2u} & & & \cdots & 0 \\
    & & \ddots & & \cdots & 0 \\
    & & & \ddots & & \vdots \\
    n_{0u} & & & & & n_{rs} \\
\end{bmatrix} \begin{bmatrix}
    n_{1s} & & & & \cdots & n_{i-r} \\
    & n_{2s} & & & \cdots & 0 \\
    & & \ddots & & \cdots & 0 \\
    & & & \ddots & & \vdots \\
    0 & & & & & n_{r} \\
\end{bmatrix} \tag{4.7} \]

On the right hand side of (4.7), if we call the left factor \( S_u[N] \) and
the right factor $S_u(N)$, then $S_u(N)$ is a uniquely normalized Smith form of $N$, and $S_u[N]$ is a unimodular matrix in $\tilde{\phi}_1(\rho_0)^{n_1 \times n_1}$ that can be absorbed by the definition of Smith form.

McMillan Form:

Given $\tilde{G} \in b(\rho_0)^{n_0 \times n_1}$. Let $d \in \tilde{\phi}_1(\rho_0)$ be a least common multiple of all denominators (obtained from the $\rho_0$-r's) of all elements of $\tilde{G}$; and let $N := d\tilde{G} \in \tilde{\phi}_1(\rho_0)^{n_0 \times n_1}$. With the Smith form $S[N]$ of $N$ defined through (4.6), we calculate

$$\tilde{G} = \frac{N}{d} = L \frac{S[N]}{d} R$$

$$L \begin{bmatrix}
\frac{\varepsilon_1}{\psi_1} & & & \\
\varepsilon_2 & & & \\
& \ddots & & \\
& & \varepsilon_r & \\
- & & & \psi_r
\end{bmatrix}
\begin{bmatrix}
1 \\
r \\
& 0 \\
& & 0
\end{bmatrix}
R$$

where $\frac{\varepsilon_i}{\psi_i}$ is a $\rho_0$-r. of $\frac{n_1}{d}$, $i = 1, 2, \ldots, r$. The second factor in (4.8),

$$M[\tilde{G}] := \frac{L}{r} \begin{bmatrix}
\frac{\varepsilon_1}{\psi_1} & & & \\
\varepsilon_2 & & & \\
& \ddots & & \\
& & \varepsilon_r & \\
- & & & \psi_r
\end{bmatrix}
\begin{bmatrix}
1 \\
r \\
& 0 \\
& & 0
\end{bmatrix}
\in b(\rho_0)^{n_0 \times n_1}$$
is called the McMillan form of \( \tilde{G} \).

Note that \( \varepsilon_i | \varepsilon_{i+1} \), \( \psi_{i+1} | \psi_i \), \( i = 1, 2, \ldots, r-1 \). \( \square \)

**Lemma 4.2.** Given the McMillan form \( M[\tilde{G}] \) of \( \tilde{G} \) in (4.8) and (4.9), let

\[
E := \begin{bmatrix}
\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_r \\
p_{n_0-r} & \cdots & 0 & 0
\end{bmatrix} \in \tilde{L}_1(\rho_0)^{n_0 \times n_i}
\quad \text{(4.10)}
\]

\[
\Psi_\tau := \begin{bmatrix}
\psi_1 & \psi_2 & \cdots & \psi_r \\
0 & \cdots & 0 & I_{n_0-r}
\end{bmatrix} \in \tilde{L}_1(\rho_0)^{n_i \times n_i}
\quad \text{(4.11)}
\]

\[
\Psi_L := \begin{bmatrix}
\psi_1 & \psi_2 & \cdots & \psi_r \\
0 & \cdots & 0 & I_{n_0-r}
\end{bmatrix} \in \tilde{L}_1(\rho_0)^{n_0 \times n_0}
\quad \text{(4.12)}
\]

Then

\[
(LE, R^{-1}\Psi_\tau) \text{ is a } \rho_0\text{-r.r. of } \tilde{G}
\quad \text{(4.13)}
\]

and

\[
(\Psi_L R^{-1}, ER) \text{ is a } \rho_0\text{-1.r. of } \tilde{G}.
\quad \text{(4.14)}
\]

**Remark 4.4.** If \((\mathcal{N}_L, D_L)\) is any \( \rho_0\)-r.r. of \( \tilde{G} \), then it is immediate from Theorem 3.3 that

\[
\mathcal{N}_L = LE, \quad D_L = R^{-1}\Psi_\tau \mod \text{ a unimodular matrix in } \tilde{L}_1(\rho_0)^{n_i \times n_i}
\]
Similarly, if \((D_\ell, N_\ell)\) is any \(\rho_0\)-l.r. of \(\tilde{G}\), then

\[
D_\ell = \psi_\ell L^{-1}, \quad N_\ell = ER \ \text{modulo} \ \text{a unimodular matrix in} \ \mathbb{C}^{n_0 \times n_0}
\]
on the left. □

Theorem 4.3. Given the McMillan form \(M[\tilde{G}]\) of \(\tilde{G} \in \mathbb{B}(\rho_0)^{n_0 \times n_0}\) in (4.8) and (4.9). Let \(\tilde{x}_G := \prod_{i=1}^r \psi_i \in \tilde{x}_{1-}(\rho_0)\). Under these conditions,

(a) \(p \in D(\rho_0)^C\) is a pole of \(\tilde{G}\) if and only if \(\tilde{x}_G(p) = 0\);

(b) if \(p \in D(\rho_0)^C\) is a pole of \(\tilde{G}\), then the order of \(p\) as a zero of \(\tilde{x}_G\) is its McMillan degree. □

4.2 Dynamic Interpretation of Poles

Given that \(\tilde{G} \in \mathbb{B}(\rho_0)^{n_0 \times n_0}\), \(\tilde{G}\) is a meromorphic function in \(D(\rho_1)^C\) (for some \(\rho_1 \in [0, \rho_0[\)), and \(\tilde{G}\) may have at most a finite number of poles in \(D(\rho_0)^C\). The following theorem gives a dynamic interpretation of such poles.

Theorem 4.4. Let \(\tilde{G} \in \mathbb{B}(\rho_0)^{n_0 \times n_0}\). Then \(p \in D(\rho_0)^C\) is a pole of \(\tilde{G}\) if and only if there exists an input sequence

\[
e \in \mathbb{L}_{1-}(\rho_1)^{n_i} \quad (4.15)
\]

for some \(\rho_1 \in ]0, \rho_0[\), such that the output sequence \(y := G*e\) satisfies

\[
y(k) = \gamma \cdot p^k + h(k) \quad \forall k \in \mathbb{N} \quad (4.16)
\]

where \(\gamma \in \mathbb{C}^{n_0}\) is nonzero, and

\[
h := (h(k))_{k=0}^{\infty} \in \mathbb{L}_{1-}(\rho_2)^{n_0} \quad (4.17)
\]

for some \(\rho_2 \in ]0, \rho_0[\). □
Remark 4.5. (i) As $k$ increases towards $+\infty$, $\gamma \cdot p^k$ is the dominant term in the output (4.16): indeed, by (4.17) $h(k)$ is at most $0(\rho_2^k)$ whereas $\gamma \cdot p^k$ is $0(|p|^k)$, and $\rho_2 < \rho_0 \leq |p|$. So for $k$ large, $|h(k)| \ll |\gamma| \cdot |p|^k = |\gamma \cdot p^k|$. Similarly, by (4.15), the output $y(k)$ also dominates the input $e(k)$.

(ii) Note that, from the proof of the theorem, both the input $e$ and the vector $\gamma$ depend on $\tilde{G}$: The point is that the input is carefully chosen so that $p$ is the only $D(\rho_0)^C$-pole of $\tilde{G}$ excited by the input.

(iii) The proof uses a $\rho_0$-r.r. of $\tilde{G}$. A slightly more involved proof can be obtained with a $\rho_0$-l.r. of $\tilde{G}$.

(iv) In the lumped case, the input sequence $e$ can be chosen so that $e$ and $h$ are identically zero except for a finite number of indices (see continuous-time analog in [Des 3; Thm. III]).

4.3 Zeros and Their Dynamic Interpretation

Let $\tilde{G} \in \tilde{B}(\rho_0)^{n_0 \times n_1}$, with a $\rho_0$-r.r. $(\mathcal{N}_\mathcal{N}, \mathcal{D}_\mathcal{D})$, i.e.

$$\tilde{G} = \mathcal{N}_\mathcal{N} \mathcal{D}_\mathcal{D}^{-1} \quad (4.18)$$

and a $\rho_0$-l.r. $(\mathcal{D}_\mathcal{D}^{-1}, \mathcal{N}_\mathcal{N})$, i.e.

$$\tilde{G} = \mathcal{D}_\mathcal{D}^{-1} \mathcal{N}_\mathcal{N} \quad (4.19)$$

Lemma 4.3. For any $z \in D(\rho_0)^C$,

$$\text{rank}[\mathcal{N}_\mathcal{N}(z)] = \text{rank}[\mathcal{N}_\mathcal{N}(z)] \quad (4.20)$$
Definition 4.4. Assume that

\[ N_{\mathcal{L}} \text{ (equivalently } N_{\mathcal{H}}) \text{ has full normal rank, i.e. } \min(n_0, n_1). \quad (4.21) \]

Then \( z_0 \in D(\rho_0)^C \) is called a (transmission) zero of \( \tilde{G} \) iff

\[ \text{rank}[N_{\mathcal{L}}(z_0)] < \min(n_0, n_1) \quad (4.22) \]

(equivalently, \( \text{rank}[N_{\mathcal{H}}(z_0)] < \min(n_0, n_1)) \).

Remark 4.6. (i) In view of Lemma 4.3, the notion of transmission zero is a property of the matrix transfer function \( \tilde{G} \), independent of any particular choice of matrix fraction representation.

(ii) Note that \( \tilde{G} \) can have a pole and a transmission zero at the same point \( z_0 \in D(\rho_0)^C \).

(iii) Let \( M[\tilde{G}] \) be the McMillan form of \( \tilde{G} \) as in (4.8)-(4.14). Then \( z_0 \in D(\rho_0)^C \) is a zero of \( \tilde{G} \) if and only if \( z_0 \) is a zero of \( \varepsilon_i \), for some \( i \in \{1, 2, \ldots, \min(n_0, n_1)\} \).

(iv) If assumption (4.21) is not satisfied, we can always ignore some redundant input or output, and consider a smaller matrix transfer function for which (4.21) is satisfied. Then the following theorems can be applied to this reduced matrix transfer function.

Theorem 4.5. Let \( \tilde{G} \in \mathcal{B}(\rho_0)^{n_0 \times n_1} \), with \( n_0 \geq n_1 \).

(a) If \( z_0 \in D(\rho_0)^C \) is a zero of \( \tilde{G} \), then there exists a nonzero \( \xi \in \mathbb{C}^{n_1} \) and a sequence

\[ m \in \mathcal{L}(\rho_1)^{n_1} \text{ for some } \rho_1 \in ]0, \rho_0[ \quad (4.23) \]

such that the input sequence \( e \in (\mathbb{C}^N)^{n_1} \) described by

\[ e_k = \begin{cases} 
\xi, & k \equiv 0 \\
0, & k \equiv 1, 2, \ldots 
\end{cases} \]
produces an output sequence \( y \in (\mathbb{C}^N)^{n_0} \) (i.e. \( y = G*e \)) such that
\[
y \in \mathbb{L}_1(\rho_2)^{n_0}
\]
for some \( \rho_2 \in ]0, \rho_0[. \)

(b) If \( v \in D(\rho_0)^c \) is neither a pole nor a zero of \( \tilde{G} \), then for all nonzero vectors \( \xi \in \mathbb{C}^{n_i} \), the input sequence \( e \in (\mathbb{C}^N)^{n_i} \) described by
\[
e(k) = \xi v^k, \quad k \in \mathbb{N}
\]
produces an output sequence \( y \in (\mathbb{C}^N)^{n_0} \) which contains the nonzero term
\[
\tilde{G}(v)\xi v^k.
\]

Remark 4.7. Consider part (a) of the theorem:

(i) In the lumped case, we can prove that the sequences \( m \) and \( y \) can be chosen to be identically zero except for a finite number of indices (see continuous-time analog in [Des 3; Thm. I]).

(ii) For \( k \) large, since \( |z_0| > \rho_0 \) and since (4.23) holds, the term \( \xi z_0^k \) in (4.24) is the dominant term in the input sequence (indeed, \( \rho_1 < \rho_0 \leq |z_0| \)); furthermore, this term also dominates the output sequence (since \( \rho_2 < \rho_0 \leq |z_0| \)). In this sense, we still have the interpretation that the zero blocks the transmission of the term \( (\xi z_0^k)_{k=0}^{\infty} \). The purpose of \( m \) in the input is to prevent any contribution in \( y \) of any of the \( D(\rho_0)^c \)-poles of \( \tilde{G} \). □

Theorem 4.6. Let \( \tilde{G} \in \tilde{D}(\rho_0)^{n_0 \times n_i} \) with \( n_0 \leq n_i \). If \( z_0 \in D(\rho_0)^c \) is a zero of \( \tilde{G} \), then there is a nonzero \( n \in \mathbb{C}^{n_0} \) such that for all
\[ \xi \in C^{n_i} \text{ there is } \tilde{m} \in \pi(\rho_0)^{n_i} \text{ so that the input sequence described by } \\
\begin{align*}
e(k) &= \xi z_0^k + m(k) \quad \forall k \in N \\
\end{align*}
\]
produces an output sequence \( y \) (i.e. \( y = G*e \)) that satisfies

\[ n^*y \in \ell_1^-(\rho_1) \quad (4.29) \]

for some \( \rho_1 \in ]0,\rho_0[ \), where \( n^*y := (n^*y(k))_{k=0}^{\infty} \).

Remark 4.8. Theorem 4.5 (which applies to cases where \( n_0 \geq n_1 \)) asserts that for some \( \xi \), the input \( e(k) = \xi z_0^k + m(k) \) produces an output \( y \) which does not have a term in \( z_0^k \), i.e. the sequence \( (z_0^k)_{k=0}^{\infty} \) is blocked for those \( \xi \)'s. Theorem 4.6 (where \( n_0 < n_1 \)) allows any \( \xi \) and asserts that, in some direction dictated by \( \eta \), \( y \) does not contain any term in \( z_0^k \).

4.4 Example

This example demonstrates Theorem 4.5(a) with a multi-input multi-output transfer function \( \tilde{G} \in \tilde{\mathcal{B}}(\rho_0)^{2 \times 2} \), where \( \rho_0 := 0.55 \), defined by

\[ \tilde{G}(z) := \begin{bmatrix}
\frac{1}{(z-1)} + z^{-2}e^{-2z} & \frac{1}{(z-4)} + \frac{5}{(2z-1)}e^{-1-3z} \\
- \frac{1}{(z-2)} - \frac{3}{z}e^{-1-3z} \\
\end{bmatrix}, \quad z \in D(\rho_0)^C. \quad (4.30) \]

Note that the set of poles of \( \tilde{G} \) in \( D(\rho_0)^C \) is

\[ P[\tilde{G}] = \{1,2,4\}. \quad (4.31) \]

A \( \rho_0 \)-l.r. \((\mathcal{D}_\xi, \mathcal{N}_\xi)\) of \( \tilde{G} \) is given by
\( D_\ell(z) := \begin{bmatrix} (z-1)(z-4) & 0 \\ z^2 & 0 \\ (z-2) \\ z \end{bmatrix} \) \hfill (4.32)

\[ N_\ell(z) := \begin{bmatrix} (z-4)[z+(z-1)e^{-2z^{-1}}] & (z-1)[2z-1+5(z-4)e^{1-3z^{-1}}] \\ z^3 & z^2(2z-1) \\ 0 & -z^3(2z-1) \end{bmatrix} \] \hfill (4.33)

\((D_\ell, N_\ell)\) are \(\rho_0\)-l.c.: indeed, they satisfy

\[ (N_\ell U_\ell + D_\ell V_\ell)(z) = I_2, \quad z \in \mathbb{D}(\rho_0)^c \] \hfill (4.34)

with \( U_\ell, V_\ell \in \mathfrak{L}_1(\rho_0)^{2 \times 2} \) described by

\[ U_\ell(z) := \begin{bmatrix} (3z^3-4z^2-32z+32) & 2(z-1) \\ 3z^3 & z^2 \\ 16(z-2)(z+4) & 2(z-4) \\ 3z^2 & z \end{bmatrix} \] \hfill (4.35)

and

\[ V_\ell(z) := \begin{bmatrix} (3z^3-4z^2-32z+32)(z-e^{-2z^{-1}}) + 80(z-2)(z+4)e^{1-3z^{-1}} \\ 3z^3 & 3z^2(2z-1) \\ -16(z+4)[z + 3(z-2)e^{1-3z^{-1}}] \\ 3z^3 \end{bmatrix} \] \hfill (4.36)
Observe that
\[
N_\mathcal{E}(3) = \begin{bmatrix}
-\frac{(3+2e^{-2/3})}{27} & 0 \\
0 & 0
\end{bmatrix}
\] (4.37)
hence, by definition, \( \tilde{G} \) has a zero at \( z = 3 \). Consequently, we choose
\[
\xi := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{C}^2
\] (4.38)
which satisfies (B.65). Next, as in (B.68), we define
\[
m(z) := -U_2(z)W_2(z)_H = ST
\]
\[
= \left[ \frac{(z-1)[(2z-1)(z-2)(3z-4)(z+4)+(15z^4-116z^3+10z^2+764z-640)e^{1-3z^{-1}}]}{3z^4(z-3)(2z-1)} \\
\frac{2[(2z-1)(5z^3+20z^2-80z+64)+(z-2)(z-4)(58z^2+111z-160)e^{1-3z^{-1}}]}{3z^3(z-3)(2z-1)} \right]
\] (4.39)

Note that \( \tilde{m} \) is analytic at \( z = 3 \), and \( m \in \mathcal{L}_1(\rho_1)^2 \) with \( \rho_1 := 0.51 \).

With the input
\[
\tilde{e}(z) := \xi * \frac{z}{(z-3)} + \tilde{m}(z)
\] (4.40)
defined as in (4.24), the output is
\[
\tilde{y}(z) := \tilde{G}(z)\tilde{e}(z) = [\tilde{y}_1(z) \quad \tilde{y}_2(z)]^T
\] (4.41)
where
\[
\tilde{y}_1(z) = \frac{1}{3z^5(z-3)(2z-1)^2}[(z-1)(2z-1)^2(z-2)(3z-4)(z+4)(z-e^{-2z^{-1}})
\]
\[
+ z(2z-1)(15z^5-181z^4-74z^3+1554z^2-2044z+640)e^{1-3z^{-1}}
\]
\[
- (z-1)(2z-1)(15z^4-116z^3+10z^2+764z-640)e^{-5z^{-1}}
\]
\[
- 10z^2(z-2)(z-4)(58z^2+111z-160)e^{2-6z^{-1}}
\]
and

\[ \tilde{y}_2(z) = \frac{(z-4)[z-3(z-2)e^{1-3z^{-1}}][ (2z-1)(3z-4)(z+4)-2(58z^2+111z-160)e^{1-3z^{-1}}]}{3z^4(z-3)(2z-1)}. \]

Observe that \( y \in \mathcal{L}_1(\rho_2)^2 \), with \( \rho_2 := 0.51 \), and \( \tilde{y} \) is analytic at \( z = 1, 2, 3, 4 \): note that \( 1, 2, 4 \in \mathcal{P}[G] \) and \( z = 3 \) is a pole of the input \( \tilde{e} \).
5. **Interconnected Systems and Characteristic Functions**

In order to discuss the stability of interconnected systems, we introduce the notion of characteristic functions. Basically the technique is very simple: we illustrate it by an example. In this process, we state without formal proof some properties that hold for more general interconnections.

**Example 5.1.** Consider the system depicted by Fig. 5-1. All transfer functions are matrices with elements in $\mathbb{C}^p$ for some $p \in [0,1[$: $\tilde{G}_p$ is the plant transfer function, $\tilde{G}_i$ is the inner-loop feedback, $\tilde{G}_c$ is a precompensator, and $\tilde{G}_0$ is the outerloop feedback. The vectors $\tilde{u}_p, \tilde{u}_i, \tilde{u}_c$ and $\tilde{u}_0$ are the respective exogenous input signals to the summing nodes of these subsystems, and $\tilde{y}_p, \tilde{y}_i, \tilde{y}_c$ and $\tilde{y}_0$ are the respective outputs of these subsystems. Let

\[
\begin{align*}
(N_{pr}, D_{pr}) & \text{ be a } p_0\text{-r.r. of } \tilde{G}_p, \\
(N_{cr}, D_{cr}) & \text{ be a } p_0\text{-r.r. of } \tilde{G}_c, \\
(D_{il}, N_{il}) & \text{ be a } p_0\text{-l.r. of } \tilde{G}_i, \\
(D_{ol}, N_{ol}) & \text{ be a } p_0\text{-l.r. of } \tilde{G}_0.
\end{align*}
\]

Let us denote by $\tilde{\xi}$ the list of output vectors from all the $D^{-1}$ matrices with the appropriate subscripts, as depicted by Fig. 5-2.

Define
\[\begin{bmatrix}
\tilde{u}_p \\
\tilde{u}_c \\
\tilde{u}_i \\
\tilde{u}_o
\end{bmatrix} \quad \begin{bmatrix}
\tilde{y}_p \\
\tilde{y}_c \\
\tilde{y}_i \\
\tilde{y}_o
\end{bmatrix} \quad \begin{bmatrix}
\tilde{\xi}_p \\
\tilde{\xi}_c \\
\tilde{\xi}_i \\
\tilde{\xi}_o
\end{bmatrix} \]

(5.5)

By equating the respective input vector of each \(D^{-1}\) matrix and the output vector of each subsystem, we describe the whole interconnected system (as in Fig. 5-2) by a set of equations in the form of

\[D\tilde{\xi} = N_\ell \tilde{u}\]

(5.6a)

\[N_\lambda \tilde{\xi} = \tilde{y}\]

(5.6b)

Specifically for this particular example, we have

\[\begin{bmatrix}
P_\ell & -N_\ell & I & 0 \\
-1 & -1 & C_\ell & -1 \\
0 & 0 & D_\ell & 0 \\
-N_\lambda N_\ell P_\ell & 0 & 0 & D_\ell
\end{bmatrix} \quad \begin{bmatrix}
\tilde{\xi}_p \\
\tilde{\xi}_c \\
\tilde{\xi}_i \\
\tilde{\xi}_o
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 \\
-1 & -1 & -1 & -1
\end{bmatrix} \begin{bmatrix}
\tilde{u}_p \\
\tilde{u}_c \\
\tilde{u}_i \\
\tilde{u}_o
\end{bmatrix} \]  

(5.7a)

\[\begin{bmatrix}
P_\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \quad \begin{bmatrix}
\tilde{\xi}_p \\
\tilde{\xi}_c \\
\tilde{\xi}_i \\
\tilde{\xi}_o
\end{bmatrix} = \begin{bmatrix}
\tilde{y}_p \\
\tilde{y}_c \\
\tilde{y}_i \\
\tilde{y}_o
\end{bmatrix} \]

(5.7b)

Now consider the matrices \(D_\ell, N_\ell\) and \(N_\lambda\) defined in (5.6) as they appear in (5.7). Using Corollary 3.1b(\(\ell\)) and the \(\rho_0\)-l.c. property of \((D_\ell, N_\ell)\) and \((D_\lambda, N_\lambda)\), it is easy to see that the pair \((D, N_\lambda)\) in (5.7a) is \(\rho_0\)-l.c. Similarly, by Corollary 3.1b(\(\lambda\)) and the \(\rho_0\)-r.c. property of \((N_\lambda, D_\ell)\) and \((N_\ell, D_\lambda)\), the pair \((N_\lambda, D)\) is \(\rho_0\)-r.c. □
We now summarize the procedure for analyzing more general interconnected systems:

**Procedure 5.1. Analysis of interconnected systems**

*Given:* subsystems each described by a matrix transfer function $\tilde{G}_k$ with elements in $\tilde{b}(\rho_0)$; the input to the transfer function $\tilde{G}_k$ is the sum of an exogenous input $\tilde{u}_k$ and outputs (modulo sign) of conformable size from other subsystems; the output of $\tilde{G}_k$ is denoted by $\tilde{y}_k$ (see Fig. 5-3).

**Step 1.** For each subsystem $\tilde{G}_k$, find either a $\rho_0$-r.c. $(N_{k\ell}, D_{k\ell})$ or a $\rho_0$-l.r. $(D_{k\ell}, N_{k\ell})$.

**Step 2.** Denote by $\tilde{\xi}_k$ the output vector from each $D_{k\ell}$ matrix.

**Step 3.** With the composite vectors $\tilde{u}$, $\tilde{y}$ and $\tilde{\xi}$ (defined as in (5.5)), equate the input vectors of each $D_{k\ell}$ matrix and the output vector of each subsystem to get a description of the interconnected system in the form of (5.6), namely

\[
D\tilde{\xi} = N_{\ell}\tilde{u}, \quad D \in \tilde{\lambda}_-^{(\rho_0)}_{n \times n}, \quad N_{\ell} \in \tilde{\lambda}_-^{n \times n_{\ell}}.
\]

\[
N_{\ell}\tilde{\xi} = \tilde{y}, \quad N_{\ell} \in \tilde{\lambda}_-^{0 \times n_{\ell}}.
\]

Then we have the following property:

**Fact 5.1.** $(D, N_{\ell})$ is $\rho_0$-l.c. and $(N_{\ell}, D)$ is $\rho_0$-r.c.

**Remark 5.1.** In this formulation, there is an additive exogenous input to each subsystem, and the output of each subsystem can be observed.
Assume a well-posedness condition\(^\dagger\) that \( \det \mathcal{D} \in \mathcal{L}_1^\infty(\rho_0) \). Then, by Cramer's rule,
\[
\tilde{G}_\ell := \mathcal{D}^{-1}N_{\ell} \quad (5.11)
\]
\[
\tilde{G}_\mathcal{N} := N_{\mathcal{N}} \mathcal{D}^{-1} \quad (5.12)
\]
and
\[
\tilde{G} := N_{\mathcal{N}} \mathcal{D}^{-1}N_{\ell} \quad (5.13)
\]
are all matrices with elements in \( \tilde{b}(\rho_0) \). In particular, \((N_{\mathcal{N}}, \mathcal{D})\) is a \( \rho_0 \)-r.r. of \( \tilde{G}_\mathcal{N} \), and \((\mathcal{D}, N_{\ell})\) is a \( \rho_0 \)-l.r. of \( \tilde{G}_\ell \).

Definition 5.1. We call \( \bar{\chi} := \det \mathcal{D} \in \mathcal{L}_1^\infty(\rho_0) \) the characteristic function of the interconnected system described in Procedure 5.1. \( \square \)

Lemma 5.1. \( p \in D(\rho_0)^C \) is a zero of the characteristic function \( \bar{\chi} \)
\[
\Leftrightarrow p \in D(\rho_0)^C \text{ is a pole of } \tilde{G}_\mathcal{N} \quad (5.14)
\]
\[
\Leftrightarrow p \in D(\rho_0)^C \text{ is a pole of } \tilde{G}_\ell \quad (5.15)
\]
\[
\Leftrightarrow p \in D(\rho_0)^C \text{ is a pole of } \tilde{G} \quad (5.16)
\]
\[
\Leftrightarrow p \in D(\rho_0)^C \text{ is a pole of } \tilde{G} \quad (5.17) \quad \square
\]

Because of Lemma 5.1, the importance of the characteristic function \( \bar{\chi} \) is obvious by the dynamic interpretation of poles of \( \tilde{G} \) in Theorem 4.4, and by the next Theorem.

Theorem 5.1. Consider the interconnected system described in Procedure 5.1. Let \( \sigma \geq \rho_0 \). The characteristic function \( \bar{\chi} \) has a zero \( p \) of absolute value \( \sigma \) if and only if there exist some \( m \in \mathbb{N}^* \) and some
\(^\dagger\)Note that if \( \lim_{|z| \to \infty} \det \mathcal{D}(z) = 0 \), then \( \mathcal{D}^{-1} \) has a pole at infinity; then for some \( \tilde{L}_1(\rho_0) \)-matrices \( U_{\ell} \) and \( V_{\ell} \), \( N_{\ell}U_{\ell} + DV_{\ell} = I \), so \( \tilde{G}_\ell U_{\ell} + V_{\ell} = \mathcal{D}^{-1} \) and \( \tilde{G}_\ell \) has a pole at infinity: hence the map \( u \mapsto \xi \) is noncausal.
input sequence \( u \) with support \( \{0\} \) such that the corresponding output sequence \( y = (y(k))_{k=0}^{\infty} := G*u \) includes a nonzero term which, for large \( k \), is \( O(k^{m-1}e^k) \).

**Remark 5.2.** In fact, a little more than Theorem 5.1 is proved: the zero \( p \) of the characteristic function \( \tilde{\chi} \) in \( D(p_0)^C \) corresponds to the mode \( p \) of the interconnected system which can be excited by some exogenous input, and observed at some subsystem output.

**Definition 5.2.** Let \( p \in [1,\infty] \). A map represented by a matrix transfer function \( \bar{G} \in \tilde{\mathcal{D}}(p_0)^{n_0 \times n_1} \) is said to be \( \ell_p \)-stable iff it takes \( \ell_p \)-input sequences to \( \ell_p \)-output sequences, and there exists some \( k \in \mathbb{R}_+ \) such that for all \( u \in \ell_p \)

\[
\|G*ull_p \leq k\|u\|_p.
\]

(Note that \( k \) may depend on \( p \).)

**Theorem 5.2.** Consider an interconnected system described by Procedure 5.1 and assumed to be well-posed. For any \( p \geq p_0 \),

\[
\tilde{G} \in \tilde{\mathcal{D}}_1(p)^{n_0 \times n_1}
\]

if and only if

\[
\tilde{\chi}(z) \neq 0 \quad \forall z \in D(p)^C .
\]

The next corollary follows from Theorem 5.2 and [Des 1, Thm. C.4.7].

**Corollary 5.2a.** Consider an interconnected system described by Procedure 5.1. Its input-output map represented by \( \bar{G} \) is \( \ell_p \)-stable \( \forall p \in [1,\infty] \) if and only if

\[
\tilde{\chi}(z) \neq 0 \quad \forall z \in D(1)^C .
\]
In such a case, we say that the interconnected system is \( p \)-stable, \( \forall p \in [1, \infty] \).

Applying Theorem 5.2 to a simple case leads to the following useful corollary.

**Corollary 5.2b.** Consider a system with input sequence \( u \) and output sequence \( y \) where \( \ddot{y} = \tilde{b}u \); let \( \tilde{b} \in b(\rho_0)_{n \times n} \), with a \( \rho_0 \)-r.r. \((N, D)\) (respectively \( \rho_0 \)-l.r. \((D, N)\)). Then for any \( \rho \geq \rho_0 \)

\[
\tilde{b} \in b_1(\rho)_{n \times n}
\]

if and only if

\[
\text{det} \ D_1(z) \neq 0 \quad \forall z \in D(\rho)^c \quad (5.22)
\]

(respectively \( \text{det} \ D_2(z) \neq 0 \quad \forall z \in D(\rho)^c \)). \( \square \)
6. Feedback System Stability

We now apply the results developed in the preceding section to analyze the multi-input multi-output feedback system $S$ depicted in Fig. 6-1. Let $\rho_0 \in ]0,1[$.

(i) Let $\tilde{p} \in \mathcal{B}(\rho_0)^{n_0 \times n_i}$ be the plant transfer function with input $u_p \in (\mathbb{C}_z)^{n_i}$ and output $y_p \in (\mathbb{C}_z)^{n_0}$; let $\tilde{c} \in \mathcal{B}(\rho_0)^{n_i \times n_0}$ be the controller transfer function with input $u_c \in (\mathbb{C}_z)^{n_0}$ and output $y_c \in (\mathbb{C}_z)^{n_i}$.

(ii) $u_s \in (\mathbb{C}_z)^{n_0}$ is the system (reference) input and $w_p \in (\mathbb{C}_z)^{n_i}$ is the plant input disturbance.

(iii) $y_s = y_p$ is the system output and $e := u_s - y_s = u_c$ is the system error.

Observe that if an additive disturbance is present at the plant output, say $w_0$, then its effect on $y_s$ is equivalent to an additional system input $-w_0$.

Next, we define the composite system input, output and error by

$$
\begin{align*}
\begin{bmatrix} u \\ y \\ e \end{bmatrix} &= \begin{bmatrix} u_s \\ y_s \\ y_p \end{bmatrix} = \begin{bmatrix} y_c \\ y_p \end{bmatrix} = \begin{bmatrix} u_c \\ u_p \end{bmatrix} \\
\end{align*}
$$

(6.1)

where $u$, $y$ and $e$ are in $(\mathbb{C}_z)^{n_0+n_i}$. Then, from Fig. 6-1, the feedback system is described by

$$
\begin{bmatrix} n_0 & n_i \\ n_0 & I_{n_0} \\ n_i & -\tilde{c} \end{bmatrix}
\begin{bmatrix} \tilde{u} \\ \tilde{e} \end{bmatrix} = \begin{bmatrix} \tilde{p} \\ \tilde{c} \end{bmatrix}
$$

(6.2)

and
Let

\[
\bar{G} := \begin{bmatrix}
0 & n_o \\
0 & 0
\end{bmatrix} \in \mathcal{B}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)}
\]  

(6.4)

\[
\tilde{J} := \begin{bmatrix}
0 & n_i \\
0 & 0
\end{bmatrix} \in \mathcal{B}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)}
\]  

(6.5)

and observe that

\[
\begin{bmatrix}
-\tilde{C} & 0 \\
-\tilde{\rho} & 0
\end{bmatrix} = J^{-1}\tilde{G}
\]  

(6.6)

Assume the well-posedness condition that

\[
\lim_{|z| \to \infty} \det[ I + PC](z) = \lim_{|z| \to \infty} \det[ I + CP] \neq 0.
\]  

(6.7)

Now the input-to-error transfer function \( \tilde{H}_{eu} : \tilde{u}l \to \tilde{e} \) and input-to-output transfer function \( \tilde{H}_{yu} : \tilde{u}l \to \tilde{y} \) satisfy

\[
\tilde{H}_{eu} = (I + \tilde{G})^{-1}
\]  

(6.8)

\[
J\tilde{H}_{yu} = \tilde{G}(I + \tilde{G})^{-1} = I - \tilde{H}_{eu}
\]  

(6.9)

Remark 6.1. (i) By assumption (6.7), \( \lim_{|z| \to \infty} \det[I + \tilde{P}C](z) \neq 0; \) hence by Theorem 2.4, \( \det[I + \tilde{P}C] = \det[I + \tilde{C}P] \) is an invertible element of \( \tilde{B}(\rho_0); \) then by applying Cramer's rule to (6.8) and (6.9), we conclude that
(i) \(\overline{H}_{eu}\) and \(\overline{H}_{yu}\) belong to \(B(p_0^{(n_1+n_0)(n_1+n_0)})\). (6.10)

(ii) For any \(p \in \mathbb{R}_+\), due to (6.8), (6.9) and the closure properties of \(\mathbb{R}_1(p)\) under addition and multiplication

\[
\overline{H}_{eu} \in \mathbb{Z}_1(p) \quad (n_1+n_0)(n_1+n_0) \quad \iff \quad \overline{H}_{yu} \in \mathbb{Z}_1(p) \quad (n_1+n_0)(n_1+n_0).
\]

Let \(D_{p}, N_{p}\) be a \(p_0\)-r. of \(\bar{p}\), (6.12)

and let \(N_{C}, D_{C}\) be a \(p_0\)-r. of \(\bar{c}\). (6.13)

Then Procedure 5.1, Definition 5.1, and simple calculations show that

\[
\tilde{\chi} := \text{det}[D_{p} D_{C} + N_{p} N_{C}]
\]

is an element of \(\mathbb{Z}_1_-^\infty (p_0)\), and is the characteristic function of the feedback system \(S\); furthermore, by assumption (6.7), \(\tilde{\chi} \in \mathbb{Z}_1_-^\infty (p_0)\).

Theorem 6.1. Consider a feedback system \(S\) described by (6.1)-(6.14).

Then

(i) \(p \in D(p_0)^C\) is a zero of \(\tilde{\chi}\) \hspace{1cm} (6.15)

\[
\iff \quad p \in D(p_0)^C\text{ is a pole of } \overline{H}_{eu}\hspace{1cm} (6.16)
\]

\[
\iff \quad p \in D(p_0)^C\text{ is a pole of } \overline{H}_{yu}\hspace{1cm} (6.17)
\]

(ii) the McMillan degree of \(p \in D(p_0)^C\) as a pole of \(\overline{H}_{eu}\) and \(\overline{H}_{yu}\) are the same and are equal to the multiplicity of \(p\) as a zero of \(\tilde{\chi}\). \hfill \square

Remark 6.2. (i) By (6.11), Theorem 5.2 and [Des 1, Thm. C.4.7], \(\overline{H}_{eu}\) (equivalently \(\overline{H}_{yu}\)) is \(\mathbb{L}_p\)-stable \(\forall p \in [1, \infty]\) if and only if \(\tilde{\chi}(z) \neq 0\) \(\forall |z| \geq 1\).
(ii) As discussed in Section 2.1, if for some $\rho \in [0,1]$ and $\forall |z| \geq \rho$, $\tilde{x}(z) \neq 0$, then the map $u \mapsto (e,y)$ will take an input sequence with finite support to an output sequence that decays exponentially to $\theta_2(n_1 + n_0)$ at a rate at least $\rho^{-1}$. \qed
7. Compensator Design for Stabilization, Tracking and Disturbance Rejection

7.1 Preliminary Algebraic Result

Suppose
\[ \tilde{\mathbf{G}} \in \mathbb{R}(p_0)^{n_0 \times n_1}, \text{ with a } p_0\text{-r.r. } (D_\mathcal{L}, N_\mathcal{L}). \quad (7.1) \]

Recall that by Corollary 3.2(\ell) there exist six matrices with elements in \( \mathcal{L}_1(p_0) \), namely,
\[ U_\mathcal{L}, V_\mathcal{L}, N_\mathcal{L}, D_\mathcal{L}, U_\mathcal{N}, V_\mathcal{N} \]
such that
\begin{align*}
(1) & \quad (N_\mathcal{L}, D_\mathcal{L}) \text{ is a } p_0\text{-r.r. of } \tilde{\mathbf{G}} \quad (7.2) \\
(2) & \quad \begin{bmatrix} n_i & n_0 \\ n_0 & n_i \end{bmatrix} \begin{bmatrix} U_\mathcal{L} & U_\mathcal{N} \\ D_\mathcal{L} & D_\mathcal{N} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}. \quad (7.3)
\end{align*}

Let us call the two matrices on the left-hand side of (7.3) \( W \) and \( W^{-1} \) respectively.

**Lemma 7.1.** Given any \( D \in \mathcal{L}_1(p_0)^{n_0 \times n_0} \).

(a) The pair \( X \in \mathcal{L}_1(p_0)^{n_i \times n_0}, Y \in \mathcal{L}_1(p_0)^{n_0 \times n_0} \) is a solution of
\[ N_\mathcal{L}X + D_\mathcal{L}Y = D \quad (7.4) \]
if and only if for some \( N \in \mathcal{L}_1(p_0)^{n_i \times n_0} \)
\[ \begin{bmatrix} X \\ -Y \end{bmatrix} = W^{-1} \begin{bmatrix} N \\ -D \end{bmatrix} \quad \text{i.e.} \quad \begin{cases} -X = D_N - U_\mathcal{L}D \\ Y = N_\mathcal{N}N + V_\mathcal{L}D \end{cases} \quad (7.5) \]
or equivalently,
\[
\begin{bmatrix}
N \\
D
\end{bmatrix} = \begin{bmatrix}
-x \\
y
\end{bmatrix}
\quad \text{i.e.} \quad \begin{cases}
N = -V_X + U_Y \\
D = N_X + D_Y
\end{cases}
\quad (7.6)
\]

Furthermore,
\[\begin{align*}
(X, Y) \text{ is } p_0 \text{-r.c. } & \iff (N, D) \text{ is a } p_0 \text{-r.c.} \\
& \quad (7.7)
\end{align*}\]

(b) If in addition
\[G(0) = \lim_{z \to \infty} \tilde{G}(z) = 0_{n_0 \times n_1} \quad (7.8)\]
then
\[\det Y \in \tilde{L}_1^{-\infty}(p_0) \iff \det D \in \tilde{L}_1^{-\infty}(p_0). \quad (7.9)\]

7.2 Problem of Stabilization, Tracking and Disturbance Rejection

Consider the feedback structure depicted in Fig. 6-1. Suppose we are given a plant \( \tilde{P} \in \tilde{G}(p_0)^{n_0 \times n_1} \) for some \( p_0 \in ]0,1[ \) and that

(i) \( p \in (\mathbb{R}^N)^{n_0 \times n_1} \subset (\mathbb{C}^N)^{n_0 \times n_1} \quad (7.10)\)

(ii) \( P(0) = \lim_{z \to \infty} \tilde{P}(z) = 0_{n_0 \times n_1} \quad (7.11)\)

Let \((D_{pl}, N_{pl})\) be a \( p_0 \)-l.r. of \( \tilde{P} \), with \( D_{pl} \in \tilde{r}(p_0)^{n_0 \times n_0} \). \( (7.12)\)

Reference signal sequences \( u_s \in (\mathbb{R}^N)^{n_0} \) (to be tracked) are generated as follows: for some fixed \( \phi_u \in \mathbb{R}[z] \) with \( Z[\phi_u] \subset D(1)^C \),
\[\tilde{u}_s = \frac{v_u}{\phi_u} \quad (7.13)\]

where \( v_u \in \mathbb{R}[z]^{n_0} \), with \( \mathcal{A}[v_u] \leq \mathcal{A}[\phi_u] \), is arbitrary.

Disturbance signal sequences \( w_p \in (\mathbb{R}^N)^{n_1} \) (to be rejected) are generated as follows: for some fixed \( \phi_w \in \mathbb{R}[z] \) with \( Z[\phi_w] \subset D(1)^C \),
\[\tilde{w}_p = \frac{v_w}{\phi_w} \quad (7.14)\]
where \( v_w \in \mathbb{R}[z]^{n_i} \), with \( \delta[v_w] \leq \delta[\phi_w] \), is arbitrary.

Define \( \phi \in \mathbb{R}[z] \) and \( q \in \mathbb{N} \) by

\[
\phi := \text{monic l.c.m. of } \phi_u \text{ and } \phi_w \tag{7.16}
\]

and

\[
q := \delta[\phi] \tag{7.17}
\]

Let \( \phi \) admit \( v \) distinct zeros; let its \( \alpha^{th} \) zero be \( z_\alpha \) with multiplicity \( m_\alpha \). Then

\[
\{z_1, z_2, \ldots, z_v\} = Z[\phi_u] \cup Z[\phi_w] \tag{7.18}
\]

\[
q = \sum_{\alpha=1}^{v} m_\alpha \tag{7.19}
\]

and \( z_\alpha \) is a zero of order \( m_\alpha \) of \( \phi \)

\[\Rightarrow z_\alpha \text{ is a zero of order } m_\alpha \text{ of } \phi . \tag{7.20}\]

In addition, the maximal order of \( z_\alpha \) as a pole of any element of \( \tilde{u}_s \) and \( \tilde{w}_p \) is \( m_\alpha \).

For tracking and disturbance rejection purposes, we assume for

\( \tilde{p} \in \mathbb{b}(\rho_0)^{n_0 \times n_i} \)

that

\[
n_i \geq n_0 \tag{7.21}
\]

\[
\text{rank}[N_{p_{\xi}}(z)] = n_0 \quad \forall z \in Z[\phi_u] \cup Z[\phi_w] . \tag{7.22}
\]

Remark 7.1. (i) To track \( n_0 \) signals, (7.21) assures that at least as many plant inputs are available to facilitate the tracking. Furthermore, (7.22) assures that \( \tilde{p} \) does not contain any transmission zeros in \( Z[\phi_u] \), thus \( \tilde{p} \) will not block the control signal required for asymptotic tracking.

(ii) To achieve asymptotic disturbance rejection, the disturbance input \( w_p \) has to be either asymptotically cancelled by the controller.
output $y_c$ or blocked by some transmission zeros (of the plant $\tilde{P}$) that lie in $Z[\phi_w]$. However, since such transmission zeros are not preserved under plant perturbation, we cannot rely on them to achieve input disturbance rejection.

**Stabilization, Tracking and Disturbance Rejection Problem (SP)**

Given data (7.10)-(7.22), and a finite list $\Lambda$ of points in the annulus $\{z | \rho_0 \leq |z| < 1\}$ such that $\lambda \in \Lambda \Rightarrow \lambda \in \Lambda$. Find a controller $\tilde{C} \in \mathbb{C}(\rho_0)^{n_i \times n_o}$, with $C \in (\mathbb{R}^N)^{n_i \times n_o} \subset (\Phi^N)^{n_i \times n_o}$ such that for the feedback system $S$ (6.1)-(6.14)

(a) $\tilde{H}_{eu}$ and $\tilde{H}_{yu}$ both are $\ell_p$-stable $\forall p \in [1, \infty]$

(b) the list of zeros of $\tilde{x}$ in $D(\rho_0)^C$, $Z[\tilde{x}; D(\rho_0)^C]$, is exactly $\Lambda$;

(c) for any $v_u$ and $v_w$ satisfying (7.13) and (7.14) respectively, the reference signals $u_s$ will be tracked asymptotically and the disturbances $w_p$ will be rejected asymptotically: more specifically, there exists $\rho \in ]0,1[$ such that

$$e_s(k) = o(\rho^k) \text{ as } k \to \infty;$$

(d) condition (c) holds for any perturbed plant $\tilde{P} \in \mathbb{C}(\rho_0)^{n_o \times n_i}$ for which the feedback system $S$ (described in (6.1)-(6.14)) still has $\tilde{H}_{eu}$ and $\tilde{H}_{yu}$ $\ell_p$-stable, $\forall p \in [1, \infty]$.

**Remark 7.2.** (i) By requiring $\tilde{C}$ to be in $\mathbb{C}(\rho_0)^{n_i \times n_o}$, $\tilde{C}$ is bounded at infinity; hence the convolution operator $C$ is causal.

(ii) By the restriction (7.11) $\lim_{|z| \to \infty} \tilde{P}(z) = 0_{n_o \times n_i}$, the well-posedness condition (6.7) is guaranteed.
(iii) By condition (b) of problem (SP) and Theorem 6.1, \( \Lambda \) is the list of (dominant) poles of \( \tilde{H}_{eu} \) and \( \tilde{H}_{yu} \) in \( D(\rho_0)^C \).

(iv) Condition (c) of problem (SP) guarantees that the feedback system \( S \) is a servomechanism; furthermore, the system error \( e_s \) decays to zero at a rate at least \( \rho^{-1} \).

(v) Condition (d) is a robustness condition that guarantees asymptotic tracking and disturbance rejection under plant perturbation, as long as the feedback system conditions (6.1)-(6.14) are satisfied and \( \tilde{H}_{yu} \) and \( \tilde{H}_{eu} \) are \( \lambda_p \)-stable \( \forall \rho \in [1, \infty] \). \( \square \)

### 7.3 Procedure for Controller Design

The problem (SP) is solved by obtaining a controller \( \hat{C} \) with the following procedure:

**Procedure 7.1**

**Data:** Plant \( \hat{P} \) with \( \rho_0 \)-l.r. \( (D, P_c, N_c) \); the polynomial \( \phi \in \mathbb{R}[z] \); the list of dominant closed-loop poles \( \Lambda \).

**Step 1.** Pick any \( d \in \mathbb{R}[z] \) monic such that

\[
\forall d = \exists \phi = q \quad \text{and} \quad d(z) \neq 0 \quad \forall z \in D(\rho_0)^C. \tag{7.23}
\]

**Comment.** (i) A simple choice of \( d \) is given by \( d(z) := z^q \).

(ii) \( \frac{\phi}{d} \in \mathcal{X}^\infty(\rho_0) \cap \mathbb{R}(z) \subset \mathcal{X}^\infty_1(\rho_0) \); furthermore, \( \phi \) and \( \frac{\phi}{d} \) have the same list of zeros.

**Step 2.** Pick \( \mathcal{D} \in \mathcal{T}_1(\rho_0)^{n_o \times n_0} \) corresponding to a matrix sequence in \( (\mathbb{R}^n_1)^{n_o \times n_0} \) such that

\[
\det \mathcal{D} \in \mathcal{X}_1^\infty(\rho_0) \tag{7.24}
\]

and such that the list of zeros of \( \det \mathcal{D} \) in \( D(\rho_0)^C \) is
\[ Z[\det \mathcal{D}; \mathcal{D}(\rho_0)^C] = \Lambda. \] (7.25)

Comment. In particular, we can choose \( \mathcal{D} \in \mathfrak{A}(\rho_0)^{n_0 \times n_0}. \)

Step 3. Observe that

\[ \bar{F} := \bar{p}d_{\phi} \in \mathfrak{B}(\rho_0)^{n_0 \times n_1} \] (7.26)

with a \( \rho_0 \)-l.r.

\[ (\mathcal{D}_\ell, \mathcal{N}_\ell) := (\mathcal{D}_{\rho_0 \phi \mathcal{D}}, \mathcal{N}_{\rho_0 \ell}). \] (7.27)

Using Corollary 3.2(\( \ell \)), find the six matrices with elements in \( \tilde{\mathfrak{X}}_{\rho_0} \),

corresponding to sequences in \( \mathbb{R}^n \), namely

\[ u_\ell, v_\ell; \mathcal{N}_\ell, \mathcal{D}_\ell, u_\ell, v_\ell \] (7.28)

such that

(i) \( (\mathcal{N}_\ell, \mathcal{D}_\ell) \) is a \( \rho_0 \)-r.r. of \( \bar{F} \) (7.29)

(ii) \[
\begin{bmatrix}
\mathcal{N}_\ell & \mathcal{D}_\ell \\
-\mathcal{N}_\ell & \mathcal{D}_\ell
\end{bmatrix}
= \begin{bmatrix}
\mathcal{I}_{n_1} & 0 \\
0 & \mathcal{I}_{n_0}
\end{bmatrix}.
\] (7.30)

Step 4. Solve, according to Lemma 7.1,

\[ \mathcal{N}_\ell X + \mathcal{D}_\ell Y = \mathcal{D} \] (7.31)

for \( X \) and \( Y \) by (i) picking \( N \in \tilde{\mathfrak{X}}_{\rho_0}^{n_1 \times n_0} \) corresponding to a sequence in \( \mathbb{R}^{n_1 \times n_0} \) such that \( (N, \mathcal{D}) \) are \( \rho_0 \)-r.c.; and (ii) setting

\[ -X := \mathcal{D}_\ell N - u_\ell \mathcal{D} \] (7.32)

\[ Y := \mathcal{N}_\ell N + v_\ell \mathcal{D}. \] (7.33)

Comment. By Corollary 3.1b(\( \lambda \)) the choice of \( N \) in (i) is equivalent to choosing \( N \) corresponding to a sequence in \( \mathbb{R}^{n_1 \times n_0} \) such that
Furthermore, \((X, Y)\) is a \(\rho_0\)-r.r. of \(X y^{-1} \in \bar{b}(\rho_0)^{n_x \times n_0}\) by Lemma 7.1.

**Step 5.** Set

\[
N_{cr} := X, \quad D_{cr} := \frac{y}{d}
\]

and

\[
\tilde{C} := N_{cr} D_{cr}^{-1}.
\]

Stop

**Theorem 7.1.** The controller \(\tilde{C}\) constructed in Procedure 7.1

(i) belongs to \(\bar{b}(\rho_0)^{n_x \times n_0}\) with \(\rho_0\)-r.r. \((N_{cr}, D_{cr})\), corresponding to a matrix sequence in \((\mathbb{R}^N)^{n_x \times n_0}\);

(ii) solves problem (SP).

**Remark 7.3.** It can be observed from the proof of Theorem 7.1 that the controller \(\tilde{C}\) constructed in Procedure 7.1 (see (7.35), (7.36)) has created blocking zeros [Fer 1] at every point in \(Z[\phi] = Z[\phi_u] \cup Z[\phi_w]\) for the transfer functions \(\tilde{H}_{es}u_s\) from \(\tilde{u}_s\) to \(\tilde{e}_s\) and for the transfer functions \(\tilde{H}_{es}w_p\) from \(\tilde{w}_p\) to \(\tilde{e}_s\).

**7.4 Example**

**Data.** The plant \(\bar{P} \in \bar{b}(\rho_0)^{1 \times 2}\), with \(\rho_0 := 0.55\), is given by

\[
\bar{P}(z) := \begin{bmatrix} 3 & 5 \\ z-1 & z-2 \end{bmatrix} \times \begin{bmatrix} 2 \times 1 & 2 \times 2 \\ z-1 & z-2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

(7.41)

which has a \(\rho_0\)-l.r. \((D_{p\ell}, N_{p\ell})\) described by

\[
D_{p\ell}(z) := (z-1)(z-2)/z^2
\]

and

\[
N_{p\ell}(z) := \begin{bmatrix} (z-2)(1z-8) & (z-1)(2z+(z-2)e+2z^{-1}) \\ z^3(2z-1) & \end{bmatrix}.
\]

The polynomial \(\phi\) and the list of dominant closed-loop poles \(\Lambda\) are
given by

\[ \phi(z) := z + 2 \quad (7.44) \]

\[ \Lambda := (0.6, -0.6) . \quad (7.45) \]

**Step 1.** Since \( q := \phi = 1 \), we pick \( d \in R[z] \) as

\[ d(z) := z . \quad (7.46) \]

**Step 2.** Choose \( D \in L_1^\infty(\rho_0) \) to have zeros at \( z = 0.6, -0.6 \), as

\[ D(z) := \frac{(z+0.6)(z-0.6)}{z^2} . \quad (7.47) \]

**Step 3.** After defining \( F := p|_{\Phi} G \in L(\rho_0)^{1 \times 2} \), we obtain a \( \rho_0 - 1.r. \) \( (D_{\Phi}, N_{\Phi}) \) for \( F \) described by

\[ D_{\Phi}(z) := D_{p \Phi}(z) \cdot \frac{\phi(z)}{d(z)} = \frac{(z-1)(z-2)(z+2)}{z^3} \quad (7.48) \]

and

\[ N_{\Phi}(z) := \frac{p_{\phi}(z)}{p_{\Phi}(z)} \quad (7.49) \]

which is given in (7.43). Next, we find the matrices \( N_{\Phi}, D_{\Phi}, U_{\Phi}, V_{\Phi} \) that satisfy (7.29) and (7.30) (note that we do not need explicit knowledge of \( U_{\Phi} \) and \( V_{\Phi} \) in our computation):

\[
N_{\Phi}(z) := \begin{bmatrix}
(3z-2)(z-1) & -(5z-2)(z-2)(e^{1+2z^{-1}} - 1) \\
6z^2(2z-1) & 6z^2(z+2)
\end{bmatrix}
\]

\[
\begin{bmatrix}
9z^4 - 2z^3 + 11z^2 - 16z + 4 + (7z-2)(z-1)(z-2)(e^{1+2z^{-1}} - 1) \\
z^4(2z-1)
\end{bmatrix}
\]

\[
D_{\Phi}(z) := \begin{bmatrix}
(3z-2)(z-1) & -(3z-1)(z-2)(z+2) \\
6z^2 & z^4
\end{bmatrix}
\]

\[
\begin{bmatrix}
(5z-2)(z-2) & (7z-2)(z-1)(z-2)(z+2) \\
6z^2 & z^4
\end{bmatrix}
\]

\[
\begin{bmatrix}
(3z-2)(z-1) & -(3z-1)(z-2)(z+2) \\
6z^2 & z^4
\end{bmatrix}
\]

\[
\begin{bmatrix}
(5z-2)(z-2) & (7z-2)(z-1)(z-2)(z+2) \\
6z^2 & z^4
\end{bmatrix}
\]

\[
\begin{bmatrix}
(3z-2)(z-1) & -(3z-1)(z-2)(z+2) \\
6z^2 & z^4
\end{bmatrix}
\]

\[
\begin{bmatrix}
(5z-2)(z-2) & (7z-2)(z-1)(z-2)(z+2) \\
6z^2 & z^4
\end{bmatrix}
\]
\[ u_\ell(z) := \begin{bmatrix} (3z-1)(z^2+4z-4) \\ 6z^3 \\ (7z-2)(z^2+4z-4) \\ 6z^3 \end{bmatrix} \] (7.52)

\[ v_\ell(z) := \frac{(12z^4-3z^3+11z^2-16z+4)}{6z^3(2z-1)} - \frac{(7z-2)(z^2+4z-4)}{6z^3(z+2)}(e^{1+2z^{-1}} - 1). \] (7.53)

Observe that these matrices are analytic in \( D(0.55)^C \), despite some denominator term \((z+2)\).

**Step 4.** We choose \( N \in \mathfrak{H}_{-1}(\rho_0)^{2 \times 1} \) by

\[ N(z) := \begin{bmatrix} (-0.42z^3-3.8z^2-1.08z+0.72) \\ z^3 \\ 0 \end{bmatrix}. \] (7.54)

Then we obtain a solution \((X,Y)\) of (7.31) by setting

\[ -X(z) := (\mathcal{D}_N - u_\ell \mathcal{D})(z) \]

\[ = \begin{bmatrix} (1.74z^2+1.7z-2.16) \\ 6z^2 \\ - (4.9z+12.04) \\ 6z \end{bmatrix} \] (7.55)

\[ Y(z) := (N_N + v_\ell \mathcal{D})(z) \]

\[ = \frac{(12z^2-4.26z-2.62)}{6z(2z-1)} - \frac{(4.9z+12.04)}{6z(z+2)}(e^{1+2z^{-1}} - 1). \] (7.56)

**Step 5.** By setting

\[ N_{cr} := X \] (7.57)

given in (7.55), and
we obtain a controller \( \tilde{C} := N_{cr}D^{-1}_{cr} \in b(\rho_0)^{2 \times 1} \) which has \((N_{cr},D_{cr})\) as a \( \rho_0 \)-r.r. Note that this controller \( \tilde{C} \) solves problem (SP) with data (7.41)-(7.45); in particular, it is easy to check that \( D_{cr} \) has a zero at \( z = -2 \) (thus creating a blocking zero for \( \bar{H}_{es}u_s \) and \( \bar{H}_{es}w_p \) at \( z = -2 \)), and

\[
\tilde{v}(z) := (D_{pe}D_{cr} + N_{pe}N_{cr})(z) = \frac{(z-0.6)(z+0.6)}{z^2}. \]
8. Decoupling Feedback Design with Square Stable Plant

8.1 Preliminary Result and Additional Notations

In this section, we study again the MIMO unity feedback system depicted in Fig. 6-1. However, we now assume that the given plant matrix transfer function \( \tilde{P} \) (with elements in \( \tilde{b}(\rho_0) \) for some \( \rho_0 \in ]0,1[ \)) is square and \( \lambda_p \)-stable \( \forall p \in [1,\infty] \), i.e.

\[
\tilde{P} \in \tilde{b}(\rho_0)^{m \times m} \cap \lambda_1^{m \times m} . \tag{8.1}
\]

Observe that if the originally given plant does not satisfy these assumptions, we can apply the Stabilization Procedure 7.1 of Section 7 and consider the resulting stable square closed-loop system as our new plant \( \tilde{P} \). For such \( \tilde{P} \) in Fig. 6-1, we propose a design method such that the transfer function \( \tilde{H}_{y_s u_s} \) from \( \tilde{u}_s \) to \( \tilde{y}_s \) is decoupled, with pole-zero assignment in each channel (subject to the constraint that every \( D(1)^c \)-zero of \( \tilde{P} \) must remain a zero of \( \tilde{H}_{y_s u_s} \), cf. continuous-time lumped-system analog in [Che 1]). The approach is based on the recent result obtained by Desoer and Chen [Des 5], which contains a refined stability theorem proposed by Zames [Zam 1].

In order to tie our description to the notations in [Des 5], we note that the algebra \( \mathcal{A} \) is here \( \tilde{b}(\rho_0)^{m \times m} \), and the radical \( \mathcal{A}_s \) of \( \mathcal{A} \) is \( \tilde{b}_s(\rho_0)^{m \times m} \), where

\[
\tilde{b}_s(\rho_0) := \{ \tilde{g} \in \tilde{b}(\rho_0) \mid \lim_{|z| \to \infty} \tilde{g}(z) = \tilde{g}(0) = 0 \} . \tag{8.2}
\]

Since we consider \( \lambda_p \)-stability for all \( p \in [1,\infty] \), we take the algebra \( \mathcal{B} \) of stable maps to be \( \lambda_1^{m \times m} \), and \( \mathcal{B}_s := \mathcal{A}_s \cap \mathcal{B} \) is hence given by \( \lambda_1^{m \times m} \), where
\[ L := \{ g \in \mathcal{L}_1 | \lim_{|z| \to \infty} \tilde{g}(z) = g(0) = 0 \} . \] (8.3)

(Note that while \( \tilde{b}_s(\rho_0) \) is a radical of \( \tilde{b}(\rho_0)^m \), \( \mathcal{L}_1 \) is not a radical of \( \mathcal{L}_1 \).) The super-ring \( \tilde{\mathcal{A}} \) of \( \mathcal{A} \) is defined as \( (\tilde{\mathcal{C}}_z^N)^{m \times m} \).

In the analysis, we need to extend the algebra \( \tilde{\mathcal{C}}_z^N \) to the field \( \tilde{\mathcal{C}}_z^Z := \{ h | h(z) = z^k \tilde{g}(z), \tilde{g} \in \tilde{\mathcal{C}}_z^N, k \in \mathbb{N} \} . \) (8.4)

Furthermore, we extend the definition of order to \( \tilde{\mathcal{C}}_z^Z \): for any nonzero \( \tilde{h} \in \tilde{\mathcal{C}}_z^Z \),

\[ \text{ord}(\tilde{h}) := k \text{ such that } \lim_{|z| \to \infty} z^k \tilde{h}(z) = \text{constant} \neq 0 , \] (8.5)

i.e. \( \text{ord}(\tilde{h}) \) picks out the first nonzero term of \( \tilde{h} \), e.g. \( \text{ord}(\tilde{h}) = -2 \) if \( \tilde{h}(z) = h_{-2}z^2 + h_1z^1 + h_0 + h_{-1}z^{-1} + \cdots \) with \( h_{-2} \neq 0 \). In addition, for \( \tilde{H} = (\tilde{h}_{ij}) \in (\tilde{\mathcal{C}}_z^Z)^{m \times m} \),

\[ \text{ord}_{c_j} [\tilde{H}] := \min_i \text{ord}[\tilde{h}_{ij}] . \] (8.6)

Let \( \rho_0 < 1 \) and consider the feedback system of Fig. 6-1 with \( \tilde{p} \in \tilde{b}(\rho_0)^{m \times m} \cap \mathcal{L}_1 \), \( \tilde{c} \in (\tilde{\mathcal{C}}_z^N)^{m \times m} \), and \( \tilde{H}_{yu} \in \tilde{b}(\rho_0)^{2m \times 2m} \) as defined in (6.9) and rewritten here

\[ \tilde{H}_{yu} = \begin{bmatrix} \tilde{c}(I+\tilde{p}\tilde{c})^{-1} & -\tilde{c}\tilde{p}(I+\tilde{c}\tilde{p})^{-1} \\ -\tilde{c}(I+\tilde{p}\tilde{c})^{-1} & \tilde{p}(I+\tilde{c}\tilde{p})^{-1} \end{bmatrix} . \] (8.7)

By defining the transfer function from \( \tilde{u}_s \) to \( \tilde{y}_c \) as

\[ \tilde{Q} := \tilde{H}_{yu}\tilde{u}_s = \tilde{c}(I+\tilde{p}\tilde{c})^{-1} \], (8.8)

\( \tilde{H}_{yu} \) in (8.7) can be rewritten as
\[ \tilde{H}_{yu} = \begin{bmatrix} \tilde{Q} & -\tilde{Q}\tilde{P} \\ \tilde{P}\tilde{Q} & \tilde{P}(I-\tilde{Q}\tilde{P}) \end{bmatrix} \]  

(8.9a)

and from (8.8),

\[ \tilde{C} = \tilde{Q}(I-\tilde{P}\tilde{Q})^{-1}. \]  

(8.9b)

We can now state a stability theorem analogous to [Des 5, Thm. 3.4].

**Theorem 8.1.** Consider the unity-feedback system of Fig. 6-1 with

\[ \rho_0 < 1, \quad \tilde{P}, \tilde{Q} \in \mathbb{B}(\rho_0)^{m\times m}, \quad \tilde{C} \in (\mathcal{C}_z^N)^{m\times m} \text{ and } \tilde{H}_{yu} \in \mathbb{B}(\rho_0)^{2m\times 2m}. \]

Under these conditions,

(i) if \[ \tilde{P} \in \mathcal{X}_l^{m\times m}, \]

then \[ \tilde{Q} \in \mathcal{X}_l^{m\times m} \Rightarrow \tilde{H}_{yu} \in \mathcal{X}_l^{2m\times 2m} \]  

(8.10)

and \[ \tilde{Q} \in \mathcal{X}_l^{m\times m} \Rightarrow \tilde{H}_{yu} \in \mathcal{X}_l^{2m\times 2m} \text{ and } \tilde{C} \in \mathbb{B}(\rho_0)^{m\times m}; \]  

(8.11)

(ii) if \[ \tilde{P} \in \mathcal{X}_l^{2m\times 2m}, \]

then \[ \tilde{Q} \in \mathcal{X}_l^{2m\times 2m} \Rightarrow \tilde{H}_{yu} \in \mathcal{X}_l^{2m\times 2m} \text{ and } \tilde{C} \in \mathbb{B}(\rho_0)^{m\times m}; \]  

(8.12)

and \[ \tilde{Q} \in \mathcal{X}_l^{2m\times 2m} \Rightarrow \tilde{H}_{yu} \in \mathcal{X}_l^{2m\times 2m} \text{ and } \tilde{C} \in \mathbb{B}(\rho_0)^{m\times m}. \]  

(8.13)

\[ \square \]

**Remark 8.1.** (i) Note that for \( \tilde{H}_{yu} \) to have elements in \( \mathcal{C}_z^N \), the transfer function \( \tilde{C} \in (\mathcal{C}_z^N)^{m\times m} \) has to satisfy

\[ \det[I + P(0)C(0)] \neq 0. \]

(ii) Based on the equivalence condition (8.14), we propose a design procedure to achieve decoupling and pole-zero assignment of the feedback system. Note that had \( \tilde{P} \) been the closed-loop system obtained through the stabilizing compensation of Section 7, it would satisfy
condition (8.13) in view of assumption (7.11). (Note that the \( \tilde{P} \) here and the \( \hat{P} \) in (7.11) are different.) Now, by (8.14), we have the following design capability.

Theorem 8.2. Suppose that we wish to design a unity-feedback system as shown in Fig. 6-1 with \( \tilde{P} \in \tilde{b}(\rho_0)^{m \times m} \cap \tilde{z}^{m \times m} \cap (\tilde{N})^{m \times m} \) and \( \tilde{H}_{yu} \in \tilde{b}(\rho_0)^{2m \times 2m} \). Then, for all \( \tilde{H}_{ys} \in \tilde{z}^{m \times m} \) such that \( \tilde{H}_{ys} = \tilde{P}\tilde{Q} \) for some \( \tilde{Q} \in \tilde{z}^{m \times m} \), there exists a \( \tilde{C} \in \tilde{b}(\rho_0)^{m \times m} \) for which

(i) the closed-loop system is \( \mathbb{L}_p \)-stable, \( \forall p \in [1, \infty] \), and

(ii) the transfer function from \( u_s \) to \( y_s \) is described by the specific \( \tilde{H}_{ys} \).

8.2 Procedure for Decoupling Feedback Design

Decoupling Problem (DP). Given a plant \( \tilde{P} \in \tilde{b}(\rho_0)^{m \times m} \cap \tilde{z}^{m \times m} \cap (\tilde{N})^{m \times m} \) such that \( \text{det} \tilde{P} \neq 0 \) in \( D(\rho_0)^c \), find a controller \( \tilde{C} \in \tilde{b}(\rho_0)^{m \times m} \cap (\tilde{N})^{m \times m} \) such that

(i) the closed-loop unity feedback system in Fig. 6-1 is \( \mathbb{L}_p \)-stable, \( \forall p \in [1, \infty] \);

(ii) the transfer function \( \tilde{H}_{ys} \) representing the I/O map from \( u_s \) to \( y_s \) is a decoupled, proper rational function matrix;

(iii) in each diagonal element of \( \tilde{H}_{ys} \), the poles and zeros (in addition to the \( D(1)^c \)-zeros imposed by \( \tilde{P} \), see [Che 1]) can be specified by the designer.

Procedure 8.1: Decoupling Feedback Design

Data. Plant \( \tilde{P} \in \tilde{b}(\rho_0)^{m \times m} \cap \tilde{z}^{m \times m} \cap (\tilde{N})^{m \times m} \), \( \text{det} \tilde{P} = 0 \) in \( D(\rho_0)^c \).

Step 1. Obtain a \( \rho_0 \)-r.r. of \( \tilde{P} \)

\[
\tilde{P} = \frac{N_{pr}D^{-1}}{pr} \tag{8.16}
\]
where \( N_{pt}, Pt \in \mathbb{C}_{\rho_0}^{m \times m} \).

**Step 2.** Calculate

\[
[\gamma_{ij}]_{m \times m} := N^{-1}_{pt} \in (\mathbb{C}_{\rho_0}^{m \times m}). \tag{8.17}
\]

**Step 3.** For \( j = 1, 2, \ldots, m \), choose a polynomial \( \hat{n}_j \in \mathbb{R}[z] \) of least degree such that for \( i = 1, 2, \ldots, m \),

\[
\gamma_{ij}(\cdot)\hat{n}_j(\cdot) \in \mathbb{C}_{\rho_0}^{Z/Z} \tag{8.18}
\]

is analytic in \( D(1)^c \). (Comment: If \( \bar{P} \) has no \( D(1)^c \)-zeros, then we can pick \( \hat{n}_j \equiv 1, \forall j \).)

**Step 4.** Choose polynomials \( n_j, d_j \in \mathbb{R}[z], j = 1, 2, \ldots, m \), in

\[
\hat{H}_{ys} := \text{diag} \left[ \frac{\hat{n}_1 n_1}{d_1}, \frac{\hat{n}_2 n_2}{d_2}, \ldots, \frac{\hat{n}_m n_m}{d_m} \right] \tag{8.19}
\]

such that for \( j = 1, 2, \ldots, m \),

(i) \( Z[d_j] \subset D(1) \tag{8.20} \)

(ii) the polynomial \( n_j \) can be chosen freely,

(iii) \( a[d_j] \geq a[n_j] + a[\hat{n}_j] - \text{ord}_c [\bar{P}^{-1}] \). \tag{8.21}

**Step 5.** Calculate the controller

\[
\tilde{C} := P_t N^{-1}_{pt} \text{diag} \left[ \frac{\hat{n}_1 n_1}{d_1-\hat{n}_1 n_1}, \frac{\hat{n}_2 n_2}{d_2-\hat{n}_2 n_2}, \ldots, \frac{\hat{n}_m n_m}{d_m-\hat{n}_m n_m} \right]. \tag{8.22}
\]

STOP.

**Theorem 8.3.** The controller \( \tilde{C} \) in (8.22) solves Problem (DP).

**Remark 8.2.** (i) Equation (8.22) shows that a "stable" controller is always possible: indeed, after the polynomials \( \hat{n}_j \) and \( n_j \) have been chosen, the polynomial \( d_j \) can always be found so that all zeros of \( d_n - \hat{n}_j n_j \) lie inside \( D(1) \).
(ii) Observe that it is not required that $\tilde{H}_{yu}$ of (8.7) be rational, but $\tilde{H}_{ysu_s}$ can be made to be rational as in (8.19).

(iii) This procedure is not a direct application of [Des 5, Alg. 4.2], because we are not restricting $n_j, \hat{n}_j, n_j, d_j$ to satisfy \( \tilde{z}_{1}(\rho_{0}) \) or \( \pi(\rho_{0}) \); instead, we choose $n_j, \hat{n}_j, n_j, d_j$ to be polynomials and we are only restricting $\hat{n}_j n_j / d_j$ to belong to \( \pi(1) \) (Note: it is easier to work with polynomial $n_j, \hat{n}_j, n_j, d_j$). A direct modification of [Des 5, Alg. 4.2] can be obtained by letting $\hat{n}_j, n_j \in \pi(\rho_{0})$ and $d_j \in \pi(\rho_{0})$, and by replacing (8.21) by

$$\text{ord}(\hat{n}_j n_j) \geq \text{ord}_{c_j} [\tilde{p}^{-1}] > 0.$$  \hspace{1cm} (8.23)

(iv) Since we are working with $d_j$ in \( \mathbb{R}[z] \) instead of in \( \pi(\rho_{0}) \), zeros of $d_j$ need not be restricted to $D(\rho_{0})^c$. In particular, if we put all these zeros at $z = 0$, then $\tilde{H}_{ysu_s}$ is a transfer function corresponding to an I/O map with finite settling time.

(v) Let $[\pi_{ij}]_{m \times m} := \tilde{p}^{-1} = D_{p} N^{-1} \in (\mathbb{Z}_2)^{m \times m}$. Since $\tilde{p} \in \mathbb{Z}_2^m$, hence $\det D_{p} (z) \neq 0, \forall z \in D(1)^c$; thus the term in (8.18), $\gamma_{ij}(\cdot) \hat{n}_j(\cdot)$, is analytic in $D(1)^c$ if and only if $\pi_{ij}(\cdot) \hat{n}_j(\cdot)$ is analytic in $D(1)^c$. \( \square \)
9. Concluding Discussion

In view of the need for a general theory to cover sampled-data systems obtained by sampling continuous-time linear time-invariant distributed systems, we have developed in Section 2 the algebra $b(p_0)$ which described a large class of such discrete-time systems. In contrast to the continuous-time distributed case which is plagued by difficult fine points of analysis, the discrete-time case can be treated by more straightforward methods: in particular, for any $\tilde{g} \in \tilde{b}(p_0)$, there is some $\rho \geq p_0$ such that $\tilde{g}$ is analytic and bounded in $|z| \geq \rho$, moreover $\tilde{g}(z)$ has a well-defined limit as $|z| \to \infty$. Such nice behavior at infinity is usually absent in transfer functions of continuous-time distributed systems (consider $\hat{g}(s) = e^{-sT}$). Consequently, this paper is essentially self-contained.

The model of system description in this paper, with transfer functions in $\tilde{B}(p_0)$, is far more general than the model with rational transfer functions (as demonstrated by the examples in Section 2.6): indeed, the algebra $\tilde{B}(p_0)$ includes, as a subalgebra, all the proper rational functions in $z$.

By generalizing in Section 3 the concept of matrix fraction representation to systems with $\tilde{B}(p_0)$-matrix transfer functions, we studied the dynamic interpretations of poles and transmission zeros for MIMO systems in Section 4. As in the rational case, each pole of $\tilde{B}(p_0)$-transfer functions can be activated individually by some appropriately chosen input signals (see Thm. 4.4). In contrast to the transmission zeros of the rational transfer function case, a transmission zero of a $\tilde{B}(p_0)$-transfer function cannot completely block out the corresponding
exponential input-signal, but it can make the output asymptotically "small" compared to the blocked exponential; hence transmission zeros of $\tilde{b}(\rho_0)$-transfer functions still pose the same kind of nuisance on the tracking problem as those in the rational transfer function case (see e.g. [Des 4], [McF 1], [Dav 1]). Note that transmission zeros of $\tilde{b}(\rho_0)$-transfer functions also impose other limitations on the design of feedback systems, parallel to the rational case: Consider a feedback system with unity feedback, some zeros of the closed-loop characteristic function approach the open-loop transmission zeros under high gain; hence high loop gains lead to instability when there are $D(1)^c$-zeros in the plant transfer function (see discussions in [McF 1], [Dav 1] about similar behavior for continuous-time rational case). In addition, for a rather general feedback system defined as in Fig. 9-1, if $z \in D(1)^c$ is a transmission zero of the plant transfer function $\tilde{p}$, then under reasonable assumptions, for any controller transfer function $\tilde{c}$ and any feedback transfer function $\tilde{f}$ such that the closed-loop system is $\ell_p$-stable $\forall p \in [1, \infty]$ (as defined in Section 5), the closed-loop transfer function $\tilde{H}_{yu}$ from input $\tilde{u}$ to output $\tilde{y}$ will have a transmission zero at $z$; thus the $D(1)^c$-transmission zeros of $\tilde{p}$ impose some fundamental limitations on the achievable closed-loop transfer function $\tilde{H}_{yu}$ (see [Che 1]). However, even though these transmission zeros cannot be removed by appropriate compensation, sometimes they can be relocated by judicious redesign of the actuators and/or sensors of the physical system [McF 1].

As for the analysis of interconnected systems using the notion of characteristic functions as described in Section 5, it is stressed that this method of analysis can be applied to any interconnection, as long
as the well-posedness condition (i.e. \( \lim_{|z| \to \infty} \det D(z) \neq 0 \)) is satisfied.

Considering MIMO feedback systems, we studied in Section 6 the problem of closed-loop stability and in Section 7 the problem of designing a robust controller to achieve stabilization, tracking, and disturbance rejection. However, we have yet to investigate the possibility of designing controllers with proper rational transfer functions that can satisfy the same or relaxed specifications. We stress that if the design procedure 7.1 is applied to systems with rational transfer functions, then the controller is guaranteed to be proper, and arbitrary "dominant" closed-loop eigenvalue assignment is achieved.

When the given plant is square and stable, we have in Section 8 a procedure to design a feedback system so that the transfer function from the reference input to the plant output is decoupled (or, equally practicable, assigned a specific structure to satisfy other specifications), with arbitrary pole and zero assignment outside \( D(\rho_0) \) (subject to, of course, the \( D(1)^C \)-zeros of the plant).

Hence, by combining the results of Sections 7 and 8, we conclude that, given any plant \( \tilde{P} \in \tilde{b}(\rho_0)^{N_0 \times N_1} \) that satisfies certain reasonable assumptions, we can design a feedback system with an inner loop to stabilize the plant (as in Section 7), and an outer loop to bring the overall system to satisfy certain specifications, e.g. decoupling (as in Section 8). We are of course aware that there are many important issues in control system design that are not addressed by the above methods.

Finally, we should point out that most of the results in this paper also apply to the continuous-time and lumped cases, by observing the similar algebraic structures of the different cases.
Appendix A: Proofs of Properties of $\ell_1(\rho_0), \ell_1^-(\rho_0)$

Proof of (2.2.1). $\ell_1(\rho_0)$ is defined by (2.4): it follows that $\ell_1(\rho_0)$ is a normed space over the field $\mathbb{C}$ with the usual definitions of addition for sequences, multiplication by scalars in $\mathbb{C}$, and a norm $\| \cdot \|_{\rho_0} : \ell_1(\rho_0) \to \mathbb{R}_+$ defined by

$$\|g\|_{\rho_0} := \sum_{k=0}^{\infty} |g(k)|\rho_0^{-k}$$ \hspace{1cm} (A.1)

By definition, $g = (g(k))_{k=0}^\infty \in (\ell_1(\rho_0),\| \cdot \|_{\rho_0})$ if and only if $\gamma = (\gamma(k))_{k=0}^\infty \in (\ell_1^-,\| \cdot \|_1)$, where $\gamma(k) := g(k)\rho_0^{-k}$ $\forall k \in \mathbb{N}$, and $|\gamma|_1 := \|\gamma\|_{\rho_0} = 1 = \sum_{k=0}^{\infty} |\gamma(k)|$ is the usual norm defined on $\ell_1$. This defines an isomorphism of $\ell_1(\rho_0)$ onto $\ell_1$ with $\|g\|_{\rho_0} = |\gamma|_1$. Hence $(\ell_1(\rho_0),\| \cdot \|_{\rho_0})$ is a Banach space, since $(\ell_1,| \cdot |_1)$ is a Banach space [Die 2, Thm. 13.11.4 (using the counting measure)]. $\ell_1(\rho_0)$ also forms a commutative ring, with a "multiplication" in $\ell_1(\rho_0)$ defined as the convolution, namely,

$$f \ast g := (\sum_{j=0}^{\infty} f(k-j)g(j))_{k=0}^\infty \text{ for } f, g \in \ell_1(\rho_0).$$ \hspace{1cm} (A.2)

Furthermore, the convolution satisfies the inequality

$$\|f \ast g\|_{\rho_0} = \sum_{k=0}^{\infty} \left| \left( \sum_{j=0}^{k} f(k-j)g(j) \right) \rho_0^{-k} \right| \leq \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \left| f(k-j) \rho_0^{-(k-j)} \right| \right) \left( \sum_{j=0}^{\infty} \left| g(j) \rho_0^{-j} \right| \right) = \|f\|_{\rho_0} \|g\|_{\rho_0}$$ \hspace{1cm} (A.3)

where the last equality follows from [Apo 1, Thm. 8.4.6]. Note that $\delta_0 := (1,0,0,...)$ is the neutral element of $\ell_1(\rho_0)$ under convolution:
indeed

\[ \delta_0 * g = g * \delta_0 = g \quad \forall g \in \mathcal{L}_1(\rho_0), \quad (A.4) \]

and \( \|\delta_0\|_{\rho_0} = 1 \). Hence \( \mathcal{L}_1(\rho_0) \) is a commutative convolution \( (A.5) \) complex Banach algebra with unit [Rud 2, pp.227-228].

**Proof of (2.2.2).** Follows immediately from (2.4) and the equivalence

\[ \rho_1 < \rho_0 \iff \rho_1^{-k} > \rho_0^{-k}, \quad \forall k \in \mathbb{N}^*. \]

**Proof of (2.2.3).** Given any two nonzero elements \( f = (f(m))_{m=0}^{\infty} \) and \( g = (g(n))_{n=0}^{\infty} \) in \( \mathbb{C}^\mathbb{N} \), let \( m_0 \) and \( n_0 \) be the least indices corresponding to a nonzero component of \( f \) and \( g \) respectively, then

\( h := f \ast g \) is nonzero because \( h(m_0+n_0) = f(m_0) \cdot g(n_0) \neq 0 \).

**Proof of (2.2.4).** (i) By assumption, \( g \in \mathcal{L}_1(\rho_0) \), then for \( |z| \geq \rho_0 \)

\[ |\bar{g}(z)| = \left| \sum_{k=0}^{\infty} g(k)z^{-k} \right| \leq \sum_{k=0}^{\infty} |g(k)||z|^{-k} \leq \sum_{k=0}^{\infty} |g(k)|\rho_0^{-k} = \|g\|_{\rho_0} \]

i.e. in \( |z| \geq \rho_0 \), the series defining \( \bar{g}(z) \) converges absolutely and is bounded by \( \|g\|_{\rho_0} \).

(ii) For any \( \epsilon > 0 \), the series defining \( \bar{g}(z) \) converges uniformly in \( |z| \geq \rho_0 + \epsilon \), hence \( \bar{g}(z) \) represents an analytic function in \( D(\rho_0 + \epsilon)^c \).

(iii) Consider the definition of \( \bar{g}(z) \), as \( |z| \to \infty \), \( \bar{g}(z) \to g(0) \).

In order to prove Property (2.2.6) we need the concept of complex homomorphism. Let \( \Delta \) denote the set of all complex homomorphisms mapping the Banach algebra \( \mathcal{L}_1(\rho_0) \) into \( \mathbb{C} \) [Rud 1, Ch. 9][Rud 2, Ch. 11].
The following lemma characterizes $\Delta$:

**Lemma A.1.** For $\phi: \ell_1(\rho_0) \to \mathbb{C}$,

\[
\phi \in \Delta \iff \text{Either (a) } \phi(f) = f(0), \forall f \in \ell_1(\rho_0) \quad (A.6)
\text{or (b) } \exists z \in D(\rho_0)^C \text{ such that } \phi(f) = \bar{f}(z), \forall f \in \ell_1(\rho_0). \quad (A.7)
\]

**Proof:** ($\Leftarrow$) By definition, $\phi: \ell_1(\rho_0) \to \mathbb{C}$ belongs to $\Delta$ if and only if $\phi(f*g) = \phi(f)\phi(g)$, $\phi(\alpha f + \beta g) = \alpha\phi(f) + \beta\phi(g)$, $\forall f, g \in \ell_1(\rho_0)$, $\forall \alpha, \beta \in \mathbb{C}$. By direct calculation, these requirements are satisfied for any $\phi$ specified by (A.6) or (A.7).

($\Rightarrow$) Let $\phi_0: f \mapsto f(0)$ be the complex homomorphism defined by (A.6). Let $\Delta_1 := \Delta \setminus \{\phi_0\}$. According to the definitions of $\delta_1$ and $\delta_k$,

\[
\delta_k = (\delta_1^*)^k = \delta_1^* \delta_1 \cdots \delta_1, \quad k \geq 1.
\]

Hence, for any $\phi \in \Delta_1$,

\[
\phi(\delta_k) = [\phi(\delta_1)]^k \quad \forall k \in \mathbb{N}. \quad (A.8)
\]

Now for any homomorphism $\phi$, $|\phi(g)| \leq \|g\|$ [Rud 1, Thm. 9.21], in particular

\[
|\phi(\delta_1)| \leq \|\delta_1\|\rho_0 = \rho_0^{-1}.
\]

Hence, for any $\phi \in \Delta_1$, there exists $z \in \mathbb{C}$ with $|z| \geq \rho_0$ such that

\[
z^{-1} = \phi(\delta_1). \quad (A.9)
\]

Now $\forall \phi \in \Delta_1$ and $\forall g \in \ell_1(\rho_0)$, we obtain successfully
\[ \phi(g) = \phi(\sum_{k=0}^{\infty} g(k)\delta_k) \]
\[ = \sum_{k=0}^{\infty} g(k)\phi(\delta_k) \text{ by linearity of homomorphism } \phi \]
\[ = \sum_{k=0}^{\infty} g(k)[\phi(\delta_1)]^k \text{ by (A.8).} \quad (A.10) \]

By using (A.9) in (A.10), we conclude that there exists \( z \in \mathcal{D}(\rho_0)^c \)
such that
\[ \phi(g) = \sum_{k=0}^{\infty} g(k)z^{-k} = \bar{g}(z). \]
\[ \square \]

**Proof of (2.2.6).** (2.7) \( \Rightarrow \) (2.8). This is obvious by noting that
\[ g(0) = \lim_{|z| \to \infty} \bar{g}(z). \]
\[ (2.6) \Rightarrow (2.8). \] By [Rud 2, Thm. 11.5(c)], \( g \in \mathcal{L}_1(\rho_0) \) is invertible in \( \mathcal{L}_1(\rho_0) \) iff \( \phi(g) \neq 0 \ \forall \phi \in \Delta. \) The result follows immediately in view of Lemma A.1. \( \square \)

**Proof of (2.2.7).** (2.10) \( \Rightarrow \) (2.11). By contraposition, suppose
\[ \inf_{|z| \geq \rho_0} |(\bar{f}(z),\bar{g}(z))| = 0. \] Then there exists a sequence \( (z_k)_{k=0}^{\infty} \) in \( \mathcal{D}(\rho_0)^c \) such that
\[ \lim_{k \to \infty} |(\bar{f}(z_k),\bar{g}(z_k))| = 0. \]

Hence, \( \forall u,v \in \mathcal{L}_1(\rho_0), \) which are necessarily bounded in \( \mathcal{D}(\rho_0)^c, \)
\[ \lim_{k \to \infty} (u\bar{f} + v\bar{g})(z_k) = 0. \]

Then (2.10) cannot hold.

(2.11) \( \Leftrightarrow \) (2.12). This is obvious by noting that \( (f(0),g(0)) = \lim_{|z| \to \infty} (\bar{f}(z),\bar{g}(z)). \)
Consider the ideal in \( \mathcal{L}_1(\rho_0) \) generated by \( f \) and \( g \)
\[
I := \{ h | h = uf + vg, u, v \in \mathcal{L}_1(\rho_0) \}.
\]
Either (i) \( I = \mathcal{L}_1(\rho_0) \), or (ii) \( I \not\subseteq \mathcal{L}_1(\rho_0) \). If (i) holds, \( \delta_0 \in I \) and we obtain (2.10). Otherwise, (ii) \( I \) is a proper ideal of \( \mathcal{L}_1(\rho_0) \), then by [Rud 2, Thm. 11.3(a)] \( I \) is contained in some maximal ideal \( M \not\subseteq \mathcal{L}_1(\rho_0) \). By [Rud 2, Thm. 11.5(a)], there exists \( \phi_M \in \Delta \) such that \( M = \phi_M^{-1}(0) \). Hence \( I \subseteq \phi_M^{-1}(0) \), i.e.
\[
\phi_M(h) = 0 \quad \forall h \in I.
\]
By Lemma A.1, either
(a) \( \phi_M = \phi_0 : f \mapsto f(0) \), then \( f, g \in I \) implies
\[
\phi_M(f) = f(0) = 0 \quad \text{and} \quad \phi_M(g) = g(0) = 0
\]
and so \( |(f(0), g(0))| = 0 \) which contradicts (2.12)(i);
or (b) if \( \phi_M \neq \phi_0 \), then there exists \( z' \in D(\rho_0)^c \) (where \( z' \) is specified by \( \phi_M \)) such that
\[
\phi_M(h) = \tilde{h}(z') \quad \forall h \in \mathcal{L}_1(\rho_0).
\]
With this particular \( z' \), since \( f, g \in I \),
\[
\phi_M(f) = \tilde{f}(z') = 0 \quad \text{and} \quad \phi_M(g) = \tilde{g}(z') = 0
\]
and so \( |(\tilde{f}(z'), \tilde{g}(z'))| = 0 \) which contradicts (2.12)(ii). Thus we conclude that \( I = \mathcal{L}_1(\rho_0) \) must hold.

Proof of (2.3.2). (i) and (ii) hold because \( \g \in \mathcal{L}_1(\rho_g) \) for some \( \rho_g < \rho_0 \).
(iii) Let \( \alpha \in \mathbb{N} \) be the least index corresponding to a nonzero component of \( g \). Then \( g(\alpha) \neq 0 \) and

\[
\tilde{g}(z) = \sum_{k=0}^{\infty} \frac{g(k)z^k}{g(\alpha)} = g(\alpha)z^{-\alpha}[1 + \sum_{k=0}^{\infty} \frac{g(\alpha+k)}{g(\alpha)} z^{-(k+1)}]. \tag{A.11}
\]

Since \( g \in \ell_1(\rho_0) \), hence \( (\frac{g(\alpha+k)}{g(\alpha)})_{k=0}^{\infty} \in \ell_1(\rho_0) \); thus there exists \( \rho \geq \rho_0 \) such that

\[
| \sum_{k=0}^{\infty} \frac{g(\alpha+k)}{g(\alpha)} z^{-(k+1)} | \leq |z|^{-\alpha} \sum_{k=0}^{\infty} \left| \frac{g(\alpha+k)}{g(\alpha)} \right| |z|^{-k} < 1 \quad \forall |z| \geq \rho.
\]

Hence by (A.11), \( \tilde{g}(z) \neq 0 \quad \forall |z| \geq \rho \); and so the zeros of \( \tilde{g}(\cdot) \) in \( D(\rho_0)^c \) are all inside the compact annulus \( \{z|\rho_0 \leq |z| \leq \rho\} \). Since the zeros of \( \tilde{g}(\cdot) \) are isolated in the region of analyticity, \( \tilde{g}(\cdot) \) can have at most a finite number of zeros in the compact set \( \{z|\rho_0 \leq |z| \leq \rho\} \), hence a finite number of zeros in \( D(\rho_0)^c \). \( \square \)

Before we prove Property (2.3.3), we consider next:

Lemma A.2. Let \( 0 \leq \rho_1 < \rho_0 \), and let \( f: D(\rho_1)^c \to \mathbb{R} \) be continuous at every point of \( S := \{z||z| = \rho_0\} \). If \( f(z) > 0 \quad \forall |z| = \rho_0 \), then \( \exists \rho_2 \in ]\rho_1,\rho_0[ \) such that

\[
f(z) > 0 \quad \forall |z| \in [\rho_2,\rho_0]. \tag*{\square}
\]

Proof. For the sake of contradiction, suppose that given any \( \rho_2 \in ]\rho_1,\rho_0[ \), \( \exists |z| \in [\rho_2,\rho_0] \) such that \( f(z) \leq 0 \). Hence we can construct a sequence \( (z_k)_{k=1}^{\infty} \) with \( |z_k| \in ]\rho_1,\rho_0[ \) such that \( |z_k| \to \rho_0 \) and \( f(z_k) \leq 0 \), \( k = 1,2,\ldots \). By compactness of the closed ball \( D(\rho_0) \), \( (z_k)_{k=1}^{\infty} \) must have a convergent subsequence specified by some index set \( K \), i.e. \( \exists K \subset \mathbb{N} \) such that \( z_k \to z^* \) for some
Furthermore, \( |z^*| = \lim_{k \to \infty} |z_k| = \rho_0 \Rightarrow z^* \in S \). Hence
\[ f(z_k) \leq 0, \quad k \in K \Rightarrow f(z^*) \leq 0, \]
and this contradicts the hypothesis.

Proof of (2.3.3). (2.16) \( \Leftrightarrow \) (2.17). This is obvious by noting that
\[ g(0) = \lim_{|z| \to \infty} g(z). \]

(2.15) \( \Rightarrow \) (2.16). \( g \in \mathcal{L}_1(\rho_0) \) implies that \( g \in \mathcal{L}_1(\rho_1) \) for some \( \rho_1 < \rho_0 \). It has an inverse in \( \mathcal{L}_1(\rho_0) \) implies that for some \( \rho_2 < \rho_0 \),
\[ \exists g^{-1} \in \mathcal{L}_1(\rho_2) \text{ with } g^{-1} = g^*g = \delta_0. \]
Hence, \( g \in \mathcal{L}_1(\rho_3) \) has an inverse in \( \mathcal{L}_1(\rho_3) \), where \( \rho_3 := \max(\rho_1, \rho_2) < \rho_0 \). By Inversion Theorem (2.2.6),
\[ \inf_{|z| > \rho_3} |g(z)| > 0. \]

(2.15) \( \Leftarrow \) (2.16). \( g \in \mathcal{L}_1(\rho_0) \) implies that \( \exists \rho_1 < \rho_0 \) such that \( g \in \mathcal{L}_1(\rho_1) \). Hence the map \( z \mapsto |\tilde{g}(z)| \) is defined for \( |z| > \rho_1 \) and is continuous at every point of \( S := \{z||z| = \rho_0\} \). By Lemma A.2,
\[ |\tilde{g}(z)| > 0 \text{ on } S \Rightarrow \exists p_2 \in ]\rho_1, \rho_0[ \text{ such that } |\tilde{g}(z)| > 0 \]
\[ \forall |z| \in [\rho_2, \rho_0]. \]
Hence \( \inf_{|z| \geq \rho_2} |\tilde{g}(z)| > 0. \) By Inversion Theorem (2.2.6),
g has an inverse in \( \mathcal{L}_1(\rho_2) \), hence in \( \mathcal{L}_1(\rho_0) \).

The next theorem is needed in some subsequent proofs.

Theorem A.1: Decomposition Theorem. Let \( \tilde{g} \in \tilde{\mathcal{L}}_1(\rho_0) \), and let \( p \in D(\rho_0)^c \). Under these conditions, \( \exists \rho_1, \rho_2 \) satisfying \( 0 < \rho_2 < \rho_1 < \rho_0 \) and such that
\[ \frac{\tilde{g}(z)}{z-p} = \frac{\tilde{g}(p) + \tilde{\zeta}(z)}{z-p} \quad \forall z \in D(\rho_2)^c \{p} \]
(A.15)
where both \( \tilde{\zeta} \) and \( z \mapsto z\tilde{\zeta}(z) \) belong to \( \tilde{\mathcal{L}}_1(\rho_1) \).
Proof. With \( \tilde{g} \in \tilde{L}_1(\rho_0) \), \( \exists \rho_2 < \rho_0 \) such that \( \tilde{g} \in \tilde{L}_1(\rho_2) \) and hence \( \tilde{g} \) is analytic in \( |z| > \rho_2 \) and bounded in \( |z| \geq \rho_2 \). Since \( \tilde{g} \) is analytic at \( p \) with \( |p| \geq \rho_0 > \rho_2 \), and since \( \tilde{g}(z) - \tilde{g}(p) \) has a zero at \( p \), we see that

\[
\zeta(z) := \frac{\tilde{g}(z) - \tilde{g}(p)}{z-p}
\]

is analytic at \( p \), hence it is analytic for \( |z| > \rho_2 \); note that \( \zeta(z) \to 0 \) as \( |z| \to \infty \). Hence, by Laurent expansion, we obtain

\[
\zeta(z) = \sum_{k=0}^{\infty} \zeta_k z^{-k} \quad \text{with} \quad \zeta(0) := \lim_{|z| \to \infty} \zeta(z) = 0
\]

which converges absolutely for \( |z| > \rho_2 \). Hence, \( \forall \rho_1 \in ]\rho_2, \rho_0] \), \( \sum_{k=0}^{\infty} |\zeta(k)| \rho_1^{-k} < \infty \), i.e. \( \zeta \in \tilde{L}_1(\rho_1) \) for any \( \rho_1 \in ]\rho_2, \rho_0] \). Since \( \zeta(0) = 0 \), we see that \( z \mapsto z\zeta(z) \in \tilde{L}_1(\rho_1) \) for the same \( \rho_1 \). \( \square \)

Remark. (i) The decomposition theorem (A.15) expresses \( \tilde{g}(z)/(z-p) \) as the sum of the principal part \( \tilde{g}(p)/(z-p) \) of the Laurent expansion at \( p \) of \( \tilde{g}(z)/(z-p) \), and of the remainder term \( \zeta \in \tilde{L}_1(\rho_1) \). Through repeated use of the decomposition theorem, for any \( \tilde{g} \in \tilde{L}_1(\rho_0) \), any \( p \in D(\rho_0)^c \), and any \( m \in \mathbb{N}^* \), the same conclusion holds for \( \tilde{g}(z)/(z-p)^m \), giving

\[
\tilde{g}(z)/(z-p)^m = \sum_{k=1}^{m} \frac{r_k}{(z-p)^k} + \zeta(z) \quad (A.16)
\]

where \( \zeta \in \tilde{L}_1(\rho_1) \) for some \( \rho_1 < \rho_0 \), and \( r_k \in \mathbb{C} \), \( k = 1,2,\ldots,m \).

Repeated application of this last result (A.16) proves the following:

For any \( \tilde{g} \in \tilde{L}_1(\rho_0) \), \( \nu \in \mathbb{N}^* \), and any \( p_\alpha \in D(\rho_0)^c \), \( m_\alpha \in \mathbb{N}^* \), \( \alpha = 1,2,\ldots,\nu \), \( \tilde{g}(z)/(z-p_\alpha)^{m_\alpha} \) can be expressed as the sum of the
principal parts of the Laurent expansions of \( \frac{g(z)}{\Pi (z-p_\alpha)^{m_\alpha}} \) at the \( p_\alpha \)'s, and of a remainder term \( \zeta \), i.e.

\[
\frac{g(z)}{\Pi (z-p_\alpha)^{m_\alpha}} = \sum_{\alpha=1}^{\nu} \sum_{k=1}^{m_\alpha} \frac{r_{\alpha k}}{(z-p_\alpha)^k} + \zeta(z), \tag{A.17}
\]

where \( \zeta \in \bar{I}_{1-}(\rho_1) \) for some \( \rho_1 < \rho_0 \), and \( r_{\alpha k} \in \mathbb{C}, \alpha = 1,2,\ldots,\nu, k = 1,2,\ldots,m_\alpha \).

(ii) If, in (A.15), \( g(p) = 0 \), then \( \tilde{f} := \frac{g(z)}{z-p} \in \bar{I}_{1-}(\rho_1) \).

Here \( f(0) = \lim_{|z| \to \infty} \tilde{f}(z) = 0 \). We can easily verify that, \( \forall a \in D(\rho_0) \),

\[
\tilde{f}(z)(z-a) = \frac{g(z)(z-a)}{z-p} \in \bar{I}_{1-}(\rho_1) \subset \bar{I}_{1-}(\rho_0). \tag{A.18}
\]

Thus \( g(z) \) can be expressed as a product of factors in \( \bar{I}_{1-}(\rho_0) \) given by

\[
g(z) = \left[\frac{(z-p)}{(z-a)}\right]\left[\frac{g(z)(z-a)}{z-p}\right]. \tag{A.19}
\]

**Corollary A.1a.** Let \( g \in \bar{I}_{1-}(\rho_0) \) and let \( p \in D(\rho_0)^C \).

(i) If \( a \in \mathbb{C} \) such that \( a \neq p \), then \( \exists 0 \leq \rho_1 < \rho_0 \) such that

\[
g(z)\frac{(z-a)}{(z-p)} = \tilde{g}(p)\frac{(p-a)}{(z-p)} + \zeta(z) \quad \forall z \in D(\rho_1)^C \setminus \{p\} \tag{A.20}
\]

where \( \zeta \in \bar{I}_{1-}(\rho_0) \).

(ii) If \( a \in D(\rho_0) \) and \( g \) has an \( m \)-th order zero at \( p \), then

\[
\frac{(z-p)}{(z-a)} \in \bar{I}_{1-}(\rho_0) \tag{A.21}
\]

and \( \tilde{g}(z)\frac{(z-a)}{(z-p)} \in \bar{I}_{1-}(\rho_0) \) and has an \( (m-1) \)-th order zero at \( p \). \( \square \)

**Proof.** (i) By Theorem A.1, multiplying equation (A.15) by \( (z-a) \), we see that \( \exists \rho_1, \rho_2 \) satisfying \( 0 \leq \rho_2 < \rho_1 < \rho_0 \) such that
\[ \tilde{g}(z) \frac{(z-a)}{(z-p)} = \tilde{g}(p) \frac{(z-a)}{(z-p)} + \tilde{e}_1(z)(z-a) \quad \forall z \in D(\rho_2) \setminus \{p\} \quad (A.22) \]

where \( \tilde{e}_1 \in \tilde{H}^\infty(\rho_1) \). Now since \( \frac{z-a}{z-p} = 1 + \frac{p-a}{z-p} \), we have, \( \forall z \in D(\rho_2) \setminus \{p\} \)

\[ \tilde{g}(z) \frac{(z-a)}{(z-p)} = \tilde{g}(p) \frac{(p-a)}{(z-p)} + \tilde{e}_1(z)(z-a) + \tilde{g}(p) . \quad (A.23) \]

Here, we define the last two terms of (A.23) as

\[ \zeta(z) := \tilde{e}_1(z)(z-a) + \tilde{g}(p) \]

\[ = \tilde{e}_1(z)(z-a) + \tilde{g}(z) - \tilde{e}_1(z) \frac{m}{z-p} \quad \text{using (A.22)} \]

\[ = \zeta_1(z)(p-a) + \tilde{g}(z) \in \tilde{H}^\infty(\rho_0) . \quad (A.24) \]

The proof is complete by defining any \( \rho_1 \in [\rho_2, \rho_0[ \).

(ii) This follows immediately from (i) and the fact that \( a \in D(\rho_0) \).

\[ \square \]

**Corollary A.1b.** Let \( \tilde{g} \in \tilde{H}^\infty(\rho_0) \) have zeros \( z_\alpha \in D(\rho_0) \) of multiplicities \( m_\alpha \), respectively for \( \alpha = 1, 2, \ldots, v \). Let \( m := \sum_\alpha m_\alpha \). Let \( a_\beta \in D(\rho_0) \), \( \beta = 1, 2, \ldots, m \). Under these conditions

(i) \[ \tilde{g} = \tilde{b} \tilde{c} \quad (A.25) \]

where \( \tilde{b} \in \tilde{H}^\infty(\rho_0) \), \( \tilde{c} \in \tilde{H}^\infty(\rho_0) \) are defined for some \( \rho_1 < \rho_0 \) by

\[ \tilde{b}(z) := \prod_\alpha^{\nu} (z-z_\alpha)^{m_\alpha} / \prod_\beta^{m} (z-a_\beta) \quad (A.26) \]

\[ \tilde{c}(z) := \tilde{g}(z) \prod_\beta^{m} (z-a_\beta) / \prod_\alpha^{\nu} (z-z_\alpha)^{m_\alpha} , \quad |z| \geq \rho_1 \quad (A.27) \]

and

\[ \tilde{c}(z_\alpha) \neq 0, \quad \alpha = 1, 2, \ldots, v . \quad (A.28) \]

(ii) If, in addition, \( \tilde{g} \in \tilde{H}^\infty(\rho_0) \) and \( z_\alpha, \alpha = 1, 2, \ldots, v \), are the only zeros of \( \tilde{g} \) in \( D(\rho_0) \), then \( \tilde{c} \) is an invertible element of \( \tilde{H}^\infty(\rho_0) \). \[ \square \]
Proof. (i) This follows immediately by repeated application of Corollary A.1a(ii).

(ii) If \( \tilde{g} \in \mathcal{L}_1^- (\rho_0) \), then

\[
\lim_{|z| \to \infty} \tilde{c}(z) = \lim_{|z| \to \infty} \tilde{g}(z)/\tilde{b}(z) = g(0) \neq 0 . \tag{A.29}
\]

Hence \( \tilde{c} \in \mathcal{L}_1^- (\rho_0) \). From (A.27), the only possible zeros of \( \tilde{c} \) in \( D(\rho_0)^c \) are the zeros of \( g \). Now that \( z_\alpha, \alpha = 1,2,\ldots,\nu \), are the only zeros of \( \tilde{g} \) in \( D(\rho_0)^c \), then in view of (A.28),

\[
\tilde{c}(z) \neq 0 \quad \forall z \in D(\rho_0)^c . \tag{A.30}
\]

Hence by Property (2.3.3), \( \tilde{c} \) is an invertible element of \( \mathcal{L}_1^- (\rho_0) \). \( \square \)

Corollary A.1c. Let \( \tilde{g} \in \mathcal{L}_1^- (\rho_0) \). Let \( \nu \in \mathbb{N}^* \) and \( p_\alpha \in D(\rho_0)^c \), \( m_\alpha \in \mathbb{N}^*, \alpha = 1,2,\ldots,\nu \). Let \( m := \sum_{\alpha=1}^{\nu} m_\alpha \), and \( a_\beta \in D(\rho_0)^c, \beta = 1,2,\ldots,m \). Under these conditions \( \exists \rho_1 \in [0,\rho_0] \) such that

\[
\tilde{g}(z) \prod_{\beta=1}^{\nu} (z-a_\beta)/\prod_{\alpha=1}^{\nu} (z-p_\alpha)^{m_\alpha} = \sum_{\alpha=1}^{\nu} \sum_{k=0}^{m_\alpha-1} r_{\alpha k}/(z-p_\alpha)^{m_\alpha-k} + \tilde{g}_p(z) \quad \forall |z| > \rho_1 \tag{A.31}
\]

where

(i) \( \tilde{g}_p \in \mathcal{L}_1^- (\rho_0) \)

(ii) for \( \alpha = 1,2,\ldots,\nu \), and \( k = 0,1,\ldots,m_\alpha-1 \), \( r_{\alpha k} \in \mathbb{C} \) is given by

\[
r_{\alpha k} = \frac{1}{k!} \left. \frac{d^k}{dz^k} \left[ \tilde{g}(z) \prod_{\beta=1}^{\nu} (z-a_\beta)/\prod_{i=1}^{\nu} (z-p_i)^{m_i} \right] \right|_{z=p_\alpha} \tag{A.32}
\]

(iii) for \( \alpha = 1,2,\ldots,\nu \)

\[
\tilde{g}(p_\alpha) \neq 0 \Rightarrow r_{\alpha0} \neq 0 . \tag{A.33}
\]
Proof: This is achieved through multiple applications of Corollary A.1a.

Proof of (2.3.4): \( \tilde{x}_1^{-}(\rho_0) \) is a Euclidean ring

Since \( \tilde{x}_1^{-}(\rho_0) \) is an integral domain (entire ring), it suffices to prove [Sig 1, p.132] [Her 1, p.143] that the gauge \( \gamma: \tilde{x}_1^{-}(\rho_0) \setminus \{0\} \to \mathbb{N} \) defined in (2.18) satisfies

(i) \( \gamma(\tilde{f}g) \geq \gamma(\tilde{f}) \quad \forall f, g \in \tilde{x}_1^{-}(\rho_0) \setminus \{0\} \) \hfill (A.35)

(ii) a Euclidean algorithm exists: \( \forall \tilde{f}, \tilde{g} \in \tilde{x}_1^{-}(\rho_0), \ \tilde{f} \neq 0, \exists \tilde{q}, \tilde{r} \in \tilde{x}_1^{-}(\rho_0) \) such that

\[ \tilde{g} = \tilde{q}\tilde{f} + \tilde{r} \] \hfill (A.36)

with either \( 0 \leq \gamma(\tilde{r}) < \gamma(\tilde{f}) \) or \( \tilde{r} = 0 \).

Observe that when \( \tilde{g} \neq 0, \ \text{ord}(\tilde{g}) \) is finite, and the last term of (2.18) is finite due to Property (2.3.2)(iii); hence the gauge \( \gamma \) in (2.18) is well-defined. Before we carry out the proof, we study the following with the gauge \( \gamma \) defined as in (2.18).

Fact A.1. For any nonzero \( \tilde{g} \in \tilde{x}_1^{-}(\rho_0) \), we can decompose

\[ \tilde{g} = \tilde{g}_u \tilde{g}_s \] \hfill (A.37)

such that

(i) \( \tilde{g}_u \in \mathbb{C}[z^{-1}] \) with \( \tilde{d}[\tilde{g}_u] = \gamma(\tilde{g}) \) and every zero \( z_0 \) of \( \tilde{g}_u \) satisfies \( |z_0^{-1}| \leq \rho_0 \) \hfill (A.38)

(ii) \( \tilde{g}_s \in \tilde{x}_1^{-}(\rho_0) \) is an invertible element of \( \tilde{x}_1^{-}(\rho_0) \). \hfill (A.39)

Proof of Fact A.1. Define \( \tilde{g}_0 \in \tilde{x}_1^{-}(\rho_0) \) by

\[ \tilde{g}(z) = z^{-\text{ord}(\tilde{g})} \tilde{g}_0(z) . \] \hfill (A.41)
Then the list of zeros of \( \tilde{g} \) (including multiplicities) and that of \( \tilde{g}_0 \) are identical. Let \( n_g \in \mathbb{C}[z] \) be a polynomial whose zeros are exactly all those of \( \tilde{g} \) in \( D(\rho_0)^c \), counting multiplicities. Then, by definition of \( n_g \) and by (2.18)

\[
\gamma(\tilde{g}) = \text{ord}(\tilde{g}) + \partial[n_g]. \tag{A.42}
\]

Note that \( n_g \) has no zero at \( z = 0 \), hence \( \frac{n_g(z)}{\partial[n_g]} \) is a polynomial in \( z^{-1} \) with \( z^{-1} \)-degree equal to \( \partial[n_g] \). Define \( \tilde{g}_s \in \tilde{L}_1^\infty(\rho_0) \) by

\[
\tilde{g}_0(z) = \frac{n_g(z)}{\partial[n_g]} \cdot \tilde{g}_s(z). \tag{A.43}
\]

Equivalently, \( \tilde{g}_s(z) := \tilde{g}_0(z) \frac{z}{n_g(z)} \), and by Corollary A.1b, \( \tilde{g}_s \in \tilde{L}_1^\infty(\rho_0) \) is an invertible element of \( \tilde{L}_1^\infty(\rho_0) \).

Define \( \tilde{g}_u \in \mathbb{C}[z^{-1}] \), a polynomial in \( z^{-1} \), by

\[
\tilde{g}_u(z) := z^{-\text{ord}(\tilde{g})} \cdot \frac{n_g(z)}{\partial[n_g]} . \tag{A.44}
\]

Then combining (A.41) and (A.43), we obtain

\[
\tilde{g} = \tilde{g}_u \cdot \tilde{g}_s
\]

and

\[
\partial[\tilde{g}_u] = \text{ord}(\tilde{g}) + \partial[n_g] = \gamma(\tilde{g}) \text{ by (A.42).} \tag{A.45}
\]

Procedure A.1: Euclidean Algorithm for \( \tilde{L}_1^\infty(\rho_0) \)

Given \( \tilde{f}, \tilde{g} \in \tilde{L}_1^\infty(\rho_0) \), \( \tilde{f} \neq 0 \).

Step 1. Decompose \( \tilde{f}, \tilde{g} \) into

\[
\tilde{f} = \tilde{f}_u \cdot \tilde{f}_s, \quad \tilde{g} = \tilde{g}_u \cdot \tilde{g}_s \tag{A.46}
\]

as in Fact A.1.
Step 2. Use the Euclidean algorithm in \( \mathbb{C}[z^{-1}] \) and find \( q_t, r_t \in \mathbb{C}[z^{-1}] \) such that
\[
\tilde{g}_u = q_t \tilde{r}_u + \tilde{r}_t \tag{A.47}
\]
where either \( \tilde{g}_u \tilde{r}_t < \tilde{g}_u \tilde{r}_u \) or \( \tilde{r}_t = 0 \).

Step 3. Define \( \tilde{q}, \tilde{r} \in \tilde{x}_1(\rho_0) \) by
\[
\tilde{q} := \tilde{q}_t \frac{\tilde{g}_s}{\tilde{r}_s} \tag{A.48}
\]
and
\[
\tilde{r} := \tilde{r}_t \tilde{g}_s \tag{A.49}
\]
Such \( \tilde{q}, \tilde{r} \) satisfy
\[
\tilde{g} = \tilde{q} \tilde{r} + \tilde{r} \tag{A.50}
\]
with either \( 0 \leq \gamma(\tilde{r}) < \gamma(\tilde{f}) \) or \( \tilde{r} = 0 \).

We now continue the proof of (2.3.4):

(i) \( \forall \) nonzero \( \tilde{f}, \tilde{g} \in \tilde{x}_1(\rho_0) \),
\[
\text{ord}(\tilde{f} \tilde{g}) = \text{ord}(\tilde{f}) + \text{ord}(\tilde{g}) \geq 0 \tag{A.51}
\]
and, by counting zeros of the analytic functions \( \tilde{f} \) and \( \tilde{g} \),
\[
\mathfrak{A}[n_{fg}] = \mathfrak{A}[n_f \cdot n_g] = \mathfrak{A}[n_f] + \mathfrak{A}[n_g] \tag{A.52}
\]
Hence
\[
\gamma(\tilde{f} \tilde{g}) = \text{ord}(\tilde{f} \tilde{g}) + \mathfrak{A}[n_{fg}]
= \text{ord}(\tilde{f}) + \text{ord}(\tilde{g}) + \mathfrak{A}[n_f] + \mathfrak{A}[n_g]
= \gamma(\tilde{f}) + \gamma(\tilde{g}) \geq \gamma(\tilde{f}) \tag{A.53}
\]

(ii) Next, we prove that the Euclidean algorithm gives the desired result: Step 1 follows from Fact A.1, and Step 2 is self-explanatory. For Step 3, since \( \tilde{g}_s, \tilde{r}_s \) are invertible elements of \( \tilde{x}_1(\rho_0) \) and \( \tilde{q}_t, \tilde{r}_t \in \mathbb{C}[z^{-1}] \subset \tilde{x}_1(\rho_0) \), hence \( \tilde{q}, \tilde{r} \) defined in
(A.48), (A.49) belong to \( \tilde{\lambda}_1(\rho_0) \). Multiplying (A.47) by \( \tilde{g}_s \), we get

\[
\tilde{g}_u \cdot \tilde{g}_s = (\tilde{g}_t \cdot \tilde{g}_s) \cdot \tilde{f}_t + \tilde{t} \cdot \tilde{g}_s \tag{A.54}
\]

which gives (A.50). Finally,

\[
\tilde{r} = 0 \text{ if } \tilde{r}_t = 0 \text{ or } \tilde{g}_s = 0 ;
\]

otherwise, by (A.47)

\[
0 \leq \gamma(\tilde{r}) \leq \tilde{3}[\tilde{r}_t] < \tilde{3}[\tilde{f}_u] = \gamma(\tilde{r}) . \tag{A.55}
\]

Proof of (2.3.6). \((2.19) \iff (2.20)\). (2.19) holds if and only if \( \delta_0 \) is a gcd(f, g), which holds if and only if (2.20) holds \([McL 1, \text{ Thm. 25, p.154}]\).

\((2.21) \iff (2.22)\). This is obvious by observing that

\[
(f(0), g(0)) = \lim_{|z| \to \infty} (\tilde{f}(z), \tilde{g}(z)).
\]

\((2.20) \Rightarrow (2.21)\). By definition, \( \exists u, v \in \ell_1(\rho_0) \) such that \( u \cdot f + v \cdot g = \delta_0 \). With \( f, g, u, v \in \ell_1(\rho_0) \), there exists \( \rho_1 < \rho_0 \) such that \( f, g, u, v \in \ell_1(\rho_1) \). Hence, \( f \) and \( g \) are coprime in \( \ell_1(\rho_1) \). By Property (2.2.7), \( \inf_{|z| \geq \rho_1} |(\tilde{f}(z), \tilde{g}(z))| > 0 \), and thus

\[
\inf_{|z| \geq \rho_0} |(\tilde{f}(z), \tilde{g}(z))| > \inf_{|z| \geq \rho_1} |(\tilde{f}(z), \tilde{g}(z))| > 0 .
\]

\((2.20) \iff (2.21)\). With \( f, g \in \ell_1(\rho_0) \), there exists \( \rho_1 < \rho_0 \) such that \( f, g \in \ell_1(\rho_1) \). Hence the map \( z \mapsto |(\tilde{f}(z), \tilde{g}(z))| \) is defined for \( |z| \geq \rho_1 \) and is continuous on \( S := \{z | |z| = \rho_0\} \). By Lemma A.2, \( |(\tilde{f}(z), \tilde{g}(z))| > 0 \) on \( S \) implies that \( \exists p_2 \in [\rho_1, \rho_0] \) such that \( |(\tilde{f}(z), \tilde{g}(z))| > 0 \ \forall |z| \in [p_2, \rho_0] \). Hence
\[ \inf_{|z| \geq \rho_2} |(\bar{f}(z), \bar{g}(z))| > 0. \] By Property (2.2.7), \( f, g \) are coprime in \( \lambda_1(\rho_2) \), i.e. \( \exists u, v \in \lambda_1(\rho_2) \subset \lambda_1(\rho_0) \) such that 

\[ u \cdot f + v \cdot g = \delta_0. \]

\[ \square \]
Appendix B: Proofs of Theorems and Lemmas

Proof of Lemma 2.1. By definition, \( g \in \tilde{b}(\rho_0) \) implies that
\[
\exists \bar{n} \in \tilde{\lambda}_1(\rho_0) \text{ and } \bar{d} \in \tilde{\lambda}_1(\rho_0) \text{ such that } g = \bar{n}/\bar{d}.
\]
If \((\bar{n}, \bar{d})\) are \(\rho_0\)-coprime, then they give a \(\rho_0\)-representation of \( g \). Otherwise, since \(\tilde{\lambda}_1(\rho_0)\) is a Euclidean ring, we can find a \(\gcd(\bar{n}, \bar{d})\) in \(\tilde{\lambda}_1(\rho_0)\): call it \(\bar{c}\). Define \( \bar{d} := \bar{d}/\bar{c} \) and \( \bar{n} := \bar{n}/\bar{c} \). By definition of the greatest common divisor \(\bar{c}\), \(\bar{d}\) and \(\bar{n}\) belong to \(\tilde{\lambda}_1(\rho_0)\) and are \(\rho_0\)-coprime. Furthermore, both \(\bar{d}, \bar{c} \in \tilde{\lambda}_1(\rho_0)\) because \(\bar{d} \in \tilde{\lambda}_1(\rho_0)\).

Proof of Lemma 2.2. \(\bar{n} \in \tilde{\lambda}_1(\rho_0)\) and \(\bar{d} \in \tilde{\lambda}_1(\rho_0)\) imply that \(\bar{n}, \bar{d}\) are both analytic in \(\text{D}(\rho_0)^{\mathbb{C}}\). Since \(\bar{n}, \bar{d}\) are \(\rho_0\)-coprime, \(\bar{n}\) and \(\bar{d}\) have no common zeros in \(\text{D}(\rho_0)^{\mathbb{C}}\). Hence statements (i) and (ii) of the lemma follow.

Proof of Theorem 2.1. The steps in Procedure 2.1 are justified in the following:

Step 1. The \(\rho_0\)-representation exists by Lemma 2.1.

Step 2. According to Property (2.3.2)(iii), \(\bar{d} \in \tilde{\lambda}_1(\rho_0)\) can have at most a finite number of zeros in \(\text{D}(\rho_0)^{\mathbb{C}}\).

Step 3. By Corollary A.1b, \(\bar{c} \in \tilde{\lambda}_1(\rho_0)\) is invertible.

Step 4. Definition 2.3 is satisfied by \((\bar{n}, \bar{d})\). In particular, since
\[
\inf \{|\bar{c}(z)| > 0 \text{ and } |\bar{c}(z)| \leq \|\bar{c}\|_{\rho_0} \forall z \in \text{D}(\rho_0)^{\mathbb{C}}, |z| \geq \rho_0
\]
\[
\text{rank}(\bar{n}(z), \bar{d}(z)) = \text{rank}(\bar{n}(z), \bar{d}(z)) \cdot \bar{c}(z)^{-1} = 1 \forall z \in \text{D}(\rho_0)^{\mathbb{C}}.
\]
Furthermore, by definition in Step 3, \(\bar{d} \in \tilde{\lambda}(\rho_0)\), \(\lim_{|z| \to \infty} \bar{d}(z) = 1\), and \(Z[\bar{d}] \subset \text{D}(\rho_0)^{\mathbb{C}}\). □
Proof of Theorem 2.2. (\(\Rightarrow\)) By assumption \(\tilde{g} \in \tilde{b}(\rho_0)\). If \(\tilde{g} \in \tilde{\lambda}_1(\rho_0)\), then (2.36) holds with \(\tilde{r} = 0\). Now suppose \(\tilde{g} \notin \tilde{\lambda}_1(\rho_0)\). By Theorem 2.1, \(\tilde{g}\) has a normalized \(\rho_0\)-representation \((\tilde{n}, \tilde{d})\) such that \(\tilde{n} \in \tilde{\lambda}_1(\rho_0)\) and \(\tilde{d} \in \tilde{r}^\infty(\rho_0)\). Since \(\tilde{g} \notin \tilde{\lambda}_1(\rho_0)\), then \(\tilde{d} \notin 1\) and hence for some \(v \in \mathbb{N}^*\), there are \(p_\alpha \in D(\rho_0)^C\) and \(m_\alpha \in \mathbb{N}^*\) where \(\alpha = 1, 2, \ldots, v\), and for \(m := \sum_{\alpha=1}^v m_\alpha\), there are \(a_\beta \in D(\rho_0)\), \(\beta = 1, 2, \ldots, m\), such that

\[
\tilde{d}(z) = \prod_{\alpha=1}^v \frac{(z-p_\alpha)^{m_\alpha}}{(z-a_\beta)}
\]

with \(\tilde{n}(p_\alpha) \neq 0\), \(\alpha = 1, 2, \ldots, v\).

Hence (2.36) with all its properties (2.37)-(2.39b) follows from Corollary A.1c, where \(\tilde{g}\) is replaced by \(\tilde{n}\) and where

\[
\tilde{n}(z) = \frac{\sum_{\alpha=1}^v \sum_{k=0}^{m_\alpha-1} r_\alpha k/(z-p_\alpha)^{m_\alpha-k}}{(z-a_\beta)} =: \tilde{r}(z).
\]

(\(\Leftarrow\)) Proof by construction, following Procedure 2.2: Step 1 is self-explanatory. The pair \((\tilde{n}, \tilde{d})\) generated by steps 2 and 3 satisfy the following:

(i) by (2.43), \(\tilde{d} \in \tilde{r}^\infty(\rho_0) \subset \tilde{\lambda}_1(\rho_0)\) with \(\lim_{|z| \to \infty} \tilde{d}(z) = 1\) and \(Z[\tilde{d}] \subset D(\rho_0)^C\); by (2.44), \(\tilde{n} \in \tilde{\lambda}_1(\rho_0)\).

(ii) \(\tilde{g} = \frac{n_r}{d_r} + \tilde{q}\) by (2.36) and (2.41)

\[= \frac{(n_r + \tilde{q}d_r)}{d_r}\]

\[= \frac{\tilde{n}}{\tilde{d}}\]

by (2.43) and (2.44)

(iii) Since \((n_r, d_r)\) are coprime polynomials, then from (2.44), \(\forall |z| \geq \rho_0\)

\[d_r(z) = 0 \Rightarrow \tilde{n}(z) = n_r(z)/z^v \neq 0\].
Hence
\[
\text{rank } \begin{bmatrix} \tilde{n}(z) \\ \tilde{d}(z) \end{bmatrix} = \text{rank } \begin{bmatrix} n_r(z) \\ d_r(z) \end{bmatrix} = 1 \quad \forall |z| \geq \rho_0,
\]
so
\[
|(\tilde{n}(z), \tilde{d}(z))| > 0 \quad \forall |z| \geq \rho_0,
\]
and by Remark 2.1, \((\tilde{n}, \tilde{d})\) are \(\rho_0\)-coprime. Therefore \((\tilde{n}, \tilde{d})\) is a normalized \(\rho_0\)-representation of \(\tilde{g}\).

**Proof of Theorem 2.3.** By Procedure 2.1 (normalization), we note that any \(\rho_0\)-representation \((\tilde{n}, \tilde{d})\) is equal to the product of its normalized form with an invertible element of \(\tilde{x}_1^-(\rho_0)\). Hence without loss of generality, we assume that both \((\tilde{n}, \tilde{d})\) and \((\tilde{n}, \tilde{d})\) are normalized. By Lemma 2.2, \(\tilde{g}\) has an \(m\)th order pole at \(p \in D(\rho_0)^c\) if and only if \(\tilde{d}\) (ditto for \(\tilde{d}\)) has an \(m\)th order zero at \(p \in D(\rho_0)^c\). Let \(d_r\) be given by the coprime factorization in (2.46). By (2.39b), \(\tilde{g}\) has an \(m\)th order pole at \(p \in D(\rho_0)^c\) if and only if \(d_r\) has an \(m\)th order zero at \(p \in D(\rho_0)^c\). Since \(\tilde{d}, \tilde{d} \in x(\rho_0)\) and \(d_r \in \mathbb{C}[z]\) have zeros only in \(D(\rho_0)^c\), they have zeros of the same order at the same locations in \(D(\rho_0)^c\), and nowhere else; furthermore, \(\lim_{|z| \to \infty} \tilde{d}(z) = \lim_{|z| \to \infty} \tilde{d}(z) = 1\), hence
\[
\tilde{d} = d_r/n_h, \quad \tilde{d} = d_r/d_h
\]
where \(n_h, d_h \in \mathbb{C}[z]\) are monic polynomials of the same degree as \(d_r\), and have zeros only in \(D(\rho_0)\). Hence \(\tilde{h} := n_h/d_h\) is an invertible element of \(\tilde{x}_1^-(\rho_0)\), and is rational. Then \(\tilde{h} = n_h/d_h = \tilde{d}/\tilde{d} = \tilde{n}/\tilde{n}\).

**Proof of Theorem 2.4.** \((\Rightarrow)\) By assumption, \(\exists \tilde{h} \in \tilde{b}(\rho_0)\) such that \(\tilde{g} \cdot \tilde{h} \equiv 1\) in \(D(\rho_0)^c\). Hence \(\lim_{|z| \to \infty} (\tilde{g} \cdot \tilde{h})(z) = 1\). Since
\[
\lim_{|z| \to \infty} |\tilde{h}(z)| = |h(0)| < \infty,
\]

then \[ g(0) = \lim_{|z| \to \infty} |g(z)| = h(0)^{-1} \neq 0. \]

(\Rightarrow) Let \((\tilde{n}, \tilde{d})\) be a \(\rho_0^\ast\)-representation of \(\tilde{g}\). Then by assumption,
\[
g(0) = \lim_{|z| \to \infty} \tilde{g}(z) = \lim_{|z| \to \infty} \tilde{n}(z)/\tilde{d}(z) = n(0)/d(0) \neq 0.
\]
Furthermore,
\[
d(0) = \lim_{|z| \to \infty} \tilde{d}(z) \neq 0
\]
because \(\tilde{d} \in \tilde{\mathbb{E}}_1^\ast(\rho_0)\). Hence \(\lim_{|z| \to \infty} \tilde{n}(z) = n(0) = g(0)d(0) \neq 0\). Thus \(\tilde{n} \in \tilde{\mathbb{E}}_1^\ast(\rho_0)\) and \(\tilde{g}^{-1} := \tilde{d}/\tilde{n}\) belongs to \(\tilde{b}(\rho_0)\) and is the inverse of \(\tilde{g}\) in \(\tilde{b}(\rho_0)\). \(\square\)

Proof of Lemma 3.1. This is immediate by Cramer's rule and Property (2.2.6) (respectively Property (2.3.3)). \(\square\)

Proof of Lemma 3.2. This is immediate by Cramer's rule and Theorem 2.4. \(\square\)

Proof of Lemma 3.3. This is immediate by applying Theorem 2.2 to every element of \(\tilde{G}\), so that for \(i = 1,2,...,n_0, j = 1,2,...,n_i,\)
\[ \tilde{g}_{ij} = \tilde{r}_{ij} + \tilde{q}_{ij} \]
where \(\tilde{g}_{ij}, \tilde{r}_{ij}, \tilde{q}_{ij}\) satisfy Theorem 2.2. \(\square\)

Proof of Lemma 3.4. \((N_\alpha, D_\alpha)\) are \(\rho_0^\ast\)-r.c. if and only if \(I_{n_i}\) is a g.c.r.d. of \((N_\alpha, D_\alpha)\), which holds if and only if (3.4) holds [McD 1, p.35]. This proves Lemma 3.4(\(\alpha\)). The proof of Lemma 3.4(\(\ell\)) is similar. \(\square\)

Proof of Theorem 3.1.

Case 1. If \(\tilde{G} \in \tilde{\mathbb{E}}_{1}^\ast(\rho_0)^{n_0 \times n_i}\), the theorem is immediately verified by choosing
\[
N_\alpha := \tilde{G}; \ D_\alpha := I_{n_i}; \ U_\alpha := 0; \ V_\alpha := I_{n_i};
\]
\[
N_\ell := \tilde{G}; \ D_\ell := I_{n_0}; \ U_\ell := 0; \ V_\ell := I_{n_0};
\]
Case 2. If $\tilde{G} \notin \tilde{L}_1(\rho_0)$, we use Procedure 3.1 to find the eight matrices that satisfy the theorem.

Step 1 is self-explanatory.

Step 2. Since all elements $r_{ij}$ of $\tilde{R}$ in (3.8) are elements of $\mathbb{C}_p(z)$ and have poles only in $D(\rho_0)^c$, they admit a rational $\rho_0$-representation $(n_{ij}, d_{ij})$ with $n_{ij} \in r(\rho_0)$ and $d_{ij} \in r^\infty(\rho_0)$ such that $(n_{ij}, d_{ij})$ are $\rho_0$-coprime, with respect to $r(\rho_0)$. It is then possible to construct a least common multiple $d_j \in r^\infty(\rho_0)$ of all denominators $d_{ij} \in r^\infty(\rho_0)$ of column $j$ [McL 1, Ch. IV, §10]. Hence

$$\tilde{\nu}_r := \frac{[n_{ij} d_j/d_{ij}]}{r(\rho_0)^{n_{ij} x n_j}} \quad \text{and} \quad \tilde{\theta}_r := \text{diag}(d_j)_{j=1}^{n_{ij} x n_i}$$

satisfy the conditions of Step 2.

Step 3. $\tilde{M}$ is full rank because $\text{det} \tilde{\theta}_r \in r^\infty(\rho_0)$ and thus $\text{det} \tilde{D}_r$ is not the zero element of $r(\rho_0)$. The rest is self-explanatory.

Step 4. Comment (i) relating to Step 4 holds as follows: Observe that all matrices in (3.10)-(3.12) have elements in $r(\rho_0) \subset \tilde{L}_1(\rho_0)$ with $\text{det} \tilde{\theta}_r$, $\text{det} \tilde{R} \in r^\infty(\rho_0)$. Moreover, from $\tilde{M} = \omega^{-1}\begin{bmatrix} -\tilde{R} \\ 0 \end{bmatrix}$,

$$\tilde{\theta}_r = \tilde{D}_r \tilde{R}, \quad \tilde{\nu}_r = \tilde{\nu}_r \tilde{R}$$

hence $\tilde{R} = \tilde{\nu}_r \tilde{D}_r^{-1} \tilde{R}$ with $\text{det} \tilde{\theta}_r \in r^\infty(\rho_0)$

and

$$\tilde{\nu}_r \tilde{D}_r + \tilde{\nu}_r \tilde{\nu}_r \equiv I_{n_i}.$$  

Hence $(\tilde{\nu}_r, \tilde{D}_r)$ is a $\rho_0$-r.r. of $\tilde{R}$, with $\tilde{R}$ a g.c.r.d. of $\tilde{\nu}_r$ and $\tilde{D}_r$. From $\omega \omega^{-1} = I$, we also get

$$\tilde{R} = \tilde{\nu}_r \tilde{D}_r^{-1} = \tilde{D}_r^{-1} \tilde{\nu}_r$$

and

$$\tilde{\nu}_r \tilde{\nu}_r + \tilde{D}_r \tilde{\nu}_r \equiv I_{n_0}.$$
Since $\tilde{W}$ is invertible in $\mathcal{R}(\rho_0)^{(n_i+n_0) \times (n_i+n_0)}$, det $\tilde{W}$ tends to a nonzero complex constant as $|z| \to \infty$. From (3.11), the matrix

$$[-\tilde{N}_\ell | \tilde{D}_\ell] = \tilde{D}_\ell [-\tilde{R} | I_{n_0}]$$

when evaluated at $z = \infty$, is a full rank matrix; hence det $\tilde{D}_\ell \in \mathcal{R}(\rho_0)$ tends to a nonzero constant at infinity. Thus $(\tilde{D}_\ell, \tilde{N}_\ell)$ is a $\rho_0$-r.r. of $\tilde{R}$.

Step 5 is self-explanatory. Comment (ii) regarding Step 5 can be verified by using (3.13) and a little computation like the preceding comment.

Proof of Corollary 3.1b. We present here the proof for Corollary 3.1b($\kappa$). The proof for Corollary 3.1b($\ell$) is similar and is omitted.

$(\Rightarrow)$ By assumption, $(N^\kappa, D^\kappa)$ are $\rho_0$-r.c., hence by (3.4)

$$\begin{bmatrix} V^\kappa & U^\kappa \end{bmatrix} \begin{bmatrix} D^\kappa \\ N^\kappa \end{bmatrix} (z) = I_{n_i} \forall z \in D(\rho_0)^c,$$

which implies

$$\text{rank} \begin{bmatrix} V^\kappa & N^\kappa \end{bmatrix} \begin{bmatrix} D^\kappa \\ N^\kappa \end{bmatrix} (z) = n_i \forall z \in D(\rho_0)^c.$$ 

Hence, by Sylvester's inequality,

$$\text{rank} \begin{bmatrix} D^\kappa \\ N^\kappa \end{bmatrix} (z) \geq n_i \forall z \in D(\rho_0)^c.$$ 

Equality holds because the matrix $[D^\kappa | N^\kappa]^T$ has only $n_i$ columns.

$(\Leftarrow)$ By Cramer's rule, $\tilde{G} := N^\kappa D^\kappa^{-1}$ belongs to $\mathcal{R}(\rho_0)^{n_0 \times n_i}$.

Hence by Theorem 3.1, $\tilde{G}$ admits a $\rho_0$-r.r. $(\tilde{N}_\kappa, \tilde{D}_\kappa)$, i.e.

$$\tilde{G} = \tilde{N}_\kappa \tilde{D}_\kappa^{-1}.$$
where \( \tilde{N}_r \in \tilde{X}_1(\rho_0)^{n_i \times n_0} \), \( \tilde{D}_r \in \tilde{X}_1(\rho_0)^{n_i \times n_i} \), \( \det \tilde{D}_r \in \tilde{X}_1^\infty(\rho_0) \), and there exist \( \tilde{u}_r \in \tilde{X}_1(\rho_0)^{n_i \times n_0} \), \( \tilde{v}_r \in \tilde{X}_1(\rho_0)^{n_i \times n_i} \) such that

\[
\tilde{u}_r \tilde{N}_r + \tilde{v}_r \tilde{D}_r = I_{n_i}.
\] (B.20)

Define \( R := \tilde{D}_r^{-1} \tilde{D}_r \). Then \( \mathcal{D}_r = \tilde{D}_r R \) and

\[
N_r = N_r \tilde{D}_r^{-1} \tilde{D}_r = \tilde{N}_r \tilde{D}_r^{-1} \tilde{D}_r = \tilde{N}_r R.
\]

Using these identities in (B.20), we obtain

\[
\tilde{u}_r N_r + \tilde{v}_r D_r = R.
\] (B.21)

Hence \( R \in \tilde{X}_1(\rho_0)^{n_i \times n_i} \), and so

\[
det R = \frac{\det D_r}{\det \tilde{D}_r} \in \tilde{X}_1^\infty(\rho_0).
\] (B.22)

By assumption,

\[
n_i = \text{rank} \begin{bmatrix} D_r(z) \\ N_r(z) \end{bmatrix} = \text{rank} \begin{bmatrix} \tilde{D}_r(z) \\ \tilde{N}_r(z) \end{bmatrix} R(z) \quad \forall z \in D(\rho_0)^c.
\]

Hence, by invoking Sylvester's inequality, \( \text{rank} R(z) = n_i \quad \forall z \in D(\rho_0)^c \), i.e.

\[
det R(z) \neq 0 \quad \forall z \in D(\rho_0)^c.
\] (B.23)

Invoking Lemma 3.1, (B.22) and (B.23) together imply that \( R \) is an invertible element of \( \tilde{X}_1(\rho_0)^{n_i \times n_i} \). Thus,

\[
U_r := R^{-1} \tilde{u}_r \in \tilde{X}_1(\rho_0)^{n_i \times n_0}
\] (B.24)

and

\[
V_r := R^{-1} \tilde{v}_r \in \tilde{X}_1(\rho_0)^{n_i \times n_i}.
\] (B.25)

Premultiply (B.21) by \( R^{-1} \) and using (B.24), (B.25), we obtain
\[ U_\nu N_\nu + V_\nu D_\nu = I_{n_1}, \]

i.e. \((N_\nu, D_\nu)\) is \(\rho_0\)-r.c.

\[ \square \]

Proof of Procedure 3.2(a). Steps 1, 2 and 3 are self-explanatory. To prove the comment in Step 4, we need to show that \(\tilde{R} = \tilde{N}_\nu \tilde{D}_\nu^{-1}\) with

(a) \(\tilde{N}_\nu \in \mathcal{R}(\rho_0)^{n_0 \times n_i}\), \(\tilde{D}_\nu \in \mathcal{R}(\rho_0)^{n_1 \times n_i}\); 
(b) \(\det \tilde{D}_\nu \in \mathcal{R}^\infty(\rho_0)\); 
(c) \((\tilde{N}_\nu, \tilde{D}_\nu)\) are \(\rho_0\)-r.c. The fact that \(\tilde{R} = \tilde{N}_\nu \tilde{D}_\nu^{-1}\) is obvious from the procedure. Now consider:

(a) By Lemma 3.3, \(\tilde{R}\) is strictly proper. Hence for 
\[ i = 1, 2, \ldots, n_i \]
\[ \mathcal{A}_c [N_\nu] \prec \mathcal{A}_c [D_\nu] = \gamma_i = \mathcal{A}_c [S]. \]  

\[ \text{(B.26)} \]

Since \(S\) is diagonal, it is column-reduced. Hence, using (B.26),
\[ \tilde{N}_\nu := N_\nu S^{-1} \in \mathcal{G}(z)^{n_0 \times n_i} \]  

is strictly proper, and \(\tilde{D}_\nu := D_\nu S^{-1} \in \mathcal{G}(z)^{n_1 \times n_i}\) is proper. Furthermore, by construction, all poles of \(\tilde{N}_\nu\) and \(\tilde{D}_\nu\) are at \(z = 0\), hence \(\tilde{N}_\nu \in \mathcal{R}(\rho_0)^{n_0 \times n_i}\) and \(\tilde{D}_\nu \in \mathcal{R}(\rho_0)^{n_1 \times n_i}\).

(b) By (3.21), \(D_\nu \in \mathbb{G}[s]^{n_1 \times n_i}\) is column-reduced, hence
\[ \mathcal{A}[\det D_\nu] = \sum_{i=1}^{n_1} \gamma_i = \mathcal{A}[\det S]. \]

Therefore \(\det \tilde{D}_\nu = \det D_\nu / \det S\) belongs to \(\mathcal{R}^\infty(\rho_0)\).

(c) Now \((N_\nu, D_\nu)\) being r.c. implies
\[ \text{rank} \left[ \begin{array}{c} D_\nu \\ N_\nu \end{array} \right] (z) = n_1 \quad \forall z \in \mathbb{C}. \]

Hence
\[ \text{rank} \left[ \begin{array}{c} D_\nu \\ N_\nu \end{array} \right] (z) = \text{rank} \left[ \begin{array}{c} D_\nu \\ N_\nu \end{array} \right] S^{-1} (z) = n_1 \quad \forall z \neq 0. \]  

\[ \text{(B.27)} \]
By Corollary 3.1b(\(\rho\)), \((\tilde{N}_\rho, \tilde{D}_\rho)\) are \(\rho_0\)-r.c. This completes the proof of the comment that \((\tilde{N}_\rho, \tilde{D}_\rho)\) is a \(\rho_0\)-r.r. of \(\tilde{R}\).

Similarly, to prove the concluding comment in Step 5, we have to show that \(\tilde{G} = N_\rho D_\rho^{-1}\) with (a) \(N_\rho \in \tilde{\varphi}_1^\times(\rho_0)^{n_0 \times n_1}\), \(D_\rho \in \tilde{\varphi}_1^\times(\rho_0)^{n_1 \times n_1}\); (b) \(\det D_\rho, P_\rho \in \tilde{\varphi}_1^\times(\rho_0)\); (c) \((N_\rho, D_\rho)\) are \(\rho_0\)-r.c. By (3.20)-(3.29),

\[
\tilde{G} = \tilde{R} + \tilde{Q} = (\tilde{N}_\rho + \tilde{P}_\rho \tilde{D}_\rho)^{-1} = N_\rho D_\rho^{-1}.
\]

(a) Furthermore, by the closure properties of \(\tilde{\varphi}_1^\times(\rho_0)\) under addition and multiplication,

\[
N_\rho \in \tilde{\varphi}_1^\times(\rho_0)^{n_0 \times n_1}, \quad D_\rho \in \tilde{\varphi}_1^\times(\rho_0)^{n_1 \times n_1}.
\]

(b) Also, \(\det D_\rho = \det \tilde{D}_\rho \in \varphi_1^\times(\rho_0) \subset \tilde{\varphi}_1^\times(\rho_0)\).

(c) Since, by (B.27), \((\tilde{N}_\rho, \tilde{D}_\rho)\) are \(\rho_0\)-r.c., there exist

\[
\tilde{u}_\rho, \tilde{v}_\rho \in \varphi(\rho_0)^{n_1 \times n_0} \subset \tilde{\varphi}_1^\times(\rho_0)^{n_1 \times n_0} \quad \text{and} \quad \tilde{v}_\rho \in \varphi(\rho_0)^{n_1 \times n_1} \subset \tilde{\varphi}_1^\times(\rho_0)^{n_1 \times n_1}
\]

such that

\[
\tilde{u}_\rho + \tilde{v}_\rho \overline{\rho} \tilde{D}_\rho = I_{n_1} \quad \text{in} \quad \varphi(\rho_0)^\times.
\]

Hence

\[
\tilde{u}_\rho (\tilde{N}_\rho + \tilde{P}_\rho \tilde{D}_\rho) + (\tilde{v}_\rho - \tilde{u}_\rho \tilde{Q}) \rho_\rho = I_{n_1} \quad \text{in} \quad \varphi(\rho_0)^\times,
\]

i.e. \((N_\rho, D_\rho)\) is \(\rho_0\)-r.c. because \(U_\rho N_\rho + V_\rho D_\rho = I_{n_1}\), with

\[
U_\rho := \tilde{u}_\rho \in \tilde{\varphi}_1^\times(\rho_0)^{n_1 \times n_0} \quad \text{and} \quad V_\rho := \tilde{v}_\rho - \tilde{u}_\rho \tilde{Q} \in \tilde{\varphi}_1^\times(\rho_0)^{n_1 \times n_1}.
\]

\(\square\)

**Proof of Theorem 3.2.** By assumption, there exist \(\bar{U}_\rho, \bar{V}_\rho \in \tilde{\varphi}_1^\times(\rho_0)^{n_1 \times n_0}\), \(\bar{V}_\rho \in \tilde{\varphi}_1^\times(\rho_0)^{n_1 \times n_1}\) such that
\[ N_r \mathcal{D}_r = \mathcal{D}_r N_r \]  
\[ \mathcal{U}_r N_r + \mathcal{U}_r \mathcal{D}_r = I_{n_1} \]  
\[ \text{and} \quad N_r \mathcal{U}_r \mathcal{D}_r + \mathcal{D}_r \mathcal{U}_r = I_{n_0}. \]

Rewriting (B.28)-(B.30), we obtain

\[
\begin{bmatrix}
\mathcal{U}_r & -\mathcal{U}_r \\
-N_r & \mathcal{D}_r & -\mathcal{U}_r
\end{bmatrix}
\begin{bmatrix}
\mathcal{D}_r & -\mathcal{U}_r \\
-I_{n_1} & \mathcal{X}
\end{bmatrix}
= \begin{bmatrix}
I_{n_1} & \mathcal{X} \\
0 & I_{n_0}
\end{bmatrix}
\]

where \( \mathcal{X} \in \mathcal{B}_{1-r}(\rho_0)^{n_1 \times n_0} \) due to the closure properties of \( \mathcal{B}_{1-r}(\rho_0) \).

Observe that the right-hand side of (B.31) has determinant unity, and

\[ (n_1 + n_0)^2 \times (n_1 + n_0) \]

is thus invertible in \( \mathcal{B}_{1-r}(\rho_0) \). Premultiplying (B.31) with the inverse \( \begin{bmatrix} I_{n_1} & -\mathcal{X} \\ 0 & -I \end{bmatrix} \), we obtain (3.30) with

\[
\mathcal{V}_r := \mathcal{V}_r + \mathcal{X} \mathcal{N}_r, \quad \mathcal{U}_r := \mathcal{U}_r - \mathcal{X} \mathcal{D}_r \\
\mathcal{V}_r := \mathcal{V}_r, \quad \mathcal{U}_r := \mathcal{U}_r.
\]

**Proof of Theorem 3.3.** We restrict the proof to the \( \rho_0 \)-r.r. case:

Define

\[ R := (\mathcal{D}_r)'^{-1} \mathcal{D}_r. \]

Since \( \mathcal{D}_r, \mathcal{D}_r' \in \mathcal{B}_{1-r}(\rho_0)^{n_1 \times n_1} \) and \( \det \mathcal{D}_r' \in \mathcal{B}_{1-s}(\rho_0) \), it follows by Cramer's rule that \( R \in \mathcal{B}(\rho_0)^{n_1 \times n_1} \). Furthermore,

\[
\mathcal{D}_r = \mathcal{D}_r' R \\
N_r = N_r \mathcal{D}_r^{-1} \mathcal{D}_r = N_r (\mathcal{D}_r')^{-1} \mathcal{D}_r = N_r R
\]

and thus (3.32) holds. By the \( \rho_0 \)-r.c. property, there exist matrices \( \mathcal{U}_r, \mathcal{V}_r, \mathcal{U}_r', \mathcal{V}_r' \) with elements in \( \mathcal{B}_{1-r}(\rho_0) \) such that
Postmultiplying (B.32) with $R^{-1}$ and (B.33) with $R$, we obtain

$$U'N' + V'D' = R^{-1}$$  \hspace{1cm} (B.34)

$$U'N + V'D = R.$$  \hspace{1cm} (B.35)

By the closure properties of $\tilde{\mathcal{L}}^{-1}(\varphi_0)$, $R, R^{-1} \in \tilde{\mathcal{L}}^{-1}(\varphi_0)^{n_x \times n_y}$ and thus (3.31) follows. \hfill \Box

Proof of Lemma 4.1.

Case 1: only one pole at $z = 0$, i.e. $v = 1, p_1 = 0$.

$$\tilde{R}(z) = \sum_{i=1}^{m} Z_i z^{-i}.$$  \hspace{1cm} (B.41)

By [Bro 1, Thm. 18-1] or [Kai 1, Lemma 6.5-7], the McMillan degree of $\tilde{R}$ is given by the rank of

$$ H := \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_m \\ Z_2 & Z_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Z_m & 0 & \cdots & 0 \end{bmatrix}. $$  \hspace{1cm} (B.41)

Case 2: only one pole at $p \in \mathbb{C}$.

$$\tilde{R}(z) = \sum_{i=1}^{m} Z_i (z-p)^{-i}.$$  \hspace{1cm} (B.41)

The change of variable $\lambda := z-p$, which defines $\tilde{R}' \in \mathbb{C}_p(\lambda)^{n_x \times n_y}$ by
\( \tilde{R}'(\lambda) := \tilde{R}(z) = \sum_{i=1}^{m} Z_i \lambda^{-i}, \)  

(B.42)

brings us back to Case 1. The conclusion follows by applying Case 1 to \( \tilde{R}'(\lambda) \) to obtain a minimal realization \((A, B, C)\) for \( \tilde{R}'(\lambda) \), thus leading to a minimal realization \((A + pI, B, C)\) for \( \tilde{R}(z) \) of dimension \( r = \text{rank } H \), with \( H \) given in (B.41).

**Case 3:** General case with \( \tilde{R} \) given by

\[
\tilde{R}(z) = \sum_{\alpha=1}^{v} \sum_{i=1}^{m_{\alpha}} Z_{\alpha i} (z-p_{\alpha})^{-i}.
\]

For \( \alpha = 1, 2, \ldots, v \), define

\[
\tilde{R}_{\alpha}(z) := \sum_{i=1}^{m_{\alpha}} Z_{\alpha i} (z-p_{\alpha})^{-i}.
\]

(B.46)

and \( r_{\alpha} := \text{rank } H_{\alpha} \) where \( H_{\alpha} \) is defined as in (4.2). Then by Case 2, \( \tilde{R}_{\alpha} \) has a minimal realization \((A_{\alpha}, B_{\alpha}, C_{\alpha})\) with \( A_{\alpha} \in \mathbb{C}^{r_{\alpha} \times r_{\alpha}} \). Letting

\[
A := \text{diag}(A_1, A_2, \ldots, A_v),
\]

(B.47)

\[
B := [B_1 | B_2 | \cdots | B_v]^T,
\]

and

\[
C := [C_1 | C_2 | \cdots | C_v],
\]

the rank tests show that \((A, B, C)\) is a minimal realization of \( \tilde{R} \), and \( A \in \mathbb{C}^{r \times r} \) with \( r := \sum_{\alpha=1}^{v} r_{\alpha} \). Furthermore, by the block diagonal structure of \( A \), the McMillan degree of \( p_{\alpha} \) as a pole of \( \tilde{R} \) is equal to the dimension of \( A_{\alpha} \), which is equal to \( r_{\alpha} = \text{rank } H_{\alpha} \). \( \square \)
Proof of Theorem 4.1. Parts (a) and (b) will only be proved for the 
ρ₀-r.r. case, the ρ₀-l.r. case is similar.

(a) (⇐) By assumption, there exist \( U ∈ \mathcal{L}(\rho₀) \) and 
\( V ∈ \mathcal{L}(\rho₀) \) such that

\[
U N + V D = I
\]

Postmultiplying both sides by \( D^{-1} \), and noting \( \tilde{G} = N D^{-1} \), we have

\[
U \tilde{G} + V = D^{-1}
\]

where, for some \( ρ₁ < ρ₀ \), both sides of (B.48) are meromorphic in 
\( D(ρ₁)^C \), and \( U \) and \( V \) are analytic and bounded in \( D(ρ₁)^C \). Hence 
if \( \det D(p) = 0 \), then \( D^{-1} \) is unbounded in any neighborhood about 
p. In view of (B.48), \( \tilde{G} \) must have a pole at \( p \).

(⇒) \( \tilde{G} \) has a pole at \( p ∈ D(ρ₀)^C \) implies that it must be 
unbounded in any neighborhood about \( p \). Since \( \tilde{G} = N D^{-1} \) and 
\( N ∈ \mathcal{L}(\rho₀) \) is analytic and bounded in \( D(ρ₁)^C \) for some 
\( ρ₁ < ρ₀ \), it follows that \( \det D(p) = 0 \).

(b) By Theorem 3.3, if \( (N, D) \) and \( (N', D') \) are two \( ρ₀-r.r. \) of 
\( \tilde{G} \), then there exists \( R ∈ \mathcal{L}(\rho₀) \) invertible in \( \mathcal{L}(\rho₀) \) 
such that \( D = D'R \). Hence

\[
\det D = \det D' \cdot \det R
\]

By Lemma 3.1, \( \det R \) is an invertible element of \( \mathcal{L}(\rho₀) \). Therefore 
\( \det D \) and \( \det D' \) are equal modulo an invertible element of \( \mathcal{L}(\rho₀) \),
which, by Property (2.3.3), has no zeros in \( D(ρ₀)^C \). Hence, to prove 
Theorem 4.1(b), it suffices to show that it holds for a particular 
\( ρ₀-r.r. \). In particular, we choose the \( ρ₀-r.r. \) \( (N, D) \) as given in
Procedure 3.2(2). Then, for $p \in \mathcal{D}(\rho_0)^c$,

McMillan degree of $p$ as a pole of $\tilde{G}

= \text{McMillan degree of } p \text{ as a pole of } \tilde{R} \text{ given in (3.20)}

(by Remark 4.1(ii))

= \text{order of } p \text{ as a zero of } \det D_r \in \mathbb{C}[z]

(by right-coprime polynomial matrix factorization)

= \text{order of } p \text{ as a zero of } \det \tilde{D}_r \in \mathcal{R}(\rho_0)

(by definition of $\tilde{D}_r$)

= \text{order of } p \text{ as a zero of } \det D_r

(since $D_r = \tilde{D}_r$).

(c) Consider $\tilde{r} := \det D_r / \det D_{\underline{L}}$. Since $\det D_r$ and $\det D_{\underline{L}}$ both belong to $\tilde{\mathcal{L}}_1^{-\infty}(\rho_0)$, it follows that $\tilde{r}$ is an invertible element of $\tilde{b}(\rho_0) = [\tilde{x}_1^{-\infty}(\rho_0)]^{-1}$. Furthermore, by Part (b), $\tilde{r}$ has neither zeros nor poles in $\mathcal{D}(\rho_0)^c$. Hence $\tilde{r}$ and $\tilde{r}^{-1}$ belong to $\tilde{\mathcal{L}}_1^{-\infty}(\rho_0)$.

Proof of Lemma 4.2. We give here the proof for (4.13); the proof for (4.14) is similar. To prove that $(LE, R^{-1}\psi_r)$ is a $\rho_0$-r.r. of $\tilde{G}$, we need to show that $\tilde{G} = LE(R^{-1}\psi_r)^{-1}$; (ii) $LE \in \tilde{x}_1^{-\infty}(\rho_0)^{n_0 \times n_1}$ and $R^{-1}\psi \in \tilde{x}_1^{-\infty}(\rho_0)^{n_1 \times n_1}$; (iii) $\det(R^{-1}\psi_r) \in \tilde{x}_1^{-\infty}(\rho_0)$; (iv) $(LE, R^{-1}\psi_r)$ are $\rho_0$-r.c. Now condition (i) follows from (4.8), (4.10)-(4.12); condition (ii) holds because $L \in \tilde{x}_1^{-\infty}(\rho_0)^{n_0 \times n_0}$ and $R \in \tilde{x}_1^{-\infty}(\rho_0)^{n_1 \times n_1}$ are unimodular; (iii) $R$ being unimodular also implies that

$\det(R^{-1}\psi_r) = (\det R)^{-1} \cdot \prod_{i=1}^{r} \psi_i \in \tilde{x}_1^{-\infty}(\rho_0)$; (iv) by construction of $E$ and $\psi_r$ in (4.10) and (4.11),
rank \[
\begin{bmatrix}
- E \\
- \psi_{\mathcal{H}}
\end{bmatrix}
\](z) = n_i \quad \forall z \in D(\rho_0)^C,
\]
hence
rank \[
\begin{bmatrix}
LE \\
R^{-1}\psi_{\mathcal{H}}
\end{bmatrix}
\](z) = rank \[
\begin{bmatrix}
L & 0 \\
0 & R^{-1}
\end{bmatrix}
\begin{bmatrix}
E \\
\psi_{\mathcal{H}}
\end{bmatrix}
\](z) = n_i \quad \forall z \in D(\rho_0)^C,
\]
and thus \((LE,R^{-1}\psi_{\mathcal{H}})\) are \(\rho_0\)-r.c. in view of Corollary 3.1b(\(\alpha\)).

**Proof of Theorem 4.3.** By Lemma 4.2, \((LE,R^{-1}\psi_{\mathcal{H}})\) is a \(\rho_0\)-r.r. Next, we note that \(\det(R^{-1}\psi_{\mathcal{H}}) = (\det R)^{-1}\tilde{x}_G\); hence \(\tilde{x}_G = \det(R^{-1}\psi_{\mathcal{H}})\) modulo an invertible element of \(\tilde{x}_1(\rho_0)\), which has neither zero nor pole in \(D(\rho_0)^C\). The proof is complete by invoking Theorem 4.1.

**Proof of Theorem 4.4.** Let \(\tilde{G} = N_{\mathcal{H}}D_{\mathcal{H}}^{-1}\) be a \(\rho_0\)-r.r.

\((\Rightarrow)\) Let \(p \in D(\rho_0)^C\) be a pole of \(\tilde{G}\). By Theorem 4.1, \(\det \mathcal{D}_{\mathcal{H}}(p) = 0\). Hence there is a nonzero \(\xi \in \mathcal{C}^{n_i}\) such that
\[
\mathcal{D}_{\mathcal{H}}(p)\xi = 0_{n_i}.
\]
Choose the input \(\tilde{e}(z) := \mathcal{D}_{\mathcal{H}}(z)\tilde{\xi}z/(z-p)\). Then since \(\mathcal{D}_{\mathcal{H}} \in \tilde{x}_1(\rho_0)^{n_i \times n_i}\), we apply Theorem A.1 to the term \(\mathcal{D}_{\mathcal{H}}(z)\tilde{\xi}/(z-p)\) and get
\[
\tilde{e}(z) = \mathcal{D}_{\mathcal{H}}(z)\tilde{\xi}z/(z-p)
= \mathcal{D}_{\mathcal{H}}(p)\tilde{\xi}(z-p) + z\tilde{e}_e(z)
= z\tilde{e}_e(z) \quad \text{by (B.51)}
\]
and hence \(e \in \tilde{x}_1(\rho_1)^{n_i}\) for some \(\rho_1 \in ]0,\rho_0[\). Next we calculate the output and apply Theorem A.1 to the term \(N_{\mathcal{H}}(z)\tilde{\xi}/(z-p)\):
\[ y(z) = N(y)D(y)^{-1}e(z) \]
\[ = N(y)\xi(z) - \frac{z}{(z-p)} \]
\[ = N(p)\xi(z) - \frac{z}{(z-p)} + z\xi_y(z) \] (B.53)

where, by defining \( \tilde{h}(z) := z\xi_y(z) \),
\[ h \in \mathbb{L}_1(-\rho_2)^{n_0} \]

for some \( \rho_2 \in [0,\rho_0[ \). Note that since \( (N, D) \) is \( \rho_0 \)-r.c., then by Corollary 3.1b(\( \rho \)), \( \text{rank}[D(p)\xi | N(p)\xi]^T = n_i \) and thus
\[ \gamma := N(p)\xi \neq \theta_{n_0} \cdot \]

(\( \Leftarrow \)) By contradiction: If \( p \in D(\rho_0)^C \) is not a pole of \( \tilde{G} \), then for any input \( u \in \mathbb{L}_1(-\rho_1)^{n_i} \) (\( \rho_1 < \rho_0 \)), \( \tilde{G}e \) is analytic at \( p \) and hence the output cannot contain a term of the form \( \gamma \cdot p^k \). \( \square \)

**Proof of Lemma 4.3.** Consider the McMillan form \( M[\tilde{G}] \) of \( \tilde{G} \) in (4.8)-(4.14). By Remark 4.4, since multiplication by unimodular matrices does not affect the rank of a matrix at any point in \( D(\rho_0)^C \), hence \( \forall z \in D(\rho_0)^C \)
\[ \text{rank}[N(z)] = \text{rank}[LE](z) = \text{rank}[E(z)] \]
\[ = \text{rank}[ER](z) = \text{rank}[N(z)] \]. \( \square \)

**Proof of Theorem 4.5.** Let \( \tilde{G} = D^{-1}N \) be a \( \rho_0 \)-l.r.

(a) Since \( (D, N) \) is \( \rho_0 \)-l.c., there exist \( V_\ell \in \mathbb{L}_1(-\rho_0)^{n_0 \times n_0} \) and \( U_\ell \in \mathbb{L}_1(-\rho_0)^{n_1 \times n_0} \) such that
\[ (D_\ell V_\ell + N_\ell U_\ell)(z) = I_{n_0} \forall z \in D(\rho_0)^C \] (B.64)
By (4.22), \( \text{rank}[N_\xi(z_0)] < n_1 \). Hence there is a nonzero \( \xi \in \xi^n \) such that
\[
N_\xi(z_0)\xi = \theta_{n_0}.
\] (B.65)

Now, since \( N_\xi(z)\xi \in \tilde{X}_1(-\rho_0)^n \), by applying Theorem A.1 to the term \( N_\xi(z)\xi/(z-z_0) \), we obtain
\[
N_\xi(z)\xi\frac{z}{(z-z_0)} = N_\xi(z_0)\xi\frac{z}{(z-z_0)} + z\xi(z)
\]
\[
= z\xi(z) \quad \text{by (B.65)}
\] (B.66)

and
\[
z \mapsto z\xi(z) \in \tilde{X}_1(-\rho_3)^n \quad \text{for some } \rho_3 \in ]0,\rho_0[ .
\] (B.67)

Choose
\[
m(z) := -U_\xi(z) \cdot z\xi(z).
\] (B.68)

By (B.67) and since \( U_\xi \in \tilde{X}_1(-\rho_0)^{n_1 \times n_0} \),
\[
m \in \tilde{X}_1(-\rho_1)^{n_1} \quad \text{for some } \rho_1 \in ]0,\rho_0[ .
\]

Using the input defined by (4.24), we calculate
\[
\tilde{y}(z) = \tilde{G}(z)\tilde{e}(z)
\]
\[
= D_\xi(z)^{-1}[z\xi(z) - N_\xi(z)U_\xi(z) \cdot z\xi(z)] \quad \text{by (B.66)}
\]
\[
= D_\xi(z)^{-1}D_\xi(z)V_\xi(z) \cdot z\xi(z) \quad \text{by (B.64)}
\]
\[
= V_\xi(z) \cdot z\xi(z) .
\] (B.69)

So, by (B.67) and since \( V_\xi \in \tilde{X}_1(-\rho_0)^{n_0 \times n_0} \),
\[
y \in \tilde{X}_1(-\rho_2)^{n_0} \quad \text{for some } \rho_2 \in ]0,\rho_0[ .
\]

(b) With the input \( e \) in (4.26), the output is given by
\[
\tilde{y}(z) = \tilde{G}(z) \cdot \frac{z}{(z-\nu)}
\]
\[ = \tilde{g}(v)\xi \frac{z}{(z-v)} + D_\ell(z)^{-1}[N_\ell(z) - D_\ell(z)D_\ell(v)^{-1}N_\ell(v)]\xi \frac{z}{(z-v)}. \] (B.70)

Since \( v \) is neither a zero nor a pole of \( \tilde{g} \),

\[ \tilde{g}(v)\xi \neq \theta_{n_0} \]

and the second term of (B.70) is analytic at \( z = v \). (This follows because \( v \in D(\rho_0)^c \) belongs to the domain of analyticity of \( D_\ell, N_\ell \), and \( D_\ell^{-1} \).)

Proof of Theorem 4.6. Let \( G = D^{-1}_\ell N_\ell \) be a \( \rho_0 \)-l.r. By (4.22), \( \text{rank}[N_\ell(z_0)] < n_0 \), hence there exists a nonzero \( \gamma \in \mathbb{C}^{n_0} \) such that

\[ \gamma^* N_\ell(z_0) = \theta_{n_0}. \] (B.71)

Define

\[ \gamma^* := \gamma^* D_\ell(z_0). \] (B.72)

Since \( (D_\ell, N_\ell) \) is \( \rho_0 \)-l.c., then by (B.71) and Corollary 3.1b(\( \ell \)),

\[ \eta \neq \theta_{n_0}. \]

Now given any \( \xi \in \mathbb{C}^{n_0} \), choose \( \pi \in \kappa(\rho_0) \) (the choice of \( \pi(z) \) will be specified below, see (B.84)), and consider the input (4.28) with \( m(z) := \pi(z)\xi \), i.e.

\[ \tilde{e}(z) = [\frac{z}{(z-z_0)} + \pi(z)]\xi. \] (B.73)

Hence

\[ \eta^* \tilde{y}(z) = \gamma^* D_\ell(z_0)D_\ell(z)^{-1}N_\ell(z)\xi[\frac{z}{(z-z_0)} + \pi(z)]. \] (B.74)

Consider

\[ \tilde{g} := \gamma^* D_\ell(z_0)D_\ell^{-1}N_\ell \xi \in \tilde{b}(\rho_0), \] (B.75)

hence, by Theorem 2.1, \( \tilde{g} \) admits a normalized \( \rho_0 \)-representation
\[ g = n/d \]  
\hspace{1cm} (B.76)

with \( n \in \tilde{k}_1_-(\rho_0) \) and \( d \in \pi_\infty(\rho_0) \). By (B.71), \( \tilde{g}(z_0) = 0 \). Hence by \( \rho_0 \)-coprimeness

\[ n(z_0) = 0 , \quad d(z_0) \neq 0 . \]  
\hspace{1cm} (B.77)

Applying Theorem A.1 to the term \( n(z)/(z-z_0) \) and using (B.77), we obtain

\[ n(z) = \frac{n(z_0)}{(z-z_0)} z + z\tilde{v}(z) = z\tilde{v}(z) \]  
\hspace{1cm} (B.78)

where \( z \mapsto z\tilde{v}(z) \in \tilde{k}_1_-(\rho_1) \) for some \( \rho_1 \in ]0,\rho_0[ \). (B.79)

Using (B.75), (B.76) and (B.78) in (B.74), we obtain

\[ n\tilde{y}(z) = z\tilde{v}(z) \cdot \left[ 1 + \frac{(z-z_0)}{z} \pi(z) \right]/d(z) . \]  
\hspace{1cm} (B.80)

We will show next that \( \pi \in \pi(\rho_0) \) can be chosen so that the second factor of (B.80) is a constant for all \( z \). Once this is done, the conclusion (4.29) of the theorem follows.

Since \( d \in \pi_\infty(\rho_0) \), let

\[ d(z) = a(z)/b(z) \]  
\hspace{1cm} (B.81)

where \( a, b \in \mathbb{C}[z] \) are coprime polynomials such that \( b \) has no zeros in \( D(\rho_0)^c \) and \( \exists a = \exists b \). Since (B.77) holds, we can pick \( \alpha \in \mathbb{C} \) such that

\[ -1 + \alpha \cdot d(z_0) = 0 , \]  
\hspace{1cm} (B.82)

hence \( (z-z_0) \) divides \([-b(z) + \alpha \cdot a(z)]\). Then with

\[ p(z) := \frac{[-b(z) + \alpha \cdot a(z)]/(z-z_0)}{b(z)} , \]  
\hspace{1cm} (B.83)

\( p \in \mathbb{C}[z] \) and \( \exists p < \exists b \). Now define
\[ \pi(z) := zp(z)/b(z) , \]  

(B.84)

hence \( \pi \in \mathcal{H}(\rho_0) \). With this choice of \( \pi \), the right-hand side of (B.80) reduces to \( \alpha \cdot z\tilde{\nu}(z) \).

Proof of Lemma 5.1. (5.14) \( \Leftrightarrow \) (5.15) \( \Leftrightarrow \) (5.16). Immediate by Theorem 4.1 because \( (N_r,D) \) is a \( \rho_0 \)-r.r. of \( \tilde{G}_r \) and \( (D,N_L) \) is a \( \rho_0 \)-l.r. of \( \tilde{G}_L \).

(5.16) \( \Rightarrow \) (5.17). Since \( (N_r,D) \) is \( \rho_0 \)-r.c., there exist matrices \( U_r, V_r \) with elements in \( \tilde{\mathcal{L}}_r(\rho_0) \) such that

\[ U_rN_r + V_rD = I . \]  

(B.90)

Postmultiply (B.90) by \( \tilde{G}_L := D^{-1}N_L \),

\[ U_r\tilde{G}_L + V_rL = \tilde{G}_L . \]  

(B.91)

Since all elements of \( U_r, V_r \) and \( N_L \) are bounded in \( D(\rho_1)^C \) for some \( \rho_1 < \rho_0 \), \( \tilde{G}_L \) has a pole at \( p \in D(\rho_0)^C \) implies that \( \tilde{G} \) has a pole at \( p \).

(5.16) \( \Leftarrow \) (5.17). Since \( \tilde{G} = N_r\tilde{G}_L \) and all elements of \( N_r \) are bounded in \( D(\rho_1)^C \) for some \( \rho_1 < \rho_0 \), \( \tilde{G} \) has a pole at \( p \in D(\rho_0)^C \) implies that \( \tilde{G}_L \) has a pole at \( p \).

Proof of Theorem 5.1. (\( \Leftarrow \)) Since \( u \) has support \( \{0\} \), then \( \tilde{u}(z) \equiv u_0 \) \( \forall z \in \mathcal{C} \). Since \( y(k) \) is \( O(k^{m-1}\sigma^k) \) for large \( k \), hence \( \tilde{y} = \tilde{G}\tilde{u} \) must have a pole of order \( m \) at some \( p \), where \( |p| = \sigma \geq \rho_0 \). Thus \( \tilde{G} \) must have a pole at \( p \) of order at least \( m \). The conclusion follows by Lemma 5.1.

(\( \Rightarrow \)) Assume \( \tilde{x}(p) = 0 \), with \( |p| = \sigma > \rho_0 \). By Lemma 5.1, \( p \) is a pole of \( \tilde{G} \). Let \( m \) be the order of \( p \) as a pole of \( \tilde{G} \). Then
there exists some \( u_0 \in \mathbb{C}^n \) such that \( \tilde{G}u_0 \) has a pole at \( p \) of order \( m \). Choose an input sequence \( u := u_0 \cdot \delta_0 \in (\mathbb{C}^N)^n \). Then the output sequence \( y \) satisfies \( \tilde{y} = \tilde{G}u_0 \), which has a pole of order \( m \) at \( p \).

Compute the Laurent expansion of \( \tilde{y} \) at \( p 
\[
\tilde{y}(z) = \frac{\xi_m}{(z-p)^m} + \frac{\xi_{m-1}}{(z-p)^{m-1}} + \ldots + \xi_0 + \xi_{-1}(z-p) + \ldots .
\]

Hence, for large \( k \), \( y(k) \) includes a term \( \xi_m \cdot \binom{k-1}{m-1} p^{k-m} \), which is \( O(k^{m-1}p^k) \).

Proof of Theorem 5.2. (\( \Rightarrow \)) By contraposition: If there is \( p \in D(p)^c \) such that \( \tilde{x}(p) = 0 \). Then by Lemma 5.1, \( \tilde{G} \) has a pole at \( p \in D(p)^c \).

Hence \( \tilde{G} \) is not bounded in \( D(p)^c \) and cannot belong to \( \tilde{\mathbb{L}}_1(\rho) \circ \mathbb{L} \).

(\( \Leftarrow \)) Since \( \tilde{x}(z) \neq 0 \) for all \( z \in D(p)^c \) and \( \tilde{x} \in \tilde{\mathbb{L}}_1^\infty(\rho_0) \) (i.e. \( \chi(0) \neq 0 \)), hence \( \tilde{x} := \det D \) is an invertible element of \( \tilde{\mathbb{L}}_1(\rho) \) by Property (2.2.6). Furthermore, since \( N_r, D, N_\ell \) all have elements in \( \tilde{\mathbb{L}}_1(\rho_0) \subset \tilde{\mathbb{L}}_1(\rho) \), it follows by Cramer's rule that \( \tilde{G} := N_r D^{-1} N_\ell \) belongs to \( \tilde{\mathbb{L}}_1(\rho) \).

Proof of Theorem 6.1. (i) follows from Lemma 5.1 and (6.9).

(ii) Since \( \tilde{p} \in \tilde{\mathbb{L}}(\rho_0)^{n_0 \times n_1} \), thus by Theorem 3.1

\[
\tilde{p} \text{ has a } \rho_0 \text{-r.r. } (N_{pr}, D_{pr}). \quad \text{ (B.100)}
\]

Hence by applying Theorem 4.1(c) to (6.12) and (B.100), there exists \( \tilde{r} \in \tilde{\mathbb{L}}_1(\rho_0) \) invertible in \( \tilde{\mathbb{L}}_1(\rho_0) \) such that

\[
\det D_{pr} = \tilde{r} \cdot \det D_{pl} . \quad \text{ (B.101)}
\]

Recall the terms defined in (6.1)-(6.14), and consider the following matrices in \( \tilde{\mathbb{L}}_1(\rho_0) \):
Then using Corollary 3.1b(α) \((N,D)\) is a \(\rho_0\)-r.r. of
\[
\tilde{\alpha} \in \tilde{b}(\rho_0)^{(n_i+n_0) \times (n_i+n_0)}.
\]
Moreover, \((N,D)\) being \(\rho_0\)-r.c. implies that
\((D,D+N)\) is \(\rho_0\)-r.c. By (B.102)

\[
D+N = \left[
\begin{array}{c|c}
D_{cr} & N_{pr} \\
\hline
-N_{cr} & D_{pr}
\end{array}
\right] = \left[
\begin{array}{c|c}
I & \bar{P} \\
\hline
I+\bar{P}\bar{C} & 0
\end{array}
\right]
\left[
\begin{array}{c|c}
D_{cr} & 0 \\
\hline
0 & I
\end{array}
\right]
\]

and thus

\[
det[D+N] = det[I + \bar{P}\bar{C}] \cdot det D_{pr} \cdot det D_{cr}.
\]

Hence using (6.7), \(det[D+N] \in \tilde{b}_{\rho_0}^{\infty} (\rho_0)\), and so \((D,D+N)\) is a \(\rho_0\)-r.r. of

\[
\tilde{H}_{eu} = (I+\tilde{\alpha})^{-1} = D(D+N)^{-1} \in \tilde{b}(\rho_0)^{(n_i+n_0) \times (n_i+n_0)}.
\]

Similarly, \((J^{-1}N,D+N)\) is a \(\rho_0\)-r.r. of

\[
\tilde{H}_{yu} = J^{-1} \tilde{G}(I+\tilde{\alpha})^{-1} = J^{-1}N(D+N)^{-1} \in \tilde{b}(\rho_0)^{(n_i+n_0) \times (n_i+n_0)}.
\]

Now by (B.103), using (6.12), (6.13) and (B.101)

\[
det[D+N] = det[I + D^{-1}_{pr} N_{pr} D^{-1}_{cr}] \cdot \tilde{r} \cdot det D_{pr} \cdot det D_{cr}
\]

\[
= \tilde{r} \cdot det[D_{pr} D_{cr} + N_{pr} D_{cr}]
\]

\[
= \tilde{r} \cdot \tilde{\gamma} \quad \text{by (6.14)}.
\]

Observe that \(\tilde{r}\) is bounded and bounded away from zero in \(D(\rho_1)^c\), for
some \(\rho_1 < \rho_0\). The conclusion follows by applying Theorem 4.1 to
(B.104) and (B.105), using (B.106).
Proof of Lemma 7.1. (7.5) ⇔ (7.6). This is immediate by (7.3).

(7.4) ⇔ (7.6). This is immediate by the second equation in (7.6).

(7.4) ⇒ (7.5). By (7.3), $(x^P, y^P) := (U_{\mathcal{L}} D, V_{\mathcal{L}} D)$ is a particular solution of (7.4). Hence $(X, Y)$ is a solution of (7.4) if and only if

$$(X^h, Y^h) := (X - U_{\mathcal{L}} D, Y - V_{\mathcal{L}} D) \quad (B.110)$$

is a solution of the homogeneous equation

$$N_{\mathcal{L}} x^h + D_{\mathcal{L}} y^h = 0_{n_0 \times n_0} \quad (B.111)$$

It remains to prove that $(x^h, y^h)$ in (B.110) is equal to $(-D_r N, N_r N)$ for some $N \in \mathcal{L}_{1-}(\rho_0)^{n_1 \times n_0}$. Define

$$N := -D_r^{-1} x^h \in \mathcal{O}(\rho_0)^{n_1 \times n_0} \quad (B.112)$$

Then by (B.111), using (7.2),

$$y^h = -D_r^{-1} N_{\mathcal{L}} x^h = -N_r D_r^{-1} x^h = N_r N \quad (B.113)$$

By (7.3), $V_{\mathcal{L}} D + U_{\mathcal{L}} N = I$; hence postmultiplying by $N$

$$-V_{\mathcal{L}} x^h + U_{\mathcal{L}} y^h = N \quad (B.114)$$

and so $N \in \mathcal{L}_{1-}(\rho_0)^{n_1 \times n_0}$ by the closure properties of $\mathcal{L}_{1-}(\rho_0)$. Thus by (B.110), (B.112) and (B.113), we obtain as required

$$X = U_{\mathcal{L}} D - D_r N, \quad Y = V_{\mathcal{L}} D + N_r N.$$ 

The proof of (7.7) proceeds as follows: ($\Leftarrow$) By Lemma 3.4(4), there exist $U \in \mathcal{L}_{1-}(\rho_0)^{n_0 \times n_1}$, $V \in \mathcal{L}_{1-}(\rho_0)^{n_0 \times n_0}$ such that $UN + VP = I$. Premultiplying (7.6) by $[U \mid V]$, we get
I = (-u_n + vN_L)x + (u_n + vD_L)y.

The matrices in parentheses have elements in $\tilde{\mathcal{L}}_1(\rho_0)$, due to the closure properties of $\tilde{\mathcal{L}}_1(\rho_0)$. Hence $(X,Y)$ are $\rho_0$-r.c. by Lemma 3.4(a).

(7.7)(\Rightarrow) follows by interchanging $(N,D)$ and $(-X,Y)$ in the above argument, and by using (7.5) instead of (7.6).

(b) Since $N_L = D_L G$ and $\det D_L \in \tilde{\mathcal{L}}_1(\rho_0)$
(i.e. \( \lim_{|z| \to \infty} \det D(z) \neq 0 \)),

\[
\lim_{|z| \to \infty} N(z) = 0_{n_0 \times n_1} \quad \text{by (7.8)}.
\]

Now by (7.6), \( D = N(z) + D(z) \in \tilde{\mathcal{L}}_1(\rho_0)^{n_0 \times n_0} \). Hence using (B.115),

\[
\lim_{|z| \to \infty} \det D(z) = \lim_{|z| \to \infty} \det D(z) \cdot \lim_{|z| \to \infty} \det V(z).
\]

Conclusion (7.9) follows because \( \lim_{|z| \to \infty} \det D(z) \neq 0 \). \( \square \)

Lemma B.1. Let \( \rho \in ]0,1[ \) and \( g, u \in \ell_1(\rho) \). Then \( y := g*u \) satisfies \( y_k = o(\rho^k) \) as \( k \to \infty \).

Proof. Since \( g, u \in \ell_1(\rho) \), \( \tilde{g} := (g(k)\rho^{-k})_{k=0}^{\infty} \) and \( \tilde{u} := (u(k)\rho^{-k})_{k=0}^{\infty} \) belong to \( \ell_1 \); hence \( \tilde{y} := \tilde{g} \ast \tilde{u} \in \ell_1 \). By simple calculations, \( \tilde{y} \) and \( y \) are related by \( \tilde{y} = (y(k)\rho^{-k})_{k=0}^{\infty} \). Consequently, since \( \tilde{y} \in \ell_1 \), \( \tilde{y}(k) = y(k)\rho^{-k} \to 0 \) as \( k \to \infty \); hence \( y(k) = o(\rho^k) \) as \( k \to \infty \). \( \square \)

Proof of Theorem 7.1. (i) We first verify Procedure 7.1 step by step:

Steps 1 and 2 are self-explanatory.

Step 3. To show that \( (D(z), N(z)) \) is a \( \rho_0 \)-l.r. of \( \tilde{F} \), note that
\[
D(z) \in \tilde{\mathcal{L}}_1(\rho_0)^{n_0 \times n_0}, \quad N(z) \in \tilde{\mathcal{L}}_1(\rho_0)^{n_0 \times n_1} \quad \text{and} \quad \det D(z) \in \tilde{\mathcal{L}}_1(\rho_0)^{n_0 \times n_0};
\]
\( \tilde{F} = D(z)^{-1} N(z) \). Furthermore, in view of (7.22), \( \text{rank}[D(z) \mid N(z)] = n_0 \), \( Vz \in D(z)^{n_0} \); hence \( (D(z), N(z)) \) is \( \rho_0 \)-l.c.

Step 4 is self-explanatory, in view of Lemma 7.1.

Step 5. To show that \( (N(z), D(z)) \) is a \( \rho_0 \)-r.r. of \( \tilde{C} \), it is immediate by definition that \( \tilde{C} = N(z)D(z)^{-1} \), where \( N(z) \in \tilde{\mathcal{L}}_1(\rho_0)^{n_1 \times n_0} \) and
\[
D(z) \in \tilde{\mathcal{L}}_1(\rho_0)^{n_0 \times n_0}.
\]
Moreover, by (7.32), (7.33) and Lemma 7.1(a)
\[ D = N_{\mathcal{L}} X + D_{\mathcal{L}} Y \]
\[ = N_{p\mathcal{L}} X + \frac{\Phi}{d} p\mathcal{L} Y; \quad (B.117) \]
hence by (7.11) and Lemma 7.1(b), \( \det D_{\mathcal{L}} = \det(\frac{\Phi}{d}) \in \Omega_{1}^\infty(\rho_0) \). Lastly, it remains to show that \( (N_{\mathcal{L}}, D_{\mathcal{L}}) \) are \( \rho_0 \)-r.c.: by (B.117) and (7.16),
\[ D(z) = N_{p\mathcal{L}}(z) X(z) \quad \forall z \in \mathbb{Z}[\phi_u] \cup \mathbb{Z}[\phi_w]; \quad (B.118) \]
by (7.25), \( \det D(z) \neq 0, \quad \forall z \in \mathbb{Z}[\phi_u] \cup \mathbb{Z}[\phi_w] \subset D(1)^C \);
hence by (B.118),
\[ \text{rank}[X(z)] = n_0 \quad \forall z \in \mathbb{Z}[\phi_u] \cup \mathbb{Z}[\phi_w]; \quad (B.119) \]
since \( (X,Y) \) are \( \rho_0 \)-r.c. by Lemma 7.1, then using (B.119)
\[ \text{rank} \left[ \begin{array}{c} X \\ \Phi_{d} \end{array} \right](z) = n_0 \quad \forall z \in D(\rho_0)^C; \quad (B.120) \]
hence according to Corollary 3.1b(\( \tau \)), \( (N_{\mathcal{L}}, D_{\mathcal{L}}) \) is \( \rho_0 \)-r.c.

Throughout the procedure, all matrices concerned have elements corresponding to sequences in \( \mathbb{Q}^\infty \): this property is preserved in \( \mathcal{C} \).

(ii) By (6.14), (7.35) and (B.117), the characteristic function of the feedback system \( S \) is
\[ \tilde{\chi} = \det D; \quad (B.121) \]
hence by (7.25), \( \mathbb{Z}[\tilde{\chi};D(\rho_0)^C] = \Lambda \) and condition (b) of Problem (SP) is satisfied. Furthermore by definition of \( \Lambda \), \( \tilde{\chi}(z) \neq 0 \quad \forall z \in D(1)^C \);
hence condition (a) is also satisfied in view of Remark 6.2. To show that condition (c) of problem (SP) holds, we calculate first the transfer functions for the maps \( u \mapsto e_s \) and \( w \mapsto e_s \), respectively,
\[
\tilde{H}_e s u_S = [I + \tilde{P} \tilde{C}]^{-1}
\]
\[
= P_c [P_{cl} P_{cr} + N_{pl} N_{cr}]^{-1} P_{pl}
\]
\[
= \frac{d}{v} P^{-1} P_{pl},
\]
(B.122)

and
\[
\tilde{H}_e s w_p = -[I + \tilde{P} \tilde{C}]^{-1} \tilde{P}
\]
\[
= -P_c [P_{cl} P_{cr} + N_{pl} N_{cr}]^{-1} N_{pl}
\]
\[
= -\frac{d}{v} P^{-1} N_{pl}.
\]
(B.123)

Since the list \( \Lambda \) is finite, there exists some \( \rho \in \rho_0 \) such that \( \det \, D(z) \neq 0 \) \( \forall z \in D(\rho)^c \); hence by applying Corollary 5.2b to \( P^{-1} \in \mathcal{B}(\rho_0)^{n_0 \times n_0} \), we have \( P^{-1} \in \mathcal{I}_1(\rho)^{n_0 \times n_0} \); consequently
\[
v P^{-1} P_{pl} \in \mathcal{I}_1(\rho)^{n_0 \times n_0}, \quad -v P^{-1} N_{pl} \in \mathcal{I}_1(\rho)^{n_0 \times n_1}.
\]
(B.124)

Also, by construction of \( \phi \) and \( d \),
\[
\frac{d}{d} u_S = \phi \frac{v}{d} u_S \in \mathcal{I}_1(\rho_0)^{n_0} \subset \mathcal{I}_1(\rho)^{n_0}
\]
(B.125)

and
\[
\frac{d}{d} \tilde{w}_p = \phi \frac{v}{d} \tilde{w}_p \in \mathcal{I}_1(\rho_0)^{n_1} \subset \mathcal{I}_1(\rho)^{n_1}.
\]
(B.126)

Now, for arbitrary \( v_u \) and \( v_w \) satisfying (7.13) and (7.14)
\[
\tilde{e}_e = \tilde{H}_e s u_S \tilde{u}_S + \tilde{H}_e s w_p \tilde{w}_p
\]
\[
= [v P^{-1} P_{pl}] [\frac{d}{d} u_S] + [-v P^{-1} N_{pl}] [\frac{d}{d} \tilde{w}_p] .
\]
(B.127)

Applying Lemma B.1 to (B.127), using (B.124), (B.125) and (B.126), we have
\[
e_e(k) = o(\rho^k) \text{ as } k \to \infty.
\]
(B.128)

To check condition (d), consider any perturbed plant \( \tilde{P} \in \mathcal{B}(\rho_0)^{n_0 \times n_1} \) satisfying (7.10), and for which the feedback system \( S \) with controller
\( \tilde{c} \) and plant \( \tilde{p} \) has transfer functions \( \tilde{H}_{eu}, \tilde{H}_{yu} \) that are \( L_p \)-stable, \( \forall p \in [1, \infty] \). By Theorem 3.1 \( \tilde{p} \) admits a \( \rho_0 \)-l.r. \( (\tilde{p}_p, \tilde{N}_p) \) and the characteristic function \( \tilde{\chi} \) becomes

\[
\tilde{\chi} = \det \tilde{\mathcal{D}}
\]

where

\[
\tilde{\mathcal{D}} := \tilde{p}_p \mathcal{D} + \tilde{N}_p \mathcal{N} \in \tilde{\mathcal{X}}_{1-}(\rho_0)^{n \times n}.
\]

By Corollary 5.2a, the \( L_p \)-stability of the perturbed feedback system implies that \( \tilde{\chi}(z) \neq 0 \ \forall z \in D(1)^c \) and, since \( \tilde{\chi} \in \tilde{\mathcal{X}}_{1-}(\rho_0) \) and \( D(1) \) is compact, there is a \( \tilde{\rho} \in [\rho_0, 1] \) such that \( \det \tilde{\mathcal{D}}(z) = \tilde{\chi}(z) \neq 0 \), \( \forall z \in D(\tilde{\rho})^c \). Calculating as above, we obtain that

\[
\tilde{H}_{es}^{u_s} = \frac{\phi_v}{d} \tilde{\mathcal{D}}^{-1} \tilde{p}_p \quad \text{and} \quad \tilde{H}_{ew}^{s_p} = -\frac{\phi_v}{d} \tilde{\mathcal{N}}_p^{-1} \tilde{N}_p
\]

where \( \tilde{\mathcal{D}}^{-1} \tilde{p}_p \in \tilde{\mathcal{X}}_{1}(\tilde{\rho})^{n \times n} \), \( -\tilde{\mathcal{N}}_p^{-1} \tilde{N}_p \in \tilde{\mathcal{X}}_{1}(\tilde{\rho})^{n \times n} \).

The arguments of (B.125)-(B.128) can be repeated here and thus condition (d) of problem (SP) is satisfied.

Proof of Theorem 8.1. The proof of this theorem with the general \( A, B \) notation can be found in [Des 5]. Just to demonstrate this theorem in terms of the system descriptions we are now concerned with, we give the proof for equivalence (8.15):

\[ (\Rightarrow) \] (8.13) and \( \tilde{Q} \in \tilde{X}_{1s}^{m \times m} \) imply that all elements of \( \tilde{H}_{yu} \) (see (8.9a)) are in \( \tilde{X}_{1s}^{m \times m} \). Since \( \tilde{p} \) is in the radical \( \tilde{b}_s(\rho_0)^{m \times m} \), \( 1 - \tilde{p}\tilde{Q} \) has an inverse in \( \tilde{b}_s(\rho_0)^{m \times m} \); and since \( \tilde{Q} \) is in \( \tilde{X}_{1s}^{m \times m} \), (8.9b) shows that \( \tilde{c} \) is in the radical \( \tilde{b}_s(\rho_0)^{m \times m} \).

\[ (\Leftarrow) \] This is immediate since \( \tilde{Q} \) is a submatrix of \( \tilde{H}_{yu} \).
Proof of Theorem 8.3. It suffices to show that $\tilde{H}_{yS}u_S$ and $\tilde{Q}$ satisfy Theorem 8.2. We note that since $\tilde{p} \in \mathbb{H}^{m \times m}$, then in (8.21)

$$\text{ord}_{c_j} [\tilde{p}^{-1}] \leq 0 , \ j = 1,2,...,m .$$

Hence in (8.19), $\tilde{H}_{yS}u_S \in \mathbb{H}(1)^{m \times m} \cap \mathbb{R}(s)^{m \times m} \subset \mathbb{H}^{m \times m}$. Furthermore, $\tilde{Q} := \tilde{p}^{-1}\tilde{H}_{yS}u_S \in \mathbb{R}^N(1)^{m \times m}$. Since all poles of $\tilde{p}^{-1}$ in $D(p_0)^C$ are zeros of $\tilde{p}$, and these poles are cancelled by $\tilde{n}_j , \ j = 1,2,...,m$ in (8.18), hence $\tilde{Q}$ is analytic in $D(1)^C$. By Remark 2.2(ii),

$$\tilde{Q} \in \mathbb{H}^{m \times m} \cap (\mathbb{R}^N)^{m \times m} . \ \square$$
References


Fig. 5.1. Example of an interconnected system.
Fig. 5-2. Interconnected system of Fig. 5-1 with coprime factorization for each subsystem.
Fig. 5-3. Model for each individual system.
Fig. 6-1. Feedback system $S$ with plant $\tilde{P}$ and controller $\tilde{C}$. 
Fig. 9-1. General feedback system.